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# On the Castelnuovo-Mumford regularity of the cohomology of fusion systems and of the Hochschild cohomology of block algebras

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## Abstract

Symonds' proof of Benson's regularity conjecture implies that the regularity of the cohomology of a fusion system and that of the Hochschild cohomology of a  $p$ -block of a finite group is at most zero. Using results of Benson, Greenlees, and Symonds, we show that in both cases the regularity is equal to zero.

Let  $p$  be a prime and  $k$  an algebraically closed field of characteristic  $p$ . Given a finite group  $G$ , a *block algebra* of  $kG$  is an indecomposable direct factor  $B$  of  $kG$  as a  $k$ -algebra. A *defect group* of a block algebra  $B$  of  $kG$  is a minimal subgroup  $P$  of  $G$  such that  $B$  is isomorphic to a direct summand of  $B \otimes_{kP} B$  as a  $B$ - $B$ -bimodule. The defect groups of  $B$  form a  $G$ -conjugacy class of  $p$ -subgroups of  $G$ . The Hochschild cohomology of  $B$  is the algebra  $HH^*(B) = \text{Ext}_{B \otimes_k B^{\text{op}}}^*(B)$ , where  $B^{\text{op}}$  is the opposite algebra of  $B$ , and where  $B$  is regarded as a  $B \otimes_k B^{\text{op}}$ -module via left and right multiplication. By a result of Gerstenhaber, the algebra  $HH^*(B)$  is graded-commutative; that is, for homogeneous elements  $\zeta \in HH^m(B)$  and  $\eta \in HH^n(B)$  we have  $\eta\zeta = (-1)^{nm}\zeta\eta$ , where  $m, n$  are nonnegative integers. In particular, if  $p = 2$ , then  $HH^*(B)$  is commutative, and if  $p$  is odd, then the even part  $HH^{\text{ev}}(B) = \bigoplus_{n \geq 0} HH^{2n}(B)$  is commutative and all homogeneous elements in odd degrees square to zero. The extension of the Castelnuovo-Mumford regularity to graded-commutative rings with generators in arbitrary positive degrees is due to Benson [2, §4]. We follow the notational conventions in Symonds [18]. In particular, if  $p$  is odd and  $T = \bigoplus_{n \geq 0} T^n$  is a finitely generated graded-commutative  $k$ -algebra and  $M$  a finitely generated graded  $T$ -module, we denote by  $\text{reg}(T, M)$  the Castelnuovo-Mumford regularity of  $M$  as a graded  $T^{\text{ev}}$ -module, where  $T^{\text{ev}} = \bigoplus_{n \geq 0} T^{2n}$  is the even part of  $T$ . We set  $\text{reg}(T) = \text{reg}(T, T)$ ; that is,  $\text{reg}(T)$  is the Castelnuovo-Mumford regularity of  $T$  as a graded  $T^{\text{ev}}$ -module. See also [3] and [8] for more background material and references. We note that Benson's definition of regularity uses the ring  $T$  instead of  $T^{\text{ev}}$ , but the two definitions are equivalent. This can be seen by noting that [18, Proposition 1.1] also holds for finitely generated graded commutative  $k$ -algebras.

**Theorem 0.1** *Let  $G$  be a finite group and  $B$  a block algebra of  $kG$ . We have  $\text{reg}(HH^*(B)) = 0$ .*

This will be shown as a consequence of a statement on Scott modules. Given a finite group  $G$  and a  $p$ -subgroup  $P$  of  $G$ , there is up to isomorphism a unique

