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COMBINATORICS AND FORMAL GEOMETRY OF THE MAURER-CARTAN EQUATION

J. CHUANG AND A. LAZAREV

ABSTRACT. We give a general treatment of the Maurer-Cartan equation in homotopy algebras and describe the operads and formal differential geometric objects governing the corresponding algebraic structures. We show that the notion of Maurer-Cartan twisting is encoded in certain automorphisms of these universal objects.

1. INTRODUCTION

Let $\mathfrak{g}$ be a differential graded associative or Lie algebra. An odd element ξ in $\mathfrak{g}$ is called *Maurer-Cartan* (or MC for short) if it satisfies the master equation (also known as the *MC equation*, sometimes with the adjectives ‘classical’ or ‘quantum’): $d\xi + \frac{1}{2}[\xi, \xi] = 0$. An MC element ξ allows one to *twist* the differential in $\mathfrak{g}$; the twisted differential has the form $d^\xi := d + [\xi, ?]$ and the MC condition ensures that $(d^\xi)^2 = 0$. The MC equation and MC twisting appear in various different contexts: deformation theory, homological algebra, rational homotopy theory, differential geometry and mathematical physics (see, e.g., [4, 11, 13]).

There is a far-reaching generalization of the notion of an MC element and MC twisting where $\mathfrak{g}$ is taken to be an L_∞ or A_∞ algebra, which could be viewed as a homotopy invariant version of a Lie or associative algebra. There are two complementary ways to treat these objects: the traditional combinatorial approach, as algebras over certain *operads* and the geometric approach, as homological vector fields on (possibly noncommutative) formal supermanifolds. The geometric viewpoint goes back to Kontsevich, [9]. In this paper we adopt both points of view.

Our aim is to describe the operadic and differential geometric structures governing MC elements and MC twisting in A_∞ and L_∞ algebras. In particular, we construct certain L_∞ algebras of vector fields whose MC elements are themselves L_∞ and A_∞ algebras together with a choice of an MC element in them; the notion of MC twisting is encoded in certain ‘twisting’ automorphisms of these L_∞ algebras. We also give versions corresponding to cyclic L_∞ and A_∞ algebras. A parallel result is also obtained on the operadic side of the picture by introducing certain dg operads $\mathcal{L}_\infty$ and $\mathcal{A}_\infty$ whose algebras are L_∞ and A_∞ algebras together with a choice of an MC element. Again, twisting corresponds to automorphisms in these operads.

Our next collection of results centers on the calculation of the homology of the operads $\mathcal{L}_\infty$ and $\mathcal{A}_\infty$ governing MC elements and we give a rather complete answer. In this context it is useful and illuminating to consider MC elements in $\mathcal{O}$ algebras where $\mathcal{O}$ is not necessarily an $\mathcal{L}_\infty$ or $\mathcal{A}_\infty$ operad. These MC elements are also governed by certain operads, which reduce to $\mathcal{L}_\infty$ and $\mathcal{A}_\infty$ in the case when $\mathcal{O} = L_\infty$ or $\mathcal{O} = A_\infty$ and we compute the homology of these operads. Additionally, our approach gives a conceptual explanation of some of the results of Willwacher [14]. In a forthcoming paper we will apply our results to construct a higher analogue of Kontsevich’s characteristic classes of homotopy algebras with values in appropriate graph complexes. This application requires extending our results in the context of modular operads and we briefly indicate how this is done in relevant places.

The paper is organized as follows. Section 1 reviews background material on operads, operadic algebras and MC elements; it contains no new results but offers a slightly less traditional

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treatment. For example, we consider two versions of the L_∞ operad: one with the unary operation m_1 and the other without, and similarly for the A_∞ operad. The reason for allowing m_1 as part of an operadic structure is that, without it, the twisting automorphisms mentioned above do not exist.

In Section 2 we construct the operads $\mathcal{L}_\infty$ and $\mathcal{A}_\infty$ associated to L_∞ and A_∞ algebras with MC elements; these operads represent the combinatorial point of view mentioned above. Section 3 is devoted to understanding the same structures from the formal geometric point of view; we construct certain big L_∞ algebras of formal vector fields that capture this geometric content of the MC elements. Of independent interest here is the notion of an extension of L_∞ algebras and classification of such extensions. The developed theory allows one, among other things, to build a large supply of L_∞ algebras out of usual graded Lie algebras.

In section 4 we apply the operadic and formal geometric frameworks to the MC twisting itself; it turns out that the latter is encoded by certain automorphisms of the corresponding operads and algebras of formal vector fields.

In Section 5 we consider an analogue of MC twisting in operads and introduce the functor $\mathcal{O} \mapsto \hat{\mathcal{O}}$ (the ‘hat-construction’) which allows one to treat the internal differential on the underlying space of an algebra over an operad $\mathcal{O}$ as part of an operadic structure of $\hat{\mathcal{O}}$. Somewhat surprisingly, the operad $\hat{\mathcal{O}}$ turns out to be always acyclic. In Section 6 this fact is used to prove that the operads $\mathcal{L}_\infty$ and $\mathcal{A}_\infty$ are likewise acyclic. This acyclicity is caused by the presence of operations of low arity in these operads and it does not, of course, imply that they have trivial categories of algebras. We also consider certain quotients of $\mathcal{A}_\infty$ and $\mathcal{L}_\infty$ which are not acyclic.

In section 7 we relate our constructions to the results of Willwacher and give an alternative construction of the operad $\text{Tw } \mathcal{O}$ whose algebras are, roughly speaking, $\mathcal{O}$ algebras on a dg vector spaces whose differential have been MC twisted. We compute the homology of $\text{Tw } \mathcal{O}$ in the case when $\mathcal{O}$ is the A_∞ or L_∞ operad.

1.1. Notation and conventions. We mainly adopt the conventions of our previous paper [2]. By a *vector space* we will mean a $\mathbb{Z}/2$ -graded vector space (also known as super-vector space) over a field $\mathbf{k}$ of characteristic zero. The adjective ‘differential graded’ will mean ‘differential $\mathbb{Z}/2$ -graded’ and will be abbreviated as ‘dg’. Thus, a dg vector space is a pair (V, d_V) where V is a $\mathbb{Z}/2$ -graded vector space and d_V is a differential on it; we will frequently omit mentioning d_V in cases when its meaning is clear from the context. While we prefer the $\mathbb{Z}/2$ -graded framework, all our results have obvious $\mathbb{Z}$ -graded analogues.

A (commutative) differential graded (Lie) algebra will be abbreviated as (c)dg(l)a. We will often invoke the notion of a *formal* (dg) vector space; this is just an inverse limit of finite-dimensional vector spaces. This terminology was adopted in [2] and some earlier works that the present paper relies on.¹ An example of a formal space is V^* , the $\mathbf{k}$ -linear dual to a discrete vector space V . A formal vector space comes equipped with a topology and whenever we deal with a topological vector space all linear maps from or into it will be assumed to be continuous; thus for a formal vector space V we always have $V^{**} \cong V$, even if V is not finite-dimensional. All of our unmarked tensors are understood to be taken over $\mathbf{k}$. If V is a discrete space and $W = \varprojlim W_i$ is a formal space we will write $V \otimes W$ for $\varprojlim V \otimes W_i$; thus for two discrete spaces V and U we have $\text{Hom}(V, U) \cong U \otimes V^*$. For a $\mathbb{Z}/2$ -graded vector space $V = V_0 \oplus V_1$ the symbol ΠV will denote the *parity reversion* of V ; thus $(\Pi V)_0 = V_1$ while $(\Pi V)_1 = V_0$. The association $V \mapsto \Pi V$ is a functor; thus to a given map $f : V \rightarrow U$ between graded spaces V and U there corresponds a map $\Pi f : \Pi V \rightarrow \Pi U$; to alleviate the notation and avoid possible confusion we will often write f for Πf . The symbol S_n stands for the symmetric group on n letters; it is understood to act on the n th tensor power of a graded vector space $V^{\otimes n}$ by permuting the tensor factors.

¹This convention is not ideal since in other contexts ‘formal’ means ‘quasi-isomorphic to cohomology’. A formal vector space is more traditionally known as a linearly compact vector space, a terminology introduced by Lefschetz [10]; we feel that in our context it is not perfect either.

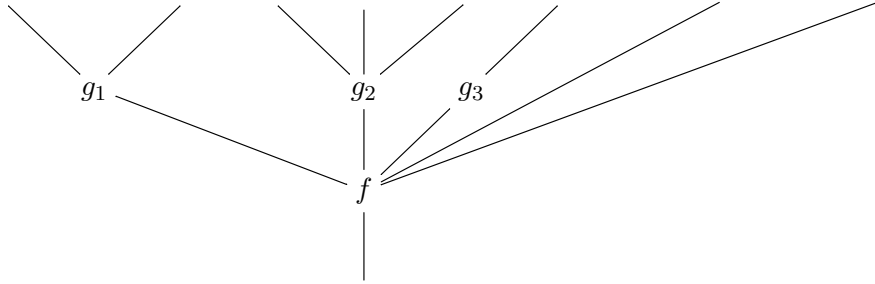
In this paper we freely use the language of operads and modular operads as developed in standard sources such as [7, 6]. An operad in this paper will always mean a non-unital operad, i.e. a collection of vector spaces $\mathcal{O}(n)$ together with actions of symmetric groups S_n on each $\mathcal{O}(n)$ and supplied with a collection of structure maps

$$\circ_i : \mathcal{O}(k) \otimes \mathcal{O}(n) \rightarrow \mathcal{O}(k+n-1), \quad i = 1, \dots, k$$

satisfying suitable axioms. If the action of symmetric groups is omitted we obtain the notion of a *non-symmetric operad*. The conventions we adopt for L_∞ and A_∞ algebras lead to the consideration of twisted operads. To avoid repeated mention of cocycle twistings, we shall use the terms cyclic operad and modular operad to mean anticyclic operad and $\mathfrak{D}_\Pi$ -twisted modular operad, respectively; see [5, 6] for details.

The *endomorphism operad* of a dg vector space V is the dg operad of the form $\mathcal{E}(V) := \{\text{Hom}(V^{\otimes n}, V)\}$, $n = 1, 2, \dots$. An *algebra* over a dg operad $\mathcal{O}$ is a map of dg operads $\mathcal{O} \rightarrow \mathcal{E}(\Pi V)$. This slightly unorthodox definition (of course, equivalent to the usual one) allows us to minimize the sign issues present. This conflicts with the more traditional usage (e.g. an associative algebra is not an algebra over the operad of associative algebras under this definition) but since we only consider it in the context of cobar-construction type operads it should not lead to confusion. If V is endowed with a symmetric inner product then, according to our convention, the operad $\mathcal{E}(\Pi V)$ is cyclic.

We also adopt the following convention. If $f \in \mathcal{O}(n)$ and $g_l \in \mathcal{O}(k_l)$ for $l = 1, \dots, i \leq n$ then we denote by $f \circ (g_1 \otimes \dots \otimes g_i)$ the operation $(\dots ((f \circ_i g_i) \circ_{i-1} g_{i-1}) \circ \dots \circ_1 g_1) \in \mathcal{O}(k_1 + \dots + k_i + n - i)$. The following picture for $f \circ (g_1 \otimes g_2 \otimes g_3)$ illustrates this notation; here $f \in \mathcal{O}(5)$, $g_1 \in \mathcal{O}(2)$, $g_2 \in \mathcal{O}(3)$ and $g_3 \in \mathcal{O}(1)$.



1.2. Linearly topologized dg algebras and operads. We often find it necessary to work with (dg) vector spaces supplied with a linear topology; in all such cases the topological dg vector spaces in question will be inverse limits of discrete dg vector spaces. For instance, an inverse limit of finite-dimensional dg vector spaces is what we call a formal dg vector space but we will see other examples as well. Correspondingly, algebraic structures on topological vector spaces have to be compatible with the topology. An instance of such an algebraic structure is a *formal non-unital cdga* which is, by definition, an inverse limit of (non-unital) finite-dimensional nilpotent cdgas; note that this condition is stronger than simply being a cdga in formal dg vector spaces. We will normally consider *unital* formal cdgas (omitting the adjective ‘unital’) that are the result of adjoining units to non-unital formal cdgas. The augmentation ideal in a formal cdga A will be denoted by A_+ .

For a dg operad $\mathcal{O}$ we will denote by $\mathcal{O}[x]$ the operad obtained from $\mathcal{O}$ by freely adjoining a variable x . The context will always make clear the arity n of the operation x , and it will be assumed that S_n acts trivially on x . The notation $\mathcal{O}[[x]]$ will stand for the completion of $\mathcal{O}[x]$ at the operadic ideal $(x) : \mathcal{O}[[x]] := \varprojlim \mathcal{O}[x]/x^n$. Note that the operad $\mathcal{O}[[x]]$ is topological, in the sense that every dg vector space $\mathcal{O}(n)$, $n = 1, 2, \dots$ is topological and all operadic compositions are continuous maps.

Whenever we have homomorphisms between algebraic objects with linear topology (dg vector spaces, algebras or operads) these will be assumed to be continuous without mentioning this

explicitly. For instance, an *algebra* over a topological dg operad $\mathcal{O}$ is a *continuous* map of operads $\mathcal{O} \rightarrow \mathcal{E}(\Pi V)$.

Next, let V be a dg vector space, A be a formal cdga and $\mathcal{O}$ be a (possibly topological) dg operad. Then we refer to a (continuous) map of operads $\mathcal{O} \rightarrow A_+ \otimes \mathcal{E}(\Pi V)$ as an *A-linear $\mathcal{O}$ algebra structure* on V ; note that $A_+ \otimes \mathcal{E}(\Pi V)$ is a topological operad since A is a topological algebra.

1.3. L_∞ and A_∞ operads and their algebras. The L_∞ and A_∞ operads are usually introduced so that the operation m_1 is treated as a differential on the underlying space of the corresponding algebra. In our treatment we view m_1 as part of the operadic structure.

Definition 1.1.

- (1) The non-symmetric operad A_∞ is freely generated by *odd* operations $m_n \in A_\infty(n)$, $n = 1, 2, \dots$, with the usual cobar-differential

$$d(m_n) = \sum_{k=1}^n \sum_{i=1}^k m_k \circ_i m_{n-k+1}.$$

Note that we allow m_1 in the definition of A_∞ ; thus for a vector space V (without a differential) an operad map $A_\infty \rightarrow \mathcal{E}(\Pi V)$ is an A_∞ algebra with possibly non-zero m_1 .

