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NUMERICAL COMPUTATION OF ESTUARINE FLOWS

by

John P. Grubert

A thesis presented for the degree of

Doctor of Philosophy

in

Civil Engineering

to The City University, London.

The experimental work was carried out in the
Department of Civil Engineering and Building,
Lanchester Polytechnic, Coventry.

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ERRATA

1. P. 40 last line. Definition of Courant number

$$\text{Courant number} = \frac{\Delta t}{\Delta x} \left[u + (gH)^{\frac{1}{2}} \right]$$

2. P. 134. In Eq.(3.189) centre row of matrix should read

$$\begin{array}{ccc} -2(a-b) & (2-2b) & 2(a+b) \end{array}$$

and in Eq.(3.191) centre row of matrix should read

$$\begin{array}{ccc} \frac{-2(a-b)}{(2+2b)} & \frac{(2-2b)}{(2+2b)} - \lambda & \frac{2(a+b)}{(2+2b)} \end{array}$$

3. P. 136, line 4. Should read "Because $b \geq 0$, $|A_f|$ is"
4. P. 264. In Eqs.(5.32)and(5.33) the constant should be 1.79 .
5. P. 266, line 3. Should read "to be 1.79 . A similar result .."
6. P. 321. In Eqs.(6.23) and (6.24) left hand side should read "0.566/1018". This correction leads to $k_m = 4.34 \times 10^{-4}$ when $(1-\epsilon) = 0.0177$, and $k'_m = 0.92 \times 10^{-5}$.
7. P. 407, line 9. Should read " 95. Miles, J.W., 'On the Stability of Heterogeneous Shear Flows', "

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ABSTRACT

A general system of one dimensional two-layer stratified flow equations is derived in terms of three dependent variables, discharge, level and density in each layer. Provision is made in these equations for bed slope, bed friction, wind friction, lateral discharges, interfacial mixing and quasi-laminar interfacial shear stress terms. These equations are solved using central implicit finite difference schemes with generalised algorithms so that any combination of subcritical and supercritical flows can be accommodated. In order that the computation can be made either in a single layer or a two-layer flow, the three lower layer equations are each split up into two halves, providing a four level fractional steps scheme. This flexibility in the computation is necessary when dealing with salt-wedge estuaries since the domain of the computation includes both single layer and two-layer flows.

An experimental programme was carried out to investigate the front conditions for a moving salt-wedge, and the conditions governing interfacial stability and mixing along salt-wedge interfaces. It was deduced theoretically that two distinct front types should exist in a salt-wedge estuary, namely wave front and St. Venant fronts. Experiments showed that although these front types exist, their flow conditions are different from that predicted by theory. Theory indicates that the onset of interfacial wave breaking can be expressed by a stability parameter which is a function of the interfacial densimetric

Froude number and the interfacial Reynolds number.

Experiments indicate that this parameter is independent of the flow Reynolds number upstream of a salt-wedge and can be scaled to give a stability parameter for prototype flows. The experiments also proved that mixing across a salt-wedge interface is a two-way process caused by the momentum exchange of turbulent flows, and that this rate of mixing is inversely proportional to the relative density.

LIST OF SYMBOLS

It has not been possible to avoid repeated use of many symbols for different purposes because of the volume of algebraic manipulation in this thesis, but each symbol is defined where it first appears with a new meaning. The following symbols are used in this thesis:

$A, A^*, B, B^*, C, C^*,$
 $D, D^*, E, E^*, F, F^*,$
 $G, G^*, H, H^*, I, I^*,$
 $J, J^*, K, K^*, L, L^*,$
 $R, R^*, S, S^*, T, T^*,$
 $U, U^*, V, V^*, W, W^* =$ dynamic implicit equation coefficients;
 $A^{**}, B^{**}, C^{**}, D^{**},$
 $E^{**}, F^{**}, G^{**}, H^{**},$
 $I^{**}, J^{**}, K^{**}, L^{**} =$ transport implicit equation coefficients;

$A, B, C, D =$ functions first used in Eq.(3.4);

$A =$ cross-sectional flow area;

$A_f =$ amplification factor;

$A_i =$ surface area of interface/wavelength;

$A_0 =$ constant in Eq.(5.6);

$a =$ base area of wave plume;

$a = 2\sin y =$ substitution used in Eq.(3.177);

$a = u \Delta t / \Delta x =$ substitution used in Eq.(3.186);

$a = Y\delta \sin^2 y =$ substitution used in Eq.(3.208);

$a' =$ substitution used in Eq.(3.90);

$a' =$ substitution used in Eq.(4.37);

$B = \frac{\partial A}{\partial x} / \frac{\partial h}{\partial x} =$ dynamical breadth;

$B' = A/h =$ mathematical breadth;

b = breadth of water surface;
 $b = r/(gH)$ = substitution used in Eq.(3.177);
 $b = 2D \Delta t / \Delta x^2$ = substitution used in Eq.(3.186);
 $b = \beta \delta i \sin y$ = substitution used in Eq.(3.208);
 b' = substitution used in Eq.(3.91);
 b' = substitution used in Eq.(4.39);
 C = Chezy friction coefficient;
 $c = (gA/b)^{\frac{1}{2}}$ = wave celerity;
 $c = -(a^2 + rb)$ = substitution used in Eq.(3.181);
 $c = (1+a+b-a)$ = substitution used in Eq.(3.210);
 c' = substitution used in Eq.(4.43);
 D = diffusion/dispersion coefficient;
 $D' = D \rho / \rho_w$ = diffusion/dispersion coefficient;
 d = depth of fluid;
 d' = substitution used in Eq.(4.44);
 d_{1c} = critical depth in upper layer;
 d_{2s} = depth of salt wedge at the sea;
 e = exponential number;
 $e = (u^2 - gA/b)$ = substitution used in Eq.(3.17);
 e' = substitution used in Eq.(4.43);
 e'' = substitution used in Eq.(4.51);
 $F = u/(gH)^{\frac{1}{2}}$ = Froude number;
 $F_d = V/(g'H)^{\frac{1}{2}}$ = densimetric Froude number;
 $F_\delta = V/(g'\delta)^{\frac{1}{2}}$ = interfacial densimetric Froude number;
 f = interfacial friction factor;
 f = substitution used in Eq.(3.17);
 $f' = (gA_1 - u_1^2 B_1)$ = substitution used in Eq.(4.48);
 $f'' = (gA_1/b_1 - u_1^2)$ = substitution used in Eq.(4.52);

f^n = Fourier coefficient at time level $n\Delta t$;
 G = substitution used in Eq.(3.30);
 g = gravity acceleration;
 $g' = g(1 - \epsilon)$ = densimetric gravity acceleration;
 H = total depth of fluid;
 H_0 = total depth of water at tip of salt wedge;
 H_s = total depth of water at the sea;
 h = level of fluid;
 I = bed slope;
 i = friction slope;
 $i = (-1)^{\frac{1}{2}}$;
 j = grid co-ordinate in x-direction;
 K' = coefficient in Manning's equation;
 K_m = theoretical coefficient of interfacial mixing;
 k = grid co-ordinate in y-direction;
 k = wave number;
 k_a = Rossby coefficient of wind shear stress;
 k_m, k'_m = experimental coefficients of interfacial mixing;
 $k'' = (b_2 h_2 - b_1 h_1)$ = substitution used in Eq.(4.50);
 L = wave length;
 L_0 = length of salt wedge;
 L_v = length of St.Venant front;
 m = mass of element of fluid;
 m = hydraulic radius exponent;
 $m' = b_2(\epsilon q_1 - q_2)$ = substitution used in Eq.(4.49);
 m'' = substitution used in Eq.(4.3);

N = number of breaking interfacial waves;
 n = time grid co-ordinate;
 n' = substitution used in Eq.(4.50);
 $n'' = q_i \rho_i$ = substitution used in Eq.(4.2);
 P = density gradient;
 p = pressure;
 Q = cross-sectional fluid discharge;
 Q_r = upland river discharge;
 q_1 = mass flux from upper to lower layer;
 q_2 = mass flux from lower to upper layer;
 q_i = lateral discharge;
 q_s = quantity of salt;
 R = hydraulic radius;
 $R^+ = u + 2(g'd)^{\frac{1}{2}}$ = positive Riemann invariant;
 $R^- = u - 2(g'd)^{\frac{1}{2}}$ = negative Riemann invariant;
 $R_1 = g'\delta/V^2$ = interfacial Richardson number;
 $R_\delta = V\delta/v_2$ = interfacial Reynolds number
 $r = gH\Delta t^2/\Delta x^2$ = substitution used in Eq.(3.131);
 $r = 4\Delta x/\Delta t$ = substitution used in Eq.(3.164);
 r' = substitution used in Eq.(4.53);
 $r'' = q_i(\rho_i - \rho)/A$ = substitution used in Eq.(4.4);
 s = salinity;
 s' = substitution used in Eq.(4.51);
 s'' = substitution used in Eq.(3.120);
 s_i = salinity of lateral inflow;
 T = wave period;
 t = time symbol;
 t' = substitution used in Eq.(4.54);

u = velocity of fluid in x-direction;
 u_a = air velocity in x-direction;
 $V = (u_1 - u_2)$ = relative velocity of fluid in x-direction;
 V_d = densimetric velocity as defined in Eq.(5.2);
 V_r = river velocity at tip of salt wedge;
 V_s = upper layer velocity at the sea;
 v = velocity of fluid in y-direction;
 $v = \partial Q / \partial t$ = substitution used in Eq.(3.143);
 v_a = air velocity in y-direction;
 v_i = velocity of lateral inflow;
 $w = (gH)^{1/2} \partial Q / \partial x$ = substitution used in Eq.(3.143);
 X = model length scale ratio;
 X = substitution for denominators of recurrence relations;
 x, y = horizontal distance symbols;
 $\hat{x}$ = characteristic direction;
 Y = model depth scale ratio;
 $y = \pi k \Delta x / L$ = substitution used in Eq.(3.138);
 y' = substitution used in Eq.(4.52);
 z = vertical distance symbol;
 $\alpha, \alpha^*, \beta, \beta^*, \gamma, \gamma^*,$
 $\delta, \delta^*, \epsilon, \epsilon^*, \zeta, \zeta^*,$
 $\eta, \eta^*, \theta, \theta^*$ = recurrence relation coefficients;
 $\alpha = 2\pi\delta/L$ = interfacial wave number;
 α = substitution used in Eq.(3.154);
 $\alpha, \beta, \gamma, \delta$ = substitutions used in Eq.(3.207);
 β, γ = symbols defined in Fig. 3.10 ;

Γ = value of control function;
 Δt = time increment;
 Δx = distance increment in x-direction;
 Δy = distance increment in y-direction;
 δ_1 = thickness of upper part of interfacial mixing layer;
 δ_2 = thickness of lower part of interfacial mixing layer;
 $\delta = \delta_1 + \delta_2$ = thickness of interfacial mixing layer;
 $\epsilon = \rho_1/\rho_2$ = density ratio;
 Θ = interfacial stability parameter;
 $\theta = 2\omega \sin \psi$;
 λ = any eigen value;
 $\lambda_1, \lambda_2, \lambda_3$ = eigen values of amplification matrix;
 ν = kinematic viscosity of fluid;
 ξ = weighting factor in difference equation;
 ρ = fluid density;
 ρ_a = air density;
 ρ_i = lateral inflow density;
 ρ_w = fresh water density;
 σ = constant in heat flow equation;
 τ = quasi-laminar interfacial shear stress;
 ϕ = angle of lateral inflow to river;
 Ψ = flow variable;
 ψ = north latitude;
 ω = angular velocity of the earth;

Subscripts

1 = upper layer;
 2 = lower layer;
 c = critical condition;
 v = St. Venant front;
 w = wave front;

Numerical Computation of Estuarine Flows

CHAPTER 1

INTRODUCTION

1.1 Estuaries

An estuary is a partly enclosed body of water that forms where river water is mixed with and diluted by seawater. In a general sense, the estuarine environment is one that is defined by salinity boundaries rather than geographic boundaries. Many coastal features that are designated by other names are in fact, estuaries. The lower reaches of rivers such as the Mersey, Thames and Humber are estuaries. Other river systems with large deltas are actually estuaries at their seaward end; the Mississippi is an example. Certain coastal embayments, such as Southampton Water, also are estuarine because fresh and salt water undergo considerable mixing. In addition, most of the fjord systems in the world are estuaries. Large, semi-enclosed tidal flat regions and coastal marshes, such as the Wadden Zee area of The Netherlands also may be termed estuaries.

The adjective estuarine can be applied to aqueous environments where seawater has become somewhat diluted by fresh water. For example, the northernmost Pacific Ocean in the vicinity of the Bering Sea and Gulf of Alaska has estuarine characteristics because of the great quantity of fresh water that flows into this region. Ocean areas off the mouths of the Mississippi and Amazon also become estuarine during the rainy season when the river discharge increases

and dilutes the coastal waters.

The tides have a very important effect on the circulation within an estuary. The volume of water within the estuary between high and low tides is known as the tidal prism. The currents generated by this volume of water flowing in and out of the estuary depend on the geometry of the tidal basin and the tidal range (distance sea level rises from maximum ebb tide to maximum flood tide). In estuaries where the tidal range is large, the tidal currents have the ability to mix both the inflowing river water and ocean water in the estuarine basin. In estuaries with smaller tidal ranges and lower current velocities, the river water may remain relatively unmixed within the basin and form a low-density layer that floats on the surface of the denser salt water.

The quantity of freshwater is important in determining the nature of the estuary. During periods of high river discharge, fresh water may force most of the salt water out of the estuary. During other seasons when river discharge is low, the estuary may be more akin to an extension of the ocean rather than an estuarine environment. Moderate river discharge may stratify, or layer, the estuary with the low-density brackish water on top and the denser, more saline waters below.

Wind can strongly affect the mixing of estuarine waters by creating local waves that break down water stratification. In addition, through stress on the surface,

the wind can cause a net transport of surface water and thus reinforce or retard the normal tidal flow.

The formation of ice within estuaries can act in many ways to affect the estuarine circulation. If large portions of the estuary are covered by ice, local wind-generated waves will have little effect in mixing the surface and bottom waters. In contrast, restricted channels caused by ice blockage may locally increase the tidal currents and mixing of the waters. Melting of estuarine ice in the spring also provides water that is of lesser density than normal estuarine water and may lead to stratification.

The physical dimensions of an estuary are extremely important with respect to the hydraulics. Broad shallow estuaries like the Humber and Hooghly are more susceptible to the effects of wind mixing by local waves than are deeper estuaries. Certain fjords with restrictive sill depths at the seaward portion of the estuary inhibit circulation and mixing to such an extent that the lower waters in the estuary become devoid of oxygen.

In order to compare different estuaries it is necessary to set up a scheme of classification. Many different schemes are possible but the two main types are classification by geomorphology and classification by estuarine circulation. The latter classification is clearly necessary not only from the hydraulic engineers point of view but also because the majority of estuaries fall within the fluvial basin category and display large differences in their circulation patterns,

density stratification and mixing processes.

1.1.1 Classification by Geomorphology

The geomorphology, or form of an estuarine basin is usually developed by fluvial erosion, fluvial and marine deposition, glacial erosion or tectonic activity. These processes are associated with four types of basins, namely, fluvial, depositional, glacial and tectonic.

Fluvial basins :

These are also known as drowned river valley or coastal plain type estuaries, and are probably the most common type of estuarine basin. These estuaries were formed during the Flandrian transgression by the flooding of previous incised valleys. Where the valleys were simple and straight with few tributaries, drowning has simply raised the level of water on the valley walls. This is true of the Hudson and St. Lawrence rivers for example.

In other river systems, especially those that were developed on softer, more unconsolidated rocks the rivers were quite complex and have an elaborate tributary system. Since sedimentation has not kept pace with the inundation the resulting estuary exhibits an extremely crenulated border that follows the contours of the former tributary valleys.

This type of estuary usually deepens and widens towards the mouth the width often having an exponential shape. The mouth of the estuary is often modified due to the formation of a spit such as in the Humber estuary.

Spits restrict the interaction between the river and sea water, allowing extensive mud flats and saltings to develop behind them and causing the talweg (deepest part of river) to be sinuous.

Fluvial estuaries are generally found in temperate climates when the river flow is small compared with the tidal prism (the volume between high and low water levels) and the amount of sediment transported by the river is also small.

Examples: Thames, Mersey, Southampton Water and Tamar in England.

Depositional basins :

These are also known as bar-built estuaries but could also be classified as fluvial basins since they have experienced incision during the ice age and subsequent inundation. However, recent sedimentation has kept pace with the inundation and they have developed a barrier that separates the sea and river waters. This barrier can take the form of offshore islands at the back of which the estuary can form, the Wadden Zee of The Netherlands is an example. Usually the barrier will be in the form of a natural levee, spit or bar and this restriction of the inlet allows an estuarine environment to develop that formerly was open to the sea. A good example of a spit protected estuary is the Humber and a bar protects the Dovey estuary in Wales. In order for this spit or bar to be well developed, the tidal range must be restricted and

large volumes of sediment available. These estuaries are only a few metres deep and remain shallow well out to sea, but have a high seasonal river discharge. Depositional basins are found either in tropical areas or in areas with substantial coastal deposition.

Examples: Humber and Dovey estuaries in Britain and the Hooghly estuary in West Bengal.

Glacial basins :

Fjords are deep, narrow, U-shaped gorges that were cut in bedrock by tongues of glacial ice during former ice ages. Many of the fjords were former river valleys that were deepened and widened by ice action. Because of the thickness and weight of the ice, the glaciers were able to erode far below sea level. In addition, the seaward end of the fjord is usually marked by a bedrock or depositional sill that forms an internal basin. This means that the fjords are extremely deep and narrow with a shallow restrictive entrance. The shallow depths of the sill prevent free exchange of the sea water outside the sill and the water in the basin. River discharge is small compared with the total fjord volume, but as many fjords have restricted tidal ranges within their mouths, the river flow is often large with respect to the tidal prism. Their occurrence is restricted to high latitudes in mountainous areas.

Examples: Loch Etive (Scotland), Oslo and Hardanger Fjords (Norway), Alberni Inlet (British Columbia) and Silver Bay (Alaska).

Tectonic basins :

Continental tectonics or movement of the land surface may create estuaries. A common form is the graben or down faulted block.

Broad local subsidence of portions of the coast can also create estuarine environments because its effect is equivalent to a rise in sealevel. The San-Francisco Bay region which has had a complex geological history, is in this category.

1.1.2 Classification on Estuarine Circulation

In most estuaries there is a certain amount of to- and fro motion of the waters as the tides flush in and out of the basin. Besides this, however, there is a net circulation pattern within estuaries that is a function primarily of the mixing of the fresh and salt water in this complex area. This is termed estuarine circulation and three distinct estuary types are recognised namely, highly stratified, partly-mixed and fully-mixed. Fjord type estuaries do not fit easily into the above classification and are best looked upon as a special form of the highly stratified estuary. Other estuaries have such a complicated topography that mixing occurs only in a portion of the estuarine system. In this case the estuary changes type both along the estuary and with time during the tidal cycle.

Highly Stratified Estuary :

(a) Salt-Wedge type

Most highly stratified estuaries occur

in regions where the tidal range is low and river discharge into the estuary is high. In these cases the less dense surface water of the river will float over the salt water forming stratification or layers of water, within the estuary. The fresh water that forms the surface layer quickly becomes brackish as it mixes with the estuarine waters, although the change from salt to fresh water is confined essentially to an interfacial mixing layer about one fifth of the total depth in thickness as shown in Fig.1.1

The velocities in a salt-wedge type estuary are usually high enough to cause the waves that develop at the interface to break. Since a circulation velocity is set up in the salt-wedge both the upper and lower layers have turbulent flows. The transfer of mass between the layers is therefore a two-way process with slightly more transfer from the salt to the fresh layer. Because the tidal variations cause the wedge to move up and down the estuary most of the mixing at the interface occurs on the flood flow when the relative velocity between layers is greatest, and can virtually cease on the ebb flow.

Conditions leading to a highly stratified estuary are enhanced in regions where there is high river discharge and low tidal flow. In addition, highly stratified estuaries are more likely to develop in deep, narrow, drowned river valleys, or artificially deepened waterways. Examples: South and South-west Passes of Mississippi delta (U.S.A.) and the Grand Rhône (France).

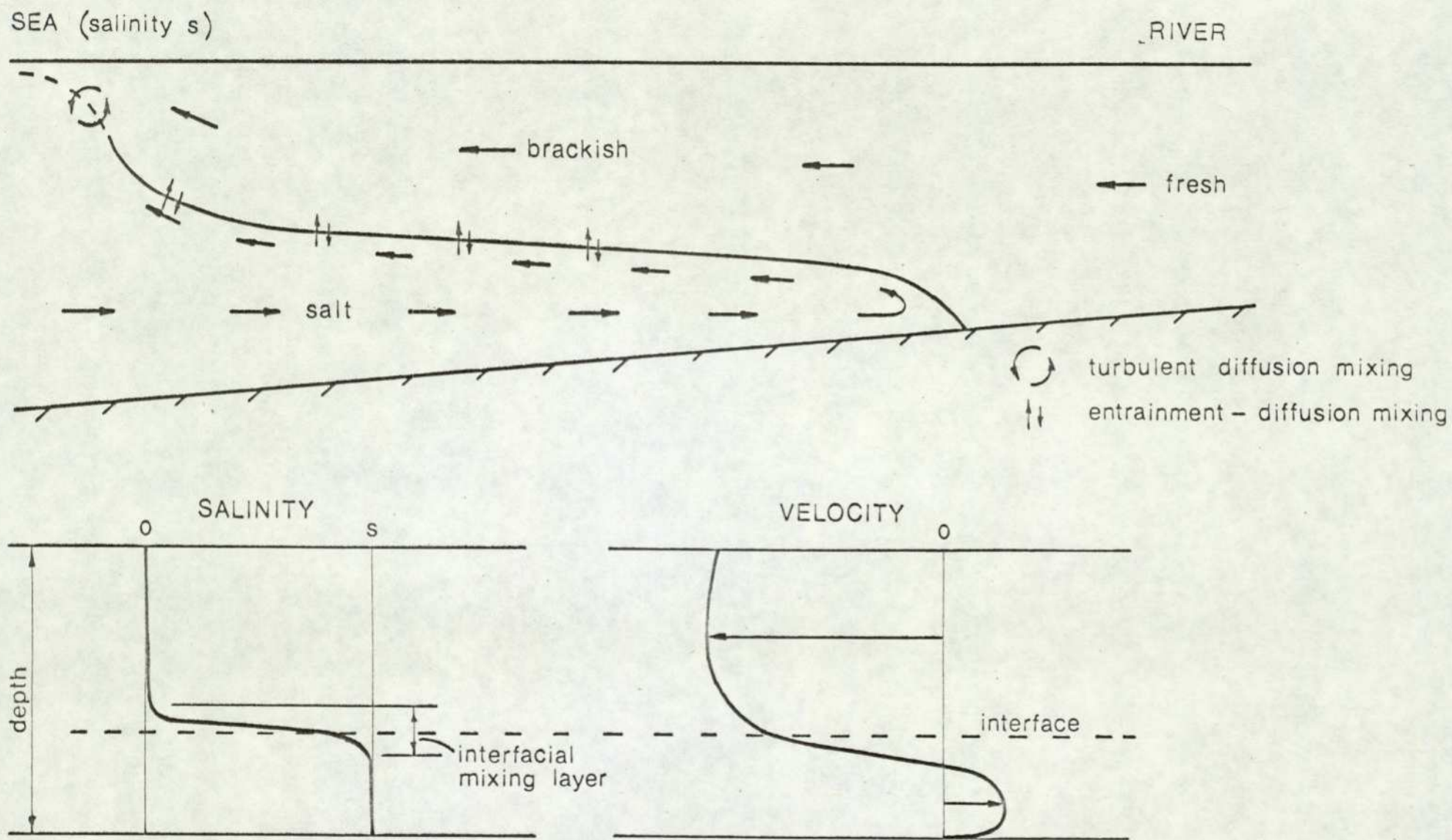


FIG. 1.1 SALT-WEDGE TYPE ESTUARY

(b) Fjord type

Fjords are highly stratified estuaries which have a very deep stagnant salt water pool over which the brackish water flows to the sea. The salt water enters the estuary over a restrictive sill and fills up the basin with water of fairly constant salinity. Since the salt layer is stagnant and therefore non-turbulent there is entrainment only from the lower to the upper turbulent layer as shown in Fig.1.2. The upper layer is usually of constant thickness from head to mouth and in some fjords its thickness is restricted to a depth equal to the sill depth.

Examples: Loch Etive (Scotland), Oslo and Hardanger Fjords (Norway), Alberni Inlet (British Columbia) and Silver Bay (Alaska).

Partly-Mixed Estuary :

As tidal flow increases or river discharge decreases mixing takes place between the upper and lower layers in an estuary not only during the flood flow but also over the ebb period as well. The greater amount of tidal mixing breaks down the interfacial mixing layer adding a larger volume of salt water to the upper layer than in the salt wedge type of estuary. This reduces the relative density between layers and encourages even more mixing. The net flow in this system is still primarily two layer, with the surface water moving seaward and the bottom water moving landward. Now, however because of mixing, circulation is much greater and there is more flow

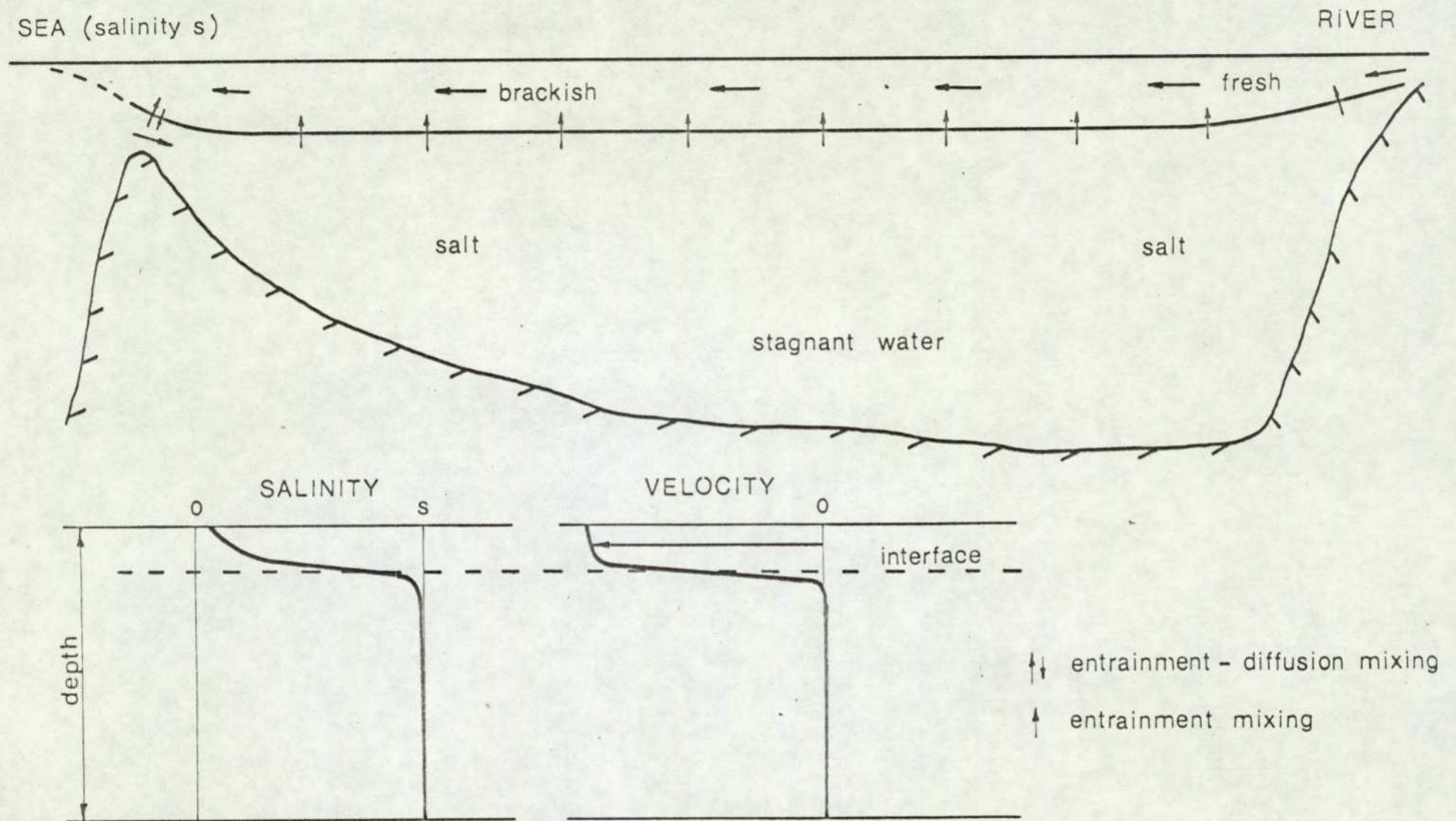


FIG. 1.2 FJORD TYPE ESTUARY

both in the upper and the lower salt waters. The volume of water circulation in this type of flow far exceeds the salt-wedge type. For example, the seaward flow in the surface layer and the inflowing water at the base of the estuary may be both twenty times the river discharge, see Fig. 1.3 .

Because of the efficient exchange of salt and fresh water the salinity structure of the estuary is different from that of the salt-wedge type. The surface salinity increases much more steadily down the estuary and undiluted fresh water only occurs at the head of the estuary. Also in the saline water of the bottom layer there is a longitudinal gradient of salinity. Consequently there is a large section in the middle reaches of this type of estuary in which the longitudinal salinity gradients are almost linear.

Examples: Thames, Mersey and Southampton Water in England, James River (U.S.A.), Fraser River (Canada), and The Rotterdam Waterway (The Netherlands).

Fully-Mixed Estuary :

If the depth of water is relatively small and the tidal range large, then the eddies formed at the bed of the estuary and those formed at the surface due to wind action mix the entire water depth. The water becomes less stratified and ultimately reaches a point at which its properties from top to bottom are relatively constant at any one location within the estuary. Although the vertical gradient is erased, a strong longitudinal

RIVER

SEA (salinity s)

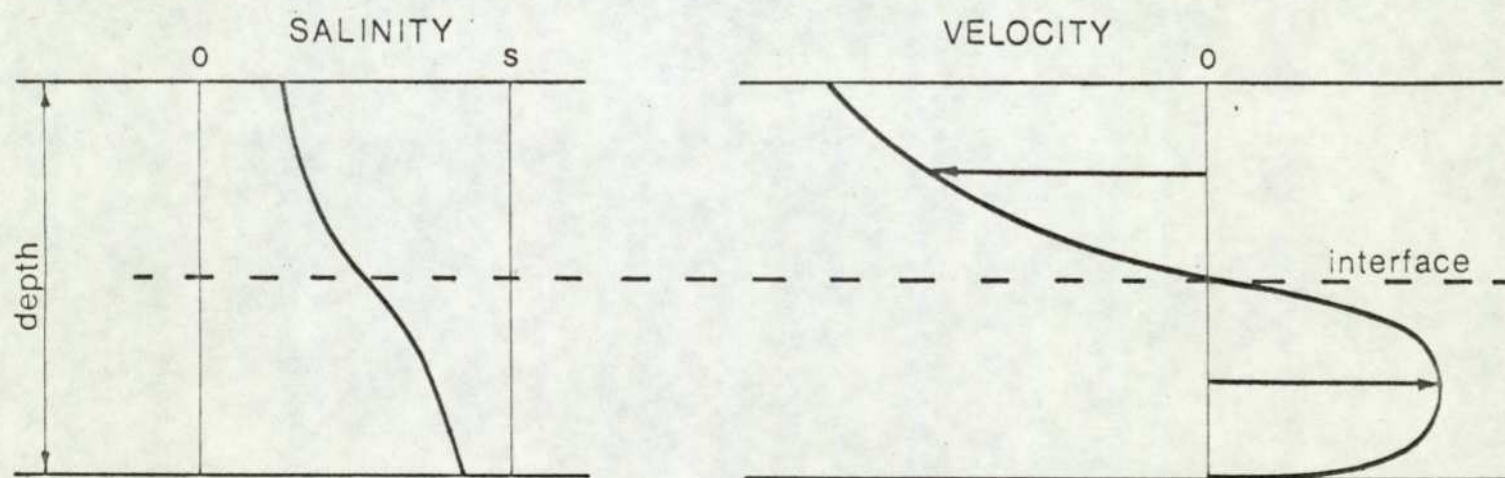
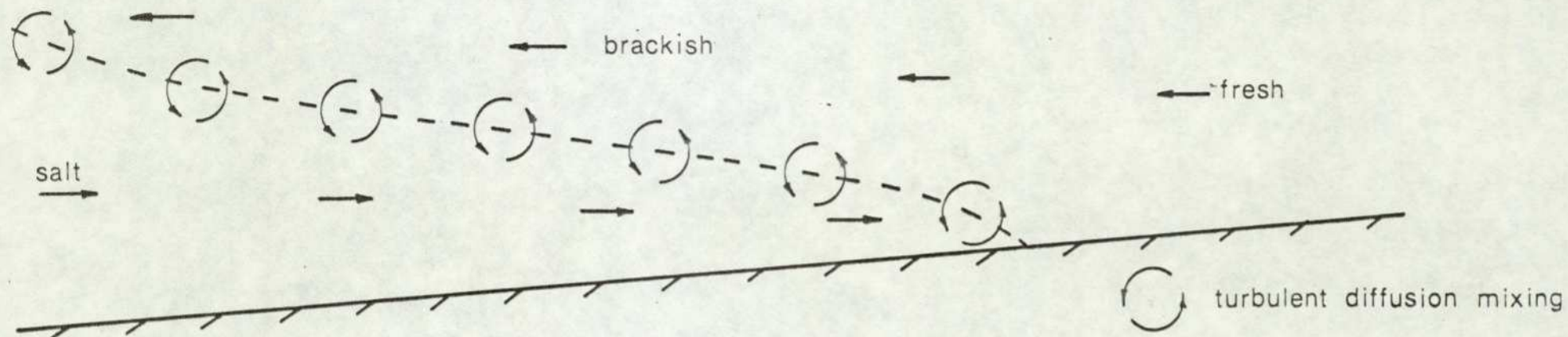


FIG. 1.3 PARTLY-MIXED ESTUARY

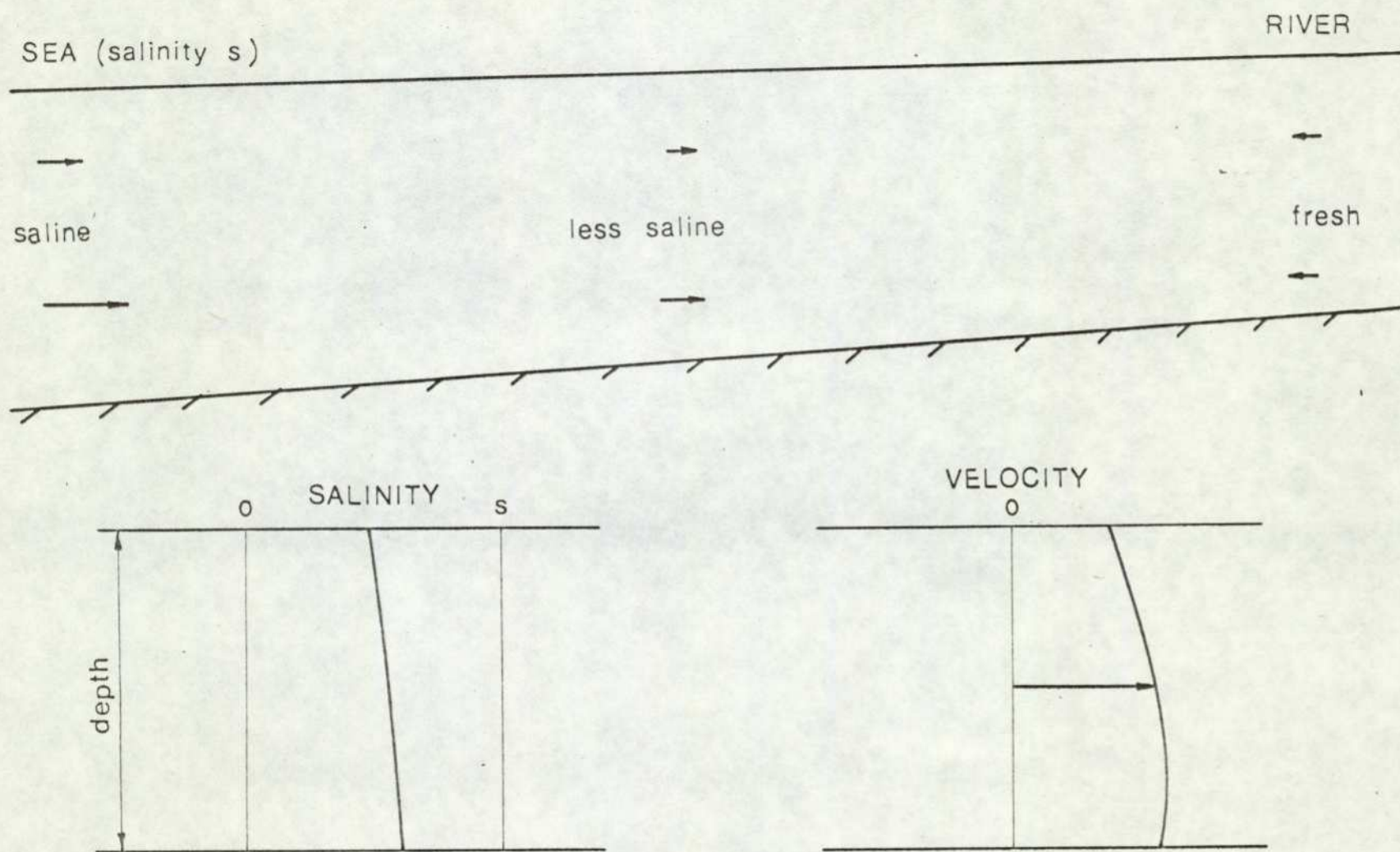


FIG. 1.4 WELL-MIXED ESTUARY

gradient extending from the fresh water at the head of the estuary to salt water near the mouth still exists as shown in Fig. 1.4 .

In certain broad, shallow estuaries there is a lateral salinity gradient across the estuary as a function of the Coriolis effect. In the Northern Hemisphere lower salinity water moves on the right side of the estuary whereas sea water enters the system on the left. The net effect is still to maintain vertical homogeneity, but a counterclockwise circulation is established within the estuary.

Examples: Severn and Humber in England and the Hooghly in West Bengal.

1.2 Historical Development of Computational Hydraulics

Because of the complicated nature of the hyperbolic partial differential equations governing tidal flows, methods of solution were limited until the availability of high speed digital computers. Application of fundamental mathematical methods was inaugurated in Holland by de Vries Broekman (147) in 1916, who demonstrated the possibility of constructing the solution of the differential equations of tidal motion, by means of finite

difference methods. Papers were published in the German language by Defant (36) in 1920 and Reineke (112) a year later applying simple finite difference methods to tidal estuaries and coastal seas. In England simple finite difference methods were used by Taylor (133) in 1921 and later by Proudman (109) in 1925 for tidal calculations in the Bristol Channel. These finite difference methods were derived from equations representing the continuity of mass and momentum but written in the form of Bessel functions. The stability theory of finite differences was discussed by Courant, Friedrichs and Lewy (30) in 1928 but serious use of these methods to tidal flows was impracticable using the manual methods of computation available at the time.

Computational hydraulics has been a recognised branch of hydraulic engineering since the commercial availability of high speed digital computers, and high level programming languages such as Fortran and Algol in the late 1950's. In the previous decade the solution of tidal problems using finite difference methods became possible due to the development of the first digital computers.

The Harvard Mark 1 computer which was built by IBM from mechanical machine parts and completed in 1944, was the first fully automatic calculator. The first all-purpose all-electric digital computer was developed at the University of Pennsylvania and built at Aberdeen in 1945.

It was called ENIAC for (Electronic Numerical Integrator and Calculator) and was used by the Los Alamos research team just after the war. However, the first commercially available computer was built by Remington Rand and completed in 1952. This first machine was used by the U.S. Bureau of the Census, and was a binary automatic computer called the UNIVAC 1 . It was on one of these machines at New York University in 1954 that finite difference methods were first seriously applied to river flood routing problems.

From about 1957, the smaller IBM 650 computer was available and this was the first truly mass-produced electronic computer. By the year 1960 about 30,000 computers were in use in the U.S.A., 400 - 500 of these being of the large UNIVAC type. Around 1965 IBM introduced their second generation electronic digital computer called the 360 series. Other companies especially in Britain and France, helped with government money also produced similar machines. These new powerful computers caused an explosion of activity in computational hydraulics which has been sustained and even revitalised by the introduction around 1975 of third generation computers such as the IBM 370 series.

Before the days of the digital computer all tidal computations were performed by manual methods. Methods suitable for manual computation were initially of the power-series and harmonic analysis types but later by the graphical characteristics method. Considerable practical progress was made in 1918 when Lorentz (92) advised the Dutch government

to make a mathematical study to compute the water levels created after the closing of the Zuiderzee. This calculation used the new Lorentz linearization of the friction losses and was carried out under the direction of Thijsse (136) using a power-series method. This power-series method was improved upon and used extensively by Dronkers (38) in Holland for all kinds of tidal calculations until the early 1950's.

In 1935 Massé (94) published a paper indicating the characteristic form of the partial differential equations, and applied the method using linearised equations to the River Seine. The characteristic solution had however been published previously by Massau (93) in Belgium thirty five years earlier but his work received no attention for almost half a century. In the following year the Russian, Khristionovich (80) produced a paper giving the characteristic form of the equations and used the linearised equations with friction to calculate simple wave movements in canals.

It was not until 1946 when Craya (32) showed how the method of characteristics could be applied graphically that the first accurate solutions for the non-linear partial differential equations were made possible. In the same year another Frenchman, Ré (111) applied Craya's method to the River Rhine to calculate wave heights due to a dam break with special attention given to the bores formed. The next year Holsters (64) adapted Massau's method of characteristics to make it more practical and called his graphical method

of characteristics the method of 'influence lines'. This method was applied to many problems, including flood routing in rivers, with friction terms representing wind shear stress and Chezy bed resistance included.

Meanwhile in Holland alternative methods of solving the partial differential equations were being investigated in order to overcome the enormous amount of work involved by the manual methods of computation. In 1947 van Veen (143) showed after a decade of basic work how an electrical network analogy could give good solutions to tidal problems. Also, Schönfeld (122) developed further the harmonic methods of tidal computation as alternatives to the iterative power-series method.

By 1948 the graphical method of characteristics was well established and the first all electric digital computers were becoming available. Mathematicians in the U.S.A. therefore directed their attention to ways of solving tidal problems on the computer, using both characteristic and explicit finite difference methods. Papers by Stoker (128), Courant, Isaacson and Rees (31) and von Neumann and Richtmeyer (100) from 1948 to 1950 advanced the basic theoretical knowledge of numerical methods in fluid mechanics. Schönfeld's (123) doctoral thesis presented to the University of Delft in July 1951, gave an extensive account of the four methods of computation available at that time. These methods were the power-series, harmonic, characteristic, and explicit finite difference methods all of which had to

be calculated manually since at that time the digital computers were too small and expensive for solving tidal problems on. Schönfeld came to the conclusion that the graphical method of characteristics was the superior of the manual methods when definite details of a tidal curve are required such as in shallow water estuaries. For deep water computations where the tidal curve is almost a pure sine wave then the simple harmonic method was found to be best.

In 1953 the Institute of Mathematical Sciences at New York University acquired a UNIVAC 1 digital computer. In the same year Stoker (129) completed the numerical procedures for an explicit finite difference computation and discarded the characteristics method. By the following year Isaacson, Stoker and Troesch (68) were running this computational method on the UNIVAC computer for a model of the River Ohio. This numerical study was the first important application of a digital computer to flood routing problems. In the same year the first application of an analogue computer to flood routing problems was reported from Japan by Ishihara, Hayami and Hayami (69) using Hayami's (59) theory of flood waves.

In 1954 Lax (81) developed an explicit finite difference scheme that could compute across a hydraulic jump without going unstable. The method introduced a pseudo-viscosity generated from the truncation error of the weak finite difference scheme. This was an improvement on the viscous term introduced by von Neumann and Richtmyer (100) but was later modified by Lax (82) in 1957 and again by

Lax and Wendroff (83) in 1960. In 1957 Richtmyer (113) published his book which gave a comprehensive analysis of the stability of difference schemes plus a new tri-diagonal algorithm, for economical scalar matrix inversion.

A two-dimensional tidal model of the North Sea using the explicit alternating direction scheme proposed by Peaceman and Rachford (102) was built in 1956 by Hansen (57). This model utilised for the first time the full shallow water equations including terms for Coriolis force and nonlinear bed and wind shear stresses. At the same time the results from shallow water tidal computations by Dronkers (39) and Schönfeld (124) on the Rijkswaterstaat's computer indicated that accurate results could now be achieved without the burden associated with manual methods of computation. This was confirmed in 1958 by Hensen (61) at Hannover using the 'initial value' explicit scheme for the Ems estuary and again in England by Otter and Day (101) in 1960 using Hensen's method for the Thames estuary.

During 1961 Abbott (1) derived the characteristic equations for a two-layer stratified flow and applied the method graphically to an oil slick in Southampton Water. Later, with the mathematician Torbe (2) he generalised the analysis to any number of fluid layers, and in 1966 published his book on the method of characteristics (4). During the early 1960's other researchers such as Preissmann and Cunge (105), Wendroff (149), Daubert and Marvaud (34) contributed much to the study of non-continuous flows using finite difference methods whilst Fromm (44) and Dronkers (40) continued the research into coastal flows. In 1961

Preissmann (106) at Grenoble introduced implicit type schemes to solve linear wave equations, but using the very uneconomical method of block matrix inversion instead of Richtmyer's scalar matrix inversion.

In January 1964, Cunge and Wegner (33) at Grenoble and Gary (46) at New York published papers showing how Richtmyer's (113) tri-diagonal algorithm could be used for solving linear wave equations economically. This new development opened up completely new possibilities for the implicit finite difference schemes which could now compete on equal terms with the explicit finite difference and characteristic methods.

By 1967 the new implicit scheme had been generalised to a pair of simultaneous tri-diagonal algorithms in order to solve non-linear wave equations. These were incorporated into working computer programs by Liggett and Woolhiser (88), Abbott and Ionescu (5) and Leendertse (86). However, in the rush to publish these papers the schemes were only tested for simple flow conditions without practical applications. It was not until the following year when Abbott and Verhoog (6) applied the method to the River Ijssel in Holland that the phenomenon of 'pseudo-instability' generated by the nonlinear friction term was discovered and suppressed. In the same year the new implicit scheme was tested by Kamphuis (71) on the St. Lawrence River and in 1971 by Verwey (145) on the River Danube, both obtaining satisfactory results. Verwey discovered the surprising fact that when using Courant numbers of order 200, to model a flood wave

rising at about two metres/day the accuracy was slightly better than with much smaller Courant numbers. The reason for this was later explained by Abbott (12), in that high Courant number working propagates fresh boundary conditions more rapidly into the interior of the calculation thereby more than compensating for the degradation of the results due to higher truncation errors. This meant that in this particular case an unconditionally stable implicit scheme was 60 times more economical than an equivalent explicit or characteristic model. Even for tidal models the implicit scheme is at least three times more economical than other schemes, but equivalent to them for Tsunami and seiching wave models. It has been shown how an implicit scheme that could only be used for linear wave equations, became in only seven years superior to all other methods of computation due to the discovery of a new algorithm.

During the last decade, developments in computational hydraulics have been made through the application of new algorithmic techniques specifically in the fields of two-dimensional flows, stratified flows and coupled dynamic-transport models. An important development was the translation in 1968 from Russian into French of Yanenko's (152) work. Yanenko had shown how it is possible to associate the splitting up of equations with a 'falsification' of the individual stages of the scheme. It was therefore possible to build a scheme in which no stage is individually consistent with its differential

equation, but only the resultant of such a scheme is so consistent. This meant that a two-dimensional implicit finite difference scheme could be solved using the alternating direction algorithm wherein the equations were split up such that unknowns appeared alternately in the x and y directions to be solved with double tri-diagonal algorithms. The schemes of Leendertse (86), Gustafsson (55), Abbott, Damsgaard and Rodenhuis (11) and Grubert (52) from 1967-1976 fall into this fractional-step category, whilst the paper by Daubert and Graffe (35) in 1967 makes an important contribution to the understanding of two-dimensional flows using the characteristics method.

The first computer solution for a stratified flow was by Boulot and Daubert (25) in 1969 applied to the River Grand Rhône. In the following year Vreugdenhil (146) applied the same technique, a Lax-Wendroff explicit finite difference scheme to the salt-wedge in the Rotterdam Waterway. The first implicit two-layer stratified flow model was built by Grubert and Abbott (51) in 1972, using a generalised algorithm to subsume the nine different characteristic structures. In 1974 Hydén (65) solved the same problem using the characteristics method as a computer solution, but this scheme in common with explicit schemes is economically inferior to implicit schemes. This is because these methods are bound by the Courant-Friedrichs-Lewy (30) stability condition which for stratified flows is particularly severe since the internal and surface wave celerities can

differ by a factor of ten or more.

A new generalised algorithm is described in this thesis for the solution of either single layer or two-layer stratified flows using coupled dynamic-transport equations. It consists of a fractional-step, implicit finite difference method in which the mass, momentum and transport-diffusion equations for the lower layer are each split up into two halves and used in a four-level scheme to compute the six dependent variables. This technique uses a double pentadiagonal algorithm plus a tri-diagonal algorithm at each stage of the computation, similar to that used by Grubert (53) for a well-mixed estuary in 1976. It has the advantage that because of its fractional-step construction each layer is calculated alternately and is therefore ideal for salt-wedge estuaries where the computation must change from a two-layer to a single layer scheme within the model.

It will be seen from this historical review that since the appearance of high speed digital computers, virtually the whole of computational hydraulics has been dominated by 'dependent variable finite difference schemes' (direct schemes). There has been speculation that in the future these methods will be superseded by finite element techniques or perhaps even 'independent variable finite difference schemes' (inverse schemes).

Weare (148) studied the relative economics of both direct finite difference schemes and finite element methods and came to the conclusion that for any enclosed

body of water finite difference methods are nearly always cheaper. Abbott and Vium (14) on the other hand studied the relative merits of inverse finite difference schemes in which the problem is to determine the values of the independent variable (distance) at which given values of the dependent variable (salinity) for example occur at given times. They concluded that although the method is very accurate and can be applied to all the main areas of computational hydraulics, the algorithms are generally more complicated than with direct schemes so that their economic outlook is dubious.

It is thought that the areas in which either finite element techniques or inverse difference schemes could replace direct difference schemes by the end of this decade are where the latter methods are weakest. These are in the modelling of two-dimensional transport-diffusion flows (13) and the propagation of seiching waves in harbours or Tsunamis waves in coastal regions with complicated boundaries.

1.3 Thesis Outline

During July 1972 two papers were published by the writer and Abbott (51,10) showing how for the first time, a two-layer stratified flow could be computed using a central implicit finite difference scheme. Previous to this the solution had been made using either a Lax-Wendroff explicit scheme or the method of characteristics. The advantage of the implicit scheme over the other two schemes is that whereas the explicit and characteristic schemes are conditionally dependent on the value of the wave celerity for stability, the implicit scheme is unconditionally stable. This property is especially important for stratified flows since the surface and internal wave celerities can differ by a factor of ten or more resulting in uneconomically high computer costs. This is because in order for the explicit and characteristic schemes to remain stable at the surface where the wave celerity is large, they must be computed using time steps one tenth of those needed for the interfacial wave computation alone.

In order to avoid these high computer costs all of the explicit schemes published to date for real estuaries have either assumed that the water surface is horizontal along the estuary or have computed the water surface profiles separately using a homogeneous flow program, thereby effectively uncoupling the layers in the computation. For short estuaries or inlets with slight tidal action these computational methods are good but contribute little to the advancement of computational hydraulics as a science

since they evade the problem at hand which is the numerical solution of a fully coupled two-layer stratified flow.

The original central implicit scheme for computing two-layer stratified flows utilised equations with two dependent variables in each layer, namely velocity and depth. These equations contained all the relevant terms including those for the transfer of fluid mass across the interface. The computation of density variations in each layer was achieved with a separate transport-diffusion central implicit program inserted between the time steps of the dynamical program.

Although the above scheme solved the two-layer stratified flow equations without uncoupling the layers or making any other assumptions it had the following three shortcomings. Firstly, because the transport-diffusion program is only lightly coupled to the dynamical program, changes in the flow direction in either layer caused non-linear instabilities to develop which could only be suppressed by using uneconomically small distance and time steps. Secondly, in order to compute for all combinations of subcritical and supercritical flows that can occur, nine different algorithmic structures (three in each layer) needed to be built into the dynamical calculation considerably increasing the complexity of the final program. Thirdly, because the recurrence relations contain dependent variables from both layers, computations from a two-layer fluid across to a single layer fluid, such as would occur in a salt-wedge estuary are very difficult to make. This is because non-linear instabilities develop at this internal

boundary, and because the central implicit scheme unlike the Lax-Wendroff scheme has no built-in dissipative mechanism, numerical instabilities destroy the computation after several prototype hours of computation. Energy dissipation can be achieved by using a backward implicit scheme but this reduces the accuracy from second to first order and should be avoided. There is unfortunately no implicit difference scheme to date which like the Lax-Wendroff explicit scheme has both second order accuracy and genuine dissipative properties.

The extended central implicit scheme developed in this thesis for a two-layer stratified flow was first published by the writer in July 1976 (53) for well-mixed estuarine flows. The extended scheme retains all the advantages of the original stratified implicit scheme, that is unconditional stability and second order accuracy, and excludes the three disadvantages. The first disadvantage mentioned, that of non-linear instabilities developing in the transport-diffusion computation is solved by treating the density as a third dependent variable together with discharge and water level in each layer. Although this increases the number of algorithmic structures in each layer from three to four, the transport-diffusion equation then becomes fully coupled to the dynamical equations. This enables a simple generalised algorithm to be used that changes the algorithmic structure as the flow direction changes. In this way the computational procedure fully conforms to the changing characteristic structure and therefore prevents non-linear instabilities from developing.

With this instability removed the full benefits of an unconditionally stable implicit scheme can be realized resulting in an economically viable computational method. The second disadvantage, that of having not only nine but now sixteen different algorithmic structures is solved by using a 'fractional-steps' method of computation whereby the three lower layer difference equations are each split up into two parts and used at different time levels in the computation. This allows the computation to be made in one layer at a time, whilst still retaining a fully coupled system of equations, therefore reducing the number of different algorithmic structures required from sixteen to four. The third disadvantage is considerably reduced by virtue of the 'fractional-steps' scheme, because now that the computation is made in one layer at a time, the upper layer calculation can proceed uninterrupted by the salt-wedge front from the sea to the tidal limit. The lower layer calculation extends from the sea to the salt-wedge front thereby avoiding most of the computational problems associated with a moving boundary on a finite grid.

For the computation of estuarine salt-wedge flows it is necessary to build into the computer system conditions at the front of the salt-wedge and equations for calculating the onset of interfacial mixing and the amount of fluid exchanged between the layers. From theoretical considerations it is known that two different frontal types should exist in a salt-wedge estuary; one called a wave front, which occurs on the flood tides and is characterised by a wall or 'head' of water and the other

a St.Venant front which should occur on the ebb tides and is characterised by zero frontal depth. Although the theoretical analysis of internal wave fronts is well documented, there has been no theoretical investigations into internal St.Venant front conditions except by the writer. Since there is no field data on salt-wedge fronts and only experimental data on lock exchange wave fronts, an experimental programme was carried out using a model estuary to investigate (a) estuarine front formation and propagation and (b) interfacial stability and mixing. The results of these experiments were published in two separate papers by the writer in June 1980, which can be found in the back of this thesis.

The experiments on salt-wedge fronts confirmed that the wave and St.Venant fronts do occur but that their front conditions are different from those predicted by theory. This is due largely to the presence of a laminar sub-layer just upstream of the St.Venant and arrested salt-wedge fronts and to interfacial turbulent mixing in the case of the wave fronts, both causing energy losses. It was deduced that the fronts of all salt-wedges effectively end at the critical flow section and that critical flow conditions should be used as boundary data here in mathematical models.

Interfacial instability has been investigated experimentally mainly by Keulegan (76) and Thorpe (138). Theoretical investigations are due largely to the efforts of Jeffreys (70), Taylor (134) and Goldstein (47). From his experimental results Keulegan suggested that a stability

parameter could be defined for interfacial waves which is equivalent to that defined by Jeffreys for wind generated surface waves. This stability parameter is a function of an interfacial Reynolds number and an interfacial Richardson number, and experiments in laboratories all suggest that it is a constant independent of the flow Reynolds number. However, the theoretical work of Taylor and Goldstein coupled with Thorpe's experimental results strongly suggested that Keulegan's stability parameter deserved closer investigation. It was found experimentally that interfacial instability occurs when the interfacial Richardson number is less than unity and therefore the value of Keulegan's stability parameter cannot be a constant independent of the flow Reynolds number but must be scaled up to the prototype. The practical significance of this finding is that the critical velocity causing interfacial instability is greater in the prototype than in the model, resulting in interfacial mixing taking place only along the seaward half of an arrested salt-wedge instead of along the whole length.

The amount of interfacial mixing was measured by Keulegan using dye concentrations and found to have a straight line relationship with the relative mean velocity between layers. The experiments described in this thesis measured the amount of mixing from the lower to the upper layer of arrested salt-wedges using modified wave monitor probes calibrated to measure density profiles. The results agreed well with those found by Keulegan, and in addition showed that the rate of interfacial mixing was inversely proportional to the relative density of the two fluids.

Also, since the results showed that intermittent mixing is initiated on both sides of the interface at the same critical velocity, then the amount of fluid transferred across the interface must be the same in either direction. This indicates that fluid is being transferred due to turbulent diffusion mixing although this is probably also associated with intermittent entrainment mixing in both directions to a greater or lesser extent. These observations are contrary to the popularly held belief that interfacial mixing takes place in an arrested saline wedge only from the lower to the upper layer, implying the false notion that the lower layer fluid is laminar.

To investigate the performance of the extended implicit scheme a computation was first made of the well-mixed Hooghly estuary in West Bengal. This estuary has a large tidal range of 5.5m and a considerable seasonal variation in upland discharge. Calculations were made over twelve tidal cycles for both extremes of upland discharge and the results for water level and salinity compared with prototype measurements. A more rigorous test on the stability of the scheme was to compute salinity profiles in the well-mixed Gambia estuary over a period of 224 days with a one metre tidal range at the sea and a variable inflow hydrograph at the tidal limit. The computed salinity profiles were compared with measured prototype values but unfortunately no measured water level curves were available for comparison.

The problem of computing salinity profiles

in well-mixed estuaries has always been done in the past using analytical solutions to the transport-diffusion equation. One of the more successful of these methods developed by Sanmuganathan (119) uses Laplace transformations together with prototype data for maximum tidal velocities. The advantage of the extended implicit finite difference scheme is that for the first time, level, discharge and density can be computed simultaneously as well as economically.

The extended implicit scheme for two-layer stratified flows was tested on the salt-wedge in the South Pass of the River Mississippi. With a half metre tidal range at the sea, tidal excursions for the wedge were computed together with the variations of depth, discharge and density in each layer. Previous finite difference computations for salt-wedge flows have always used uncoupled equations either by assuming a horizontal water surface or by computing surface water profiles using a separate homogeneous flow model. The models by Boulet and Daubert (25) for the River Grand Rhône and Vreugdenhil (146) for the Rotterdam Waterway use the first approximation while Hodgins, Osborn and Quick's (63) model of the River Fraser used the second. Also, by ignoring interfacial mixing these models make no attempt to compute the distribution of density in each layer, even though in the partly-mixed Rotterdam Waterway and River Fraser considerable quantities of water are exchanged between the upper and lower layers.

CHAPTER 2

PARTIAL DIFFERENTIAL EQUATIONS

2.1 The Continuity Equation for Well-Mixed Flows

This equation expresses the balance of quantities of water. The following items are involved, referring to Fig. 2.1 .

(a) Accumulation by the differences in the transport by the river. ρQ is the mass flow rate of water transmitted per unit time through the cross-section at x . It must be counted as a supply for the section of the river between x and $x + dx$. At $x + dx$ there is a mass flow rate $\left[\rho Q + \frac{\partial}{\partial x} (\rho Q) dx \right]$. Hence,

$$- \frac{\partial}{\partial x} (\rho Q) dx$$

is the excess of the supply over the mass flow rate, that is the mass of water accumulated per unit time between x and $x + dx$.

(b) Lateral discharge or supply. There may be a lateral discharge (supply) out of (into) the river, for instance when flood plains are flooded. Here we consider only a discharge distributed along an appreciable length of the river. Let q_i be the lateral discharge per unit time and per unit length of the river, then if ρ_i is the density of this water, then

$$\rho_i q_i dx$$

is formally a mass of water per unit time, entering the

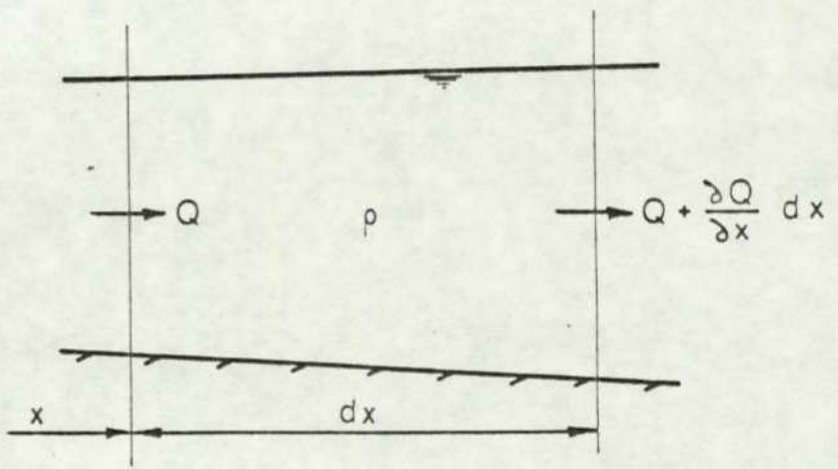
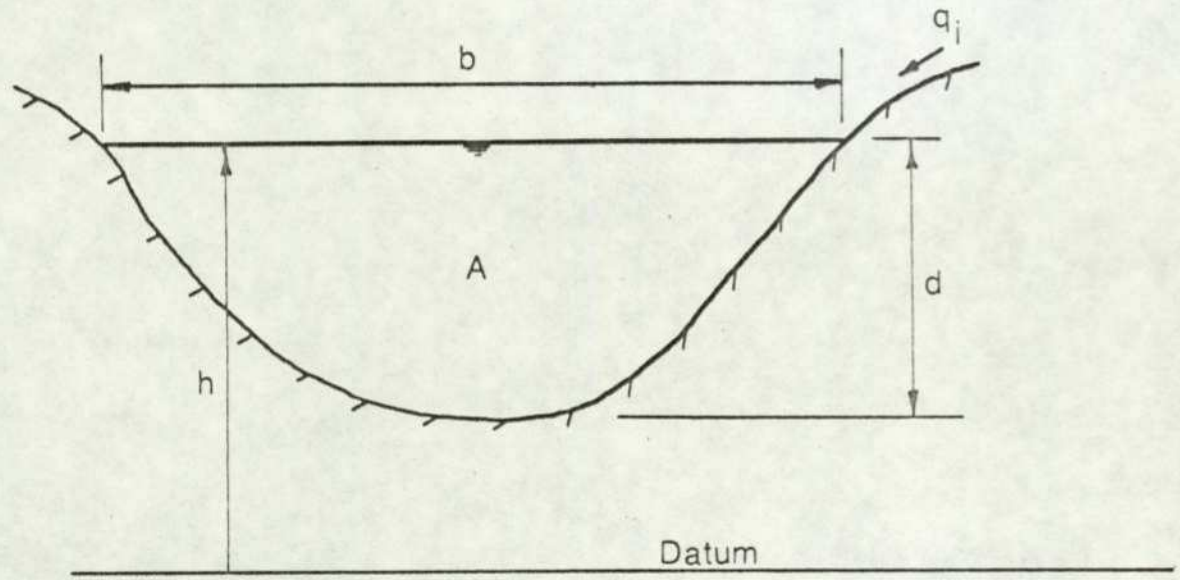


FIG 2.1 CHANNEL SECTION FOR WELL-MIXED FLOWS

river between x and $x + dx$.

The sum of these items must be equal to:

(c) The storage by the river. By rising of the water surface, water is stored in the river between x and $x + dx$. This amounts to

$$b \frac{\partial}{\partial t} (\rho h) dx$$

per unit time.

Hence we obtain after some reduction:

$$\frac{\partial}{\partial x} (\rho Q) + b \frac{\partial}{\partial t} (\rho h) - q_i \rho_i = 0 \quad (2.1)$$

2.2 The Dynamical Equation for Well-Mixed Flows

This equation expresses the balance of forces and inertia. The following items are involved:

(a) Gravity. The slope of the water surface is $-\frac{\partial h}{\partial x}$. This results in a force

$$- g \frac{\partial h}{\partial x} \rho A dx$$

acting upon the water between x and $x + dx$.

(b) Resistance, wind force, etc. Shearing stresses work along the perimeter of the cross-section. Let i be the slope of the water surface sufficient to compensate for these stresses. Then we may put

$$- g i \rho A dx$$

for the force acting upon the water between x and $x + dx$, resulting from these stresses. The slope may be specified

as follows: The perimeter of the cross-section consists of (i) bottom and embankments (the so-called wetted perimeter), and (ii) the water surface. When the air is at rest, the stress along the surface is negligible, and according to Chézy's formula we may put

$$i = \frac{Q |Q|}{A^2 C^2 R}$$

where R is the hydraulic radius and C is Chézy's coefficient. When there is a wind, we may add a term for the stress along the water surface, as follows:

$$i = \frac{Q |Q|}{A^2 C^2 R} - k_a \rho_a u_a |u_a| b / (\rho g A)$$

where u_a is the wind velocity 10m above the water surface in the positive direction of water flow, ρ_a the air density, and k_a the Rossby coefficient of wind shear stress.

(c) Lateral discharge or supply. If a lateral discharge per unit time and per unit length q_i of density ρ_i enters the river at an angle of ϕ to the streamlines with a velocity v_i then the equivalent force acting on the water between sections x and $x + dx$ is

$$v_i \cos \phi q_i \rho_i dx$$

The sum of these items is the resulting force which balances:

(d) The inertia. The change of momentum of the water between x and $x + dx$ is

$$d(mu) = \frac{\partial}{\partial x} (mu) dx + \frac{\partial}{\partial t} (mu) dt$$

Since $u = Q/A$ and $m = \rho A dx = \rho Q dt$ then

$$d(mu) = \frac{\partial}{\partial x}(\rho Q dt Q/A) dx + \frac{\partial}{\partial t}(\rho A dx Q/A) dt$$

Now $d(mu)/dt$ is the rate of change of momentum, hence the resulting force becomes

$$\left[\frac{\partial}{\partial x} \left(\frac{\rho Q^2}{A} \right) + \frac{\partial}{\partial t} (\rho Q) \right] dx$$

Hence we obtain after some reduction:

$$\begin{aligned} \frac{\partial}{\partial t} (\rho Q) + \frac{\partial}{\partial x} \left(\frac{\rho Q^2}{A} \right) + \rho g A \frac{\partial h}{\partial x} + \frac{\rho g Q |Q|}{C^2 A R} - k_a \rho_a u_a |u_a| b \\ - v_i \cos \phi q_i \rho_i = 0 \end{aligned} \quad (2.2)$$

2.3 The Continuity Equation for the Solute

This equation is a transport-diffusion equation which completes the system of equations for a well-mixed estuary, in which there are three dependent variables namely discharge Q , level h and density ρ .