indecomposable  $kG$ -module  $Sc(G; P)$  with vertex  $P$  and trivial source having a quotient (or equivalently, a submodule) isomorphic to the trivial  $kG$ -module  $k$ . The module  $Sc(G; P)$  is called the *Scott module of  $kG$  with vertex  $P$* . It is constructed as follows: Frobenius reciprocity implies that  $\text{Hom}_{kG}(\text{Ind}_P^G(k), k) \cong \text{Hom}_{kP}(k, k) \cong k$ , and hence  $\text{Ind}_P^G(k)$  has up to isomorphism a unique direct summand  $Sc(G; P)$  having  $k$  as a quotient. Since  $\text{Ind}_P^G(k)$  is selfdual, the uniqueness of  $Sc(G; P)$  implies that  $Sc(G; P)$  is also selfdual, and hence  $Sc(G; P)$  can also be characterised as the unique summand, up to isomorphism, of  $\text{Ind}_P^G(k)$  having a nonzero trivial submodule. Moreover, it is not difficult to see that  $Sc(G; P)$  has  $P$  as a vertex. See [7] for more details on Scott modules, as well as [11] for connections between Scott modules and fusion systems. For a finitely generated graded module  $X$  over  $H^*(G; k)$  we denote by  $H_m^{*,*}(X)$  the local cohomology with respect to the maximal ideal of  $H^*(G; k)$  generated by all elements in positive degree. The first grading is here the local cohomological grading, and the second is induced by the grading of  $X$ .

**Theorem 0.2** *Let  $G$  be a finite group and  $P$  a  $p$ -subgroup of  $G$ . We have*

$$\text{reg}(H^*(G; k); H^*(G; Sc(G; P))) = 0 .$$

**Remark 0.3** Using Benson's reinterpretation in [1, §4], of the 'last survivor' from [5, §7], applied to the Scott module instead of the trivial module, one can show more precisely that

$$H_m^{r,-r}(H^*(G; Sc(G, P))) \neq \{0\} ,$$

where  $r$  is the rank of  $P$ . It is not clear whether this property, or even the property of having cohomology with regularity zero, characterises Scott modules amongst trivial source modules.

For  $\mathcal{F}$  a saturated fusion system on a finite  $p$ -group  $P$ , we denote by  $H^*(P; k)^{\mathcal{F}}$  the graded subalgebra of  $H^*(P; k)$  consisting of all elements  $\zeta$  satisfying  $\text{Res}_Q^P(\zeta) = \text{Res}_\varphi(\zeta)$  for any subgroup  $Q$  of  $P$  and any morphism  $\varphi : Q \rightarrow P$  in  $\mathcal{F}$ . If  $\mathcal{F}$  is the fusion system of a finite group  $G$  on one of its Sylow- $p$ -subgroups  $P$ , then  $H^*(P; k)^{\mathcal{F}}$  is isomorphic to  $H^*(G; k)$  through the restriction map  $\text{Res}_P^G$ , by the characterisation of  $H^*(G; k)$  in terms of stable elements due to Cartan and Eilenberg. In that case we have  $\text{reg}(H^*(P; k)^{\mathcal{F}}) = 0$  by [18, Corollary 0.2]. If  $\mathcal{F}$  is the fusion system of a block algebra  $B$  of  $kG$  on a defect group  $P$ , then  $H^*(P; k)^{\mathcal{F}}$  is the block cohomology  $H^*(B)$  as defined in [14, Definition 5.1]. It is not known whether all block fusion systems arise as fusion systems of finite groups. There are examples of fusion systems which arise neither from finite groups nor from blocks; see [10], [13].

**Theorem 0.4** *Let  $\mathcal{F}$  be a saturated fusion system on a finite  $p$ -group  $P$ . We have*

$$\text{reg}(H^*(P; k)^{\mathcal{F}}) = 0 .$$

The key ingredients for proving the above results are Greenlees' local cohomology spectral sequence [9, Theorem 2.1], results and techniques in work of Benson [1], [2], [4], and Symonds' proof in [18] of Benson's regularity conjecture. We use the properties of the regularity from [18, §1] and [19, §2].