In common with any non-symmetric operad, A_∞ gives rise to a usual (symmetric) operad $\{A_\infty(n) \otimes \mathbf{k}[S_n]\}$. Abusing notation, we will denote the latter operad also by A_∞ ; the context will always makes clear which of the two versions is being considered.

The operad a_∞ is defined as the quotient of A_∞ by the ideal (m_1) .

- (2) The operad L_∞ is a suboperad in (the symmetric version of) the operad A_∞ generated by the elements $m'_n := \sum_{\sigma \in S_n} \sigma m_n$, so that the action of $\sigma \in S_n$ on m'_n is trivial. The differential in L_∞ is given by the formula

$$d(m'_n) = \sum_{k=1}^n \sum_{\sigma \in \text{Sh}(n-k+1, k-1)} \sigma (m'_k \circ_1 m'_{n-k+1}),$$

where the inner sum is taken over the set of $(n-k+1, k-1)$ shuffles in S_n . Note that we allow m'_1 in the definition of L_∞ ; thus for a vector space V (without a differential) an operad map $L_\infty \rightarrow \mathcal{E}(\Pi V)$ is an L_∞ algebra with possibly non-zero m'_1 .

Later on we will omit the superscript $'$ and use the symbols m_n to denote the L_∞ as well as the A_∞ operations; the context will always allow one to recover the correct meaning. Note that any A_∞ algebra can be viewed as an L_∞ algebra via this map; this is an infinity version of the commutator Lie algebra associated to an associative algebra.

The operad l_∞ is defined as the quotient of L_∞ by the ideal (m_1) .

Remark 1.2.

- (1) The operad A_∞ is the cobar-construction of the operad ass_e , the unital version of the operad governing associative algebras. Recall that the (symmetric) operad ass_e has $\text{ass}_e(n) = \mathbf{k}[S_n]$, $n = 1, 2, \dots$. The operad ass_e maps to the (unital) operad comm_e for which $\text{comm}_e(n) = \mathbf{k}$, $n = 1, 2, \dots$, governing commutative algebras. It is clear from this description that the operad L_∞ is the cobar-construction of comm_e and thus, is free on the S_n -invariant operations $m'_n \in L_\infty(n)$ that are dual to the operators in $\text{comm}_e(n)$ representing the n -fold iterated multiplication.
- (2) Since A_∞ and L_∞ are cobar-constructions of unital operads, they are acyclic, in contrast with a_∞ and l_∞ (which are cobar-constructions of *non-unital* ass and comm respectively). This fact, however, *does not* mean that the homotopy theory of algebras over the operads A_∞ and L_∞ is trivial. Indeed, even though the operads A_∞, L_∞ are themselves homologically trivial, they are not *cofibrant* (owing to their unary part) and thus, maps out of them cannot be viewed as homotopically trivial.

- (3) The operads A_∞ and L_∞ can be obtained from a_∞ and l_∞ with the help of a certain general construction which will be discussed in Section 5. The relationship between A_∞ and L_∞ algebras and their lower case counterparts will also be considered in some detail there.

Specializing our general definitions of operadic algebras we arrive at the notions of A_∞ and L_∞ algebras, and of A -linear A_∞ and L_∞ algebras over a cdga.

Definition 1.3. Let V be a dg vector space and A be a formal cdga.

- (1) (a) An L_∞ algebra structure on V is a map of operads $L_\infty \rightarrow \mathcal{E}(\Pi V)$; it is thus a collection of multilinear maps

$$m = \{m_n : (\Pi V)^{\otimes n} \rightarrow \Pi V, n = 1, 2, \dots\}$$

that are symmetric and satisfy the usual L_∞ identities. The pair (V, m) will be referred to as an L_∞ algebra.

- (b) An A_∞ algebra structure on V is a map of operads $A_\infty \rightarrow \mathcal{E}(\Pi V)$; it is thus a collection of multilinear maps

$$m = \{m_n : (\Pi V)^{\otimes n} \rightarrow \Pi V, n = 1, 2, \dots\}$$

that satisfy the usual A_∞ identities. The pair (V, m) will be referred to as an A_∞ algebra.

- (2) (a) An A -linear L_∞ algebra structure on V is a map of operads $L_\infty \rightarrow A_+ \otimes \mathcal{E}(\Pi V)$; it is thus a collection of multilinear maps

$$m = \{m_n : (\Pi V)^{\otimes n} \rightarrow A_+ \otimes \Pi V, n = 1, 2, \dots\}$$

that are symmetric and satisfy the usual L_∞ identities. The pair (V, m) will be referred to as an A -linear L_∞ algebra.

- (b) An A -linear A_∞ algebra structure on V is a map of operads $A_\infty \rightarrow A_+ \otimes \mathcal{E}(\Pi V)$; it is thus a collection of multilinear maps

$$m = \{m_n : (\Pi V)^{\otimes n} \rightarrow A_+ \otimes \Pi V, n = 1, 2, \dots\}$$

that satisfy the usual A_∞ identities. The pair (V, m) will be referred to as an A -linear A_∞ algebra.

Remark 1.4. The operation m_1 gives V the structure of a dg vector space, which is distinct from the one associated with the internal differential on V . Thus, our definition of an L_∞ algebra is seemingly more general than the usual one (unless the internal differential on V vanishes). On the other hand, if m_1 acts trivially on V the given L_∞ algebra structure on V descends to an l_∞ algebra structure recovering the usual notion. We shall return to the question of the distinction between L_∞ and l_∞ algebras in Section 5. Similar remarks apply to A_∞ and a_∞ structures.

One can also view an L_∞ or A_∞ structure on V as a homological vector field on ΠV ; this point of view will be recalled in section 3. We refer for a more detailed analysis of this to [2] and references therein.

Remark 1.5. The operads L_∞ and A_∞ are *cyclic*, cf. [5] concerning this notion, and every statement about them made in this subsection has an obvious cyclic analogue. For example, a cyclic L_∞ algebra is a map of cyclic operads $L_\infty \rightarrow \mathcal{E}(\Pi V)$ where V is a vector space with a linear symplectic structure giving $\mathcal{E}(\Pi V)$ the structure of a cyclic operad.

1.4. Maurer-Cartan elements in L_∞ and A_∞ algebras.

Definition 1.6. Let (V, d) be a dg vector space and A be a formal cdga.

- (1) Let m be an A -linear L_∞ -structure on V . Then an even element $\xi \in A_+ \otimes \Pi V$ is *Maurer-Cartan* (MC) if $(d_A \otimes \text{id} + \text{id} \otimes d_{\Pi V})(\xi) + \sum_{i=1}^{\infty} \frac{1}{i!} m_i(\xi^{\otimes i}) = 0$.
- (2) Let m be an A -linear A_∞ -structure on V and A . Then an even element $\xi \in A_+ \otimes \Pi V$ is *Maurer-Cartan* (MC) if $(d_A \otimes \text{id} + \text{id} \otimes d_{\Pi V})(\xi) + \sum_{i=1}^{\infty} m_i(\xi^{\otimes i}) = 0$.

The set of MC elements in $A_+ \otimes \text{IIV}$ is denoted by $\text{MC}((V, d), m), A)$ but we will usually shorten it to $\text{MC}(V, m), A)$ or even $\text{MC}(V, A)$ leaving d and m understood.

Remark 1.7. It would, perhaps, be notationally more natural (in the context of the above definition) to consider the set $\text{MC}(V, B)$ where B is a formal cdga *over* A , however we will never need this level of generality.

We refer to [2] for some basic facts on MC elements in L_∞ and A_∞ algebras. In some situations it is useful to consider MC elements with values in not necessarily formal cdgas, particularly in $\mathbf{k}$, however in general it makes no sense since the MC condition involves a possibly divergent series. Note that in the case when V is a dgla or a dga such a problem does not arise and the MC constraint takes the form of the equation of a flat connection $d(\xi) + \frac{1}{2}[\xi, \xi] = 0$.

2. OPERADS ASSOCIATED TO MC ELEMENTS

Let (V, m) be an L_∞ or A_∞ algebra and ξ be an MC element in it with values in a formal cdga A . We will describe the operad governing this collection of data.

Definition 2.1.

- (1) Let $\mathcal{L}_\infty$ be the operad $L_\infty[[x]]$, where the generator $x \in \mathcal{L}_\infty(0)$. The differential in $L_\infty \subset \mathcal{L}_\infty$ is the standard cobar differential whereas

$$(2.1) \quad d(x) = \sum_{n=1}^{\infty} \frac{1}{n!} m_n \circ \underbrace{(x \otimes \dots \otimes x)}_n$$

- (2) Let $\mathcal{A}_\infty$ be the non-symmetric operad $A_\infty[[x]]$, where the generator $x \in \mathcal{A}_\infty(0)$. The differential in $A_\infty \subset \mathcal{A}_\infty$ is the cobar differential whereas

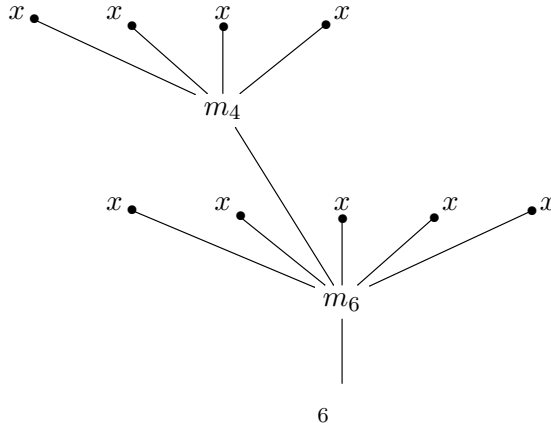
$$(2.2) \quad d(x) = \sum_{n=1}^{\infty} m_n \circ \underbrace{(x \otimes \dots \otimes x)}_n$$

Lemma 2.2. *The differential d in both $\mathcal{L}_\infty$ and $\mathcal{A}_\infty$ satisfies $d^2 = 0$.*

Proof. Let us treat the $\mathcal{A}_\infty$ case first. It suffices to prove that $d^2(x) = 0$. We have, taking into account that the operations m_n are odd,

$$\begin{aligned} d^2(x) &= d \left(\sum_{n=1}^{\infty} m_n \circ (x \otimes \dots \otimes x) \right) \\ &= \sum_{n=1}^{\infty} d(m_n) \circ (x \otimes \dots \otimes x) - \sum_{n=1}^{\infty} m_n \circ (d(x) \otimes x \otimes \dots \otimes x + \dots + x \otimes \dots \otimes x \otimes d(x)). \end{aligned}$$

The last expression involves sums of planar trees having precisely one edge connecting vertices of valence greater than one. An example of such a tree is depicted below.



Moreover every such tree contributes precisely one summand to $\sum_{n=1}^{\infty} d(m_n) \circ (x \otimes \dots \otimes x)$ and one summand to $m_n \circ (d(x) \otimes x \otimes \dots \otimes x + \dots + x \otimes \dots \otimes x \otimes d(x))$. These contributions cancel and we get $d^2(x) = 0$.

Let us now consider the $\mathcal{L}_{\infty}$ case. Note that the canonical injection of operads $L_{\infty} \rightarrow A_{\infty}$ extends naturally to an inclusion $L_{\infty}[[x]] \rightarrow A_{\infty}[[x]]$ and this inclusion is compatible with the operators d on $L_{\infty}[[x]]$ and $A_{\infty}[[x]]$ defined by formulas (2.1) and (2.2). Since d squares to zero in $A_{\infty}[[x]]$ it follows that it also squares to zero in $L_{\infty}[[x]]$. $\square$

The following result describes explicitly algebras over $\mathcal{A}_{\infty}$ and $\mathcal{L}_{\infty}$.

Theorem 2.3. *Let A be a formal cdga.*

- (1) *Let V be an A -linear L_{∞} algebra and $\xi \in \text{MC}(V, A)$. Then V has the structure of an A -linear $\mathcal{L}_{\infty}$ algebra. Conversely, an A -linear algebra over $\mathcal{L}_{\infty}$ gives rise to an A -linear L_{∞} algebra V together with a choice of an element in $\text{MC}(V, A)$.*
- (2) *Let V be an A -linear A_{∞} algebra and $\xi \in \text{MC}(V, A)$. Then V has the structure of an A -linear $\mathcal{A}_{\infty}$ algebra. Conversely, an A -linear algebra over $\mathcal{A}_{\infty}$ gives rise to an A -linear A_{∞} algebra V together with a choice of an element in $\text{MC}(V, A)$.*

Proof. The structure of an A -linear L_{∞} algebra over $\mathcal{L}_{\infty}$ on V is the same as a continuous map of operads

$$f : \mathcal{L}_{\infty} \rightarrow A_+ \otimes \mathcal{E}(\text{IV}).$$

Having such a map is, in turn, equivalent to having a continuous map of operads $L_{\infty} \rightarrow A_+ \otimes \mathcal{E}(\text{IV})$ (that is, an A -linear L_{∞} structure on V) together with an element $f(x) \in A_+ \otimes \text{IV}$ for which $df(x) = f(dx)$. The latter equality is precisely the condition that $f(x)$ is an MC element in $A_+ \otimes \text{IV}$. The A_{∞} case is considered similarly. $\square$

Remark 2.4. Note that the operads $\mathcal{L}_{\infty}$ and $\mathcal{A}_{\infty}$ are cyclic operads in an obvious way and Theorem 2.3 continues to hold in the context of cyclic L_{∞} and A_{∞} algebras.

We next discuss the analogues of the above results in the context of modular operads, cf. [6]. We do not insist that our modular operads $\{\mathcal{O}((g, n))\}$ satisfy the stability condition $2g + n - 2 \leq 0$. All our conventions concerning (topological) dg operads extend in an obvious way to (topological) dg modular operads. The cyclic operads L_{∞} and A_{∞} governing the L_{∞} and A_{∞} algebras admit modular closures $\overline{L}_{\infty}$ and $\overline{A}_{\infty}$ respectively. Then $\overline{L}_{\infty}((n))$ is spanned by isomorphism classes of graphs with n legs and $\overline{A}_{\infty}((n))$ is spanned by isomorphism classes of ribbon graphs with n legs. Modular algebras over $\overline{L}_{\infty}$ and $\overline{A}_{\infty}$ are the same as cyclic algebras over L_{∞} and A_{∞} ; therefore one can speak about MC elements in them with values in formal cdgas. The modular closures of the operads $\mathcal{L}_{\infty}$ and $\mathcal{A}_{\infty}$ will be denoted by $\overline{\mathcal{L}}_{\infty}$ and $\overline{\mathcal{A}}_{\infty}$ respectively.