Deriving the transport-diffusion equation macroscopically, we write

$$\begin{aligned} \frac{\partial}{\partial t} (\text{salt level}) + \frac{\partial}{\partial x} (\text{convected salt flux}) + \\ \frac{\partial}{\partial x} (\text{diffused salt flux}) = (\text{lateral salt inflow}) \end{aligned} \quad (2.3)$$

$$\text{or } \frac{\partial}{\partial t} (s A) + \frac{\partial}{\partial x} (s Q) + \frac{\partial}{\partial x} (q_s A) = q_i s_i \quad (2.4)$$

where s is salinity and q_s is the salt passing unit control areas orthogonal to the flow direction in unit time due to diffusion. Using the well known relation

$$q_s = - D' \frac{\partial s}{\partial x} \quad (2.5)$$

where D' is a diffusion coefficient which is always positive, we obtain

$$\frac{\partial}{\partial t}(s A) + \frac{\partial}{\partial x}(s Q) - \frac{\partial}{\partial x}(D' A \frac{\partial s}{\partial x}) = q_i s_i \quad (2.6)$$

Neglecting products of derivatives and treating D' as a constant

$$\frac{\partial s}{\partial t} + \frac{Q}{A} \frac{\partial s}{\partial x} - D' \frac{\partial^2 s}{\partial x^2} + \frac{s}{A} (\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x}) - \frac{q_i s_i}{A} = 0 \quad (2.7)$$

since $s = (\rho - \rho_w)/\rho_s$ then

$$\frac{\partial \rho}{\partial t} + \frac{Q}{A} \frac{\partial \rho}{\partial x} - D' \frac{\partial^2 \rho}{\partial x^2} + \frac{(\rho - \rho_w)}{A} (\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x}) - q_i (\rho_i - \rho_w)/A = 0 \quad (2.8)$$

but from Eq.(2.1)

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = - \frac{Q}{\rho} \frac{\partial \rho}{\partial x} + \frac{b h}{\rho} \frac{\partial \rho}{\partial t} + \frac{q_i \rho_i}{\rho} \quad (2.9)$$

from Eqs.(2.8) and (2.9)

$$\frac{\partial \rho}{\partial t} + \frac{Q}{A} \frac{\partial \rho}{\partial x} - D \frac{\partial^2 \rho}{\partial x^2} + \frac{q_i}{A} (\rho - \rho_i) = 0 \quad (2.10)$$

where $D = \rho D'/\rho_w$

2.4 Equations for Two-Layer Stratified Flows

For a two-layer flow shown in Fig.2.2 the subscript 1 refers to the upper layer and the subscript 2 to the lower layer, also for stability $\rho_1 < \rho_2$. A mass flux q_1 , leaves the upper layer to enter the lower layer

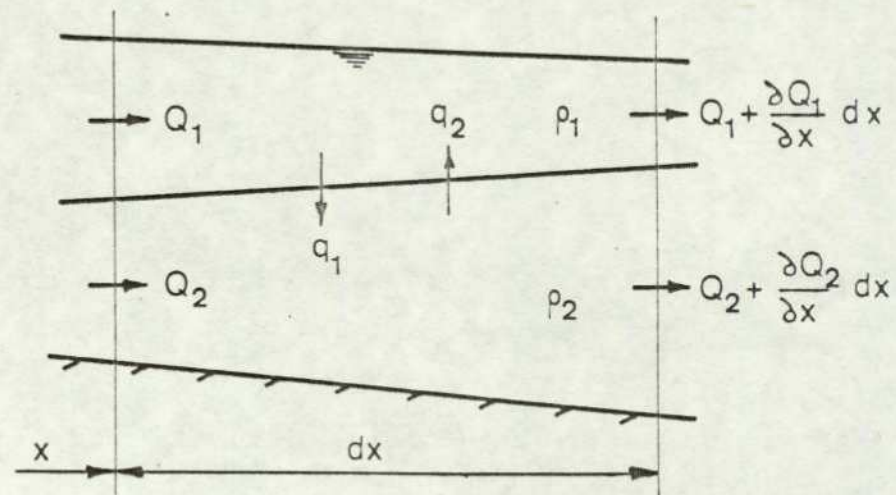
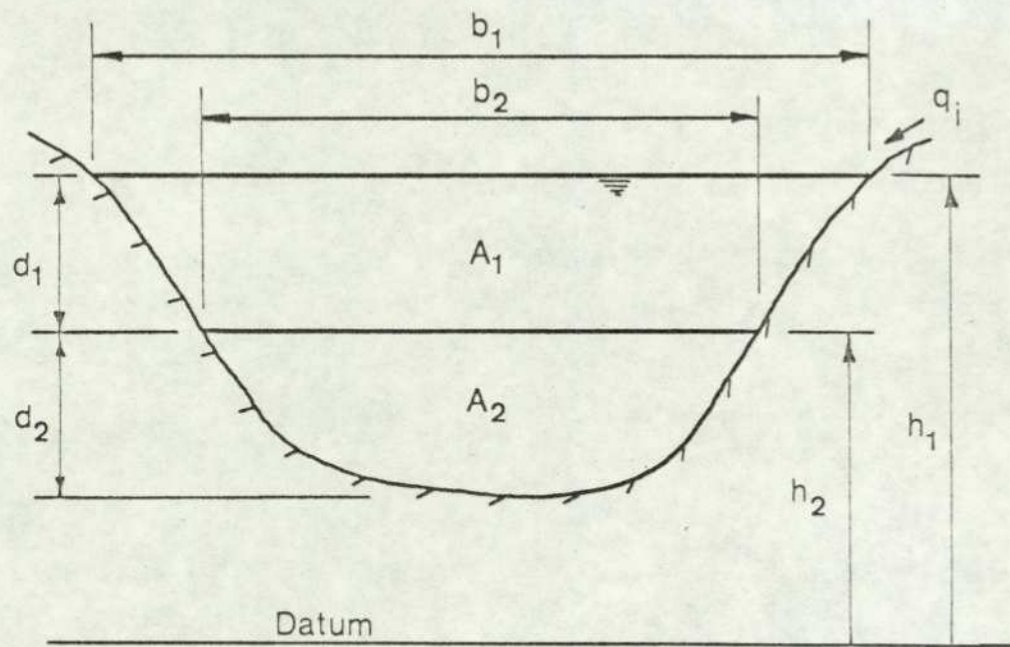


FIG. 2.2 CHANNEL SECTION FOR TWO-LAYER FLOWS

and a mass flux q_2 leaves the lower layer to enter the upper layer. The interfacial shear stresses are accounted for by momentum fluxes between the layers and a quasi-laminar shear stress τ as indicated in Section 6.4 .

The Continuity Equation for the Lower Layer

Referring to Section 2.1 the mass of water accumulated per unit time between x and $x + dx$ in the lower layer is

$$- \frac{\partial}{\partial x} (\rho_2 q_2) dx$$

The mass of water entering the lower layer per unit time, between sections x and $x + dx$ is

$$b_2 (\rho_1 q_1 - \rho_2 q_2) dx$$

These two quantities cause the interfacial surface to rise per unit time, storing the mass of water

$$b_2 \frac{\partial}{\partial t} (\rho_2 h_2) dx$$

Hence we obtain after some reduction.

$$\frac{\partial}{\partial x} (\rho_2 q_2) + b_2 \frac{\partial}{\partial t} (\rho_2 h_2) - b_2 (\rho_1 q_1 - \rho_2 q_2) = 0 \quad (2.11)$$

The Continuity Equation for the Upper Layer

The mass of water accumulated per unit time between x and $x + dx$ in the upper layer is

$$- \frac{\partial}{\partial x} (\rho_1 q_1) dx$$

The mass of water entering the upper layer per unit time, between sections x and $x + dx$ is

$$b_2 (\rho_2 q_2 - \rho_1 q_1) dx + \rho_1 q_1 dx$$

These two quantities can cause both the free surface to rise and the interface to fall per unit time, storing the mass

$$b_1 \frac{\partial}{\partial t} (\rho_1 h_1) dx - b_2 \frac{\partial}{\partial t} (\rho_1 h_2) dx$$

Hence, we obtain after some reduction:

$$\begin{aligned} \frac{\partial}{\partial x} (\rho_1 q_1) + b_1 \frac{\partial}{\partial t} (\rho_1 h_1) - b_2 \frac{\partial}{\partial t} (\rho_1 h_2) - \\ b_2 (\rho_2 q_2 - \rho_1 q_1) - q_1 \rho_1 = 0 \end{aligned} \quad (2.12)$$

The Dynamical Equation for the Lower Layer

Referring to Section 2.2 the force acting on the water in the lower layer due to gravity between sections x and $x + dx$ consists of two components. Firstly the slope of the interface $-\frac{\partial h_2}{\partial x}$, results in the force

$$- g \frac{\partial h_2}{\partial x} \rho_2 A_2 dx$$

acting upon the lower layer between x and $x + dx$.

Secondly the variation in the thickness of the upper layer $-\frac{\partial}{\partial x} (h_1 - h_2) dx$, results in a force on the lower layer of

$$- g \frac{\partial}{\partial x} (h_1 - h_2) \rho_1 A_1 dx$$

The shearing force due to friction along the solid perimeter of the lower layer, plus the shear forces at the interface due to mixing and quasi-laminar friction are

$$- \frac{\rho_2 g Q_2 |Q_2|}{C^2 A_2 R_2} dx + b_2 (\rho_1 q_1 u_1 - \rho_2 q_2 u_2) dx - \tau b_2 dx$$

where $\tau = \frac{1}{8} f (u_2 - u_1) |(u_2 - u_1)| \rho_{1,2}$

The sum of these items is the resultant force on

the lower layer which balances the rate of change of momentum which from Section 2.2 is given by

$$\left[\frac{\partial}{\partial x} (p_2 Q_2^2 / A_2) + \frac{\partial}{\partial t} (p_2 Q_2) \right] dx$$

Hence we obtain after some reduction:

$$\begin{aligned} & \frac{\partial}{\partial t} (p_2 Q_2) + \frac{\partial}{\partial x} (p_2 Q_2^2 / A_2) + p_2 g (1 - \epsilon) A_2 \frac{\partial h_2}{\partial x} + p_2 g \epsilon A_2 \frac{\partial h_1}{\partial x} \\ & + \frac{p_2 g Q_2 |Q_2|}{c^2 A_2 R_2} - b_2 (p_1 q_1 u_1 - p_2 q_2 u_2) + \tau b_2 = 0 \end{aligned} \quad (2.13)$$

where $\epsilon = p_1 / p_2$

The Dynamical Equation for the Upper Layer

The force acting on the water in the upper layer due to gravity is due to the slope of the water surface $-\frac{\partial h_1}{\partial x}$, and equals

$$-g \frac{\partial h_1}{\partial x} p_1 A_1 dx$$

Shearing forces due to friction along the side walls of the upper layer, wind forces at the surface and at the interface due to mixing and laminar friction are

$$\begin{aligned} & - \frac{p_1 g Q_1 |Q_1|}{c^2 A_1 R_1} dx + k_a p_a u_a |u_a| b_1 dx + b_2 (p_2 q_2 u_2 - p_1 q_1 u_1) dx \\ & + \tau b_2 dx \end{aligned}$$

The force acting on the water between sections x and $x + dx$ due to a lateral discharge q_i is

$$v_i \cos \phi q_i p_i dx$$

The sum of these items is the resultant force

on the upper layer which has to balance the following rate of change of momentum.

$$\left[\frac{\partial}{\partial x} (\rho_1 Q_1^2 / A_1) + \frac{\partial}{\partial t} (\rho_1 Q_1) \right] dx$$

Hence, after some reduction

$$\begin{aligned} \frac{\partial}{\partial t} (\rho_1 Q_1) + \frac{\partial}{\partial x} (\rho_1 Q_1^2 / A_1) + \rho_1 g A_1 \frac{\partial h_1}{\partial x} + \frac{\rho_1 g Q_1 |Q_1|}{C^2 A_1 R_1} - v_i \cos \phi q_i \rho_i \\ - k_a \rho_a u_a |u_a| b_1 - b_2 (\rho_2 q_2 u_2 - \rho_1 q_1 u_1) - \tau b_2 = 0 \end{aligned} \quad (2.14)$$

The Continuity Equations for the Solute

It follows directly from Eq.(2.10) that the continuity equation for the solute in the lower layer must be

$$\frac{\partial \rho_2}{\partial t} + u_2 \frac{\partial \rho_2}{\partial x} - D_2 \frac{\partial^2 \rho_2}{\partial x^2} + \frac{b_2 q_1}{A_2} (\rho_2 - \rho_1) = 0 \quad (2.15)$$

and for the upper layer

$$\frac{\partial \rho_1}{\partial t} + u_1 \frac{\partial \rho_1}{\partial x} - D_1 \frac{\partial^2 \rho_1}{\partial x^2} + \frac{b_2 q_2}{A_1} (\rho_1 - \rho_2) + \frac{q_i}{A_1} (\rho_1 - \rho_i) = 0 \quad (2.16)$$

2.5 Equations for Well-Mixed Flows in Two Dimensions

Some estuarine environments unlike the lower reaches of rivers and fjords do not have a dominant one dimensional flow system. In such cases a two dimensional grid needs to be employed, the equation system for the one dimensional case being extended to the two dimensional situation.

The following equations for two dimensional well-mixed flows omit all tidal generating forces, inflow terms, and pressure variations. They do however contain terms representing the Coriolis force. The dependent variables used are those of water depth H , density ρ , and velocities u and v in the x and y directions respectively. An elemental volume in a two dimensional flow is shown in Fig. 2.3 .

The Continuity Equation in Two Dimensions

The mass of water accumulated per unit time between sections x and $x + dx$ of the elemental volume shown in Fig. 2.3 is

$$- \frac{\partial}{\partial x} (\rho u H) dx dy$$

The mass of water accumulated per unit time between sections y and $y + dy$ is

$$- \frac{\partial}{\partial y} (\rho v H) dy dx$$

The sum of these items must be equal to the mass of water stored within the element due to the surface rising. This amounts to

$$\frac{\partial}{\partial t} (\rho H) dx dy$$

per unit time.

Hence after some reduction

$$\frac{\partial}{\partial t} (\rho H) + \frac{\partial}{\partial x} (\rho u H) + \frac{\partial}{\partial y} (\rho v H) = 0 \quad (2.17)$$

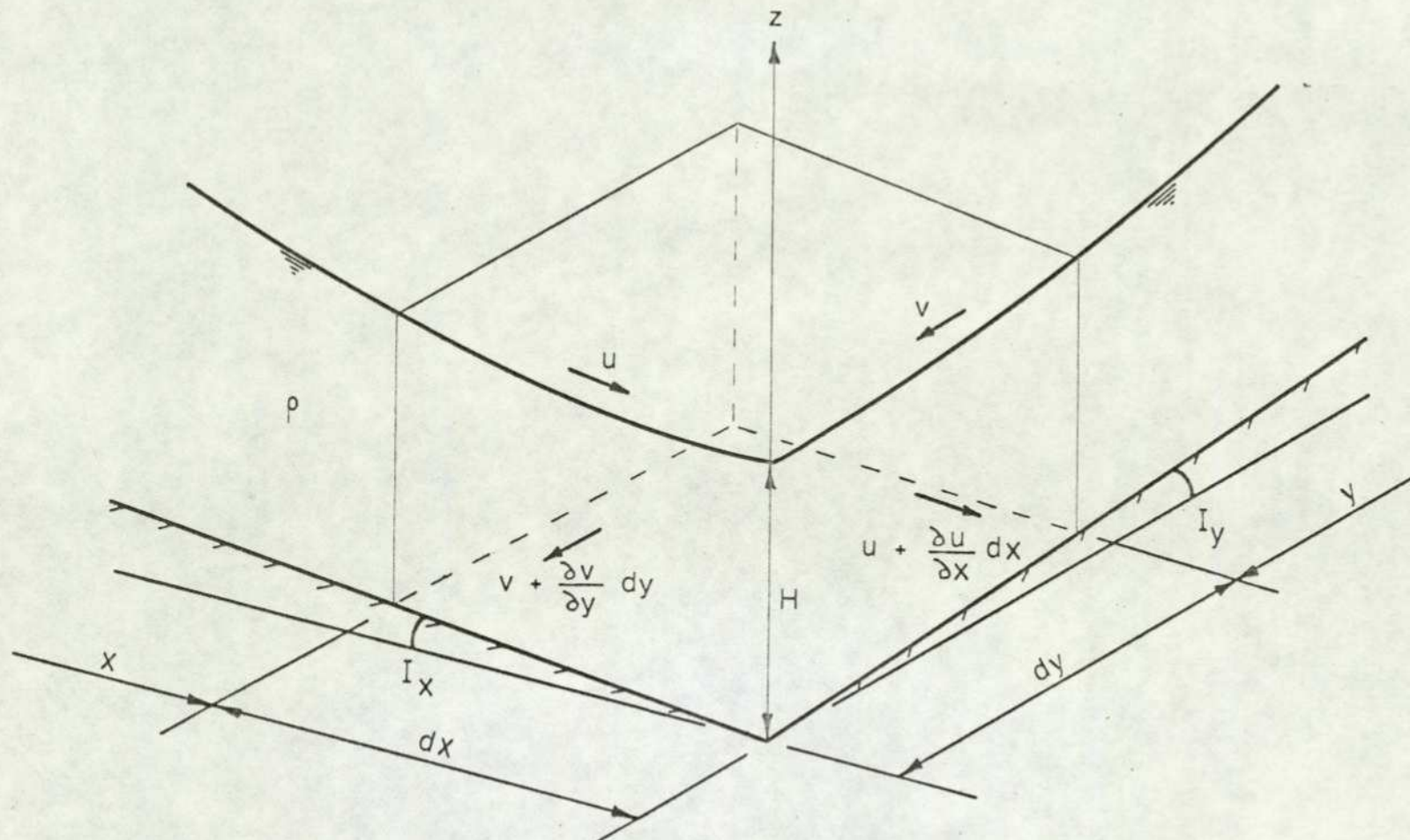


FIG. 2.3 DEFINITION SKETCH FOR TWO-DIMENSIONAL FLOWS

The Dynamical Equation in the x direction

The force acting on the water due to gravity in the x direction is due to the slope of the water surface in the x direction which is $-\frac{\partial H}{\partial x} + I_x$, where I_x is the bed slope in the x direction. This results in a force

$$\left(-\frac{\partial H}{\partial x} + I_x \right) \rho g H dy dx$$

There are also forces due to the friction on the bed, the wind shear force at the surface, and the Coriolis force. In the x direction these are

$$\left[-\frac{\rho g u(u^2+v^2)^{\frac{1}{2}}}{C^2} + k_a \rho_a u_a (u_a+v_a)^{\frac{1}{2}} + \rho \theta H v \right] dx dy$$

where $\theta = 2\omega \sin \psi$

and $\omega =$ angular velocity of the earth

$\psi =$ north latitude

The sum of these items is the resultant force which balances the rate of change of momentum in the x direction.

The momentum change between sections x and x + dx in the x direction is

$$d(mu) = \frac{\partial}{\partial x}(m_x u) dx + \frac{\partial}{\partial y}(m_y u) dy + \frac{\partial}{\partial t}(mu) dt$$

where $m = \rho H dx dy$; $m_x = \rho u H dy dt$; $m_y = \rho v H dx dt$

$$d(mu) = \left[\frac{\partial}{\partial x}(\rho u^2 H) + \frac{\partial}{\partial y}(\rho uv H) + \frac{\partial}{\partial t}(\rho u H) \right] dx dy dt$$

Since the resultant force = $d(mu)/dt$, we obtain after some reduction

$$\begin{aligned} & \frac{\partial}{\partial t}(\rho uH) + \frac{\partial}{\partial x}(\rho u^2H) + \frac{\partial}{\partial y}(\rho uvH) + \rho gH \frac{\partial H}{\partial x} - \rho gHI_x \\ & + \frac{\rho g u(u^2 + v^2)^{\frac{1}{2}}}{c^2} - k_a \rho_a u_a (u_a^2 + v_a^2)^{\frac{1}{2}} - \rho \theta Hv = 0 \end{aligned} \quad (2.18)$$

The Dynamical Equation in the y direction

This follows directly from Eq.(2.18) by turning the horizontal co-ordinate axis through 90° .

$$\begin{aligned} & \frac{\partial}{\partial t}(\rho vH) + \frac{\partial}{\partial y}(\rho v^2H) + \frac{\partial}{\partial x}(\rho vuH) + \rho gH \frac{\partial H}{\partial y} - \rho gHI_y \\ & + \frac{\rho g v(u^2 + v^2)^{\frac{1}{2}}}{c^2} - k_a \rho_a v_a (u_a^2 + v_a^2)^{\frac{1}{2}} - \rho \theta Hu = 0 \end{aligned} \quad (2.19)$$

The Conservation Equation for the Solute

Deriving the transport-diffusion equation macroscopically, we write

$$\begin{aligned} & \frac{\partial}{\partial t}(\text{salt level}) + \frac{\partial}{\partial x}(\text{convected salt flux in x direction}) \\ & + \frac{\partial}{\partial y}(\text{convected salt flux in y direction}) + \frac{\partial}{\partial x}(\text{diffused salt flux in x direction}) \\ & + \frac{\partial}{\partial y}(\text{diffused salt flux in y direction}) = 0 \end{aligned} \quad (2.20)$$

$$\begin{aligned} \text{or } & \frac{\partial}{\partial t}(sH) + \frac{\partial}{\partial x}(suH) + \frac{\partial}{\partial y}(svH) + \frac{\partial}{\partial x}(q_{sx}H) \\ & + \frac{\partial}{\partial y}(q_{sy}H) = 0 \end{aligned} \quad (2.21)$$

where q_{sx} and q_{sy} are the quantities of salt passing unit control areas orthogonal to the flow direction in unit time due to diffusion in the x and y directions respectively.

$$\text{Since } q_{sx} = - D'_x \frac{\partial s}{\partial x} \quad \text{and} \quad q_{sy} = - D'_y \frac{\partial s}{\partial y}$$

where D'_x and D'_y are the diffusion coefficients in the x and y directions respectively, we obtain

$$\begin{aligned} & \frac{\partial}{\partial t}(sH) + \frac{\partial}{\partial x}(suH) + \frac{\partial}{\partial y}(svH) - \frac{\partial}{\partial x}(D'_x H \frac{\partial s}{\partial x}) \\ & - \frac{\partial}{\partial y}(D'_y H \frac{\partial s}{\partial y}) = 0 \end{aligned} \quad (2.22)$$

Neglecting products of derivatives and treating D'_x and D'_y as constants

$$\begin{aligned} & \frac{\partial s}{\partial t} + u \frac{\partial s}{\partial x} + v \frac{\partial s}{\partial y} - D'_x \frac{\partial^2 s}{\partial x^2} - D'_y \frac{\partial^2 s}{\partial y^2} \\ & + \frac{s}{H} \left[\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(uH) + \frac{\partial}{\partial y}(vH) \right] = 0 \end{aligned} \quad (2.23)$$

since $s = (\rho - \rho_w)/\rho_s$ then

$$\begin{aligned} & \left[\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} - D'_x \frac{\partial^2 \rho}{\partial x^2} - D'_y \frac{\partial^2 \rho}{\partial y^2} \right] \\ & + \frac{(\rho - \rho_w)}{H} \left[\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(uH) + \frac{\partial}{\partial y}(vH) \right] = 0 \end{aligned} \quad (2.24)$$

from Eq.(2.17)

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(uH) + \frac{\partial}{\partial y}(vH) = - \frac{H}{\rho} \left(\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} \right) \quad (2.25)$$

from Eqs.(2.24) and (2.25)

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} - D_x \frac{\partial^2 \rho}{\partial x^2} - D_y \frac{\partial^2 \rho}{\partial y^2} = 0 \quad (2.26)$$

where $D_x = D'_x \rho / \rho_w$ and $D_y = D'_y \rho / \rho_w$

CHAPTER 3

INTRODUCTION TO COMPUTATIONAL HYDRAULICS

3.1 Equation Types and Solution Procedures

The calculation of flow variables (ψ) such as depth, discharge, density or pressure involves the solution of partial differential equations. These equations are grouped into three types called elliptic, parabolic and hyperbolic.

Elliptic are those equations of the Laplacian type:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad (3.1)$$

that describe a system which is in equilibrium. Such an equation describes the steady (constant state) flow of ground-water, hence the absence of any time derivative.

Parabolic equations are those of the 'heat flow' type:

$$\frac{\partial \psi}{\partial t} - \sigma \frac{\partial^2 \psi}{\partial x^2} = 0 \quad (3.2)$$

and describe a system which is in transition from one equilibrium state to another, due to molecular movement or turbulence. This type of equation describes the drawdown process in ground-water flow, the diffusion of density

gradients in an estuary and the degradation of beach deposits.

Hyperbolic equations are of the form:

$$\frac{\partial^2 \psi}{\partial t^2} - c^2 \frac{\partial^2 \psi}{\partial x^2} = 0 \quad (3.3)$$

and describe a system in which a propagation of state takes place. These equations describe the propagation of long water waves in rivers and coastal seas and also water hammer waves in pipes.

Characteristics

If the independent variable 't' in Eqs.(3.2) and (3.3) be represented by the symbol 'y', then Eqs.(3.1, 3.2 and 3.3) can be combined into the generalised partial differential equation of second order:

$$A \frac{\partial^2 \psi}{\partial y^2} + B \frac{\partial^2 \psi}{\partial x \partial y} + C \frac{\partial^2 \psi}{\partial x^2} = D \quad (3.4)$$

where A,B,C and D are functions of x,y, ψ , $\frac{\partial \psi}{\partial x}$ and $\frac{\partial \psi}{\partial y}$.

Since Eq.(3.4) has dependent variables $\frac{\partial \psi}{\partial x}$ and $\frac{\partial \psi}{\partial y}$,

and independent variables x and y, its equations of variation are:

$$\frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial x} \right) dx + \frac{\partial}{\partial y} \left(\frac{\partial \psi}{\partial y} \right) dy = d \left(\frac{\partial \psi}{\partial x} \right) \quad (3.5)$$

$$\frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial y} \right) dx + \frac{\partial}{\partial y} \left(\frac{\partial \psi}{\partial y} \right) dy = d \left(\frac{\partial \psi}{\partial y} \right) \quad (3.6)$$

Eqs. (3.4, 3.5 and 3.6) can be arranged in matrix form as:-

$$\begin{bmatrix} C & B & A \\ dx & dy & 0 \\ 0 & dx & dy \end{bmatrix} \begin{bmatrix} \frac{\partial^2 \psi}{\partial x^2} \\ \frac{\partial^2 \psi}{\partial x \partial y} \\ \frac{\partial^2 \psi}{\partial y^2} \end{bmatrix} = \begin{bmatrix} D \\ d\left(\frac{\partial \psi}{\partial x}\right) \\ d\left(\frac{\partial \psi}{\partial y}\right) \end{bmatrix} \quad (3.7)$$

The derivatives of the equation system (3.7) become indeterminate when the determinant of the left hand side coefficient matrix equals zero.

$$\begin{vmatrix} C & B & A \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix} = 0$$

$$\text{or} \quad C dy^2 - B dx dy + A dx^2 = 0 \quad (3.8)$$

$$A \left(\frac{dx}{dy}\right)^2 - B \frac{dx}{dy} + C = 0$$

$$\frac{dx}{dy} = \left[B \pm (B^2 - 4AC)^{\frac{1}{2}} \right] / (2A) \quad (3.9)$$

Elliptic Equations

From Eq.(3.1), $A = C = 1$ and $B = D = 0$,
therefore from Eq.(3.9) :-

$$\frac{dx}{dy} = \pm (-1)^{\frac{1}{2}} = \pm i \quad (3.10)$$

Eq.(3.10) indicates that the two roots of the elliptic equation are both imaginary, therefore elliptic equations have no real characteristics or propagation paths. Information concerning discontinuities in the derivatives at the boundaries cannot propagate into the system since there are no paths for this to take place along. The solution surface is known instantaneously and is completely defined by the boundary values. Within the system the solution surface is analytic and every part of the solution surface depends on every other part.

Parabolic Equations

From Eq.(3.2), $A = B = 0$, $C = -\sigma$ and $D = -\frac{\partial \psi}{\partial y}$, therefore since $(B^2 - 4AC)^{\frac{1}{2}} = 0$, then

$$\frac{dx}{dy} = \frac{0}{0} \quad (3.11)$$

and the two roots are real and infinite. Since the two roots are equal there is effectively only one root and this is real. Parabolic equations therefore have in general one real characteristic along which discontinuities in derivatives can propagate. If the equations are similar to the heat flow equation in which the characteristics are infinite, then there are no true propagation paths and like elliptic equations the solution surface is analytic.

Hyperbolic Equations

From Eq.(3.3), $A = 1$, $B = D = 0$ and $C = -c^2$, hence

$$\frac{dx}{dy} = \pm c \quad (3.12)$$

The two roots of hyperbolic equations are real and different therefore these equations have two real characteristics or propagation paths. Thus, every point on the solution surface is traversed by two characteristics both carrying information concerning the discontinuities of derivatives at boundaries. Hence, derivatives are not always defined within the system and therefore the solution surface is not analytic.

It has been shown that the three different types of partial differential equations are recognised by the characteristics associated with them. For an elliptic equation there are no real characteristics and therefore the solution can be obtained analytically. Parabolic equations have effectively only one real root and a solution can generally be obtained analytically. For hyperbolic equations however, which have two different real roots, it is not possible to obtain a solution analytically and non-analytical techniques such as finite difference, finite element or the method of characteristics must be used.

The equations used for the computation of water levels and discharges in rivers, estuaries and coastal waters are known as shallow water equations. These equations have two different real roots and are therefore hyperbolic equations. The equation used to compute density changes along an estuary is known as a transport-diffusion equation. This is a mixed equation of the parabolic type which has in general a non-analytical solution. All the equations used for estuarine flow computations are non-analytical

and because of their complicated nature a solution must be used which is readily programmable on a digital computer.

There are only three general methods of solution which are used today in computational hydraulics, namely characteristic, finite difference and finite element methods. Other methods of flood routing, such as the Muskingum method, which do not use the basic partial differential equations of unsteady flow will not be considered in this thesis.

The characteristic method can be applied on the computer in a similar way to the manual graphical method. When this is done a plotter-routine is required to draw the characteristic lines, otherwise a fixed-grid method can be used which has the appearance of a finite difference technique. Since the characteristic method uses a purely mathematical solution of the differential equations, it can be very accurate and is often used as a control solution for other more economical methods.

Finite difference methods by their very nature are more suitable for solving on a digital computer than the characteristics method. There are two broad methods of solution namely, explicit and implicit techniques. The explicit solution progresses from one boundary to the other, grid point by grid point using known values to make a calculation. The implicit solution utilises recurrence relationships between the unknown variables at the time level under calculation, in order to sweep into the calculation known boundary values at both ends of the model. Since the accuracy and stability of all finite difference

methods depends on how the finite differences are weighted between forward and backward time levels, it is meaningless to make comparisons between these two groups without studying individual schemes. Also, the accuracy of the characteristics method depends on the manner in which the characteristics are interpolated between grid points. Since there are many ways of obtaining a characteristics solution on the computer, the accuracy of this method depends entirely on the technique used.

Finite element methods have been rarely used for estuarine or river models and are only in a very early stage of development. From the information published to date on finite element techniques (45) their future appears to lie especially with two-dimensional deep water models having complicated boundaries. The reason for this is that with the finite difference method the approximation of the differential equations is relatively straight forward, although treatment of the boundary conditions is not so simple. The opposite is true with the finite element method, which is therefore well suited to approximate irregular boundaries but which produces difficulties in the approximation of the differential equations.

The method of solution chosen to solve the differential equations depends entirely on the individual. Usually a novice to computational hydraulics begins with the characteristics method, since this is a technique which can be learnt graphically before being applied on a computer. The learner will then advance after studying some stability

theory to finite difference methods, beginning with explicit schemes since these are the easier to programme. After mastering the explicit methods, conversion may be made to the more complicated implicit methods, thereby discovering that not only can they do everything that the characteristic and explicit methods can, but do it more economically. A hydraulic engineer with several years of programming experience may be involved in the development of very large programs suitable for commercial use. At this level of application implicit schemes must be used out of necessity rather than choice since with computer time measured in hours rather than minutes only economical schemes are viable.

3.2 Characteristic Solution for the Wave Equations

If the density is constant then the equations for homogeneous flows in one-dimension can be written from Eqs.(2.1) and (2.2) as:

$$\frac{\partial Q}{\partial x} + b \frac{\partial h}{\partial t} - q_i = 0 \quad (3.13)$$

$$\begin{aligned} \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left(\frac{Q^2}{A} \right) + gA \frac{\partial h}{\partial x} + \frac{gQ|Q|}{C^2 A R} \\ - k_a \rho_a u_a |u_a| b / \rho - v_i \cos \phi q_i = 0 \end{aligned} \quad (3.14)$$

Since $\frac{\partial A}{\partial t} = b \frac{\partial h}{\partial t}$ and $\frac{\partial A}{\partial x} = b \left(\frac{\partial h}{\partial x} + I \right)$

where I is the bed slope, then Eqs.(3.13) and (3.14) can be rewritten in terms of dependent variables Q and A as:

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} - q_i = 0 \quad (3.15)$$

$$\begin{aligned} \frac{\partial Q}{\partial t} + 2u \frac{\partial Q}{\partial x} - \left(u^2 - \frac{gA}{b} \right) \frac{\partial A}{\partial x} - gAI + \frac{gQ|Q|}{C^2_A R} \\ - k_a \rho_a u_a |u_a| b / \rho - v_i \cos \phi q_i = 0 \end{aligned} \quad (3.16)$$

let $e = \left(u^2 - \frac{gA}{b} \right)$ and

$$f = \left(-gAI + \frac{gQ|Q|}{C^2_A R} - k_a \rho_a u_a |u_a| b - v_i \cos \phi q_i \right)$$

then Eq.(3.16) becomes

$$\frac{\partial Q}{\partial t} + 2u \frac{\partial Q}{\partial x} - e \frac{\partial A}{\partial x} + f = 0 \quad (3.17)$$

Eqs.(3.15) and (3.17) plus the two equations of variation:

$$\frac{\partial Q}{\partial x} dx + \frac{\partial Q}{\partial t} dt = dQ \quad (3.18)$$

$$\frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial t} dt = dA \quad (3.19)$$

can be written in the following matrix as:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 2u & 1 & -e & 0 \\ dx & dt & 0 & 0 \\ 0 & 0 & dx & dt \end{bmatrix} \begin{bmatrix} \frac{\partial Q}{\partial x} \\ \frac{\partial Q}{\partial t} \\ \frac{\partial A}{\partial x} \\ \frac{\partial A}{\partial t} \end{bmatrix} = \begin{bmatrix} q_i \\ -f \\ dQ \\ dA \end{bmatrix} \quad (3.20)$$

If this matrix is rearranged and $\dot{x}$ is written for $\frac{dx}{dt}$, $\dot{Q}$ for $\frac{dQ}{dt}$ and $\dot{A}$ for $\frac{dA}{dt}$, it may be expressed as:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 2u & 1 & -e & 0 \\ \dot{x} & 1 & 0 & 0 \\ 0 & 0 & \dot{x} & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial Q}{\partial x} \\ \frac{\partial Q}{\partial t} \\ \frac{\partial A}{\partial x} \\ \frac{\partial A}{\partial t} \end{bmatrix} = \begin{bmatrix} q_i \\ -f \\ \dot{Q} \\ \dot{A} \end{bmatrix} \quad (3.21)$$

The condition that the 4x4 determinant of this matrix be zero defines the characteristics directly.

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 2u & 1 & -e & 0 \\ \dot{x} & 1 & 0 & 0 \\ 0 & 0 & \dot{x} & 1 \end{vmatrix} = 0 \quad (3.22)$$

After simplifying and excluding rows and columns containing a single unity.

$$\begin{vmatrix} 1 & 0 & 1 \\ (2u-\dot{x}) & -e & 0 \\ 0 & \dot{x} & 1 \end{vmatrix} = 0 \quad (3.23)$$

giving $-e + \dot{x}(2u - \dot{x}) = 0$

$$\dot{x} = u \pm (u^2 - e)^{\frac{1}{2}} \quad (3.24)$$

hence since $e = (u^2 - \frac{gA}{b})$

$$\dot{x} = u \pm (\frac{gA}{b})^{\frac{1}{2}} \quad (3.25)$$

It is clear that Eq.(3.25) has two different real roots giving two real characteristics or propagation paths. Thus, equations that describe the propagation of long water waves are of hyperbolic type.

The wave characteristics just derived, define the paths along which energy is propagated. If this energy is conserved along these propagation paths then certain functions should remain constant along these paths. These functions are called Riemann Invariants and may be obtained by taking any 4x4 determinant of the equation system (3.20) except the one used to find the characteristics, and setting it to zero. Hence:

$$\begin{vmatrix} 1 & 0 & -q_i & 1 \\ 2u & 1 & f & 0 \\ \dot{x} & 1 & -\dot{Q} & 0 \\ 0 & 0 & -\dot{A} & 1 \end{vmatrix} = 0 \quad (3.26)$$

Simplifying and excluding rows and columns containing a single unity gives.

$$\begin{vmatrix} 1 & -q_i & 1 \\ (2u-\dot{x}) & (f+\dot{Q}) & 0 \\ 0 & -\dot{A} & 1 \end{vmatrix} = 0 \quad (3.27)$$

$$-(2u - \dot{x})\dot{A} + (f + \dot{Q}) + q_i(2u - \dot{x}) = 0 \quad (3.28)$$

$$- \left[2u - u \mp \left(\frac{gA}{b} \right)^{\frac{1}{2}} \right] dA + dQ = q_i(\dot{x} - 2u)dt - f dt$$

but $dQ = d(Au) = Adu + u dA$ hence

$$du \pm \left(\frac{g}{Ab} \right)^{\frac{1}{2}} dA = \frac{q_i}{A}(\dot{x} - 2u) dt - \frac{f}{A} dt \quad (3.29)$$

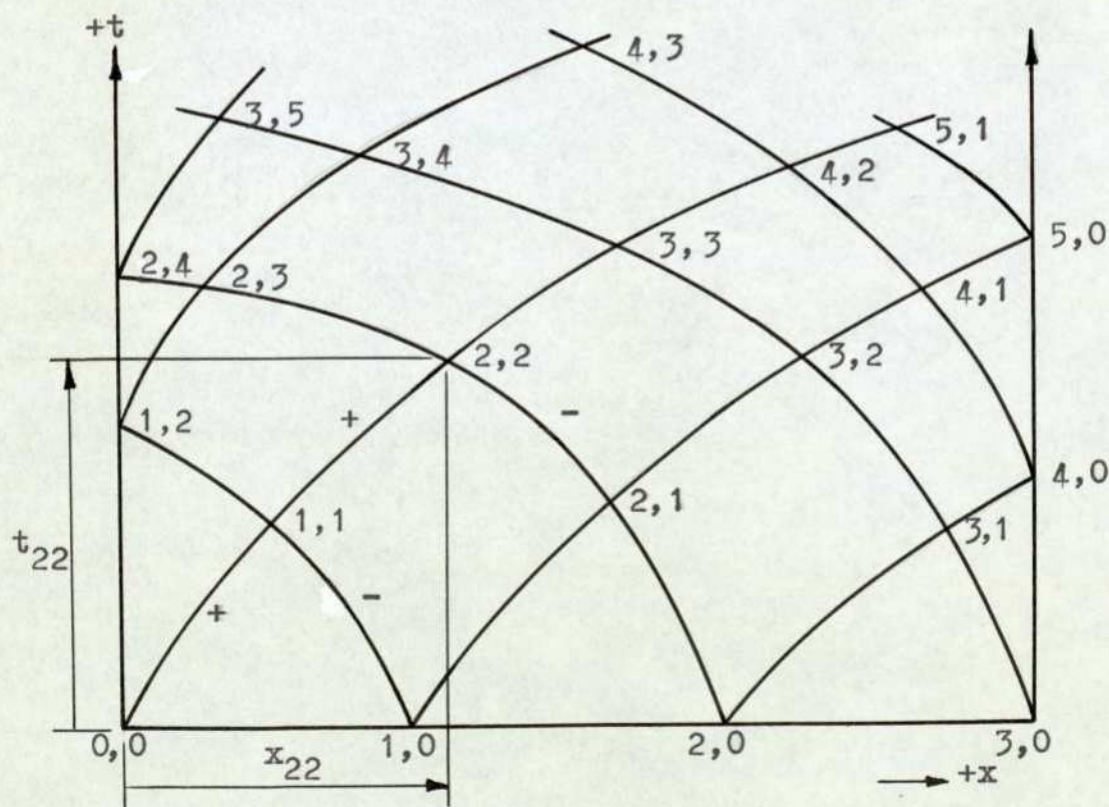
Integrating Eq.(3.29) between states 1 at time t_1 and state 2 at time t_2 gives.

$$\begin{aligned} \left[u \pm 2 \left(\frac{gA}{b} \right)^{\frac{1}{2}} \right]_1^2 &= \left\{ gI - \frac{g u |u|}{C^2 R} + k_a \rho_a u_a |u_a| b / (\rho A) \right. \\ &\left. + \frac{q_i}{A} \left[-u \pm \left(\frac{gA}{b} \right)^{\frac{1}{2}} + v_1 \cos \phi \right] \right\} (t_2 - t_1) \end{aligned} \quad (3.30)$$

Eq.(3.30) together with Eq.(3.25) can be used to calculate a solution for the equation system represented by Eqs.(3.13) and (3.14). Since the direction of the characteristics defines the number of data points required at boundaries, it can be seen from Eq.(3.25) that for subcritical flows there will be one positive and one

If the celerity $(\frac{gA}{b})^{\frac{1}{2}}$ in Eqs.(3.25) and (3.30) is represented by the symbol 'c', and the term on the right of Eq.(3.30) by 'G', then the equations required to find the four unknowns u, c, x and t are.

$$u_2 \pm 2c_2 = u_1 \pm 2c_1 + G_{\pm} \quad (3.32)$$



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Using Eqs.(3.31) and (3.32) alternately an iterative solution is advanced in time as shown in Fig.3.1. Consider a typical point within the computation such as (2,2), then by using a three point calculation the unknowns at point (2,2) can be found as follows. Let the velocity at point (2,2) be denoted by u_{22} , the time by t_{22} , distance by x_{22} and celerity by c_{22} . If the characteristics are assumed to vary linearly between intersecting points then from Eq.(3.31)

$$(x_{22} - x_{11}) = (t_{22} - t_{11})(u_{11} + c_{11} + u_{22} + c_{22})/2 \quad (3.33)$$

$$(x_{22} - x_{21}) = (t_{22} - t_{21})(u_{21} - c_{21} + u_{22} - c_{22})/2 \quad (3.34)$$

giving

$$t_{22} = \left[2(x_{21} - x_{11}) + t_{11}(u_{11} + c_{11} + u_{22} + c_{22}) - t_{21}(u_{21} - c_{21} + u_{22} - c_{22}) \right] / (u_{11} + c_{11} + c_{21} + 2c_{22} - u_{21}) \quad (3.35)$$

The values of x , t , u and c are known at points (1,1) and (2,1) from the previous calculations, but not at point (2,2). As an initial estimate of t_{22} , the slopes of the positive and negative characteristics at point (2,2) are set equal to those at points (1,1) and (2,1) respectively, giving.

$$t_{22} = \left[x_{21} - x_{11} + (u_{11} + c_{11})t_{11} - (u_{21} - c_{21})t_{21} \right] / (u_{11} + c_{11} - u_{21} + c_{21}) \quad (3.36)$$

Approximate values of G can now be calculated for the positive and negative characteristics using u and c values at points (1,1) and (2,1) respectively. Eq.(3.32) is then used to calculate values for u and c at point (2,2), giving.

$$u_{22} + 2c_{22} = u_{11} + 2c_{11} + G_+ \quad (3.37)$$

$$u_{22} - 2c_{22} = u_{21} - 2c_{21} + G_- \quad (3.38)$$

hence

$$u_{22} = (u_{11} + u_{21} + 2c_{11} - 2c_{21} + G_+ + G_-)/2 \quad (3.39)$$

$$c_{22} = (u_{11} - u_{21} + 2c_{11} + 2c_{21} + G_+ - G_-)/4 \quad (3.40)$$

Using these values for u and c at point (2,2), Eq.(3.35) is applied to calculate an improved value for t_{22} . This new time is used together with the average values of u and c along the positive and negative characteristics in order to calculate improved values for G_+ and G_- . Equations (3.34) and (3.35) are applied again to obtain improved values of u and c at point (2,2) and the whole process repeated until the values of u , c and t at point (2,2) converge to within acceptable limits. The position of point (2,2) is now calculated from Eqs.(3.33) and (3.34) to give.

$$\begin{aligned}
 x_{22} = & (x_{11} + x_{21})/2 + \frac{1}{4} [t_{22} (u_{11} + c_{11} + 2u_{22} + u_{21} - c_{21}) \\
 & - t_{11} (u_{11} + c_{11} + u_{22} + c_{22}) \\
 & - t_{21} (u_{21} - c_{21} + u_{22} - c_{22})] \quad (3.41)
 \end{aligned}$$

At a boundary point such as (1,2) either u or c must be given or a relationship between them. Let it be assumed that u_{12} is given, then the procedure to find t_{12} and c_{12} is as follows. From Eq.(3.31)

$$(x_{12} - x_{11}) = (t_{12} - t_{11})(u_{11} - c_{11} + u_{12} - c_{12})/2 \quad (3.42)$$

giving

$$t_{12} = t_{11} + 2(x_{12} - x_{11})/(u_{11} - c_{11} + u_{12} - c_{12}) \quad (3.43)$$

Since c_{12} is not known, assume that $(u_{11} - c_{11}) = (u_{12} - c_{12})$ giving

$$t_{12} = t_{11} + (x_{12} - x_{11})/(u_{11} - c_{11}) \quad (3.44)$$

The value of G_- is calculated using t_{12} and the average value of u along the negative characteristic plus the c value at point (1,1); then from Eq.(3.32)

$$u_{12} - 2c_{12} = u_{11} - 2c_{11} + G_- \quad (3.45)$$

hence

$$c_{12} = (u_{12} - u_{11} + 2c_{11} - G_-)/2 \quad (3.46)$$

Eq.(3.43) is then used to calculate an improved value for t_{12} , and G_- recalculated using this new time plus an average value for c along the negative characteristic.

Eq.(3.46) is again used to calculate an improved value for c_{12} and the process repeated until t_{12} and c_{12} converge to within acceptable limits.

A similar boundary procedure is required for points (4,0), (5,0) etc., plus a plotter routine to draw the characteristic curves shown in Fig.3.1 . Although a characteristic method requiring a plotter routine is generally inconvenient , it can be very useful when discontinuities in the flow are likely to develop, since then a visual record of the formation of the discontinuity is available.

The method of characteristics just outlined would be called a three-point variable mesh method, but many other techniques exist both variable mesh (9,16,88) and fixed mesh (17, 127). It follows from the procedure just described that any correctly formulated method of characteristics, whether variable or fixed mesh must have an iterative solution. Only in this way can average values for the characteristic slopes and G values between two grid points be utilised. There are however much simpler techniques called direct methods which use characteristic slopes and G values calculated from known backward grid points only. These direct methods have been shown by Sivaloganathan (127) to be of comparable accuracy to the more correctly posed iterative solutions and therefore should be used whenever possible.

3.3 Explicit Solutions for the Wave Equations

The partial differential equations describing shallow water flows, can be solved using finite differences. This is achieved by using one term of the Taylor series expansion of the derivatives of the differential equation, thereby introducing a truncation error. Normally a rectangular grid of equal distance and time steps is used and in explicit methods the unknown value is expressed directly in terms of known values. There are two kinds of explicit finite difference schemes, referred to as forward and central types.

Lax Scheme: The forward type is schematized in Fig.3.2 where unknown values at the $(n+1) \Delta t$ time level are calculated explicitly from known values at the $n \Delta t$ time level. This particular scheme is called the 'Lax scheme' (82) and can be shown (see Section 3.7) to have only first order accuracy with a truncation error of $O(\Delta x^2, \Delta t)$. It has the advantage that it is amplitude dissipative and can be used for modelling flows with hydraulic jumps. However, since the dissipation is indiscriminate one can never be sure of the accuracy of the results, therefore such schemes must be used with caution. For the wave equations (3.15) and 3.17) the following linearly stable difference scheme is used.

From Eq.(3.15)

$$\frac{1}{2\Delta x} (Q_{j+1}^n - Q_{j-1}^n) + \frac{1}{\Delta t} [A_j^{n+1} - (A_{j+1}^n + A_{j-1}^n)/2] = q_{1j}^n \quad (3.47)$$

From Eq.(3.17)

$$\begin{aligned} \frac{1}{\Delta t} \left[Q_j^{n+1} - (Q_{j+1}^n + Q_{j-1}^n)/2 \right] + \frac{u_j^n}{\Delta x} (Q_{j+1}^n - Q_{j-1}^n) \\ - \frac{e_j^n}{2 \Delta x} (A_{j+1}^n - A_{j-1}^n) = -f_j^n \end{aligned} \quad (3.48)$$

Since two-point boundary data is required, and only one-point data is allowed by characteristics theory for sub-critical flows, the scheme is physically unrealistic and can become nonlinearly unstable. If however, the scheme is simplified so that $(A_{j+1}^n + A_{j-1}^n)/2$ becomes A_j^n in Eq.(3.47) and $(Q_{j+1}^n + Q_{j-1}^n)/2$ becomes Q_j^n in Eq.(3.48) then the scheme is unconditionally unstable and should not be used.

From Eqs.(3.47) and (3.48) we obtain

$$A_j^{n+1} = (A_{j+1}^n + A_{j-1}^n)/2 - \frac{\Delta t}{2 \Delta x} (Q_{j+1}^n - Q_{j-1}^n) + q_{ij}^n \Delta t \quad (3.49)$$

$$\begin{aligned} Q_j^{n+1} = (Q_{j+1}^n + Q_{j-1}^n)/2 + \frac{e_j^n \Delta t}{2 \Delta x} (A_{j+1}^n - A_{j-1}^n) \\ - \frac{u_j^n \Delta t}{\Delta x} (Q_{j+1}^n - Q_{j-1}^n) - f_j^n \Delta t \end{aligned} \quad (3.50)$$

By using Eqs.(3.49) and (3.50) a solution can be obtained explicitly on the staggered grid of Fig. 3.2 at the new time level $(n+1) \Delta t$. Since both dependent variables are

required at every grid point, average values can be assigned at the intermediate points. The second boundary value at point $(j-1, n+2)$ can be calculated iteratively from point $(j, n+1)$ using the backward characteristic solution given by Eq.(3.32).

Leap-Frog Scheme: The central type explicit scheme shown schematized in Fig. 3.3 is known as the 'leap-frog scheme'. It has second order accuracy with a truncation error of $O(\Delta x^2, \Delta t^2)$, but as with the Lax scheme, since it requires two-point boundary data it can become nonlinearly unstable. For the wave equations (3.15) and (3.17) the following linearly stable difference scheme is used.

From Eq.(3.15)

$$\frac{1}{2\Delta x} (Q_{j+1}^n - Q_{j-1}^n) + \frac{1}{2\Delta t} (A_j^{n+1} - A_j^{n-1}) = q_{ij}^n \quad (3.51)$$

From Eq.(3.17)

$$\begin{aligned} & \frac{1}{2\Delta t} (Q_j^{n+1} - Q_j^{n-1}) + \frac{u_j^n}{\Delta x} (Q_{j+1}^n - Q_{j-1}^n) \\ & - \frac{e_j^n}{2\Delta x} (A_{j+1}^n - A_{j-1}^n) = -f_j^n \end{aligned} \quad (3.52)$$

From Eqs.(3.51) and (3.52) we obtain

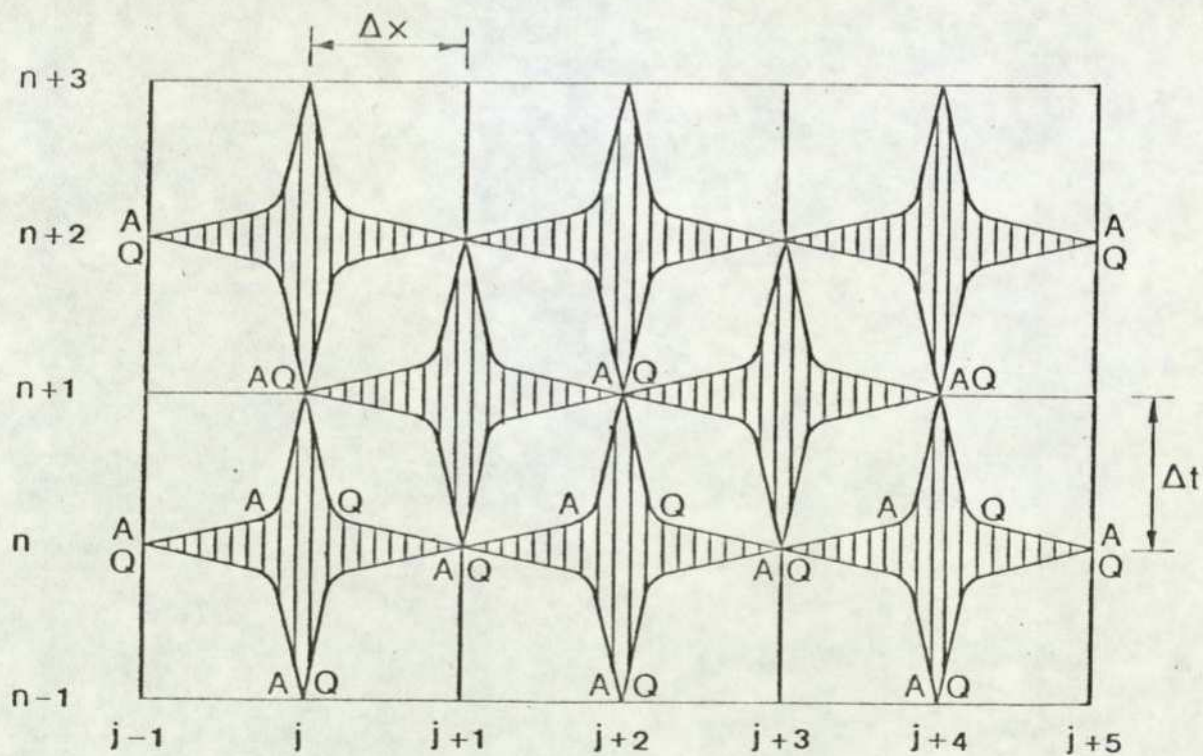
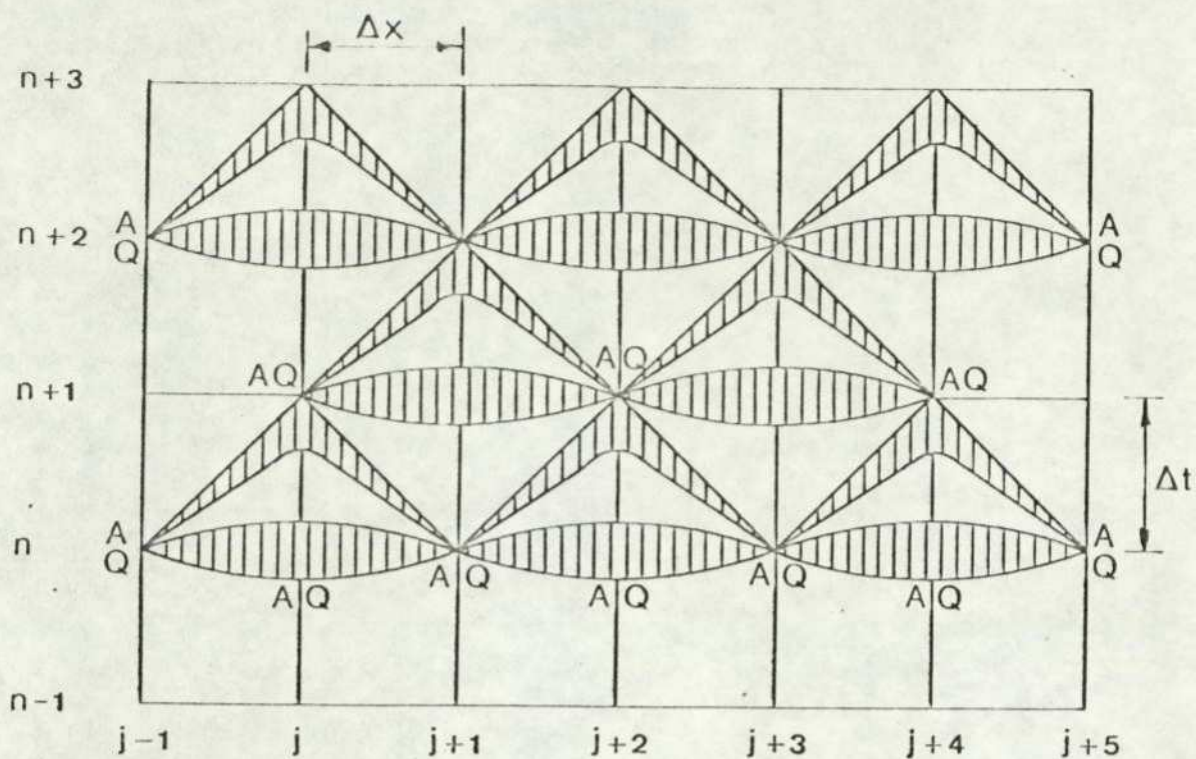
$$A_j^{n+1} = A_j^{n-1} - \frac{\Delta t}{\Delta x} (Q_{j+1}^n - Q_{j-1}^n) + 2 q_{ij}^n \Delta t \quad (3.53)$$

$$\begin{aligned} Q_j^{n+1} &= Q_j^{n-1} - \frac{2u_j^n \Delta t}{\Delta x} (Q_{j+1}^n - Q_{j-1}^n) \\ &+ \frac{e_j^n \Delta t}{\Delta x} (A_{j+1}^n - A_{j-1}^n) - 2 f_j^n \Delta t \end{aligned} \quad (3.54)$$

It can be seen from the above equations that the leap-frog method assumes that two rows of points are known and the solution is then projected to the third row. By using Eqs.(3.53) and (3.54) a solution can be obtained explicitly on the staggered grid shown in Fig. 3.3 . Like the Lax scheme both dependent variables are obtained at every grid point by averaging and the second boundary value by iteration from the characteristic solution.

Two-Step Lax-Wendroff Scheme

The accuracy of the Lax scheme can be improved by interposing leap-frog operators as schematized in Fig. 3.4 to become of second order with a truncation error of $O(\Delta x^2, \Delta t^2)$. This scheme is known as the 'two-step Lax-Wendroff scheme' (83,114) and has similar dissipative properties as the Lax scheme so that it can be used with caution for modelling flows with hydraulic jumps. Abbott, Marshall and Ohno (7) studied the dissipative properties of both the Lax and Lax-Wendroff schemes and came to the conclusion that the more accurate Lax-Wendroff scheme was not entirely superior to the simpler Lax scheme. From the point of view of dissipation since the Lax-Wendroff scheme dissipates more of the short waves than the Lax scheme then in this respect it is the better scheme. However, it was found that for stationary hydraulic jumps, the Lax-Wendroff scheme unlike the Lax scheme had the serious defect of becoming unstable. Boundary conditions are the same as for the leap-frog and Lax schemes and since they are physically unrealistic nonlinear instabilities can develop.



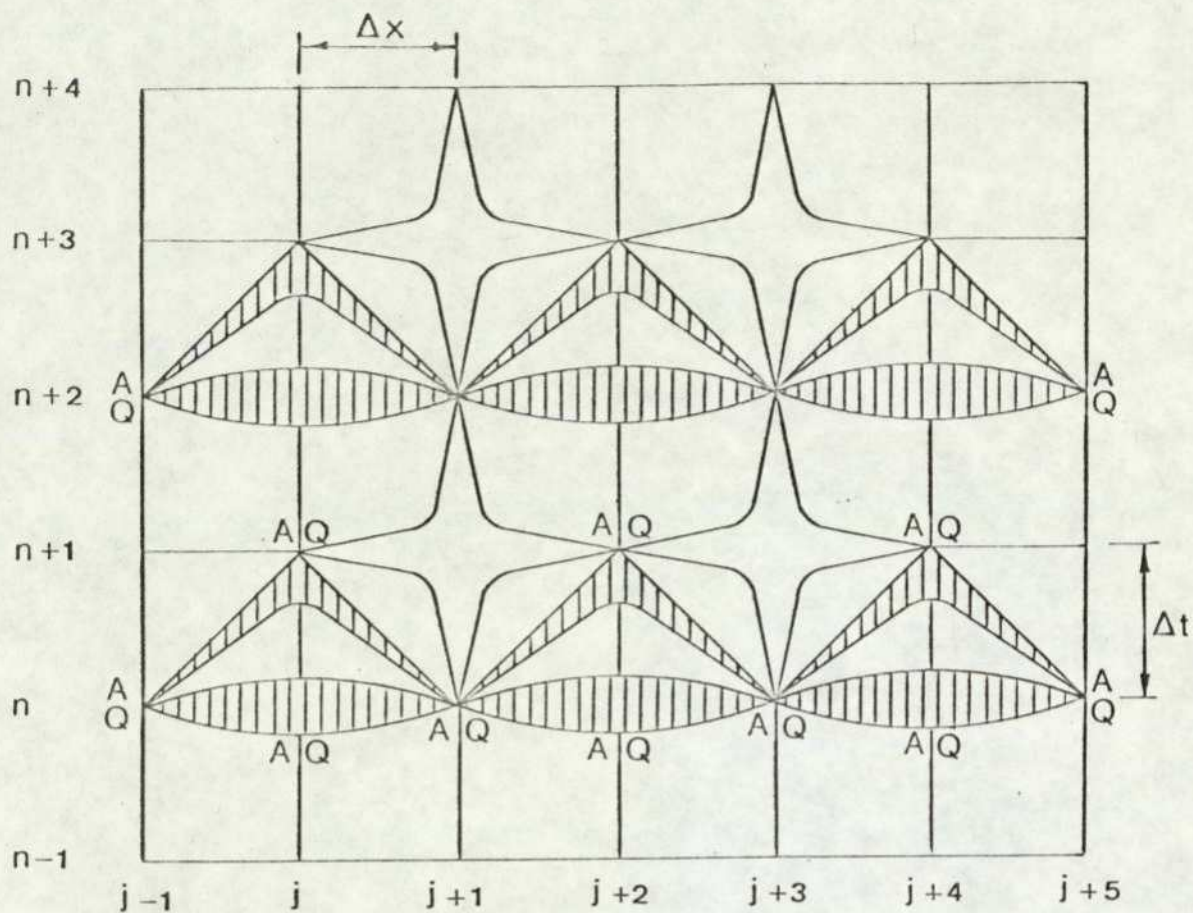


FIG. 3.4 TWO-STEP LAX-WENDROFF SCHEME

3.4 Implicit Solutions for the Wave Equations

There are two kinds of implicit scheme, the backward implicit and the central implicit. For implicit schemes the finite difference structure is different from that of the explicit schemes. The value of the dependent variables to be obtained are calculated from known backward values plus unknown values at the same time level as the solution. Because of this the equations cannot be solved directly but must be solved simultaneously, using either iteration, block matrix inversion or Richtmyer's (113) algorithm for scalar matrix inversion. Since iterative techniques are expensive and block matrix inversion very expensive only Richtmyer's algorithm will be considered. For the full shallow water momentum equation (3.17) only a central implicit scheme can be constructed but if the small non-linear term $u \frac{\partial Q}{\partial x}$ in Eq.(3.17) is neglected then a backward implicit scheme can be used.

Backward Implicit Scheme: This scheme is schematized in Fig.3.5 where unknown values at time level $(n+1) \Delta t$ are calculated simultaneously from known values at time level $n \Delta t$ and known boundary values at time level $(n+1) \Delta t$. This scheme can be shown to have only first order accuracy with a truncation error of $O(\Delta x^2, \Delta t)$. For the wave equations (3.15) and (3.17) with $u \frac{\partial Q}{\partial x}$ set to zero the following linearly unconditionally stable difference scheme is used.