**Lemma 0.5** *Let  $G$  be a finite group and  $V$  an indecomposable trivial source  $kG$ -module. Then  $\text{reg}(H^*(G; k); H^*(G; V)) \leq 0$ .*

**Proof** Since  $V$  is a direct summand of  $\text{Ind}_P^G(k)$ , we have

$$\text{reg}(H^*(G; k); H^*(G; V)) \leq \text{reg}(H^*(G; k); H^*(G; \text{Ind}_P^G(k))) .$$

By [12, Lemma 4], the right side is equal to  $\text{reg}(H^*(P; k))$ , hence zero by [18, Corollary 0.2].  $\square$

**Lemma 0.6** *Let  $G$  be a finite group and  $V$  a finitely generated  $kG$ -module. If  $H_0(G; V) \neq \{0\}$ , then  $\text{reg}(H^*(G; k); H^*(G; V)) \geq 0$ .*

**Proof** It follows from the assumption  $H_0(G; V) \neq \{0\}$  and Greenlees' spectral sequence [9, Theorem 2.1] that there is an integer  $s$  such that  $H_m^{s, -s}(H^*(G; V)) \neq \{0\}$ , which implies the result.  $\square$

**Proof of Theorem 0.2** Set  $V = \text{Sc}(G; P)$ . By Lemma 0.5 we have

$$\text{reg}(H^*(G; k); \text{Ext}_{kG}^*(k; V)) \leq 0.$$

Since  $V$  has a nonzero trivial submodule, we have  $H_0(G; V) \neq \{0\}$ , and hence the other inequality follows from Lemma 0.6.  $\square$

Theorem 0.1 will be a consequence of Theorem 0.2 and the following well-known observation (for which we include a proof for the convenience of the reader; the block theoretic background material can be found in [20]).

**Lemma 0.7** *Let  $G$  be a finite group,  $B$  a block algebra of  $kG$  and  $P$  a defect group of  $B$ . As a module over  $kG$  with respect to the conjugation action of  $G$  on  $B$ , the  $kG$ -module  $B$  has an indecomposable direct summand isomorphic to the Scott module  $\text{Sc}(G; P)$ .*

**Proof** Since the conjugation action of  $G$  on  $B$  induces the trivial action on  $Z(B)$  and since  $Z(B) \neq \{0\}$ , it follows that the  $kG$ -module  $B$  has a nonzero trivial submodule. Moreover,  $B$  is a direct summand of  $kG$ , hence  $B$  is a  $p$ -permutation  $kG$ -module, and the vertices of the indecomposable direct summands of  $B$  are conjugate to subgroups of  $P$ . Thus  $B$  has a Scott module with a vertex contained in  $P$  as a direct summand. Since  $Z(B)$  is not contained in the kernel of the Brauer homomorphism  $\text{Br}_P$ , it follows that  $B$  has a direct summand isomorphic to the Scott module  $\text{Sc}(G; P)$ .  $\square$

**Proof of Theorem 0.1** By [12, Proposition 5] we have  $\text{reg}(HH^*(B)) \leq 0$ . Recall that  $HH^*(kG)$  is an  $H^*(G; k)$ -module via the diagonal induction map, and we have a canonical graded isomorphism  $HH^*(B) \cong H^*(G; B)$  as  $H^*(G; B)$ -modules where  $G$  acts on  $B$  by conjugation; see e. g. [17, (3.2)]. It follows from [12, Lemma 4] that

$$\text{reg}(HH^*(B)) = \text{reg}(H^*(G; k); H^*(G; B)) .$$

By Lemma 0.7, the  $kG$ -module  $B$  has a direct summand isomorphic to  $V = Sc(G; P)$ , where  $P$  is a defect group of  $B$ . Thus as an  $H^*(G; k)$ -module,  $H^*(G; B)$  has a direct summand isomorphic to  $H^*(G; V)$ . It follows that

$$\text{reg}(HH^*(B)) \geq \text{reg}(H^*(G; k); H^*(G; V)) = 0,$$

where the last equality is from Theorem 0.2. This completes the proof of Theorem 0.1.  $\square$