We then have the following analogue of Theorem 2.3.

Theorem 2.5. *Let A be a formal cdga.*

- (1) *Let V be an A -linear $\overline{L}_{\infty}$ algebra and $\xi \in \text{MC}(V, A)$. Then V has the structure of an A -linear $\overline{\mathcal{L}}_{\infty}$ algebra. Conversely, an A -linear algebra over $\overline{\mathcal{L}}_{\infty}$ gives rise to an A -linear $\overline{L}_{\infty}$ algebra V together with a choice of an element in $\text{MC}(V, A)$.*
- (2) *Let V be an A -linear $\overline{A}_{\infty}$ algebra and $\xi \in \text{MC}(V, A)$. Then V has the structure of an A -linear $\overline{\mathcal{A}}_{\infty}$ algebra. Conversely, an A -linear algebra over $\overline{\mathcal{A}}_{\infty}$ gives rise to an A -linear $\overline{A}_{\infty}$ algebra V together with a choice of an element in $\text{MC}(V, A)$.*

$\square$

3. L_{∞} ALGEBRAS OF FORMAL VECTOR FIELDS

The purpose of this section is to describe $\mathcal{A}_{\infty}$ and $\mathcal{L}_{\infty}$ algebras in more geometric terms. Recall that A_{∞} and L_{∞} algebras are MC elements in certain (infinite dimensional) Lie algebras. It turns out that analogous structures governing $\mathcal{A}_{\infty}$ and $\mathcal{L}_{\infty}$ algebras are themselves L_{∞} algebras.

3.1. Extensions of L_∞ algebras. In order to introduce L_∞ algebras of formal vector fields in a conceptual way we need the notion of an *extension* of L_∞ algebras which is of independent interest. First recall the geometric definition of an L_∞ algebra, and of an A_∞ algebra which we also have the opportunity to use later on.

Definition 3.1.

- (1) Let (V, d_V) be a dg vector space. An L_∞ structure on V is an odd element $m \in \text{Der}(\hat{S}\Pi V^*)$ which has no constant term and satisfies the MC equation $d_V(m) + \frac{1}{2}[m, m] = 0$. The pair (V, m) will be referred to as an L_∞ algebra and the algebra $\hat{S}\Pi V^*$, supplied with the differential $d_V + m$, as its representing formal cdga. The internal differential d_V will often be suppressed from the notation. For two L_∞ algebras (V, m_V) and (U, m_U) an L_∞ map $V \rightarrow U$ is, by definition, a continuous map of its representing cdgas $\hat{S}\Pi U^* \rightarrow \hat{S}\Pi V^*$.
- (2) Let (V, d_V) be a dg vector space. An A_∞ structure on V is an odd element $m \in \text{Der}(\hat{T}\Pi V^*)$ which has no constant term and satisfies the MC equation $d_V(m) + \frac{1}{2}[m, m] = 0$. The pair (V, m) will be referred to as an A_∞ algebra and the algebra $\hat{T}\Pi V^*$, supplied with the differential $d_V + m$, as its representing formal dga. The internal differential d_V will often be suppressed from the notation. For two A_∞ algebras (V, m_V) and (U, m_U) an A_∞ map $V \rightarrow U$ is, by definition, a continuous map of its representing dgas $\hat{T}\Pi U^* \rightarrow \hat{T}\Pi V^*$.

Remark 3.2. We would like to make several comments on the above definition of A_∞ and L_∞ structures; to avoid repeating ourselves we confine those comments to the L_∞ context, however it is clear that they also apply with appropriate modifications to the A_∞ case.

- (1) The above definition of an L_∞ structure is, of course, equivalent to the one via the operad L_∞ (and so, generalizes the traditional one to which it reduces when the internal differential d_V on V vanishes). The derivation $m : \hat{S}\Pi V^* \rightarrow \hat{S}\Pi V^*$ determining an L_∞ structure on V has the form $m = m_1 + m_2 + \dots$ where $m_n, n = 1, 2, \dots$ is the component of m landing in $\hat{S}^n \Pi V^*$, so that $m_n : \Pi V^* \rightarrow (\Pi V^*)^{\otimes n}$. By abuse of notation we have the same symbol for this map as for the structure map $(\Pi V)^{\otimes n} \rightarrow \Pi V$ (to which it is dual). The notion of an L_∞ morphism does not have a straightforward interpretation in terms of operads.
- (2) Note that an L_∞ structure, as defined above, is an *odd* derivation. It follows that if one adopts our convention to view an MC element in a dgla $\mathfrak{g}$ as an *even* element in $\Pi \mathfrak{g}$ (which ameliorates the sign issues) then one cannot consider an L_∞ structure as an MC element. It is, of course, possible to recover consistency by applying the parity shift to the Lie algebra of formal vector fields but this creates other notational complications which do not justify that. This discrepancy underscores the different roles played by the ‘big’ L_∞ algebras (such as Lie algebras of formal vector fields) and ‘small’ L_∞ algebras which are themselves MC elements in those big Lie algebras.
- (3) Just as in the operadic context the above definition admits an extension covering A -linear L_∞ algebras where A is a formal cdga. Namely, an A -linear L_∞ algebra on V is a continuous A -linear odd derivation of $A \otimes \hat{S}\Pi V^*$ having its image in $A_+ \otimes \hat{S}\Pi V^*$, which satisfies the MC equation, and has no constant term; the notion of an A -linear L_∞ algebra is defined similarly.
- (4) If $\hat{S}\Pi U^* \rightarrow \hat{S}\Pi V^*$ is a map of formal cdgas representing an L_∞ map $V \rightarrow U$ then f can be written as $f = f_1 + f_2 + \dots$ where $f_n : \Pi U^* \rightarrow (\Pi V^*)^{\otimes n}, n = 1, 2, \dots$ is the component of f of degree $n - 1$ with respect to the tensor grading on $\hat{S}\Pi U^*$ and $\hat{S}\Pi V^*$. Note that the component f_0 is absent since there are no nontrivial algebra maps $\hat{S}\Pi U^* \rightarrow \mathbf{k}$. However if U and V are A -linear L_∞ algebras for a formal cdga A then an A -linear L_∞ map represented by a map of cdgas $f : \hat{S}\Pi U^* \rightarrow A \otimes \hat{S}\Pi V^*$ *might* have a nontrivial f_0 -part (corresponding to a cdga map $\hat{S}\Pi U^* \rightarrow A$). An L_∞ map f having nonzero f_0 will be referred to as a *curved* L_∞ map.

Definition 3.3.

- (1) Let (V, m) be an L_∞ algebra; then $I \subset V$ is called an L_∞ ideal in V if for any n we have $m_n(\Pi v_1, \dots, \Pi v_n) \in \Pi I$ as long as at least one of the v_i 's belongs to I . It is clear that in that case both I and $U = V/I$ inherit structures of L_∞ algebras.
- (2) If, in addition, the sequence of L_∞ algebras

$$I \rightarrow V \rightarrow U$$

splits as a sequence of dg vector spaces then it is called an *extension* of U by I ; the choice of a splitting is thus part of the data of an extension. The set of such extensions will be denoted by $\text{Ex}_{L_\infty}(U, I)$.

Remark 3.4. It is useful to reformulate the notion of an extension of L_∞ algebras in terms of its representing formal cdgas. Given an extension $e : I \rightarrow V \rightarrow U$ consider the formal cdgas $\hat{S}\Pi V^*$ and $\hat{S}\Pi U^*$ representing the L_∞ algebras V and U . Then the condition that I is an L_∞ ideal in V is equivalent to saying that $\hat{S}\Pi U^*$ is a sub cdga in $\hat{S}\Pi V^*$.

The set $\text{Ex}_{L_\infty}(U, I)$ is contravariantly functorial with respect to L_∞ maps in the variable U . Indeed, suppose that $e : I \rightarrow V \rightarrow U$ is an extension and let an L_∞ map $W \rightarrow U$ be represented by a map of formal cdgas $f : \hat{S}\Pi U^* \rightarrow \hat{S}\Pi W^*$ and consider the sequence of maps of formal cdgas

$$(3.1) \quad \hat{S}\Pi W^* \rightarrow \hat{S}\Pi W^* \otimes_{\hat{S}\Pi U^*} \hat{S}\Pi V^* \rightarrow \hat{S}\Pi I^*.$$

Here $\hat{S}\Pi W^*$ is regarded as a $\hat{S}\Pi U^*$ -module via f . Note that (3.1) represents an L_∞ extension of W by I . To see that note that there is a decomposition $V \cong I \oplus U$ associated to the extension e and it follows that the representing cdga of V then has the form $\hat{S}\Pi V^* \cong \hat{S}\Pi I^* \otimes \hat{S}\Pi U^*$. Thus, the sequence (3.1) can be rewritten as

$$\hat{S}\Pi W^* \rightarrow \hat{S}\Pi I^* \otimes \hat{S}\Pi W^* \rightarrow \hat{S}\Pi I^*$$

which clearly does represent an L_∞ extension of W by I . This extension will be denoted by $f^*(e)$.

The functor $U \mapsto \text{Ex}_{L_\infty}(U, I)$ is analogous to the functor associating to a topological space the set of fibrations with a typical fiber having a fixed homotopy type (the L_∞ algebra I playing the role of the said fiber). The extension $f^*(e)$ described above should be viewed as an analogue of the induced fibration. In light of this analogy the following result is not unexpected.

Proposition 3.5. *The functor $U \mapsto \text{Ex}_{L_\infty}(U, I)$ is represented by the dgla $\text{Der}(\hat{S}\Pi I^*)$. In more detail: there is natural bijective correspondence between the elements in $\text{Ex}_{L_\infty}(U, I)$ and L_∞ maps $U \rightarrow \text{Der}(\hat{S}\Pi I^*)$.*

Proof. Let $I \rightarrow V \rightarrow U$ be an extension; its dg splitting gives an isomorphism of dg vector spaces $V \cong I \oplus U$. We have the following sequence of representing formal cdgas: $\hat{S}\Pi U^* \rightarrow \hat{S}\Pi U^* \otimes \hat{S}\Pi I^* \rightarrow \hat{S}\Pi I^*$. The L_∞ structure m_V on $V = \hat{S}\Pi U^* \otimes \hat{S}\Pi I^*$ is determined by its restriction onto $\hat{S}\Pi I^*$. This restriction can be viewed as an $\hat{S}\Pi U^*$ -linear odd derivation of $\hat{S}\Pi U^* \otimes \hat{S}\Pi I^*$, i.e. an element $\xi \in \hat{S}\Pi U^* \otimes \text{Der}(\hat{S}\Pi I^*)$. The condition $m_V^2 = 0$ gets translated into the MC condition for ξ . This establishes a 1-1 correspondence between L_∞ extensions of U by I and elements in $\text{MC}(\text{Der}(\hat{S}\Pi I^*), \hat{S}\Pi U^*)$. It only remains to observe that an L_∞ map $U \rightarrow \text{Der}(\hat{S}\Pi I^*)$ is also naturally identified with an element in $\text{MC}(\text{Der}(\hat{S}\Pi I^*), \hat{S}\Pi U^*)$. $\square$

Let us see in more concrete terms how an L_∞ map $U \rightarrow \text{Der}(\hat{S}\Pi I^*)$ gives rise to an L_∞ structure on $V = U \oplus I$. The latter is specified by the collection of structure maps $m_n : (\Pi V)^{\otimes n} \rightarrow V$. Since m_n is symmetric it suffices to specify its value on the collection $(\Pi u_1, \dots, \Pi u_k, \Pi x_1, \dots, \Pi x_l)$ where $u_1, \dots, u_k \in U, x_1, \dots, x_l \in I$ and $k + l = n$.

The proof of Proposition 3.5 yields the following corollary. Here we adopt the convention of Remark 3.2(1): a derivation $\eta \in \text{Der}(\hat{S}\Pi I^*)$ is determined by a collection of maps $\Pi I^* \rightarrow (\Pi I^*)^{\otimes l}$, $l = 0, 1, 2, \dots$, and we denote by $\eta_l : (\Pi I)^{\otimes l} \rightarrow \Pi I$ the dual maps.

Corollary 3.6. *Let $f = (f_1, f_2, \dots)$ be an L_∞ map $U \rightarrow \text{Der}(\hat{S}\Pi I^*)$. Then the L_∞ structure $m^f = (m_1^f, m_2^f, \dots)$ on $U \oplus I$ associated with f has the form:*

$$m_n^f(w_1, \dots, w_k, x_1, \dots, x_l) = (\Pi f_k(w_1, \dots, w_k))_l(x_1, \dots, x_l)$$

where $w_1, \dots, w_k \in \Pi U$, $x_1, \dots, x_l \in I$, $1 \leq l \leq n$, $0 \leq k \leq n-1$ and $k+l = n$; the value of m_n^f on other collections of arguments is zero. In the case $l = 0$, $k = n$ the formula is modified as follows:

$$m_n^f(w_1, \dots, w_n) = (\Pi f_n(w_1, \dots, w_n))_0 + m_n^U(w_1, \dots, w_n)$$

where m_n^U is the corresponding L_∞ structure map for U .

□

Remark 3.7.

- (1) Note that the dgla $\text{Der}(\hat{S}\Pi I^*)$ is just the Chevalley-Eilenberg complex $\text{CE}(I, I)$ of the L_∞ algebra I with coefficients in itself, cf. [2].
- (2) The classical notion of an extension of a Lie algebra by an ideal is, of course, a (very) special case of an L_∞ extension. Indeed, suppose that $I \rightarrow V \rightarrow U$ is such an extension. It corresponds, via Proposition 3.5 to an L_∞ map $f = f_1 + f_2 + \dots : U \rightarrow \text{Der}(\hat{S}\Pi I^*)$ for which, firstly, all higher components $f_3, f_4, \dots$ are zero, secondly, the image of $f_1 : U \rightarrow \text{Der}(\hat{S}\Pi I^*)$ consists of linear derivations $\text{End}(\Pi I^*) \in \text{Der}(\hat{S}\Pi I^*)$ and thirdly, the image of $f_2 : U \otimes U \rightarrow \text{Der}(\hat{S}\Pi I^*)$ consists of constant derivations $\Pi I^* \in \text{Der}(\hat{S}\Pi I^*)$. The subspace consisting of linear and constant derivations forms a Lie subalgebra in $\text{Der}(\hat{S}\Pi I^*)$ but it is not closed with respect to the differential unless the Lie bracket in I is zero. In the latter case f_1 gives I the structure of a U -module and f_2 determines a 2-cocycle of U with values in I . Extensions of this kind are classified, as expected, in terms of the second Chevalley-Eilenberg cohomology of U with coefficients in I but we will not elaborate this point further.