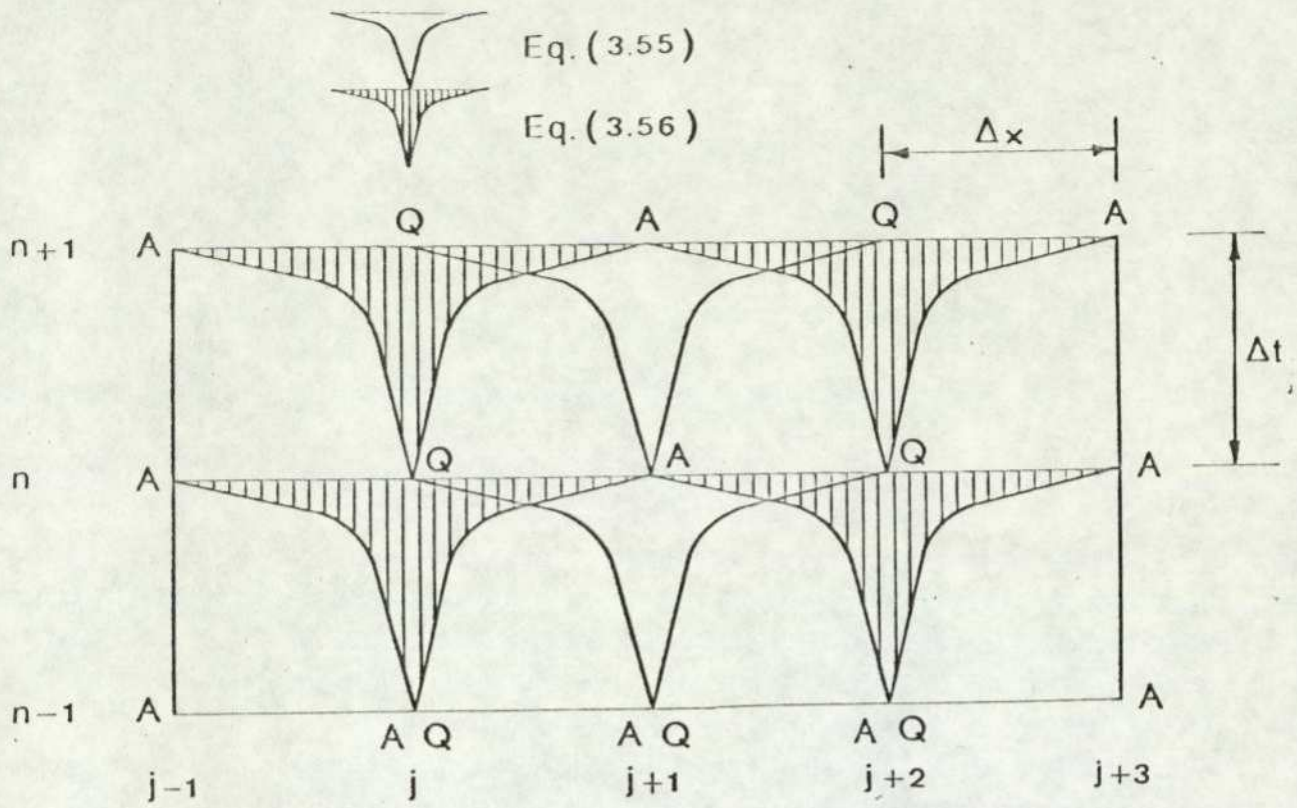


FIG. 3.5 BACKWARD IMPLICIT SCHEME

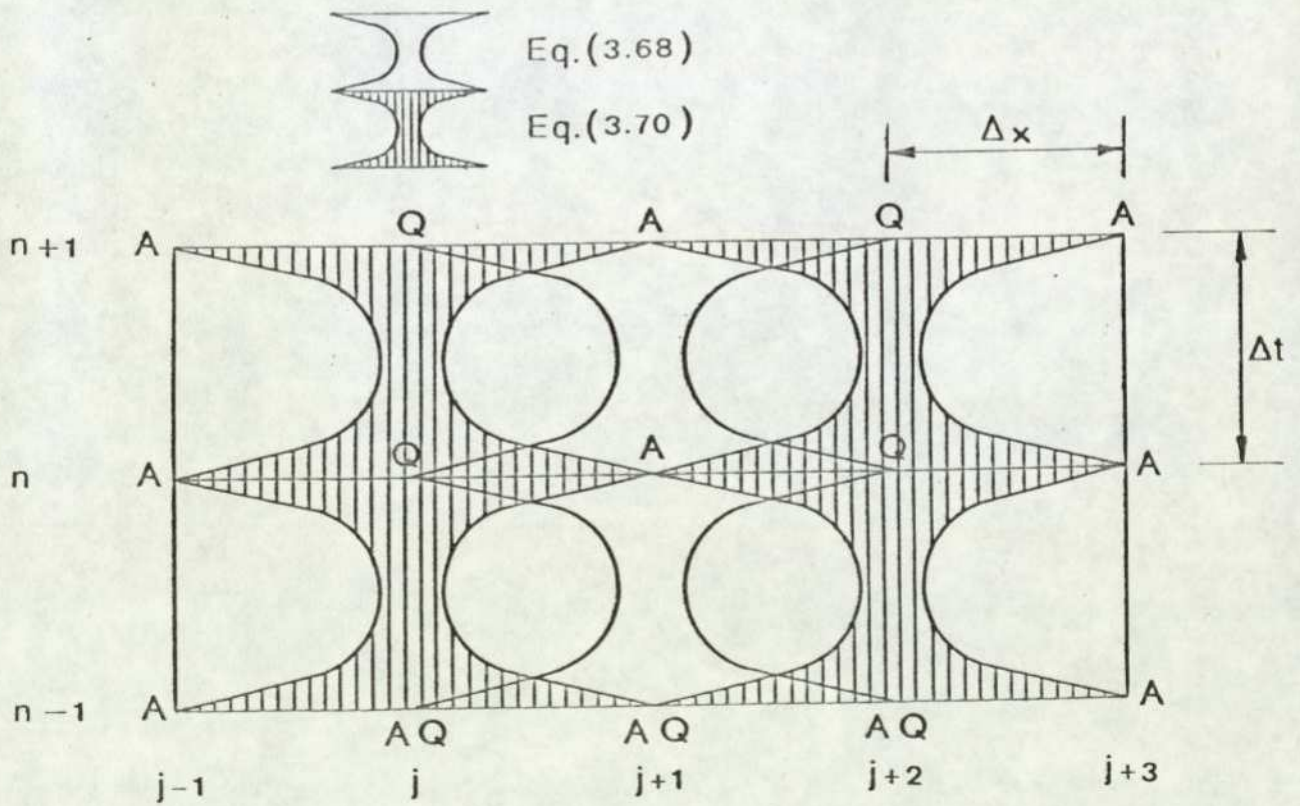


FIG. 3.7 CENTRAL IMPLICIT SCHEME

From Eq.(3.15)

$$\frac{1}{2\Delta x} (Q_{j+1}^{n+1} - Q_{j-1}^{n+1}) + \frac{1}{\Delta t} (A_j^{n+1} - A_j^n) = q_{i,j}^{n+\frac{1}{2}} \quad (3.55)$$

From Eq.(3.17)

$$\frac{1}{\Delta t} (Q_j^{n+1} - Q_j^n) - \frac{e_j^{n+\frac{1}{2}}}{2\Delta x} (A_{j+1}^{n+1} - A_{j-1}^{n+1}) + f_j^{n+\frac{1}{2}} = 0 \quad (3.56)$$

Eqs.(3.55) and (3.56) can be written as

$$\begin{aligned} \left[\frac{1}{2\Delta x} \right] Q_{j+1}^{n+1} + \left[\frac{1}{\Delta t} \right] A_j^{n+1} + \left[-\frac{1}{2\Delta x} \right] Q_{j-1}^{n+1} \\ = \left[\frac{A_j^n}{\Delta t} + q_{i,j}^{n+\frac{1}{2}} \right] \end{aligned} \quad (3.57)$$

$$\begin{aligned} \left[-\frac{e_j^{n+\frac{1}{2}}}{2\Delta x} \right] A_{j+1}^{n+1} + \left[\frac{1}{\Delta t} \right] Q_j^{n+1} + \left[\frac{e_j^{n+\frac{1}{2}}}{2\Delta x} \right] A_{j-1}^{n+1} \\ = \left[\frac{Q_j^n}{\Delta t} - f_j^{n+\frac{1}{2}} \right] \end{aligned} \quad (3.58)$$

Eqs.(3.57) and (3.58) can be written as

$$G_j Q_{j+1}^{n+1} + I_j A_j^{n+1} + K_j Q_{j-1}^{n+1} = L_j \quad (3.59)$$

$$G_j^* A_{j+1}^{n+1} + I_j^* Q_j^{n+1} + K_j^* A_{j-1}^{n+1} = L_j^* \quad (3.60)$$

Eqs. (3.59) and (3.60) can be solved directly using an

iterative solution, but this is uneconomical and it is better to use an algorithm similar to that first used by Richtmyer.

Introducing auxiliary variables a , β , a^* , β^* defined for subcritical flows only by

$$Q_{j-1}^{n+1} = a_j A_j^{n+1} + \beta_j \quad (3.61)$$

$$A_{j-1}^{n+1} = a_j^* Q_j^{n+1} + \beta_j^* \quad (3.62)$$

Then substituting Eq.(3.61) into Eq.(3.59) and Eq.(3.62) into Eq.(3.60) gives respectively

$$A_j^{n+1} = \frac{-G_j}{(I_j + K_j a_j)} Q_{j+1}^{n+1} + \frac{(L_j - K_j \beta_j)}{(I_j + K_j a_j)} \quad (3.63)$$

$$Q_j^{n+1} = \frac{-G_j^*}{(I_j^* + K_j^* a_j^*)} A_{j+1}^{n+1} + \frac{(L_j^* - K_j^* \beta_j^*)}{(I_j^* + K_j^* a_j^*)} \quad (3.64)$$

Comparing Eq.(3.63) with Eq.(3.62) and Eq.(3.64) with Eq.(3.61) gives the recurrence relations

$$a_{j+1}^* = \frac{-G_j}{(I_j + K_j a_j)} ; \quad \beta_{j+1}^* = \frac{(L_j - K_j \beta_j)}{(I_j + K_j a_j)} \quad (3.65)$$

$$a_{j+1} = \frac{-G_j^*}{(I_j^* + K_j^* a_j^*)} ; \quad \beta_{j+1} = \frac{(L_j^* - K_j^* \beta_j^*)}{(I_j^* + K_j^* a_j^*)} \quad (3.66)$$

The calculation is started at the left hand side of the grid shown in Fig.3.6, by setting the known boundary value Q_0^{n+1} at $j=0$, to the auxiliary variable β at $j=1$. Hence

$$\alpha_1 = 0 \quad ; \quad \beta_1 = Q_0^{n+1} \quad (3.67)$$

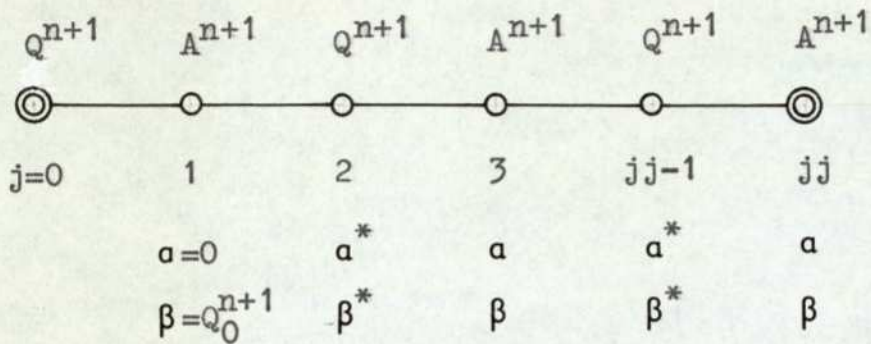


FIG. 3.6 IMPLICIT COMPUTATION GRID

The recurrence coefficients α^* and β^* at $j=2$ are calculated from Eq.(3.65), then α and β at $j=3$ from Eq.(3.66). Eqs.(3.65) and (3.66) are used alternately until the right hand side of the grid is reached. Using the known boundary value A^{n+1} at $j=jj$, Q_{jj-1}^{n+1} is calculated from Eq.(3.61), then A_{jj-2}^{n+1} from Eq.(3.62). Eqs.(3.61) and (3.62) are used alternately until all the dependent variables at time level $(n+1) \Delta t$ have been computed. Since the equation coefficients should be centred at the $(n+\frac{1}{2}) \Delta t$ time level, two complete computations are required, the first using values for the

equation coefficients calculated from the $n\Delta t$ time level. The newly found dependent variables are now assigned to the $n\Delta t$ time level and the process repeated in order to advance the solution in time.

Central Implicit Scheme: This scheme is schematized in Fig.3.7 and because of its central weighting has a second order truncation error of $O(\Delta x^2, \Delta t^2)$. However, because this scheme develops a pseudo-instability, first noticed by Verhoog (6), if the friction term is centred between forward and backward time levels, this must be put together with other 'explicit' terms in the 'implicit' part of the tri-diagonal algorithm. When this is done the truncation error remains $O(\Delta x^2, \Delta t^2)$ but the scheme then becomes unconditionally stable.

From Eq.(3.15)

$$\frac{1}{4\Delta x}(Q_{j+1}^{n+1} - Q_{j-1}^{n+1} + Q_{j+1}^n - Q_{j-1}^n) + \frac{1}{\Delta t}(A_j^{n+1} - A_j^n) = q_i^{n+\frac{1}{2}} \quad (3.68)$$

Substituting Eq.(3.15) into Eq.(3.17) to remove $\frac{\partial Q}{\partial x}$ we obtain

$$\frac{\partial Q}{\partial t} + 2u(-\frac{\partial A}{\partial t} + q_i) - e\frac{\partial A}{\partial x} + f = 0 \quad (3.69)$$

From Eq.(3.69)

$$\frac{1}{\Delta t}(Q_j^{n+1} - Q_j^n) - \frac{u_j^{n+\frac{1}{2}}}{\Delta t}(A_{j+1}^{n+1} - A_{j+1}^n + A_{j-1}^{n+1} - A_{j-1}^n)$$

$$\begin{aligned}
& - \frac{e_j^{n+\frac{1}{2}}}{4 \Delta x} (A_{j+1}^{n+1} - A_{j-1}^{n+1} + A_{j+1}^n - A_{j-1}^n) - \frac{gI}{4} (A_{j+1}^{n+1} + A_{j-1}^{n+1} \\
& + A_{j+1}^n + A_{j-1}^n) + \left(\frac{g}{C^2 A R} \right)_j^{n+\frac{1}{2}} Q_j^{n+1} |Q|_j^n \\
& + (2u_{q_i} - k_a \rho_a u_a |u_a| b - v_i \cos \phi q_i)_j^{n+\frac{1}{2}} = 0 \quad (3.70)
\end{aligned}$$

Eqs.(3.68) and (3.70) can be rearranged as

$$\begin{aligned}
& \left[\frac{1}{4 \Delta x} \right] Q_{j+1}^{n+1} + \left[\frac{1}{\Delta t} \right] A_j^{n+1} + \left[-\frac{1}{4 \Delta x} \right] Q_{j-1}^{n+1} \\
& = \left[\frac{1}{4 \Delta x} (Q_{j-1}^n - Q_{j+1}^n) + \frac{A_j^n}{\Delta t} + q_i^{n+\frac{1}{2}} \right] \quad (3.71)
\end{aligned}$$

$$\begin{aligned}
& \left[-\frac{u_j^{n+\frac{1}{2}}}{\Delta t} - \frac{e_j^{n+\frac{1}{2}}}{4 \Delta x} - \frac{gI}{4} \right] A_{j+1}^{n+1} + \left[\frac{1}{\Delta t} + \left(\frac{g}{C^2 A R} \right)_j^{n+\frac{1}{2}} |Q|_j^n \right] Q_j^{n+1} \\
& + \left[-\frac{u_j^{n+\frac{1}{2}}}{\Delta t} + \frac{e_j^{n+\frac{1}{2}}}{4 \Delta x} - \frac{gI}{4} \right] A_{j-1}^{n+1} = \left\{ \frac{Q_j^n}{\Delta t} + \left[\frac{e_j^{n+\frac{1}{2}}}{4 \Delta x} - \frac{u_j^{n+\frac{1}{2}}}{\Delta t} \right. \right. \\
& + \left. \frac{gI}{4} \right] A_{j+1}^n + \left[-\frac{e_j^{n+\frac{1}{2}}}{4 \Delta x} - \frac{u_j^{n+\frac{1}{2}}}{\Delta t} + \frac{gI}{4} \right] A_{j-1}^n \\
& \left. - (2u_{q_i} - k_a \rho_a u_a |u_a| b - v_i \cos \phi q_i)_j^{n+\frac{1}{2}} \right\} \quad (3.72)
\end{aligned}$$

Eqs.(3.71) and (3.72) can be written as

$$G_j Q_{j+1}^{n+1} + I_j A_j^{n+1} + K_j Q_{j-1}^{n+1} = L_j \quad (3.73)$$

$$G_j^* A_{j+1}^{n+1} + I_j^* Q_j^{n+1} + K_j^* A_{j-1}^{n+1} = L_j^* \quad (3.74)$$

These equations have the same tri-diagonal form as Eqs.(3.59) and (3.60) and can be solved with the same algorithm as was used for the backward implicit scheme.

3.5 Implicit Solution for the Transport-Diffusion Equation

From Eq.(2.10) the partial differential equation for the mass conservation of the solute in a well-mixed estuary was found to be

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} - D \frac{\partial^2 \rho}{\partial x^2} + \frac{q_i}{A} (\rho - \rho_i) = 0 \quad (3.75)$$

where $D = \rho D' / \rho_w$

For Eq.(3.75) the following linearly unconditionally stable difference scheme is used:

$$\begin{aligned} & \frac{1}{4 \Delta t} (\rho_{j+1}^{n+1} - \rho_{j+1}^n) + \frac{1}{2 \Delta t} (\rho_j^{n+1} - \rho_j^n) + \frac{1}{4 \Delta t} (\rho_{j-1}^{n+1} - \rho_{j-1}^n) \\ & + \frac{u}{4 \Delta x} (\rho_{j+1}^{n+1} - \rho_{j-1}^{n+1} + \rho_{j+1}^n - \rho_{j-1}^n) \\ & - \frac{D}{2 \Delta x^2} (\rho_{j+1}^{n+1} - 2 \rho_j^{n+1} + \rho_{j-1}^{n+1} + \rho_{j+1}^n - 2 \rho_j^n + \rho_{j-1}^n) \\ & + \frac{q_i}{2A} (\rho_j^{n+1} + \rho_j^n) - q_i \rho_i / A = 0 \end{aligned} \quad (3.76)$$

Eq.(3.76) can be rearranged as:

$$\begin{aligned} & \left[\frac{1}{4 \Delta t} + \frac{u}{4 \Delta x} - \frac{D}{2 \Delta x^2} \right] \rho_{j+1}^{n+1} + \left[\frac{1}{2 \Delta t} + \frac{D}{\Delta x^2} + \frac{q_i}{2A} \right] \rho_j^{n+1} \\ & + \left[\frac{1}{4 \Delta t} - \frac{u}{4 \Delta x} - \frac{D}{2 \Delta x^2} \right] \rho_{j-1}^{n+1} = \left\{ \left[\frac{1}{4 \Delta t} - \frac{u}{4 \Delta x} + \frac{D}{2 \Delta x^2} \right] \rho_{j+1}^n \right. \end{aligned}$$

$$+ \left[\frac{1}{2 \Delta t} - \frac{D}{\Delta x^2} - \frac{q_i}{2A} \right] \rho_j^n + \left[\frac{1}{4 \Delta t} + \frac{u}{4 \Delta x} + \frac{D}{2 \Delta x^2} \right] \rho_{j-1}^n + q_i \rho_i / A \} \quad (3.77)$$

Eq.(3.77) can be written in the tri-diagonal form

$$A_j^{**} \rho_{j+1}^{n+1} + B_j^{**} \rho_j^{n+1} + C_j^{**} \rho_{j-1}^{n+1} = D_j^{**} \quad (3.78)$$

This equation can be solved using Richtmyer's algorithm by introducing auxiliary variables α and β defined by

$$\rho_{j-1}^{n+1} = \alpha_j \rho_j^{n+1} + \beta_j \quad (3.79)$$

By substituting Eq.(3.79) into Eq.(3.78) gives

$$\rho_j^{n+1} = \frac{-A_j^{**}}{(B_j^{**} + C_j^{**} \alpha_j)} \rho_{j+1}^{n+1} + \frac{(D_j^{**} - C_j^{**} \beta_j)}{(B_j^{**} + C_j^{**} \alpha_j)} \quad (3.80)$$

Comparing Eq.(3.80) with Eq.(3.79) gives the recurrence relations

$$\alpha_{j+1} = \frac{-A_j^{**}}{(B_j^{**} + C_j^{**} \alpha_j)} ; \quad \beta_{j+1} = \frac{(D_j^{**} - C_j^{**} \beta_j)}{(B_j^{**} + C_j^{**} \alpha_j)} \quad (3.81)$$

The calculation is started at the left hand side of the grid shown in Fig. 3.8, by setting the known boundary value ρ^{n+1} at $j=0$ to the auxiliary variable β at $j=1$. Hence

$$\alpha_1 = 0 ; \quad \beta_1 = \rho_0^{n+1} \quad (3.82)$$

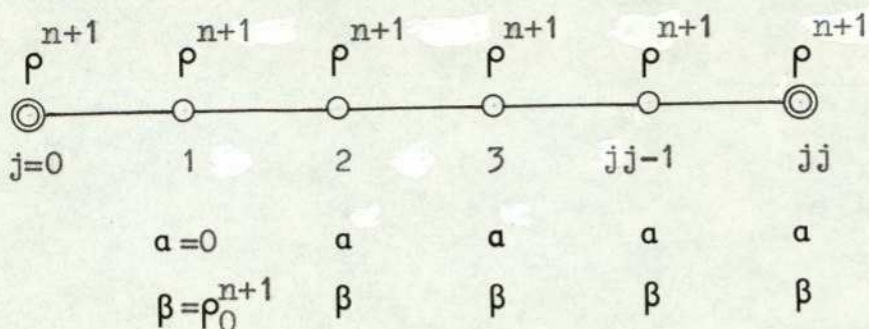


FIG. 3.8 IMPLICIT COMPUTATION GRID (SOLUTE)

The recurrence coefficients α and β are calculated using Eq.(3.81) from $j=2$ till $j=jj$. The dependent variable ρ at time level $(n+1) \Delta t$ is then calculated from $j=jj-1$ till $j=1$ using Eq.(3.79) and the known boundary value ρ_{jj}^{n+1} .

This particular scheme can be shown to be unconditionally consistent with Eq.(3.75) to truncation error $O(\Delta x^2, \Delta t^2)$, while its amplification factors tend to unity as dispersion effects tend to zero. Because Eq.(3.75) is of mixed parabolic type it has two sets of characteristics associated with the transport and diffusive parts of the equation respectively. Since it is impossible to satisfy all the characteristic structures with Eq.(3.79) non-linear instabilities will develop with this scheme unless the distance and time steps are kept within the limits calculated in Section 3.7, and shown below.

$$\frac{2D\Delta t}{\Delta x^2} = 1 \quad \text{and} \quad \left| u \frac{\Delta t}{\Delta x} \right| < 0.25 \quad (3.83)$$

3.6 Implicit Solution for Two-Dimensional Flows

The partial differential equations for two-dimensional well-mixed flows, when the density is a constant, are from Section 2.5 .

Continuity Equation

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(uH) + \frac{\partial}{\partial y}(vH) = 0 \quad (3.84)$$

Dynamical Equation in the x-direction

$$\begin{aligned} & \frac{\partial}{\partial t}(uH) + \frac{\partial}{\partial x}(u^2H) + gH \frac{\partial H}{\partial x} - gHI_x + \frac{\partial}{\partial y}(uvH) \\ & + \frac{gu}{c^2}(u^2+v^2)^{\frac{1}{2}} - \frac{k_a \rho_a u_a}{\rho} (u_a^2+v_a^2)^{\frac{1}{2}} - \theta H v = 0 \end{aligned} \quad (3.85)$$

Dynamical Equation in the y-direction

$$\begin{aligned} & \frac{\partial}{\partial t}(vH) + \frac{\partial}{\partial y}(v^2H) + gH \frac{\partial H}{\partial y} - gHI_y + \frac{\partial}{\partial x}(vuH) \\ & + \frac{gv}{c^2}(u^2+v^2)^{\frac{1}{2}} - \frac{k_a \rho_a v_a}{\rho} (u_a^2+v_a^2)^{\frac{1}{2}} + \theta H u = 0 \end{aligned} \quad (3.86)$$

Eqs.(3.84),(3.85) and (3.86) can be reduced respectively to the following Eulerian forms:

$$\left(\frac{\partial H}{\partial t} + u \frac{\partial H}{\partial x} + v \frac{\partial H}{\partial y} \right) + H \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0 \quad (3.87)$$

$$\left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) + g \frac{\partial H}{\partial x} = a' \quad (3.88)$$

$$\left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) + g \frac{\partial H}{\partial y} = b' \quad (3.89)$$

where

$$a' = \theta v - \frac{gu}{HC^2}(u^2+v^2)^{\frac{1}{2}} + \frac{k_a \rho_a u_a}{\rho_H}(u_a^2+v_a^2)^{\frac{1}{2}} + gI_x \quad (3.90)$$

$$b' = -\theta u - \frac{gv}{HC^2}(u^2+v^2)^{\frac{1}{2}} + \frac{k_a \rho_a v_a}{\rho_H}(u_a^2+v_a^2)^{\frac{1}{2}} + gI_y \quad (3.91)$$

Also $\theta = 2\omega \sin\psi$ and $k_a = 0.0021$

Equations (3.87) to (3.89) can be solved using an implicit tri-diagonal alternating direction algorithm. This utilises the fractional steps technique of Yanenko (152) whereby the continuity equation is split up into two halves. Each half of the continuity equation is combined with its respective dynamical equation enabling unknowns to be calculated in triples alternately in the x and y directions. In this way a solution can be generated using a three level scheme each level being solved with Richtmyer's tri-diagonal algorithm. With known values for u,v and H at the $n \Delta t$ time level the second level computes in the x-direction for u^{n+1} and $H^{n+\frac{1}{2}}$ and the third level in the y-direction for v^{n+1} and H^{n+1} .

Three-Level Implicit Difference Scheme

Eqs.(3.87) and (3.88) are written in the following proposed difference forms for the second level of a three-level scheme. From Eq.(3.87)

$$\begin{aligned} & \frac{1}{\Delta t} (H_{j,k}^{n+\frac{1}{2}} - H_{j,k}^n) + \frac{u}{4\Delta x} (H_{j+1,k}^{n+\frac{1}{2}} - H_{j-1,k}^{n+\frac{1}{2}}) \\ & + \frac{v}{4\Delta y} (H_{j,k+1}^n - H_{j,k-1}^n) + \frac{H}{4\Delta x} (u_{j+1,k}^{n+1} - u_{j-1,k}^{n+1}) \end{aligned}$$

$$+ \frac{H}{4 \Delta y} (v_{j,k+1}^n - v_{j,k-1}^n) = 0 \quad (3.92a)$$

From Eq.(3.88)

$$\begin{aligned} & \frac{1}{\Delta t} (u_{j,k}^{n+1} - u_{j,k}^n) + \frac{u}{2 \Delta x} (u_{j+1,k}^{n+1} - u_{j-1,k}^{n+1}) \\ & + \frac{v}{2 \Delta y} (u_{j,k+1}^{n+\frac{1}{2}} - u_{j,k-1}^{n+\frac{1}{2}}) + \frac{g}{2 \Delta x} (H_{j+1,k}^{n+\frac{1}{2}} - H_{j-1,k}^{n+\frac{1}{2}}) = a' \quad (3.92b) \end{aligned}$$

Eqs.(3.87) and (3.89) provide the following third level of the three-level scheme. From Eq.(3.87)

$$\begin{aligned} & \frac{1}{\Delta t} (H_{j,k}^{n+1} - H_{j,k}^{n+\frac{1}{2}}) + \frac{u}{4 \Delta x} (H_{j+1,k}^{n+\frac{1}{2}} - H_{j-1,k}^{n+\frac{1}{2}}) \\ & + \frac{v}{4 \Delta y} (H_{j,k+1}^{n+1} - H_{j,k-1}^{n+1}) + \frac{H}{4 \Delta x} (u_{j+1,k}^n - u_{j-1,k}^n) \\ & + \frac{H}{4 \Delta y} (v_{j,k+1}^{n+1} - v_{j,k-1}^{n+1}) = 0 \quad (3.93a) \end{aligned}$$

From Eq.(3.89)

$$\begin{aligned} & \frac{1}{\Delta t} (v_{j,k}^{n+1} - v_{j,k}^n) + \frac{u}{2 \Delta x} (v_{j+1,k}^{n+\frac{1}{2}} - v_{j-1,k}^{n+\frac{1}{2}}) \\ & + \frac{v}{2 \Delta y} (v_{j,k+1}^{n+1} - v_{j,k-1}^{n+1}) + \frac{g}{2 \Delta y} (H_{j,k+1}^{n+1} - H_{j,k-1}^{n+1}) = b' \quad (3.93b) \end{aligned}$$

The unsubscripted variables in Eqs.(3.92) and (3.93) must be centred at $(n+\frac{1}{2}) \Delta t$ by iteration, as must the $\partial u / \partial y$ term from Eq.(3.88) and the $\partial v / \partial x$ term from Eq.(3.89).

This iteration is done by making two complete computations. The first computation using values at the $n \Delta t$ time level and the second, the average of the $n \Delta t$ and the new $(n+1) \Delta t$ values. Note that adding the two halves of the mass equation together gives the difference form of Eq.(3.87) centred exactly both in space and time. Also the difference equations for Eqs.(3.88) and (3.89) are centred in space and weighted on the $(n+1) \Delta t$ time level for increased stability.

Eq.(3.92) can be rearranged in the following tri-diagonal forms:

$$\begin{aligned}
 & \left[\frac{H}{4 \Delta x} - \frac{u^2}{4g \Delta x} - \frac{u}{4g \Delta t} - \frac{u(u^2+v^2)^{\frac{1}{2}}}{4HC^2} \right] u_{j+1,k}^{n+1} \\
 & + \left[\frac{1}{\Delta t} + \frac{1}{4 \Delta y} (v_{j,k+1}^n - v_{j,k-1}^n) \right] H_{j,k}^{n+\frac{1}{2}} \\
 & + \left[-\frac{H}{4 \Delta x} + \frac{u^2}{4g \Delta x} - \frac{u}{4g \Delta t} - \frac{u(u^2+v^2)^{\frac{1}{2}}}{4HC^2} \right] u_{j-1,k}^{n+1} \\
 & = \frac{1}{\Delta t} H_{j,k}^n + \frac{v}{4 \Delta y} (H_{j,k-1}^n - H_{j,k+1}^n) - \frac{u}{2g} \left[\frac{1}{\Delta t} u_{j,k}^n \right. \\
 & \left. + \frac{v}{2 \Delta y} (u_{j,k-1}^{n+\frac{1}{2}} - u_{j,k+1}^{n+\frac{1}{2}}) + \theta v + g I_x - \frac{k_a \rho_a u_a (u_a^2+v_a^2)^{\frac{1}{2}}}{\rho H} \right] \quad (3.94) \\
 & \left[\frac{g}{2 \Delta x} - \frac{u^2}{2H \Delta x} - \frac{u}{H \Delta t} \right] H_{j+1,k}^{n+\frac{1}{2}} + \left[\frac{1}{\Delta t} + \frac{g(u^2+v^2)^{\frac{1}{2}}}{HC^2} \right] u_{j,k}^{n+1} \\
 & + \left[-\frac{g}{2 \Delta x} + \frac{u^2}{2H \Delta x} - \frac{u}{H \Delta t} \right] H_{j-1,k}^{n+\frac{1}{2}} = \frac{1}{\Delta t} u_{j,k}^n
 \end{aligned}$$

$$\begin{aligned}
& + \frac{v}{2 \Delta y} (u_{j,k-1}^{n+\frac{1}{2}} - u_{j,k+1}^{n+\frac{1}{2}}) + \theta v + g I_x - \frac{k_a \rho_a u_a (u_a + v_a)^{\frac{1}{2}}}{\rho H} \\
& + \frac{u}{2 \Delta y} (v_{j,k+1}^n - v_{j,k-1}^n) - \frac{2u}{H} \left[\frac{1}{\Delta t} H_{j,k}^n + \right. \\
& \left. \frac{v}{4 \Delta y} (H_{j,k-1}^n - H_{j,k+1}^n) \right] \quad (3.95)
\end{aligned}$$

Which can be written as:

$$A_{j,k} u_{j+1,k}^{n+1} + B_{j,k} H_{j,k}^{n+\frac{1}{2}} + C_{j,k} u_{j-1,k}^{n+1} = D_{j,k} \quad (3.96)$$

$$A_{j,k}^* H_{j+1,k}^{n+\frac{1}{2}} + B_{j,k}^* u_{j,k}^{n+1} + C_{j,k}^* H_{j-1,k}^{n+\frac{1}{2}} = D_{j,k}^* \quad (3.97)$$

Similarly Eq.(3.93) can be rearranged in the following tri-diagonal forms:

$$\begin{aligned}
& \left[\frac{H}{4 \Delta y} - \frac{v^2}{4g \Delta y} - \frac{v}{4g \Delta t} - \frac{v(u^2+v^2)^{\frac{1}{2}}}{4HC^2} \right] v_{j,k+1}^{n+1} + \left[\frac{1}{\Delta t} \right] H_{j,k}^{n+1} \\
& + \left[-\frac{H}{4 \Delta y} + \frac{v^2}{4g \Delta y} - \frac{v}{4g \Delta t} - \frac{v(u^2+v^2)^{\frac{1}{2}}}{4HC^2} \right] v_{j,k-1}^{n+1} \\
& = \left[\frac{1}{2 \Delta t} - \frac{u}{4 \Delta x} \right] H_{j+1,k}^{n+\frac{1}{2}} + \left[\frac{1}{2 \Delta t} + \frac{u}{4 \Delta x} \right] H_{j-1,k}^{n+\frac{1}{2}} \\
& + \frac{H}{4 \Delta x} (u_{j-1,k}^n - u_{j+1,k}^n) - \frac{v}{2g} \left[\frac{u}{2 \Delta x} (v_{j-1,k}^{n+\frac{1}{2}} - v_{j+1,k}^{n+\frac{1}{2}}) \right. \\
& \left. + \frac{1}{\Delta t} v_{j,k}^n - \theta u + g I_y - \frac{k_a \rho_a v_a (u_a^2+v_a^2)^{\frac{1}{2}}}{\rho H} \right] \quad (3.98)
\end{aligned}$$

$$\begin{aligned}
& \left[\frac{g}{2 \Delta y} - \frac{v^2}{2H \Delta y} - \frac{v}{H \Delta t} \right] H_{j,k+1}^{n+1} + \left[\frac{1}{\Delta t} + \frac{g(u^2+v^2)^{\frac{1}{2}}}{HC^2} \right] v_{j,k}^{n+1} \\
& + \left[-\frac{g}{2 \Delta y} + \frac{v^2}{2H \Delta y} - \frac{v}{H \Delta t} \right] H_{j,k-1}^{n+1} = \frac{1}{\Delta t} v_{j,k}^n \\
& + \frac{u}{2 \Delta x} (v_{j-1,k}^{n+\frac{1}{2}} - v_{j+1,k}^{n+\frac{1}{2}}) - \theta u + g I_y - \frac{k_a \rho_a v_a (u_a^2 + v_a^2)^{\frac{1}{2}}}{\rho H} \\
& - \frac{2v}{H} \left[\left(\frac{1}{2 \Delta t} - \frac{u}{4 \Delta x} \right) H_{j+1,k}^{n+\frac{1}{2}} + \left(\frac{1}{2 \Delta t} + \frac{u}{4 \Delta x} \right) H_{j-1,k}^{n+\frac{1}{2}} \right. \\
& \left. + \frac{H}{4 \Delta x} (u_{j-1,k}^n - u_{j+1,k}^n) \right] \quad (3.99)
\end{aligned}$$

Which can be written as:

$$G_{j,k} v_{j,k+1}^{n+1} + I_{j,k} H_{j,k}^{n+1} + K_{j,k} v_{j,k-1}^{n+1} = L_{j,k} \quad (3.100)$$

$$G_{j,k}^* H_{j,k+1}^{n+1} + I_{j,k}^* v_{j,k}^{n+1} + K_{j,k}^* H_{j,k-1}^{n+1} = L_{j,k}^* \quad (3.101)$$

To obtain Eqs.(3.94) and (3.95) it was assumed that:

$$H_{j,k}^{n+\frac{1}{2}} = 0.5(H_{j+1,k}^{n+\frac{1}{2}} + H_{j-1,k}^{n+\frac{1}{2}}) \text{ and}$$

$$u_{j,k}^{n+1} = 0.5(u_{j+1,k}^{n+1} + u_{j-1,k}^{n+1}) \quad (3.102)$$

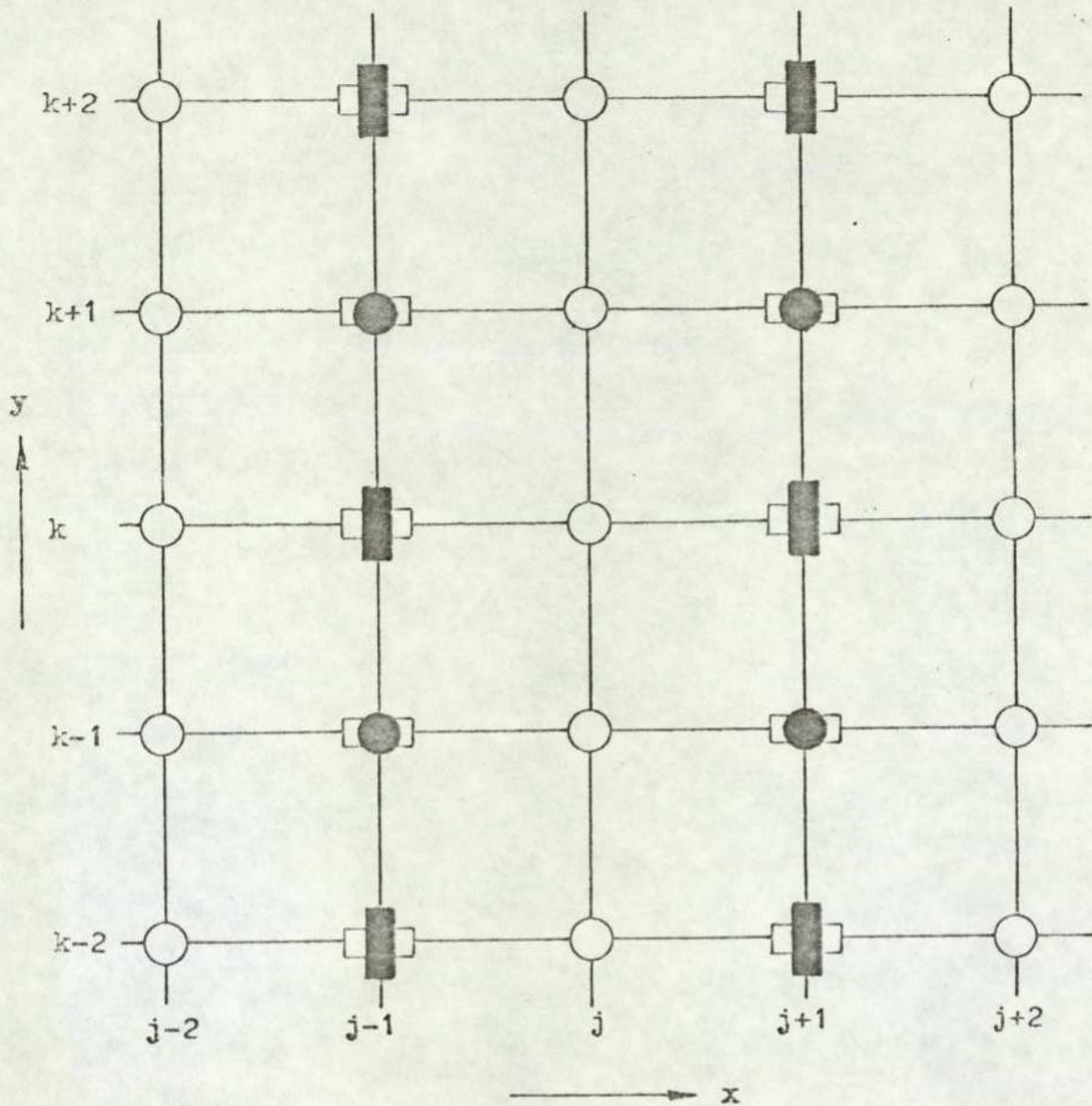
while for Eqs.(3.98) and (3.99) it was assumed that:

$$H_{j,k}^{n+1} = 0.5(H_{j,k+1}^{n+1} + H_{j,k-1}^{n+1}) \text{ and}$$

$$v_{j,k}^{n+1} = 0.5(v_{j,k+1}^{n+1} + v_{j,k-1}^{n+1}) \quad (3.103)$$

These are considered valid since Eqs.(3.94) and (3.95) are used in the x-direction for a calculation based on a system of straight parallel canals, and Eqs.(3.98) and (3.99) for a similar calculation in the y-direction. Eqs.(3.96) and (3.97) are solved in the x-direction for $H^{n+\frac{1}{2}}$ and u^{n+1} using the double tri-diagonal algorithm described in Section 3.4, whilst Eqs.(3.100) and (3.101) are likewise solved for H^{n+1} and v^{n+1} in the y-direction, but on alternate grid lines. The use of alternate grid lines for the second calculation is necessary, as this calculation must use the $H^{n+\frac{1}{2}}$ values found in the previous calculation; otherwise, the two halves of the mass equation will not add together exactly. The two-dimensional computation grid is shown illustrated in Fig. 3.9 with black symbols representing the variables computed in the y-direction.

The resulting truncation error is $O(\Delta x^2, \Delta y^2, \Delta t)$ most of which is phase error. Non-linear instabilities are kept to a minimum by utilising a central implicit scheme but at boundaries non-linear instabilities can be generated when the scheme is run for long periods under gale conditions. However, the author (52) found when running a model of Torquay harbour with gale force winds that the iteration used to centre the equation coefficients acted as an effective dissipative mechanism.



- Depth (H) at $(n+\frac{1}{2}) \Delta t$
- Depth (H) at $(n+1) \Delta t$
- Velocity (v) at $(n+1) \Delta t$
- Velocity (u) at $(n+1) \Delta t$

FIG. 3.9 TWO-DIMENSIONAL COMPUTATION GRID

3.7 Accuracy and Stability of Difference Schemes

The continuity equation is from Eq. (3.15)

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = q_i \quad (3.104)$$

This partial differential equation was approximated using the Lax scheme illustrated in Fig. 3.2 with the finite differences.

$$\left(\frac{\partial Q}{\partial x} \right)_j^n \approx \frac{1}{2\Delta x} (Q_{j+1}^n - Q_{j-1}^n) \quad (3.105)$$

$$\left(\frac{\partial A}{\partial t} \right)_j^n \approx \frac{1}{\Delta t} \left[A_j^{n+1} - (A_{j+1}^n + A_{j-1}^n)/2 \right] \quad (3.106)$$

The dynamical equation is from Eq. (3.17)

$$\frac{\partial Q}{\partial t} + 2u \frac{\partial Q}{\partial x} - e \frac{\partial A}{\partial x} + f = 0 \quad (3.107)$$

This equation was approximated using the Lax scheme with the following finite differences.

$$\left(\frac{\partial Q}{\partial t} \right)_j^n \approx \frac{1}{\Delta t} \left[Q_j^{n+1} - (Q_{j+1}^n + Q_{j-1}^n)/2 \right] \quad (3.108)$$

$$\left(\frac{\partial Q}{\partial x} \right)_j^n \approx \frac{1}{2\Delta x} (Q_{j+1}^n - Q_{j-1}^n) \quad (3.109)$$

$$\left(\frac{\partial A}{\partial x} \right)_j^n \approx \frac{1}{2\Delta x} (A_{j+1}^n - A_{j-1}^n) \quad (3.110)$$

Eq. (3.104) is seen to be approximated by a central difference in the x - direction and a forward difference in the t - direction. Eq. (3.107)

is seen to be approximated by central differences in the x - direction and a forward difference in the t - direction. The resulting finite difference equations are given by Eqs. (3.49) and (3.50) and these provide values for the dependent variables at one time level from their values at a lower time level. When these difference schemes are used, a solution to these difference schemes is obtained and not to the differential equations after which they are modelled. It is hoped in practice that the results obtained using difference schemes are close to the solution of the differential equations. They are however only approximate solutions of the true solution generated by the partial differential equations.

There are many ways of transforming a set of partial differential equations into finite difference equations by using either backward, central or forward operators. However, only a small fraction of the combinations possible will be of any practical value. It is the job of the computational hydraulic engineer to choose the best scheme for a particular purpose and to run this scheme at its optimal accuracy without it going unstable. This optimal accuracy implies that the computer costs should be as low as possible for the accuracy required, since contrary to popular belief mathematical models can frequently be more expensive than physical models and therefore not commercially viable.

3.7.1 Approximations

In order to determine how well the finite difference equations approximate to the differential equations a Taylor series expansion of the derivatives of the differential equations is used. Normally only one term of the Taylor expansion is used for each finite difference

thereby introducing a truncation error. If the finite difference equation is subtracted from the Taylor series expansion then an order of magnitude for the truncation error can be made. The order of magnitude of the truncation error is an indication of the relative accuracy of finite difference methods. The following calculations are made in order to derive the order of magnitude of the leading truncation error for the schemes mentioned in section 3.6.

(a) Truncation Error for the Lax Scheme

The shallow water finite difference wave equations are from Eqs. (3.47) and (3.48).

$$\frac{1}{2\Delta x} (Q_{j+1}^n - Q_{j-1}^n) + \frac{1}{\Delta t} \left[A_j^{n+1} - (A_{j+1}^n + A_{j-1}^n)/2 \right] = q_{1j}^n \quad (3.111)$$

$$- \frac{e_j^n}{2\Delta x} (A_{j+1}^n - A_{j-1}^n) + \frac{1}{\Delta t} \left[Q_j^{n+1} - (Q_{j+1}^n + Q_{j-1}^n)/2 \right] + \frac{u_j^n}{\Delta x} (Q_{j+1}^n - Q_{j-1}^n) + f_j^n = 0 \quad (3.112)$$

Using a Taylor series expansion for the dependent variables in Eq. (3.112) in order to calculate their values at different locations and times we find that:-

$$Q_j^{n+1} = Q_j^n + \left(\frac{\partial Q}{\partial t} \right)_j^n \Delta t + \left(\frac{\partial^2 Q}{\partial t^2} \right)_j^n \frac{\Delta t^2}{2} + \left(\frac{\partial^3 Q}{\partial t^3} \right)_j^n \frac{\Delta t^3}{6} + \dots \quad (3.113a)$$

$$Q_{j+1}^n = Q_j^n + \left(\frac{\partial Q}{\partial x} \right)_j^n \Delta x + \left(\frac{\partial^2 Q}{\partial x^2} \right)_j^n \frac{\Delta x^2}{2} + \left(\frac{\partial^3 Q}{\partial x^3} \right)_j^n \frac{\Delta x^3}{6} + \dots \quad (3.113b)$$

$$Q_{j-1}^n = Q_j^n - \left(\frac{\partial Q}{\partial x}\right)_j^n \Delta x + \left(\frac{\partial^2 Q}{\partial x^2}\right)_j^n \frac{\Delta x^2}{2} - \left(\frac{\partial^3 Q}{\partial x^3}\right)_j^n \frac{\Delta x^3}{6} + \dots (3.113c)$$

$$A_{j+1}^n = A_j^n + \left(\frac{\partial A}{\partial x}\right)_j^n \Delta x + \left(\frac{\partial^2 A}{\partial x^2}\right)_j^n \frac{\Delta x^2}{2} + \left(\frac{\partial^3 A}{\partial x^3}\right)_j^n \frac{\Delta x^3}{6} + \dots (3.113d)$$

$$A_{j-1}^n = A_j^n - \left(\frac{\partial A}{\partial x}\right)_j^n \Delta x + \left(\frac{\partial^2 A}{\partial x^2}\right)_j^n \frac{\Delta x^2}{2} - \left(\frac{\partial^3 A}{\partial x^3}\right)_j^n \frac{\Delta x^3}{6} + \dots (3.113e)$$

Substituting Eqs. (3.113) into Eq. (3.112) we obtain

$$\left[\frac{\partial Q}{\partial t} + 2u \frac{\partial Q}{\partial x} - e \frac{\partial A}{\partial x} + f \right]_j^n + \left(\frac{\partial^2 Q}{\partial t^2}\right)_j^n \frac{\Delta t}{2} + \left(\frac{\partial^3 Q}{\partial t^3}\right)_j^n \frac{\Delta t^2}{6} + \left(2u \frac{\partial^3 Q}{\partial x^3} - e \frac{\partial^3 A}{\partial x^3}\right)_j^n \frac{\Delta x^2}{6} - \left(\frac{\partial^2 Q}{\partial x^2}\right)_j^n \frac{\Delta x^2}{2\Delta t} + \dots = 0 \quad (3.114)$$

Comparing Eq. (3.114) with the differential equation (3.17), the leading truncation error is seen to be $O(\Delta x^2, \Delta t)$ which is also the order of truncation error for Eq. (3.111). $O(\Delta t)$ indicates a first order error in time and means that if Δt is halved the error in the term Δt is halved. $O(\Delta x^2)$ indicates a second order error in distance, and means that if Δx is halved the error in the term Δx^2 is quartered.

(b) Truncation Error for the Leap-Frog Scheme

The shallow water finite difference equations are from Eqs. (3.51) and (3.52)

$$\frac{1}{2\Delta x} (Q_{j+1}^n - Q_{j-1}^n) + \frac{1}{2\Delta t} (A_j^{n+1} - A_j^{n-1}) = q_{1j}^n \quad (3.115)$$

$$\begin{aligned}
& - \frac{e_j^n}{2\Delta x} (A_{j+1}^n - A_{j-1}^n) + \frac{1}{2\Delta t} (Q_j^{n+1} - Q_j^{n-1}) \\
& + \frac{u_j^n}{\Delta x} (Q_{j+1}^n - Q_{j-1}^n) + f_j^n = 0
\end{aligned} \tag{3.116}$$

From a Taylor series expansion

$$Q_j^{n-1} = Q_j^n - \left(\frac{\partial Q}{\partial t}\right)_j^n \Delta t + \left(\frac{\partial^2 Q}{\partial t^2}\right)_j^n \frac{\Delta t^2}{2} - \left(\frac{\partial^3 Q}{\partial t^3}\right)_j^n \frac{\Delta t^3}{6} + \dots \tag{3.117}$$

Substituting Eqs. (3.113) and (3.117) into Eq. (3.116) we obtain

$$\begin{aligned}
& \left[\frac{\partial Q}{\partial t} + 2u \frac{\partial Q}{\partial x} - e \frac{\partial A}{\partial x} + f \right]_j^n + \left(2u \frac{\partial^3 Q}{\partial x^3} - e \frac{\partial^3 A}{\partial x^3} \right)_j^n \frac{\Delta x^2}{6} \\
& + \left(\frac{\partial^3 Q}{\partial t^3} \right)_j^n \frac{\Delta t^2}{6} + \dots = 0
\end{aligned} \tag{3.118}$$

Comparing Eq. (3.118) with Eq. (3.17) the leading truncation error is seen to be $O(\Delta x^2, \Delta t^2)$ which is also the truncation error for Eq. (3.115).

(c) Truncation Error for the Central Implicit Scheme

The shallow water finite difference wave equations are from Eqs. (3.68) and (3.70)

$$\begin{aligned}
& \frac{1}{4\Delta x} (Q_{j+1}^{n+1} - Q_{j-1}^{n+1} + Q_{j+1}^n - Q_{j-1}^n) + \frac{1}{\Delta t} (A_j^{n+1} - A_j^n) = q_{1j}^{n+\frac{1}{2}} \tag{3.119} \\
& - \frac{e_j^{n+\frac{1}{2}}}{4\Delta x} (A_{j+1}^{n+1} - A_{j-1}^{n+1} + A_{j+1}^n - A_{j-1}^n) + \frac{1}{\Delta t} (Q_j^{n+1} - Q_j^n)
\end{aligned}$$

$$\begin{aligned}
& - \frac{u_j^{n+\frac{1}{2}}}{\Delta t} (A_{j+1}^{n+1} - A_{j+1}^n + A_{j-1}^{n+1} - A_{j-1}^n) - g \frac{I}{4} (A_{j+1}^{n+1} + A_{j-1}^{n+1} + A_{j+1}^n + A_{j-1}^n) \\
& + \left(\frac{g}{C^2 AR} \right)_j^{n+\frac{1}{2}} Q_j^{n+1} |Q|_j^n + (2uq_i - k_a \rho_a u_a |u_a| b - v_i \cos \emptyset q_i)_j^{n+\frac{1}{2}} = 0 \quad (3.120)
\end{aligned}$$

The equation coefficients are centred at the $(n+\frac{1}{2})\Delta t$ time level by making two complete computations. In the first computation they are taken at the $n\Delta t$ time level and in the second computation the mean of the $n\Delta t$ values and the newly found $(n+1)\Delta t$ values.

Let $(2uq_i - k_a \rho_a u_a |u_a| b - v_i \cos \emptyset q_i) = s''$ in Eq. (3.120), then $s_j^{n+\frac{1}{2}} = (s_j^{n+1} + s_j^n)/2$.

From a Taylor series expansion

$$\begin{aligned}
A_{j+1}^{n+1} &= A_j^n + \left(\frac{\partial A}{\partial x} \right)_j^n \Delta x + \left(\frac{\partial A}{\partial t} \right)_j^n \Delta t + \left(\frac{\partial^2 A}{\partial x^2} \right)_j^n \frac{\Delta x^2}{2} \\
&+ \left(\frac{\partial^2 A}{\partial x \partial t} \right)_j^n \Delta x \Delta t + \left(\frac{\partial^2 A}{\partial t^2} \right)_j^n \frac{\Delta t^2}{2} + \left(\frac{\partial^3 A}{\partial x^3} \right)_j^n \frac{\Delta x^3}{6} \\
&+ \left(\frac{\partial^3 A}{\partial x^2 \partial t} \right)_j^n \frac{\Delta x^2 \Delta t}{2} + \left(\frac{\partial^3 A}{\partial x \partial t^2} \right)_j^n \frac{\Delta x \Delta t^2}{2} + \left(\frac{\partial^3 A}{\partial t^3} \right)_j^n \frac{\Delta t^3}{6} + \dots \quad (3.121a)
\end{aligned}$$

$$\begin{aligned}
A_{j-1}^{n+1} &= A_j^n - \left(\frac{\partial A}{\partial x} \right)_j^n \Delta x + \left(\frac{\partial A}{\partial t} \right)_j^n \Delta t + \left(\frac{\partial^2 A}{\partial x^2} \right)_j^n \frac{\Delta x^2}{2} \\
&- \left(\frac{\partial^2 A}{\partial x \partial t} \right)_j^n \Delta x \Delta t + \left(\frac{\partial^2 A}{\partial t^2} \right)_j^n \frac{\Delta t^2}{2} - \left(\frac{\partial^3 A}{\partial x^3} \right)_j^n \frac{\Delta x^3}{6} + \left(\frac{\partial^3 A}{\partial x^2 \partial t} \right)_j^n \frac{\Delta x^2 \Delta t}{2} \\
&- \left(\frac{\partial^3 A}{\partial x \partial t^2} \right)_j^n \frac{\Delta x \Delta t^2}{2} + \left(\frac{\partial^3 A}{\partial t^3} \right)_j^n \frac{\Delta t^3}{6} + \dots \quad (3.121b)
\end{aligned}$$

$$(s_j^{n+1} + s_j^n)/2 = s_j^n + \left(\frac{\partial s''}{\partial t}\right)_j^n \frac{\Delta t}{2} + \left(\frac{\partial^2 s''}{\partial t^2}\right)_j^n \frac{\Delta t^2}{4} + \left(\frac{\partial^3 s''}{\partial t^3}\right)_j^n \frac{\Delta t^3}{12} + \dots$$

(3.121c)

Substituting Eqs. (3.113) and (3.121) into Eq. (3.120) we obtain

$$\begin{aligned} & \left[\frac{\partial Q}{\partial t} - 2u \frac{\partial A}{\partial t} - e \frac{\partial A}{\partial x} - gIA + \frac{g}{C_{AR}^2} Q |Q| + s'' \right]_j^n \\ & + \frac{\partial}{\partial t} \left[\frac{\partial Q}{\partial t} - 2u \frac{\partial A}{\partial t} - e \frac{\partial A}{\partial x} - gIA + \frac{g}{C_{AR}^2} Q |Q| + s'' \right]_j^n \frac{\Delta t}{2} \\ & + \frac{\partial^2}{\partial t^2} \left[\frac{\partial Q}{\partial t} - 2u \frac{\partial A}{\partial t} - e \frac{\partial A}{\partial x} - gIA + \frac{g}{C_{AR}^2} Q |Q| + s'' \right]_j^n \frac{\Delta t^2}{6} \\ & + \frac{\partial^2}{\partial x^2} \left[-\frac{e}{3} \frac{\partial A}{\partial x} - 2u \frac{\partial A}{\partial t} - gIA \right]_j^n \frac{\Delta x^2}{2} \\ & - \frac{1}{12} \left[e \frac{\partial^3 A}{\partial x \partial t^2} + gI \frac{\partial^2 A}{\partial t^2} - \frac{4g}{C_{AR}^2} |Q| \frac{\partial^2 Q}{\partial t^2} - \frac{\partial^2 s''}{\partial t^2} \right]_j^n \Delta t^2 \\ & - \left[\frac{gI}{4} \frac{\partial^3 A}{\partial x^2 \partial t} \right]_j^n \Delta x^2 \Delta t + \frac{1}{12} \left[\frac{2g}{C_{AR}^2} |Q| \frac{\partial^3 Q}{\partial t^3} - gI \frac{\partial^3 A}{\partial t^3} + \frac{\partial^3 s''}{\partial t^3} \right]_j^n \Delta t^3 = 0 \end{aligned}$$

(3.122)

The differential equation (3.69) can be written as:-

$$\frac{\partial Q}{\partial t} - 2u \frac{\partial A}{\partial t} - e \frac{\partial A}{\partial x} - gIA + \frac{g}{C_{AR}^2} Q |Q| + s'' = 0 \quad (3.123)$$

Comparing Eq. (3.122) with Eq. (3.123) the leading truncation error is seen to be $O(\Delta t^2)$. It follows that for Eq.(3.119) the error is $O(\Delta x^2)$ giving a truncation error for the scheme of $O(\Delta x^2, \Delta t^2)$.

(d) Truncation Error for the Implicit Transport - Diffusion Scheme

The finite difference equation is from Eq. (3.76)

$$\begin{aligned}
 & \frac{1}{4\Delta t} (\rho_{j+1}^{n+1} - \rho_{j+1}^n) + \frac{1}{2\Delta t} (\rho_j^{n+1} - \rho_j^n) + \frac{1}{4\Delta t} (\rho_{j-1}^{n+1} - \rho_{j-1}^n) \\
 & + \frac{u}{4\Delta x} (\rho_{j+1}^{n+1} - \rho_{j-1}^{n+1} + \rho_{j+1}^n - \rho_{j-1}^n) \\
 & - \frac{D}{2\Delta x^2} (\rho_{j+1}^{n+1} - 2\rho_j^{n+1} + \rho_{j-1}^{n+1} + \rho_{j+1}^n - 2\rho_j^n + \rho_{j-1}^n) \\
 & + \frac{q_i}{2A} (\rho_j^{n+1} + \rho_j^n) - \left[\left(\frac{q_i \rho_i}{2A} \right)_j^{n+1} + \left(\frac{q_i \rho_i}{2A} \right)_j^n \right] = 0 \quad (3.124)
 \end{aligned}$$

Using a Taylor series expansion for the density levels in Eq. (3.124) given by Eqs. (3.113) and (3.121) with density as dependent variable leads to:-

$$\begin{aligned}
 & \left[\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} - D \frac{\partial^2 \rho}{\partial x^2} + \frac{q_i}{A} (\rho - \rho_i) \right]_j^n \\
 & + \frac{\partial}{\partial t} \left[\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} - D \frac{\partial^2 \rho}{\partial x^2} + \frac{q_i}{A} (\rho - \rho_i) \right]_j^n \frac{\Delta t}{2} \\
 & + \frac{\partial^2}{\partial t^2} \left[\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} - D \frac{\partial^2 \rho}{\partial x^2} + \frac{q_i}{A} (\rho - \rho_i) \right]_j^n \frac{\Delta t^2}{6} \\
 & + \left[3 \frac{\partial^3 \rho}{\partial x^2 \partial t} + 2u \frac{\partial^3 \rho}{\partial x^3} \right]_j^n \frac{\Delta x^2}{12} \\
 & + \left[u \frac{\partial^3 \rho}{\partial x \partial t^2} - D \frac{\partial^4 \rho}{\partial x^2 \partial t^2} + \frac{q_i}{A} (\rho - \rho_i) \right]_j^n \frac{\Delta t^2}{12} = 0 \quad (3.125)
 \end{aligned}$$

Comparing Eq. (3.125) with the differential equation Eq. (3.75),

the leading truncation error is seen to be $O(\Delta x^2, \Delta t^2)$.

3.7.2 Consistency and Convergence

In the previous sub-section the amount by which the difference equation departed from the differential equation was called the truncation error. Different difference schemes were shown to produce different truncation errors and in general it was shown that centred schemes give a second order truncation error whilst forward and backward schemes give a first order error.

If this truncation error tends to zero as $\Delta x, \Delta t \rightarrow 0$ we say that the difference equation is consistent with the differential equation. If the truncation error tends to zero regardless of how $\Delta x, \Delta t \rightarrow 0$ we further say that the difference equation is unconditionally consistent with the differential equation. If the truncation error only tends to zero if Δx and Δt are taken to zero in a particular way, the difference equation is said to be conditionally consistent with the differential equation. On the other hand if the error does not vanish as $\Delta x, \Delta t \rightarrow 0$ then the difference solution will not be the solution of the differential equation but of another and then the difference equation is said to be inconsistent with the differential equation.

If the shallow water wave equations have to be used for calculating across hydraulic jumps, surges, fronts or any other localised region of rapid energy losses then it is necessary to use either conditionally consistent or even inconsistent finite difference schemes. The reason for this is that when the flow is no longer represented by its differential equations, some terms in the truncation error rapidly increase in value and act as pseudo-viscosities. This effectively 'burns up' enough energy to satisfy the mass and momentum equations enabling a continuous calculation to be made for the long water waves without it going unstable.

The truncation error for the shallow water wave equations resulting from using the Lax scheme is shown in Eq. (3.114). The last term in the equation is a function of $\Delta x^2/\Delta t$, hence it tends to zero relative to how Δx and Δt tend to zero. It is this term which provides the Lax scheme with its dissipative properties and resulting weak solutions to the differential equation. This Lax scheme is therefore conditionally consistent with its differential equation and can be shown to be amplitude dissipative. The Lax scheme together with the Lax - Wendroff scheme both generate their pseudo-viscosities from the truncation error of their difference equations and are therefore ideal schemes for computing across discontinuities in the flow.

Both the explicit and implicit centred difference schemes can be seen from their truncation errors given by Eqs. (3.118), (3.122) and (3.125) to be all of second order accuracy and also unconditionally consistent with their differential equations. Since it is desirable for models of estuarine flows to conserve mass and momentum, unconditionally consistent finite difference schemes should be used and in order to maintain second order accuracy these should be centred schemes.

If the truncation error tended to zero as $\Delta x, \Delta t \rightarrow 0$ then the difference equation was said to be consistent with the differential equation. No mention was made as to whether or not the solution of the difference equation converged to the true solution as $\Delta x, \Delta t \rightarrow 0$, only about the consistency of the equations. Clearly we are interested in the solution to the difference equation as $\Delta x, \Delta t \rightarrow 0$ since although a difference scheme may be consistent with its differential equation, the solution

to the former need not necessarily converge to the latter. It often happens that certain combinations of Δx and Δt give numerical solutions that are unstable or even converge to the wrong analytical solution. Clearly the convergence properties of any new numerical scheme need to be tested analytically to discover if there are restrictions on the values of Δx and Δt . In practice such a calculation is laborious and use is made instead of the Equivalent theorem. This states that if the difference scheme is stable and consistent with its differential equation then its solution will converge to that of the differential equation.

3.7.3 Linear Stability

It has been shown that as $\Delta x, \Delta t \rightarrow 0$ the truncation error tends to zero but only if the scheme is stable will the solution of the difference equations tend to the solution of the differential equation. When the computation is unstable, these solutions differ more and more as $\Delta x, \Delta t \rightarrow 0$, even though the truncation error tends to zero. It is a fact that the concept of stability makes no reference to the differential equation that is to be solved, but is a property solely of the sequence of difference equations.

If a difference scheme is stable it should be possible to define a fixed bound such that with a certain fixed relationship between Δx and Δt , the solution will always be contained within this bound no matter how many computation steps are executed. Expressed differently, because a computer cannot calculate to an infinite number of decimal places, each calculation step produces a round-off error. If these round-off errors do not cancel each other out and remain within a certain fixed bound then they will grow exponentially and swamp the calculation producing the phenomenon of instability.

In order to investigate analytically the stability of finite difference schemes, either of the following two methods can be used. The first method expresses the equations of motion in matrix form and examines the eigen values of the associated amplification matrix; in the second method the eigen values are determined by using a Fourier series. The first method is more rigorous since it includes the boundary conditions but the second method is simpler and especially useful for non-linear stability analysis where the attenuation of high wave number disturbances is of importance. The following calculations using both matrix and Fourier methods examine the linear stability properties of the finite difference schemes mentioned in section 3.6.

(a) Linear Stability of the Lax Scheme

The linearised wave equations become from Eqs. (3.104) and (3.107), when all minor terms are omitted.

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0 \quad (3.126)$$

$$\frac{\partial Q}{\partial t} + gH \frac{\partial A}{\partial x} = 0 \quad (3.127)$$

where H is the average depth.

From Eqs. (3.126) and (3.127)

$$\frac{\partial^2 A}{\partial t^2} - gH \frac{\partial^2 A}{\partial x^2} = 0 \quad (3.128)$$

$$\frac{\partial^2 Q}{\partial t^2} - gH \frac{\partial^2 Q}{\partial x^2} = 0 \quad (3.129)$$

Using Eq. (3.129) the Lax scheme is equivalent to:-

$$\frac{(Q_{j+1}^n - 2Q_j^{n+1} + Q_{j-1}^n)}{\Delta t^2} - gH \frac{(Q_{j+1}^n - 2Q_j^n + Q_{j-1}^n)}{\Delta x^2} = 0 \quad (3.130)$$

$$\text{or } Q_j^{n+1} = \left[\frac{1}{2} - \frac{gH}{2} \frac{\Delta t^2}{\Delta x^2} \right] Q_{j+1}^n + \left[\frac{gH}{2} \frac{\Delta t^2}{\Delta x^2} \right] Q_j^n + \left[\frac{1}{2} - \frac{gH}{2} \frac{\Delta t^2}{\Delta x^2} \right] Q_{j-1}^n$$

$$\text{let } gH \frac{\Delta t^2}{\Delta x^2} = r$$

$$Q_j^{n+1} = \left(\frac{1}{2} - \frac{1}{2}r \right) Q_{j+1}^n + r Q_j^n + \left(\frac{1}{2} - \frac{1}{2}r \right) Q_{j-1}^n \quad (3.131)$$

If we consider just three grid points in the x- direction that is $j = 1, 2$ and 3 , then if we can prove that the above scheme is stable on three grid points it follows that it must be stable on any number of grid points.

Expressing Eq. (3.131) in matrix form, together with its boundary conditions gives

$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}^{n+1} = \begin{bmatrix} 1 & 0 & 0 \\ (\frac{1}{2}-\frac{1}{2}r) & r & (\frac{1}{2}-\frac{1}{2}r) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}^n \quad (3.132)$$

If the eigen values are represented by the symbol λ then

$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}^{n+1} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}^n \quad (3.133)$$

Subtracting Eq. (3.133) from Eq. (3.132) gives

$$\begin{bmatrix} (1 - \lambda) & 0 & 0 \\ (\frac{1}{2} - \frac{1}{2}r) & (r - \lambda) & (\frac{1}{2} - \frac{1}{2}r) \\ 0 & 0 & (1 - \lambda) \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}^n = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (3.134)$$

we get $(1 - \lambda)^2 (r - \lambda) = 0$

hence $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = r$

If $r \leq 1$ then none of the eigen values is greater than unity therefore none of the eigen vectors are amplified. This is a necessary but not a sufficient condition for linear stability. A sufficient condition for linear stability is when the sum of the absolute values of the elements in the rows of the matrix in Eq. (3.132) are ≤ 1 .

$$\text{hence } \left| \frac{1}{2} - \frac{1}{2}r \right| + |r| + \left| \frac{1}{2} - \frac{1}{2}r \right| \leq 1 \quad (3.135)$$

Eq. (3.135) gives the result that $r \leq 1$. Hence the Lax scheme is conditionally stable with a necessary and sufficient condition for linear stability of

$$\frac{\Delta t}{\Delta x} (gH)^{\frac{1}{2}} \leq 1 \quad (3.136)$$

Since dependent variables in the wave equation can be replaced by periodicity conditions, they can be written as a Fourier series, i.e.

$$Q_{j+1}^{n+1} = \sum f^{n+1} e^{i\pi k (k+1) \Delta x/L} \quad (3.137)$$

in which k = wave number; L = wavelength; f^{n+1} = Fourier coefficient at time level $(n+1)\Delta t$. Let $y = \pi k \Delta x / L$, then substituting Eq. (3.137) into Eq. (3.131) gives

$$f^{n+1} e^{iyj} = \left(\frac{1}{2} - \frac{1}{2}r\right) f^n e^{iy(j+1)} + r f^n e^{iyj} + \left(\frac{1}{2} - \frac{1}{2}r\right) f^n e^{iy(j-1)} \quad (3.138)$$

$$f^{n+1} = \left(\frac{1}{2} - \frac{1}{2}r\right) f^n e^{iy} + r f^n + \left(\frac{1}{2} - \frac{1}{2}r\right) f^n e^{-iy}$$

$$f^{n+1} = f^n \left[r + \left(\frac{1}{2} - \frac{1}{2}r\right)(\cos y + i \sin y + \cos(-y) + i \sin(-y)) \right]$$

$$f^{n+1} = f^n \left[r + \left(\frac{1}{2} - \frac{1}{2}r\right) 2 \cos y \right] = f^n A_f \quad (3.139)$$

where A_f is the amplification factor.

To satisfy the von Neumann necessary condition for linear stability

$$|A_f| \leq 1.$$

$$|A_f| = |r + (1 - r) \cos y| \quad (3.140)$$

$$\text{if } y = \frac{\pi}{2} \text{ then } \cos y = 0 \text{ and } |A_f| = |r|$$

$$\text{if } y = \pi \text{ then } \cos y = -1 \text{ and } |A_f| = |2r-1|$$

Therefore as before the Lax scheme is shown to be conditionally stable with a necessary condition for linear stability of

$$|r| \leq 1 \text{ or}$$

$$\frac{\Delta t}{\Delta x} (gH)^{\frac{1}{2}} \leq 1 \quad (3.141)$$

Since $|A_f| < 1$ for $|r| < 1$, the Lax scheme can be used as an amplitude dissipative scheme. When it is used in this way the dissipation is generated from the $\frac{\Delta x^2}{\Delta t}$ term in the truncation error.