**Remark 0.8** The above proof can be adapted to show that the regularity of the stable quotient  $\overline{HH}^*(B)$  of  $HH^*(B)$  also equals zero. Recall that  $\overline{HH}^*(B)$  is the quotient of  $HH^*(B)$  by the ideal  $Z^{\text{pr}}(B) = \text{Tr}_1^G(B)$  of  $Z(B) \cong HH^0(B)$ . Note that  $Z^{\text{pr}}(B)$  is concentrated in degree 0. Alternatively,  $\overline{HH}^*(B)$  may be defined as the non-negative part of the Tate Hochschild cohomology of  $B$ . Our interest in  $\overline{HH}^*(B)$  comes from the fact that Tate Hochschild cohomology of symmetric algebras is an invariant of stable equivalence of Morita type. We briefly indicate how the regularity of  $\overline{HH}^*(B)$  may be calculated. Let  $B = \oplus_i M_i$  be a decomposition of  $B$  into a direct sum of indecomposable  $kG$ -modules  $M_i$ , where  $G$  acts by conjugation on  $B$ . The canonical graded  $H^*(G; k)$ -module isomorphism  $HH^*(B) \cong H^*(G; B)$  induces an isomorphism

$$HH^0(B) \cong H^0(G; B) = \oplus_i H^0(G; M_i)$$

in degree zero. Composing this with the canonical isomorphisms  $Z(B) \cong HH^0(B)$  and  $H^0(G; M_i) \cong M_i^G$ , it is easy to check that the image of  $Z^{\text{pr}}(B)$  in  $\oplus_i M_i^G$  is  $\oplus_i \text{Tr}_1^G(M_i)$ . Since  $B$  is a  $p$ -permutation  $kG$ -module,  $\text{Tr}_1^G(M_i)$  is non-zero precisely if  $M_i$  is isomorphic to the Scott module  $Sc(G; 1)$  (which is a projective cover of the trivial  $kG$ -module). Let  $M'$  denote the sum of all  $M_i$ 's in the above decomposition which are isomorphic to  $Sc(G, 1)$  and let  $M''$  be the complement of  $M'$  in  $B$  with respect to the above decomposition. Since  $Z^{\text{pr}}(B)$  is concentrated in degree zero, we have a direct sum decomposition  $HH^*(B) \cong \oplus H^*(G; M'') \oplus Z^{\text{pr}}(B)$  as  $H^*(G; k)$ -modules. In particular,

$$\text{reg}(H^*(G; k); HH^*(B)) = \max\{\text{reg}(H^*(G; k); H^*(G; M'')), \text{reg}(H^*(G; k); Z^{\text{pr}}(B))\}.$$

We may assume that a defect group  $P$  of  $B$  is non-trivial. By Lemma 0.7,  $M''$  contains a direct summand isomorphic to  $Sc(G; P)$ . Hence by Theorem 0.2  $\text{reg}(H^*(G; k); H^*(G; M'')) \geq 0$ . It follows from Theorem 0.1 and the above displayed equation that  $\overline{HH}^*(B) \cong H^*(G; M'')$  has regularity zero.

**Proof of Theorem 0.4** By [18, Proposition 6.1] we have  $\text{reg}(H^*(P; k)^{\mathcal{F}}) \leq 0$ . For the other inequality we follow the arguments in [1, §3, §4], applied to transfer maps using fusion stable bisets. For  $Q$  a subgroup of  $P$  and  $\varphi : Q \rightarrow P$  an injective group homomorphism, we denote by  $P \times_{(Q, \varphi)} P$  the  $P$ - $P$ -biset of equivalence classes in  $P \times P$  with respect to the relation  $(uw, v) \sim (u, \varphi(w)v)$ , where  $u, v \in P$ , and  $w \in Q$ . The  $kP$ - $kP$ -bimodule having  $P \times_{(Q, \varphi)} P$  as a  $k$ -basis is canonically isomorphic to  $kP \otimes_{kQ} (\varphi kP)$ . This biset gives rise to a transfer map  $\text{tr}_{P \times_{(Q, \varphi)} P}$  on  $H^*(P; k)$