3.2. Extensions related to Lie algebras or formal vector fields and MC elements.

The following example of an L_∞ extension is what prompted us to develop the general theory in the first place.

Example 3.8. Let V be a dg vector space, and $\mathfrak{c}(V)$ be the dgla of continuous derivations of $\hat{S}\Pi V^*$ vanishing at zero. Let us consider the tautological embedding $f : \mathfrak{c}(V) \hookrightarrow \text{Der}(\hat{S}\Pi V^*)$; note that this is a usual map between two dglas. Associated to f is an L_∞ algebra structure on $\tilde{\mathfrak{c}}(V) := \mathfrak{c}(V) \oplus V$ defined as follows.

On the subspace $\Pi \mathfrak{c}(V) \subset \Pi \tilde{\mathfrak{c}}(V)$ the product m_2 is just (the parity shift of) the bracket on the dgla $\mathfrak{c}(V)$ and the higher L_∞ products are zero. On the subspace V the L_∞ structure is zero. The remaining non-zero L_∞ products are of the form $m_n(\Pi g, w_1, \dots, w_{n-1})$ with $g \in \text{Hom}((\Pi V)^{\otimes n-1}, \Pi V) \subset \mathfrak{g}(V)$ and $w_i \in \Pi V, i = 1, 2, \dots, n-1$. Namely, $m_n(\Pi g, w_1, \dots, w_{n-1}) = g(w_1, \dots, w_{n-1})$.

Varying the choice of a Lie subalgebra in $\text{Der}(\hat{S}\Pi V^*)$ one can construct many other examples of nontrivial L_∞ structures. We list some natural examples; the L_∞ structure maps are given by the same formulas as in Example 3.8.

Example 3.9.

- (1) Associated to the identity map $\text{Der}(\hat{S}\Pi V^*) \rightarrow \text{Der}(\hat{S}\Pi V^*)$ is an L_∞ structure on $\text{Der}(\hat{S}\Pi V^*) \oplus V$. This is, perhaps the most natural special case; however it will not play a role in the present paper.
- (2) Suppose that the dg vector space V is endowed with a symmetric non-degenerate scalar product; this is equivalent to having a linear (super)symplectic structure on ΠV . Denote by $\mathfrak{cs}(V)$ the dgla of *symplectic derivations* of $\hat{S}\Pi V^*$ vanishing at zero. Then the embedding of dglas $\mathfrak{cs}(V) \hookrightarrow \text{Der}(\hat{S}\Pi V^*)$ gives an L_∞ structure on $\tilde{\mathfrak{cs}}(V) := \mathfrak{cs}(V) \oplus V$.

- (3) Consider the dgla $\mathfrak{a}(V)$ consisting of continuous derivations of $\hat{T}\Pi V^*$ vanishing at zero; since a derivation preserves the commutator ideal any such derivation gives rise to a derivation of the corresponding dgla $\text{Der}(\hat{S}\Pi V^*)$. This determines a dgla map $\mathfrak{a}(V) \rightarrow \text{Der}(\hat{S}\Pi V^*)$ and thus it gives rise to an L_∞ structure on $\tilde{a}(V) := \mathfrak{a}(V) \oplus V$.
- (4) Suppose that the dg vector space V is endowed with a symmetric non-degenerate scalar product and denote by $\mathfrak{as}(V)$ the corresponding dgla of symplectic derivations of $\hat{T}\Pi V^*$ (cf. for example [8] concerning this notion). There is a dgla map $\mathfrak{as}(V) \rightarrow \text{Der}(\hat{S}\Pi V^*)$ defined as the composition $\mathfrak{as}(V) \hookrightarrow \mathfrak{a}(V) \rightarrow \text{Der}(\hat{S}\Pi V^*)$. We will denote the corresponding L_∞ structure on $\mathfrak{as}(V) \oplus V$ by $\tilde{\mathfrak{as}}(V)$.

For us the dglas $\mathfrak{c}(V)$, $\mathfrak{cs}(V)$, $\mathfrak{a}(V)$ and $\mathfrak{as}(V)$ are important because MC elements in these are (by definition), respectively, L_∞ , cyclic L_∞ , A_∞ and cyclic A_∞ structures on the dg vector space V . It turns out that the MC elements in the corresponding L_∞ extensions of these Lie algebras also admit a very natural description.

Theorem 3.10. *Let A be a formal cgda and V is a dg vector space.*

- (1) *An element in $\text{MC}(\tilde{\mathfrak{c}}(V), A)$ is an A -linear L_∞ structure on V together with a choice of an element in $\text{MC}(V, A)$.*
- (2) *An element in $\text{MC}(\tilde{\mathfrak{cs}}(V), A)$ is an A -linear cyclic L_∞ structure on V together with a choice of an element in $\text{MC}(V, A)$.*
- (3) *An element in $\text{MC}(\tilde{\mathfrak{a}}(V), A)$ is an A -linear A_∞ structure on V together with a choice of an element in $\text{MC}(V, A)$.*
- (4) *An element in $\text{MC}(\tilde{\mathfrak{as}}(V), A)$ is an A -linear cyclic A_∞ structure on V together with a choice of an element in $\text{MC}(V, A)$.*

Proof. The proofs in all four cases are completely analogous and so we will restrict ourselves to considering part (1). Let $g + \xi \in (A_+ \otimes \Pi \mathfrak{c}(V)) \oplus (A_+ \otimes \Pi V) = A_+ \otimes \Pi \tilde{\mathfrak{c}}(V)$ be an element in $\text{MC}(\tilde{\mathfrak{c}}(V), A)$. Taking into account the formulas for the L_∞ structure maps in $\tilde{\mathfrak{c}}(V)$ given in Example 3.8 we see that the MC equation for $g + \xi$,

$$d(g + \xi) + m_1(g + \xi) + \frac{1}{2!}m_2(g + \xi, g + \xi) + \dots + \frac{1}{n!}m_n(g + \xi, \dots, g + \xi) + \dots = 0,$$

is equivalent to two equations:

$$d(g) + \frac{1}{2!}m_2(g, g) = 0;$$

$$d(\xi) + m_2(g, \xi) + \frac{1}{2!}m_3(g, \xi, \xi) + \dots + \frac{1}{(n-1)!}m_n(g, \xi, \dots, \xi) + \dots = 0.$$

The first of these equations is precisely the condition that $g = (g_1, g_2, \dots) \in A_+ \otimes \mathfrak{c}(V)$ is an A -linear L_∞ structure on V (here $g_i \in \text{Hom}(V^{\otimes i}, A_+ \otimes V)$, whereas the second can be rewritten as

$$d(\xi) + g_1(\xi) + \frac{1}{2!}g_2(\xi, \xi) + \dots + \frac{1}{(n-1)!}g_{n-1}(\xi, \dots, \xi) = 0,$$

which is precisely the condition specifying ξ as an MC element in the L_∞ algebra with underlying space $A_+ \otimes V$ whose L_∞ structure is given by the collection $g = (g_1, g_2, \dots)$. $\square$

4. TWISTING IN L_∞ AND A_∞ ALGEBRAS

Let us recall the standard definition of MC twisting following [2]. Our usual convention is to let the underlying graded vector space V carry its own differential d_V ; this results in a slight generalization of the treatment in op.cit.

Definition 4.1.

- (1) Let A be a formal cdga, $((V, d_V), m)$ be an A -linear L_∞ algebra structure on a dg vector space (V, d_V) and $\xi \in \text{MC}(V, A)$, viewed as an even A -linear derivation of $A \otimes \hat{S}\Pi V^*$. Then $m^\xi := e^\xi m e^{-\xi} - d_{A \otimes \Pi V}(\xi)$, considered as an odd A -linear derivation of $A \otimes \hat{S}\Pi V^*$,

satisfies the MC equation and is, therefore, an A -linear L_∞ structure on (V, d_V) . This structure is called the *twisted* L_∞ structure (by the MC element ξ).

- (2) Let A be a formal cdga, $((V, d_V), m)$ be an A -linear A_∞ algebra structure on a dg vector space (V, d_V) , and $\xi \in \text{MC}(V, A)$, viewed as an even A -linear derivation of $A \otimes \hat{T}\Pi V^*$. Then $m^\xi := e^\xi m e^{-\xi} - d_{A \otimes \Pi V}(\xi)$, considered as an odd A -linear derivation of $A \otimes \hat{T}\Pi V^*$ satisfies the MC equation and is, therefore, an A -linear A_∞ structure on (V, d_V) . This structure is called the *twisted* A_∞ structure (by the MC element ξ).

We now give explicit formulas for the structure maps of twisted L_∞ or A_∞ algebras. As usual, we denote by A a fixed formal cdga.

- (1) Let V be an L_∞ algebra and $\xi \in \text{MC}(V, A)$. Then for $x_1, \dots, x_n \in \Pi V$,

$$(4.1) \quad m_n^\xi(x_1, \dots, x_n) = \sum_{i=0}^{\infty} \frac{1}{i!} m_{n+i}(\xi, \dots, \xi, x_1, \dots, x_n).$$

- (2) Let V be an A_∞ algebra and $\xi \in \text{MC}(V, A)$. Then for $x_1, \dots, x_n \in \Pi V$,

$$(4.2) \quad m_n^\xi(x_1, \dots, x_n) = \sum_{i=0}^{\infty} m_{n+i}(\xi, \dots, \xi | x_1, \dots, x_n),$$

where

$$m_{n+i}(\xi, \dots, \xi | x_1, \dots, x_n) := \sum m_{n+i}(z_1, \dots, z_{n+i}),$$

the sum running over all $\binom{n+i}{n}$ sequences $z_1, \dots, z_{n+i}$ containing $x_1, \dots, x_n$ in order, together with i copies of ξ .

Remark 4.2. It is easy to see that $\eta \mapsto \eta - \xi$ establishes a one-to-one correspondence $\text{MC}((V, m), A) \rightarrow \text{MC}((V, m^\xi), A)$. In particular, $-\xi \in \text{MC}((V, m^\xi), A)$.

4.1. Non-formal twisting in L_∞ and A_∞ algebras. We would like to outline briefly another approach to twisting in L_∞ and A_∞ algebras, which makes no use of a formal ‘coefficient’ cdga. As far as we know this approach is new and it is of independent interest; however it plays a very minor role in the present paper and so we will not give too detailed a treatment.

Let (V, d_V) be a dg vector space and $((V, d_V), m)$ be a corresponding L_∞ structure or A_∞ structure. Let $\xi \in \Pi V$ and consider, in the L_∞ case, the MC equation

$$(4.3) \quad d_V(\xi) + \sum_{i=1}^{\infty} \frac{1}{i!} m_i(\xi^{\otimes i}) = 0,$$

which in the A_∞ case should be replaced with

$$(4.4) \quad d_V(\xi) + \sum_{i=1}^{\infty} m_i(\xi^{\otimes i}) = 0.$$

Note that equations (4.3) and (4.4) only make sense if the sums appearing in them are finite; a solution of either will be called an MC element in V and the set of MC elements will be denoted by $\text{MC}(V)$. We will now describe some natural conditions on V which are necessary for the existence of MC elements and for the notion of MC twisting. To this end consider any line (i.e. a one-dimensional subspace) $l \subset \Pi V$ and the ideal $I = I_l \subset S\Pi V^*$ generated by the annihilator of l in V^* . For example, if we choose coordinates x_i in ΠV so that $S\Pi V^* \cong \mathbf{k}[x_1, x_2, \dots]$ and l consists of the elements $(\alpha, 0, 0, \dots)$, $\alpha \in \mathbf{k}$, then the ideal I is generated by $x_2, x_3, \dots$. We denote by $\hat{S}_I \Pi V^*$ the completion of $S\Pi V^*$ at the ideal I and by $\text{Der}(\hat{S}_I \Pi V^*)$ the corresponding dgl of continuous derivations.

In the A_∞ case we similarly consider the two-sided ideal $I = I_l$ in $T\Pi V^*$ generated by the annihilator of l in V^* , the corresponding completion $\hat{T}_I \Pi V^*$ and the dgl $\text{Der}(\hat{T}_I \Pi V^*)$. Note in passing that the dga $\hat{T}_I \Pi V^*$ and the cdga $\hat{S}_I \Pi V^*$ are both topological, but *not* formal, owing to the fact that the ideal I is not maximal. Because of that, nontrivial dga maps $\hat{T}_I \Pi V^* \rightarrow \mathbf{k}$ and $\hat{S}_I \Pi V^* \rightarrow \mathbf{k}$ may exist.

Definition 4.3.

- (1) A *restricted* L_∞ structure on V is an odd element $m \in \text{Der}(\hat{S}_I \Pi V^*)$ which has no constant term and satisfies the MC equation $d_V(m) + \frac{1}{2}[m, m] = 0$. The triple (V, l, m) will be referred to as a restricted L_∞ algebra and the algebra $\hat{S}_I \Pi V^*$, supplied with the differential $d_V + m$, as its representing cdga.
- (2) A *restricted* A_∞ structure on V is an odd element $m \in \text{Der}(\hat{T}_I \Pi V^*)$ which has no constant term and satisfies the MC equation $d_V(m) + \frac{1}{2}[m, m] = 0$. The triple (V, l, m) will be referred to as a restricted A_∞ algebra and the algebra $\hat{T}_I \Pi V^*$, supplied with the differential $d_V + m$, as its representing dga.

Then we have the following standard result whose proof is virtually identical to the proof of Proposition 2.2 of [2].

Proposition 4.4.

- (1) Let (V, l, m) be a restricted L_∞ algebra. Then the set of solutions of (4.3) with $\xi \in l$ is in 1-1 correspondence with (continuous) cdga maps $\hat{S}_I \Pi V^* \rightarrow \mathbf{k}$.
- (2) Let (V, l, m) be a restricted A_∞ algebra. Then the set of solutions of (4.3) with $\xi \in l$ is in 1-1 correspondence with (continuous) dga maps $\hat{T}_I \Pi V^* \rightarrow \mathbf{k}$.