(b) Linear Stability of the Leap-Frog Scheme

Using Eq. (3.129) the Leap-Frog scheme is equivalent to:-

$$\frac{(Q_j^{n+1} - 2Q_j^n + Q_j^{n-1}))}{\Delta t^2} - gH \frac{(Q_{j+1}^n - 2Q_j^n + Q_{j-1}^n)}{\Delta x^2} = 0 \quad (3.142)$$

With a three level scheme such as this we introduce additional variables

$$v = \frac{\partial Q}{\partial t} ; \quad w = (gH)^{\frac{1}{2}} \frac{\partial Q}{\partial x} \quad (3.143)$$

$$\text{Since} \quad \frac{\partial^2 Q}{\partial t^2} - gH \frac{\partial^2 Q}{\partial x^2} = 0$$

$$\frac{\partial v}{\partial t} = (gH)^{\frac{1}{2}} \frac{\partial w}{\partial x} \quad (3.144)$$

and

$$\frac{\partial w}{\partial t} = (gH)^{\frac{1}{2}} \frac{\partial v}{\partial x} \quad (3.145)$$

The difference equations for Eq. (3.143) are

$$v_j^n = \frac{(Q_j^n - Q_j^{n-1}))}{\Delta t} \quad \text{and} \quad w_{j-\frac{1}{2}}^n = (gH)^{\frac{1}{2}} \frac{(Q_j^n - Q_{j-1}^n)}{\Delta x}$$

Substituting into Eq. (3.142) we obtain the difference equations in our new variables.

$$\frac{(v_j^{n+1} - v_j^n)}{\Delta t} - (gH)^{\frac{1}{2}} \frac{(w_{j+\frac{1}{2}}^n - w_{j-\frac{1}{2}}^n)}{\Delta x} = 0 \quad (3.146)$$

$$\frac{(w_j^{n+1} - w_j^n)}{\Delta t} - (gH)^{\frac{1}{2}} \frac{(v_{j+\frac{1}{2}}^{n+1} - v_{j-\frac{1}{2}}^{n+1})}{\Delta x} = 0 \quad (3.147)$$

Eqs. (3.146) and (3.147) are equivalent to Eq. (3.142) and represent forward and backward operators respectively.

Using a Fourier analysis to find the stability condition of Eqs. (3.146) and (3.147)

$$v_{j+1}^{n+1} = \sum f_1^{n+1} e^{i\pi k(j+1)\Delta x/L} = \sum f_1^{n+1} e^{iy(j+1)} \quad (3.148)$$

$$w_{j+1}^{n+1} = \sum f_2^{n+1} e^{i\pi k(j+1)\Delta x/L} = \sum f_2^{n+1} e^{iy(j+1)} \quad (3.149)$$

From Eqs. (3.146) and (3.147) respectively

$$(f_1^{n+1} - f_1^n) e^{iyj} - (gH)^{\frac{1}{2}} \frac{\Delta t}{\Delta x} \left[e^{iy(j+\frac{1}{2})} - e^{iy(j-\frac{1}{2})} \right] f_2^n = 0 \quad (3.150)$$

$$(f_2^{n+1} - f_2^n) e^{iyj} - (gH)^{\frac{1}{2}} \frac{\Delta t}{\Delta x} \left[e^{iy(j+\frac{1}{2})} - e^{iy(j-\frac{1}{2})} \right] f_1^{n+1} = 0 \quad (3.151)$$

Hence

$$f_1^{n+1} - f_1^n - (gH)^{\frac{1}{2}} \frac{\Delta t}{\Delta x} \left[2 \sin\left(\frac{y}{2}\right) \right] f_2^n = 0 \quad (3.152)$$

$$f_2^{n+1} - f_2^n - (gH)^{\frac{1}{2}} \frac{\Delta t}{\Delta x} \begin{bmatrix} 2 \sin(\frac{y}{2}) \end{bmatrix} f_1^{n+1} \quad (3.153)$$

$$\text{let } (gH)^{\frac{1}{2}} \frac{\Delta t}{\Delta x} \begin{bmatrix} 2 \sin(\frac{y}{2}) \end{bmatrix} = \alpha$$

$$\begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}^{n+1} = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}^n \quad (3.154)$$

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}^{n+1} = \begin{bmatrix} 1 & 0 \\ -\alpha & 1 \end{bmatrix} \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}^n \quad (3.155)$$

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}^{n+1} = \begin{bmatrix} 1 & \alpha \\ -\alpha & (1-\alpha^2) \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}^n \quad (3.156)$$

If the eigen values are represented by λ then

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}^{n+1} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}^n \quad (3.157)$$

Subtracting Eq. (3.157) from Eq. (3.156)

$$\begin{bmatrix} (1-\lambda) & \alpha \\ -\alpha & (1-\alpha^2-\lambda) \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}^n = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (3.158)$$

$$\text{hence } (1-\lambda)(1-\alpha^2-\lambda) + \alpha^2 = 0$$

$$\lambda^2 + (\alpha^2-2)\lambda + 1 = 0 \quad (3.159)$$

when $\alpha^2 = 0$, $\lambda = 1$; $\alpha^2 = 2$, $\lambda = \pm i$

$\alpha^2 = 4$, $\lambda = -1$; $\alpha^2 \geq 4$, $\lambda \leq -1$

If $\lambda = \gamma i + \beta$, the following Argand diagram can be drawn.

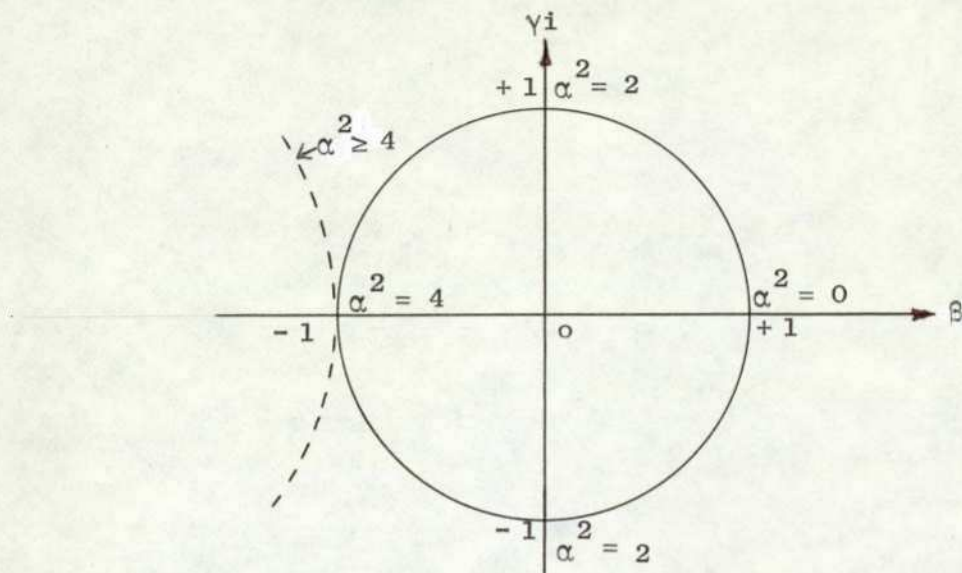


FIG. 3.10 ARGAND DIAGRAM FOR LINEAR STABILITY

The von Neumann necessary condition for linear stability is satisfied when

$$\alpha^2 \leq 4 \quad \text{or}$$

$$gH \left(\frac{\Delta t}{\Delta x} \right)^2 \sin^2 \left(\frac{y}{2} \right) \leq 1 \quad (3.160)$$

Hence the Leap-Frog scheme is conditionally stable with a necessary condition for linear stability of

$$\frac{\Delta t}{\Delta x} (gH)^{\frac{1}{2}} \leq 1 \quad (3.161)$$

The Leap-Frog scheme is also non-dissipative under this condition since $|\lambda| = 1$.

(c) Linear Stability of the Central Implicit Scheme

From the linearised Eqs.(3.126) and (3.127) the central implicit scheme can be written as

$$\frac{1}{4\Delta x} (Q_{j+1}^{n+1} - Q_{j-1}^{n+1} + Q_{j+1}^n - Q_{j-1}^n) + \frac{1}{\Delta t} (A_j^{n+1} - A_j^n) = 0 \quad (3.162)$$

$$\frac{1}{\Delta t} (Q_j^{n+1} - Q_j^n) + \frac{gH}{4\Delta x} (A_{j+1}^{n+1} - A_{j-1}^{n+1} + A_{j+1}^n - A_{j-1}^n) = 0 \quad (3.163)$$

If $\frac{4\Delta x}{\Delta t} = r$ then from Eqs.(3.162) and (3.163)

$$Q_{j+1}^{n+1} + rA_j^{n+1} - Q_{j-1}^{n+1} = -Q_{j+1}^n + rA_j^n + Q_{j-1}^n \quad (3.164)$$

$$A_{j+1}^{n+1} + \frac{r}{gH} Q_j^{n+1} - A_{j-1}^{n+1} = -A_{j+1}^n + \frac{r}{gH} Q_j^n + A_{j-1}^n \quad (3.165)$$

Expressing Eq.(3.164) in matrix form together with its boundary conditions gives

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & r & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Q_3 \\ A_2 \\ Q_1 \end{bmatrix}^{n+1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & r & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Q_3 \\ A_2 \\ Q_1 \end{bmatrix}^n \quad (3.166)$$

The inverse of the left hand side matrix is multiplied by the right hand side matrix to give

$$\begin{bmatrix} Q_3 \\ A_2 \\ Q_1 \end{bmatrix}^{n+1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & r & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{r} & \frac{1}{r} & \frac{1}{r} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Q_3 \\ A_2 \\ Q_1 \end{bmatrix}^n \quad (3.167)$$

$$\begin{bmatrix} Q_3 \\ A_2 \\ Q_1 \end{bmatrix}^{n+1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Q_3 \\ A_2 \\ Q_1 \end{bmatrix}^n \quad (3.168)$$

If the eigen values are represented by the symbol λ then

$$\begin{bmatrix} Q_3 \\ A_2 \\ Q_1 \end{bmatrix}^{n+1} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} Q_3 \\ A_2 \\ Q_1 \end{bmatrix}^n \quad (3.169)$$

Subtracting Eq.(3.169) from Eq.(3.168) gives

$$\begin{bmatrix} (1-\lambda) & 0 & 0 \\ -2 & (1-\lambda) & 2 \\ 0 & 0 & (1-\lambda) \end{bmatrix} \begin{bmatrix} Q_3 \\ A_2 \\ Q_1 \end{bmatrix}^n = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (3.170)$$

We obtain $(1-\lambda)^3 = 0$ or $\lambda_1 = \lambda_2 = \lambda_3 = 1$

Exactly the same result is obtained from Eq.(3.165). Because none of the eigen values are greater than unity none of the eigen vectors are amplified. Since the eigen values are independent of the equation coefficients, this central implicit scheme is linearly unconditionally stable. Also, since all the eigen values equal unity, this is a non-dissipative scheme, all error being phase error.

Writing the dependent variables in the wave equations as periodicity conditions, using Fourier series:-

$$Q_{j+1}^{n+1} = \sum f_1^{n+1} e^{i\pi k(j+1)\Delta x/L} = \sum f_1^{n+1} e^{iy(j+1)} \quad (3.171)$$

$$A_{j+1}^{n+1} = \sum f_2^{n+1} e^{i\pi k(j+1)\Delta x/L} = \sum f_2^{n+1} e^{iy(j+1)} \quad (3.172)$$

From Eqs. (3.164) and (3.165) respectively

$$\begin{aligned} f_1^{n+1} e^{iy(j+1)} + r f_2^{n+1} e^{iyj} - f_1^{n+1} e^{iy(j-1)} = \\ - f_1^n e^{iy(j+1)} + r f_2^n e^{iyj} + f_1^n e^{iy(j-1)} \end{aligned} \quad (3.173)$$

$$\begin{aligned} f_2^{n+1} e^{iy(j+1)} + \frac{r}{gH} f_1^{n+1} e^{iyj} - f_2^{n+1} e^{iy(j-1)} = \\ - f_2^n e^{iy(j+1)} + \frac{r}{gH} f_1^n e^{iyj} + f_2^n e^{iy(j-1)} \end{aligned} \quad (3.174)$$

From Eq. (3.173)

$$f_1^{n+1} (2i \sin y) + r f_2^{n+1} = f_1^n (-2i \sin y) + r f_2^n \quad (3.175)$$

From Eq. (3.174)

$$f_2^{n+1} (2i \sin y) + \frac{r}{gH} f_1^{n+1} = f_2^n (-2i \sin y) + \frac{r}{gH} f_1^n \quad (3.176)$$

$$\text{Let } 2 \sin y = a ; \frac{r}{gH} = b$$

$$\begin{bmatrix} ai & r \\ b & ai \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}^{n+1} = \begin{bmatrix} -ai & r \\ b & -ai \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}^n \quad (3.177)$$

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}^{n+1} = \frac{-1}{(a^2 + rb)} \begin{bmatrix} ai & -r \\ -b & ai \end{bmatrix} \begin{bmatrix} -ai & r \\ b & -ai \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}^n \quad (3.178)$$

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}^{n+1} = - \frac{1}{(a^2 + rb)} \begin{bmatrix} (a^2 - rb) & 2rai \\ 2bai & (a^2 - rb) \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}^n \quad (3.179)$$

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}^{n+1} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}^n \quad (3.180)$$

$$\text{let } -(a^2 + rb) = c$$

$$\begin{bmatrix} \frac{(a^2 - rb)}{c} - \lambda & \frac{2rai}{c} \\ \frac{2bai}{c} & \frac{(a^2 - rb)}{c} - \lambda \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}^n = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (3.181)$$

$$\lambda^2 - \frac{2(a^2 - rb)}{c} \lambda + \frac{4ba^2 r}{c^2} + \frac{(a^2 - rb)^2}{c^2} = 0$$

$$\lambda^2 - \frac{2(a^2 - rb)}{c} \lambda + 1 = 0 \quad (3.182)$$

$$\lambda = - \frac{(a^2 - rb)}{(a^2 + rb)} \pm \left[\frac{(a^2 - rb)^2}{(a^2 + rb)^2} - 1 \right]^{\frac{1}{2}}$$

$$\lambda = - \frac{(a^2 - rb) \pm 2ia(rb)^{\frac{1}{2}}}{(a^2 + rb)}$$

$$\lambda = \frac{[ia \pm (rb)^{\frac{1}{2}}]^2}{(a^2 + rb)} = - \frac{[ia \pm (rb)^{\frac{1}{2}}]^2}{[ia - (rb)^{\frac{1}{2}}][ia + (rb)^{\frac{1}{2}}]}$$

$$\lambda_1 = - \left[\frac{ia + (rb)^{\frac{1}{2}}}{ia - (rb)^{\frac{1}{2}}} \right] = -1 \quad (3.183)$$

$$\lambda_2 = - \left[\frac{ia - (rb)^{\frac{1}{2}}}{ia + (rb)^{\frac{1}{2}}} \right] = -1 \quad (3.184)$$

Since $|\lambda_{1,2}| = 1$, the scheme is linearly unconditionally stable and also non-dissipative: all truncation error is phase error.

(d) Linear Stability of the Implicit Transport - Diffusion Scheme

From Eq.(3.76) the finite difference equation, with inflow terms set to zero is written as:-

$$\begin{aligned} & \rho_{j+1}^{n+1} \left[\frac{1}{4\Delta t} + \frac{u}{4\Delta x} - \frac{D}{2\Delta x^2} \right] + \rho_j^{n+1} \left[\frac{1}{2\Delta t} + \frac{D}{\Delta x^2} \right] \\ & + \rho_{j-1}^{n+1} \left[\frac{1}{4\Delta t} - \frac{u}{4\Delta x} - \frac{D}{2\Delta x^2} \right] + \rho_{j+1}^n \left[-\frac{1}{4\Delta t} + \frac{u}{4\Delta x} - \frac{D}{2\Delta x^2} \right] \\ & + \rho_j^n \left[-\frac{1}{2\Delta t} + \frac{D}{\Delta x^2} \right] + \rho_{j-1}^n \left[-\frac{1}{4\Delta t} - \frac{u}{4\Delta x} - \frac{D}{2\Delta x^2} \right] = 0 \quad (3.185) \end{aligned}$$

$$\text{let } a = \frac{u\Delta t}{\Delta x} \text{ and } b = \frac{2D\Delta t}{\Delta x^2}$$

$$\begin{aligned} & \rho_{j+1}^{n+1} (1 + a - b) + \rho_j^{n+1} (2 + 2b) + \rho_{j-1}^{n+1} (1 - a - b) = \\ & + \rho_{j+1}^n (1 - a + b) + \rho_j^n (2 - 2b) + \rho_{j-1}^n (1 + a + b) \end{aligned} \quad (3.186)$$

Expressing Eq.(3.186) in matrix form together with its boundary conditions gives

$$\begin{bmatrix} 1 & 0 & 0 \\ (1+a-b) & (2+2b) & (1-a-b) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho_3 \\ \rho_2 \\ \rho_1 \end{bmatrix}^{n+1} = \begin{bmatrix} 1 & 0 & 0 \\ (1-a+b) & (2-2b) & (1+a+b) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho_3 \\ \rho_2 \\ \rho_1 \end{bmatrix}^n \quad (3.187)$$

The inverse of the left hand side matrix is multiplied by the right hand side matrix to give

$$\begin{bmatrix} \rho_3 \\ \rho_2 \\ \rho_1 \end{bmatrix}^{n+1} = \frac{1}{(2+2b)} \begin{bmatrix} 1 & 0 & 0 \\ (1-a+b) & (2-2b) & (1+a+b) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} (2+2b) & 0 & 0 \\ -(1-a-b) & 1 & -(1-a-b) \\ 0 & 0 & (2+2b) \end{bmatrix} \begin{bmatrix} \rho_3 \\ \rho_2 \\ \rho_1 \end{bmatrix}^n \quad (3.188)$$

$$\begin{bmatrix} \rho_3 \\ \rho_2 \\ \rho_1 \end{bmatrix}^{n+1} = \frac{1}{(2+2b)} \begin{bmatrix} (2+2b) & 0 & 0 \\ -4(a-2b) & (2-2b) & 4(a+2b) \\ 0 & 0 & (2+2b) \end{bmatrix} \begin{bmatrix} \rho_3 \\ \rho_2 \\ \rho_1 \end{bmatrix}^n \quad (3.189)$$

If the eigen values are represented by the symbol λ then

$$\begin{bmatrix} \rho_3 \\ \rho_2 \\ \rho_1 \end{bmatrix}^{n+1} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} \rho_3 \\ \rho_2 \\ \rho_1 \end{bmatrix}^n \quad (3.190)$$

Subtracting Eq.(3.190) from Eq.(3.189) gives

$$\begin{bmatrix} (1-\lambda) & 0 & 0 \\ -4\frac{(a-2b)}{(2+2b)} & \frac{(2-2b)}{(2+2b)} - \lambda & \frac{4(a+2b)}{(2+2b)} \\ 0 & 0 & (1-\lambda) \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix}^n = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (3.191)$$

We obtain

$$(1-\lambda)^2 \left[\frac{(2-2b)}{(2+2b)} - \lambda \right] = 0 \quad (3.192)$$

$$\lambda_1 = \lambda_2 = 1 ; \lambda_3 = \frac{1-b}{1+b} \quad (3.193)$$

Since $b \geq 0$ then $|\lambda_3| \leq 1$, and therefore all three eigen values have an absolute value less than or equal to unity. Because the von Neumann necessary condition for linear stability is satisfied this difference scheme is unconditionally stable.

Writing the dependent variable as a periodicity condition, using a Fourier series:-

$$\rho_{j+1}^{n+1} = \sum f^{n+1} e^{i\pi k(j+1)\Delta x/L} = \sum f^{n+1} e^{iy(j+1)} \quad (3.194)$$

From Eq. (3.186)

$$f^{n+1} \left[(1+a-b) e^{iy} + (2+2b) + (1-a-b) e^{-iy} \right] = f^n \left[(1-a+b) e^{iy} + (2-2b) + (1+a+b) e^{-iy} \right] \quad (3.195)$$

$$f^{n+1} (2 \cos y + 2ai \sin y - 2b \cos y + 2+2b) =$$

$$f^n (2 \cos y - 2ai \sin y + 2b \cos y + 2-2b)$$

$$f^{n+1} = f^n \left[\frac{\cos y(1+b) + 1-b - ai \sin y}{\cos y(1-b) + 1+b + ai \sin y} \right] = f^n A_f \quad (3.196)$$

where A_f is the amplification factor.

To satisfy the von Neumann necessary condition for linear stability

$$|A_f| \leq 1, \text{ hence}$$

$$|A_f| = \left| \frac{\cos y(1+b) + 1-b-ai \sin y}{\cos y(1-b) + 1+b+ai \sin y} \right| \leq 1 \quad (3.197)$$

Because $b \geq 1$, $|A_f|$ is always less than or equal to unity for all wave numbers, therefore this scheme is unconditionally stable. It will also be seen that since $|A_f| = 1$ only for the longest wave length, this scheme is amplitude dissipative.

(e) Influence on Stability of Minor Terms

Minor constant terms in the equations of motion such as wind shear stress, Coriolis force and inflow-outflow terms have an unstabilising effect on central difference schemes. This can be readily seen from Eq.(3.170), where the left hand side matrix no longer equals zero, but a negative number causing the amplification factor to be greater than unity. Fortunately this instability is weak and is shown later to be counterbalanced by the strong stabilising effect of the larger bed friction term.

3.7.4 Non-Linear Instability

Non-linear terms in the equations of motion such as the convective acceleration term and the bed friction term were neglected in the previous linear stability analysis. For shallow water estuarine flows, and especially stratified estuarine flows these terms are not negligible and can influence the overall stability of the difference schemes used.

Non-linear instability can be recognised from linear instability by reducing the time and distance steps by the same ratio. If the scheme is linearly unstable this will not alter the instability. If non-linearly unstable, matters will be made worse since the increase in calculation steps will generate even more higher wave number components.

The occurrence of non-linear instability in numerical methods is due to the physical process of transferring turbulent energy from longer to shorter wave motions through the non-uniform characteristics of the flow. The numerical model endeavours to copy this process, but it is limited by the fact that it cannot transfer energy to waves smaller than twice the grid size. For this reason energy tends to accumulate at a wave length of $2\Delta x$ and this results in an explosive growth of oscillations which completely swamp the correct solution after some time. This process by which non-linear terms such as $u \frac{\partial u}{\partial x}$, give squared terms in the difference equations thereby halving the wavelength or doubling the wave number is called aliasing. As an illustration consider the following difference form of the term $u \frac{\partial u}{\partial x}$

$$u \frac{\partial u}{\partial x} \approx u_j^n \frac{(u_{j+1}^n - u_{j-1}^n)}{2\Delta x} \quad (3.198)$$

Written as a Fourier series where

$$u_{j+1}^n = \sum f^n e^{i\pi k(j+1)\Delta x/L}$$

$$u_j^n (u_{j+1}^n - u_{j-1}^n) = f^{2n} e^{2i\pi k j \Delta x/L} \cdot 2 \cos(\pi k \Delta x/L) \quad (3.199)$$

Thus, components of wave number k , breed higher wave number components. In the physical process, this aliasing produces steeper and steeper waves which eventually break to form jumps, which at an interface in a stratified fluid exhibit the phenomenon of entrainment.

It is not only non-linear terms in the equations of motion that produce non-linear instabilities. An important contributor to the generation of higher wave number components and consequent non-linear instabilities is when a difference scheme is used with incorrect algorithmic structures or boundary equations. For example, data must be supplied to every grid point in a difference scheme depending on the number of positive and negative characteristics at that point. If data correct for a subcritical flow is supplied where the flow is supercritical, higher wave numbers will be generated and the numerical scheme will become non-linearly unstable. Similarly if the equations of motion are not correctly represented at boundaries, again higher wave numbers will be generated and the calculation will eventually become unstable.

One method used to suppress the growth of these parasitic waves is to introduce a pseudo-viscosity type energy dissipation into the mathematical model. This has the effect of artificially smoothing the numerical solution enabling the computation to continue. The procedure adopted by Vreugdenhil (146) and others is to spatially average the computed velocities at each time step using a weighting factor ξ and an equation of the type:-

$$\bar{u}_j^{n+1} = (1-\xi) u_j^{n+1} + \frac{\xi}{2} (u_{j+1}^{n+1} + u_{j-1}^{n+1}) \quad (3.200)$$

where $\xi \approx 0.01$.

Central implicit finite difference schemes of the type described in this chapter, require for second order accuracy two complete computations at every time step in order to centre the equation coefficients. This iteration process was shown by the author (52) to be an effective dissipative mechanism for small scale damping of parasitic waves. Since this dissipation is achieved by the difference scheme itself, it does not have the disadvantage of the pseudo-viscosity method where one is never sure whether it is non-linear instabilities that are being damped or the small scale turbulent energy.

Another method used to suppress the growth of parasitic waves is to use dissipative difference schemes. With these schemes the amplification factor is usually unity for the longest wavelength but decreases as the wavelength decreases. This means that parasitic waves and thus non-linear instabilities can be suppressed while the longer wave components may be very little influenced. Such a difference scheme that dissipates the shorter wavelength components is called a dissipative difference scheme. As remarked before with the pseudo-viscosity method, this dissipative behaviour can either suppress unwanted parasitic waves or cause an incorrect solution due to it diffusing the small scale turbulent energy.

Dissipative difference schemes therefore have the property of simulating higher order terms in the equations of motion when the flow approaches a discontinuity. Since these dissipative terms are generated from the truncation error of the difference equations, they should be small when the flow is smooth and rapidly increase near a discontinuity in the flow such as a hydraulic jump. A solution of this type is said to constitute a weak solution of the equations of nearly horizontal flow. It should be

remembered as pointed out by Abbott, Marshall and Ohno (7) that since dissipative difference schemes possess both amplitude and phase errors, the phase errors can either contribute to the dissipative process or they may spoil it.

In order to study the properties of an amplitude dissipative scheme the amplification factor is plotted against the wave number. As an example, consider the amplitude portraits for the implicit transport-diffusion scheme given by Eq.(3.197) and shown in Fig.3.11. It can be seen that for $b = 1$ and $a < 0.1$, the scheme attains its maximum amplitude attenuation for the highest wave numbers while for $b < 0.1$ and $a > 1$ the amplification factor tends to unity for all wave numbers.

It follows that in order to prevent the formation of parasitic waves caused by the generation of higher wave numbers and consequent non-linear instability, 'b' must be chosen near to unity and 'a' less than about 0.25 . Thus

$$\frac{2D\Delta t}{\Delta x^2} \approx 1 \quad \text{and} \quad \left| \frac{u\Delta t}{\Delta x} \right| < 0.25 \quad (3.201)$$

The influence of non-linear terms on the stability of difference schemes is not easy to prove. This is because different components of the Fourier series will interact with one another through the non-linear terms. However, following Richtmyer and Morton some idea of their influence can be obtained by converting non-linear terms into linear components with constant coefficients. The quasi-linearised momentum equation becomes

$$\frac{\partial Q}{\partial t} - 2u_o \frac{\partial A}{\partial t} - e_o \frac{\partial A}{\partial x} + \frac{gQ |Q_o|}{C^2_{A_o} R_o} = 0 \quad (3.202)$$

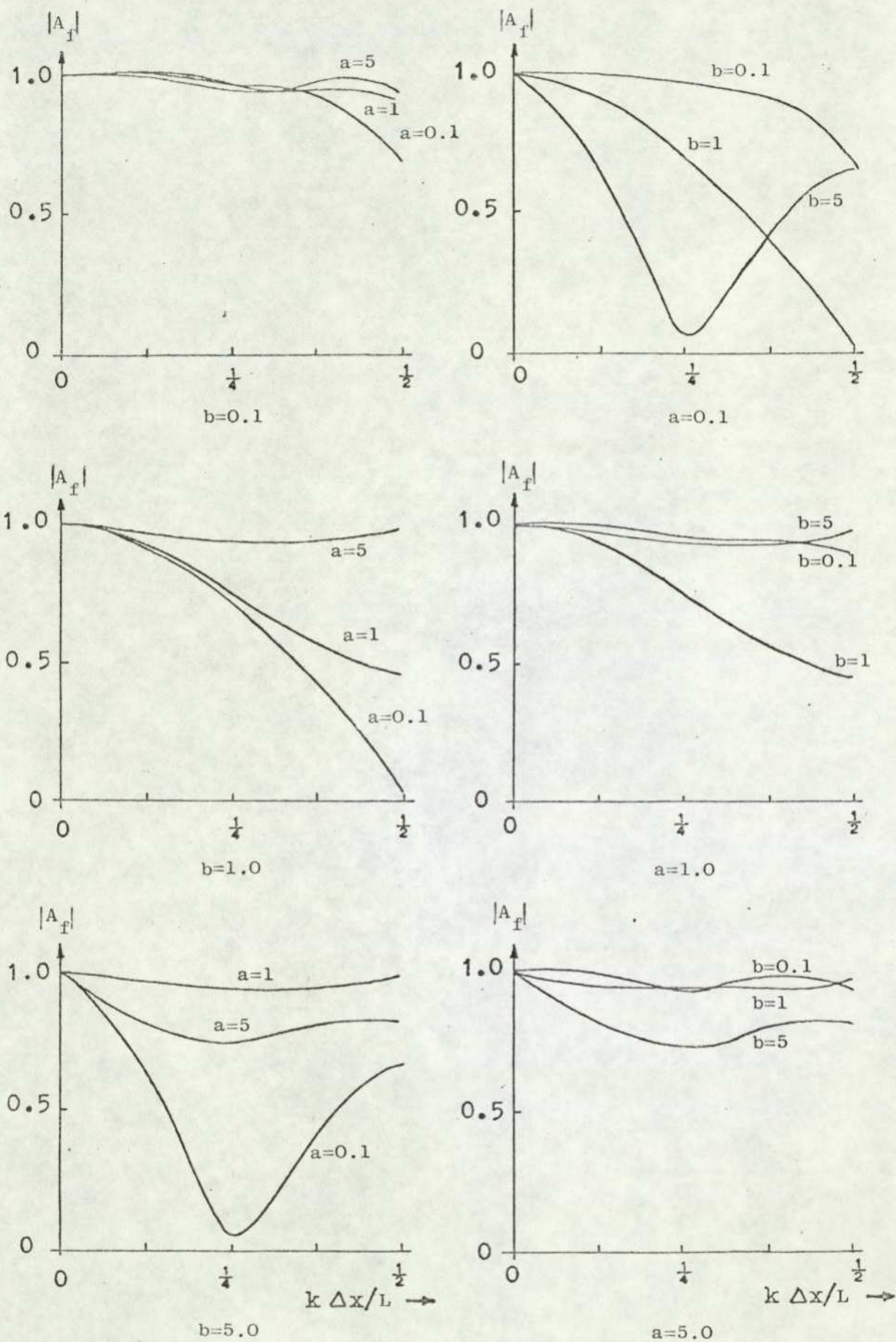


FIG. 3.11 CURVES OF AMPLIFICATION FACTOR VERSUS WAVE NUMBER

where suffix o denotes constant terms. The centred implicit difference form of Eq.(3.202) is

$$\begin{aligned} \frac{1}{\Delta t} (Q_j^{n+1} - Q_j^n) - \frac{u_o}{\Delta t} (A_{j+1}^{n+1} - A_{j+1}^n + A_{j-1}^{n+1} - A_{j-1}^n) \\ - \frac{e_o}{4\Delta x} (A_{j+1}^{n+1} - A_{j-1}^{n+1} + A_{j+1}^n - A_{j-1}^n) + \frac{g |Q_o|}{C_{A_o}^2 R_o} Q_j^{n+1} = 0 \end{aligned} \quad (3.203)$$

The centred implicit difference form of the mass equation is from Eq.(3.162),

$$\frac{1}{4\Delta x} (Q_{j+1}^{n+1} - Q_{j-1}^{n+1} + Q_{j+1}^n - Q_{j-1}^n) + \frac{1}{\Delta t} (A_j^{n+1} - A_j^n) = 0 \quad (3.204)$$

Expressing as Fourier series where

$$Q_{j+1}^{n+1} = \sum f_1^{n+1} e^{i\pi j(j+1)\Delta x/L} = \sum f_1^{n+1} e^{iy(j+1)}$$

$$A_{j+1}^{n+1} = \sum f_2^{n+1} e^{i\pi k(j+1)\Delta x/L} = \sum f_2^{n+1} e^{iy(j+1)}$$

Eq.(3.203) becomes

$$\begin{aligned} f_1^{n+1} \left[1 + \frac{g |Q_o| \Delta t}{C_{A_o}^2 R_o} \right] + f_2^{n+1} \left[-2u_o \cos y - \frac{e_o \Delta t}{2\Delta x} i \sin y \right] \\ - f_1^n + f_2^n \left[2u_o \cos y - \frac{e_o \Delta t}{2\Delta x} i \sin y \right] = 0 \end{aligned} \quad (3.205)$$

Eq. (3.204) becomes

$$f_1^{n+1} \left[\frac{\Delta t}{2\Delta x} i \sin y \right] + f_2^{n+1} + f_1^n \left[\frac{\Delta t}{2\Delta x} i \sin y \right] - f_2^n = 0 \quad (3.206)$$

$$\text{Let } \alpha = \frac{g |Q_o| \Delta t}{C_o^2 A_o R_o} ; \quad \beta = 2u_o \cos y ; \quad \gamma = \frac{e_o \Delta t}{2\Delta x} ; \quad \delta = \frac{\Delta t}{2\Delta x}$$

Expressing Eqs. (3.205) and (3.206) in matrix form

$$\begin{bmatrix} (1 + \alpha) & -(\beta + \gamma i \sin y) \\ \delta i \sin y & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}^{n+1} = \begin{bmatrix} 1 & -(\beta - \gamma i \sin y) \\ -\delta i \sin y & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}^n \quad (3.207)$$

$$\text{Let } a = \gamma \delta \sin^2 y \text{ and } b = \beta \delta i \sin y$$

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}^{n+1} = \frac{1}{(1 + \alpha + b - a)} \begin{bmatrix} (1 + b + a) & \gamma i \sin y (2 + \alpha) - \alpha \beta \\ (-2\delta i \sin y) & (-b + a + 1 + \alpha) \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}^n \quad (3.208)$$

Also if λ are the eigen values

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}^{n+1} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}^n \quad (3.209)$$

Subtracting Eq. (3.209) from Eq. (3.208) and letting $c = (1 + \alpha + b - a)$ gives

$$\lambda^2 - \frac{\lambda}{c} (2 + 2a + \alpha) + \frac{1}{2c} (-b^2 + a^2 + 1 + \alpha - 2a - \alpha - b\alpha) = 0 \quad (3.210)$$

Solving Eq. (3.210) to find the eigen values we obtain

$$2c\lambda = 2 + 2a + \alpha \pm (\alpha^2 + 8a\alpha + 16a + 4b^2 + 4b\alpha)^{\frac{1}{2}} \quad (3.211)$$

For the longest wavelengths when $y = 0$, $a = b = 0$ and $c = 1 + \alpha$.

$$2(1 + \alpha)\lambda = 2 + \alpha \pm \alpha$$

$$A_f = \lambda_1 \lambda_2 = \frac{1}{1 + \alpha} \quad (3.212)$$

For the shortest wavelengths when $y = \frac{\pi}{2}$, $a = \gamma\delta$, $b = 0$ and $c = 1 + \alpha - \gamma\delta$.

$$(2 + 2\alpha - 2\gamma\delta)\lambda = 2 + 2\gamma\delta + \alpha \pm (\alpha^2 + 8\alpha\gamma\delta + 16\gamma\delta)^{\frac{1}{2}} \quad (3.213)$$

$$A_f = \lambda_1 \lambda_2 = \frac{(1 + \alpha + \gamma^2\delta^2 - \alpha\gamma\delta - 2\gamma\delta)}{(1 + \alpha + \gamma^2\delta^2 - \alpha\gamma\delta - 2\gamma\delta) + (\alpha - \gamma\delta\alpha + \alpha^2)} \quad (3.214)$$

If $\alpha = 0$ $A_f = 1$. If $\gamma\delta < 0$ the flow is subcritical since $e_o = u_o^2 - gH_o$ and $|A_f| < 1$ because $\alpha > 0$. If $\gamma\delta > 0$ the flow is supercritical and we have the stability condition as $\alpha \rightarrow 0$, that $\gamma\delta < 1$ or

$$(u_o^2 - gH_o) \frac{\Delta t^2}{4\Delta x} < 1 \quad (3.215)$$

In the limit if $u_o^2 \gg gH_o$ then approximately

$$\left| \frac{u_o \Delta t}{2 \Delta x} \right| < 1 \quad (3.216)$$

The above analysis shows that provided the bed friction is zero, non-linear terms in the equations of motion do not alter the unconditional, non-dissipative properties of the scheme. It can be seen that for all wavelengths, bed friction increases the stability of the scheme. This is an important property of the centred implicit scheme since this effectively counterbalances the unstabling effects of wind shear stress and Coriolis forces.

For the highest resolvable wave numbers to be continuously dissipated in a real supercritical flow the non-linear stability condition given by Eq.(3.215) is necessary.

CHAPTER 4

EXTENDED IMPLICIT FINITE DIFFERENCE SCHEMES

4.1 Characteristic Structure of Well-Mixed Equations

The characteristic directions for well-mixed flows using the extended equations with density as a dependent variable will be obtained by matrix analysis after expansion and rearranging of the equation system (2.1), (2.2) and (2.10).

Substituting Eq.(2.1) into Eq.(2.2) the dynamical equation becomes

$$\begin{aligned} \frac{\partial}{\partial t} (\rho Q) - 2ub \frac{\partial}{\partial t} (\rho h) - (u^2 b - gA) \rho \frac{\partial h}{\partial x} - u^2 b I \rho \\ + \frac{\rho g Q |Q|}{C^2 A R} + (2u - v \cos \phi) q_i \rho_i - \frac{Q^2}{A} \frac{\partial \rho}{\partial x} \\ - k_a \rho_a u_a |u_a| b = 0 \end{aligned} \quad (4.1)$$

From Eq.(2.1)

$$Q \frac{\partial \rho}{\partial x} + \rho \frac{\partial Q}{\partial x} + \rho b \frac{\partial h}{\partial t} + b h \frac{\partial \rho}{\partial t} = q_i \rho_i = n'' \quad (4.2)$$

from Eq.(4.1)

$$\begin{aligned}
& \left(\frac{Q}{\rho} - \frac{2ub}{\rho} \right) \frac{\partial \rho}{\partial t} - \frac{Q^2}{\rho A} \frac{\partial \rho}{\partial x} + \frac{\partial Q}{\partial t} - (u^2 b - gA) \frac{\partial h}{\partial x} \\
& - 2ub \frac{\partial h}{\partial t} = - \frac{gQ|Q|}{C^2 AR} - (2u - v \cos \phi) \frac{q_i \rho_i}{\rho} + u^2 b I \\
& + \frac{k_a \rho_a u_a |u_a| b}{\rho} = m''
\end{aligned} \tag{4.3}$$

Equations (4.2) and (4.3) together with the equation for the mass convection of the solute (Eq. (2.10) with the diffusion coefficient set to zero), that is

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} = \frac{q_i}{A} (\rho_i - \rho) = r'' \tag{4.4}$$

plus the following three equations of variation

$$\frac{\partial Q}{\partial x} dx + \frac{\partial Q}{\partial t} dt = dQ \tag{4.5}$$

$$\frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial t} dt = dh \tag{4.6}$$

$$\frac{\partial \rho}{\partial x} dx + \frac{\partial \rho}{\partial t} dt = d\rho \tag{4.7}$$

can be written in the following matrix as:

$$\begin{bmatrix}
 \rho & 0 & 0 & \rho b & Q & bh \\
 0 & 1 & -(u^2b-gA) & -2ub \left(-\frac{Q^2}{\rho A} \right) \frac{1}{\rho}(Q-2uhb) \\
 0 & 0 & 0 & 0 & u & 1 \\
 dx & dt & 0 & 0 & 0 & 0 \\
 0 & 0 & dx & dt & 0 & 0 \\
 0 & 0 & 0 & 0 & dx & dt
 \end{bmatrix}
 \begin{bmatrix}
 \frac{\partial Q}{\partial x} \\
 \frac{\partial Q}{\partial t} \\
 \frac{\partial h}{\partial x} \\
 \frac{\partial h}{\partial t} \\
 \frac{\partial \rho}{\partial x} \\
 \frac{\partial \rho}{\partial t}
 \end{bmatrix}
 =
 \begin{bmatrix}
 n'' \\
 m'' \\
 r'' \\
 dQ \\
 dh \\
 d\rho
 \end{bmatrix} \quad (4.8)$$

If this matrix is rearranged and $\dot{x}$ is written for dx/dt , $\dot{Q}$ for dQ/dt , etc., it may be expressed in the form

$$\begin{bmatrix}
 \rho & 0 & 0 & \rho b & Q & bh & n'' \\
 -\dot{x} & 0 & -(u^2b-gA) & -2ub \left(-\frac{Q^2}{\rho A} \right) \frac{1}{\rho}(Q-2uhb) & (m''-\dot{Q}) \\
 0 & 0 & 0 & 0 & (u-\dot{x}) & 0 & (r''-\dot{\rho}) \\
 \dot{x} & 1 & 0 & 0 & 0 & 0 & \dot{Q} \\
 0 & 0 & \dot{x} & 1 & 0 & 0 & \dot{h} \\
 0 & 0 & 0 & 0 & \dot{x} & 1 & \dot{\rho}
 \end{bmatrix} \quad (4.9)$$

The condition that the first 6x6 determinant of this matrix be zero defines the characteristics directly, this condition reducing, by excluding rows and columns containing a single unity, to

$$\begin{bmatrix} \rho & 0 & \rho b & Q & bh \\ -\dot{x} & -(u^2b - gA) & -2ub & \left(-\frac{Q^2}{\rho A}\right) \frac{1}{\rho} (Q - 2uhb) \\ 0 & 0 & 0 & (u - \dot{x}) & 0 \\ 0 & \dot{x} & 1 & 0 & 0 \\ 0 & 0 & 0 & \dot{x} & 1 \end{bmatrix} = 0 \quad (4.10)$$

we obtain

$$\begin{aligned} \rho \left[(u^2b - gA) (u - \dot{x}) + \dot{x} (-2ub) (u - \dot{x}) \right] + \dot{x}^2 [\rho b (u - \dot{x})] &= 0 \\ \rho b (u - \dot{x}) \left[\dot{x}^2 - 2u\dot{x} + \left(\frac{u^2b}{b} - \frac{gA}{b} \right) \right] &= 0 \end{aligned} \quad (4.11)$$

hence

$$\left[1 - \frac{(u - \dot{x})^2 b}{gA} \right] (u - \dot{x}) = 0 \quad (4.12)$$

The characteristic directions are seen to be

$$\dot{x} = u; \quad \dot{x} = u \pm (gA/b)^{\frac{1}{2}} \quad (4.13)$$

For a rectangular channel where $A = bd$, the characteristic directions become

$$\dot{x} = u; \quad \dot{x} = u \pm (gd)^{\frac{1}{2}} \quad (4.14)$$

Equation (4.13) indicates that there are three characteristic directions associated with well-mixed flows, for the convection of the three dependent variables Q , h and ρ . Two of these characteristics are associated with the convection of level and discharge and the third $\dot{x} = u$ is associated with the convection of the density. There are two more characteristics to be added to the above three, namely those associated with the diffusion of the solute. Since the signal velocity of the diffusive shock wave travels at the speed of sound, which is effectively infinite, these characteristic directions are

$$\dot{x} = \pm \infty \quad (4.15)$$

The direction of the characteristics define the data input to the field of calculation both at the boundaries of the model and internally. Since it is necessary to know when this data input or algorithmic structure changes a scan procedure which monitors the number of positive and negative characteristics is used. Clearly the algorithmic structure changes when one of the characteristic directions passes

through zero. From Eq.(4.12) the control function becomes

$$u(1 - \frac{u^2 b}{gA}) = \Gamma \quad (4.16)$$

If Γ changes sign then the algorithmic structure has changed, and the new characteristic directions are then calculated from Eq.(4.13).

4.2 Extended Implicit Solution for Well-Mixed Flows

Equation System

The three relevant equations for well-mixed flows are Eqs.(4.2), (4.3) and (2.10) which are the continuity, dynamical and solute continuity equations respectively. Writing these equations down again we have

$$\rho b \frac{\partial h}{\partial t} + \rho \frac{\partial Q}{\partial x} + Q \frac{\partial \rho}{\partial x} + hb \frac{\partial \rho}{\partial t} - q_i \rho_i = 0 \quad (4.17)$$

$$\begin{aligned} \frac{\partial Q}{\partial t} - (u^2 B - gA) \frac{\partial h}{\partial x} - 2ub \frac{\partial h}{\partial t} + \left(\frac{Q}{\rho} - \frac{2uhb}{\rho} \right) \frac{\partial \rho}{\partial t} - \frac{Q^2}{\rho A} \frac{\partial \rho}{\partial x} \\ + \frac{gQ|Q|}{C^2 AR} + (2u - v \cos \phi) \frac{q_i \rho_i}{\rho} - \frac{k_a \rho_a u_a |u_a| b}{\rho} = 0 \end{aligned} \quad (4.18)$$

where $A = B'h$ and $\frac{\partial A}{\partial x} = (B' + h \frac{dB'}{dh}) \frac{\partial h}{\partial x} = B \frac{\partial h}{\partial x}$

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} - D \frac{\partial^2 \rho}{\partial x^2} + \frac{q_i}{A} (\rho - \rho_i) = 0 \quad (4.19)$$

B' and B are called mathematical and dynamical breadths respectively.

Implicit Difference Schemes

The difference schemes corresponding to Eqs. (4.17) and (4.18) are:

$$\begin{aligned}
 & \rho b \left(\frac{h_j^{n+1} - h_j^n}{\Delta t} \right) + \frac{\rho}{2} \left(\frac{Q_{j+2}^{n+1} - Q_{j-2}^{n+1}}{4\Delta x} + \frac{Q_{j+2}^n - Q_{j-2}^n}{4\Delta x} \right) \\
 & + \frac{Q}{2} \left(\frac{\rho_{j+1}^{n+1} - \rho_{j-1}^{n+1}}{2\Delta x} + \frac{\rho_{j+1}^n - \rho_{j-1}^n}{2\Delta x} \right) \\
 & + \frac{hb}{2} \left(\frac{\rho_{j+1}^{n+1} - \rho_{j+1}^n}{\Delta t} + \frac{\rho_{j-1}^{n+1} - \rho_{j-1}^n}{\Delta t} \right) = q_i \rho_i \quad (4.20)
 \end{aligned}$$

$$\begin{aligned}
 & \left(\frac{Q_j^{n+1} - Q_j^n}{\Delta t} \right) - \left(\frac{u^2 B}{2} - \frac{gA}{2} \right) \left(\frac{h_{j+2}^{n+1} - h_{j-2}^{n+1}}{4\Delta x} + \frac{h_{j+2}^n - h_{j-2}^n}{4\Delta x} \right) \\
 & - ub \left(\frac{h_{j+2}^{n+1} - h_{j+2}^n}{\Delta t} + \frac{h_{j-2}^{n+1} - h_{j-2}^n}{\Delta t} \right) \\
 & + \left(\frac{Q}{2\rho} - \frac{uhb}{\rho} \right) \left(\frac{\rho_{j+1}^{n+1} - \rho_{j+1}^n}{\Delta t} + \frac{\rho_{j-1}^{n+1} - \rho_{j-1}^n}{\Delta t} \right) \\
 & - \frac{Q^2}{2\rho A} \left(\frac{\rho_{j+1}^{n+1} - \rho_{j+1}^n}{2\Delta x} + \frac{\rho_{j-1}^n - \rho_{j-1}^n}{2\Delta x} \right) + Q_j^{n+1} \left(\frac{|Q_j^n|g}{C^2 AR} + \frac{q_i \rho_i}{A\rho} \right)
 \end{aligned}$$

$$+ Q_j^n \frac{q_i \rho_i}{A \rho} - \frac{v \cos \phi q_i \rho_i}{\rho} - \frac{k_a \rho_a u_a |u_a| b}{\rho} = 0 \quad (4.21)$$

Eqs. (4.20) and (4.21) are written in the pentadiagonal forms

$$A_j Q_{j+2}^{n+1} + B_j \rho_{j+1}^{n+1} + C_j h_j^{n+1} + D_j \rho_{j-1}^{n+1} + E_j Q_{j-2}^{n+1} = F_j \quad (4.22)$$

$$A_j^* h_{j+2}^{n+1} + B_j^* \rho_{j+1}^{n+1} + C_j^* \rho_j^{n+1} + D_j^* \rho_{j-1}^{n+1} + E_j^* h_{j-2}^{n+1} = F_j^* \quad (4.23)$$

where

$$A_j = \frac{\rho}{8\Delta x}; \quad B_j = \frac{Q}{4\Delta x} + \frac{bh}{2\Delta t}; \quad C_j = \frac{\rho b}{\Delta t};$$

$$D_j = -\frac{Q}{4\Delta x} + \frac{bh}{2\Delta t}; \quad E_j = -\frac{\rho}{8\Delta x};$$

$$F_j = E_j Q_{j+2}^n + C_j h_j^n + A_j Q_{j-2}^n + D_j \rho_{j+1}^n + B_j \rho_{j-1}^n - q_i \rho_i$$

$$A_j^* = -\frac{u^2 B}{8\Delta x} + \frac{gA}{8\Delta x} - \frac{ub}{\Delta t};$$

$$B_j^* = \frac{Q}{2\rho\Delta t} - \frac{ubh}{\rho\Delta t} - \frac{Q^2}{4\rho A\Delta x};$$

$$C_j^* = \frac{1}{\Delta t} + \frac{|Q_j^n|g}{C^2 AR} + \frac{q_i \rho_i}{\rho A};$$

$$D_j^* = \frac{Q}{2\rho\Delta t} - \frac{ubh}{\rho\Delta t} + \frac{Q^2}{4\rho A\Delta x};$$

$$E_j^* = \frac{u^2 B}{8\Delta x} - \frac{gA}{8\Delta x} - \frac{ub}{\Delta t} ;$$

$$F_j^* = E_j^{*h} \rho_{j+2}^n + D_j^{*\rho} \rho_{j+1}^n + \left(\frac{1}{\Delta t} - \frac{q_i \rho_i}{\rho A} \right) Q_j^n + B_j^{*\rho} \rho_{j-1}^n \\ + A_j^{*h} \rho_{j-2}^n + \frac{v \cos \phi q_i \rho_i}{\rho} + \frac{k_a \rho_a u_a |u_a| b}{\rho} ;$$

For Eq.(4.19) the following difference scheme is used:

$$\frac{1}{4} \left(\frac{\rho_{j+2}^{n+1} - \rho_{j+2}^n}{\Delta t} \right) + \frac{1}{2} \left(\frac{\rho_j^{n+1} - \rho_j^n}{\Delta t} \right) + \frac{1}{4} \left(\frac{\rho_{j-2}^{n+1} - \rho_{j-2}^n}{\Delta t} \right) \\ + \frac{u}{2} \left(\frac{\rho_{j+2}^{n+1} - \rho_{j-2}^{n+1}}{4\Delta x} + \frac{\rho_{j+2}^n - \rho_{j-2}^n}{4\Delta x} \right) \\ - \frac{D}{2} \left(\frac{\rho_{j+2}^{n+1} - 2\rho_j^{n+1} + \rho_{j-2}^{n+1}}{4\Delta x^2} + \frac{\rho_{j+2}^n - 2\rho_j^n + \rho_{j-2}^n}{4\Delta x^2} \right) \\ + \frac{q_i}{2A} \left(\rho_j^{n+1} + \rho_j^n \right) - \frac{q_i \rho_i}{A} = 0 \quad (4.24)$$

Eq.(4.24) is written in the tridiagonal form

$$A_j^{**} \rho_{j+2}^{n+1} + B_j^{**} \rho_j^{n+1} + C_j^{**} \rho_{j-2}^{n+1} = D_j^{**} \quad (4.25)$$

where

$$A_j^{**} = \frac{1}{2\Delta t} + \frac{u}{4\Delta x} - \frac{D}{4\Delta x^2}$$

$$B_j^{**} = \frac{1}{\Delta t} + \frac{D}{2\Delta x^2} + \frac{q_i}{2A} ;$$

$$C_j^{**} = \frac{1}{2\Delta t} - \frac{u}{4\Delta x} - \frac{D}{4\Delta x^2}$$

$$D_j^{**} = \left(\frac{1}{2\Delta t} - \frac{u}{4\Delta x} + \frac{D}{4\Delta x^2} \right) \rho_{j+2}^n + \left(\frac{1}{\Delta t} - \frac{D}{2\Delta x^2} - \frac{q_i}{2A} \right) \rho_j^n \\ + \left(\frac{1}{2\Delta t} + \frac{u}{4\Delta x} + \frac{D}{4\Delta x^2} \right) \rho_{j-2}^n + \frac{q_i \rho_i}{A} ;$$

It can be shown (Section 3.7), that Eqs.(4.20) and (4.21) are unconditionally consistent with Eqs.(4.17) and (4.18) to truncation error $O(\Delta x^2, \Delta t^2)$ and have almost zero amplification error for all wave numbers, so that they are linearly unconditionally stable. Also Eq.(4.24) has been shown in Section 3.7 to be unconditionally consistent with Eq.(4.19) to truncation error $O(\Delta x^2, \Delta t^2)$, while its amplification factors tend to unity as dispersion effects tend to zero.

Algorithmic Structures

In well-mixed estuaries the flow conditions are always subcritical therefore as shown in Section 4.1, the dynamical equation system provides one positive and one negative characteristic. At every grid point in time and space there will therefore be two negative characteristics, $\dot{x} = u - (gA/b)^{1/2}$ and $\dot{x} = -\infty$, the former transmitting information concerning the level or discharge convected and the latter concerning the density diffused. Likewise there will be two positive

characteristics, $\dot{x} = u + (gA/b)^{1/2}$ and $\dot{x} = \infty$, the former again transmitting information concerning the level or discharge convected and the latter concerning the density diffused. As well as these four characteristics there is one more, $\dot{x} = u$ associated with the convection of the density. Since the direction of this characteristic depends upon the flow direction, a generalized algorithm must be introduced such that when the flow direction changes so does the algorithm.

Consider Eqs.(4.22), (4.23) and (4.25) on the grid of Fig.4.1 and the algebraic relations:

$$h_{j-2}^{n+1} = \alpha_j \rho_{j-1}^{n+1} + \beta_j Q_j^{n+1} + \delta_j \quad (4.26)$$

$$Q_{j-2}^{n+1} = \epsilon_j \rho_{j-1}^{n+1} + \zeta_j h_j^{n+1} + \theta_j \quad (4.27)$$

$$\rho_{j-2}^{n+1} = \alpha_j^* h_{j-1}^{n+1} + \beta_j^* \rho_j^{n+1} + \delta_j^* \quad (4.28)$$

$$\rho_{j-2}^{n+1} = \epsilon_j^* Q_{j-1}^{n+1} + \zeta_j^* \rho_j^{n+1} + \theta_j^* \quad (4.29)$$

As is indicated in Fig.4.1 by the double circles, two-point boundary data are required at both ends of the grid, one data point being an elevation or discharge and the other a density. However, if the flow direction is positive at a grid point, no data concerning the density being convected can enter that grid point from the right. Thus, in the generalized equations, Eqs.(4.28) and (4.29), the recurrence coefficients, α_j^* and ϵ_j^* , must be set to zero at that point.

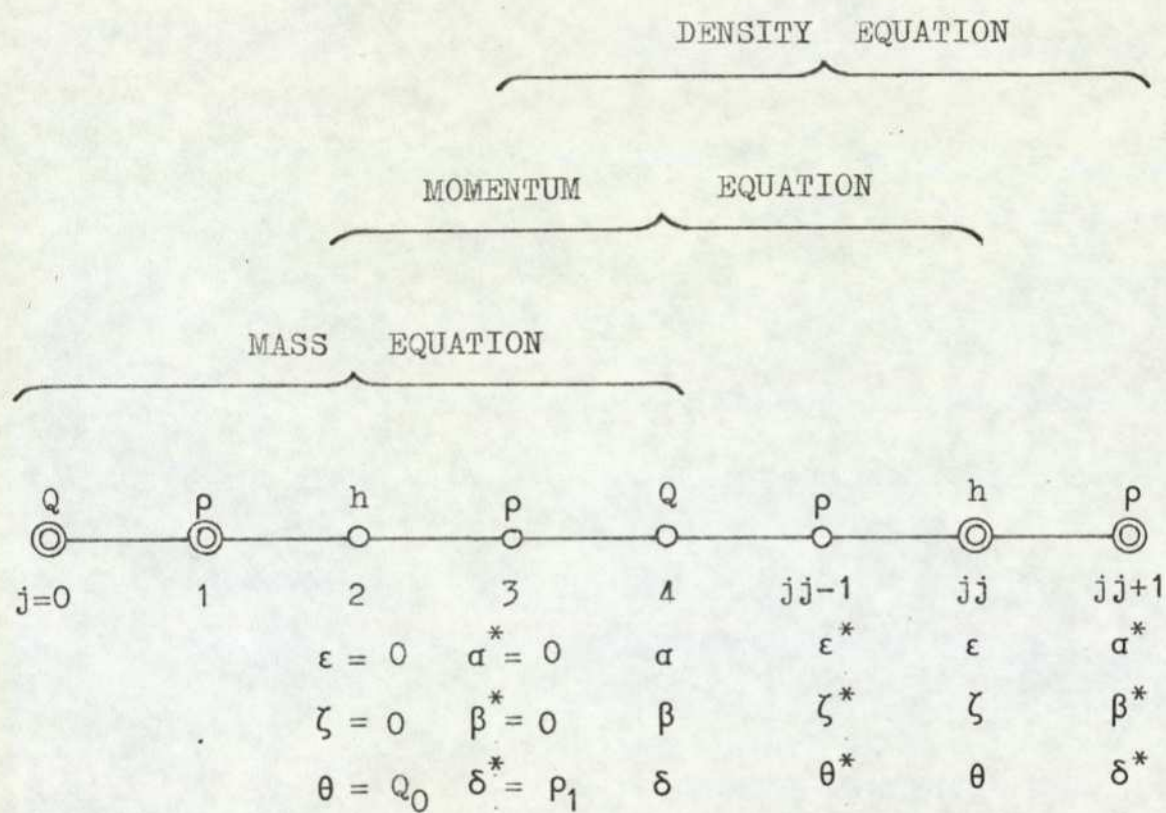


FIG. 4.1 COMPUTATION GRID FOR WELL-MIXED FLOWS

Recurrence Relations

By substituting Eqs.(4.26), (4.27), (4.28) and (4.29) into Eqs.(4.22), (4.23) and (4.25), recurrence relations can be built between the coefficients of Eqs.(4.26), (4.27), (4.28) and (4.29) and from these a double sweep algorithm can be constructed.

From Eq.(4.23)

$$A_j^* h_{j+2} + B_j^* \rho_{j+1} + C_j^* Q_j + D_j^* \rho_{j-1} + E_j^* h_{j-2} = F_j^*$$

substitute Eq.(4.26) for h_{j-2}

$$A_j^* h_{j+2} + B_j^* \rho_{j+1} + C_j^* Q_j + D_j^* \rho_{j-1} + E_j^* (\alpha_j \rho_{j-1} + \beta_j Q_j + \delta_j) = F_j^*$$

substitute Eq.(4.29) for ρ_{j-1}

$$A_j^* h_{j+2} + B_j^* \rho_{j+1} + Q_j (C_j^* + E_j^* \beta_j) + (D_j^* + E_j^* \alpha_j)$$

$$(\zeta_{j+1}^* Q_j + \zeta_{j+1}^* \rho_{j+1} + \theta_{j+1}^*) = F_j^* - E_j^* \delta_j$$

$$Q_j [C_j^* + E_j^* \beta_j + (D_j^* + E_j^* \alpha_j) \zeta_{j+1}^*] = \rho_{j+1} [-B_j^* - (D_j^* + E_j^* \alpha_j) \zeta_{j+1}^*]$$

$$- h_{j+2} A_j^* + [F_j^* - E_j^* \delta_j - (D_j^* + E_j^* \alpha_j) \theta_{j+1}^*]$$

comparing with Eq.(4.27) the recurrence relations at the h sections are

$$\begin{aligned}\epsilon_{j+2} &= \left[-B_j^* - (D_j^* + E_j^* \alpha_j) \zeta_{j+1}^* \right] / X ; \quad \zeta_{j+2} = -A_j^* / X ; \\ \theta_{j+2} &= \left[F_j^* - E_j^* \delta_j - (D_j^* + E_j^* \alpha_j) \theta_{j+1}^* \right] / X ;\end{aligned}\quad (4.30)$$

in which $X = C_j^* + E_j^* \beta_j + (D_j^* + E_j^* \alpha_j) \epsilon_{j+1}^*$ or if the

discharge is positive $X = (C_j^* + E_j^* \beta_j)$.

From Eq.(4.22)

$$A_j Q_{j+2} + B_j \rho_{j+1} + C_j h_j + D_j \rho_{j-1} + E_j Q_{j-2} = F_j$$

substitute Eq.(4.27) for Q_{j-2}

$$\begin{aligned}A_j Q_{j+2} + B_j \rho_{j+1} + C_j h_j + D_j \rho_{j-1} \\ + E_j (\epsilon_j \rho_{j-1} + \zeta_j h_j + \theta_j) = F_j\end{aligned}$$

substitute Eq.(4.28) for ρ_{j-1}

$$\begin{aligned}A_j Q_{j+2} + B_j \rho_{j+1} + (C_j + E_j \zeta_j) h_j \\ + (D_j + E_j \epsilon_j) (\alpha_{j+1}^* h_j + \beta_{j+1}^* \rho_{j+1} + \delta_{j+1}^*) = F_j - E_j \theta_j\end{aligned}$$

$$\begin{aligned}
& h_j [C_j + E_j \zeta_j + \alpha_{j+1}^* (D_j + E_j \varepsilon_j)] \\
& = \rho_{j+1} [-B_j - (D_j + E_j \varepsilon_j) \beta_{j+1}^*] - A_j Q_{j+2} \\
& + [F_j - E_j \theta_j - (D_j + E_j \varepsilon_j) \delta_{j+1}^*]
\end{aligned}$$

comparing with Eq. (4.26)

$$\begin{aligned}
\alpha_{j+2} &= [-B_j - (D_j + E_j \varepsilon_j) \beta_{j+1}^*] / X ; \quad \beta_{j+2} = -A_j / X ; \\
\delta_{j+2} &= [F_j - E_j \theta_j - (D_j + E_j \varepsilon_j) \delta_{j+1}^*] / X \quad (4.31)
\end{aligned}$$

in which $X = C_j + E_j \zeta_j + \alpha_{j+1}^* (D_j + E_j \varepsilon_j)$ or if the

discharge is positive $X = (C_j + E_j \zeta_j)$.

From Eqs. (4.26) and (4.28)

$$\rho_{j-2} = \alpha_j^* (\alpha_{j+1} \rho_j + \beta_{j+1} Q_{j+1} + \delta_{j+1}) + \beta_j^* \rho_j + \delta_j^*$$

substitute for ρ_{j-2} from Eq. (4.25)

$$\begin{aligned}
& - (A_j^{**} \rho_{j+2} + B_j^{**} \rho_j - D_j^{**}) \frac{1}{C_j^{**}} \\
& = \alpha_j^* (\alpha_{j+1} \rho_j + \beta_{j+1} Q_{j+1} + \delta_{j+1}) + \beta_j^* \rho_j + \delta_j^*
\end{aligned}$$

$$\rho_j \left(-\frac{B_j^{**}}{C_j^{**}} - \alpha_j^* \alpha_{j+1} - \beta_j^* \right) = Q_{j+1} (\alpha_j^* \beta_{j+1})$$

$$+ \frac{A_j^{**}}{C_j^{**}} \rho_{j+2} + (\alpha_j^* \delta_{j+1} + \delta_j^* - \frac{D_j^{**}}{C_j^{**}})$$

comparing with Eq. (4.29)

$$\epsilon_{j+2}^* = \alpha_j^* \beta_{j+1} / X ; \quad \zeta_{j+2}^* = \left(\frac{A_j^{**}}{C_j^{**}} \right) / X ;$$

$$\theta_{j+2}^* = (\alpha_j^* \delta_{j+1} + \delta_j^* - \frac{D_j^{**}}{C_j^{**}}) / X ; \quad (4.32)$$

in which $X = \left(-\frac{B_j^{**}}{C_j^{**}} - \alpha_j^* \alpha_{j+1} - \beta_j^* \right)$ or if the discharge

is positive

$$\epsilon_{j+2}^* = 0 ; \quad \zeta_{j+2}^* = -A_j^{**} / (B_j^{**} + C_j^{**} \beta_j^*) ;$$

$$\theta_{j+2}^* = (D_j^{**} - C_j^{**} \delta_j^*) / (B_j^{**} + C_j^{**} \beta_j^*) ; \quad (4.33)$$

From Eqs. (4.27) and (4.29)

$$\rho_{j-2} = \epsilon_j^* (\epsilon_{j+1} \rho_j + \zeta_{j+1} h_{j+1} + \theta_{j+1}) + \zeta_j^* \rho_j + \theta_j^*$$

substitute for ρ_{j-2} from Eq. (4.25)

$$- (A_j^{**} \rho_{j+2} + B_j^{**} \rho_j - D_j^{**}) \frac{1}{C_j^{**}}$$

$$= \epsilon_j^* (\epsilon_{j+1} \rho_j + \zeta_{j+1} h_{j+1} + \theta_{j+1}) + \zeta_j^* \rho_j + \theta_j^*$$

$$\rho_j \left(-\frac{B_j^{**}}{C_j^{**}} - \epsilon_j^* \epsilon_{j+1} - \zeta_j^* \right) = h_{j+1} (\epsilon_j^* \zeta_{j+1})$$

$$+ \frac{A_j^{**}}{C_j^{**}} \rho_{j+2} + (\epsilon_j^* \theta_{j+1} + \theta_j^* - \frac{D_j^{**}}{C_j^{**}})$$

comparing with Eq. (4.28)

$$\alpha_{j+2}^* = \epsilon_j^* \zeta_{j+1} / X ; \quad \beta_{j+2}^* = \left(\frac{A_j^{**}}{C_j^{**}} \right) / X ;$$

$$\delta_{j+2}^* = (\epsilon_j^* \theta_{j+1} + \theta_j^* - \frac{D_j^{**}}{C_j^{**}}) / X ; \quad (4.34)$$

in which $X = \left(-\frac{B_j^{**}}{C_j^{**}} - \epsilon_j^* \epsilon_{j+1} - \zeta_j^* \right)$, or if the discharge

is positive

$$\alpha_{j+2}^* = 0 ; \quad \beta_{j+2}^* = A_j^{**} / (B_j^{**} + C_j^{**} \zeta_j^*) ;$$

$$\delta_{j+2}^* = (D_j^{**} - C_j^{**} \theta_j^*) / (B_j^{**} + C_j^{**} \zeta_j^*) ; \quad (4.35)$$

The recurrence coefficients are calculated from the left-hand side of the grid by setting the boundary data there to the coefficients at grid sections 2 and 3, as shown in Fig.4.1. At the seaward end of the grid the density is a function of the upland discharge and therefore cannot be used directly as a boundary condition. It is found, however, that the density gradient at the mouth of an estuary is independent of the upland discharge, and this is used instead. If the density gradient here is denoted by P_{jj} , in which

$P_{jj} = (\rho_{jj+1}^{n+1} - \rho_{jj-1}^{n+1}) / (2\Delta x)$ then from Eq. (4.28)

$$\rho_{jj+1}^{n+1} = (\alpha_{jj+1} h_{jj}^{n+1} + 2\Delta x P_{jj} + \delta_{jj+1}) / (1 - \beta_{jj+1}) \quad (4.36)$$

Using this boundary data plus the vertical tide at section jj , Eqs. (4.26), (4.27), (4.28) and (4.29) are used alternately to calculate the dependent variables. For second-order accuracy, this computation is executed twice in order to centre the equation coefficients.

4.3 Characteristic Structure of Two-Layer Stratified Equations

The characteristic directions for the two-layer stratified flow equations will be derived, by matrix analysis, and their use as data or algorithmic structure controls in stratified models explained. It is necessary first to expand and rearrange the Eqs. (2.11) to (2.14) into a form suitable for matrix analysis.