obtained by composing the restriction map  $\text{res}_{\varphi(Q)}^P : H^*(P; k) \rightarrow H^*(\varphi(Q); k)$ , the isomorphism  $H^*(\varphi(Q); k) \cong H^*(Q; k)$  induced by  $\varphi$ , and the transfer map  $\text{tr}_Q^P : H^*(Q; k) \rightarrow H^*(P; k)$ . Let  $X$  be an  $\mathcal{F}$ -stable  $P$ - $P$ -biset satisfying the conclusions of [6, Proposition 5.5]. That is, every transitive subbiset of  $X$  is isomorphic to  $P \times_{(Q, \varphi)} P$  for some subgroup  $Q$  of  $P$  and some group homomorphism  $\varphi : Q \rightarrow P$  belonging to  $\mathcal{F}$ , the integer  $|X|/|P|$  is prime to  $p$ , and for any subgroup  $Q$  of  $P$  and any group homomorphism  $\varphi : Q \rightarrow P$  in  $\mathcal{F}$ , the  $Q$ - $P$ -bisets  ${}_{\varphi}X$  and  ${}_QX$  (resp. the  $P$ - $Q$ -bisets  $X_Q$  and  $X_{\varphi}$ ) are isomorphic. By taking the sum, over the transitive subbisets  $P \times_{(Q, \varphi)} P$ , of the transfer maps  $\text{tr}_{P \times_{(Q, \varphi)} P}$ , we obtain a transfer map  $\text{tr}_X$  on  $H^*(P; k)$ . Following [15, Proposition 3.2], the map  $\text{tr}_X$  acts as multiplication by  $\frac{|X|}{|P|}$  on  $H^*(P; k)^{\mathcal{F}}$ , hence  $\text{Im}(\text{tr}_X) = H^*(P; k)^{\mathcal{F}}$ , and we have a direct sum decomposition

$$H^*(P; k) = H^*(P; k)^{\mathcal{F}} \oplus \ker(\text{tr}_X)$$

as  $H^*(P; k)^{\mathcal{F}}$ -modules. A similar decomposition holds for Tate cohomology, and for homology (using either the canonical duality  $H_n(P; k) \cong H^n(P; k)^{\vee}$  or the isomorphism  $H_n(P; k) \cong \hat{H}^{-n-1}(P; k)$  obtained from composing the previous duality with Tate duality). By [1, Equation (4.1)], the transfer map  $\text{tr}_Q^P$  induces a homomorphism of Greenlees' local cohomology spectral sequences

$$\begin{array}{ccc} H_m^{i,j} H^*(Q; k) & \Longrightarrow & H_{-i-j}(Q; k) \\ (\text{tr}_Q^P)_* \downarrow & & \downarrow (\text{res}_Q^P)_* \\ H_m^{i,j} H^*(P; k) & \Longrightarrow & H_{-i-j}(P; k) \end{array}$$

where  $(\text{tr}_Q^P)_*$  and  $(\text{res}_Q^P)_*$  are the maps induced by  $\text{tr}_Q^P$  and the inclusion  $Q \rightarrow P$ , respectively. The isomorphism  $\varphi : Q \rightarrow \varphi(Q)$  induces an obvious isomorphism of spectral sequences

$$\begin{array}{ccc} H_m^{i,j} H^*(\varphi(Q); k) & \Longrightarrow & H_{-i-j}(\varphi(Q); k) \\ \cong \downarrow & & \downarrow \cong \\ H_m^{i,j} H^*(Q; k) & \Longrightarrow & H_{-i-j}(Q; k) \end{array}$$