□

Given an MC element ξ in a restricted L_∞ or A_∞ algebra V , the latter can be twisted by ξ .

Definition 4.5.

- (1) Let (V, l, m) be a restricted L_∞ algebra structure on a dg vector space V and $\xi \in l$ be an MC-element in V , viewed as an even derivation of $\hat{S}_I \Pi V^*$. Then $m^\xi := e^\xi m e^{-\xi} - d_{\Pi V}(\xi)$ considered as an odd derivation of $\hat{S}_I \Pi V^*$, satisfies the MC equation and therefore determines a (restricted) L_∞ structure on V . This structure is called the *twisted* L_∞ structure (by the MC element ξ).
- (2) Let (V, l, m) be a restricted A_∞ algebra structure on a dg vector space V and $\xi \in l$ be an MC-element in V , viewed as an even derivation of $\hat{T}_I \Pi V^*$. Then $m^\xi := e^\xi m e^{-\xi} - d_{\Pi V}(\xi)$ considered as an odd derivation of $\hat{T}_I \Pi V^*$, satisfies the MC equation and therefore determines a (restricted) L_∞ structure on V . This structure is called the *twisted* A_∞ structure (by the MC element ξ).

It is easy to see that formulas (4.1) and (4.2) still hold in the restricted context.

4.2. Twisting automorphisms of Lie algebras of formal vector fields. Twisting in infinity-algebras by MC elements naturally gives rise to L_∞ automorphisms in the L_∞ algebras governing the corresponding structures.

Theorem 4.6. *Let V be a dg vector space.*

- (1) Consider the automorphism tw of the functor $A \rightarrow \text{MC}(\tilde{\mathfrak{c}}(V), A)$ given by $\text{tw} : (m, \xi) \mapsto (m^\xi, -\xi)$ where A is a formal cdga. This automorphism induces an L_∞ automorphism of $\tilde{\mathfrak{c}}(V)$ (which will be denoted by the same symbol), with components $\text{tw}_n : (\Pi \tilde{\mathfrak{c}}(V))^{\otimes n} \rightarrow \Pi \tilde{\mathfrak{c}}(V)$, $n = 1, 2, \dots$. For any $f \in \text{Hom}(S^{k+n-1} \Pi V, \Pi V)$ with $k \geq 1$ and $w_1, \dots, w_{n-1} \in \Pi V$, we have

$$\text{tw}_n(\Pi f, w_1, \dots, w_{n-1}) \in \Pi \text{Hom}(S^k \Pi V, \Pi V) \subset \Pi \tilde{\mathfrak{c}}(V),$$

given by the formula

$$(4.5) \quad \text{tw}_n(\Pi f, w_1, \dots, w_{n-1})(x_1, \dots, x_k) = \Pi f(w_1, \dots, w_{n-1}, x_1, \dots, x_k).$$

In addition $\text{tw}_1(w) = -w$ for all $w \in \Pi V$. The value of tw_n on any other collection of homogeneous elements is zero.

- (2) Consider the automorphism tw of the functor $A \rightarrow \text{MC}(\mathfrak{cs}(V), A)$ given by $\text{tw} : (m, \xi) \mapsto (m^\xi, -\xi)$. This automorphism induces an L_∞ automorphism of $\tilde{\mathfrak{cs}}(V)$ which is simply the restriction of the corresponding L_∞ automorphism in part (1).

- (3) Consider the automorphism tw of the functor $A \rightarrow \text{MC}(\tilde{\mathfrak{a}}(V), A)$ given by $\text{tw} : (m, \xi) \mapsto (m^\xi, -\xi)$. This automorphism induces an L_∞ automorphism of $\tilde{\mathfrak{a}}(V)$ (which will be denoted by the same symbol), with components $\text{tw}_n : (\Pi \tilde{\mathfrak{a}}(V))^{\otimes n} \rightarrow \Pi \tilde{\mathfrak{a}}(V), n = 1, 2, \dots$. For any $f \in \text{Hom}((\Pi V)^{\otimes n-1+k}, \Pi V)$ with $k \geq 1$ and $w_1, \dots, w_{n-1} \in \Pi V$, we have

$$\text{tw}_n(\Pi f, w_1, \dots, w_{n-1}) \in \Pi \text{Hom}((\Pi V)^{\otimes k}, \Pi V) \subset \Pi \tilde{\mathfrak{a}}(V),$$

given by the formula

$$(4.6) \quad \text{tw}_n(\Pi f, w_1, \dots, w_{n-1})(x_1, \dots, x_{k-n+1}) = \sum_{\sigma \in S_{n-1}} \Pi f(\sigma(w_1 \otimes \dots \otimes w_{n-1}) \mid x_1, \dots, x_{k-n+1}).$$

In addition $\text{tw}_1(w) = -w$ for all $w \in \Pi V$. The value of tw_n on any other collection of homogeneous elements is zero.

- (4) Consider the automorphism tw of the functor $A \rightarrow \text{MC}(\mathfrak{a}\mathfrak{s}(V), A)$ given by $\text{tw} : (m, \xi) \mapsto (m^\xi, -\xi)$. This automorphism induces an L_∞ automorphism of $\mathfrak{a}\mathfrak{s}(V)$ which is simply the restriction of the corresponding L_∞ automorphism in part (3).

Proof. Parts (2) and (4) are immediate consequences of (1) and (3) respectively. Let us prove (1). Denote temporarily by tw' the automorphism of $\hat{S}\Pi[\tilde{\mathfrak{c}}(V)]^*$ given by formula (4.5). It is sufficient to prove that for any formal cdga A the automorphism tw' induces the same map on A -linear MC elements $(m, \xi) \in \text{MC}(\tilde{\mathfrak{c}}(V), A)$ as tw does. It would follow then that tw and tw' coincide. We have

$$\text{tw}'(m, \xi) = \sum_{n=1}^{\infty} \frac{1}{n!} \text{tw}'_n((m, \xi)^{\otimes n}) = \left(\sum_{n=1}^{\infty} \frac{1}{(n-1)!} \text{tw}'_n(m, \underbrace{\xi, \dots, \xi}_{n-1}), -\xi \right).$$

On the other hand

$$\text{tw}(m, \xi) = (m^\xi, -\xi) = (m_1^\xi + m_2^\xi + \dots, -\xi) \in \text{Der}(\hat{S}\Pi V^*) \oplus V.$$

Similarly

$$\text{tw}'(m, \xi) = (\text{tw}'(m, \xi)_1 + \text{tw}'(m, \xi)_2 + \dots, -\xi) \in \text{Der}(\hat{S}\Pi V^*) \oplus V,$$

where $\text{tw}'(m, \xi)_i \in \text{Hom}(S^i \Pi V, \Pi V) \subset \text{Der}(\hat{S}\Pi V^*)$. We have the following formula for the structure maps of twisted L_∞ structures taking into account (4.1):

$$\begin{aligned} \text{tw}'(m, \xi)_i(x_1, \dots, x_i) &= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} m_{i+n-1}(\underbrace{\xi, \dots, \xi}_{n-1}, x_1, \dots, x_i) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} m_{i+n}(\underbrace{\xi, \dots, \xi}_n, x_1, \dots, x_i) \\ &= m_i^\xi(x_1, \dots, x_i), \end{aligned}$$

where $x_1, \dots, x_i \in \Pi V$. This shows that tw' and tw agree and finishes the proof of part (1).

The proof of part (3) is similar. We denote temporarily by tw' an automorphism of $\hat{S}\Pi[\tilde{\mathfrak{a}}(V)]^*$ given by formula (4.6). It is sufficient to prove that for any formal cdga A the automorphisms tw' and tw induce the same maps on A -linear MC elements $(m, \xi) \in \text{MC}(\tilde{\mathfrak{a}}(V), A)$. It would follow then that tw and tw' coincide. We have

$$\text{tw}'(m, \xi) = \sum_{n=1}^{\infty} \frac{1}{n!} \text{tw}'_n((m, \xi)^{\otimes n}) = \left(\sum_{n=1}^{\infty} \frac{1}{(n-1)!} \text{tw}'_n(m, \underbrace{\xi, \dots, \xi}_{n-1}), -\xi \right).$$

On the other hand $\text{tw}(m, \xi) = (m^\xi, -\xi) = (m_1^\xi + m_2^\xi + \dots, -\xi)$ and we have the following formula for the structure maps of twisted L_∞ structures taking into account (4.2):

$$\begin{aligned} \text{tw}'(m, \xi)_i(x_1, \dots, x_i) &= \sum_{n=1}^{\infty} \sum_{\sigma \in S_{n-1}} \frac{1}{(n-1)!} m_{i+n-1}(\underbrace{\sigma(\xi \otimes \dots \otimes \xi)}_{n-1} | x_1, \dots, x_i) \\ &= \sum_{n=0}^{\infty} m_{i+n}(\underbrace{\xi, \dots, \xi}_n | x_1, \dots, x_i) \\ &= m_i^\xi(x_1, \dots, x_i), \end{aligned}$$

where $x_1, \dots, x_i \in \Pi V$. This shows that tw' and tw agree and finishes the proof of part (3). $\square$

4.3. Twisting automorphisms in operads $\mathcal{L}_\infty$ and $\mathcal{A}_\infty$. We will now describe how MC twisting in infinity algebras gives rise to automorphisms in the operads $\mathcal{L}_\infty$ and $\mathcal{A}_\infty$.

Theorem 4.7.

- (1) *There exists an automorphism tw of the dg operad $\mathcal{L}_\infty$ given by the formulas*

$$\begin{aligned} \text{tw}(x) &= -x; \\ \text{tw}(m_n) &= \sum_{i=0}^{\infty} \frac{1}{i!} m_{n+i} \circ (\underbrace{x \otimes \dots \otimes x}_i) \end{aligned}$$

such that for any formal cdga A , an A -linear L_∞ algebra (V, m) and $\xi \in \text{MC}(V, A)$, the A -linear $\mathcal{L}_\infty$ -algebra given by the composition of $\text{tw} : \mathcal{L}_\infty \rightarrow \mathcal{L}_\infty$ with the operadic action map $\mathcal{L}_\infty \rightarrow A_+ \otimes \mathcal{E}(\Pi V)$ determines the pair $\text{tw}(m, \xi) = (m^\xi, -\xi)$.

- (2) *There exists an automorphism tw of the dg modular operad $\overline{\mathcal{L}}_\infty$ given by the same formula as in part (1) and such that for any formal cdga A , a cyclic A -linear L_∞ algebra $(\underline{V}, m)$ and $\xi \in \text{MC}(V, A)$, the A -linear $\overline{\mathcal{L}}_\infty$ -algebra given by the composition of $\text{tw} : \overline{\mathcal{L}}_\infty \rightarrow \overline{\mathcal{L}}_\infty$ with the action map of modular operads $\overline{\mathcal{L}}_\infty \rightarrow A_+ \otimes \mathcal{E}(\Pi V)$ determines the pair $\text{tw}(m, \xi) = (m^\xi, -\xi)$.*
- (3) *There exists an automorphism tw of the dg operad $\mathcal{A}_\infty$ given by the formula*

$$\begin{aligned} \text{tw}(x) &= -x; \\ \text{tw}(m_n) &= \sum_{i=0}^{\infty} \sum_{\sigma \in \text{Sh}(i, n)} \sigma(m_{i+n}) \circ (\underbrace{x \otimes \dots \otimes x}_i) \end{aligned}$$

such that for any formal cdga A , an A -linear A_∞ algebra (V, m) and $\xi \in \text{MC}(V, A)$, the A -linear $\mathcal{A}_\infty$ -algebra given by the composition of $\text{tw} : \mathcal{A}_\infty \rightarrow \mathcal{A}_\infty$ with the operadic action map $\mathcal{A}_\infty \rightarrow A_+ \otimes \mathcal{E}(\Pi V)$ determines the pair $\text{tw}(m, \xi) = (m^\xi, -\xi)$. Here the notation $\text{Sh}(i, n)$ stands for the set of (i, n) -shuffles.

- (4) *There exists an automorphism tw of the dg modular operad $\overline{\mathcal{A}}_\infty$ given by the same formula as in part (3) and such that for any formal cdga A , a cyclic A -linear A_∞ algebra $(\underline{V}, m)$ and $\xi \in \text{MC}(V, A)$, the A -linear $\overline{\mathcal{A}}_\infty$ -algebra given by the composition of $\text{tw} : \overline{\mathcal{A}}_\infty \rightarrow \overline{\mathcal{A}}_\infty$ with the action map of modular operads $\overline{\mathcal{A}}_\infty \rightarrow A_+ \otimes \mathcal{E}(\Pi V)$ determines the pair $\text{tw}(m, \xi) = (m^\xi, -\xi)$.*

Proof. The main issue is proving the compatibility of the automorphism tw with differentials; we start with part (3). Suppose that a discrete dg vector space V has the structure of an $\mathcal{A}_\infty$ algebra given by a (continuous) map of operads $f : \mathcal{A}_\infty \rightarrow \mathcal{E}(\Pi V)$ and consider the composite map

$$\mathcal{A}_\infty \xrightarrow{\text{tw}} \mathcal{A}_\infty \xrightarrow{f} \mathcal{E}(\Pi V).$$

The images $f \circ \text{tw}(m_n), f \circ \text{tw}(x)$ in $\mathcal{E}(\text{IIV})$ endow V with a structure of a A_∞ algebra twisted by an MC element (in the sense of Section 4.1) and it follows that $f \circ \text{tw}$ is a dg map although tw is not yet known to be compatible with the differential. To prove the desired compatibility it suffices to find such a V on which $\mathcal{A}_\infty$ acts faithfully. This is of course impossible, since $\mathcal{A}_\infty$ itself is not discrete.

To circumvent this minor difficulty, we note that $\mathcal{A}_\infty = \varprojlim_m \mathcal{A}_\infty/(x^m)$, an inverse limit of dg operads, and that tw descends to an automorphism of $\mathcal{A}_\infty/(x^m)$. Thus it suffices to prove that tw , viewed as an automorphism of $\mathcal{A}_\infty/(x^m)$, commutes with the differential in $\mathcal{A}_\infty/(x^m)$. Since $\mathcal{A}_\infty/(x^m)$ is discrete, its regular representation $V = \oplus_{n=0}^\infty (\mathcal{A}_\infty/(x^m))(n)$ is also discrete (and of course faithful), and then the argument suggested above applies, giving the desired conclusion.