From Eq. (2.11)

$$\begin{aligned} b_2 \rho_2 \frac{\partial h_2}{\partial t} + b_2 h_2 \frac{\partial \rho_2}{\partial t} + \rho_2 \frac{\partial Q_2}{\partial x} + Q_2 \frac{\partial \rho_2}{\partial x} - b_2 (\rho_1 q_1 - \rho_2 q_2) &= 0 \\ b_2 \frac{\partial h_2}{\partial t} + \frac{\partial Q_2}{\partial x} &= \frac{b_2}{\rho_2} (\rho_1 q_1 - \rho_2 q_2) - \frac{b_2 h_2}{\rho_2} \frac{\partial \rho_2}{\partial t} - \frac{Q_2}{\rho_2} \frac{\partial \rho_2}{\partial x} \\ &= a' \end{aligned} \quad (4.37)$$

From Eq. (2.12)

$$\begin{aligned}
 & b_1 \rho_1 \frac{\partial h_1}{\partial t} + b_1 h_1 \frac{\partial \rho_1}{\partial t} - b_2 \rho_1 \frac{\partial h_2}{\partial t} - b_2 h_2 \frac{\partial \rho_1}{\partial t} + \rho_1 \frac{\partial Q_1}{\partial x} \\
 & + Q_1 \frac{\partial \rho_1}{\partial x} - b_2 (\rho_2 q_2 - \rho_1 q_1) - q_1 \rho_1 = 0
 \end{aligned} \quad (4.38)$$

Substituting for $\frac{\partial h_2}{\partial t}$ from Eq. (4.37) into Eq. (4.38)

$$\begin{aligned}
 & b_1 \frac{\partial h_1}{\partial t} + \frac{\partial Q_2}{\partial x} + \frac{\partial Q_1}{\partial x} = \frac{b_2}{\rho_1} (\rho_2 q_2 - \rho_1 q_1) (1-\epsilon) + \frac{q_1 \rho_1}{\rho_1} \\
 & - \frac{b_2 h_2}{\rho_2} \frac{\partial \rho_2}{\partial t} + \frac{1}{\rho_1} (b_2 h_2 - b_1 h_1) \frac{\partial \rho_1}{\partial t} - \frac{Q_1}{\rho_1} \frac{\partial \rho_1}{\partial x} \\
 & - \frac{Q_2}{\rho_2} \frac{\partial \rho_2}{\partial x} = b'
 \end{aligned} \quad (4.39)$$

From Eq. (2.13)

$$\begin{aligned}
 & \rho_2 \frac{\partial Q_2}{\partial t} + Q_2 \frac{\partial \rho_2}{\partial t} + 2\rho_2 u_2 \frac{\partial Q_2}{\partial x} + \frac{Q_2^2}{A_2} \frac{\partial \rho_2}{\partial x} - u_2^2 \rho_2 \frac{\partial A_2}{\partial x} \\
 & + \rho_2 g (1-\epsilon) A_2 \frac{\partial h_2}{\partial x} + \rho_2 g A_2 \frac{\partial h_1}{\partial x} - b_2 (\rho_1 q_1 u_1 - \rho_2 q_2 u_2) \\
 & + \tau b_2 + \frac{\rho_2 g Q_2 |Q_2|}{C^2 A_2 R_2} = 0
 \end{aligned} \quad (4.40)$$

let $A_2 = B_2' h_2$ and $(A_1 + A_2) = B_1' h_1$ where B_1' and B_2' are called mathematical breadths, then

$$\frac{\partial A_2}{\partial x} = (B_2' + h_2 \frac{dB_2'}{dh_2}) \frac{\partial h_2}{\partial x} = B_2 \frac{\partial h_2}{\partial x} \quad (4.41)$$

$$\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial x} = (B_1' + h_1 \frac{dB_1'}{dh_1}) \frac{\partial h_1}{\partial x} = B_1 \frac{\partial h_1}{\partial x} \quad (4.42)$$

where B_1 and B_2 are called the dynamical breadths of the channel.

Substituting Eq. (4.41) into Eq. (4.40) to remove $\frac{\partial A_2}{\partial x}$

$$\begin{aligned} & \frac{\partial Q_2}{\partial t} + 2u_2 \frac{\partial Q_2}{\partial x} + g\epsilon A_2 \frac{\partial h_1}{\partial x} + \frac{\partial h_2}{\partial x} [g(1-\epsilon)A_2 - u_2^2 B_2] \\ &= -\frac{Q_2}{\rho_2} \frac{\partial \rho_2}{\partial t} - \frac{Q_2^2}{\rho_2 A_2} \frac{\partial \rho_2}{\partial x} + b_2 (\epsilon q_1 u_1 - q_2 u_2) \\ & - \frac{gQ_2 |Q_2|}{C^2 A_2 R_2} - \frac{\tau b_2}{\rho_2} = c' \end{aligned} \quad (4.43)$$

$$\text{let } [g(1-\epsilon)A_2 - u_2^2 B_2] = e'$$

From Eq. (2.14)

$$\begin{aligned} & \rho_1 \frac{\partial Q_1}{\partial t} + Q_1 \frac{\partial \rho_1}{\partial t} + 2\rho_1 u_1 \frac{\partial Q_1}{\partial x} + \frac{Q_1^2}{A_1} \frac{\partial \rho_1}{\partial x} - u_1^2 \rho_1 (B_1 \frac{\partial h_1}{\partial x} - B_2 \frac{\partial h_2}{\partial x}) \\ & + \rho_1 g A_1 \frac{\partial h_1}{\partial x} - b_2 (\rho_2 q_2 u_2 - \rho_1 q_1 u_1) + \frac{\rho_1 g Q_1 |Q_1|}{C^2 A_1 R_1} \\ & - \tau b_2 - v_1 \cos \phi q_i \rho_i = 0 \end{aligned}$$

$$\begin{aligned} & \frac{\partial Q_1}{\partial t} + 2u_1 \frac{\partial Q_1}{\partial x} + \frac{\partial h_1}{\partial x} (gA_1 - u_1^2 B_1) + u_1^2 B_2 \frac{\partial h_2}{\partial x} \\ &= -\frac{Q_1}{\rho_1} \frac{\partial \rho_1}{\partial t} - \frac{Q_1^2}{\rho_1 A_1} \frac{\partial \rho_1}{\partial x} + b_2 (\frac{q_2 u_2}{\epsilon} - q_1 u_1) \\ & - \frac{gQ_1 |Q_1|}{C^2 A_1 R_1} + \frac{\tau b_2}{\rho_1} + \frac{v_1 \cos \phi q_i \rho_i}{\rho_1} = d' \end{aligned} \quad (4.44)$$

$$\text{let } (gA_1 - u_1^2 B_1) = f'$$

Hence, the equations for the two-layer stratified flow can be rewritten from Eqs.(4.37), (4.39), (4.43) and (4.44) as

Continuity Equation (Lower Layer)

$$b_2 \frac{\partial h_2}{\partial t} + \frac{\partial Q_2}{\partial x} = a' \quad (4.45)$$

Continuity Equation (Upper Layer)

$$b_1 \frac{\partial h_1}{\partial t} + \frac{\partial Q_1}{\partial x} + \frac{\partial Q_2}{\partial x} = b' \quad (4.46)$$

Dynamical Equation (Lower Layer)

$$\frac{\partial Q_2}{\partial t} + 2u_2 \frac{\partial Q_2}{\partial x} + g\epsilon A_2 \frac{\partial h_1}{\partial x} + e' \frac{\partial h_2}{\partial x} = c' \quad (4.47)$$

Dynamical Equation (Upper Layer)

$$\frac{\partial Q_1}{\partial t} + 2u_1 \frac{\partial Q_1}{\partial x} + f' \frac{\partial h_1}{\partial x} + u_1^2 B_2 \frac{\partial h_2}{\partial x} = d' \quad (4.48)$$

If level h in Eqs.(4.45) to (4.48) is replaced by flow cross-sectional area A , then since

$$\frac{\partial A_2}{\partial t} = b_2 \frac{\partial h_2}{\partial t} ; \quad \frac{\partial A_1}{\partial t} = b_1 \frac{\partial h_1}{\partial t} - b_2 \frac{\partial h_2}{\partial t} \quad \text{and}$$

$$\frac{\partial A_2}{\partial x} = b_2 \frac{\partial h_2}{\partial x} + b_2 I ; \quad \frac{\partial A_1}{\partial x} = b_1 \frac{\partial h_1}{\partial x} - b_2 \frac{\partial h_2}{\partial x}$$

Eqs.(4.45) to (4.48) become:

Continuity Equation (Lower Layer)

$$\frac{\partial A_2}{\partial t} + \frac{\partial Q_2}{\partial x} + \frac{b_2 h_2}{\rho_2} \frac{\partial \rho_2}{\partial t} + \frac{\partial Q_2}{\rho_2} \frac{\partial \rho_2}{\partial x} = b_2 (\epsilon q_1 - q_2) = m' \quad (4.49)$$

Continuity Equation (Upper Layer)

$$\begin{aligned} \frac{\partial A_1}{\partial t} + \frac{\partial Q_1}{\partial x} + \frac{Q_1}{\rho_1} \frac{\partial \rho_1}{\partial x} - \frac{1}{\rho_1} (b_2 h_2 - b_1 h_1) \frac{\partial \rho_1}{\partial t} \\ = b_2 \left(\frac{q_2}{\epsilon} - q_1 \right) + \frac{q_1 \rho_1}{\rho_1} = n' \end{aligned} \quad (4.50)$$

and let $(b_2 h_2 - b_1 h_1) = k''$.

Dynamical Equation (Lower Layer)

$$\begin{aligned} \frac{\partial Q_2}{\partial t} + 2u_2 \frac{\partial Q_2}{\partial x} + \left[-u_2^2 + \frac{g(1-\epsilon)A_2}{b_2} + \frac{g\epsilon A_2}{b_1} \right] \frac{\partial A_2}{\partial x} \\ + \frac{g\epsilon A_2}{b_1} \frac{\partial A_1}{\partial x} + \frac{Q_2}{\rho_2} \frac{\partial \rho_2}{\partial t} + \frac{Q_2^2}{\rho_2 A_2} \frac{\partial \rho_2}{\partial x} \\ = b_2 (\epsilon q_1 u_1 - q_2 u_2) - \frac{gQ_2 |Q_2|}{C^2 A_2 R_2} - \frac{\tau b_2}{\rho_2} + gA_2 I = s' \end{aligned} \quad (4.51)$$

and let $\left[-u_2^2 + \frac{g(1-\epsilon)A_2}{b_2} + \frac{g\epsilon A_2}{b_1} \right] = e''$

Dynamical Equation (Upper Layer)

$$\begin{aligned} \frac{\partial Q_1}{\partial t} + 2u_1 \frac{\partial Q_1}{\partial x} + \left(\frac{gA_1}{b_1} - u_1^2 \right) \frac{\partial A_1}{\partial x} + \frac{gA_1}{b_1} \frac{\partial A_2}{\partial x} \\ + \frac{Q_1}{\rho_1} \frac{\partial \rho_1}{\partial t} + \frac{Q_1^2}{\rho_1 A_1} \frac{\partial \rho_1}{\partial x} = b_2 \left(\frac{q_2 u_2}{\epsilon} - q_1 u_1 \right) + gA_1 I \end{aligned}$$

$$-\frac{gQ_1|Q_1|}{C^2A_1R_1} + \frac{v_1 \cos\phi q_i\rho_i}{\rho_1} + \frac{\tau b_2}{\rho_1} = y' \quad (4.52)$$

and let $(\frac{gA_1}{b_1} - u_1^2) = f''$.

Equations (4.49) to (4.52) together with the equations for the mass convection of the solute Eqs.(2.15) and (2.16) with diffusion coefficients set to zero , that is

$$\frac{\partial \rho_2}{\partial t} + u_2 \frac{\partial \rho_2}{\partial x} = \frac{b_2 q_2}{A_2} (\rho_1 - \rho_2) = r' \quad (4.53)$$

$$\frac{\partial \rho_1}{\partial t} + u_1 \frac{\partial \rho_1}{\partial x} = \frac{b_2 q_2}{A_1} (\rho_2 - \rho_1) + \frac{q_i}{A_1} (\rho_i - \rho_1) = t' \quad (4.54)$$

plus the following six equations of variation

$$\frac{\partial Q_1}{\partial x} dx + \frac{\partial Q_1}{\partial t} dt = dQ_1 \quad (4.55)$$

$$\frac{\partial A_1}{\partial x} dx + \frac{\partial A_1}{\partial t} dt = dA_1 \quad (4.56)$$

$$\frac{\partial Q_2}{\partial x} dx + \frac{\partial Q_2}{\partial t} dt = dQ_2 \quad (4.57)$$

$$\frac{\partial A_2}{\partial x} dx + \frac{\partial A_2}{\partial t} dt = dA_2 \quad (4.58)$$

$$\frac{\partial \rho_1}{\partial x} dx + \frac{\partial \rho_1}{\partial t} dt = d\rho_1 \quad (4.59)$$

$$\frac{\partial \rho_2}{\partial x} dx + \frac{\partial \rho_2}{\partial t} dt = d\rho_2 \quad (4.60)$$

can be written in the following matrix as:

$$\begin{bmatrix}
 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{Q_1}{\rho_1} & -\frac{k''}{\rho_1} & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \frac{Q_2}{\rho_2} & \frac{b_2 h_2}{\rho_2} \\
 2u_1 & 1 & f'' & 0 & 0 & 0 & \frac{gA_1}{b_1} & 0 & \frac{Q_1^2}{\rho_1 A_1} & \frac{Q_1}{\rho_1} & 0 & 0 \\
 0 & 0 & \frac{g\epsilon A_2}{b_1} & 0 & 2u_2 & 1 & e'' & 0 & 0 & 0 & \frac{Q_2^2}{\rho_2 A_2} & \frac{Q_2}{\rho_2} \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_1 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_2 & 1 \\
 dx & dt & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & dx & dt & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & dx & dt & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & dx & dt & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & dx & dt & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & dx & dt
 \end{bmatrix}
 \begin{bmatrix}
 \frac{\partial Q_1}{\partial x} \\
 \frac{\partial Q_1}{\partial t} \\
 \frac{\partial A_1}{\partial x} \\
 \frac{\partial A_1}{\partial t} \\
 \frac{\partial Q_2}{\partial x} \\
 \frac{\partial Q_2}{\partial t} \\
 \frac{\partial A_2}{\partial x} \\
 \frac{\partial A_2}{\partial t} \\
 \frac{\partial \rho_1}{\partial x} \\
 \frac{\partial \rho_1}{\partial t} \\
 \frac{\partial \rho_2}{\partial x} \\
 \frac{\partial \rho_2}{\partial t}
 \end{bmatrix}
 =
 \begin{bmatrix}
 n' \\
 m' \\
 y' \\
 s' \\
 t' \\
 r' \\
 dQ_1 \\
 dA_1 \\
 dQ_2 \\
 dA_2 \\
 d\rho_1 \\
 d\rho_2
 \end{bmatrix}
 \quad (4.61)$$

If this matrix is rearranged, and $\dot{x}$ is written for dx/dt , $\dot{Q}_1$ for dQ_1/dt , etc., it may be expressed in the form:

$$\begin{bmatrix}
 1 & 0 & -\dot{x} & 0 & 0 & 0 & 0 & 0 & \frac{Q_1}{\rho_1} & -\frac{k''}{\rho_1} & 0 & 0 & (n' - \dot{A}_1) \\
 0 & 0 & 0 & 0 & 1 & 0 & -\dot{x} & 0 & 0 & 0 & \frac{Q_2}{\rho_2} & \frac{b_2 h_2}{\rho_2} & (m' - \dot{A}_2) \\
 (2u_1 - \dot{x}) & 0 & f'' & 0 & 0 & 0 & \frac{gA_1}{b_1} & 0 & \frac{Q_1^2}{\rho_1 A_1} & \frac{Q_1}{\rho_1} & 0 & 0 & (y' - \dot{Q}_1) \\
 0 & 0 & \frac{g\epsilon A_2}{b_1} & 0 & (2u_2 - \dot{x}) & 0 & e'' & 0 & 0 & 0 & \frac{Q_2^2}{\rho_2 A_2} & \frac{Q_2}{\rho_2} & (s' - \dot{Q}_2) \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (u_1 - \dot{x}) & 0 & 0 & 0 & (t' - \dot{\rho}_1) \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (u_2 - \dot{x}) & 0 & (r' - \dot{\rho}_2) \\
 \dot{x} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dot{Q}_1 \\
 0 & 0 & \dot{x} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dot{A}_1 \\
 0 & 0 & 0 & 0 & \dot{x} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dot{Q}_2 \\
 0 & 0 & 0 & 0 & 0 & 0 & \dot{x} & 1 & 0 & 0 & 0 & 0 & \dot{A}_2 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dot{x} & 1 & 0 & 0 & \dot{\rho}_1 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dot{x} & 1 & \dot{\rho}_2
 \end{bmatrix}$$

(4.62)

The condition that the first 12x12 determinant of this matrix be zero defines the characteristics directly, this condition reducing, by excluding rows and columns containing a single unity, to

we obtain

$$\left[\dot{x}^2 (2u_1 - \dot{x}) (2u_2 - \dot{x}) + e'' \dot{x} (2u_1 - \dot{x}) + f'' \dot{x} (2u_2 - \dot{x}) + f'' e'' - \frac{g^2 \epsilon A_1 A_2}{b_1^2} \right] (u_1 - \dot{x}) (u_2 - \dot{x}) = 0$$

or

$$\left\{ \left[(2u_1 - \dot{x}) \dot{x} + f'' \right] \left[(2u_2 - \dot{x}) \dot{x} + e'' \right] - \frac{g^2 \epsilon A_1 A_2}{b_1^2} \right\}$$

$$(u_1 - \dot{x}) (u_2 - \dot{x}) = 0$$

$$\left\{ \left[(2u_1 - \dot{x}) \dot{x} + \frac{gA_1}{b_1} - u_1^2 \right] \left[(2u_2 - \dot{x}) \dot{x} + \frac{g(1-\epsilon)A_2}{b_2} + \frac{g\epsilon A_2}{b_1} - u_2^2 \right] - \frac{g^2 \epsilon A_1 A_2}{b_1^2} \right\} (u_1 - \dot{x}) (u_2 - \dot{x}) = 0$$

$$\left\{ \left[1 - \frac{(u_1 - \dot{x})^2 b_1}{gA_1} \right] \left[1 - \frac{(u_2 - \dot{x})^2 b_2}{gA_2} + \epsilon \left(\frac{b_2}{b_1} - 1 \right) \right] - \epsilon \frac{b_2}{b_1} \right\}$$

$$(u_1 - \dot{x}) (u_2 - \dot{x}) = 0 \quad (4.64)$$

Eq. (4.64) gives the characteristic directions for a two-layer fully stratified flow in a channel with vertical side walls. This equation can be simplified for a rectangular channel, where $b_1 = b_2$; $A_1 = b_1 d_1$ and $A_2 = b_2 d_2$ to

$$\left\{ \left[1 - \frac{(u_1 - \dot{x})^2}{gd_1} \right] \left[1 - \frac{(u_2 - \dot{x})^2}{gd_2} \right] - \varepsilon \right\} (u_1 - \dot{x})(u_2 - \dot{x}) = 0 \quad (4.65)$$

Eq.(4.65) indicates that there are six characteristic directions associated with a two-layer stratified flow, for the convection of the dependent variables. Four of these characteristics are associated with the convection of level and discharge (two in each layer) and the other two which are seen to be $\dot{x} = u_1$ and $\dot{x} = u_2$, are associated with the convection of the density in the upper and lower layers respectively. The characteristic directions associated with the diffusion of the solute are $\dot{x} = \pm \infty$ in each layer, since the signal velocity being the speed of sound is effectively infinite.

Since the direction of the characteristics defines the data input to the scheme we need to know when this data input or algorithmic structure changes. Clearly it changes when an $\dot{x}$ passes through zero, thus from Eq.(4.65) for rectangular channels

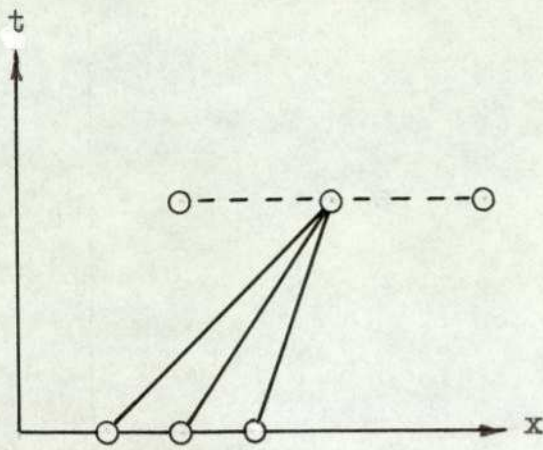
$$u_1 u_2 \left[\left(1 - \frac{u_1^2}{gd_1} \right) \left(1 - \frac{u_2^2}{gd_2} \right) - \varepsilon \right] = \Gamma \quad (4.66)$$

Hence, if Γ changes sign then the algorithmic structure has changed, Eq.(4.66) is called the control function of the equation system (4.49, 4.50, 4.51, 4.52, 4.53, 4.54).

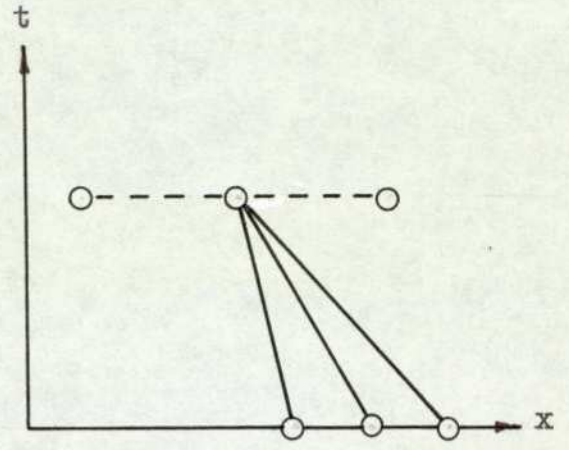
In each layer the three convective characteristics can assume one of the four algorithmic structures shown in Fig.4.2. Fig.4.2a shows a situation when the flow is supercritical flowing from left to right so that the three characteristic directions are positive. Fig.4.2b shows that the flow is supercritical and flowing from right to left. In Fig.4.2c the flow is subcritical flowing from left to right whereas in Fig.4.2d the flow is again subcritical but flowing from right to left.

Considering both layers coupled together, it is clear that there are sixteen possible algorithmic structures for a fully stratified two-layer flow. These various algorithmic structures are shown tabulated in Table 4.1, using the following abbreviations: sub for subcritical; sup for supercritical; l for left; and r for right.

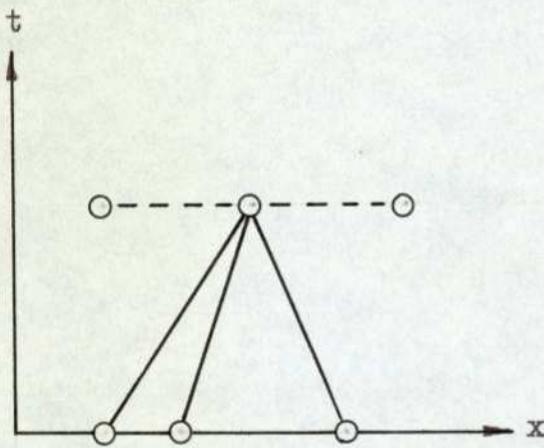
A computational system for the calculation of two-layer stratified flows requires both a scan procedure and also an eigen value procedure. The scan procedure scans all the grid points at each time level using the control function Γ to see if any of the characteristics have changed sign. If a change of characteristics is found then the values of the new characteristic speeds are calculated from the eigen value procedure and a new data-point matrix assigned to those points. Using this data-point matrix information, the computation can be directed to the correct computational procedure at each grid point in turn.



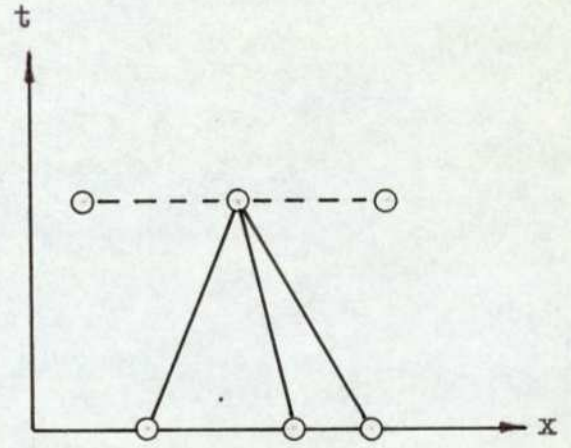
(a)



(b)



(c)



(d)

○—○ CONVECTIVE CHARACTERISTIC DIRECTIONS

○---○ DIFFUSIVE CHARACTERISTIC DIRECTIONS

FIG. 4.2 CHARACTERISTIC CONVENTION FOR EACH LAYER

DATA - POINT MATRIX (1)	LOWER LAYER (2)	UPPER LAYER (3)
3 0	sup l $\leftrightarrow$ r	sup l $\leftrightarrow$ r
3 0		
3 0	sup r $\rightarrow$ l	sup l $\leftrightarrow$ r
0 3		
3 0	sub l $\leftrightarrow$ r	sup l $\rightarrow$ r
2 1		
3 0	sub r $\rightarrow$ l	sup l $\rightarrow$ r
1 2		
0 3	sup r $\rightarrow$ l	sup r $\rightarrow$ l
0 3		
0 3	sup l $\leftrightarrow$ r	sup r $\rightarrow$ l
3 0		
0 3	sub l $\rightarrow$ r	sup r $\rightarrow$ l
2 1		
0 3	sub r $\rightarrow$ l	sup r $\rightarrow$ l
1 2		
1 2	sub r $\rightarrow$ l	sub r $\rightarrow$ l
1 2		
2 1	sub l $\rightarrow$ r	sub r $\rightarrow$ l
1 2		
0 3	sup r $\rightarrow$ l	sub r $\rightarrow$ l
1 2		
3 0	sup l $\rightarrow$ r	sub r $\rightarrow$ l
2 1		
3 0	sup l $\rightarrow$ r	sub l $\leftrightarrow$ r
2 1		
0 3	sup r $\rightarrow$ l	sub l $\leftrightarrow$ r
2 1		
2 1	sub l $\rightarrow$ r	sub l $\leftrightarrow$ r
2 1		
2 1	sub r $\rightarrow$ l	sub l $\rightarrow$ r
1 2		

TABLE 4.1 DATA STRUCTURES

The determinant given by Eq. (4.63) can be reduced to the following determinant, when the characteristics associated with the convection of the density are removed and a rectangular channel is considered.

$$\begin{bmatrix} u_1 - \dot{x} & g & 0 & g \\ d_1 & u_1 - \dot{x} & 0 & 0 \\ 0 & \epsilon g & u_2 - \dot{x} & g \\ 0 & 0 & d_2 & u_2 - \dot{x} \end{bmatrix} = 0 \quad (4.67)$$

The four characteristic directions can be found accurately using a standard eigen value procedure. However, even a simple eigen value procedure needed to solve this problem uses considerable time and computer storage, therefore it is better to use the following approximate equations if the flow conditions allow this.

$$\dot{x} = \frac{d_1 u_1 + d_2 u_2}{(d_1 + d_2)} \pm \left[g(d_1 + d_2) \right]^{\frac{1}{2}} \quad (4.68)$$

$$\dot{x} = \frac{d_1 u_2 + d_2 u_1}{(d_1 + d_2)} \pm \left[\frac{g(1-\epsilon) d_1 d_2}{(d_1 + d_2)} - \frac{(u_1 - u_2)^2 d_1 d_2}{(d_1 + d_2)^2} \right]^{\frac{1}{2}} \quad (4.69)$$

These equations are valid if u_1 and $u_2 \ll \left[g(d_1 + d_2) \right]^{\frac{1}{2}}$ and $\epsilon \approx 1$.

Eqs.(4.68) and (4.69) give the characteristic speeds of the surface and internal long waves respectively.

4.4 Extended Implicit Solution for Two-Layer Stratified Flows

It was shown in the previous section that for a coupled two-layer stratified flow there are a total of sixteen different algorithmic structures. However, if each layer is considered separately there are only four different algorithmic structures. It is therefore prudent to use a fractioned steps technique so that each layer can be calculated separately, without uncoupling the two-layer equation system.

This is achieved by splitting the centred implicit finite difference equations for the lower layer into two halves, so that a four level scheme is obtained. The second level of this scheme then calculates new values for $(h_2, Q_2 \text{ and } \rho_2)$ at time level $(n + \frac{1}{2})\Delta t$, from known values at time level $n\Delta t$. The third level calculates new values of $(h_1, Q_1 \text{ and } \rho_1)$ at time level $(n + 1)\Delta t$ and finally the fourth level calculates new values of $(h_2, Q_2 \text{ and } \rho_2)$ at time level $(n + 1)\Delta t$.

Equation System

The equation system for a two-layer stratified flow is given by Eqs.(4.45), (4.46), (4.47) and (4.48), which represents the dynamic part, plus the two mass equations for the solute

Eqs.(2.15) and (2.16). Eqs.(4.47) and (4.48) need to be rearranged before the finite difference equations can be written. Substituting Eq.(4.45) into Eq.(4.47) to remove $\frac{\partial Q_2}{\partial x}$

$$\frac{\partial Q_2}{\partial t} + g\epsilon A_2 \frac{\partial h_1}{\partial x} + e' \frac{\partial h_2}{\partial x} - 2u_2 b_2 \frac{\partial h_2}{\partial t} = c' - 2u_2 a' \quad (4.70)$$

Substitute Eq.(4.46) into Eq.(4.48) to remove $\frac{\partial Q_1}{\partial x}$

$$\frac{\partial Q_1}{\partial t} + f' \frac{\partial h_1}{\partial x} - 2u_1 b_1 \frac{\partial h_1}{\partial t} - 2u_1 \frac{\partial Q_2}{\partial x} + u_1^2 B_2 \frac{\partial h_2}{\partial x} = d' - 2u_1 b' \quad (4.71)$$

From Eqs.(4.45), (4.46), (4.70) and (4.71)

Continuity Equation (Lower Layer)

$$b_2 \frac{\partial h_2}{\partial t} + \frac{\partial Q_2}{\partial x} + \frac{b_2 h_2}{\rho_2} \frac{\partial \rho_2}{\partial t} + \frac{Q_2}{\rho_2} \frac{\partial \rho_2}{\partial x} = b_2 (\epsilon q_1 - q_2) \quad (4.72)$$

Continuity Equation (Upper Layer)

$$\begin{aligned} b_1 \frac{\partial h_1}{\partial t} + \frac{\partial Q_1}{\partial x} + \frac{\partial Q_2}{\partial x} + \frac{b_2 h_2}{\rho_2} \frac{\partial \rho_2}{\partial t} - \frac{1}{\rho_1} (b_2 h_2 - b_1 h_1) \frac{\partial \rho_1}{\partial t} \\ + \frac{Q_1}{\rho_1} \frac{\partial \rho_1}{\partial x} + \frac{Q_2}{\rho_2} \frac{\partial \rho_2}{\partial x} = b_2 \left(\frac{q_2}{\epsilon} - q_1 \right) (1 - \epsilon) + \frac{q_1 \rho_1}{\rho_1} \end{aligned} \quad (4.73)$$

Dynamical Equation (Lower Layer)

$$\begin{aligned}
 & \frac{\partial Q_2}{\partial t} + g \epsilon A_2 \frac{\partial h_1}{\partial x} + e' \frac{\partial h_2}{\partial x} - 2u_2 b_2 \frac{\partial h_2}{\partial t} + \frac{Q_2}{\rho_2} \left(1 - \frac{2b_2 h_2}{A_2}\right) \frac{\partial \rho_2}{\partial t} \\
 & - \frac{Q_2^2}{\rho_2 A_2} \frac{\partial \rho_2}{\partial x} = u_2 (b_2 q_2 - 2b_2 \epsilon q_1) + u_1 b_2 q_1 \epsilon \\
 & - \frac{g Q_2 |Q_2|}{C^2 A_2 R_2} - \frac{\tau b_2}{\rho_2}
 \end{aligned} \tag{4.74}$$

Dynamical Equation (Upper Layer)

$$\begin{aligned}
 & \frac{\partial Q_1}{\partial t} + f' \frac{\partial h_1}{\partial x} - 2u_1 b_1 \frac{\partial h_1}{\partial t} - 2u_1 \frac{\partial Q_2}{\partial x} + u_1^2 B_2 \frac{\partial h_2}{\partial x} \\
 & - \frac{Q_1^2}{\rho_1 A_1} \frac{\partial \rho_1}{\partial x} + \frac{1}{\rho_1} \left[Q_1 + 2u_1 (b_2 h_2 - b_1 h_1) \right] \frac{\partial \rho_1}{\partial t} \\
 & - \frac{2u_1 b_2 h_2}{\rho_2} \frac{\partial \rho_2}{\partial t} - \frac{2u_1 Q_2}{\rho_2} \frac{\partial \rho_2}{\partial x} = u_1 b_2 \left[q_1 (1 - 2\epsilon) \right. \\
 & \left. + 2q_2 \left(1 - \frac{1}{\epsilon}\right) - \frac{2q_i \rho_i}{b_2 \rho_1} \right] - \frac{g Q_1 |Q_1|}{C^2 A_1 R_1} + \frac{\tau b_2}{\rho_1} \\
 & + \frac{u_2 b_2 q_2}{\epsilon} + \frac{v_1 \cos \phi q_i \rho_i}{\rho_1}
 \end{aligned} \tag{4.75}$$

Implicit Difference Schemes (Dynamic)

The equation system (4.72), (4.73), (4.74) and (4.75) is written in the following finite difference forms.

From Eq. (4.72)

$$\begin{aligned}
 & b_2 \left[\frac{h_{2j}^{n+\frac{1}{2}} - h_{2j}^n}{\Delta t} + \frac{h_{2j}^{n+1} - h_{2j}^{n+\frac{1}{2}}}{\Delta t} \right] \\
 & + \left[\frac{Q_{2j+2}^{n+1} - Q_{2j-2}^{n+1}}{16\Delta x} + 2 \frac{(Q_{2j+2}^{n+\frac{1}{2}} - Q_{2j-2}^{n+\frac{1}{2}})}{16\Delta x} + \frac{Q_{2j+2}^n - Q_{2j-2}^n}{16\Delta x} \right] \\
 & + \frac{b_2 h_2}{\rho_2} \left[\frac{\rho_{2j+1}^{n+\frac{1}{2}} - \rho_{2j+1}^n}{2\Delta t} + \frac{\rho_{2j-1}^{n+\frac{1}{2}} - \rho_{2j-1}^n}{2\Delta t} \right] \\
 & + \frac{\rho_{2j+1}^{n+1} - \rho_{2j+1}^{n+\frac{1}{2}}}{2\Delta t} + \frac{\rho_{2j-1}^{n+1} - \rho_{2j-1}^{n+\frac{1}{2}}}{2\Delta t} \left] \right. \\
 & + \frac{Q_2}{\rho_2} \left[\frac{\rho_{2j+1}^{n+1} - \rho_{2j-1}^{n+1}}{8\Delta x} + 2 \frac{(\rho_{2j+1}^{n+\frac{1}{2}} - \rho_{2j-1}^{n+\frac{1}{2}})}{8\Delta x} \right. \\
 & \left. + \frac{\rho_{2j+1}^n - \rho_{2j-1}^n}{8\Delta x} \right] = 2 \left[\frac{b_2 (\epsilon q_1 - q_2)}{2} \right] \quad (4.76)
 \end{aligned}$$

From Eq. (4.74)

$$\begin{aligned}
& \left[\frac{Q_2^{n+\frac{1}{2}} - Q_2^n}{\Delta t} + \frac{Q_2^{n+1} - Q_2^{n+\frac{1}{2}}}{\Delta t} \right] \\
& + g \epsilon A_2 \left[\frac{h_{1j+2}^n - h_{1j-2}^n}{8\Delta x} + \frac{h_{1j+2}^{n+1} - h_{1j-2}^{n+1}}{8\Delta x} \right] \\
& + e' \left[\frac{h_{2j+2}^{n+1} - h_{2j-2}^{n+1}}{16\Delta x} + 2 \frac{(h_{2j+2}^{n+\frac{1}{2}} - h_{2j-2}^{n+\frac{1}{2}})}{16\Delta x} \right. \\
& + \left. \frac{h_{2j+2}^n - h_{2j-2}^n}{16\Delta x} \right] - 2u_2 b_2 \left[\frac{h_{2j+2}^{n+\frac{1}{2}} - h_{2j+2}^n}{2\Delta t} \right. \\
& + \left. \frac{h_{2j-2}^{n+\frac{1}{2}} - h_{2j-2}^n}{2\Delta t} + \frac{h_{2j+2}^{n+1} - h_{2j+2}^{n+\frac{1}{2}}}{2\Delta t} + \frac{h_{2j-2}^{n+1} - h_{2j-2}^{n+\frac{1}{2}}}{2\Delta t} \right] \\
& + \frac{Q_2}{\rho_2} \left(1 - \frac{2b_2 h_2}{A_2} \right) \left[\frac{\rho_2^{n+\frac{1}{2}} - \rho_2^n}{2\Delta t} + \frac{\rho_2^{n+\frac{1}{2}} - \rho_2^n}{2\Delta t} \right. \\
& + \left. \frac{\rho_2^{n+1} - \rho_2^{n+\frac{1}{2}}}{2\Delta t} + \frac{\rho_2^{n+1} - \rho_2^{n+\frac{1}{2}}}{2\Delta t} \right] \\
& - \frac{Q_2}{\rho_2 A_2} \left[\frac{\rho_2^{n+1} - \rho_2^{n+1}}{8\Delta x} + 2 \frac{(\rho_2^{n+\frac{1}{2}} - \rho_2^{n+\frac{1}{2}})}{8\Delta x} \right. \\
& + \left. \frac{\rho_2^n - \rho_2^n}{8\Delta x} \right]
\end{aligned}$$

$$\begin{aligned}
&= (b_2 q_2 - 2b_2 \varepsilon q_1) \left[\frac{Q_{2j}^{n+\frac{1}{2}} + Q_{2j}^n}{4A_2} + \frac{Q_{2j}^{n+1} + Q_{2j}^{n+\frac{1}{2}}}{4A_2} \right] \\
&+ \frac{b_2 q_1 \varepsilon}{A_1} \frac{(Q_{1j}^n + Q_{1j}^{n+1})}{2} - \frac{g}{2C^2 A_2 R_2} \left(|Q_{2j}|^n Q_{2j}^{n+\frac{1}{2}} + |Q_{2j}|^{n+\frac{1}{2}} Q_{2j}^{n+1} \right) \\
&- \frac{\tau b_2}{\rho_2} \quad (4.77)
\end{aligned}$$

From Eq. (4.73)

$$\begin{aligned}
&b_1 \left[\frac{h_{1j}^{n+1} - h_{1j}^n}{\Delta t} \right] + \left[\frac{Q_{1j+2}^{n+1} - Q_{1j-2}^{n+1}}{8\Delta x} + \frac{Q_{1j+2}^n - Q_{1j-2}^n}{8\Delta x} \right] \\
&+ \left[\frac{Q_{2j+2}^{n+\frac{1}{2}} - Q_{2j-2}^{n+\frac{1}{2}}}{4\Delta x} \right] + \frac{2b_2 h_2}{\rho_2} \left[\frac{\rho_{2j}^{n+\frac{1}{2}} - \rho_{2j}^n}{\Delta t} \right] \\
&- \frac{1}{\rho_1} (b_2 h_2 - b_1 h_1) \left[\frac{\rho_{1j+1}^{n+1} - \rho_{1j+1}^n}{2\Delta t} + \frac{\rho_{1j-1}^{n+1} - \rho_{1j-1}^n}{2\Delta t} \right] \\
&+ \frac{Q_1}{\rho_1} \left[\frac{\rho_{1j+1}^{n+1} - \rho_{1j-1}^{n+1}}{4\Delta x} + \frac{\rho_{1j+1}^n - \rho_{1j-1}^n}{4\Delta x} \right] \\
&+ \frac{Q_2}{\rho_2} \left[\frac{\rho_{2j+1}^{n+\frac{1}{2}} - \rho_{2j-1}^{n+\frac{1}{2}}}{2\Delta x} \right] = b_2 \left(\frac{q_2}{\varepsilon} - q_1 \right) (1 - \varepsilon) + \frac{q_1 \rho_1}{\rho_1} \quad (4.78)
\end{aligned}$$

From Eq. (4.75)

$$\begin{aligned}
& \left[\frac{Q_1^{n+1} - Q_1^n}{\Delta t} \right] + f' \left[\frac{h_1^{n+1} - h_1^{n+1}}{8\Delta x} + \frac{h_1^n - h_1^n}{8\Delta x} \right] \\
& - 2u_1 b_1 \left[\frac{h_1^{n+1} - h_1^n}{2\Delta t} + \frac{h_1^{n+1} - h_1^n}{2\Delta t} \right] \\
& - 2u_1 \left[\frac{Q_2^{n+\frac{1}{2}} - Q_2^{n+\frac{1}{2}}}{4\Delta x} \right] + \frac{1}{\rho_1} \left[Q_1 + 2u_1 (b_2 h_2 - b_1 h_1) \right] \\
& \left[\frac{\rho_1^{n+1} - \rho_1^n}{2\Delta t} + \frac{\rho_1^{n+1} - \rho_1^n}{2\Delta t} \right] \\
& - \frac{Q_1^2}{\rho_1 A_1} \left[\frac{\rho_1^{n+1} - \rho_1^{n+1}}{4\Delta x} + \frac{\rho_1^n - \rho_1^n}{4\Delta x} \right] \\
& - \frac{4u_1 b_2 h_2}{\rho_2} \left[\frac{\rho_2^{n+\frac{1}{2}} - \rho_2^n}{\Delta t} \right] \\
& - \frac{2u_1 Q_2}{\rho_2} \left[\frac{\rho_2^{n+\frac{1}{2}} - \rho_2^{n+\frac{1}{2}}}{2\Delta x} \right] + u_1^2 B_2 \left[\frac{h_2^{n+\frac{1}{2}} - h_2^{n+\frac{1}{2}}}{4\Delta x} \right] \\
& = \frac{b_2}{A_1} \left[q_1 (1-2\epsilon) + 2q_2 (1 - \frac{1}{\epsilon}) - \frac{2q_i \rho_i}{b_2 \rho_1} \right] \left[\frac{Q_1^{n+1} + Q_1^n}{2} \right] \\
& + \frac{b_2 q_2}{\epsilon A_2} Q_2^{n+\frac{1}{2}} - \frac{g}{c^2 A_1 R_1} (|Q_1|_j^n Q_1^{n+1}) + \frac{\tau b_2}{\rho_1} + \frac{v_1 \cos \phi q_i \rho_i}{\rho_1}
\end{aligned}$$

(4.79)

Four-Level Implicit Difference Scheme (Dynamic)

The difference schemes for the lower layer that is Eqs.(4.76) and (4.77), are now split up into two halves. The first half forms the second level calculation from time level $n\Delta t$ to $(n+\frac{1}{2})\Delta t$ and the remaining half, the fourth level calculation from time level $(n+\frac{1}{2})\Delta t$ to $(n+1)\Delta t$.

Eq.(4.72) is written in the following proposed difference form for the second level of a four-level scheme.

From Eq.(4.76)

$$\begin{aligned}
 & b_2 \left[\frac{h_{2j}^{n+\frac{1}{2}} - h_{2j}^n}{\Delta t} \right] + \left[\frac{Q_{2j+2}^{n+\frac{1}{2}} - Q_{2j-2}^{n+\frac{1}{2}}}{16\Delta x} + \frac{Q_{2j+2}^n - Q_{2j-2}^n}{16\Delta x} \right] \\
 & + \frac{b_2 h_2}{\rho_2} \left[\frac{\rho_{2j+1}^{n+\frac{1}{2}} - \rho_{2j+1}^n}{2\Delta t} + \frac{\rho_{2j-1}^{n+\frac{1}{2}} - \rho_{2j-1}^n}{2\Delta t} \right] \\
 & + \frac{Q_2}{\rho_2} \left[\frac{\rho_{2j+1}^{n+\frac{1}{2}} - \rho_{2j-1}^{n+\frac{1}{2}}}{8\Delta x} + \frac{\rho_{2j+1}^n - \rho_{2j-1}^n}{8\Delta x} \right] \\
 & = \frac{b_2}{2} (\epsilon q_1 - q_2) \quad (4.80)
 \end{aligned}$$

Eq.(4.74) is written in the following proposed difference form for the second level of a four-level scheme.

From Eq.(4.77)

$$\begin{aligned}
& \left[\frac{Q_{2j}^{n+\frac{1}{2}} - Q_{2j}^n}{\Delta t} \right] + g\epsilon A_2 \left[\frac{h_{1j+2}^n - h_{1j-2}^n}{8\Delta x} \right] \\
& + e' \left[\frac{h_{2j+2}^{n+\frac{1}{2}} - h_{2j-2}^{n+\frac{1}{2}}}{16\Delta x} + \frac{h_{2j+2}^n - h_{2j-2}^n}{16\Delta x} \right] \\
& - 2u_2 b_2 \left[\frac{h_{2j+2}^{n+\frac{1}{2}} - h_{2j+2}^n}{2\Delta t} + \frac{h_{2j-2}^{n+\frac{1}{2}} - h_{2j-2}^n}{2\Delta t} \right] \\
& + \frac{Q_2}{\rho_2} \left(1 - \frac{2b_2 h_2}{A_2} \right) \left[\frac{\rho_{2j+1}^{n+\frac{1}{2}} - \rho_{2j+1}^n}{2\Delta t} + \frac{\rho_{2j-1}^{n+\frac{1}{2}} - \rho_{2j-1}^n}{2\Delta t} \right] \\
& - \frac{Q_2^2}{\rho_2 A_2} \left[\frac{\rho_{2j+1}^{n+\frac{1}{2}} - \rho_{2j-1}^{n+\frac{1}{2}}}{8\Delta x} + \frac{\rho_{2j+1}^n - \rho_{2j-1}^n}{8\Delta x} \right] \\
& = (b_2 q_2 - 2b_2 \epsilon q_1) \left[\frac{Q_{2j}^{n+\frac{1}{2}} + Q_{2j}^n}{4A_2} \right] + \frac{b_2 q_1 \epsilon}{2A_1} Q_{1j}^n \\
& - \frac{g|Q_{2j}|^n}{2C^2 A_2 R_2} Q_{2j}^{n+\frac{1}{2}} - \frac{\tau b_2}{2\rho_2}
\end{aligned} \tag{4.81}$$

Eqs.(4.73) and (4.75) provide the third level of the four level scheme. The proposed difference schemes are given by Eqs.(4.78) and (4.79) respectively.

Eq.(4.72) is written in the following proposed difference form for the fourth level of a four-level scheme.

From Eq.(4.76)

$$\begin{aligned}
 & b_2 \left[\frac{h_{2j}^{n+1} - h_{2j}^{n+\frac{1}{2}}}{\Delta t} \right] + \left[\frac{Q_{2j+2}^{n+\frac{1}{2}} - Q_{2j-2}^{n+\frac{1}{2}}}{16\Delta x} \right] + \left[\frac{Q_{2j+2}^{n+1} - Q_{2j-2}^{n+1}}{16\Delta x} \right] \\
 & + \frac{b_2 h_2}{\rho_2} \left[\frac{\rho_{2j+1}^{n+1} - \rho_{2j+1}^{n+\frac{1}{2}}}{2\Delta t} + \frac{\rho_{2j-1}^{n+1} - \rho_{2j-1}^{n+\frac{1}{2}}}{2\Delta t} \right] \\
 & + \frac{Q_2}{\rho_2} \left[\frac{\rho_{2j+1}^{n+\frac{1}{2}} - \rho_{2j-1}^{n+\frac{1}{2}}}{8\Delta x} + \frac{\rho_{2j+1}^{n+1} - \rho_{2j-1}^{n+1}}{8\Delta x} \right] \\
 & = \frac{b_2}{2} (\epsilon q_1 - q_2) \quad (4.82)
 \end{aligned}$$

Eq.(4.74) is written in the following proposed difference form for the fourth level of a four-level scheme

From Eq. (4.77)

$$\begin{aligned}
& \left[\frac{Q_2^{n+1} - Q_2^{n+\frac{1}{2}}}{\Delta t} \right] + g\epsilon A_2 \left[\frac{h_{1j+2}^{n+1} - h_{1j-2}^{n+1}}{8\Delta x} \right] \\
& + e' \left[\frac{h_{2j+2}^{n+\frac{1}{2}} - h_{2j-2}^{n+\frac{1}{2}}}{16\Delta x} + \frac{h_{2j+2}^{n+1} - h_{2j-2}^{n+1}}{16\Delta x} \right] \\
& - 2u_2 b_2 \left[\frac{h_{2j+2}^{n+1} - h_{2j+2}^{n+\frac{1}{2}}}{2\Delta t} + \frac{h_{2j-2}^{n+1} - h_{2j-2}^{n+\frac{1}{2}}}{2\Delta t} \right] \\
& + \frac{Q_2}{\rho_2} \left(1 - \frac{2b_2 h_2}{A_2} \right) \left[\frac{\rho_{2j+1}^{n+1} - \rho_{2j+1}^{n+\frac{1}{2}}}{2\Delta t} + \frac{\rho_{2j-1}^{n+1} - \rho_{2j-1}^{n+\frac{1}{2}}}{2\Delta t} \right] \\
& - \frac{Q_2^2}{\rho_2^2 A_2} \left[\frac{\rho_{2j+1}^{n+\frac{1}{2}} - \rho_{2j-1}^{n+\frac{1}{2}}}{8\Delta x} + \frac{\rho_{2j+1}^{n+1} - \rho_{2j-1}^{n+1}}{8\Delta x} \right] \\
& = (b_2 q_2 - 2b_2 \epsilon q_1) \left[\frac{Q_2^{n+1} + Q_2^{n+\frac{1}{2}}}{4A_2} \right] + \frac{b_2 q_1 \epsilon}{2A_1} Q_1^{n+1} \\
& - \frac{g|Q_2|^{n+\frac{1}{2}}}{2C^2 A_2 R_2} Q_2^{n+1} - \frac{b_2 \tau}{2\rho_2}
\end{aligned} \tag{4.83}$$

Eqs. (4.80) and (4.81) can be arranged in the following pentadiagonal forms for the second level calculation of a four-level scheme.

From Eq. (4.80)

$$\begin{aligned}
& \frac{1}{16\Delta x} Q_2^{n+\frac{1}{2}} + \left(\frac{b_2 h_2}{2\rho_2 \Delta t} + \frac{Q_2}{8\rho_2 \Delta x} \right) \rho_2^{n+\frac{1}{2}} + \frac{b_2}{\Delta t} h_2^{n+\frac{1}{2}} \\
& + \left(\frac{b_2 h_2}{2\rho_2 \Delta t} - \frac{Q_2}{8\rho_2 \Delta x} \right) \rho_2^{n+\frac{1}{2}} + \left(-\frac{1}{16\Delta x} \right) Q_2^{n+\frac{1}{2}} \\
& = \left(-\frac{1}{16\Delta x} \right) Q_2^n + \left(\frac{b_2 h_2}{2\rho_2 \Delta t} - \frac{Q_2}{8\rho_2 \Delta x} \right) \rho_2^n \\
& + \frac{b_2}{\Delta t} h_2^n + \left(\frac{b_2 h_2}{2\rho_2 \Delta t} + \frac{Q_2}{8\rho_2 \Delta x} \right) \rho_2^n \\
& + \frac{1}{16\Delta x} Q_2^n + \frac{b_2}{2} (\epsilon q_1 - q_2) \tag{4.84}
\end{aligned}$$

From Eq. (4.81)

$$\begin{aligned}
& \left(\frac{e'}{16\Delta x} - \frac{u_2 b_2}{\Delta t} \right) h_2^{n+\frac{1}{2}} + \left[\frac{Q_2}{2\rho_2 \Delta t} \left(1 - \frac{2b_2 h_2}{A_2} \right) - \frac{Q_2^2}{8\rho_2 A_2 \Delta x} \right] \rho_2^{n+\frac{1}{2}} \\
& + \left[\frac{1}{\Delta t} - \frac{(b_2 q_2 - 2b_2 \epsilon q_1)}{4A_2} + \frac{g|Q_2|^n}{2C^2 A_2 R_2} \right] Q_2^{n+\frac{1}{2}} \\
& + \left[\frac{Q_2}{2\rho_2 \Delta t} \left(1 - \frac{2b_2 h_2}{A_2} \right) + \frac{Q_2^2}{8\rho_2 A_2 \Delta x} \right] \rho_2^{n+\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
& + \left(-\frac{e'}{16\Delta x} - \frac{u_2 b_2}{\Delta t} \right) h_{2,j-2}^{n+\frac{1}{2}} \\
& = \left(-\frac{e'}{16\Delta x} - \frac{u_2 b_2}{\Delta t} \right) h_{2,j+2}^n + \left[\frac{Q_2}{2\rho_2 \Delta t} \left(1 - \frac{2b_2 h_2}{A_2} \right) \right. \\
& \quad \left. + \frac{Q_2^2}{8\rho_2 A_2 \Delta x} \right] \rho_{2,j+1}^n + \left[\frac{1}{\Delta t} + \frac{b_2 q_2 - 2b_2 \epsilon q_1}{4A_2} \right] Q_{2,j}^n \\
& \quad + \left[\frac{Q_2}{2\rho_2 \Delta t} \left(1 - \frac{2b_2 h_2}{A_2} \right) - \frac{Q_2^2}{8\rho_2 A_2 \Delta x} \right] \rho_{2,j-1}^n \\
& \quad + \left(\frac{e'}{16\Delta x} - \frac{u_2 b_2}{\Delta t} \right) h_{2,j-2}^n + \frac{b_2 q_1 \epsilon}{2A_1} Q_{1,j}^n \\
& \quad - g\epsilon A_2 \left[\frac{h_{1,j+2}^n - h_{1,j-2}^n}{8\Delta x} \right] - \frac{\tau b_2}{2\rho_2}
\end{aligned} \tag{4.85}$$

Eqs.(4.84) and (4.85) can be written as:

$$A_j Q_{2,j+2}^{n+\frac{1}{2}} + B_j \rho_{2,j+1}^{n+\frac{1}{2}} + C_j h_{2,j}^{n+\frac{1}{2}} + D_j \rho_{2,j-1}^{n+\frac{1}{2}} + E_j Q_{2,j-2}^{n+\frac{1}{2}} = F_j \tag{4.86}$$

$$A_j^* h_{2,j+2}^{n+\frac{1}{2}} + B_j^* \rho_{2,j+1}^{n+\frac{1}{2}} + C_j^* Q_{2,j}^{n+\frac{1}{2}} + D_j^* \rho_{2,j-1}^{n+\frac{1}{2}} + E_j^* h_{2,j-2}^{n+\frac{1}{2}} = F_j^* \tag{4.87}$$

Eqs. (4.78) and (4.79) can be arranged in the following pentadiagonal forms for the third level calculation of a four-level scheme.

From Eq. (4.78)

$$\begin{aligned}
& \frac{1}{8\Delta x} Q_{1,j+2}^{n+1} + \left[\frac{Q_1}{4\rho_1\Delta x} - \frac{b_2 h_2 - b_1 h_1}{2\rho_1\Delta t} \right] \rho_{1,j+1}^{n+1} + \frac{b_1}{\Delta t} h_{1,j}^{n+1} \\
& + \left[-\frac{Q_1}{4\rho_1\Delta x} - \frac{b_2 h_2 - b_1 h_1}{2\rho_1\Delta t} \right] \rho_{1,j-1}^{n+1} + \left(-\frac{1}{8\Delta x} \right) Q_{1,j-2}^{n+1} \\
& = \left(-\frac{1}{8\Delta x} \right) Q_{1,j+2}^n + \left[-\frac{Q_1}{4\rho_1\Delta x} - \frac{b_2 h_2 - b_1 h_1}{2\rho_1\Delta t} \right] \rho_{1,j+1}^n \\
& + \frac{b_1}{\Delta t} h_{1,j}^n + \left[\frac{Q_1}{4\rho_1\Delta x} - \frac{b_2 h_2 - b_1 h_1}{2\rho_1\Delta t} \right] \rho_{1,j-1}^n + \frac{1}{8\Delta x} Q_{1,j-2}^n \\
& - \frac{Q_{2,j+2}^{n+\frac{1}{2}} - Q_{2,j-2}^{n+\frac{1}{2}}}{4\Delta x} - \frac{2b_2 h_2}{\rho_2\Delta t} \left(\rho_{2,j}^{n+\frac{1}{2}} - \rho_{2,j}^n \right) \\
& - \frac{Q_2}{2\rho_2\Delta x} \left(\rho_{2,j+1}^{n+\frac{1}{2}} - \rho_{2,j-1}^{n+\frac{1}{2}} \right) + b_2 \left(\frac{q_2}{\varepsilon} - q_1 \right) (1 - \varepsilon) \\
& + \frac{q_i \rho_i}{\rho_1}
\end{aligned} \tag{4.88}$$

From Eq. (4.79)

$$\begin{aligned}
& \left(\frac{f'}{8\Delta x} - \frac{u_1 b_1}{\Delta t} \right) h_{1,j+2}^{n+1} + \left[\frac{Q_1 + 2u_1(b_2 h_2 - b_1 h_1)}{2\rho_1 \Delta t} \right. \\
& \left. - \frac{Q_1^2}{4\rho_1 A_1 \Delta x} \right] \rho_{1,j+1}^{n+1} + \left\{ \frac{1}{\Delta t} - \frac{b_2}{2A_1} \left[q_1(1-2\varepsilon) + 2q_2 \left(1 - \frac{1}{\varepsilon}\right) \right. \right. \\
& \left. \left. - \frac{2q_1 \rho_1}{b_2 \rho_1} \right] + \frac{g|Q_1|^n}{C^2 A_1 R_1} \right\} Q_{1,j}^{n+1} + \left[\frac{Q_1 + 2u_1(b_2 h_2 - b_1 h_1)}{2\rho_1 \Delta t} \right. \\
& \left. + \frac{Q_1^2}{4\rho_1 A_1 \Delta x} \right] \rho_{1,j-1}^{n+1} + \left(-\frac{f'}{8\Delta x} - \frac{u_1 b_1}{\Delta t} \right) h_{1,j-2}^{n+1} \\
& = \left(-\frac{f'}{8\Delta x} - \frac{u_1 b_1}{\Delta t} \right) h_{1,j+2}^n + \left[\frac{Q_1 + 2u_1(b_2 h_2 - b_1 h_1)}{2\rho_1 \Delta t} \right. \\
& \left. + \frac{Q_1^2}{4\rho_1 A_1 \Delta x} \right] \rho_{1,j+1}^n + \left\{ \frac{1}{\Delta t} - \frac{b_2}{2A_1} \left[q_1(1-2\varepsilon) + 2q_2 \left(1 - \frac{1}{\varepsilon}\right) \right. \right. \\
& \left. \left. - \frac{2q_1 \rho_1}{b_2 \rho_1} \right] \right\} Q_{1,j}^n + \left[\frac{Q_1 + 2u_1(b_2 h_2 - b_1 h_1)}{2\rho_1 \Delta t} - \frac{Q_1^2}{4\rho_1 A_1 \Delta x} \right] \rho_{1,j-1}^n \\
& + \left(\frac{f'}{8\Delta x} - \frac{u_1 b_1}{\Delta t} \right) h_{1,j-2}^n + \frac{2u_1}{4\Delta x} \left(Q_{2,j+2}^{n+\frac{1}{2}} - Q_{2,j-2}^{n+\frac{1}{2}} \right) \\
& + \frac{4u_1 b_2 h_2}{\rho_2 \Delta t} \left(\rho_{2,j}^{n+\frac{1}{2}} - \rho_{2,j}^n \right) + \frac{u_1 Q_2}{\rho_2 \Delta x} \left(\rho_{2,j+1}^{n+\frac{1}{2}} - \rho_{2,j-1}^{n+\frac{1}{2}} \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{b_2 q_2}{\epsilon A_2} Q_{2j}^{n+1/2} - u_1^2 B_2 \left[\frac{h_{2j+2}^{n+1/2} - h_{2j-2}^{n+1/2}}{4\Delta x} \right] + \frac{v_1 \cos \phi q_i \rho_i}{\rho_1} \\
& + \frac{\tau b_2}{\rho_1}
\end{aligned} \tag{4.89}$$

Eqs.(4.88) and (4.89) can be written as:

$$G_j Q_{1j+2}^{n+1} + H_j \rho_{1j+1}^{n+1} + I_j h_{1j}^{n+1} + J_j \rho_{1j-1}^{n+1} + K_j Q_{1j-2}^{n+1} = L_j \tag{4.90}$$

$$G_j^* h_{1j+2}^{n+1} + H_j^* \rho_{1j+1}^{n+1} + I_j^* Q_{1j}^{n+1} + J_j^* \rho_{1j-1}^{n+1} + K_j^* h_{1j-2}^{n+1} = L_j^* \tag{4.91}$$

Eqs.(4.82) and (4.83) can be arranged in the following pentadiagonal forms for the fourth level calculation of a four-level scheme.

From Eq.(4.82)

$$\frac{1}{16\Delta x} Q_{2j+2}^{n+1} + \left(\frac{b_2 h_2}{2\rho_2 \Delta t} + \frac{Q_2}{8\rho_2 \Delta x} \right) \rho_{2j+1}^{n+1} + \frac{b_2}{\Delta t} h_{2j}^{n+1}$$

$$\begin{aligned}
& + \left(\frac{b_2 h_2}{2\rho_2 \Delta t} - \frac{Q_2}{8\rho_2 \Delta x} \right) \rho_{2,j-1}^{n+1} + \left(-\frac{1}{16\Delta x} \right) Q_{2,j-2}^{n+1} \\
& = \left(-\frac{1}{16\Delta x} \right) Q_{2,j+2}^{n+1} + \left(\frac{b_2 h_2}{2\rho_2 \Delta t} - \frac{Q_2}{8\rho_2 \Delta x} \right) \rho_{2,j+1}^{n+1} \\
& + \frac{b_2}{\Delta t} h_{2,j}^{n+1} + \left(\frac{b_2 h_2}{2\rho_2 \Delta t} + \frac{Q_2}{8\rho_2 \Delta x} \right) \rho_{2,j-1}^{n+1} \\
& + \frac{1}{16\Delta x} Q_{2,j-2}^{n+1} + \frac{b_2}{2} (\epsilon q_1 - q_2) \tag{4.92}
\end{aligned}$$

From Eq. (4.83)

$$\begin{aligned}
& \left(\frac{e'}{16\Delta x} - \frac{u_2 b_2}{\Delta t} \right) h_{2,j+2}^{n+1} + \left[\frac{Q_2}{2\rho_2 \Delta t} \left(1 - \frac{2h_2 b_2}{A_2} \right) \right. \\
& \left. - \frac{Q_2^2}{8\rho_2 A_2 \Delta x} \right] \rho_{2,j+1}^{n+1} + \left[\frac{1}{\Delta t} + \frac{g|Q_2|_j^{n+1}}{2C^2 A_2 R_2} - \frac{b_2 q_2 - 2b_2 \epsilon q_1}{4A_2} \right] Q_{2,j}^{n+1} \\
& + \left[\frac{Q_2}{2\rho_2 \Delta t} \left(1 - \frac{2b_2 h_2}{A_2} \right) + \frac{Q_2^2}{8\rho_2 A_2 \Delta x} \right] \rho_{2,j-1}^{n+1} \\
& + \left(-\frac{e'}{16\Delta x} - \frac{u_2 b_2}{\Delta t} \right) h_{2,j-2}^{n+1} = \left(-\frac{e'}{16\Delta x} - \frac{u_2 b_2}{\Delta t} \right) h_{2,j+2}^{n+1} \\
& + \left[\frac{Q_2}{2\rho_2 \Delta t} \left(1 - \frac{2b_2 h_2}{A_2} \right) + \frac{Q_2^2}{4\rho_2 A_2 \Delta x} \right] \rho_{2,j+1}^{n+1}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\Delta t} + \frac{b_2 q_2 - 2b_2 \epsilon q_1}{4A_2} Q_2^{n+1/2} \\
& + \frac{Q_2}{2\rho_2 \Delta t} \left(1 - \frac{2b_2 h_2}{A_2}\right) - \frac{Q_2^2}{4\rho_2 A_2 \Delta x} \rho_2^{n+1/2} \\
& + \frac{e'}{16\Delta x} - \frac{u_2 b_2}{\Delta t} h_2^{n+1/2} + \frac{b_2 q_1 \epsilon}{2A_1} Q_1^{n+1} \\
& - \frac{g\epsilon A_2}{8\Delta x} h_1^{n+1} - h_1^{n+1} - \frac{\tau b_2}{2\rho_2}
\end{aligned} \tag{4.93}$$

Eqs.(4.92) and (4.93) can be written as:

$$R_j Q_2^{n+1} + S_j \rho_2^{n+1} + T_j h_2^{n+1} + U_j \rho_2^{n+1} + V_j Q_2^{n+1} = W_j \tag{4.94}$$

$$R_j^* h_2^{n+1} + S_j^* \rho_2^{n+1} + T_j^* Q_2^{n+1} + U_j^* \rho_2^{n+1} + V_j^* h_2^{n+1} = W_j^* \tag{4.95}$$

Implicit Difference Schemes (Solute)

The two equations representing the mass conservation of the solute in the lower and upper layers are Eqs.(2.15) and (2.16) respectively. The difference scheme for the upper layer is from Eq.(2.16)

$$\begin{aligned}
 & \frac{1}{4} \left[\frac{\rho_{1,j+2}^{n+1} - \rho_{1,j+2}^n}{\Delta t} \right] + \frac{1}{2} \left[\frac{\rho_{1,j}^{n+1} - \rho_{1,j}^n}{\Delta t} \right] + \frac{1}{4} \left[\frac{\rho_{1,j-2}^{n+1} - \rho_{1,j-2}^n}{\Delta t} \right] \\
 & + \frac{u_1}{2} \left[\frac{\rho_{1,j+2}^{n+1} - \rho_{1,j-2}^{n+1}}{4\Delta x} + \frac{\rho_{1,j+2}^n - \rho_{1,j-2}^n}{4\Delta x} \right] \\
 & - \frac{D_1}{2} \left[\frac{\rho_{1,j+2}^{n+1} - 2\rho_{1,j}^{n+1} + \rho_{1,j-2}^{n+1}}{4\Delta x^2} + \frac{\rho_{1,j+2}^n - 2\rho_{1,j}^n + \rho_{1,j-2}^n}{4\Delta x^2} \right] \\
 & + \frac{b_2 q_2 + q_i}{2A_1} \left(\rho_{1,j}^{n+1} + \rho_{1,j}^n \right) - \frac{b_2 q_2}{A_1} \rho_{2,j}^{n+\frac{1}{2}} \\
 & - \frac{q_i \rho_i}{A_1} = 0
 \end{aligned} \tag{4.96}$$

From Eq.(2.15)

$$\left[\frac{\rho_{2,j+2}^{n+1} - \rho_{2,j+2}^{n+\frac{1}{2}}}{4\Delta t} + \frac{\rho_{2,j+2}^{n+\frac{1}{2}} - \rho_{2,j+2}^n}{4\Delta t} \right]$$

$$\begin{aligned}
& + 2 \left[\frac{\rho_{2j}^{n+1} - \rho_{2j}^{n+\frac{1}{2}}}{4\Delta t} + \frac{\rho_{2j}^{n+\frac{1}{2}} - \rho_{2j}^n}{4\Delta t} \right] \\
& + \left[\frac{\rho_{2j-2}^{n+1} - \rho_{2j-2}^{n+\frac{1}{2}}}{4\Delta t} + \frac{\rho_{2j-2}^{n+\frac{1}{2}} - \rho_{2j-2}^n}{4\Delta t} \right] \\
& + u_2 \left[\frac{\rho_{2j+2}^{n+1} - \rho_{2j-2}^{n+1}}{16\Delta x} + 2 \frac{(\rho_{2j+2}^{n+\frac{1}{2}} - \rho_{2j-2}^{n+\frac{1}{2}})}{16\Delta x} + \frac{\rho_{2j+2}^n - \rho_{2j-2}^n}{16\Delta x} \right] \\
& - \frac{D_2}{4} \left[\frac{\rho_{2j+2}^{n+1} - 2\rho_{2j}^{n+1} + \rho_{2j-2}^{n+1}}{4\Delta x^2} + 2 \frac{(\rho_{2j+2}^{n+\frac{1}{2}} - 2\rho_{2j}^{n+\frac{1}{2}} + \rho_{2j-2}^{n+\frac{1}{2}})}{4\Delta x^2} \right. \\
& \left. + \frac{\rho_{2j+2}^n - 2\rho_{2j}^n + \rho_{2j-2}^n}{4\Delta x^2} \right] + \frac{b_2 q_1}{4A_2} (\rho_{2j}^{n+1} + 2\rho_{2j}^{n+\frac{1}{2}} + \rho_{2j}^n) \\
& - \frac{b_2 q_1}{2A_2} (\rho_{1j}^{n+1} + \rho_{1j}^n) = 0 \tag{4.97}
\end{aligned}$$

Four-Level Implicit Difference Scheme (Solute)

The difference scheme for the lower layer given by Eq.(4.97) is split up into two halves. The first half forms the second level calculation from time level $n\Delta t$ to $(n+\frac{1}{2})\Delta t$ and the remaining half, the fourth level calculation from time level $(n+\frac{1}{2})\Delta t$ to $(n+1)\Delta t$.