Restriction and transfer on Tate cohomology are dual to each other under Tate duality, and hence the dual version of [1, Equation (4.1)] implies that the restriction  $\text{res}_{\varphi(Q)}^P$  induces a homomorphism of spectral sequences

$$\begin{array}{ccc} H_m^{i,j} H^*(P; k) & \Longrightarrow & H_{-i-j}(P; k) \\ (\text{res}_{\varphi(Q)}^P)_* \downarrow & & \downarrow (\text{tr}_{\varphi(Q)}^P)_* \\ H_m^{i,j} H^*(\varphi(Q); k) & \Longrightarrow & H_{-i-j}(\varphi(Q); k) \end{array}$$

Composing the three diagrams above yields a homomorphism induced by  $\text{tr}_{P \times_{(Q, \varphi)} P}$  on the spectral sequence for  $P$ , and taking the sum over all transitive subbisets of

$X$  yields a homomorphism of spectral sequences

$$\begin{array}{ccc} H_m^{i,j} H^*(P, k) & \Longrightarrow & H_{-i-j}(P; k) \\ (\mathrm{tr}_X)_* \downarrow & & \downarrow (\mathrm{tr}_{X^\vee})_* \\ H_m^{i,j} H^*(P; k) & \Longrightarrow & H_{-i-j}(P; k) \end{array}$$

where  $X^\vee$  is the  $P$ - $P$ -biset  $X$  with the opposite action  $u \cdot x \cdot v = v^{-1}xu^{-1}$  for all  $u, v \in P$  and  $x \in X$ . One easily checks that  $X^\vee$  is isomorphic to a dual basis of  $X$  in the dual bimodule  $\mathrm{Hom}_k(kX, k)$ . By [6, Proposition 5.2],  $H^*(P; k)$  is finitely generated as a module over  $H^*(P; k)^\mathcal{F}$ . Thus the local cohomology spaces  $H_m^{i,j} H^*(P; k)$  can be calculated using for  $m$  the maximal ideal of positive degree elements in  $H^*(P; k)^\mathcal{F}$  instead of  $H^*(P; k)$ . It follows that  $\mathrm{tr}_X$  induces a homomorphism of spectral sequences

$$\begin{array}{ccc} H_m^{i,j} H^*(P, k) & \Longrightarrow & H_{-i-j}(P; k) \\ (\mathrm{tr}_X)_* \downarrow & & \downarrow (\mathrm{tr}_{X^\vee})_* \\ H_m^{i,j} H^*(P; k)^\mathcal{F} & \Longrightarrow & H_{-i-j}(P; k)^\mathcal{F} \end{array}$$

For  $i = -j = r$ , where  $r$  is the rank of  $P$ , the edge homomorphism yields a commutative diagram of the form

$$\begin{array}{ccccc} H_m^{r,-r} H^*(P; k) & \xrightarrow{\gamma_P} & H_0(P; k) & \xrightarrow{\cong} & k \\ (\mathrm{tr}_X)_* \downarrow & & (\mathrm{tr}_{X^\vee})_* \downarrow & & \downarrow \cdot \frac{|X|}{|P|} \\ H_m^{r,-r} H^*(P; k)^\mathcal{F} & \xrightarrow{\delta_\mathcal{F}} & H_0(P; k)^\mathcal{F} & \xrightarrow{\cong} & k \end{array}$$

where the right vertical map is multiplication on  $k$  by  $\frac{|X|}{|P|}$ . By [1, Theorem 4.1], the map  $\gamma_P$  is surjective, and hence so is the map  $\delta_\mathcal{F}$ . In particular,  $H_m^{r,-r} H^*(P; k)^\mathcal{F} \neq \{0\}$ , whence the result.  $\square$

**Remark 0.9** The fact that transfer and restriction on Tate cohomology are dual to each other under Tate duality can be deduced from a more general duality for transfer maps on Tate-Hochschild cohomology of symmetric algebras induced by bimodules which are finitely generated projective as left and right modules (cf. [16]).

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