Let us now show that tw preserves the cyclic structure. In the $\mathcal{L}_\infty$ case this boils down to proving that the element $\sum_{i=0}^\infty \frac{1}{i!} m_{n+i} \circ \underbrace{(x \otimes \dots \otimes x)}_i$ is cyclically invariant but this is clear since by the cyclicity of L each individual term $m_{n+i} \circ \underbrace{(x \otimes \dots \otimes x)}_i$ is cyclically invariant. Therefore tw is an automorphism of a cyclic operad $\mathcal{L}_\infty$ and thus determines an automorphism of its modular closure $\overline{\mathcal{L}}_\infty$.

Similarly, in the $\mathcal{A}_\infty$ case we are required to prove that $\sum_{\sigma \in \text{Sh}(i,n)} \sigma(m_{i+n}) \circ \underbrace{(x \otimes \dots \otimes x)}_i$ is invariant with respect to the action of the n -cycle and this follows from the fact that the latter determines a bijection of the set of (i, n) -shuffles onto a conjugate set of shuffles.

This completes the proof that tw is a dg automorphism in all four cases. The claim that the A -linear $\mathcal{L}_\infty$ algebra or $\mathcal{A}_\infty$ algebra structure on $A \otimes V$ obtained by pulling back along tw is the MC-twisted one is a direct consequence of formulas (4.1) and (4.2). The cyclic analogue is likewise clear. $\square$

5. TWISTING IN GENERAL OPERADS

Until now we studied the concept of MC twisting in infinity-algebras and operads governing twisted structures. Since an operad itself is a generalization of an algebra it is natural to ask whether it is possible to twist operads themselves. It is not clear that a full-fledged generalization is possible; here we only consider a very restricted notion of twisting by MC elements corresponding to unary operations. Our main example of such a twisting is a functor $\mathcal{O} \mapsto \hat{\mathcal{O}}$ which allows one to consider the underlying differential on a given dg vector space as part of an operadic structure.

5.1. Operad twisting. Let $\mathcal{O}$ be a dg operad. Associated to $\mathcal{O}$ is the dgla $\text{Der}(\mathcal{O})$ of derivations of $\mathcal{O}$ consisting of collections of linear maps $\xi_n : \mathcal{O}(n) \rightarrow \mathcal{O}(n)$, $n = 0, 1, \dots$ such that $\xi_n(f \circ_i g) = (\xi_n f) \circ_i g + (-1)^{|f||\xi_n|} f \circ_i \xi_n(g)$ where $f \in \mathcal{O}(k)$, $g \in \mathcal{O}(l)$, $i = 1, \dots, l$. It is clear that the commutator of derivations is again a derivation and the differential in $\mathcal{O}$ induces one on $\text{Der}(\mathcal{O})$; thus the latter is indeed a dgla. An MC element ξ in it allows one to twist the differential $d_{\mathcal{O}}$ in $\mathcal{O}$ by the formula $d^\xi := d_{\mathcal{O}} + \xi$.

Consider a special case of this construction corresponding to an *inner* derivation of $\mathcal{O}$. Note that the dga $\mathcal{O}(1)$ acts on $\mathcal{O}$ as operad derivations according to the formula

$$\delta_a(z) = a \circ_1 z - (-1)^{|a||z|} \sum_{i=1}^n z \circ_i a$$

where $z \in \mathcal{O}(n)$ and $a \in \mathcal{O}(1)$. Then the correspondence $a \mapsto \delta_a$ determines a dgla map $\mathcal{O}(1) \rightarrow \text{Der}(\mathcal{O})$. If m is an MC element in $\mathcal{O}(1)$, i.e. m satisfies the equation $dm + \frac{1}{2}[m, m] = 0$, then we obtain the twisted dg operad $\mathcal{O}^m$ having the same underlying operad as $\mathcal{O}$ and whose differential is d^{δ_m} .

Example 5.1. Let (V, d_V) be a dg vector space. Then an MC element in the dga $\mathcal{E}(V, d_V)(1)$ is an odd endomorphism $m : V \rightarrow V$ such that $d_V m + m d_V + m^2 = 0$, i.e. such that $(d_V + m)^2 = 0$. The correspondence $m \mapsto d_V + m$ determines a bijection between the set of MC elements in $\mathcal{E}(V, d_V)(1)$ and the set of differentials on the graded vector space V . Furthermore,

$$[\mathcal{E}(V, d_V)]^m = \mathcal{E}(V, d_V + m).$$

Now let $\mathcal{O} \rightarrow \mathcal{E}(\Pi V)$ be an $\mathcal{O}$ algebra structure on a dg space (V, d_V) and $m \in \mathcal{O}(1)$ be an MC element which is also viewed as an odd operator $\Pi V \rightarrow \Pi V$. Then we have an induced map

$$(5.1) \quad \mathcal{O}^m \rightarrow [\mathcal{E}(\Pi V)]^m = \mathcal{E}(\Pi V, d_V + m)$$

Conversely, such a map (5.1) of twisted operads can be ‘untwisted’ by the MC element $-m \in \mathcal{O}^m(1)$ giving an $\mathcal{O}$ -algebra structure on (V, d_V) . We obtain the following result.

Proposition 5.2. *Let $\mathcal{O}$ be a dg operad and m be an MC element in $\mathcal{O}(1)$. Then an $\mathcal{O}$ -algebra structure on a dg vector space (V, d_V) is equivalent to an $\mathcal{O}^m$ algebra structure on $(V, d_V + m)$.*

□

5.2. The hat construction.

Definition 5.3. Let $\mathcal{O}$ be a formal dg operad and consider the formal operad $\mathcal{O}[m]$ generated freely by $\mathcal{O}$ and $m \in \mathcal{O}[m](1)$. The differential in $\mathcal{O}[m]$ is fixed by requiring that $\mathcal{O}$ is a dg suboperad in $\mathcal{O}[m]$ and $d(m) = -m^2$. Note this makes m an MC element of $\mathcal{O}[m]$.

Then define the hat-construction $\hat{\mathcal{O}}$ as $\hat{\mathcal{O}} := (\mathcal{O}[m])^m$, the twisting of $\mathcal{O}[m]$ by the MC element m .

Remark 5.4. Let $\mathcal{O} = \mathbf{B}\mathcal{P}$ be the cobar-construction operad of a dg operad $\mathcal{P}$. Then it follows directly from the definition that

- $\mathcal{O}[m] \cong \mathbf{B}(\mathcal{P} \oplus \mathbf{k}\langle e \rangle)$ where $\mathcal{P} \oplus \mathbf{k}\langle e \rangle$ is the operad $\mathcal{P}$ with one extra generator e in arity one having zero compositions with all elements of $\mathcal{P}$ and such that $x \circ_1 x = x$;
- $\hat{\mathcal{O}} \cong \mathbf{B}(\mathcal{P}_e)$ where $\mathcal{P}_e$ is the operad $\mathcal{P}$ with an adjoined unit (i.e. the underlying space of $\mathcal{P}_e$ is the same as that of $\mathcal{P} \oplus \mathbf{k}\langle e \rangle$ but this time operadic compositions make e a unit). In particular:

- (1) $\hat{l}_\infty \cong L_\infty$.
- (2) $\hat{a}_\infty \cong A_\infty$.

Proposition 5.5. *Let $\mathcal{O}$ be a dg operad and (V, d) be a dg vector space. Then the set of $\hat{\mathcal{O}}$ algebra structures on (V, d) is in one-to-one correspondence with the set of $\mathcal{O}$ algebra structures on (V, d') where d' is any differential on V .*

Proof. Let $f : \hat{\mathcal{O}} \rightarrow \mathcal{E}(V, d)$ be a dg operad map giving (V, d) the structure of an $\hat{\mathcal{O}}$ algebra. Then $g := f|_{\mathcal{O}}$ determines a map of dg operads $\mathcal{O} \rightarrow \mathcal{E}(V, d - f(m))$. Conversely, any dg operad map $g : \mathcal{O} \rightarrow \mathcal{E}(V, d')$ can be extended to $f : \hat{\mathcal{O}} \rightarrow \mathcal{E}(V, d)$ by setting $f(m) = d - d'$. □

Remark 5.6. We conclude that the set of $\hat{\mathcal{O}}$ algebra structures on (V, d) does not depend on d and thus, d can be taken to be equal to zero. In other words, any $\hat{\mathcal{O}}$ structure on (V, d) determines, and is determined by, an $\hat{\mathcal{O}}$ algebra structure on $(V, 0)$. It is, thus, sufficient to consider only $\hat{\mathcal{O}}$ algebra structures on graded vector spaces (i.e. with vanishing differential). Equivalently, one can consider algebra structures on V over the smaller operad $\mathcal{O}$, but this time varying a differential on V leads to genuinely different structures. In particular, l_∞ and a_∞ structures on dg vector spaces are equivalent, respectively, to L_∞ and A_∞ structures on graded vector spaces.

Theorem 5.7. *Let $\mathcal{O}$ be a dg operad.*

- (1) *The inclusion of operads $\mathcal{O} \hookrightarrow \mathcal{O}[m]$ is a quasi-isomorphism.*
- (2) *The operad $\hat{\mathcal{O}}$ is acyclic.*

Proof. The operad $\mathcal{O}[m]$ may be understood as the free product of $\mathcal{O}$ with the acyclic nonunital dga $m\mathbf{k}[m] \subset \mathbf{k}[m]$ (with differential $d(m) = -m^2$). Since $\mathbf{k}$ is of characteristic zero, it follows that the operad inclusion $\mathcal{O} \hookrightarrow \mathcal{O}[m]$ is a quasi-isomorphism. For part (2), first note that $\hat{\mathcal{O}}$ is a direct sum of $m\mathbf{k}[m]$ (with the twisted differential $d^m(m) = m^2$) and the two-sided operadic ideal $\mathcal{Q}$ generated by $\mathcal{O}$. Let $\mathcal{P}$ be the operadic right ideal generated by $\mathcal{O}$; then there is a decomposition

$$\begin{aligned}\mathcal{Q} &= \mathcal{P} \oplus (m\mathbf{k}[m] \circ \mathcal{P}) \\ &\cong \mathcal{P} \oplus (m\mathbf{k}[m] \otimes \mathcal{P}) \\ &\cong (\mathbf{k} \oplus m\mathbf{k}[m]) \otimes \mathcal{P} \\ &\cong \mathbf{k}[m] \otimes \mathcal{P}.\end{aligned}$$

Here, by $\mathbf{k}[m] \otimes \mathcal{P}$ we understand a collection (not an operad) such that $(\mathbf{k}[m] \otimes \mathcal{P})(n) = \mathbf{k}[m] \otimes \mathcal{P}(n)$, and similarly for the other tensor products appearing in the above isomorphisms. Under this identification, the differential on $\mathcal{Q}$ is a $(\mathbf{k}[m], d^m)$ -linear derivation such that for any $p \in \mathcal{P}$ we have $d_{\mathcal{Q}}(1 \otimes p) = 1 \otimes p' + m \otimes p$ for some $p' \in \mathcal{P}$. Hence

$$\begin{aligned}d_{\mathcal{Q}}(m^r \otimes p) &= (-1)^r m^r d_{\mathcal{Q}}(1 \otimes p) + d^m(m^r) \otimes p \\ &= (-1)^r m^r (1 \otimes p' + m \otimes p) + d^m(m^r) \otimes p \\ &= (-1)^r (m^r \otimes p' + m^{r+1} \otimes p) + \begin{cases} 0 & \text{if } r \text{ is even} \\ m^{r+1} \otimes p & \text{if } r \text{ is odd} \end{cases} \\ &= (-1)^r m^r \otimes p' + \begin{cases} m^{r+1} \otimes p & \text{if } r \text{ is even} \\ 0 & \text{if } r \text{ is odd} \end{cases}\end{aligned}$$

We see, therefore, that $d_{\mathcal{Q}} = d_1 \otimes \text{id} + \text{id} \otimes d_2$, where d_1 is an acyclic differential on $\mathbf{k}[m]$, and d_2 is a differential on $\mathcal{P}$ whose precise form is unimportant. It follows that $\mathcal{Q}$, and therefore $\hat{\mathcal{O}}$, is acyclic, as required. $\square$

Remark 5.8.

- (1) If $\mathcal{O}$ is a cobar-construction operad then $\hat{\mathcal{O}}$ is a cobar-construction of a *unital* operad which is, thus, acyclic. The argument showing acyclicity of the hat-construction in general (part (2) of Theorem 5.7) turns out to be considerably more involved.
- (2) If $\mathcal{O}$ is a dg modular operad then one constructs a dg modular operad $\hat{\mathcal{O}} := \mathcal{O}[m]^m$ where $m \in \mathcal{O}((0, 2))$ and m is S_2 -invariant. All results of this section concerning $\hat{\mathcal{O}}$ have obvious analogues in the modular context, with one important exception: the vacuum part $\hat{\mathcal{O}}((0))$ of the hat-construction of $\mathcal{O}$ is *not* acyclic but quasi-isomorphic to the direct sum of $\mathcal{O}((0))$ and the complex consisting of bivalent graphs (polygons) with the usual expanding differential. The reason for this is that the twisting by m in $\mathcal{O}[m]((0))$ amounts to doing nothing as the unary operation m acts trivially on the vacuum part; thus $\hat{\mathcal{O}}((0)) \cong \mathcal{O}[m]((0))$. In particular, this explains a well-known fact that the graph complex (commutative, ribbon or Lie) containing bivalent vertices is a direct sum of the usual graph complex (with vertices of valence three or higher) and the polygonal complex.

6. MC ELEMENTS IN OPERADIC ALGEBRAS

In this section we assume that $\mathcal{O}$ is a dg operad supplied with a dg operad map $\phi : L_{\infty} \rightarrow \mathcal{O}$. In this situation there is a notion of an MC element in an $\mathcal{O}$ algebra.

Definition 6.1. Let A be a formal cdga and V be an A -linear $\mathcal{O}$ algebra. An MC element (more precisely, a ϕ -MC element) in V with coefficients in A is an MC element in $A_+ \otimes V$ viewed as an L_{∞} algebra by pulling back along the map $\phi : L_{\infty} \rightarrow \mathcal{O}$. The set of such MC elements

will be denoted by $\text{MC}_\phi(V, A)$ or simply by $\text{MC}(V, A)$ if the map $\phi : L_\infty \rightarrow \mathcal{O}$ is clear from the context.