Eq.(2.15) is written in the following proposed difference form for the second level of a four-level scheme.

From Eq.(4.97)

$$\begin{aligned}
 & \frac{\rho_{2,j+2}^{n+\frac{1}{2}} - \rho_{2,j+2}^n}{4\Delta t} + 2 \left(\frac{\rho_{2,j}^{n+\frac{1}{2}} - \rho_{2,j}^n}{4\Delta t} \right) + \frac{\rho_{2,j-2}^{n+\frac{1}{2}} - \rho_{2,j-2}^n}{4\Delta t} \\
 & + u_2 \left[\frac{\rho_{2,j+2}^{n+\frac{1}{2}} - \rho_{2,j-2}^{n+\frac{1}{2}}}{16\Delta x} + \frac{\rho_{2,j+2}^n - \rho_{2,j-2}^n}{16\Delta x} \right] \\
 & - \frac{D_2}{4} \left[\frac{\rho_{2,j+2}^{n+\frac{1}{2}} - 2\rho_{2,j}^{n+\frac{1}{2}} + \rho_{2,j-2}^{n+\frac{1}{2}}}{4\Delta x^2} + \frac{\rho_{2,j+2}^n - 2\rho_{2,j}^n + \rho_{2,j-2}^n}{4\Delta x^2} \right] \\
 & + \frac{b_2 q_1}{4A_2} \left(\rho_{2,j}^{n+\frac{1}{2}} + \rho_{2,j}^n \right) = \frac{b_2 q_1}{2A_2} \rho_{1,j}^n \quad (4.98)
 \end{aligned}$$

Eq.(2.16) provides the third level of a four level scheme, and the proposed difference scheme is given by Eq.(4.96). Eq.(2.15) is written in the following proposed difference form for the fourth level of a four-level scheme.

From Eq.(4.97)

$$\begin{aligned}
& \frac{\rho_{2,j+2}^{n+1} - \rho_{2,j+2}^{n+\frac{1}{2}}}{4\Delta t} + 2 \left(\frac{\rho_{2,j}^{n+1} - \rho_{2,j}^{n+\frac{1}{2}}}{4\Delta t} \right) + \frac{\rho_{2,j-2}^{n+1} - \rho_{2,j-2}^{n+\frac{1}{2}}}{4\Delta t} \\
& + u_2 \left[\frac{\rho_{2,j+2}^{n+1} - \rho_{2,j-2}^{n+1}}{16\Delta x} + \frac{\rho_{2,j+2}^{n+\frac{1}{2}} - \rho_{2,j-2}^{n+\frac{1}{2}}}{16\Delta x} \right] \\
& - \frac{D_2}{4} \left[\frac{\rho_{2,j+2}^{n+1} - 2\rho_{2,j}^{n+1} + \rho_{2,j-2}^{n+1}}{4\Delta x^2} + \frac{\rho_{2,j+2}^{n+\frac{1}{2}} - 2\rho_{2,j}^{n+\frac{1}{2}} + \rho_{2,j-2}^{n+\frac{1}{2}}}{4\Delta x^2} \right] \\
& + \frac{b_2 q_1}{4A_2} \left(\rho_{2,j}^{n+1} + \rho_{2,j}^{n+\frac{1}{2}} \right) = \frac{b_2 q_1}{2A_2} \rho_{1,j}^{n+1} \quad (4.99)
\end{aligned}$$

Eq.(4.98) is written in the following tridiagonal form.

$$\begin{aligned}
& \left[\frac{1}{4\Delta t} + \frac{u_2}{16\Delta x} - \frac{D_2}{16\Delta x^2} \right] \rho_{2,j+2}^{n+\frac{1}{2}} + \left[\frac{1}{2\Delta t} + \frac{D_2}{8\Delta x^2} + \frac{b_2 q_1}{4A_2} \right] \rho_{2,j}^{n+\frac{1}{2}} \\
& + \left[\frac{1}{4\Delta t} - \frac{u_2}{16\Delta x} - \frac{D_2}{16\Delta x^2} \right] \rho_{2,j-2}^{n+\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
&= \left[\frac{1}{4\Delta t} - \frac{u_2}{16\Delta x} + \frac{D_2}{16\Delta x^2} \right] \rho_{2,j+2}^n + \left[\frac{1}{2\Delta t} - \frac{D_2}{8\Delta x^2} - \frac{b_2 q_1}{4A_2} \right] \rho_{2,j}^n \\
&+ \left[\frac{1}{4\Delta t} + \frac{u_2}{16\Delta x} + \frac{D_2}{16\Delta x^2} \right] \rho_{2,j-2}^n + \frac{b_2 q_1}{2A_2} \rho_{1,j}^n \quad (4.100)
\end{aligned}$$

which can be written as:

$$A_j^{**} \rho_{2,j+2}^{n+\frac{1}{2}} + B_j^{**} \rho_{2,j}^{n+\frac{1}{2}} + C_j^{**} \rho_{2,j-2}^{n+\frac{1}{2}} = D_j^{**} \quad (4.101)$$

Eq.(4.96) is written in the following tridiagonal form

$$\begin{aligned}
&\left[\frac{1}{4\Delta t} + \frac{u_1}{8\Delta x} - \frac{D_1}{8\Delta x^2} \right] \rho_{1,j+2}^{n+1} + \left[\frac{1}{2\Delta t} + \frac{D_1}{4\Delta x^2} + \frac{b_2 q_2 - q_i}{2A_1} \right] \rho_{1,j}^{n+1} \\
&+ \left[\frac{1}{4\Delta t} - \frac{u_1}{8\Delta x} - \frac{D_1}{8\Delta x^2} \right] \rho_{1,j-2}^{n+1} = \left[\frac{1}{4\Delta t} - \frac{u_1}{8\Delta x} + \frac{D_1}{8\Delta x^2} \right] \rho_{1,j+2}^n \\
&+ \left[\frac{1}{2\Delta t} - \frac{D_1}{4\Delta x^2} - \frac{b_2 q_2 + q_i}{2A_1} \right] \rho_{1,j}^n \\
&+ \left[\frac{1}{4\Delta t} + \frac{u_1}{8\Delta x} + \frac{D_1}{8\Delta x^2} \right] \rho_{1,j-2}^n + \frac{b_2 q_2}{A_1} \rho_{2,j}^{n+\frac{1}{2}} + \frac{q_i \rho_i}{A_1} \quad (4.102)
\end{aligned}$$

which can be written as:

$$E_j^{**} \rho_{1_{j+2}}^{n+1} + F_j^{**} \rho_{1_j}^{n+1} + G_j^{**} \rho_{1_{j-2}}^{n+1} = H_j^{**} \quad (4.103)$$

Eq.(4.99) is written in the following tridiagonal form

$$\begin{aligned} & \left[\frac{1}{4\Delta t} + \frac{u_2}{16\Delta x} - \frac{D_2}{16\Delta x^2} \right] \rho_{2_{j+2}}^{n+1} + \left[\frac{1}{2\Delta t} + \frac{D_2}{8\Delta x^2} + \frac{b_2 q_1}{4A_2} \right] \rho_{2_j}^{n+1} \\ & + \left[\frac{1}{4\Delta t} - \frac{u_2}{16\Delta x} - \frac{D_2}{16\Delta x^2} \right] \rho_{2_{j-2}}^{n+1} \\ & = \left[\frac{1}{4\Delta t} - \frac{u_2}{16\Delta x} + \frac{D_2}{16\Delta x^2} \right] \rho_{2_{j+2}}^{n+1} + \left[\frac{1}{2\Delta t} - \frac{D_2}{8\Delta x^2} - \frac{b_2 q_1}{4A_2} \right] \rho_{2_j}^{n+1} \\ & + \left[\frac{1}{4\Delta t} + \frac{u_2}{16\Delta x} + \frac{D_2}{16\Delta x^2} \right] \rho_{2_{j-2}}^{n+1} + \frac{b_2 q_1}{2A_2} \rho_{1_j}^{n+1} \end{aligned} \quad (4.104)$$

which can be written as:

$$I_j^{**} \rho_{2_{j+2}}^{n+1} + J_j^{**} \rho_{2_j}^{n+1} + K_j^{**} \rho_{2_{j-2}}^{n+1} = L_j^{**} \quad (4.105)$$

Algorithmic Structures

For stratified estuarine flows unlike well-mixed flows, the possibility of supercritical flow conditions in either layer is very high and must always be considered. It therefore follows that for the convection of the three dependent variables there can be three characteristics of the same sign in each layer as is indicated in Table 4.1. Since the number of characteristic directions of the same sign dictates the number of data points in that direction, this number must be reflected in the algebraic relations required for the double sweep algorithm.

For calculation at the second level consider the Eqs.(4.86), (4.87) and (4.101) on the grid of Fig.4.3 and the algebraic relations:

$$h_{2,j-2}^{n+1/2} = \alpha_j \rho_{2,j-1}^{n+1/2} + \beta_j Q_{2,j}^{n+1/2} + \gamma_j h_{2,j+2}^{n+1/2} + \delta_j \quad (4.106)$$

$$Q_{2,j-2}^{n+1/2} = \epsilon_j \rho_{2,j-1}^{n+1/2} + \zeta_j h_{2,j}^{n+1/2} + \eta_j Q_{2,j+2}^{n+1/2} + \theta_j \quad (4.107)$$

$$\rho_{2,j-2}^{n+1/2} = \alpha_j^* h_{2,j-1}^{n+1/2} + \beta_j^* \rho_{2,j}^{n+1/2} + \gamma_j^* Q_{2,j+1}^{n+1/2} + \delta_j^* \quad (4.108)$$

$$\rho_{2,j-2}^{n+1/2} = \epsilon_j^* Q_{2,j-1}^{n+1/2} + \zeta_j^* \rho_{2,j}^{n+1/2} + \eta_j^* h_{2,j+1}^{n+1/2} + \theta_j^* \quad (4.109)$$

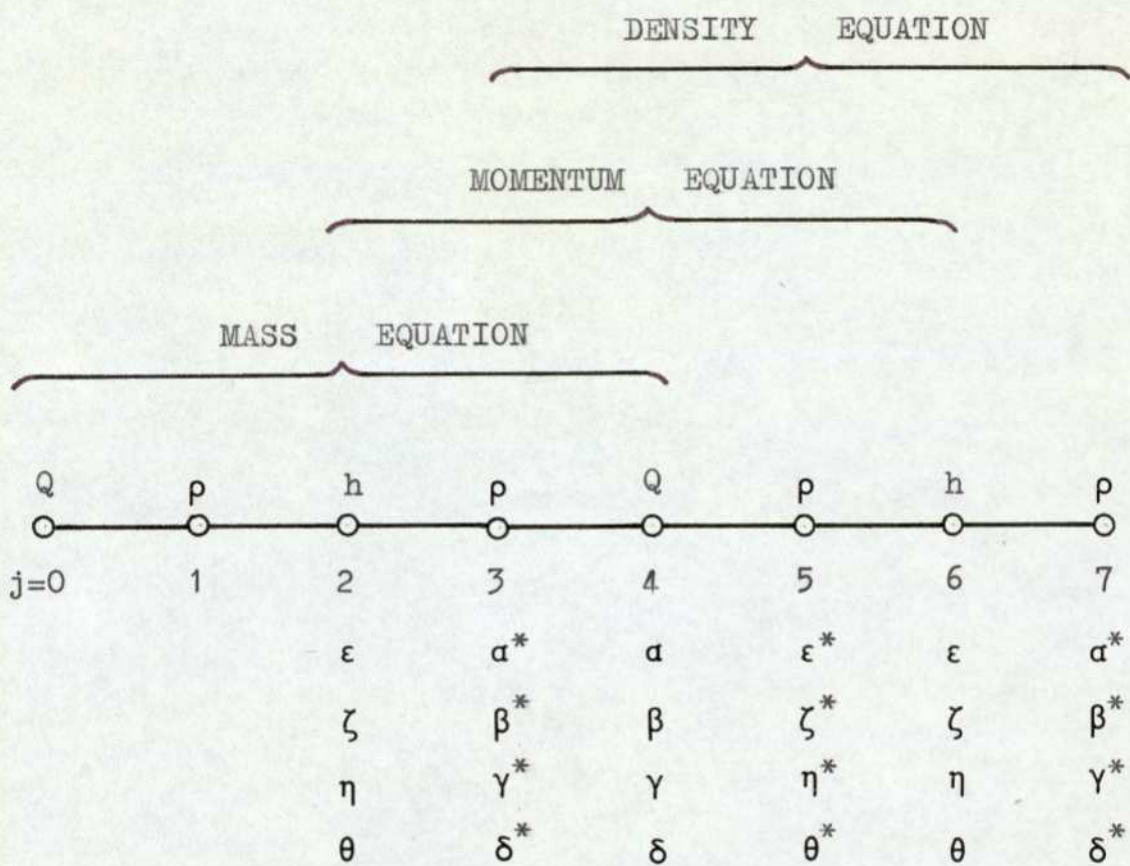


FIG. 4.3 COMPUTATION GRID FOR STRATIFIED FLOWS

According to conditions at the left hand side of the calculation grid point for the lower layer at time level $(n+\frac{1}{2})\Delta t$, appropriate coefficients can be set to zero as shown in Table 4.2. Then, by substituting Eqs.(4.106), (4.107), (4.108) and (4.109) into Eqs.(4.86), (4.87) and (4.101) recurrence relations can be built between the coefficients of Eqs.(4.106) to (4.109) for each data structure. From these a double sweep algorithm can be constructed which can be used in an identical way for calculations not only at the second level but at the third and fourth levels also.

Table 4.2 indicates that there are four algorithmic structures corresponding to the four characteristic structures for each layer, compared with three algorithmic structures when the density is not included in the algorithm. Thus instead of nine data structures being associated with a two-layer stratified flow (51), there is now sixteen data structures as indicated in Table 4.1. Since the computation uses a fractional steps technique enabling each layer to be calculated separately, these sixteen data structures are effectively reduced to four.

For calculation at the third level consider the Eqs.(4.90), (4.91) and (4.103) on the grid of Fig.4.3 and the algebraic relations Eqs.(4.106) to (4.109) with the time level changed from $(n+\frac{1}{2})\Delta t$ to $(n+1)\Delta t$ and the dependent variables (h_2, Q_2, ρ_2) changed to (h_1, Q_1, ρ_1) . For calculation at the fourth level

FLOW TYPE (1)	DATA POINTS GIVEN AT LEFT BOUNDARY (2)	COEFFICIENTS SET TO ZERO (3)
sup $r \rightarrow l$	1 (ρ)	NONE
sub $r \rightarrow l$	2 (ρ, h or Q)	$\gamma \gamma^* \eta \eta^*$
sub $l \rightarrow r$	2 (ρ, h or Q)	$\gamma \gamma^* \eta \eta^* \alpha^* \epsilon^*$
sup $l \rightarrow r$	3 (ρ, h and Q)	$\gamma \gamma^* \eta \eta^* \alpha^* \epsilon^* \beta \zeta$

TABLE 4.2 APPROPRIATE COEFFICIENTS SET TO ZERO

consider the Eqs.(4.94), (4.95) and (4.105) on the grid of Fig.4.3 and the algebraic relations Eqs.(4.106) to (4.109) with the time level changed from $(n+\frac{1}{2})\Delta t$ to $(n+1)\Delta t$.

Recurrence Relations

Because of the four different algorithmic structures in each layer shown in Table 4.2, each algorithmic structure or flow type will be treated separately. Also since the algorithmic structure of the equations is the same at each level of the calculation, the indices denoting time level and layer number will be deleted.

(a) Supercritical Flow (Right to Left)

Eqs.(4.106) to (4.109) become

$$h_{j-2} = \alpha_j \rho_{j-1} + \beta_j Q_j + \gamma_j h_{j+2} + \delta_j \quad (4.110)$$

$$Q_{j-2} = \epsilon_j \rho_{j-1} + \zeta_j h_j + \eta_j Q_{j+2} + \theta_j \quad (4.111)$$

$$\rho_{j-2} = \alpha_j^* h_{j-1} + \beta_j^* \rho_j + \gamma_j^* Q_{j+1} + \delta_j^* \quad (4.112)$$

$$\rho_{j-2} = \epsilon_j^* Q_{j-1} + \zeta_j^* \rho_j + \eta_j^* h_{j+1} + \theta_j^* \quad (4.113)$$

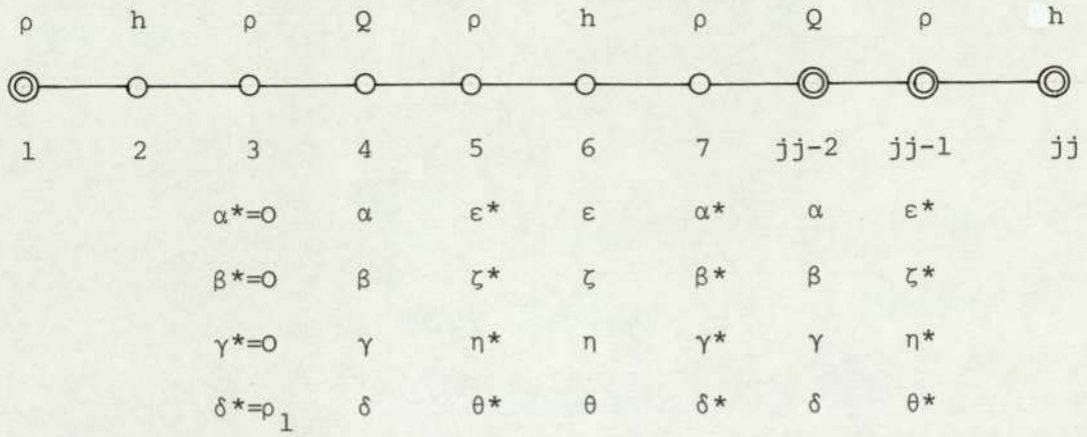


Fig.4.4 Boundary Conditions and Starting Values for Case (a)

From Eq. (4.86)

$$A_j Q_{j+2} + B_j \rho_{j+1} + C_j h_j + D_j \rho_{j-1} + E_j Q_{j-2} = F_j$$

substitute Eq.(4.111) for Q_{j-2}

$$A_j Q_{j+2} + B_j \rho_{j+1} + C_j h_j + D_j \rho_{j-1} + E_j (\epsilon_j \rho_{j-1} + \zeta_j h_j + \eta_j Q_{j+2} + \theta_j) = F_j$$

$$\begin{aligned} \rho_{j-1} (D_j + E_j \epsilon_j) &= -h_j (C_j + E_j \zeta_j) - B_j \rho_{j+1} \\ &\quad - Q_{j+2} (A_j + E_j \eta_j) + (F_j - E_j \theta_j) \end{aligned}$$

comparing with Eq.(4.112)

$$\begin{aligned}\alpha_{j+1}^* &= - \frac{(C_j + E_j \zeta_j)}{(D_j + E_j \epsilon_j)} ; \quad \beta_{j+1}^* = \frac{-B_j}{(D_j + E_j \epsilon_j)} ; \\ \gamma_{j+1}^* &= - \frac{(A_j + E_j \eta_j)}{(D_j + E_j \epsilon_j)} ; \quad \delta_{j+1}^* = \frac{(F_j - E_j \theta_j)}{(D_j + E_j \epsilon_j)} .\end{aligned}\quad (4.114)$$

From Eq. (4.87)

$$A_j^* h_{j+2} + B_j^* \rho_{j+1} + C_j^* Q_j + D_j^* \rho_{j-1} + E_j^* h_{j-2} = F_j^*$$

substitute Eq. (4.110) for h_{j-2}

$$\begin{aligned}A_j^* h_{j+2} + B_j^* \rho_{j+1} + C_j^* Q_j + D_j^* \rho_{j-1} \\ + E_j^* (\alpha_j \rho_{j-1} + \beta_j Q_j + \gamma_j h_{j+2} + \delta_j) = F_j^*\end{aligned}$$

$$\begin{aligned}\rho_{j-1} (D_j^* + E_j^* \alpha_j) &= -Q_j (C_j^* + E_j^* \beta_j) - B_j^* \rho_{j+1} \\ - h_{j+2} (A_j^* + E_j^* \gamma_j) &+ (F_j^* - E_j^* \delta_j)\end{aligned}$$

comparing with Eq. (4.113)

$$\begin{aligned}
\varepsilon_{j+1}^* &= - \frac{(C_j^* + E_j^* \beta_j)}{(D_j^* + E_j^* \alpha_j)} ; \quad \zeta_{j+1}^* = \frac{B_j^*}{(D_j^* + E_j^* \alpha_j)} ; \\
\eta_{j+1}^* &= - \frac{(A_j^* + E_j^* \gamma_j)}{(D_j^* + E_j^* \alpha_j)} ; \quad \theta_{j+1}^* = \frac{(F_j^* - E_j^* \delta_j)}{(D_j^* + E_j^* \alpha_j)} \quad (4.115)
\end{aligned}$$

From Eq. (4.87)

$$A_{j+1}^* h_{j+3} + B_{j+1}^* \rho_{j+2} + C_{j+1}^* Q_{j+1} + D_{j+1}^* \rho_j + E_{j+1}^* h_{j-1} = F_{j+1}^*$$

substitute Eq. (4.101) for ρ_{j+2}

$$A_{j+1}^* h_{j+3} + \frac{B_{j+1}^*}{A_j^{**}} (D_j^{**} - B_j^{**} \rho_j - C_j^{**} \rho_{j-2}) + C_{j+1}^* Q_{j+1}$$

$$+ D_{j+1}^* \rho_j + E_{j+1}^* h_{j-1} = F_{j+1}^*$$

substitute Eq. (4.112) for ρ_{j-2}

$$\begin{aligned}
&A_{j+1}^* h_{j+3} + C_{j+1}^* Q_{j+1} + \rho_j \left(D_{j+1}^* - \frac{B_{j+1}^{**} B_{j+1}^*}{A_j^{**}} \right) + \frac{D_{j+1}^{**} B_{j+1}^*}{A_j^{**}} \\
&- F_{j+1}^* - \frac{B_{j+1}^* C_{j+1}^{**}}{A_j^{**}} (\alpha_j^* h_{j-1} + \beta_j^* \rho_j + \gamma_j^* Q_{j+1} + \delta_j^*) = - E_{j+1}^* h_{j-1}
\end{aligned}$$

$$\begin{aligned}
h_{j-1} \left[\frac{B_{j+1}^* C_{j,j}^{**\alpha*}}{A_j^{**}} - E_{j+1}^* \right] &= \rho_j \left[D_{j+1}^* - \frac{B_{j,j+1}^{**} B_{j+1}^*}{A_j^{**}} - \frac{B_{j+1}^* C_{j,j}^{**\beta*}}{A_j^{**}} \right] \\
+ Q_{j+1} \left[C_{j+1}^* - \frac{B_{j+1}^* C_{j,j}^{**\gamma*}}{A_j^{**}} \right] &+ h_{j+3} A_{j+1}^* \\
+ \left[\frac{D_{j,j+1}^{**} B_{j+1}^*}{A_j^{**}} - F_{j+1}^* - \frac{B_{j+1}^* C_{j,j}^{**\delta*}}{A_j^{**}} \right]
\end{aligned}$$

comparing with Eq. (4.110)

$$\alpha_{j+1} = \left[D_{j+1}^* A_j^{**} - B_{j+1}^* (B_{j,j}^{**} + C_{j,j}^{**\beta*}) \right] / X ;$$

$$\beta_{j+1} = (C_{j+1}^* A_j^{**} - B_{j+1}^* C_{j,j}^{**\gamma*}) / X ;$$

$$\gamma_{j+1} = A_{j+1}^* A_j^{**} / X ;$$

$$\delta_{j+1} = \left[B_{j+1}^* (D_{j,j}^{**} - C_{j,j}^{**\delta*}) - F_{j+1}^* A_j^{**} \right] / X \quad (4.116)$$

where

$$X = B_{j+1}^* C_{j,j}^{**\alpha*} - E_{j+1}^* A_j^{**}$$

From Eq. (4.86)

$$A_{j+1} Q_{j+3} + B_{j+1} \rho_{j+2} + C_{j+1} h_{j+1} + D_{j+1} \rho_j + E_{j+1} Q_{j-1} = F_{j+1}$$

substitute Eq.(4.101) for ρ_{j+2}

$$A_{j+1}Q_{j+3} + \frac{B_{j+1}}{A_j^{**}} (D_j^{**} - B_j^{**}\rho_j - C_j^{**}\rho_{j-2}) + C_{j+1}h_{j+1}$$

$$+ D_{j+1}\rho_j + E_{j+1}Q_{j-1} = F_{j+1}$$

substitute Eq.(4.113) for ρ_{j-2}

$$A_{j+1}Q_{j+3} + C_{j+1}h_{j+1} + \rho_j \left(D_{j+1} - \frac{B_{j+1}B_j^{**}}{A_j^{**}} \right) + \frac{D_j^{**}B_{j+1}}{A_j^{**}}$$

$$- F_{j+1} - \frac{B_{j+1}C_j^{**}}{A_j^{**}} \left(\epsilon_j^* Q_{j-1} + \zeta_j^* \rho_j + \eta_j^* h_{j+1} + \theta_j^* \right)$$

$$= - E_{j+1}Q_{j-1}$$

$$Q_{j-1} \left(\frac{B_{j+1}C_j^{**}\epsilon_j^*}{A_j^{**}} - E_{j+1} \right) = \rho_j \left(D_{j+1} - \frac{B_{j+1}B_j^{**}}{A_j^{**}} - \frac{B_{j+1}C_j^{**}\zeta_j^*}{A_j^{**}} \right)$$

$$+ h_{j+1} \left(C_{j+1} - \frac{B_{j+1}C_j^{**}\eta_j^*}{A_j^{**}} \right) + Q_{j+3}A_{j+1}$$

$$+ \left(\frac{D_j^{**}B_{j+1}}{A_j^{**}} - F_{j+1} - \frac{B_{j+1}C_j^{**}\theta_j^*}{A_j^{**}} \right)$$

comparing with Eq.(4.111)

$$\epsilon_{j+1} = \left[D_{j+1} A_j^{**} - B_{j+1} (B_j^{**} + C_j^{**} \zeta_j^*) \right] / X ;$$

$$\zeta_{j+1} = (C_{j+1} A_j^{**} - B_{j+1} C_j^{**} \eta_j^*) / X ;$$

$$\eta_{j+1} = A_{j+1} A_j^{**} / X ;$$

$$\theta_{j+1} = \left[B_{j+1} (D_j^{**} - C_j^{**} \theta_j^*) - F_{j+1} A_j^{**} \right] / X \quad (4.117)$$

$$\text{where } X = B_{j+1} C_j^{**} \epsilon_j^* - E_{j+1} A_j^{**} .$$

(b) Subcritical Flow (Right to Left)

Eqs.(4.106) to (4.109) become

$$h_{j-2} = \alpha_j \rho_{j-1} + \beta_j Q_j + \delta_j \quad (4.118)$$

$$Q_{j-2} = \epsilon_j \rho_{j-1} + \zeta_j h_j + \theta_j \quad (4.119)$$

$$\rho_{j-2} = \alpha_j^* h_{j-1} + \beta_j^* \rho_j + \delta_j^* \quad (4.120)$$

$$\rho_{j-2} = \epsilon_j^* Q_{j-1} + \zeta_j^* \rho_j + \theta_j^* \quad (4.121)$$

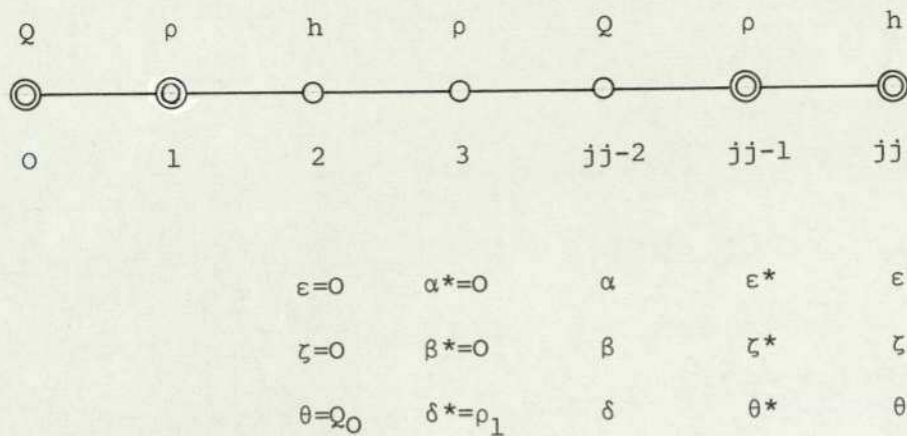


Fig.4.5 Boundary Conditions and Starting Values for Case (b)

From Eq. (4.87)

$$A_j^* h_{j+2} + B_j^* \rho_{j+1} + C_j^* Q_j + D_j^* \rho_{j-1} + E_j^* h_{j-2} = F_j^*$$

substitute Eq. (4.118) for h_{j-2}

$$A_j^* h_{j+2} + B_j^* \rho_{j+1} + C_j^* Q_j + D_j^* \rho_{j-1}$$

$$+ E_j^* (\alpha_j \rho_{j-1} + \beta_j Q_j + \delta_j) = F_j^*$$

substitute Eq. (4.121) for ρ_{j-1}

$$A_j^* h_{j+2} + B_j^* \rho_{j+1} + C_j^* Q_j + (\epsilon_{j+1}^* Q_j + \zeta_{j+1}^* \rho_{j+1} + \theta_{j+1}^*)$$

$$(D_j^* + E_j^* \alpha_j) + E_j^* (\beta_j Q_j + \delta_j) = F_j^*$$

$$Q_j \left[C_j^* + E_j^* \beta_j + \epsilon_{j+1}^* (D_j^* + E_j^* \alpha_j) \right] =$$

$$\rho_{j+1} \left[-B_j^* - \zeta_{j+1}^* (D_j^* + E_j^* \alpha_j) \right] - h_{j+2} A_j^*$$

$$+ \left[F_j^* - E_j^* \delta_j - \theta_{j+1}^* (D_j^* + E_j^* \alpha_j) \right]$$

comparing with Eq. (4.119)

$$\epsilon_{j+2} = \left[-B_j^* - \zeta_{j+1}^* (D_j^* + E_j^* \alpha_j) \right] / X ;$$

$$\zeta_{j+2} = -A_j^* / X ;$$

$$\theta_{j+2} = \left[F_j^* - E_j^* \delta_j - \theta_{j+1}^* (D_j^* + E_j^* \alpha_j) \right] / X \quad (4.122)$$

where $X = C_j^* + E_j^* \beta_j + \epsilon_{j+1}^* (D_j^* + E_j^* \alpha_j) .$

From Eq. (4.86)

$$A_j Q_{j+2} + B_j \rho_{j+1} + C_j h_j + D_j \rho_{j-1} + E_j Q_{j-2} = F_j$$

substitute Eq. (4.119) for Q_{j-2}

$$A_j Q_{j+2} + B_j \rho_{j+1} + C_j h_j + D_j \rho_{j-1} + E_j (\epsilon_j \rho_{j-1} + \zeta_j h_j + \theta_j) = F_j$$

substitute Eq. (4.120) for ρ_{j-1}

$$A_j Q_{j+2} + B_j \rho_{j+1} + C_j h_j + D_j (\alpha_{j+1}^* h_j + \beta_{j+1}^* \rho_{j+1} + \delta_{j+1}^*)$$

$$+ E_j [\epsilon_j (\alpha_{j+1}^* h_j + \beta_{j+1}^* \rho_{j+1} + \delta_{j+1}^*) + \zeta_j h_j + \theta_j] = F_j$$

$$h_j [C_j + E_j \zeta_j + \alpha_{j+1}^* (D_j + E_j \epsilon_j)]$$

$$= \rho_{j+1} [-B_j - (E_j \epsilon_j + D_j) \beta_{j+1}^*] - A_j Q_{j+2}$$

$$+ [F_j - D_j \delta_{j+1}^* - (\theta_j + \epsilon_j \delta_{j+1}^*) E_j]$$

comparing with Eq.(4.118)

$$\alpha_{j+2} = \left[-B_j - (E_j \varepsilon_j + D_j) \beta_{j+1}^* \right] / X ;$$

$$\beta_{j+2} = -A_j / X ;$$

$$\delta_{j+2} = \left[F_j - D_j \delta_{j+1}^* - (\theta_j + \varepsilon_j \delta_{j+1}^*) E_j \right] / X \quad (4.123)$$

$$\text{where } X = C_j + E_j \zeta_j + \alpha_{j+1}^* (D_j + E_j \varepsilon_j) .$$

From Eqs.(4.118) and (4.120)

$$\rho_{j-2} = \alpha_j^* (\alpha_{j+1} \rho_j + \beta_{j+1} Q_{j+1} + \delta_{j+1}) + \beta_j^* \rho_j + \delta_j^*$$

substitute for ρ_{j-2} from Eq.(4.101)

$$- (A_j^{**} \rho_{j+2} + B_j^{**} \rho_j - D_j^{**}) / C_j^{**}$$

$$= \alpha_j^* (\alpha_{j+1} \rho_j + \beta_{j+1} Q_{j+1} + \delta_{j+1}) + \beta_j^* \rho_j + \delta_j^*$$

$$\rho_j \left(-\frac{B_j^{**}}{C_j^{**}} - \alpha_j^* \alpha_{j+1} - \beta_j^* \right) = Q_{j+1} \alpha_j^* \beta_{j+1} + \frac{A_j^{**}}{C_j^{**}} \rho_{j+2} \\ + \left(\alpha_j^* \delta_{j+1} + \delta_j^* - \frac{D_j^{**}}{C_j^{**}} \right)$$

comparing with Eq.(4.121)

$$\epsilon_{j+2}^* = \alpha_j^* \beta_{j+1} / X ; \quad \zeta_{j+2}^* = A_j^{**} / (C_j^{**} X) ;$$

$$\theta_{j+2}^* = \left(\alpha_j^* \delta_{j+1} + \delta_j^* - D_j^{**} / C_j^{**} \right) / X \quad (4.124)$$

$$\text{where } X = -B_j^{**} / C_j^{**} - \alpha_j^* \alpha_{j+1} - \beta_j^* .$$

From Eqs.(4.119) and (4.121)

$$\rho_{j-2} = \epsilon_j^* (\epsilon_{j+1} \rho_j + \zeta_{j+1} h_{j+1} + \theta_{j+1}) + \zeta_j^* \rho_j + \theta_j^*$$

substitute for ρ_{j-2} from Eq.(4.101)

$$- (A_j^{**} \rho_{j+2} + B_j^{**} \rho_j - D_j^{**}) / C_j^{**}$$

$$= \epsilon_j^* (\epsilon_{j+1} \rho_j + \zeta_{j+1} h_{j+1} + \theta_{j+1}) + \zeta_j^* \rho_j + \theta_j^*$$

$$\rho_j (-B_j^{**} / C_j^{**} - \epsilon_j^* \epsilon_{j+1} - \zeta_j^*) = h_{j+1} \epsilon_j^* \zeta_{j+1}$$

$$+ \frac{A_j^{**}}{C_j^{**}} \rho_{j+2} + (\epsilon_j^* \theta_{j+1} + \theta_j^* - D_j^{**} / C_j^{**})$$

comparing with Eq.(4.120)

$$\alpha_{j+2}^* = \epsilon_j^* \zeta_{j+1} / X ; \quad \beta_{j+2}^* = A_j^{**} / (C_j^{**} X) ;$$

$$\delta_{j+2}^* = (\epsilon_j^* \theta_{j+1} + \theta_j^* - D_j^{**} / C_j^{**}) / X \quad (4.125)$$

$$\text{where } X = -B_j^{**} / C_j^{**} - \epsilon_j^* \epsilon_{j+1} - \zeta_j^* .$$

(c) Subcritical Flow (Left to Right)

Eqs.(4.106) to (4.109) become

$$h_{j-2} = \alpha_j \rho_{j-1} + \beta_j Q_j + \delta_j \quad (4.126)$$

$$Q_{j-2} = \epsilon_j \rho_{j-1} + \zeta_j h_j + \theta_j \quad (4.127)$$

$$\rho_{j-2} = \beta_j^* \rho_j + \delta_j^* \quad (4.128)$$

$$\rho_{j-2} = \zeta_j^* \rho_j + \theta_j^* \quad (4.129)$$

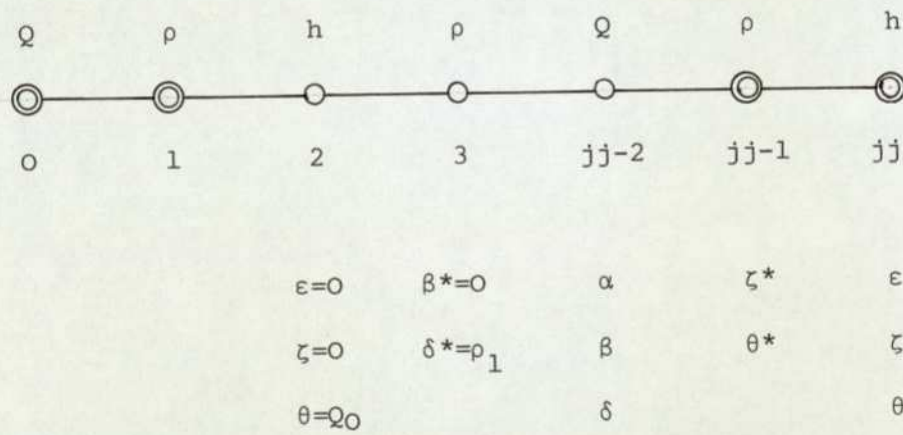


Fig.4.6 Boundary Conditions and Starting Values for Case (c)

Since α_j^* and ϵ_j^* in Eqs.(4.120) and (4.121) respectively equal zero, this gives from case (b)

$$\epsilon_{j+2} = \left[-B_j^* - \zeta_{j+1}^* (D_j^* + E_j^* \alpha_j) \right] / X ;$$

$$\zeta_{j+2} = -A_j^* / X ;$$

$$\theta_{j+2} = \left[F_j^* - E_j^* \delta_j - \theta_{j+1}^* (D_j^* + E_j^* \alpha_j) \right] / X \quad (4.130)$$

where $X = C_j^* + E_j^* \beta_j$

similarly

$$\alpha_{j+2} = \left[-B_j - (E_j \epsilon_j + D_j) \beta_{j+1}^* \right] / X ;$$

$$\beta_{j+2} = -A_j / X ;$$

$$\delta_{j+2} = \left[F_j - D_j \delta_{j+1}^* - (\theta_j + \epsilon_j \delta_{j+1}^*) E_j \right] / X \quad (4.131)$$

where $X = C_j + E_j \zeta_j$

From Eqs.(4.101) and (4.128)

$$A_j^{**}\rho_{j+2} + B_j^{**}\rho_j + C_j^{**}(\beta_j^*\rho_j + \delta_j^*) = D_j^{**}$$

$$\rho_j(B_j^{**} + C_j^{**}\beta_j^*) = -A_j^{**}\rho_{j+2} + (D_j^{**} - C_j^{**}\delta_j^*)$$

comparing with Eq.(4.129)

$$\zeta_{j+2}^* = \frac{-A_j^{**}}{(B_j^{**} + C_j^{**}\beta_j^*)} ; \quad \theta_{j+2}^* = \frac{(D_j^{**} - C_j^{**}\delta_j^*)}{(B_j^{**} + C_j^{**}\beta_j^*)} \quad (4.132)$$

From Eqs.(4.101) and (4.129)

$$A_j^{**}\rho_{j+2} + B_j^{**}\rho_j + C_j^{**}(\zeta_j^*\rho_j + \theta_j^*) = D_j^{**}$$

$$\rho_j(B_j^{**} + C_j^{**}\zeta_j^*) = -A_j^{**}\rho_{j+2} + (D_j^{**} - C_j^{**}\theta_j^*)$$

comparing with Eq.(4.128)

$$\beta_{j+2}^* = \frac{-A_j^{**}}{(B_j^{**} + C_j^{**}\zeta_j^*)} ; \quad \delta_{j+2}^* = \frac{(D_j^{**} - C_j^{**}\theta_j^*)}{(B_j^{**} + C_j^{**}\zeta_j^*)} \quad (4.133)$$

(d) Supercritical Flow (Left to Right)

Eqs. (4.106) to (4.109) become

$$h_{j-2} = \alpha_j \rho_{j-1} + \delta_j \quad (4.134)$$

$$Q_{j-2} = \epsilon_j \rho_{j-1} + \theta_j \quad (4.135)$$

$$\rho_{j-2} = \beta_j^* \rho_j + \delta_j^* \quad (4.136)$$

$$\rho_{j-2} = \zeta_j^* \rho_j + \theta_j^* \quad (4.137)$$

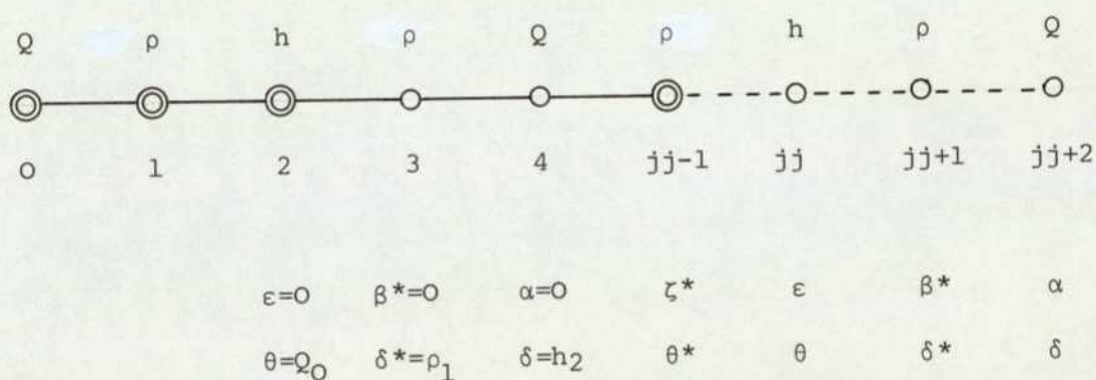


Fig.4.7 Boundary Conditions and Starting Values for Case (d)

From Eq. (4.86)

$$A_{j-3} Q_{j-1} + B_{j-3} \rho_{j-2} + C_{j-3} h_{j-3} + D_{j-3} \rho_{j-4} + E_{j-3} Q_{j-5} = F_{j-3}$$

substitute Eq.(4.134) for h_{j-3} , Eq.(4.135) for Q_{j-5} and
Eq.(4.136) for ρ_{j-4}

$$A_{j-3}Q_{j-1} + B_{j-3}\rho_{j-2} + C_{j-3}(\alpha_{j-1}\rho_{j-2} + \delta_{j-1}) \\ + D_{j-3}(\beta_{j-2}^*\rho_{j-2} + \delta_{j-2}^*) + E_{j-3}(\epsilon_{j-3}\rho_{j-4} + \theta_{j-3}) = F_{j-3}$$

substitute Eq.(4.136) for ρ_{j-4}

$$A_{j-3}Q_{j-1} + B_{j-3}\rho_{j-2} + C_{j-3}(\alpha_{j-1}\rho_{j-2} + \delta_{j-1}) \\ + D_{j-3}(\beta_{j-2}^*\rho_{j-2} + \delta_{j-2}^*) + E_{j-3}\epsilon_{j-3}(\beta_{j-2}^*\rho_{j-2} + \delta_{j-2}^*) \\ + E_{j-3}\theta_{j-3} = F_{j-3} \\ A_{j-3}Q_{j-1} = -\rho_{j-2}(B_{j-3} + C_{j-3}\alpha_{j-1} + D_{j-3}\beta_{j-2}^* + E_{j-3}\epsilon_{j-3}\beta_{j-2}^*) \\ - (C_{j-3}\delta_{j-1} + D_{j-3}\delta_{j-2}^* + E_{j-3}\epsilon_{j-3}\delta_{j-2}^* + E_{j-3}\theta_{j-3} - F_{j-3})$$

substitute Eq.(4.137) for ρ_{j-2}

$$A_{j-3}Q_{j-1} = - (\zeta_j^* \rho_j + \theta_j^*) (B_{j-3} + C_{j-3}\alpha_{j-1} + D_{j-3}\beta_{j-2}^*$$

$$+ E_{j-3}\epsilon_{j-3}\beta_{j-2}^*) - [C_{j-3}\delta_{j-1} + D_{j-3}\delta_{j-2}^*$$

$$+ E_{j-3}(\epsilon_{j-3}\delta_{j-2}^* + \theta_{j-3}) - F_{j-3}]$$

$$A_{j-3}Q_{j-1} = - \rho_j \zeta_j^* (B_{j-3} + C_{j-3}\alpha_{j-1} + D_{j-3}\beta_{j-2}^*$$

$$+ E_{j-3}\epsilon_{j-3}\beta_{j-2}^*) - [\theta_j^* (B_{j-3} + C_{j-3}\alpha_{j-1} + D_{j-3}\beta_{j-2}^*$$

$$+ E_{j-3}\epsilon_{j-3}\beta_{j-2}^*) + C_{j-3}\delta_{j-1} + D_{j-3}\delta_{j-2}^*$$

$$+ E_{j-3}(\epsilon_{j-3}\delta_{j-2}^* + \theta_{j-3}) - F_{j-3}]$$

comparing with Eq.(4.135)

$$\epsilon_{j+1} = - \frac{\zeta_j^*}{A_{j-3}} (B_{j-3} + C_{j-3}\alpha_{j-1} + D_{j-3}\beta_{j-2}^* + E_{j-3}\epsilon_{j-3}\beta_{j-2}^*) ;$$

$$\begin{aligned}
\theta_{j+1} = & -\frac{1}{A_{j-3}} \left[\theta_j^* (B_{j-3} + C_{j-3} \alpha_{j-1} + D_{j-3} \beta_{j-2}^* \right. \\
& + E_{j-3} \epsilon_{j-3} \beta_{j-2}^*) + C_{j-3} \delta_{j-1} + D_{j-3} \delta_{j-2}^* \\
& \left. + E_{j-3} (\epsilon_{j-3} \delta_{j-2}^* + \theta_{j-3}) - F_{j-3} \right] \quad (4.138)
\end{aligned}$$

From Eq. (4.87)

$$A_{j-3}^* h_{j-1} + B_{j-3}^* \rho_{j-2} + C_{j-3}^* Q_{j-3} + D_{j-3}^* \rho_{j-4} + E_{j-3}^* h_{j-5} = F_{j-3}^*$$

substitute Eq. (4.135) for Q_{j-3} , Eq. (4.134) for h_{j-5} and
Eq. (4.137) for ρ_{j-4}

$$\begin{aligned}
& A_{j-3}^* h_{j-1} + B_{j-3}^* \rho_{j-2} + C_{j-3}^* (\epsilon_{j-1} \rho_{j-2} + \theta_{j-1}) \\
& + D_{j-3}^* (\zeta_{j-2}^* \rho_{j-2} + \theta_{j-2}^*) + E_{j-3}^* (\alpha_{j-3} \rho_{j-4} + \delta_{j-3}) = F_{j-3}^*
\end{aligned}$$

substitute Eq. (4.137) for ρ_{j-4}

$$\begin{aligned}
& A_{j-3}^* h_{j-1} + B_{j-3}^* \rho_{j-2} + C_{j-3}^* (\epsilon_{j-1} \rho_{j-2} + \theta_{j-1}) \\
& + D_{j-3}^* (\zeta_{j-2}^* \rho_{j-2} + \theta_{j-2}^*) + E_{j-3}^* \alpha_{j-3} (\zeta_{j-2}^* \rho_{j-2} + \theta_{j-2}^*) \\
& + E_{j-3}^* \delta_{j-3} = F_{j-3}^*
\end{aligned}$$

substitute Eq.(4.136) for ρ_{j-2}

$$\begin{aligned}
A_{j-3}^* h_{j-1} &= - \rho_{j-2} (B_{j-3}^* + C_{j-3}^* \epsilon_{j-1} + D_{j-3}^* \zeta_{j-2}^* \\
&+ E_{j-3}^* \alpha_{j-3} \zeta_{j-2}^*) - (C_{j-3}^* \theta_{j-1} + D_{j-3}^* \theta_{j-2}^* + E_{j-3}^* \alpha_{j-3} \theta_{j-2}^* \\
&+ E_{j-3}^* \delta_{j-3} - F_{j-3}^*)
\end{aligned}$$

substitute Eq.(4.136) for ρ_{j-2}

$$\begin{aligned}
A_{j-3}^* h_{j-1} &= - (\beta_j^* \rho_j + \delta_j^*) (B_{j-3}^* + C_{j-3}^* \epsilon_{j-1} + D_{j-3}^* \zeta_{j-2}^* \\
&+ E_{j-3}^* \alpha_{j-3} \zeta_{j-2}^*) - (C_{j-3}^* \theta_{j-1} + D_{j-3}^* \theta_{j-2}^*
\end{aligned}$$

$$+ E_{j-3}^* \alpha_{j-3} \theta_{j-2}^* + E_{j-3}^* \delta_{j-3} - F_{j-3}^*)$$

$$A_{j-3}^* h_{j-1} = - \rho_j \beta_j^* (B_{j-3}^* + C_{j-3}^* \epsilon_{j-1} + D_{j-3}^* \zeta_{j-2}^*$$

$$+ E_{j-3}^* \alpha_{j-3} \zeta_{j-2}^*) - \left[\delta_j^* (B_{j-3}^* + C_{j-3}^* \epsilon_{j-1} + D_{j-3}^* \zeta_{j-2}^*$$

$$+ E_{j-3}^* \alpha_{j-3} \zeta_{j-2}^*) + C_{j-3}^* \theta_{j-1} + D_{j-3}^* \theta_{j-2}^*$$

$$+ E_{j-3}^* (\alpha_{j-3} \theta_{j-2}^* + \delta_{j-3}) - F_{j-3}^* \quad]$$

comparing with Eq. (4.134)

$$\alpha_{j+1} = - \frac{B_j^*}{A_{j-3}^*} (B_{j-3}^* + C_{j-3}^* \epsilon_{j-1} + D_{j-3}^* \zeta_{j-2}^* + E_{j-3}^* \alpha_{j-3} \zeta_{j-2}^*) ;$$

$$\delta_{j+1} = - \frac{1}{A_{j-3}^*} \left[\delta_j^* (B_{j-3}^* + C_{j-3}^* \epsilon_{j-1} + D_{j-3}^* \zeta_{j-2}^*$$

$$+ E_{j-3}^* \alpha_{j-3} \zeta_{j-2}^*) + C_{j-3}^* \theta_{j-1} + D_{j-3}^* \theta_{j-2}^*$$

$$+ E_{j-3}^* (\alpha_{j-3} \theta_{j-2}^* + \delta_{j-3}) - F_{j-3}^* \quad] \quad (4.139)$$

From Eqs.(4.101) and (4.136)

$$A_j^{**}\rho_{j+2} + B_j^{**}\rho_j + C_j^{**} (\beta_j^*\rho_j + \delta_j^*) = D_j^{**}$$

$$\rho_j (B_j^{**} + C_j^{**}\beta_j^*) = -A_j^{**}\rho_{j+2} + (D_j^{**} - C_j^{**}\delta_j^*)$$

comparing with Eq.(4.137)

$$\zeta_{j+2}^* = \frac{-A_j^{**}}{(B_j^{**} + C_j^{**}\beta_j^*)} ; \quad \theta_{j+2}^* = \frac{(D_j^{**} - C_j^{**}\delta_j^*)}{(B_j^{**} + C_j^{**}\beta_j^*)} \quad (4.140)$$

From Eqs.(4.101) and (4.137)

$$A_j^{**}\rho_{j+2} + B_j^{**}\rho_j + C_j^{**} (\zeta_j^*\rho_j + \theta_j^*) = D_j^{**}$$

$$\rho_j (B_j^{**} + C_j^{**}\zeta_j^*) = -A_j^{**}\rho_{j+2} + (D_j^{**} - C_j^{**}\theta_j^*)$$

comparing with Eq.(4.136)

$$\beta_{j+2}^* = \frac{-A_j^{**}}{(B_j^{**} + C_j^{**}\zeta_j^*)} ; \quad \delta_{j+2}^* = \frac{(D_j^{**} - C_j^{**}\theta_j^*)}{(B_j^{**} + C_j^{**}\zeta_j^*)} \quad (4.141)$$

Depending upon the algorithmic structure recurrence coefficients are calculated from the left-hand side of the grid by setting the boundary data there to the starting recurrence coefficients as shown in the Figs.(4.4) to (4.7). Eqs.(4.106) to (4.109) are then used alternately (with the appropriate coefficients set to zero) from the right-hand side of the grid in order to calculate the dependent variables.

At the seaward end of the grid the density is a function of the upland discharge and the degree of interfacial mixing, therefore it cannot be used directly as a boundary condition. The density gradient at the mouth of an estuary is however, independent of the upstream flow conditions for well-mixed estuaries. Assuming that density gradients of different magnitude occur in both the upper and lower layers at the mouth of an estuary, the density in either layer is calculated as follows.

Let the density gradient in a layer be denoted by P, then

$$P_{jj} = (\rho_{jj+1} - \rho_{jj-1}) / (2\Delta x) \quad (4.142)$$

For Subcritical flows

From Eqs.(4.120) and (4.142)

$$\rho_{jj-1} = (\alpha_{jj+1}^* h_{jj} + 2\Delta x P_{jj} \beta_{jj+1}^* + \delta_{jj+1}^*) / (1 - \beta_{jj+1}^*) \quad (4.143)$$

For Supercritical Flows

From Eqs. (4.113) and (4.142)

$$\begin{aligned} \rho_{jj-1} = & (\epsilon_{jj-1}^* Q_{jj-2} + \eta_{jj-1}^* h_{jj} + \theta_{jj-1}^* \\ & + 2\Delta x P_{jj-2}) / (1 - \zeta_{jj-1}^*) \end{aligned} \quad (4.144)$$

If the flow direction is from left to right then in Eq. (4.143) $\alpha_{jj+1}^* = 0$, and in Eq. (4.144) ϵ_{jj-1}^* and $\eta_{jj-1}^* = 0$.

4.5 Computations for Partly-Mixed Flows

Estuaries like the Thames which have a low upland flow and follow a meandering course fall into the category of partly-mixed estuaries. In such estuaries the interface formed between the salt water wedge at the bed and the overflowing brackish water is partly eroded away due to mixing between the layers. This mixing takes place when the relative velocity between the layers exceeds a certain critical value, and occurs especially at the bends in the river where secondary currents cause highly turbulent flow conditions. The degree of mixing is however not sufficient to form a homogeneous vertical density distribution, and a partly-mixed estuary still exhibits the characteristics of a two-layer flow.

The main differences between an estuary of the fully stratified type and that of the partly-mixed type are the flow directions in the upper and lower layers during a tidal cycle. In a fully stratified flow, because of the high upland discharge and low tidal range, the fresh water velocity is downstream throughout the tidal cycle while the salt wedge moves slowly upstream and downstream as the tide changes from flood to ebb flow. In a partly-mixed estuary however, such as the Fraser River in British Columbia (63) both layers can ebb and flood at almost the same time, the maximum velocities in the upper layer being almost double those in the lower layer. Clearly flow conditions in the Fraser River resemble those of a well-mixed estuary except that the two-layer stratification is still prominent.

For the mathematical modelling of a partly-mixed estuary a decision has to be made as to whether a single layer well-mixed model is to be used or a two-layer stratified model. The actual program used would be the two-layer stratified program since this is built so as to operate also as a single layer model, as it must do upstream of the wedge. However, the boundary conditions used in the model necessitate a decision as to whether the partly-mixed estuary behaves more like a well-mixed estuary than as a two-layer stratified estuary.

A third alternative exists, and that is to treat the partly-mixed estuary as consisting of several layers. Although this may appear reasonable at first sight, in reality it is not for the following reasons. Firstly, the computer time greatly increases as the number of layers increase. For example, whereas a two-layer flow requires a four level computational scheme, a three-layer flow would require an eight level scheme and a four-layer flow a twelve level scheme. This increase in computer costs would have to be justified by a substantial improvement in the accuracy of the model. However, the accuracy of any mathematical model is only as good as the boundary data available and even for a two-layer model this is rather meagre, especially in regard to interfacial mixing. For a multi-layer stratified model it is economically impossible to supply boundary data of sufficient accuracy to justify using such a model for river estuarine flows.

In conclusion, a partly-mixed river estuary should be treated mathematically as a two-layer stratified estuary provided boundary data can be supplied for each layer. If the interface cannot be easily located and therefore boundary conditions for each layer cannot be reliably determined then it is best to treat the estuary as well-mixed.

4.6 Advantages of Extended Implicit Difference Schemes

It was shown in sections 3.4 and 3.5 that in order to compute the average longitudinal density variations in a well-mixed estuary two separate models are required. Firstly a dynamic model is needed which computes water levels and discharges along the estuary, and secondly a transport-diffusion model which calculates density variations. By inserting the density model between the time steps of the dynamic model a crude coupling was achieved which enabled computed velocities to be fed into the density model and computed densities to the dynamic model. In this way values of water level, discharge and density could be calculated for the estuary. This method was used again by the author (51) to calculate the average longitudinal density variations in each layer of a two-layer stratified flow.

In section 3.7, the stability conditions of this density model were calculated and found to be unconditionally stable as far as the von Neumann linear stability was concerned but conditionally stable for non-linear stability. Since this method of computing density must ignore the direction of the convective density characteristic, non-linear instabilities in the form of 'parasitic waves' are generated by the computation. In order to suppress these parasitic waves, which are caused by the generation of higher wave numbers, the computation is restricted to very small distance and time steps. As an example, for the well-mixed River Hooghly in

West Bengal, the maximum distance and time steps were calculated to be 100m and 25s respectively.

When extended implicit finite difference schemes are used the direction of the convective density characteristic is taken into account by changing the algorithmic structure. In this way there are no 'parasitic waves' developed by the computation and the scheme is governed only by its linear stability conditions, which for both the dynamic and transport-diffusion equation systems are unconditionally stable. Applying this new scheme to the River Hooghly (53), distance and time steps of 5km and 900s respectively were used giving a Courant number of 2.

It can be concluded that although the extended schemes are mathematically more complicated than existing schemes, especially for stratified flows, the fact that they can be run thousands of times faster on the computer than existing schemes proves their worth. In fact for the River Hooghly with a tidal limit 280km from the sea, the computer time required to calculate water level, discharge and salinity variations was five minutes on an IBM 360/65 machine at a cost of about one hundred pounds. To make the same computation using existing finite difference schemes would have required about six days computer time at a cost exceeding one hundred thousand pounds.

CHAPTER 5

ANALYSIS OF SALT WEDGE FLOWS

5.1 The Arrested Saline Wedge

In deep estuaries where the tidal range is very small and the upland discharge constant a wedge of salt water intrudes into the estuary from the sea and remains fixed in position. The fresh water flowing over the wedge imparts a velocity to the salt wedge just below the interface and the resulting velocity gradients if large enough can induce a transfer of mass across the interface in both directions. In many cases the velocity of the fresh water over the landward half of the wedge is not high enough to induce entrainment across the interface and entrainment and mixing are usually confined to the seaward half of the wedge where the velocities are higher.

The removal of salt water near the interface is balanced by an inflow of salt water from the sea near the bottom of the salt wedge. In this way the salt wedge remains constant in position even though there is a circulation velocity set up within the wedge, as shown in Fig.1.1.

At any cross-section the average velocity in the salt wedge is almost zero and for a typical arrested salt wedge shown diagrammatically in Fig.5.1, the important variables can be expressed in the following dimensionless groups.

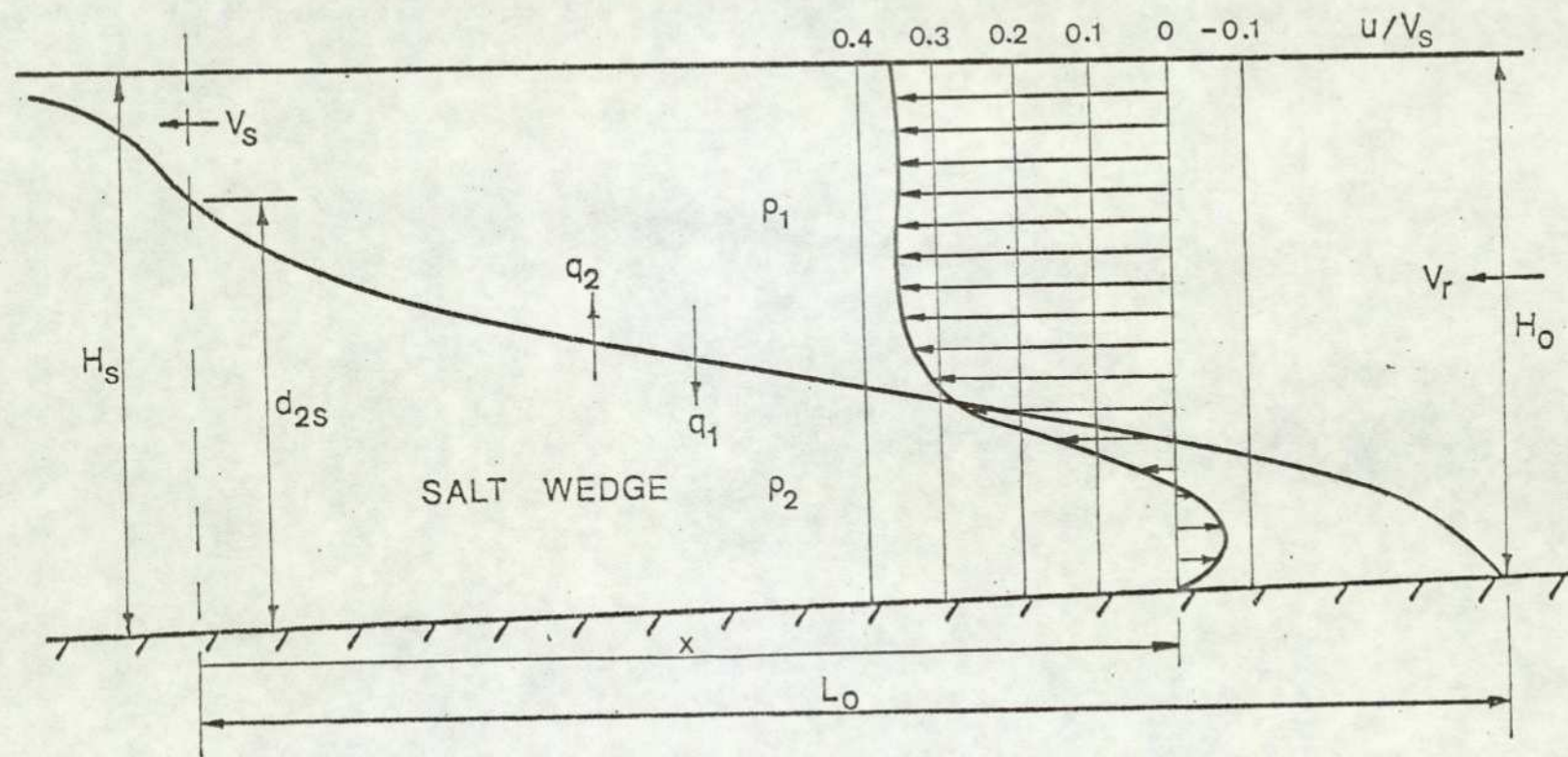


FIG. 5.1 DIAGRAM OF ARRESTED SALT WEDGE

$$\frac{L_0}{H_0} = \Phi \left[\left(\frac{H_0 V_r}{H_s V_d} \right), \left(\frac{V_d H_0}{v_1} \right), \left(\frac{H_0}{b} \right) \right] \quad (5.1)$$

$$\text{where } V_d = \left[g H_0 (1 - \epsilon) \right]^{1/2} \quad (5.2)$$

is called the densimetric velocity; $\frac{V_d H_0}{v_1}$ the densimetric Reynolds number, and $\epsilon = \rho_1 / \rho_2$.

If the flow is critical at the river mouth then

$$V_s^2 = g (H_s - d_{2s}) (1 - \epsilon) \quad (5.3)$$

$$\text{also } V_s (H_s - d_{2s}) = V_r H_0 \quad (5.4)$$

$$\text{thus } d_{2s}/H_s = 1 - \left[(H_0 V_r) / (H_s V_d) \right]^{2/3} \quad (5.5)$$

Experiments were carried out to find an expression for the length of an arrested saline wedge in terms of the variables shown in Eq.(5.1).