Remark 6.2. Recall that there is a canonical map $L_\infty \rightarrow A_\infty$; it is then straightforward to see that the above definition of an MC element in an A_∞ algebra is consistent with one considered earlier in the present paper.

We now define the operad $\mathcal{O}_{\text{MC}}$ governing $\mathcal{O}$ algebras together with a choice of an MC element.

Definition 6.3. The dg operad $\mathcal{O}_{\text{MC}}$ has $\mathcal{O}[[x]]$ as the underlying graded operad, where $x \in \mathcal{O}[[x]](0)$. The differential of $\mathcal{O}_{\text{MC}}$ is specified by requiring that $\mathcal{O} \subset \mathcal{O}_{\text{MC}}$ be a dg suboperad and by the formula

$$d_{\mathcal{O}_{\text{MC}}}(x) = \sum_{n=1}^{\infty} \frac{1}{n!} \phi(m_n) \circ x^{\otimes n}.$$

Remark 6.4. Let $\mathcal{O} = L_\infty$ and ϕ be the identity morphism. Then $(L_\infty)_{\text{MC}}$ is by definition the operad $\mathcal{L}_\infty$ introduced in Definition 2.1. This observation also implies that $d_{\mathcal{O}_{\text{MC}}}$ squares to zero. Indeed, by construction we have a unique morphism $\phi' : \mathcal{L}_\infty \rightarrow \mathcal{O}_{\text{MC}}$ extending ϕ and sending $x \in \mathcal{L}_\infty$ to $x \in \mathcal{O}_{\text{MC}}$. Then $d_{\mathcal{O}_{\text{MC}}} \circ \phi' = \phi' \circ d_{\mathcal{L}_\infty}$ and it follows that $d_{\mathcal{O}_{\text{MC}}}^2(x) = 0$ from which the desired conclusion follows.

We now have the following straightforward generalization of Theorem 2.3.

Theorem 6.5. *Let V be an A -linear $\mathcal{O}$ algebra and $\xi \in \text{MC}(V, A)$. Then V has the structure of an A -linear $\mathcal{O}_{\text{MC}}$ algebra. Conversely, an A -linear $\mathcal{O}_{\text{MC}}$ algebra gives rise to an A -linear $\mathcal{O}$ algebra V together with a choice of an element in $\text{MC}(V, A)$.*

□

We now consider the special case of the above construction in which the given operad map $\phi : L_\infty \rightarrow \mathcal{O}$ factors through l_∞ ; abusing notation, the ensuing map $l_\infty \rightarrow \mathcal{O}$ will also be denoted by ϕ . We will see that in this situation the functor $\mathcal{O}_{\text{MC}}$ has better homotopical properties. This special case is also the basis of our interpretation of Willwacher's construction $\text{Tw } \mathcal{O}$ given in the next section.

Proposition 6.6. *Let $\phi : l_\infty \rightarrow \mathcal{O}$ and $\psi : l_\infty \rightarrow \mathcal{P}$ be maps of dg operads, and let $\rho : \mathcal{O} \rightarrow \mathcal{P}$ be a map of dg operads such that $\rho \circ \phi = \psi$. If ψ is a quasi-isomorphism, then the induced map $\psi_{\text{MC}} : \mathcal{O}_{\text{MC}} \rightarrow \mathcal{P}_{\text{MC}}$ is a quasi-isomorphism.*

Proof. The dg operad $\mathcal{O}_{\text{MC}}$ is complete with respect to the x -adic filtration, and the associated graded dg operad is isomorphic to $\mathcal{O}[[x]]$, with differential equal to $d_{\mathcal{O}}$ on $\mathcal{O}$ and vanishing on x . The map $\psi_{\text{MC}} : \mathcal{O}_{\text{MC}} \rightarrow \mathcal{P}_{\text{MC}}$ induces the map $\psi[[x]] : \mathcal{O}[[x]] \rightarrow \mathcal{P}[[x]]$ on associated graded dg operads. Under our assumption that the characteristic of $\mathbf{k}$ is zero, the fact that ψ is a quasi-isomorphism implies that $\psi[[x]]$ is a quasi-isomorphism. It follows that ψ_{MC} is also a quasi-isomorphism. □

Remark 6.7. Proposition 6.6 could be phrased by saying that the functor $\mathcal{O} \mapsto \mathcal{O}_{\text{MC}}$ is quasi-isomorphism invariant. Note however, that this holds only for operads $\mathcal{O}$ for which the structure map $L_\infty \rightarrow \mathcal{O}$ factors through l_∞ . Consider, by contrast, the operad L_∞ itself. We shall see that $(L_\infty)_{\text{MC}}$ is acyclic. On the other hand L_∞ itself is acyclic, i.e. it is quasi-isomorphic to the zero operad 0 while 0_{MC} is clearly not acyclic – it is one-dimensional in arity zero.

Let us consider the case $\mathcal{O} = l_\infty$. Note that l_∞ is quasi-isomorphic to lie , the (non-unital) operad governing Lie algebras. It follows that $(l_\infty)_{\text{MC}}$ is quasi-isomorphic to lie_{MC} . Similarly $(a_\infty)_{\text{MC}}$ is quasi-isomorphic to ass_{MC} where ass is the operad of associative algebras. However, more is true.

Theorem 6.8.

- (1) The natural split inclusions $a_\infty \hookrightarrow (a_\infty)_{\text{MC}}$ and $l_\infty \hookrightarrow (l_\infty)_{\text{MC}}$ are quasi-isomorphisms of dg operads.
- (2) The natural split inclusions $\text{ass} \hookrightarrow \text{ass}_{\text{MC}}$ and $\text{lie} \hookrightarrow \text{lie}_{\text{MC}}$ are quasi-isomorphisms of dg operads.

Proof. First note that part (1) is a direct consequence of part (2) by virtue of Proposition 6.6. Let us now prove part (2) in the *lie* case. The dg vector space $\text{lie}_{\text{MC}}(0)$ is spanned by the elements x and $m_2(x, x)$ with the differential acting as $x \mapsto m_2(x, x)$; it is, thus, acyclic.

Now consider the case $n > 0$. Recall that $\text{lie}(1) = 0$ while for $n \geq 2$ the space $\text{lie}(n)$ can be identified with the space spanned inside the free algebra on n generators $a_1, \dots, a_n$ by the Lie monomials in these generators containing each of them exactly once. Using the Lyndon basis inside a free Lie algebra (see e.g. [12] concerning this notion) we see that $\text{lie}(n)$ has a basis consisting of operations of the form $[a_{\sigma(1)}[a_{\sigma(2)} \dots [a_{\sigma(n-1)}, a_n] \dots]]$. Here $\sigma \in S_{n-1}$. To alleviate the notation we will write $a \cdot b$ for $[a, b]$ and omit the brackets in monomials. Our convention will be that $a \cdot b \cdot \dots \cdot c \cdot d := a \cdot (b \cdot (\dots (c \cdot d) \dots))$.

Thus, we write our basis in $\text{lie}(n)$, $n \geq 2$ as $\{a_{\sigma(1)} \cdot \dots \cdot a_{\sigma(n-1)} \cdot a_n\}_{\sigma \in S_{n-1}}$; moreover the subgroup $S_{n-1} \subset S_n$ acts on such expressions by permuting the first $n-1$ symbols. This does not specify the full S_n -action on $\text{lie}(n)$ but is sufficient for our purposes. We have

$$\text{lie}[[x]](n) = \text{lie}(n) \oplus \prod_{k=1}^{\infty} \text{lie}(k+n)_{S_k}.$$

Recall that a basis in $\text{lie}(n+k)$ consists of monomials in $a_1, \dots, a_{n+k}$ where a_{n+k} is placed last and the remaining a_i s can be arbitrarily permuted. This gives rise to a basis in $\text{lie}(k+n)_{S_k}$ for $k \geq 1$. Namely, it consists of monomials in a_i s as above, with each of the k symbols $a_1, \dots, a_k$ replaced by the symbol x . Redenoting the symbol a_{n+i} as a_i for $1 \leq i \leq k$ we can write a typical element in the constructed basis as $x^{\cdot k_1} \cdot a_{\sigma(1)} \cdot x^{\cdot k_2} \cdot a_{\sigma(2)} \cdot \dots \cdot x^{\cdot k_l} \cdot a_{\sigma(n-1)} \cdot a_n$ where $k_i \geq 0$, $\sum_{i=1}^l k_i = k$ and $\sigma \in S_{n-1}$.

In other words all our monomials are obtained from $x^{\cdot k} \cdot a_1 \cdot \dots \cdot a_{n-1} \cdot a_n$ by permuting the symbols x and a_i , $i = 1, \dots, n-1$ in all possible ways. This specifies a (topological) basis in $\text{lie}[[x]](n)$.

Further, the differential in $\text{lie}_{\text{MC}}(n) \cong \text{lie}[[x]](n)$ is given formally by $d(x) = x \cdot x$ and $d(a_i) = 0$, $i = 1, \dots, n$ and the Leibniz rule with respect to the product $\cdot$. The differential vanishes on the direct summand $\text{lie}(n) \subset \text{lie}_{\text{MC}}(n)$ as it should. It suffices, therefore, to prove that the dg space $\prod_{k=1}^{\infty} \text{lie}(k+n)_{S_k}$ is acyclic. We define a contracting homotopy s on the basis consisting of monomials in x and a_i , $i = 1, \dots, n$ described above as follows. Given a monomial $M = x^{\cdot k_1} \cdot a_{\sigma(1)} \cdot x^{\cdot k_2} \cdot a_{\sigma(2)} \cdot \dots \cdot x^{\cdot k_l} \cdot a_{\sigma(n-1)} \cdot a_n$ choose s such that k_s is the first positive integer among $k_1, \dots, k_l$ and set

$$s(M) = x^{\cdot k_1} \cdot a_{\sigma(1)} \cdot x^{\cdot k_2} \cdot a_{\sigma(2)} \cdot \dots \cdot x^{\cdot k_s-1} \cdot \dots \cdot x^{\cdot k_l} \cdot a_{\sigma(n-1)} \cdot a_n$$

if k_s is even and $s(M) = 0$ if k_s is odd. In other words the cluster $\underbrace{x \cdot \dots \cdot x}_{k_s}$ in the monomial M gets replaced by the cluster $\underbrace{x \cdot \dots \cdot x}_{k_s-1}$, if k_s is even. This concludes the proof in the *lie* case.

The proof in the *ass* case is similar but simpler. The case $n = 0$ does not have to be considered separately. The space $\text{ass}(0)$ is zero while for $n > 1$ the space $\text{ass}(n)$ is spanned by tensor monomials in n generators $a_1, \dots, a_n$ which contain every such generator exactly once; they are therefore of the form $a_{\sigma(1)} \cdot \dots \cdot a_{\sigma(n)}$ where $\sigma \in S_n$. Further, we have an isomorphism $\text{ass}[[x]](n) \cong \prod_{k=0}^{\infty} \text{ass}(k+n)_{S_k}$ and the space $\text{ass}(k+n)_{S_k}$ has a basis of monomials of length $n+k$ containing the symbols $a_1, \dots, a_n$ and k copies of x . The differential in ass_{MC} is specified on such monomials by the same formula as in the *lie* case and the same contracting homotopy can be used to show that the direct complement of ass inside ass_{MC} is acyclic. □

Remark 6.9. The statement of Theorem 6.8 can be restated as the existence of commutative diagrams of operads where all arrows are quasi-isomorphisms.

$$\begin{array}{ccc} a_\infty & \longrightarrow & (a_\infty)_{\text{MC}} \\ \downarrow & & \downarrow \\ \text{ass} & \longrightarrow & \text{ass}_{\text{MC}} \end{array} \quad \begin{array}{ccc} l_\infty & \longrightarrow & (l_\infty)_{\text{MC}} \\ \downarrow & & \downarrow \\ \text{lie} & \longrightarrow & \text{lie}_{\text{MC}} \end{array}$$

We will now see that the dg operads $(A_\infty)_{\text{MC}}$ and $(L_\infty)_{\text{MC}}$ are acyclic.

Lemma 6.10. *Let $\phi : l_\infty \rightarrow \mathcal{O}$ be a map of dg operads. Then we have an isomorphism of dg operads $\widehat{\mathcal{O}}_{\text{MC}} \cong \widehat{\hat{\mathcal{O}}}_{\text{MC}}$, where the MC construction on the right-hand side is taken with respect to the induced map $\hat{\phi} : L_\infty = \hat{l}_\infty \rightarrow \hat{\mathcal{O}}$.*

Proof. Both $\widehat{\mathcal{O}}_{\text{MC}}$ and $\widehat{\hat{\mathcal{O}}}_{\text{MC}}$ are isomorphic, as graded operads, to $\mathcal{O}[m][[x]]$, the dg operad obtained from $\mathcal{O}$ by adjoining a 0-ary operator x and a 1-ary operator m and completing at the ideal generated by x . A straightforward calculation shows that their differentials agree on $\mathcal{O}$, x and m , and hence they are isomorphic as dg operads. $\square$

Corollary 6.11. *The operads $\mathcal{L}_\infty$ and $\mathcal{A}_\infty$ are acyclic.*

Proof. We have the isomorphism $\mathcal{L}_\infty \cong (L_\infty)_{\text{MC}} \cong \widehat{(l_\infty)_{\text{MC}}}$. Since by Theorem 5.7 the hat-construction of any operad is acyclic we conclude that $\mathcal{L}_\infty$ is acyclic. The proof for $\mathcal{A}_\infty$ is similar. $\square$

7. TWISTING IN OPERADIC ALGEBRAS

Given an A -linear $\mathcal{O}$ algebra structure on V together with an MC element $\xi \in \text{MC}(V, A)$ we can consider the twist of the underlying A -linear L_∞ structure by ξ . If there is an endomorphism of $\mathcal{O}_{\text{MC}}$ extending the automorphism tw of $\mathcal{L}_\infty$ then the twisted higher L_∞ products on $A \otimes V$ are themselves part of an $\mathcal{O}$ algebra structure. In general, such an endomorphism does not exist. For example, there is no endomorphism of $(l_\infty)_{\text{MC}}$ extending the endomorphism tw on $(L_\infty)_{\text{MC}} = \mathcal{L}_\infty$. However, we can still twist the differential on $A \otimes V$ and ask whether this new dg vector space supports an operadic structure. This answer to this is provided by the notion of Willwacher's twisting of the operad $\mathcal{O}$ introduced in [14], cf. also a detailed exposition in [3]; we give an alternative, hopefully more transparent, treatment.

Lemma 7.1. *Let $\alpha := -\sum_{n=1}^{\infty} \frac{1}{n!} m_{n+1} \circ x^{\otimes n} \in [l_\infty]_{\text{MC}}(1)$. Then α is an MC element in the dga $[l_\infty]_{\text{MC}}(1)$.*

Proof. Note that $\alpha = m_1 - \text{tw}(m_1)$. Since $-m_1$ is an MC element in the dga $[\hat{l}_\infty]_{\text{MC}}(1) = \mathcal{L}_\infty(1)$ and tw is an automorphism of $\mathcal{L}_\infty(1)$ we conclude that $\text{tw}(-m_1) = -\text{tw}(m_1)$ is also an MC element. Upon twisting by $-m_1$ in the dga $\mathcal{L}_\infty(1)$ the MC element $-\text{tw}(m_1)$ becomes an MC element $m_1 - \text{tw}(m_1) = \alpha$ inside $\mathcal{L}_\infty^{-m_1}$ and it remains to observe that α belongs to the dg subalgebra $[l_\infty]_{\text{MC}}(1) \subset \mathcal{L}_\infty^{-m_1}$. $\square$

The following result is an immediate consequence.

Corollary 7.2. *Let $\alpha := -\sum_{n=1}^{\infty} \frac{1}{n!} \phi(m_{n+1}) \circ x^{\otimes n} \in \mathcal{O}_{\text{MC}}(1)$. Then α is an MC element in the dga $\mathcal{O}_{\text{MC}}(1)$.*

$\square$

We can now give a definition of an operad ‘governing’ $\mathcal{O}$ algebras with an MC twisted differential.

Definition 7.3. Let $\mathcal{O}$ be a dg operad supplied with a dg operad map $\phi : l_\infty \rightarrow \mathcal{O}$. The dg operad $\text{Tw}_\phi \mathcal{O}$ is defined as $\text{Tw}_\phi \mathcal{O} = [\mathcal{O}_{\text{MC}}]^\alpha$. When the map ϕ is clear from the context it will be suppressed from notation.

Remark 7.4. Our version of $\text{Tw } \mathcal{O}$ is almost the same as Willwacher's in [14], which we denote here by $\text{Tw}^W \mathcal{O}$. Namely, $\text{Tw}^W \mathcal{O}$ is a dg suboperad in $\text{Tw } \mathcal{O}$ and under this inclusion $\text{Tw}^W \mathcal{O}(n) \cong \text{Tw } \mathcal{O}(n)$ for $n > 0$. In arity zero we have

$$\text{Tw}^W \mathcal{O}(0) = \mathcal{O}(0) \hookrightarrow \mathcal{O}(0) \oplus \mathbf{k}\langle x \rangle = \text{Tw } \mathcal{O}(0)$$

where $\mathbf{k}\langle x \rangle$ stands for the one-dimensional space spanned by x .

Willwacher's construction starts with the operad $\widetilde{\text{Tw } \mathcal{O}}$ which is our $\mathcal{O}[[x]]$ in arities > 0 and $\mathcal{O}(0)$ in arity zero. In order to arrive at $\mathcal{O}_{\text{MC}}$ the differential in $\widetilde{\text{Tw } \mathcal{O}}$ is twisted by the MC element ϕ in the convolution Lie algebra of maps from l_∞ to $\mathcal{O}$ which is shown to act on the Lie algebra of derivations of $\widetilde{\text{Tw } \mathcal{O}}$. Finally, the twisting by α is performed to obtain Tw^W .

The following result is analogous to Proposition 6.6 and is proved using the same argument.

Proposition 7.5. *Let $\phi : l_\infty \rightarrow \mathcal{O}$ and $\psi : l_\infty \rightarrow \mathcal{P}$ be maps of dg operads, and let $\rho : \mathcal{O} \rightarrow \mathcal{P}$ be a map of dg operads such that $\rho \circ \phi = \psi$. If ψ is a quasi-isomorphism, then the induced map $\psi_{\text{MC}} : \text{Tw}_\phi \mathcal{O} \rightarrow \text{Tw}_\phi \mathcal{P}$ is a quasi-isomorphism.*

□

In particular, we see that operads of the form $\text{Tw } \hat{\mathcal{O}}$ (such as $\text{Tw } L_\infty$) are acyclic. The following result computes the homology of $\text{Tw } l_\infty$ and $\text{Tw } a_\infty$; it is analogous to the statement of Theorem 6.8 and relies on it.

Theorem 7.6.

- (1) *The natural projections $\text{Tw } a_\infty \rightarrow a_\infty$ and $\text{Tw } l_\infty \rightarrow l_\infty$ sending $x \in \text{Tw } l_\infty(0)$ or $x \in \text{Tw } a_\infty(0)$ to zero are quasi-isomorphisms of dg operads.*
- (2) *The natural projections $\text{Tw } \text{ass} \rightarrow \text{ass}$ and $\text{Tw } \text{lie} \rightarrow \text{lie}$ sending $x \in \text{Tw } \text{ass}(0)$ or $x \in \text{Tw } \text{ass}(0)$ to zero are quasi-isomorphisms of dg operads.*

Proof. First note that part (1) is a direct consequence of part (2) by virtue of Proposition 7.5. We now prove part (2), restricting ourselves to the *lie* case, the proof of the *ass* case being essentially the same.

Recall that $\text{Tw } \text{lie} := [\text{lie}_{\text{MC}}]^\alpha$ where $\alpha = -m_2(x, -) \in \text{lie}_{\text{MC}}(1)$ and $m_2 \in \text{lie}_{\text{MC}}(2)$. Further, lie_{MC} is isomorphic to $\text{lie}[[x]]$ where $x \in \text{lie}(0)$ and the differential acts as $d(x) = \frac{1}{2}m_2(x, x)$; here m_2 is the binary generator of *lie* (and, of course, $d(m_2) = 0$). The twisted differential will have the form

$$d^\alpha(x) = \frac{1}{2}m_2(x, x) - m_2(x, x) = -\frac{1}{2}m_2(x, x).$$

Next, using the symbols a, b as placeholders we find

$$d^\alpha(m_2)(a, b) = [x, [a, b]] - [[x, a], b] - [a, [x, b]].$$

The last expression is zero by the Jacobi identity and so $d^\alpha(m_2) = 0$. It follows that the map $\text{Tw } l_\infty \cong (l_\infty)_{\text{MC}}^\alpha \rightarrow (l_\infty)_{\text{MC}}$ with $m_2 \mapsto -m_2$ and $x \mapsto x$ is an isomorphism of dg operads and the conclusion follows from Theorem 6.8. □

Remark 7.7. Theorem 7.6 could be phrased as saying that there exists a commutative diagrams of dg operads all arrows of which are quasi-isomorphisms.

$$\begin{array}{ccc} \text{Tw } l_\infty & \longrightarrow & l_\infty \\ \downarrow & & \downarrow \\ \text{Tw } \text{lie} & \longrightarrow & \text{lie} \end{array} \qquad \begin{array}{ccc} \text{Tw } a_\infty & \longrightarrow & a_\infty \\ \downarrow & & \downarrow \\ \text{Tw } \text{ass} & \longrightarrow & \text{ass} \end{array}$$

7.1. Block modules and algebras. To understand properly algebras over the operad $\text{Tw}_\phi \mathcal{O}$ we need a certain generalization of the notion of an A -linear $\mathcal{O}$ algebra which is of some independent interest. This generalization is obtained by treating A consistently as the ‘ground’ ring, so that it completely supplants $\mathbf{k}$. We note that all our results involving A -linear structures could be formulated in this more general framework but for most applications it is not needed which is the reason it has not been adopted throughout the paper.

Definition 7.8. Let (A, d_A) be a cdga and (V, d_V) be a dg vector space. Let $d \in \text{End}(A \otimes V)$ be an A -linear derivation, i.e. $d(a \otimes v) = d_A(a) \otimes v + (-1)^{|a|} a \otimes d_V(v)$; additionally assume that $d^2 = 0$. The pair $(A \otimes V, d)$ is called a *Block module*. It is called a *formal Block module* if, in addition, A is a formal cdga and $d(v) - d_V(v) \in A_+ \otimes V$ for all $v \in V$.

Remark 7.9. Block modules were introduced (in somewhat greater generality) in [1]. We will only be concerned with formal Block modules. Given a formal cdga A and a dg vector space V , the formal Block module structures on $A \otimes V$ are in 1-1 correspondence with elements in $\text{MC}(\text{End}(V), A)$; namely, given $\xi \in \text{MC}(\text{End}(V), A)$, the endomorphism $d = d_A \otimes \text{id} + \text{id} \otimes d_V + \xi$ is a Block differential on $A \otimes V$.

Definition 7.10. Let $(A \otimes V, d)$ be a formal Block module. Form the suboperad $\mathcal{E}_A(A \otimes V, d) \subset \mathcal{E}(A \otimes V, d)$ consisting of A -multilinear maps $\{(A \otimes V)^{\otimes n} \rightarrow A_+ \otimes V\}$; note that $\mathcal{E}_A(A \otimes V, d)(n) \cong A_+ \otimes \text{Hom}(V^{\otimes n}, V)$.

Further, given a dg operad $\mathcal{O}$, a Block $\mathcal{O}$ algebra structure on $(A \otimes V, d)$ is an operad map $\mathcal{O} \rightarrow \mathcal{E}_A(\Pi A \otimes V, d) \cong \mathcal{E}_A(A \otimes \Pi V, d)$.

Remark 7.11. Note that the operad $\mathcal{E}_A(A \otimes V, d)$ is obtained from $A_+ \otimes \mathcal{E}(V, d_V)$ by twisting with the MC-element $\xi \in \text{MC}(\mathcal{E}(V, d_V)(1), A)$ associated to the Block module $(A \otimes V, d)$. In particular, the zero MC element gives a trivial twisting of the operad $A_+ \otimes \mathcal{E}(V, d_V)$; the associated notion of a Block $\mathcal{O}$ algebra then reduces to the notion of an A -linear $\mathcal{O}$ algebra considered earlier in the paper.

Proposition 7.12. Let $\mathcal{O}$ be a dg operad and m be an MC element in $\mathcal{O}(1)$. Then the structure of a Block $\mathcal{O}$ algebra on a formal Block module $(A \otimes V, d)$ is equivalent to an $\mathcal{O}^m$ algebra structure on the Block module $(A \otimes V, d + m)$.

Proof. Let $\xi \in A_+ \otimes \mathcal{E}(V)(1)$ be the MC-element associated to a Block module $(A \otimes V, d)$. Suppose we are given an operad map $\mathcal{O} \rightarrow \mathcal{E}_A(A \otimes \Pi V, d)$. Then the image of $m \in \mathcal{O}(1)$ in $\mathcal{E}_A(A \otimes \Pi V, d)(1) = [A_+ \otimes \mathcal{E}(\Pi V)(1)]^\xi$ is an MC-element which we denote by the same symbol. So we obtain an induced operad map $\mathcal{O}^m \rightarrow [A_+ \otimes \mathcal{E}(\Pi V)]^{\xi+m} = \mathcal{E}_A(A \otimes \Pi V, d + m)$. This establishes the desired 1-1 correspondence. $\square$

Definition 7.13. Let $\mathcal{O}$ be a dg operad supplied with an operad map $\phi : L_\infty \rightarrow \mathcal{O}$ and $(A \otimes V, d)$ be a Block $\mathcal{O}$ algebra. Then an even element $\xi \in A_+ \otimes \Pi V$ is called a ϕ -MC element if $d(\xi) + \sum_{i=1}^\infty \frac{1}{i!} \phi(m_i)(\xi^{\otimes i}) = 0$. The set of such MC elements will be denoted by $\text{MC}_\phi(V, A)$ or simply by $\text{MC}(V, A)$ if the map $\phi : L_\infty \rightarrow \mathcal{O}$ is clear from the context.

We have the following straightforward analogue of Theorem 6.5.

Proposition 7.14. Let $(A \otimes V, d)$ be an Block $\mathcal{O}$ algebra and $\xi \in \text{MC}(V, A)$. Then $(A \otimes V, d)$ has the structure of a Block $\mathcal{O}_{\text{MC}}$ algebra. Conversely, any Block $\mathcal{O}_{\text{MC}}$ algebra gives rise to an Block $\mathcal{O}$ algebra $(A \otimes V, d)$ together with a choice of an element in $\text{MC}(V, A)$. $\square$

Then Proposition 7.12 and Proposition 7.14 yield the following characterization of Block $\mathcal{O}$ algebras over $\text{Tw } \mathcal{O}$.

Theorem 7.15. A Block $\text{Tw } \mathcal{O}$ algebra structure on $(A \otimes V, d)$ is equivalent to a Block $\mathcal{O}$ algebra structure on $(A \otimes V, d')$ together with $\xi \in \text{MC}_\phi(V, A)$ such that d is the MC-twisted differential, i.e. $d = d' + \sum_{n=1}^\infty \frac{1}{n!} m_{n+1}(\xi, \dots, \xi, -)$. $\square$

Remark 7.16.

- (1) Since an A -linear $\text{Tw } \mathcal{O}$ algebra is a special case of a Block algebra the above result also gives a characterization of A -linear $\text{Tw } \mathcal{O}$ algebra structures. However, even if the original Block $\text{Tw } \mathcal{O}$ algebra happens to be A -linear, the associated Block $\mathcal{O}$ algebra, in general, will not be. This was our chief reason for introducing Block modules and algebras.

- (2) Since the dg operad $\mathrm{Tw} \mathcal{O}$ acts on $\mathcal{O}$ algebras with MC twisted differentials it follows that the suboperad $\mathrm{Tw}^W \mathcal{O} \subset \mathrm{Tw} \mathcal{O}$ also acts on such objects. However, it is unclear whether any action of $\mathrm{Tw}^W \mathcal{O}$ can be extended to an action of $\mathrm{Tw} \mathcal{O}$; this indicates that, perhaps, the construction Tw is more ‘natural’ than Tw^W .

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