The apparatus used is shown in Figs. 5.2a,b and consisted of a perspex channel 2.5m long by 3.5cm wide gradually enlarging into an experimental sea 0.9m wide by 0.6m long and 0.4m deep, all mounted on top of a tilting flume. Fresh water was introduced at the one end and withdrawn at the seaward end using a system of syphon tubes supported in a polystyrene block. The horizontal dimensions of the model were scaled such that a typical arrested saline wedge of length 20km became 2m in the model, and a river width of 350m became 3.5cm giving a scale ratio $X = 10^4$. The vertical scale was chosen such that a depth of 15m became 6cm in the model, giving a scale ratio $Y = 250$. These

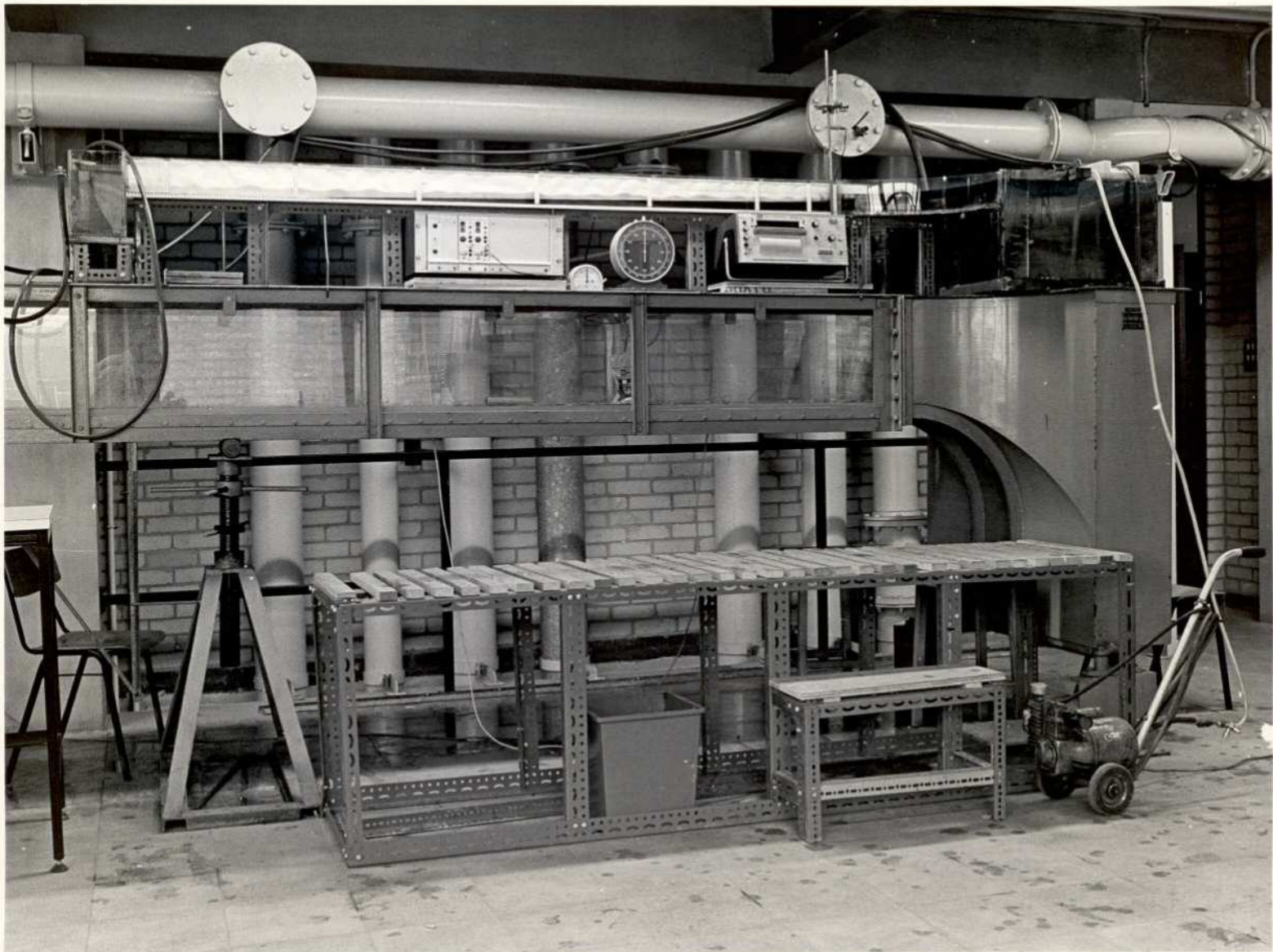


FIG. 5.2a APPARATUS USED IN EXPERIMENTS

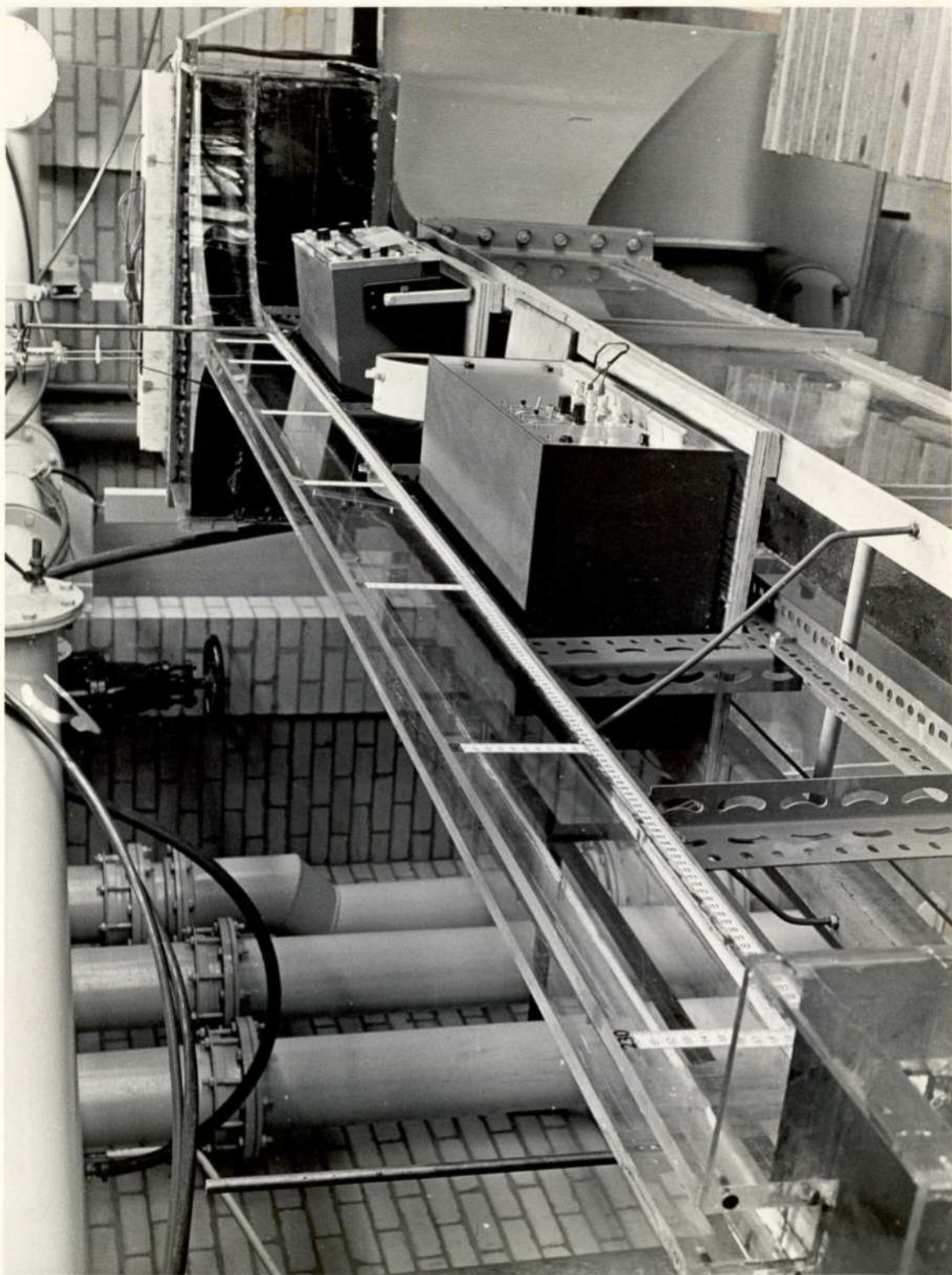


FIG. 5.2b DOWNSTREAM VIEW OF CHANNEL

prototype values correspond roughly to those for the salt wedge in the South Pass of the Mississippi River (79). For the successful modelling of a stratified flow it is necessary that the densimetric Froude number should be the same in both model and prototype. Therefore the first dimensionless term on the right hand side of Eq.(5.1), should be the same in model and prototype. This is equivalent to saying that the characteristic directions should correspond in model and prototype and such a condition can be shown to be sufficient as well as necessary for correct scaling (3). Using these horizontal and vertical scales the velocity ratio becomes $Y^{0.5} = 15.8$; the discharge ratio $XY^{1.5} = 4.0 \times 10^7$, and the time ratio $X/Y^{0.5} = 633$. Hence a river discharge of $2200\text{m}^3/\text{s}$ in the South Pass of the Mississippi River becomes 3.3 litres/min, which is a convenient flow rate to control with syphon tubes. These syphon tubes did not pass right through the polystyrene block but ended up in slots 1.5cm deep just above the base so that salt water could not be sucked up into the syphon tubes through the fresh water. By hanging suitable weights beneath the block, it could be made to float rigidly at a level such that the water surface was just beneath the roof of the slots.

The procedure for creating an arrested salt wedge in the apparatus was to first ensure that the salt water in the sea end was at a density of at least 1032kg/m^3 . The reason for maintaining this high salinity for the salt

water is that the interfacial stability is proportional to the relative density of the two liquids. By maintaining a high relative density between the fresh and salt water perfectly stable arrested salt wedges could be created which were free from interfacial waves. This salt water was then coloured with potassium permanganate solution and with the flume tilted back a wedge of salt water initially lay in the dry channel. By slowly increasing the fresh water supply and gradually lowering the flume to the required slope a salt wedge was formed in the channel with the fresh water flowing over it. By adjusting the syphon tubes they could be made to eject fresh water at the seaward end of the model at the same rate as the inflow, thereby ensuring that an arrested salt wedge was formed. After several minutes of constant flow to ensure that the salt wedge did not move in the channel the depth of the interface and surface were recorded along the whole length of the channel by direct measurement from scales on the side of the channel. The temperature of the fresh water was then recorded together with the slope of the channel. After a block had been wedged into the channel to prevent the salt water from flowing into the fresh water inlet the fresh water supply pipes were removed and their discharge measured volumetrically using a two-litre measuring jar. This procedure was repeated for different channel slopes, fresh water discharges and depths of water at the seaward end until enough results had been obtained to draw the required graphs. In all, one hundred and forty

eight stable arrested salt wedges were examined and the data from these used to plot Figs. 5.3 and 5.4 .

From Eq.(5.1) the ratio L_0/H_0 is seen to be a function of the densimetric Froude number $(H_0 V_r)/(H_s V_d)$; the densimetric Reynolds number $(V_d H_0)/\nu_1$, and the depth to channel width ratio H_0/b . Because only one channel was available for these experiments the influence of H_0/b , which varied from 0.5-1.5 was impossible to assess. From the results of Keulegan (79) who experimented on three channel widths namely 5.2, 11.3 and 22.9cm; for a change in H_0/b ratio from 1-2 the length of the arrested wedge decreased by 30% .

Figure 5.3 indicates that L_0/H_0 is strongly dependent on the densimetric Froude number but only weakly dependent on the densimetric Reynolds number. The slight dependence on the densimetric Reynolds number can be seen in Fig.5.4 , and from these two graphs the equation for the length of an arrested saline wedge becomes:-

$$\frac{L_0}{H_0} = A_0 \left(\frac{V_d H_0}{\nu_1} \right)^{0.45} \left(\frac{H_0 V_r}{H_s V_d} \right)^{-2.5} \quad (5.6)$$

for $H_0/b = 0.5 - 1.5$

The constant A_0 was found from the gradient of the line in Fig.5.5 , to be 1/30, giving a final equation for the length to depth ratio of a model arrested saline wedge of :-

$$\frac{L_0}{H_0} = \frac{1}{30} \left(\frac{V_d H_0}{\nu_1} \right)^{0.45} \left(\frac{H_0 V_r}{H_s V_d} \right)^{-2.5} \quad (5.7)$$

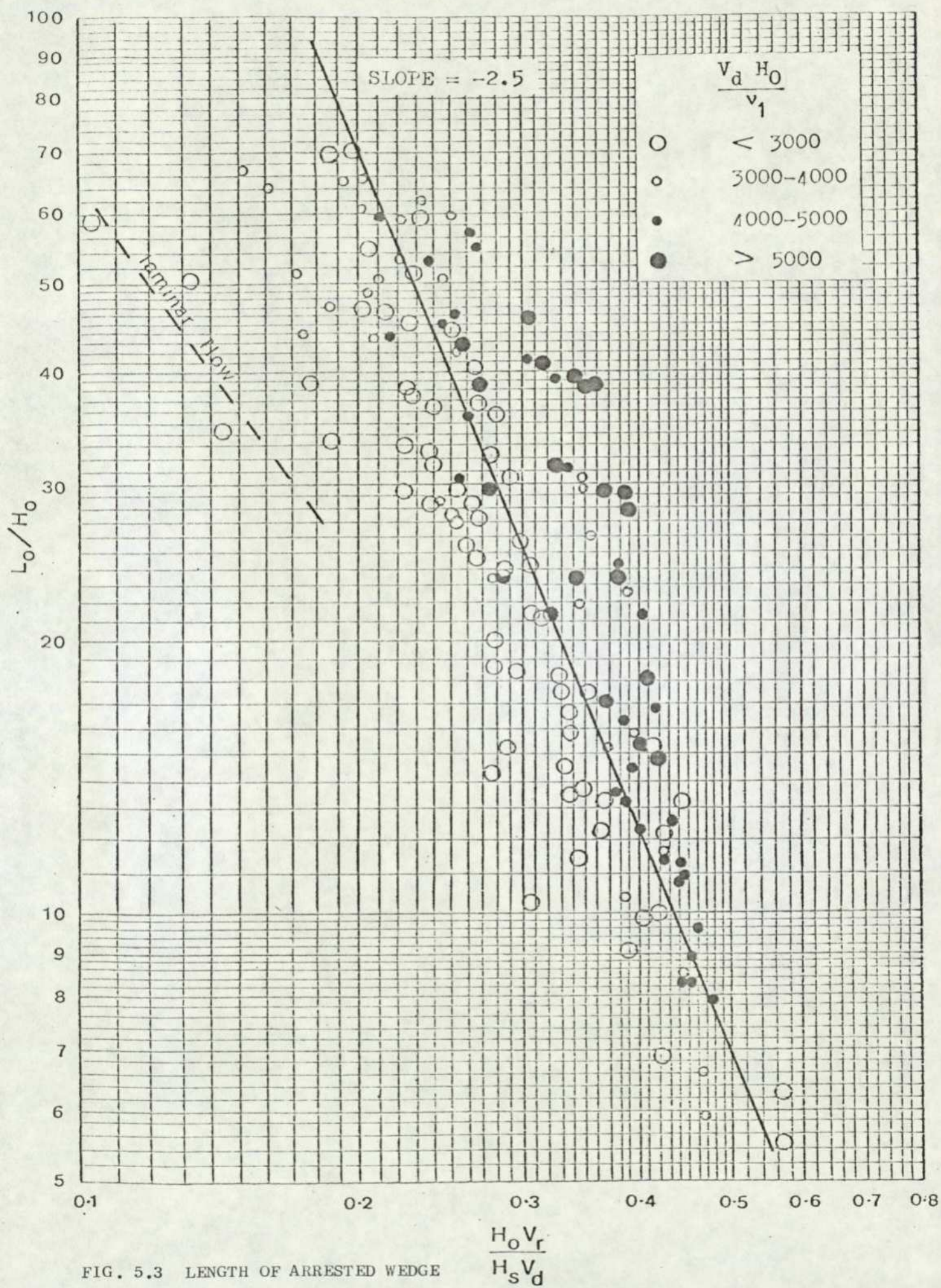


FIG. 5.3 LENGTH OF ARRESTED WEDGE

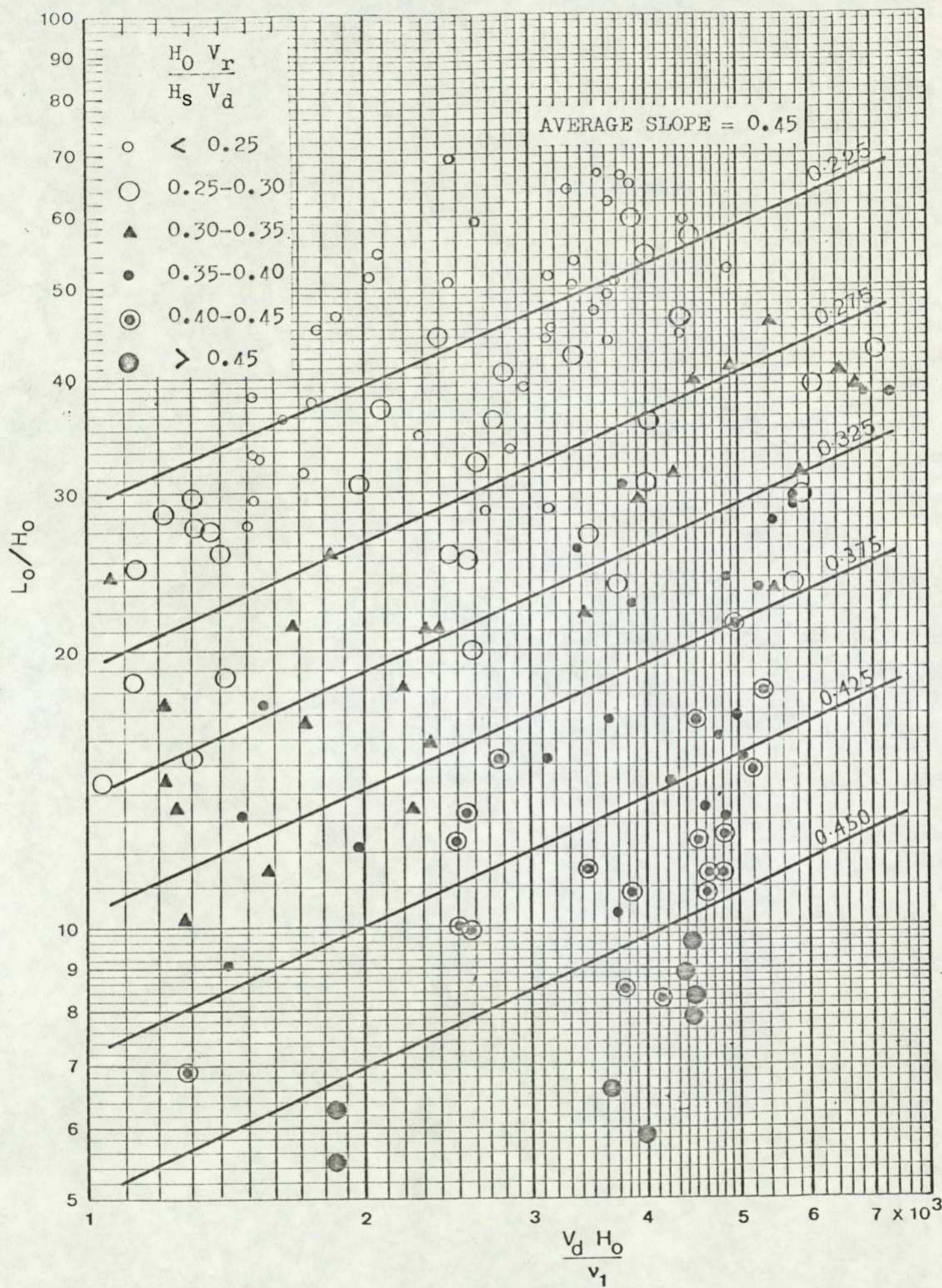


FIG. 5.4 LENGTH AGAINST DENSIMETRIC REYNOLDS NUMBER

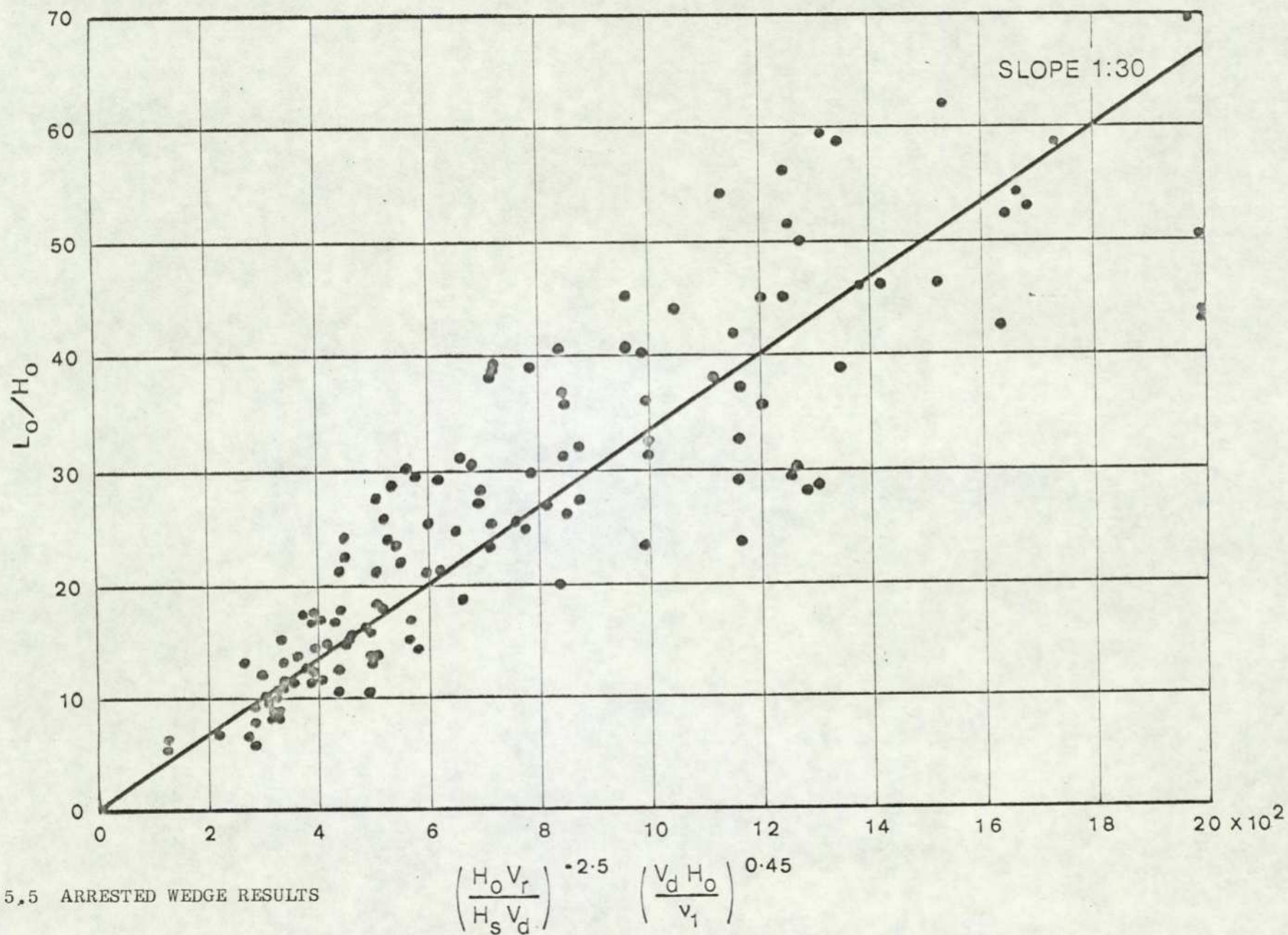


FIG. 5.5 ARRESTED WEDGE RESULTS

Although the above expression applies to a laboratory channel maintaining large flows, providing the interfacial and bottom shear stresses in the model are scaled with the prototype this equation should predict the length of a salt wedge in the prototype. The narrow laboratory channel is an advantage since the side walls provide increased frictional resistance to the flow. This is a necessary requirement in a physical river model since the viscous stresses are undervalued when scaling is made to Froude numbers alone.

As an example to the relevance of Eq.(5.7), the length of the salt wedge in the South Pass of the Mississippi River, using the data of Keulegan(79) will be computed. With a river discharge of 100,000 cfs the length of the arrested saline wedge is 14 miles. The width of the river is 1,500ft depth 45ft, relative density $(1 - \epsilon) = 0.02$ and water temperature 20°C .

$$V_r = 1.48 \text{ ft/s} , V_d = 5.38 \text{ ft/s} , \frac{V_d H_0}{v_1} = 2.25 \times 10^7$$

$$L_0/H_0 = \frac{1}{30} (2.25 \times 10^7)^{0.45} (1.48/5.38)^{-2.5} = 1708.7$$

$$L_0 = 1708.7 \times 45/5280 = \underline{14.56 \text{ miles}}$$

Thus the agreement between the observed and computed values is close.

Another equation for the length of an arrested saline wedge has been derived theoretically by Poggi(104). This equation depends on the value chosen for the interfacial friction coefficient f , defined by the relationship:-

$$\tau = \frac{1}{8} f \rho_2 V^2 \quad (5.8)$$

where τ is the interfacial shear stress.

The equation is:-

$$L_0 = \frac{8H_0}{f} \left[\frac{1}{20F_d} - \frac{1}{2} + \frac{3}{4} F_d^{1/3} - \frac{3}{10} F_d^{2/3} \right] \quad (5.9)$$

$$\text{where } F_d = (V_r/V_d)^2 \quad (5.10)$$

When the data for the South Pass of the River Mississippi is inserted into this equation a value of $f = 2.1 \times 10^{-3}$ is required to obtain the actual length of 14 miles. Since Keulegan's (79) experiments predicted interfacial friction factors of 2.0×10^{-3} for the South Pass of the River Mississippi the accuracy of Eq.(5.9) is worthy of note.

The condition for the flow in the upper layer to be critical at the mouth of the estuary is given by Eq.(5.5). The left and right hand sides of this equation were plotted as shown in Fig.5.6 for all the tests made on the arrested saline wedge. The results indicate that for most of the tests, flow conditions were critical at the mouth of the experimental river. The average affine form of the experimental arrested saline wedges is shown in Fig.5.7, which agrees exactly with the shape found by Keulegan(79). The actual shape for the wedge in the South West Pass of the River Mississippi is slightly different and is shown in Fig.5.7 as a broken line. The reason why the shapes are different is that all the experimental arrested salt wedges tested were completely stable whereas in a

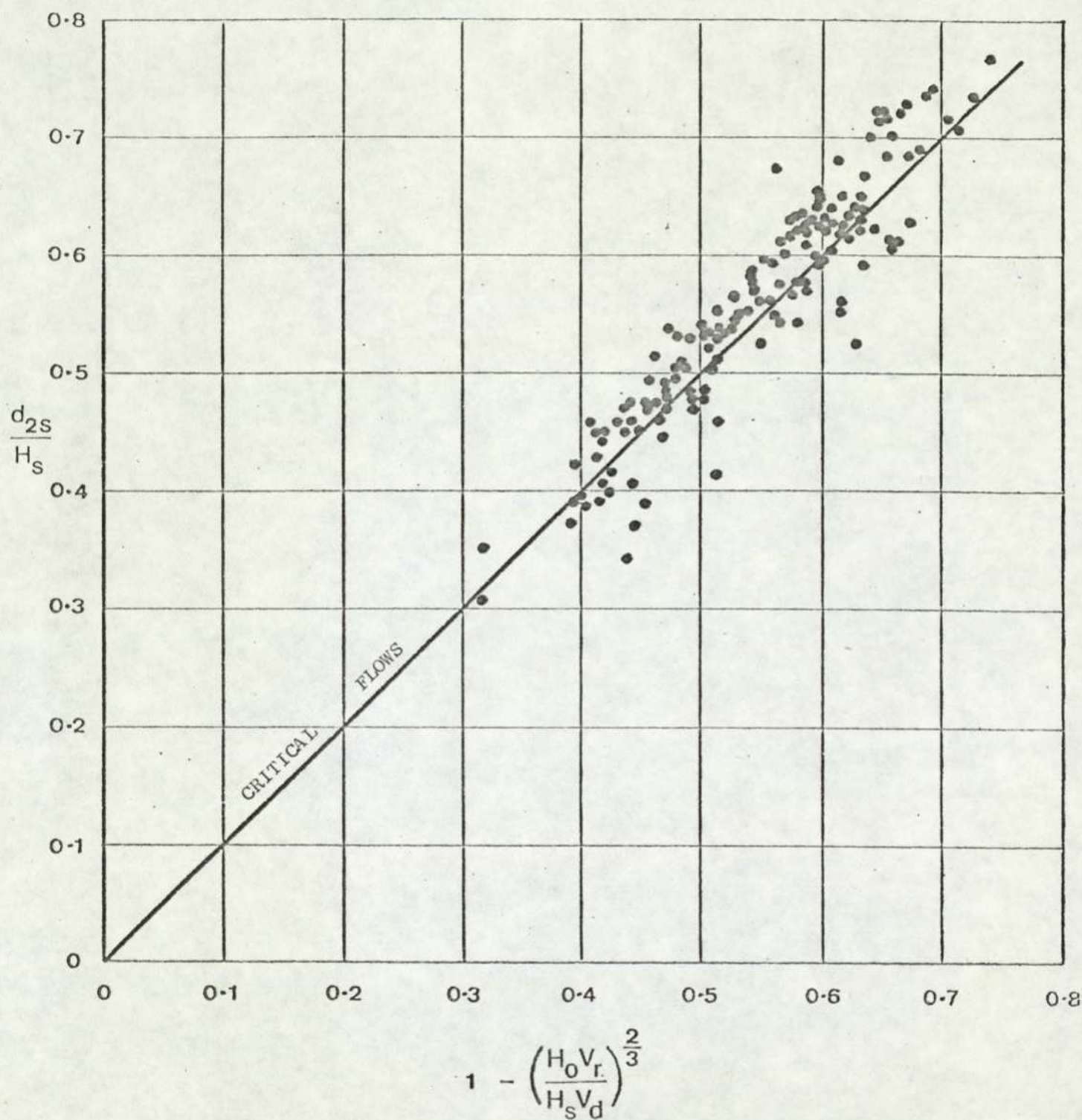


FIG. 5.6 DEPTH OF SALINE WATER AT RIVER MOUTH

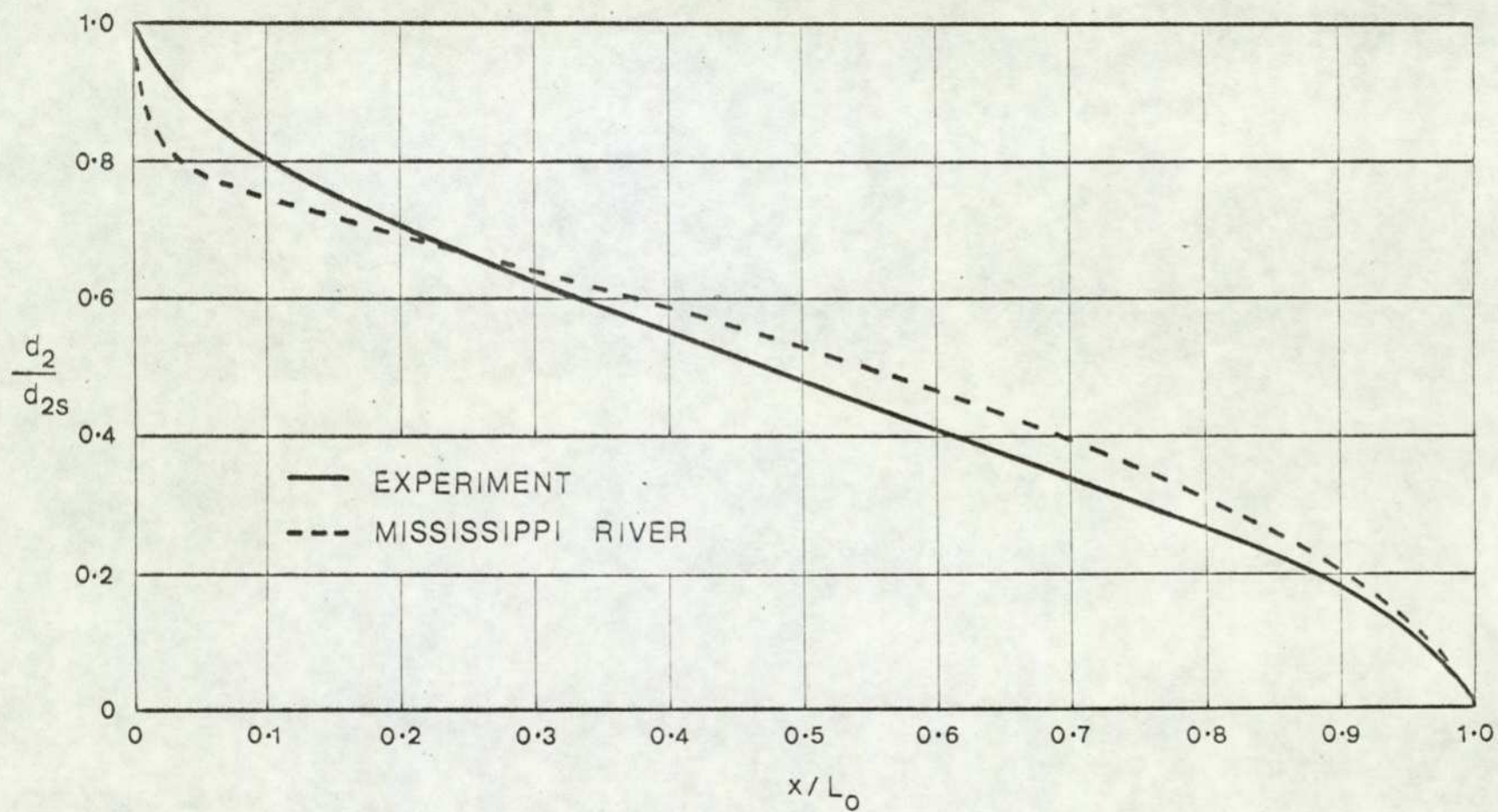


FIG. 5.7 AFFINE FORM OF ARRESTED SALINE WEDGE

real river the level of turbulence is higher and mixing between layers is present. Stommel and Farmer (1930) have shown that at the mouth of an estuary flow conditions in the upper layer are kept critical due to turbulent diffusion mixing eroding away the interface and consequently increasing the thickness of the upper layer. At the other end of the wedge critical conditions in the lower layer occur from $x/L_0 = 0.8 - 1.0$. Even though the velocities here are much smaller than at the mouth of the estuary, the velocities are still high enough to cause some entrainment. This mixing erodes the interface causing the lower layer thickness to increase thereby keeping the densimetric Froude numbers below unity.

5.2 Review of Existing Front Theory

In this section it is necessary to investigate the line or surface of demarcation which separates a stratified flow of two layers from an adjacent one layer flow. This line of separation between the two conditions of flow is called a fluid front and depending on the front conditions it can be conveniently classified as will be indicated later as either a St. Venant or a wave front.

The St. Venant front is characterised by the coincidence of two characteristics at zero depth and was first investigated by St. Venant to describe the fluid front caused by the sudden failure of a dam. From experimental studies of flows following the sudden failure of dams it is known that such a front cannot exist for this condition,

in fact another type of front characterised by a vertical wall of water exists which because of its physical resemblance to a bore or travelling wave is called a wave front.

The occurrence of these two types of fronts can be best illustrated when considering the motion of a salt wedge in an estuary. When the wedge is advancing upstream so that the front is invading its ambient fluid then the front is a wave front having a front celerity which is proportional to the relative velocity between the layers. As the relative velocity decreases due to a decrease in upland discharge for instance then the height of the front decreases. When the relative velocity reaches zero the front height becomes zero and since now two of the characteristics coincide in the time stationary position there is a transition from the wave front to the St. Venant front as shown by Abbott and Torbe (2). This St. Venant front persists as long as the wedge is retreating in its ambient fluid and can only become a wave front after two of the characteristics again coincide in the time stationary position.

The above discussion of the two different fronts indicate that the St. Venant front is always associated with supercritical flows retreating in their ambient fluid and the wave front to subcritical flows invading their ambient fluid. Because of the resemblance of the wave front to a hydraulic jump one can guess that

one of its characteristics must be close to the time stationary position. Rayleigh (110) in fact investigated the circumstances in which a two layer fluid becomes unstable and found that this was indeed when one of the characteristics became time stationary. It is now necessary to review various experiments and theories associated with the formation of fronts in order to discover if front conditions coincide with the stability conditions for real flows.

Wave fronts: The front portion of a moving underflow has been observed to have a characteristic 'head' or bulge of greater thickness than the uniform current behind it. Since the head represents a negligible length of a saline wedge, and it is the depth of the current following the head which influences the speed of the front, the shape of the head is of no significance in estuarine saline wedges. For underflows initially released into a shallow river the shape of the head may be of interest and experiments have been carried out by Braucher (26) to find this shape. The reason for the formation of this head is that a greater driving force is required at the front of the density current in order to overcome the work needed to accelerate the ambient fluid over the front and to overcome the interfacial and boundary resistance. Consequently this increased driving force for the head calls for an increase in the initial depth of the density current.

The speed of a gravity current advancing along the bed in an ambient fluid of infinite extent was

investigated by von Kármán (74) in 1940. Figure 5.8 shows a typical underflow density current, the depth of the current is denoted by d_2 and this advances at speed V , under an infinite ambient fluid of density ρ_1 . The relative velocity between layers is therefore V . If the density current is brought to rest with a Galilean frame moving at velocity V in the same direction to the lower layer, then by applying the Bernoulli equation to points A and B on the boundary of the layers yields.

$$\frac{p_A}{\rho_1 g} = \frac{p_B}{\rho_1 g} + d_2 + \frac{V^2}{2g} \quad (5.11)$$

but, since the lower layer is now at rest

$$\frac{p_A}{\rho_2 g} = \frac{p_B}{\rho_2 g} + d_2 \quad (5.12)$$

where p_A and p_B are the pressures at points A and B respectively. From Eqs.(5.11) and (5.12)

$$V = [2gd_2(\rho_2 - \rho_1) / \rho_1]^{1/2} \quad (5.13)$$

The same result was found by Engelund and Christensen (43) if the equation for an internally advancing density current is modified to that of a front advancing at the free surface.

If the ambient fluid is not of infinite extent the result is quite different, although for small differences in density between the layers, numerically the same (155).

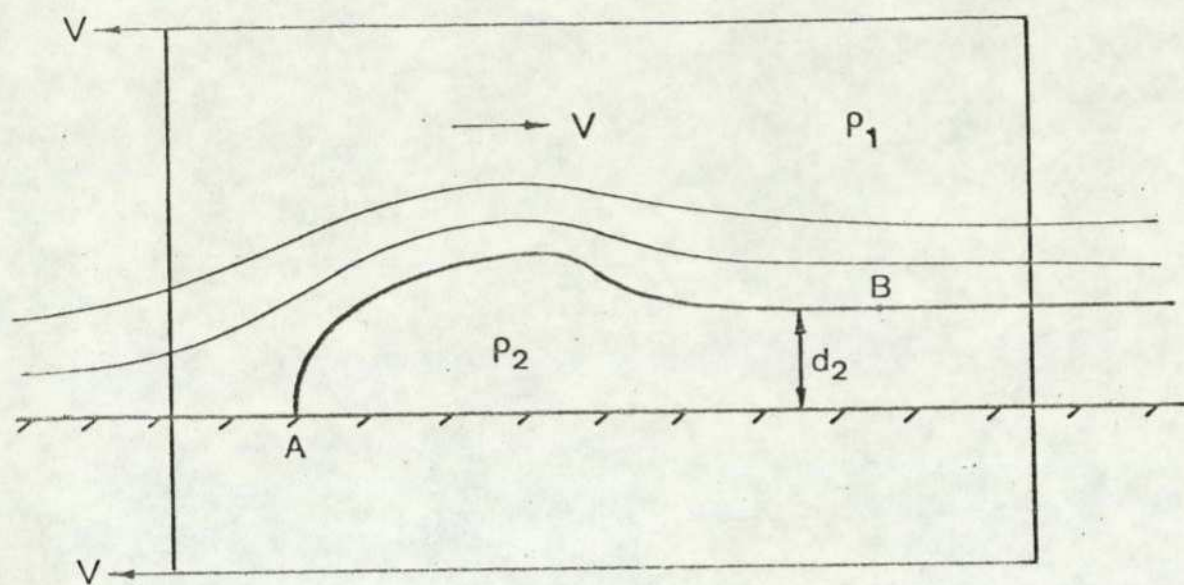


FIG. 5.8 UNDERFLOW DENSITY CURRENT

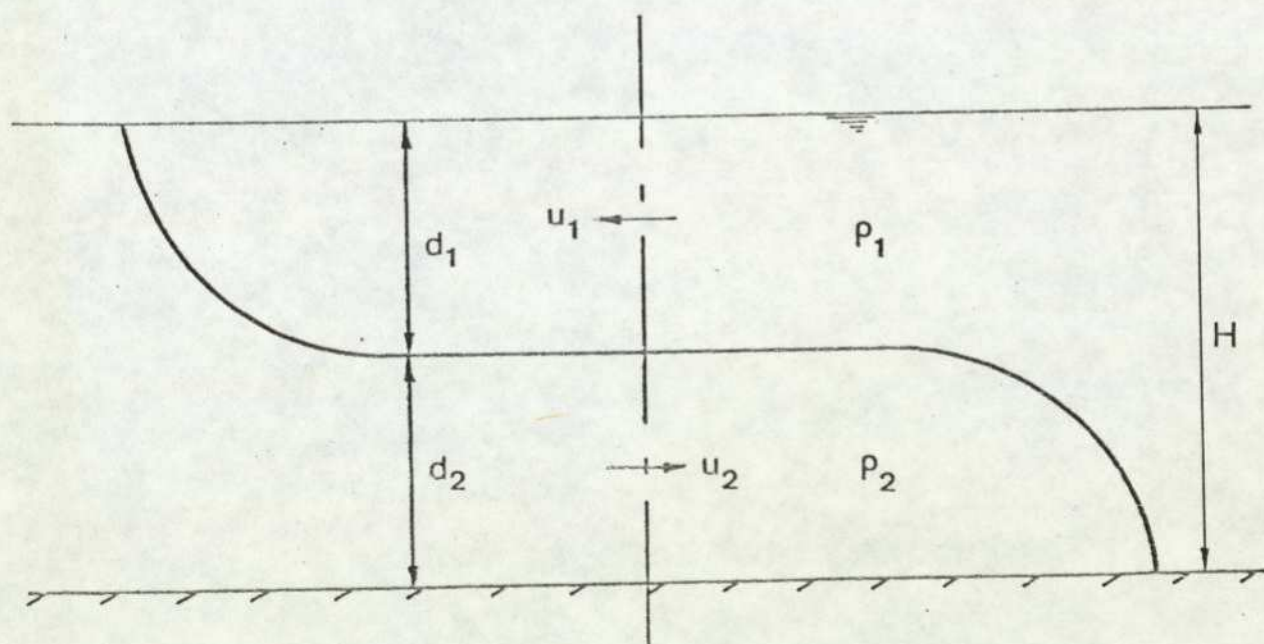


FIG. 5.9 EXCHANGE FLOW

Consider the exchange flow shown in Fig.5.9, where $d_1=d_2=d$, and $u_1=u_2=u$. The relative velocity V between layers is equal to $2u$. In time Δt the wedges will have advanced a distance $u \Delta t$, since for all practical purposes the two wedges can be assumed antisymmetric. The kinetic energy gained by both fluids per unit width is

$$0.5(\rho_1 + \rho_2) u^2 (u \Delta t d) \quad (5.14)$$

The lighter fluid gains potential energy of

$$[g \rho_1 (d/2)] (u \Delta t d) \quad (5.15)$$

and the heavier fluid loses potential energy of

$$[g \rho_2 (d/2)] (u \Delta t d) \quad (5.16)$$

From the principle of conservation of energy

$$(\rho_1 + \rho_2) u^2 = g d (\rho_1 - \rho_2) \quad (5.17)$$

$$\text{or} \quad V = [4 g d_2 (\rho_2 - \rho_1) / (\rho_2 + \rho_1)]^{1/2} \quad (5.18)$$

It can be seen that Eq.(5.18) differs from Eq.(5.13) only in that $2 \rho_1$ in the denominator of Eq.(5.13) is replaced by $(\rho_2 + \rho_1)$ in Eq.(5.18). This is due to the fact that the heavier fluid has to accelerate from a velocity of zero to u .

If $\rho_1 \approx \rho_2$ then since $(\rho_2 - \rho_1) / \rho_2 = (1 - \epsilon)$ and $g' = g (1 - \epsilon)$, Eqs.(5.13) and (5.18) become

$$V = 1.41 (g' d_2)^{1/2} \quad (5.19)$$

With d_2 in Eq.(5.19) replaced by $H/2$, since for an initially

formed exchange flow, H is the depth behind the front driving it forward.

$$V = (g' H)^{1/2} \quad (5.20)$$

Since $V/(g' H)^{1/2}$ has the dimensions of a densimetric Froude number the coefficient 1.41 in Eq.(5.19) and unity in Eq.(5.20) can be looked upon as densimetric Froude numbers describing their particular fronts.

Experiments by Yih (153) on exchange flows in 1947 and Keulegan (78) whose work was done before 1947, on saline fronts, both found a densimetric Froude number at the front of between 0.9 - 1.0 . Experiments by Abbott (1) with oil over water gave values for the densimetric Froude number of 1.3 when the oil slick was stationary and 1.0 when the ambient fluid was stationary. It was stated that the turbulence in the frontal zone was less for the later case suggesting that this experimental procedure was the more accurate. Tests by Barr (19) with warm water exchange flows suggested values for the densimetric Froude number of between 0.85 - 0.95 for underflow fronts and between 0.95 - 1.05 for overflow fronts. Other experiments on a smaller scale were those conducted by Bata (23) with gasoline fronts over water which gave values for the densimetric Froude number of between 0.8 - 1.0 .

Lean and Whillock (84) studied the two-dimensional motion of a warm-water layer flowing over calm water.

Values for the front densimetric Froude number calculated using integral quantities are reported for thirteen experiments. Wright and Coleman (151) studied mixing of freshwater river outflow near a river delta, including consideration of lateral front motion. Both investigations found that the assumption of an initial front densimetric Froude number equal to 1.41 locally yielded a result after integration of the layer depth, that the average densimetric Froude number for the front is very close to unity. Finally, formulas are derived theoretically by Kao (73) for the velocity of propagation of a variety of gravity motivated density currents. These include flow of surface warm discharge, spreading of a slug of sewage sludge at a thermocline, and the spreading of an oil slick on the water surface.

If one now considers free surface flows then one can relate the front conditions of the free surface flow to that of exchange flows by noting that the only real difference between the two is the manner in which decelerating forces act on the fluid in order to produce a critical condition at the front. The densimetric Froude number is approximately squared in the free surface flow to what it would be for an exchange flow, this fact being confirmed by many experiments performed by Bata (23) and others. All the available data on dam break experiments performed by Schoklitsch (121), Trifonov (139) and Levin (87) are grouped together in a graph by Bata (23). This graph indicates a

densimetric Froude number at the front immediately after dam break of between 1.8 - 2.0 , which for exchange flows would be between 1.3 - 1.4 .

Since the wave front suggests an instability in the stratified system it can be compared again with the same phenomena that would occur in a free surface flow. In this case one can use the Vedernikov (142) stability criterion which separates long wave motion from short wave motion. The stability criterion if one considers wide rectangular channels is, for stability.

$$m F \leq 1 \quad (5.21)$$

Where m = index of hydraulic radius in the uniform flow law $u = K'R^m i^{\frac{1}{2}}$, i.e. $m = 0.5$ and 0.67 respectively for Chézy and Manning resistance laws. Using the equations of Chézy and Manning the Froude numbers at which roll waves or instabilities are initiated are 2.0 and 1.5 respectively. This suggests that a wave front in an exchange flow may have a densimetric Froude number of between 1.4 - 1.2 , if one assumes that the resistance law at the front is similar to that in open channel flows.

The above data concerning exchange flows, free surface flows and instabilities all suggest that the initial densimetric Froude number of an exchange flow is $(2)^{1/2}$. After the current has travelled a short distance so that friction effects influence its speed then it has a limiting densimetric Froude number of unity. Barr and Hassan(21) have shown that dilution of the front also reduces its speed.

Thus, the limiting wave front condition coincides with the stability condition, and that in this limiting case a wave front can be defined as one invading its ambient fluid with one of its two characteristics time stationary.

St. Venant fronts: It remains now to consider the velocity at the St. Venant front, for which there appears to be no experimental data. However, according to the theory of fluid behaviour following the failure of a dam, as originally proposed by St. Venant the equation governing the motion of long waves are assumed to hold over the entire range of flow, including the fluid front, for all times following the failure. The physical characteristics corresponding to this assumption are shown in Fig.5.10, whilst in Fig. 5.11 the equivalent shape of actual dam break water surface profiles (116) compared with the St. Venant front profile, and a wave front profile at a front Froude number of two are shown.

It can be seen that the St. Venant front is formed due to the coincidence of positive and negative characteristics so that the front velocity u_v is identical with the characteristic speed. By following a positive characteristic such as AB from the upstream reservoir boundary through the critical flow section c, to the negative characteristic of the downstream front boundary, we find equating positive Riemann invariants that.

$$2(g H)^{1/2} = u_c + 2(g d_c)^{1/2} = u_v \quad (5.22)$$

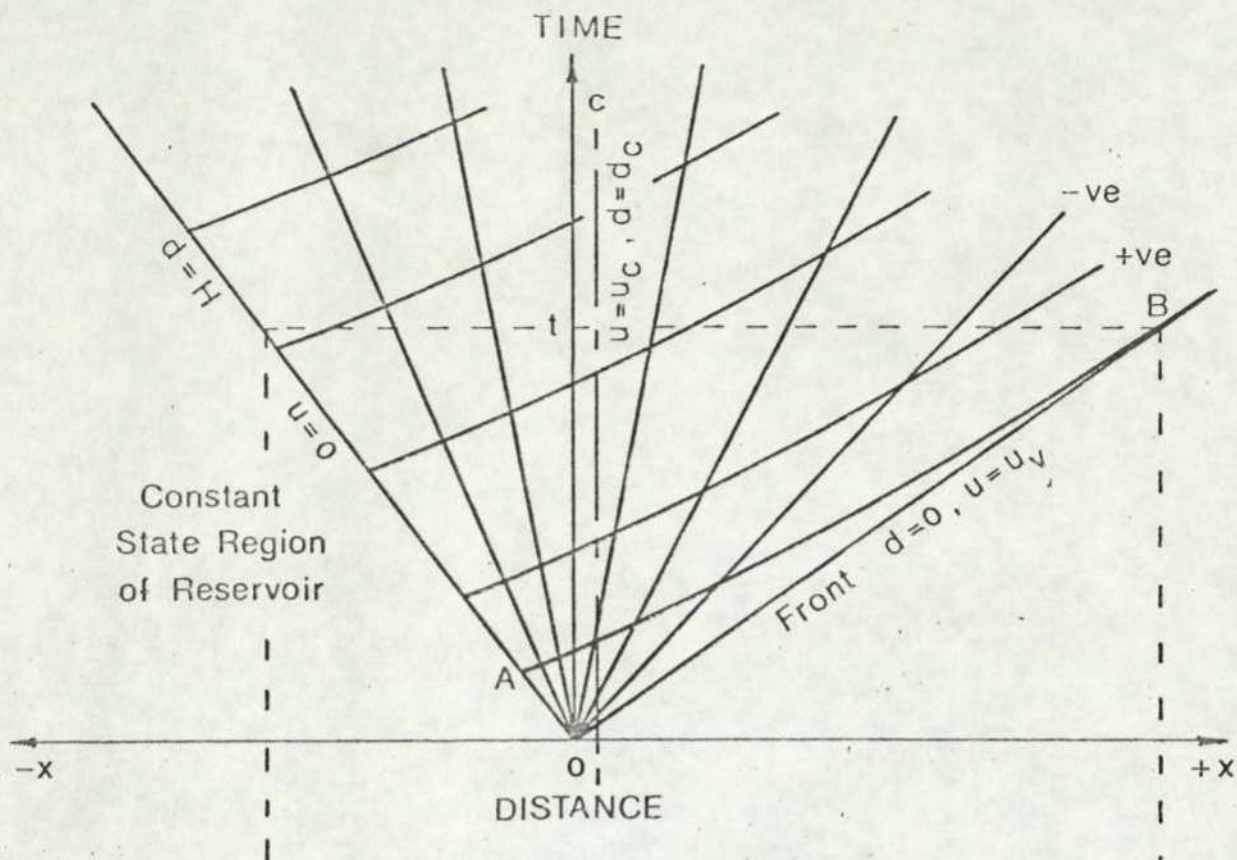


FIG. 5.10 PHYSICAL CHARACTERISTICS OF DAM BREAK FLOW

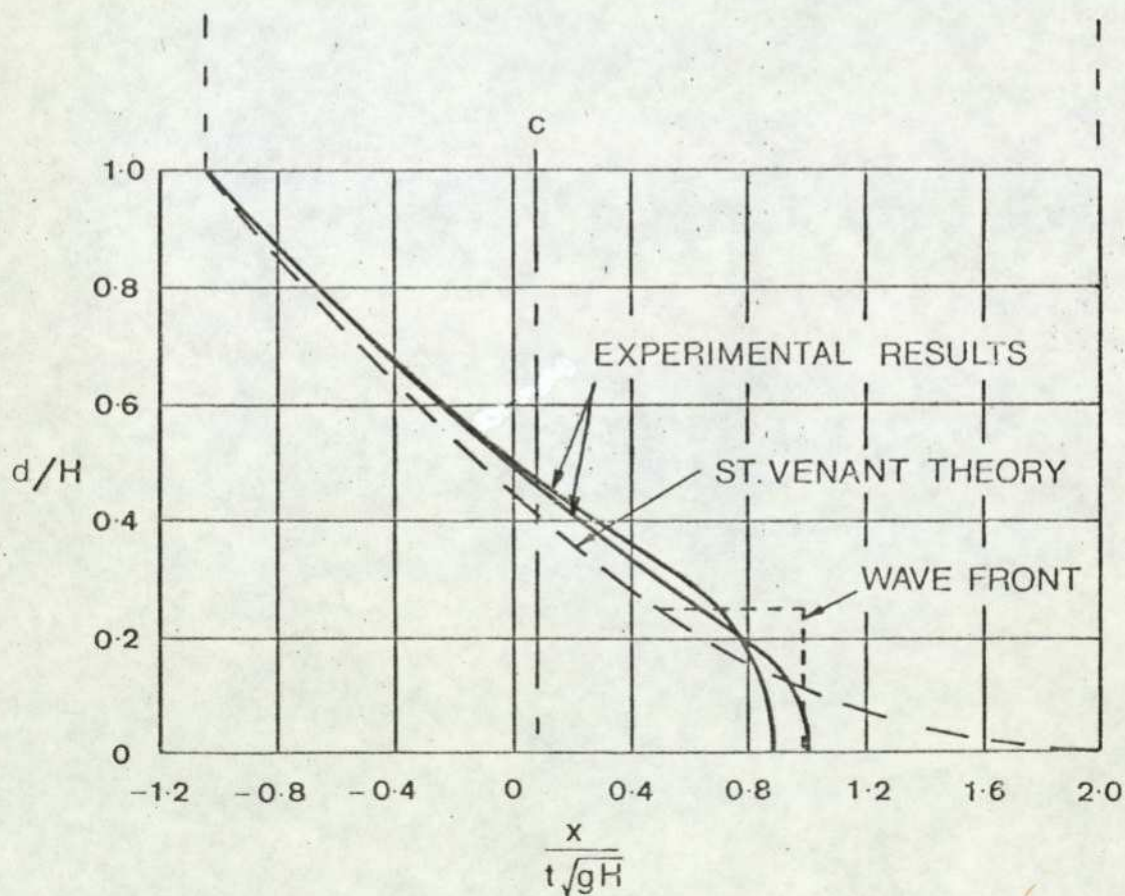


FIG. 5.11 DAM BREAK WATER SURFACE PROFILES

Since at the critical flow section $u_c = (g d_c)^{1/2}$ then

$$d_c = 4/9 H \quad (5.23)$$

and
$$u_v = 3(g d_c)^{1/2} \quad (5.24)$$

Hence, the front velocity for the dam break St. Venant front is three times the fluid velocity at the critical section. It must be made perfectly clear at this stage that the dam break St. Venant front does not exist in reality, since it breaks down into a wave front with a Froude number of about two. How this imaginary dam break St. Venant front corresponds to the exchange flow St. Venant front is not known, but assuming that the retreating stable front can be treated as a reflection of the unstable invading front, then the same front celerity applies to both.

5.3 Additional Front Theory

In order to ascertain the St. Venant front velocity for the retreating front consider the front shown in Fig.5.12 , where the lower layer depth is small compared with the upper layer depth. Also, by using relative velocities in the lower layer, the top layer remains roughly a constant state and does not affect the lower layer. Using the method of characteristics it is shown in Fig.5.12 that the effect of this front with zero depth is the same as if the fluid is extended to the left with depths and velocities the same as those on the right. The reflection front on the left has the form of an invading St. Venant front, and since the characteristic structure is continuous over both front types, the relative velocity of the retreating

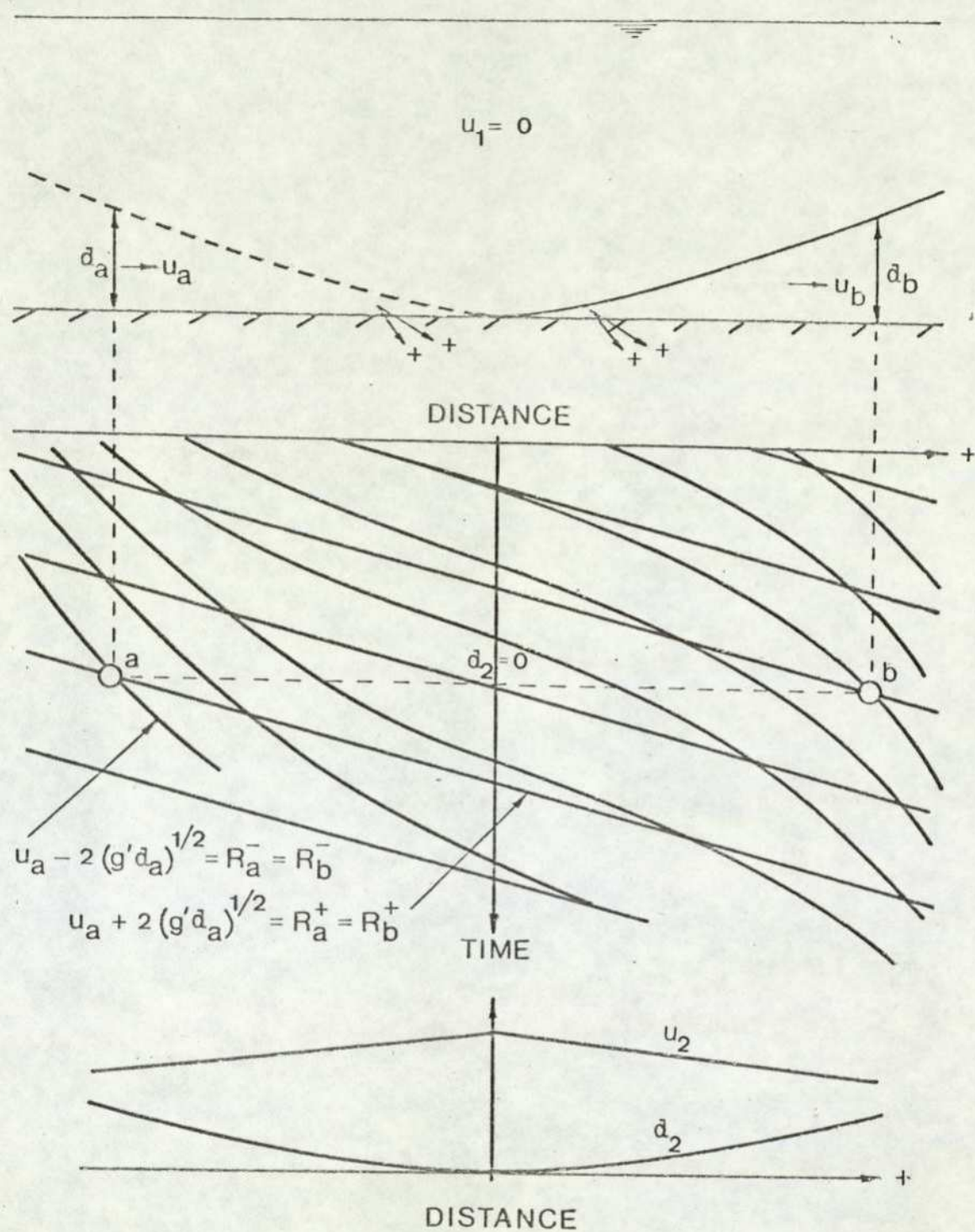


FIG. 5.12 CHARACTERISTIC STRUCTURE OF ST. VENANT FRONTS

front is equal to the relative velocity of the advancing front with equal Riemann invariants R^+ and R^- .

For the case of a free surface flow the relative velocity at the front is three times the velocity at the critical section. If one assumes that the relationship between Froude numbers for a theoretical free surface St. Venant front and an exchange flow St. Venant front is the same as for the wave front, then the densimetric Froude number becomes $(3)^{1/2}$, instead of a Froude number of 3.0 .

In an exchange flow the driving potential is the hydrostatic pressure difference which in the case of Fig. 5.13 is $H(1 - \epsilon)$, and therefore a constant for all front types. This driving force is responsible for all motions of the front and if relative velocities are used it is very simple to calculate the front velocities. For half the exchange flow the driving potential is $H(1 - \epsilon)/2$ and this equals the kinetic energy in the front.

For the wave front :-

$$H(1 - \epsilon)/2 = (2u_w)^2/(2g) = V_w^2/(2g) \quad (5.25)$$

where V_w is the relative velocity of the wave front.

$$\text{or} \quad V_w = (g' H)^{1/2} \quad (5.26)$$

which is the same result as found before in Eq.(5.20) .

For the St. Venant front :-

$$H(1 - \epsilon)/2 = 1/(2g) \int_0^{L_v} \frac{V^2}{L_v} dx \quad (5.27)$$

where V is the relative velocity at distance x from origin.

For the St. Venant front the critical flow section must be at the centre line for an exchange flow. Also, the velocity between the centre line and the front tip varies linearly with distance, evident from the fan shape of the centred simple wave shown in Fig.5.10 . Therefore :-

$$V = V_c + (V_v - V_c) x / L_v \quad (5.28)$$

where V_v and V_c are the relative velocities at the front tip and critical section respectively.

$$\begin{aligned} \int_0^{L_v} \frac{V^2}{L_v} dx &= \frac{1}{L_v} \int_0^{L_v} \left[V_c^2 + (V_v - V_c)^2 x^2 / L^2 + 2V_c(V_v - V_c)x/L_v \right] dx \\ &= \frac{1}{L_v} \left[V_c^2 x + (V_v - V_c)^2 x^3 / (3L^2) + V_c(V_v - V_c)x^2/L_v \right]_0^{L_v} \\ &= V_c^2 + (V_v - V_c)^2/3 + V_c V_v - V_c^2 = (V_v^2 + V_c V_v + V_c^2)/3 \end{aligned}$$

$$\text{hence } (V_v^2 + V_c V_v + V_c^2) = 3 g H (1 - \epsilon) \quad (5.29)$$

but at the critical flow section :-

$$V_c^2 = g H (1 - \epsilon)/2 \quad (5.30)$$

$$\text{thus } V_v^2 + V_c V_v - 5 V_c^2 = 0 \quad (5.31)$$

$$V_v = \frac{-V_c \pm (V_c^2 + 20 V_c^2)^{1/2}}{2} = \frac{-V_c \pm 4.5 V_c}{2}$$

$$V_v = 1.75 V_c \quad (5.32)$$

since $d_c = H/2$

$$V_v = 1.75 (g' d_c)^{1/2} \quad (5.33)$$

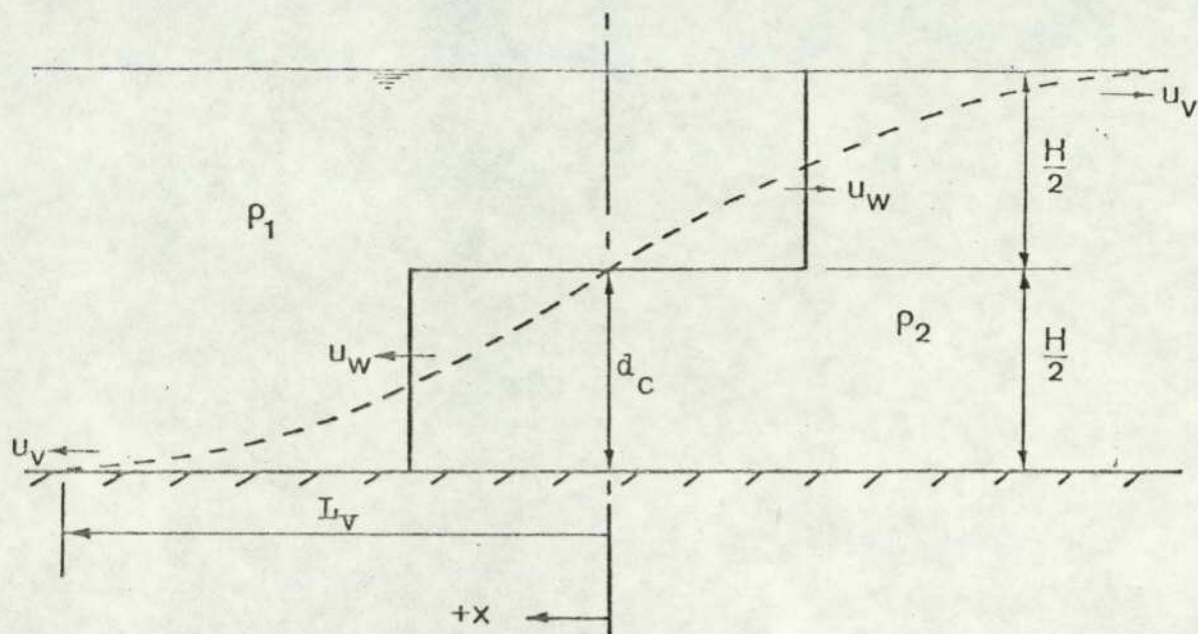


FIG. 5.13 SCHEMATISED EXCHANGE FLOW

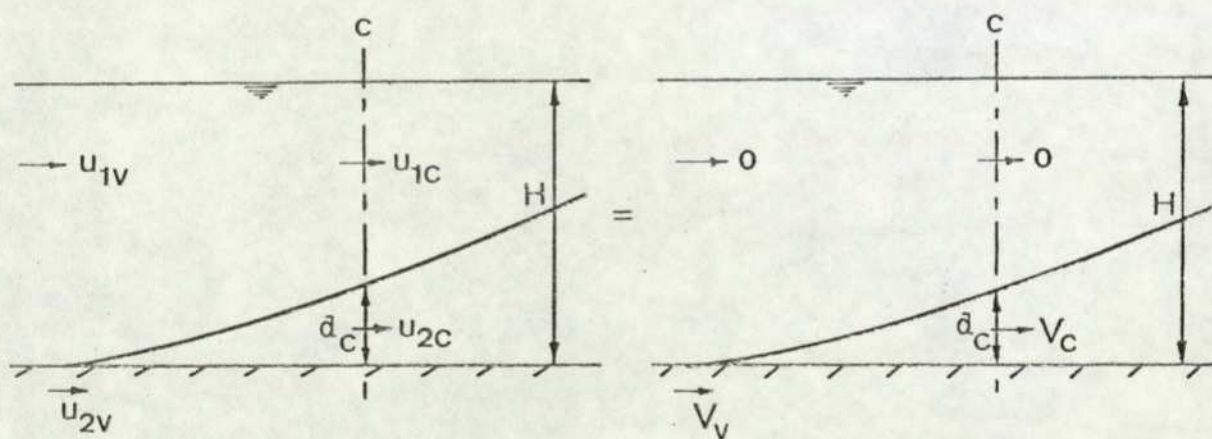


FIG. 5.14 SCHEMATISED ST. VENANT FLOW

From this calculation the densimetric Froude number of the St. Venant front in terms of the critical depth is found to be 1.75 . The same result can be obtained by equating specific energies between the critical section and the front tip for any St. Venant front.

Equating the specific energy head between the front tip and the critical section shown in Fig. 5.14 gives.

$$(H - d_c)\epsilon + d_c + V_c^2/(2g) = H\epsilon + V_v^2/(2g) \quad (5.34)$$

also at the critical section

$$V_c^2 = g d_c (1 - \epsilon) \quad (5.35)$$

from Eqs.(5.34) and (5.35) $3/2 V_c^2 = 1/2 V_v^2$

therefore $V_v = (3)^{1/2} V_c$ (5.36)

and $V_v = 1.73 (g' d_c)^{1/2}$ (5.37)

It is concluded from Eqs.(5.32) and (5.36) that the relative velocity for a retreating St. Venant front in a stratified fluid is 1.73 times the relative velocity at the critical flow section, in the absence of friction. With friction the front velocity is reduced and if one compares the reduction in the wave front densimetric Froude number from $(2)^{1/2}$ to unity or less, then the St. Venant front coefficient should reduce from $(3)^{1/2}$ to about 1.2 or less. Because the St. Venant front has zero depth the front velocity will be influenced by laminar shear stresses, which will probably reduce the densimetric Froude number to unity.

It has been shown that the St. Venant front retreating in its ambient fluid is a reflection of the St. Venant front advancing in its ambient fluid. The reason why the former is physically stable and why the latter decomposes into a wave front is now suggested. It was shown by Rayleigh (110) that for all flows, short wave motions are developed whenever one of the characteristics becomes time stationary. It is the propagation of these short waves which determines whether or not a St. Venant front remains stable.

In Fig. 5.15(a) a dam break type St. Venant front is shown with its velocities and rates of energy propagation. Short waves developed near the critical section c travel at the speed of energy propagation ($2u_c$) which is less than the actual front velocity ($3u_c$), therefore in the absence of friction this front is stable. With friction the front travels at less speed enabling the short waves to reach shallower depths. Here they break or are reflected causing considerable turbulence so that the resulting front takes the form of a wall of water, the depth of which is some function of the velocity depending on the friction factors involved.

Fig. 5.15(b) shows a St. Venant front advancing in its ambient fluid. The short waves developed at the critical section travel at a speed ($2V_c$), which is greater than the fluid at the front ($1.73V_c$). Therefore even in the absence of friction this front is unstable, friction

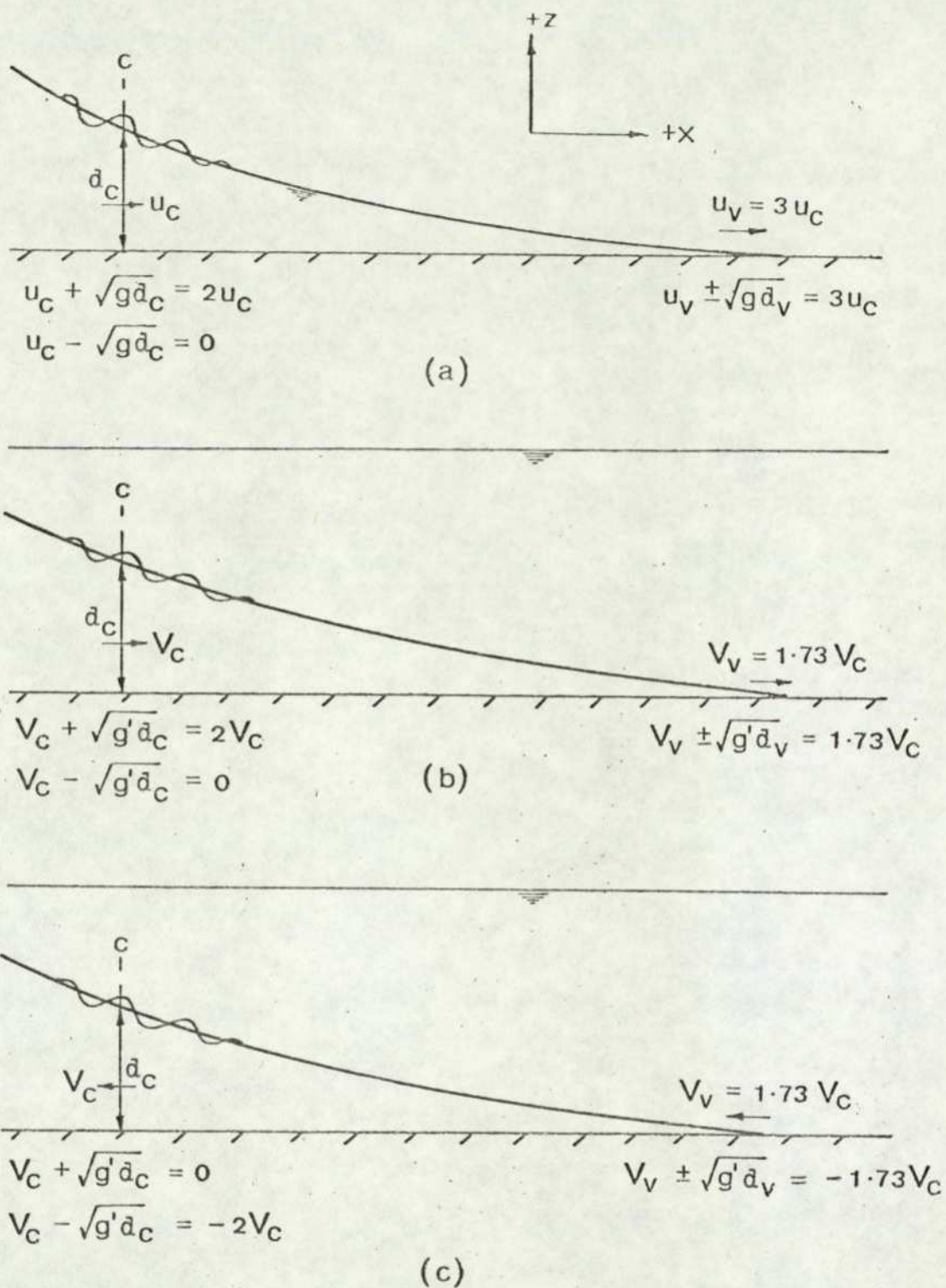


FIG. 5.15 ST. VENANT FRONT TYPES

increasing its instability. Fig. 5.15(c) shows a St. Venant front retreating in its ambient fluid. The short waves developed at the critical section travel at a speed greater than the fluid at the front and away from the front, so that they can never be reflected or amplified. With friction the front velocity will decrease just the same as with the advancing front, so with friction the stability of the front increases and is therefore perfectly stable.

5.4 Experimental Determination of Front Conditions

In the previous section, theoretical and experimental results were reviewed for the densimetric Froude number of wave fronts, and found to agree with one another. Also, the St. Venant front was analysed theoretically but since there were no known experimental tests performed on St. Venant fronts, the theoretical results could not be checked. This section records the experimental front results obtained from tests carried out in the channel shown in Fig.5.2, firstly with exchange underflows and secondly with fronts formed due to saline wedges moving up and down the channel due to tidal action.

The progress of exchange flows in the channel was recorded photographically. Results of one such test are shown with the times elapsed after raising the barrier in Fig.5.16 . This particular flow (Test 2) had an initial depth H_s at the sea of 80mm and a channel slope I of 9.6×10^{-3} . All the tests used fresh water for the lighter fluid and salt water of density 1032kg/m^3 , coloured with

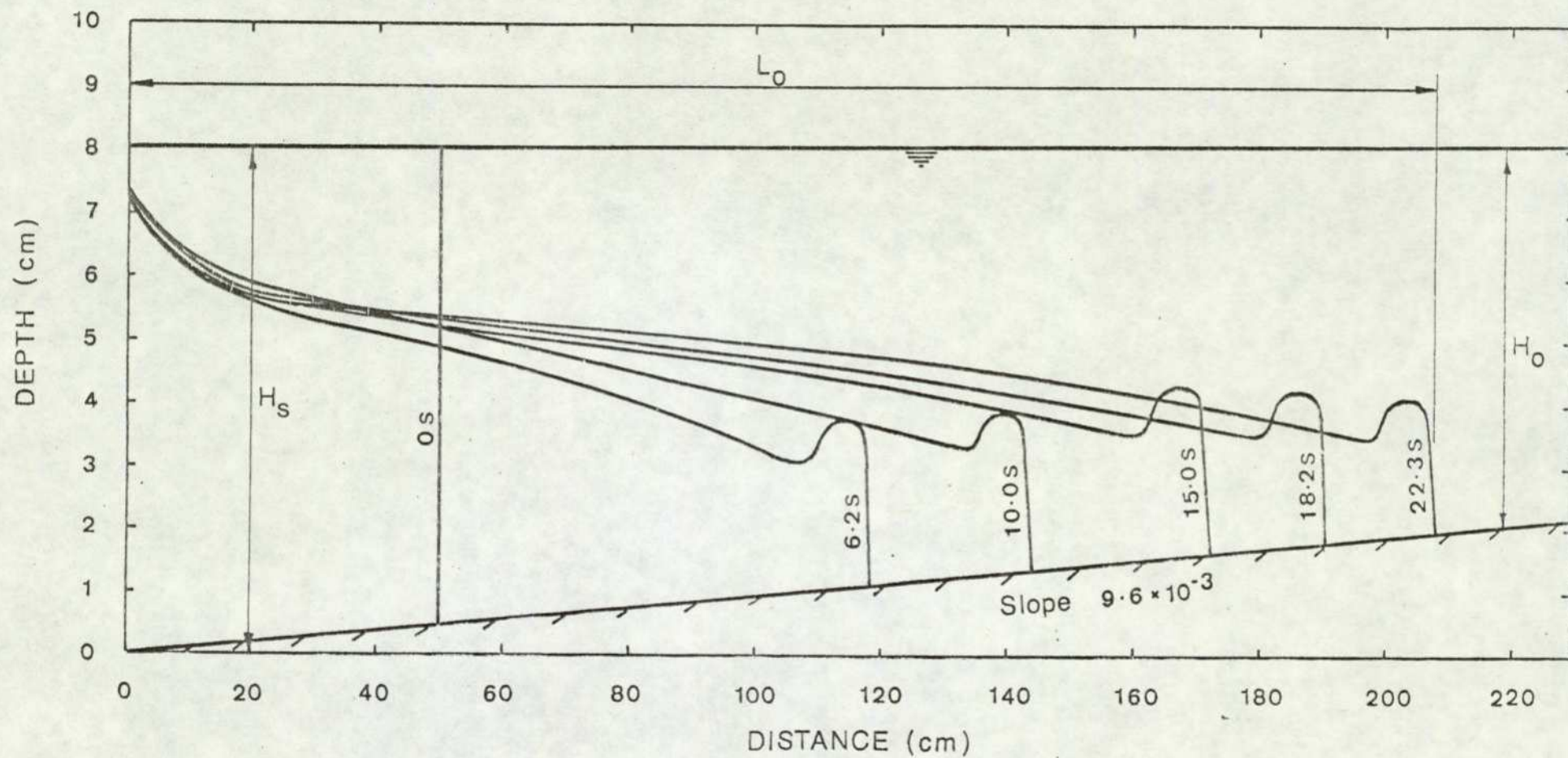


FIG. 5.16 PROGRESS OF EXPERIMENTAL EXCHANGE FLOW

potassium permanganate solution for the heavier fluid. The results from four exchange underflows at different initial depths and channel slopes is shown in Fig.5.17 where the front densimetric Froude number is plotted against the ratio of the wedge length L_0 to the depth H_0 . It can be seen that the fronts have an initial densimetric Froude number of about 1.4 reducing to about 0.75 as L_0/H_0 becomes large. For the arrested saline wedge in the South Pass of the River Mississippi, L_0/H_0 between critical sections is about 1400. Since the horizontal scale of the model $X = 10^4$, and the vertical scale $Y = 250$ the equivalent model L_0/H_0 ratio for the River Mississippi is 35. This suggests from Fig.5.17 that the front densimetric Froude number for the saline wedge in the River Mississippi is between 0.89 - 0.97 .

To form St. Venant fronts experimentally it is necessary to reproduce the tidal motion of the sea in the model, so that the salt wedge is drawn back towards the sea on the ebb flow. This was achieved by placing a car tyre inner-tube on the bed of the experimental sea, weighted down and loosely held to a stout board to which three 10 kg weights were firmly bolted. The open valve of the inner-tube was connected by a pipe through a simple gate valve to a portable electric compressor shown in Fig. 5.18 . With the gate valve set to a fixed opening the compressor was alternately switched on and off for set periods, and as the tube inflated and deflated so the

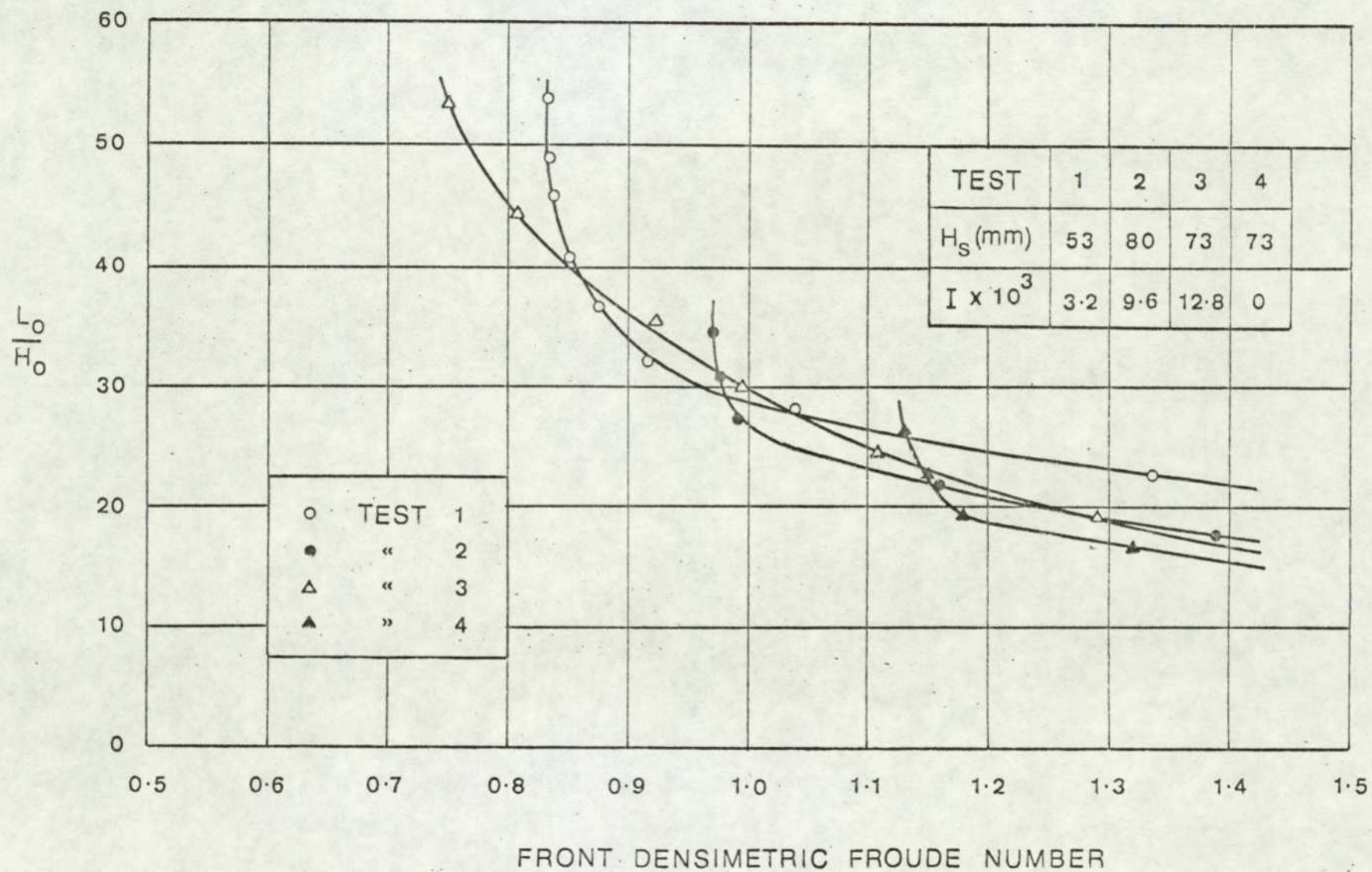


FIG. 5.17 LENGTH OF EXPERIMENTAL EXCHANGE FLOWS

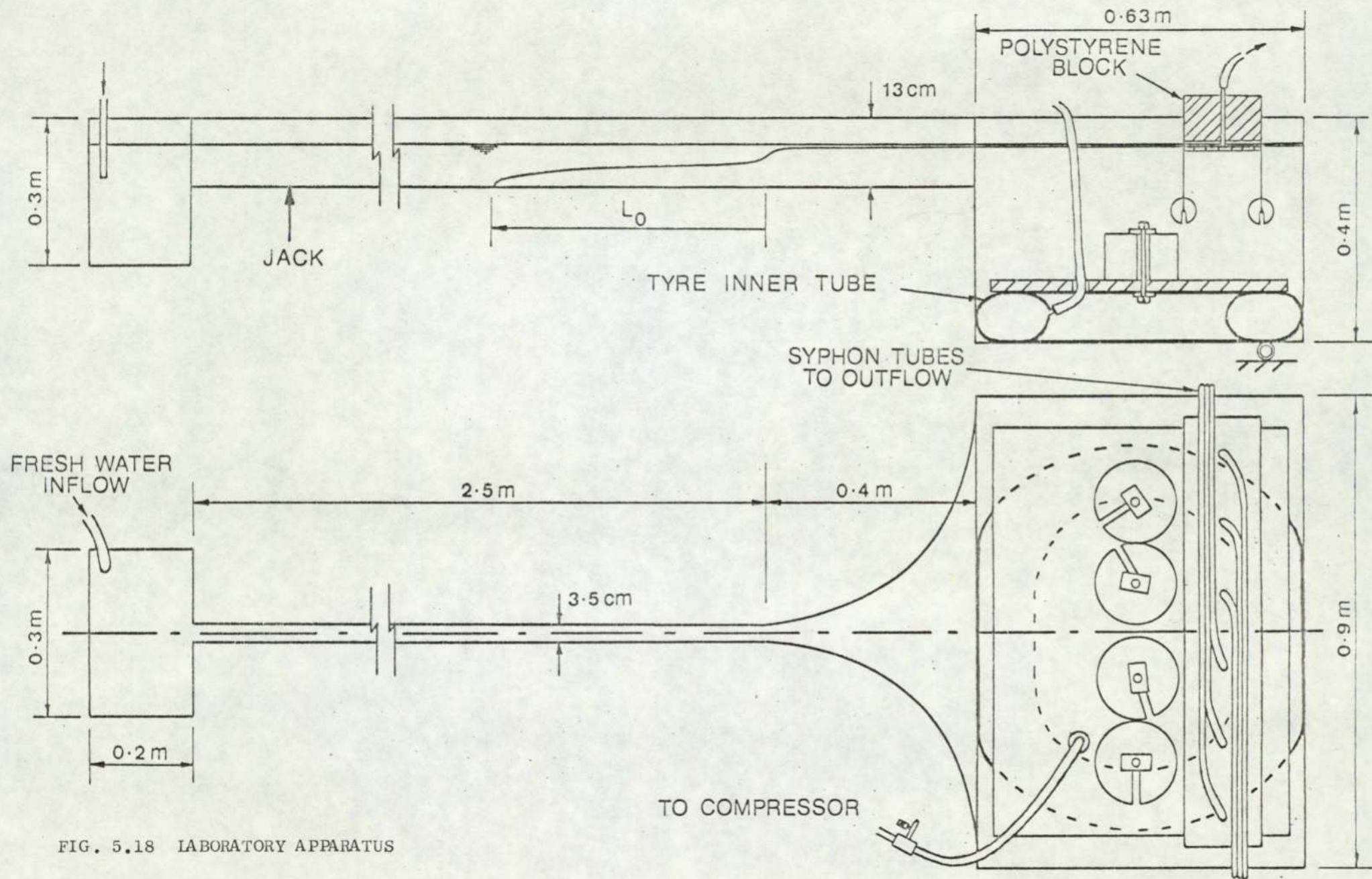


FIG. 5.18 LABORATORY APPARATUS

water level in the sea rose and fell producing a rather good copy of a shallow water tidal wave, without any valve regulation. The insert of Fig.5.19 shows the vertical tides produced at the seaward end of the channel when the compressor was switched on for periods of 30s, alternating with periods of 90s when it was switched off. The insert of Fig.5.21 shows the vertical tides produced when the compressor was switched alternately on for 60s and off for 90s. The time scale for the model is $X/Y^{0.5} = 633$, therefore a tidal period of 12.5 hrs in the prototype is equivalent to 71s in the model. The actual model tidal periods of 120s and 150s are longer than this because it was necessary to move the fronts over sufficient distance to be able to calculate their velocities accurately.

The progress of the saline wedge in the first test is shown in Fig.5.19, which was recorded photographically after a stabilising period of five tidal cycles. The head of the wave fronts has been omitted since this is not relevant to the wedge shape and only confuses the diagram. The channel slope for this test was 1.28×10^{-2} , the density of the saline water 1023kg/m^3 , with a constant upland fresh water discharge of $67.7\text{cm}^3/\text{s}$. Fig.5.19 clearly shows that the wedge advances as a wave front on the flood flow and retreats as a St. Venant front with zero depth on the ebb flow. The front height and position with time, plus the resulting upper layer velocity u_1 and lower layer velocity u_2 at the front with time, and the calculated

front densimetric Froude number with time are shown in Fig.5.20 . For this particular tide, the initial densimetric Froude number for the wave front has a value greater than 2.0 reducing to a value of about 1.2 for most of the flood period. The St. Venant front has a densimetric Froude number in terms of the critical depth varying from 0.9-1.0 .

For the second test the flood period was increased to 60s, the bed slope set at 0.96×10^{-2} , the salt water density at 1024 kg/m^3 and a constant upland fresh water discharge of $98.5 \text{ cm}^3/\text{s}$. The progress of this new wedge is shown in Fig.5.21 and illustrated graphically in Fig.5.22 . It can be seen from Fig.5.22 that the initial densimetric Froude number for the wave front is about 1.4 decreasing to 1.0 for the latter half of the flood period. The densimetric Froude number of the St. Venant front in terms of the critical depth has a constant value of about 0.95 over the whole of the ebb period.

The conclusions drawn from these experiments concerning the front types formed when a salt wedge advances and retreats in an estuary, is that a wave front exists during the flood period and a St. Venant front during the ebb period. The densimetric Froude number for the wave front for most of the flood period is close to unity. For the St. Venant front the densimetric Froude number in terms of the critical depth is also close to unity, indicating that for real flows both front types are governed by stability conditions.

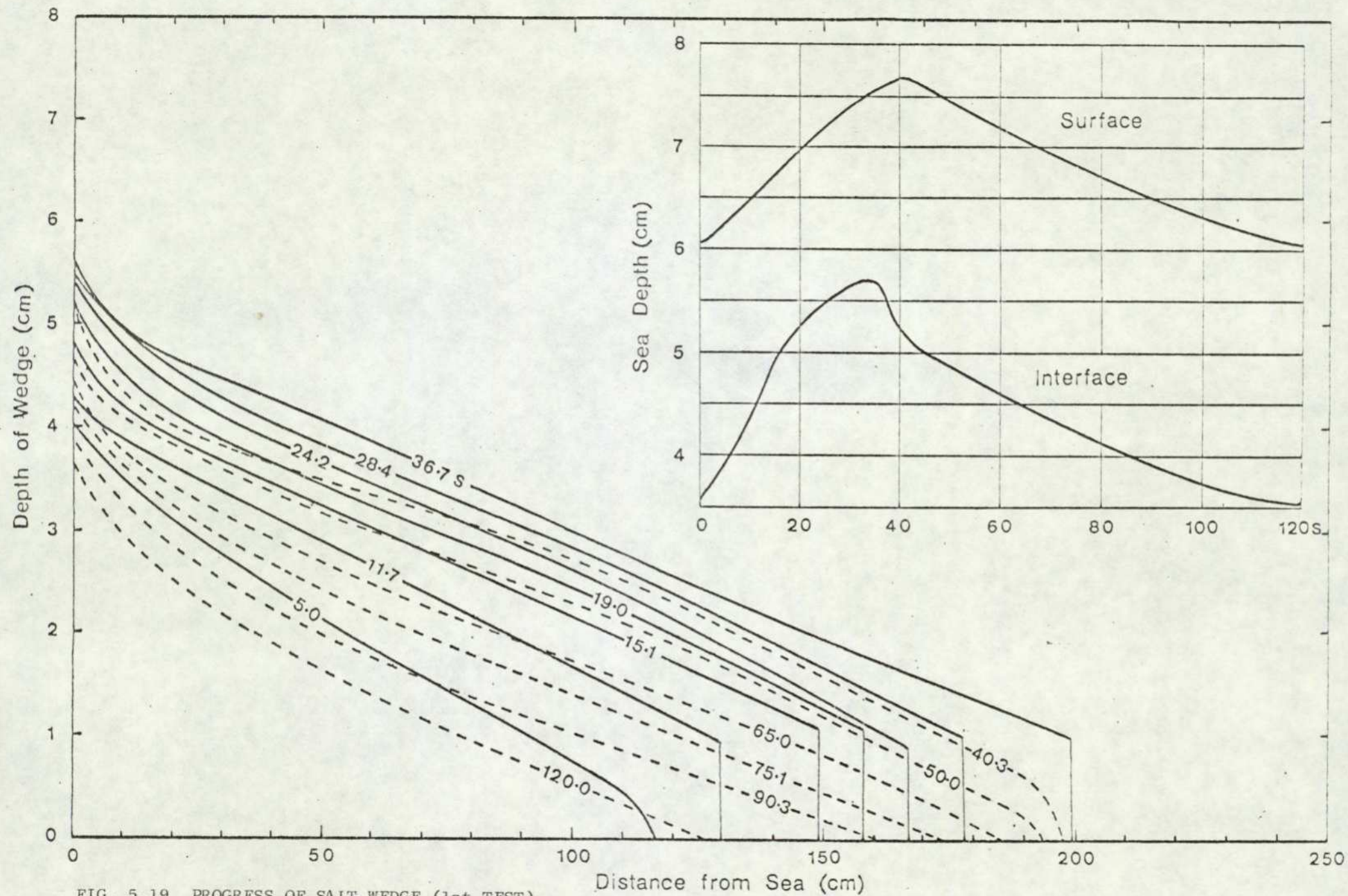


FIG. 5.19 PROGRESS OF SALT WEDGE (1st TEST)

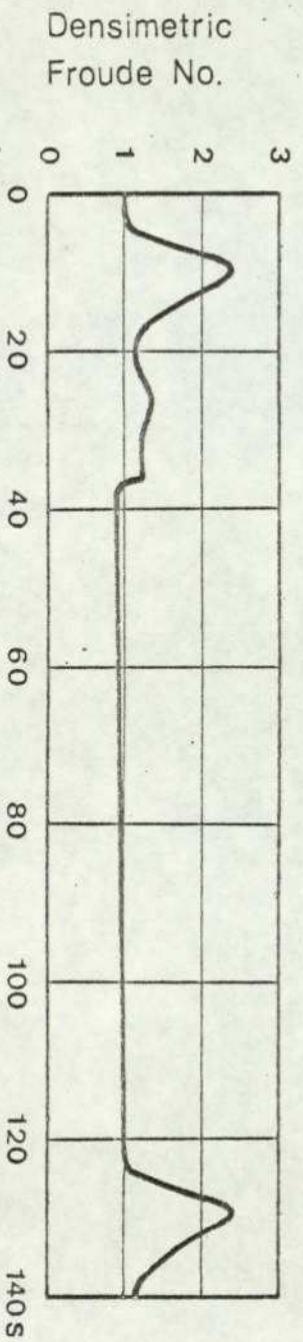
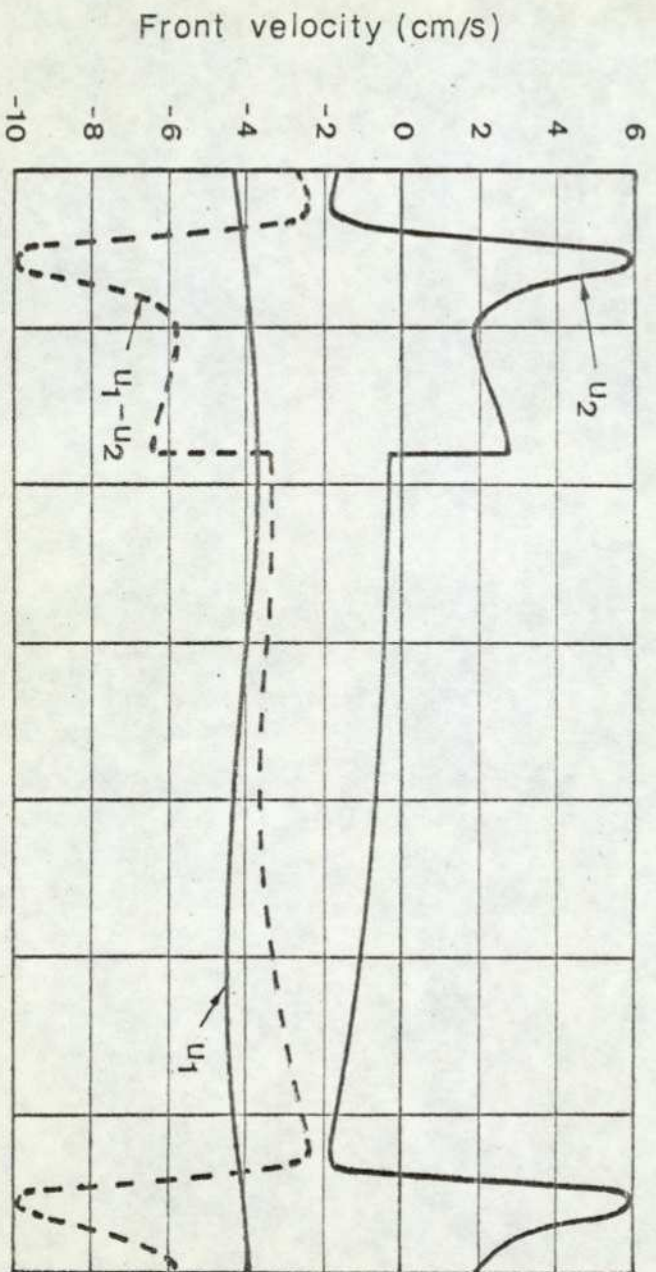
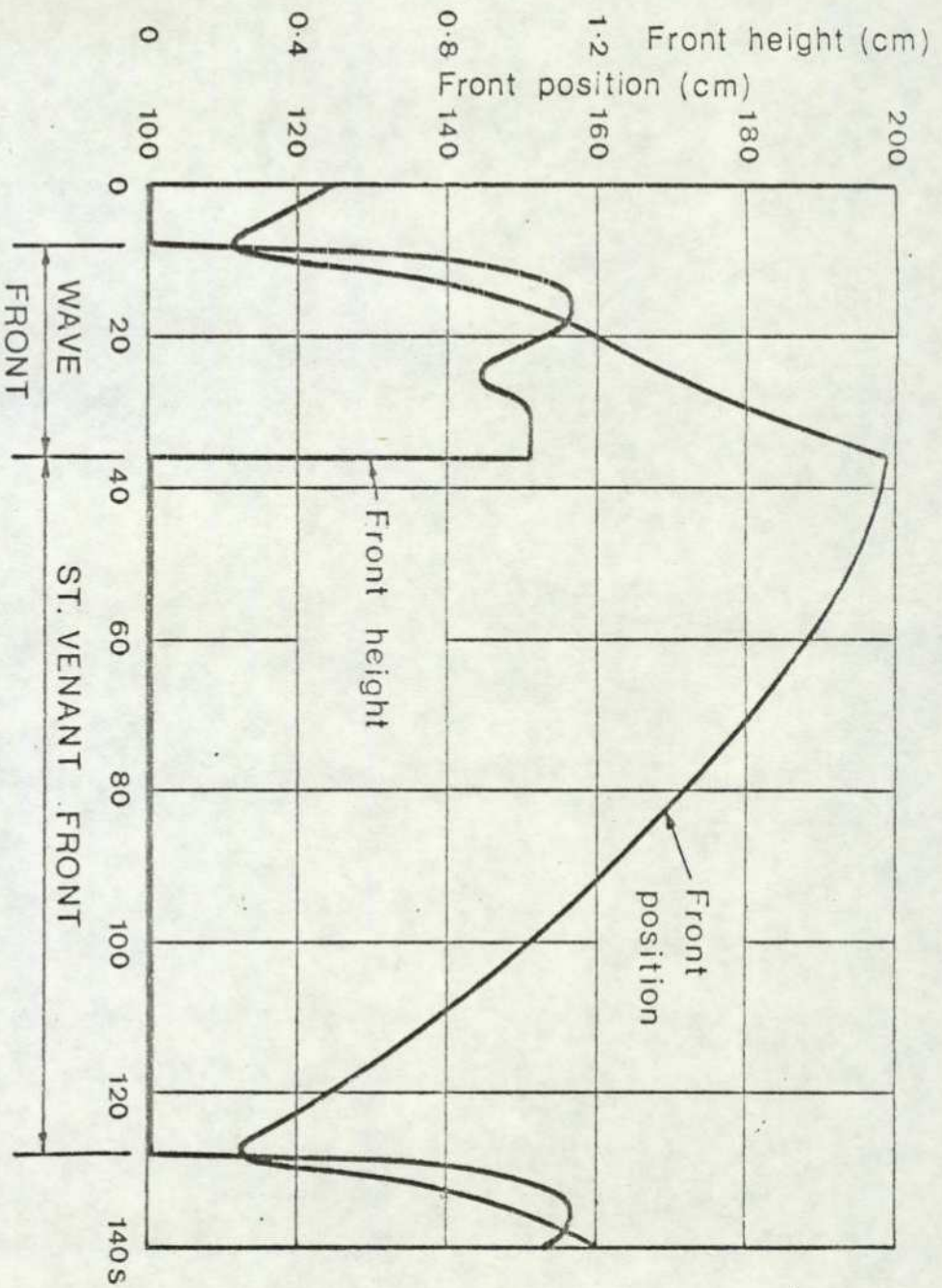


FIG. 5.20 VARIATION OF DENSIMETRIC FROUDE NUMBER (1st TEST)

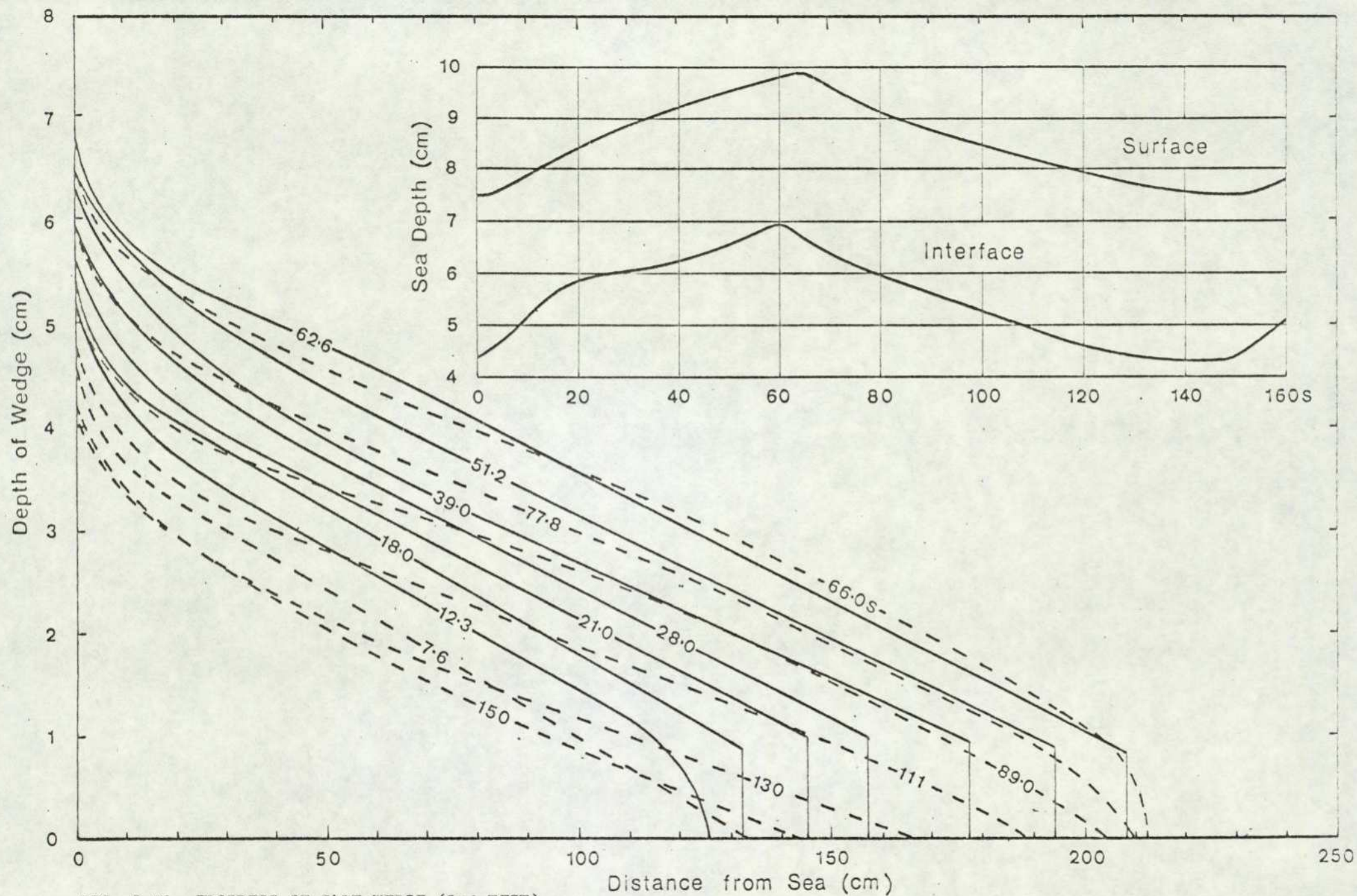


FIG. 5.21 PROGRESS OF SALT WEDGE (2nd TEST)

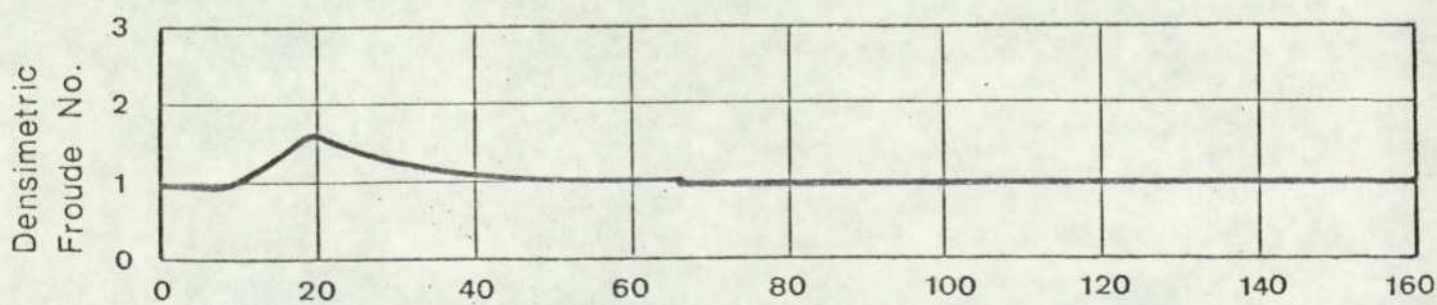
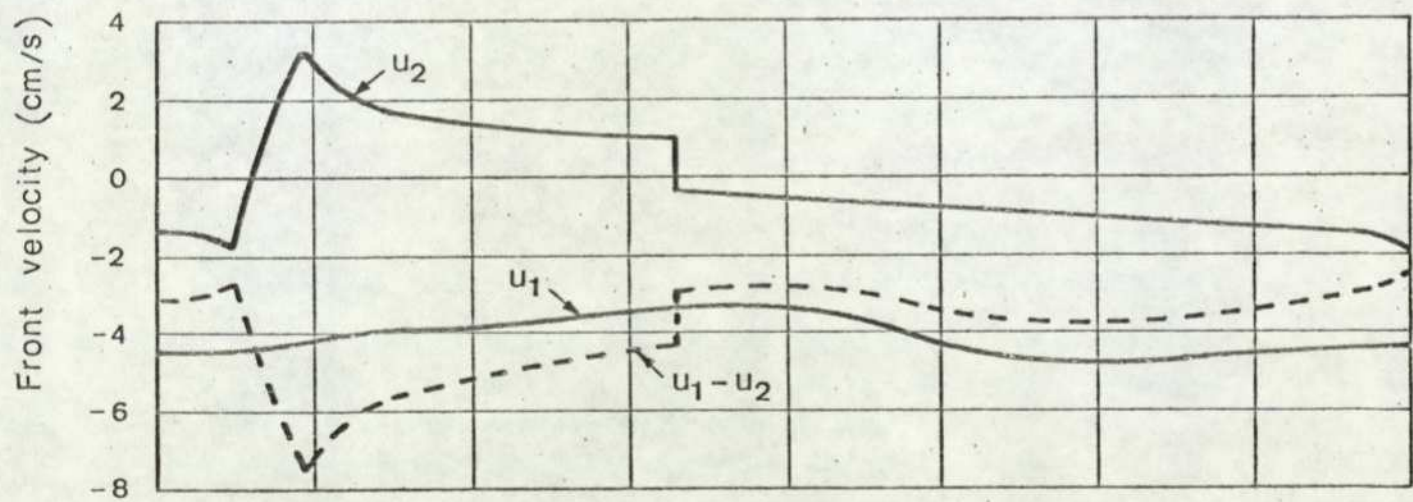
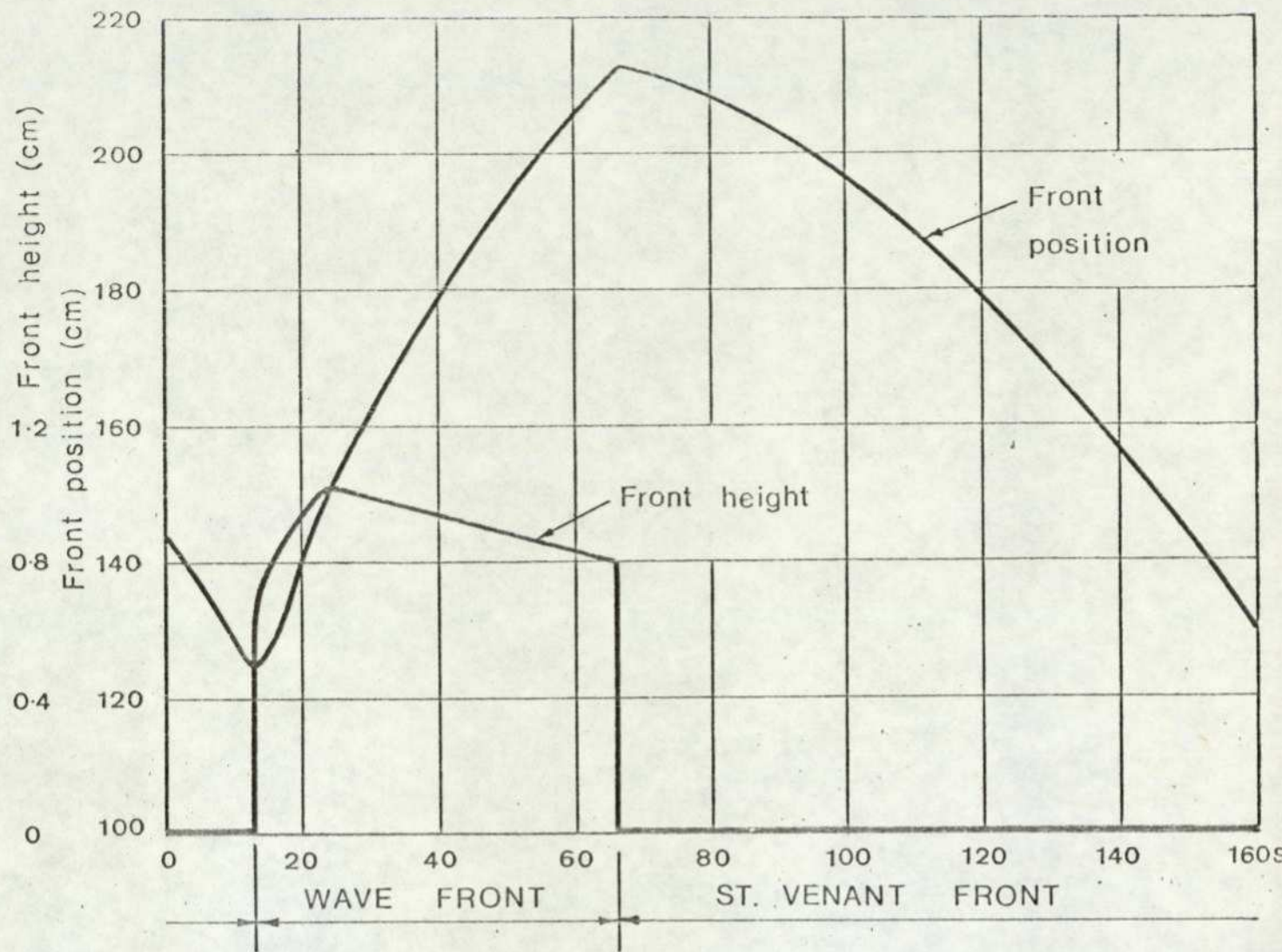


FIG. 5.22 VARIATION OF DENSIMETRIC FROUDE NUMBER (2nd TEST)

5.5 Application of Front Conditions to Mathematical Models

In a mathematical model of a stratified estuary the computation is usually made from the seaward end of the estuary to the tidal limit. The computation must therefore compute flow conditions within the two-layer regime of the estuary occupied by the salt wedge and also in the one-layer regime upstream of the wedge. The connecting link between the two different computational models is the front condition of the salt wedge.

It has been shown experimentally that both front types are governed by stability conditions, therefore for both fronts irrespective of their particular frontal shapes, the front can be taken as a vertical wall of water located at the critical flow section. This is reasonable since although the actual St. Venant front has the visual appearance of a theoretical exchange flow St. Venant front, the fluid velocity between the critical section and the front tip is constant. It follows that this fluid is not really part of the front at all, since relative to the critical section it is stagnant. The various frontal shapes ranging from the 'head' of the wave front to the wedge of the St. Venant front are therefore different facies of the same front, which effectively ends at the critical flow section. Since these various front facies do not influence the flow conditions in the subcritical fluid beyond the critical section, there is no need to include them in the two-layer flow computations.

With the fronts ending at the critical flow section the programming of the St. Venant front is simplified since supercritical flow conditions close to the frontal tip are avoided. It is also necessary computationally since there is a rapid change from one front type to the other as the tide reverses, and trying to follow this change would only lead to numerical instabilities. The actual wedge shape of the St. Venant front can be calculated separately and added on after the main computation is completed.

5.6 Conditions at the Mouth of an Estuary

Since the mathematical model must have a boundary at the mouth of the estuary, the location and flow conditions of this boundary need to be considered. In the case of a highly stratified salt wedge estuary with little or no tidal motion, the end of the estuary is according to Schijf and Schönfeld (120) that point at which critical flow conditions occur. For a waterway which has a fairly constant cross-section and then rapidly widens or deepens at its mouth, critical flow conditions will occur at the change of section. This is confirmed by the results shown in Fig. 5.6, from the experiments on arrested saline wedges. At the mouth of a fjord the fresh water surface layer develops a shape resembling the 'head' of an underflow wave front. It has been shown by Long (90) that, allowing for mixing, the end of a fjord can only reasonably occur just upstream of the critical flow section. Also, Stommel and Farmer (130) have shown that at the mouth of an estuary without tidal motion, the flow conditions in the upper layer are kept

critical due to turbulent diffusion mixing eroding away the interface and increasing the thickness of the upper layer as shown in Fig.5.7 .

If the estuary is tidal so that the salt wedge moves up and down the estuary as the flow floods and ebbs then the boundary conditions are more complicated. This is because the celerity of the interfacial long waves is much smaller than that of the surface waves. Consequently the interface cannot change its level fast enough at every moment to maintain critical conditions at the mouth of the estuary. Hodgins Osborn and Quick (63) found that for the partially-mixed Fraser River estuary the flow at the mouth in the upper layer remained critical for all flow conditions. For the lower layer however, supercritical outflows as well as subcritical were common although on the flood tide they recorded only one case of supercritical inflow.

The inability of the interface to adjust itself to critical flow at each moment is clearly recorded in the experimental tidal profiles shown in Figs. 5.19 and 5.21 . For the first test the critical depth in the upper layer at the sea is given approximately as:-

$$d_{1c} = \left[Q_r^2 / (g' b^2) \right]^{1/3} = 2.57 \text{ cm} \quad (5.38)$$

where Q_r is the upland fresh water discharge.

This particular value of critical depth is underestimated since no account has been taken of mixing reducing the relative density between layers.

It can be seen from the insert of Fig. 5.19 that critical conditions exist throughout the ebb period but since the depth is less than critical throughout the flood period, supercritical inflow conditions exist during this part of the tide. Exactly the same conditions are seen to occur during the second test shown in Fig. 5.21 where the critical depth in the upper layer at the sea is calculated to be 3.25 cm.

It is evident from the changing boundary conditions at the mouth of a stratified estuary that the mathematical program must have the capability of scanning the whole estuary to see if the flow is subcritical or supercritical. From this information the correct algorithmic structure can be selected for the computation, and in this way instabilities due to incorrect data presentation are avoided.

CHAPTER 6

INTERFACIAL INSTABILITY AND MIXING IN STRATIFIED FLOWS

6.1 Interfacial Instability

Inviscid Flow — It has been shown by Groen (49) and Heyna and Groen(62) that for simple harmonic small amplitude gravity waves within vertically inhomogeneous, stable stratified, incompressible fluids, either unbounded or bounded by one or two rigid horizontal planes the following two theorems hold.

1. The periods of the waves are monotonic increasing functions of the wave length.
2. A positive lower limit exists for the period T of the waves, which is the minimum value given by:-

$$T = 2\pi \left[-\rho / \left(g \frac{d\rho}{dz} \right) \right]^{1/2} \quad (6.1)$$

where g = gravitational acceleration; ρ = local fluid density at position z ; and $d\rho/dz$ = rate of change of density with elevation at position z .

Brunt (28) and Väisälä (141) independently determined that the period of oscillation of a fluid particle displaced from its position of neutral buoyancy would be given by theorem 2 and hence shear generated internal gravity waves are referred to as Brunt-Väisälä waves.

For inviscid flows, criterion for interfacial wave stability have been worked out by Taylor(134), Goldstein(47) and others since (37,96,137,60,85). For a

two-layer stratified flow with an interfacial mixing layer of thickness δ in which the velocity varies linearly and the density is the mean of the bottom and top layers, Taylor and Goldstein defined a region of instability shown in Fig.6.1 . From this figure a convenient condition for instability is that $R_i/\alpha < 0.5$ (Helmholtz 1868), where R_i is the Richardson number, $R_i = g'\delta / V^2$ and α is the dimensionless wave number in which $L =$ the wave length and $V = u_1 - u_2$; see Fig.6.2 .

If the interfacial layer has an exponential density distribution whilst the velocity distribution is linear, Goldstein obtained a result that harmonic oscillations are stable for $R_i > 1/4$ and unstable for all wavelengths for $0 < R_i < 1/4$. If the thickness of the interfacial layer becomes smaller, Miles(95) showed that this model approaches the Helmholtz model. Miles and Howard(96) showed that for a linear variation of density within the interfacial layer a much smaller region of instability exists, see Fig.6.3, than is predicted by the Taylor - Goldstein model.

It can be concluded from the work on hydrodynamic stability of stratified flows that when the interfacial Richardson number is less than some critical value, the flow will be unstable to disturbances that fall within a certain range of wave lengths.

The celerity c of short period internal gravity waves is given by Keulegan(77) as

$$c = \left[g' L / (4 \pi) \right]^{1/2} \quad (6.2)$$

where $g' = g(1-\epsilon)$

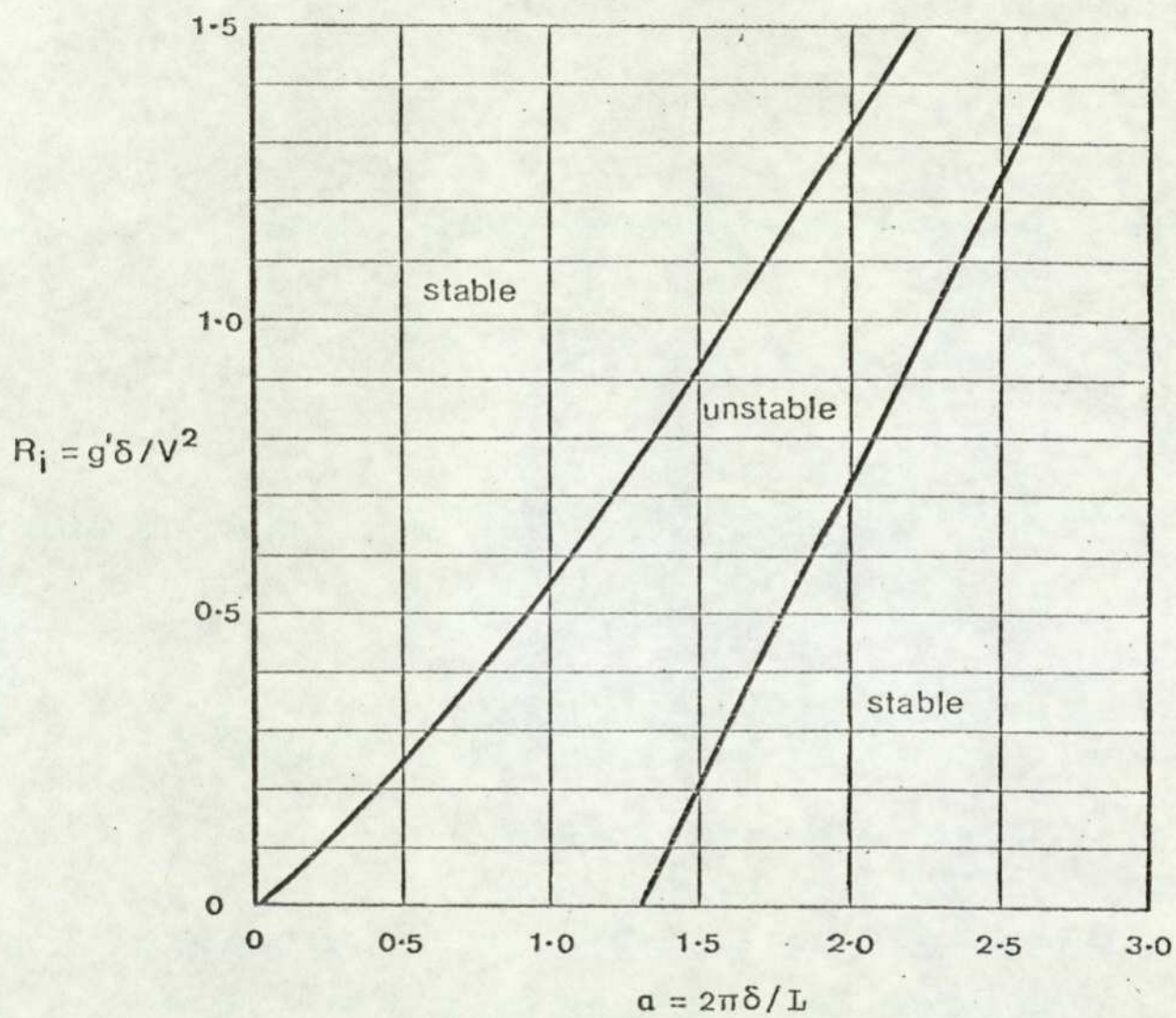


FIG. 6.1 TAYLOR - GOLDSTEIN STABILITY DIAGRAM

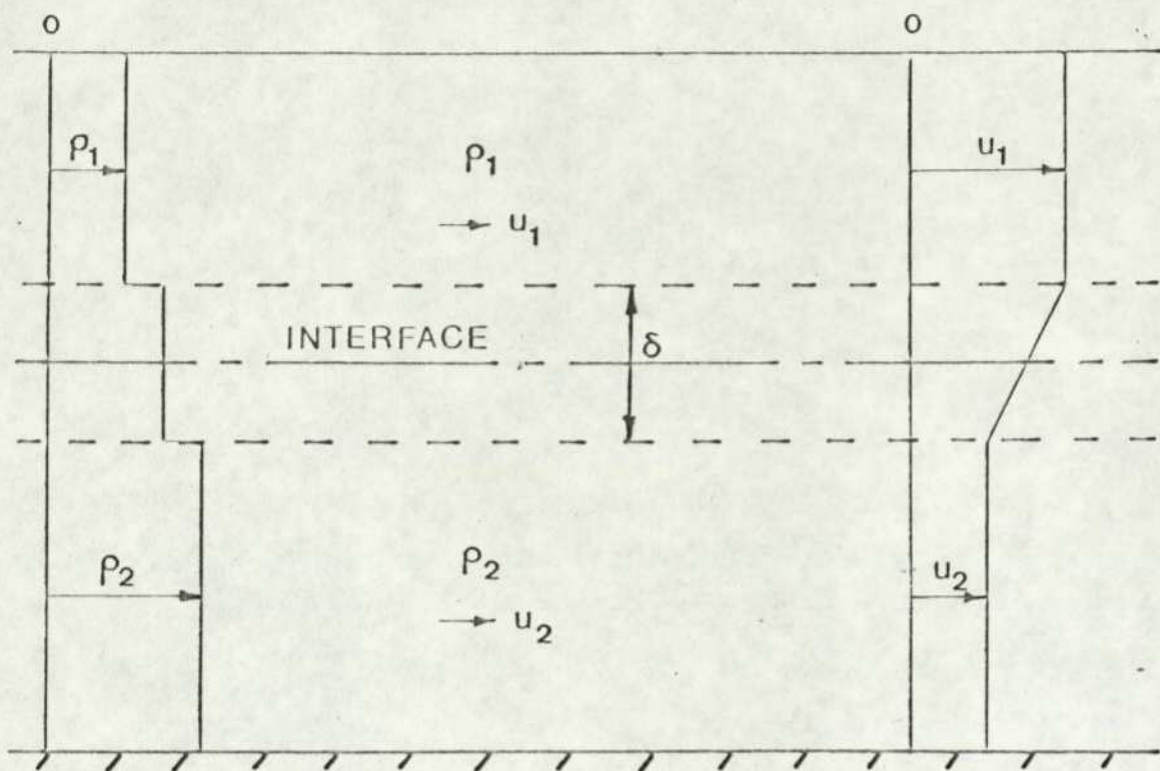


FIG. 6.2 TAYLOR-GOLDSTEIN INTERFACIAL MIXING LAYER

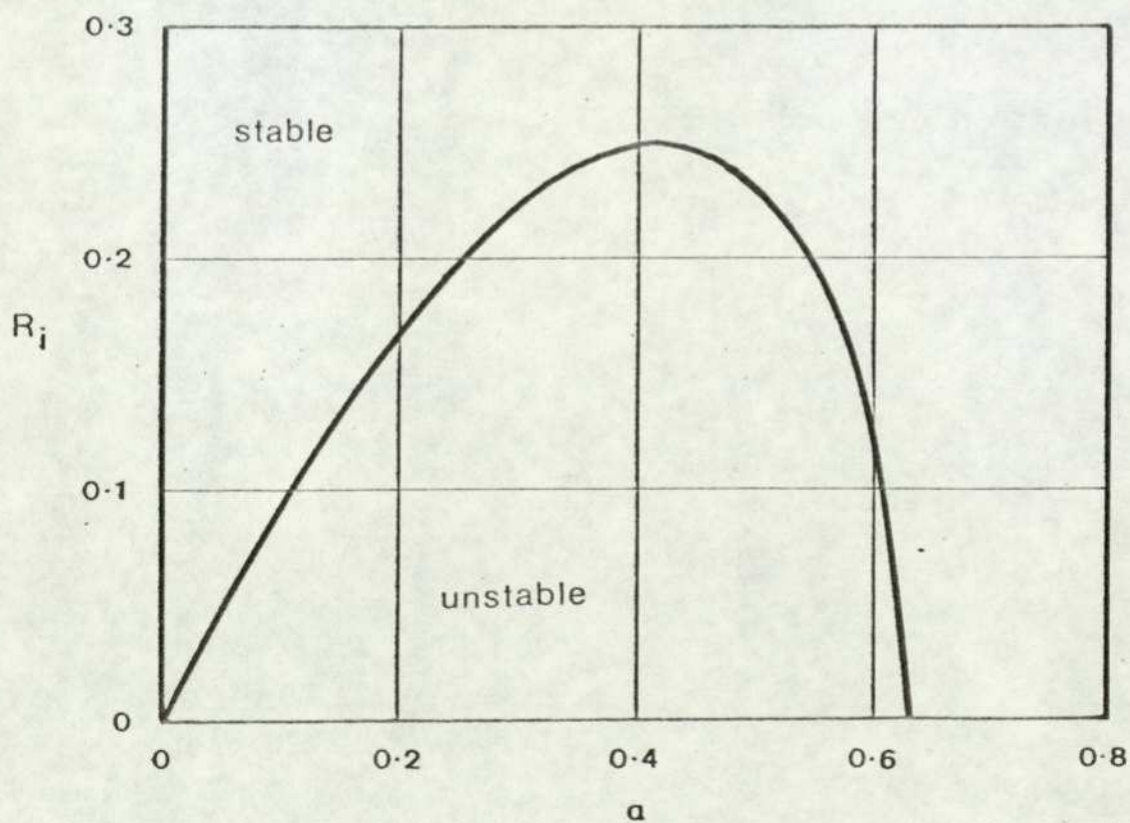


FIG. 6.3 MILES-HOWARD STABILITY DIAGRAM

The celerity of the unstable waves forming at an interface were measured by the writer from experiments to move at 0.4 times the relative velocity between layers (i.e. $c=0.4V$). If it is assumed that the period of these waves is given by Eq.(6.1) then

$$T = 2\pi \left[\frac{\delta}{g(1 - \epsilon)} \right]^{1/2} = L/c \quad (6.3)$$

and $L = \pi \delta \quad (6.4)$

It also follows that the critical interfacial densimetric Froude number is

$$F_{\delta} = V_c / (g' \delta)^{1/2} = 1.25 \quad (6.5)$$

Based on this analysis it would be expected that when the interfacial densimetric Froude number is greater than 1.25 or the Richardson number less than 0.64 interfacial mixing due to turbulent diffusion will occur. Before a mixing layer and hence turbulent diffusion develops there is however interfacial instability and mixing due to the entrainment of breaking waves. Therefore the critical interfacial densimetric Froude number is best given as

$$1.0 < F_{\delta} < 1.25 \quad (6.6)$$

especially since Keulegan's(76) experiments on entrainment gave the celerity of the unstable interfacial waves as $c = 0.5V$. The foregoing analysis does not include the effects of fluid viscosity in the determination of critical

conditions for incipient wave breaking.

Viscous Flow — The conditions of instability governing waves at an interface between two liquids must be similar to that of wind generated surface gravity waves. The latter were studied by Jeffreys(70) who obtained a criterion of stability in the form of a dimensionless parameter Θ equal to 0.215, where

$$\Theta = (v_2 g')^{1/3} / V \quad (6.7)$$

From experiments with fresh water and sugar solutions, where a light surface layer flows over a stationary lower density layer, Keulegan(76) found that with a flow Reynolds number > 450 , the interfacial waves broke when the upper layer had a velocity V_c given by the value of $\Theta = 0.178$.

From dimensional considerations if δ is the thickness of the interfacial mixing layer, then any stability parameter should be a function of the interfacial Reynolds number $R_\delta = (V \delta) / v_2$ and the interfacial densimetric Froude number $F_\delta = V / (g' \delta)^{1/2}$. Since $\Theta = 1 / (F_\delta^2 R_\delta)^{1/3}$, this parameter is of considerable practical importance as a criterion for interfacial wave instability.

From all the above theories the important facts that emerge concerning the stability of a density interface are as follows. The density difference between fresh and salt water is capable of suppressing turbulent mixing at the interface, by converting kinetic energy into

potential energy. This is explained in simple language by Schijf and Schönfeld (120) as follows. If, for example, a volume element of denser water from a lower layer is raised to a higher layer, and in the lower layer replaced by an element of less dense water from the higher layer, both elements acquire an amount of potential energy. If then the elements dissolve in the new environment by molecular diffusion, the potential energy acquired dissipates. Hence, in a flow with a density gradient, the equilibrium between energy supply and dissipation is attained by less developed turbulence than in a flow without a density gradient. This means that the density gradient attenuates friction as well as mixing.

Lee (85) has indicated that the important physical mechanisms for the existence of a salt wedge flow are.

1. Strong stabilizing influence of the large density change at the surface.
2. Weak stabilizing influence of the small density change across the interface.
3. The destabilizing influence of the shearing parallel flow.

The stabilizing influence of the free surface and the interface have been found to be dependent on the wave number of the disturbances. Also, that the instability at large wave numbers is not of practical interest because of the stabilizing effects of viscosity.

6.2 Interfacial Mixing

Mixing of fluid between strata of slightly different densities plays a critical role in various phenomena of meteorology, hydraulics, and oceanography. These usually involve the motion of one stratum relative to another, such as that of warm or cold layers in the atmosphere, a sediment laden current beneath the clear water of a reservoir, superposed fresh and salt water in estuaries, and thermal streams in the ocean. In each instance the rate of intermixing depends upon an as yet uncertain relationship between the velocity, viscosity, and density characterizing the relative motion.

Rouse and Dodu (117) found from experiments on a two-layer stratified fluid where turbulence was produced artificially in one of the layers only, two pertinent facts. First, fluid is entrained from the surface of the non-turbulent layer and diffused throughout the turbulent layer. Second, so long as the turbulence is produced by some external mechanism, there is a finite rate of entrainment across the interface for any Froude and Reynolds numbers. Pedersen (103) carried out similar experiments on a fresh water saline water interface and concluded that entrainment is a one-way transport process directed from the nonturbulent to the turbulent fluid.

In the preceding section it was seen that for inviscid flows, the criterion for interfacial wave stability depended upon the formation of an interfacial

mixing layer across which the density changed. This mixing layer can only be formed when turbulent diffusion mixing is taking place at the interface. There are in fact two distinct types of mixing, one is called entrainment mixing and the other turbulent diffusion mixing. When one of the layers is laminar flow, the lack of turbulence means that the mixing is confined to the interface and therefore plumes of fluid are ejected from the non-turbulent to the turbulent layer only. This type of mixing in which plumes from the non-turbulent fluid are entrained within the turbulent fluid is called entrainment mixing. If both layers are in a turbulent state then plumes of fluid are ejected across both sides of the interface, the rate depending on the level of turbulence. If both layers are equally turbulent an interfacial mixing layer is formed which has the effect of maintaining a high density gradient at the interface and thereby suppressing turbulent mixing. Eddy motions are now observed within the interfacial mixing layer and the mixing is described as turbulent diffusion mixing. In many natural stratified flow situations such as a saline wedge estuary and especially fjord flows where the velocities in the lower layer can be very low, both types of mixing can be present at the same time to a greater or lesser extent.

The rate of interfacial mixing has for many years been thought to be a function of the overall Richardson number, this is however no longer the case.

Experiments by Kantha, Phillips and Azad (72) were unable to show that the rate of entrainment was a function of the overall Richardson number as other experimenters such as Turner (140), Kato-Phillips (75) and Moore-Long (97) had suggested, and concluded that it certainly has no simple power law dependence on the overall Richardson number. Experiments by Sumer and Fischer (132) on natural channels found that the rate of transverse mixing increased by density stratification if the channel cross-section was irregular. They suggested that the same probably applies to vertical mixing and that in natural channels no relationship exists between the rate of mixing and the overall Richardson number, or if it does, it may be in the opposite direction from what has usually been assumed. Barr (20) also states that the condition for similarity between model and prototype for mixing does not directly involve the Richardson criterion, only the relative density between layers. Keulegan (76) suggested from experience and showed by experiment that:-

$$q = k_m (V - V_c) \quad (6.8)$$

where q = rate of mixing, V = relative mean velocity between layers, V_c = relative mean velocity at critical conditions, and k_m = coefficient of mixing. This equation can be shown to have a theoretical as well as an experimental basis even though it is not entirely satisfactory; since as Barr (20) has stated, the mixing is a function of the

relative density between layers, and should therefore appear in the equation.

From the velocity distribution across an arrested salt wedge shown in Fig. 5.1 it is found that if V is the average velocity in the upper layer then referring to Fig. 6.4a, $u_1 = 0.8V$ and $u_2 = 0.3V$. The velocity at the interface is about $0.55V$. If it is assumed that this velocity gradient creates eddies from the breaking plumes as shown in Fig. 6.4a, then this mixing between layers should be responsible for the average interfacial shear stress τ . The velocity gradient of the flow $\frac{\partial u}{\partial z} = (u_1 - u_2)/\delta$ since $\delta_1 \approx \delta_2$ and $\delta_1 + \delta_2 = \delta$.

The eddy motion across the flow relative to the mean motion is $\frac{1}{2} \delta \frac{\partial u}{\partial z}$. Every unit mass of fluid changes its momentum by an amount $\delta \frac{\partial u}{\partial z}$. The total rate of change of momentum between layers equals the shear force at the interface, thus

$$N(a \rho_2 \frac{1}{2} \delta \frac{\partial u}{\partial z} + a \rho_1 \frac{1}{2} \delta \frac{\partial u}{\partial z}) \delta \frac{\partial u}{\partial z} = A_i \tau \quad (6.9)$$

where a = base area of plume.

A_i = surface area of interface/wave length.

N = number of waves breaking/wave length at any particular instant.

$$\text{since } \left(\delta \frac{\partial u}{\partial z} \right)^2 = (u_1 - u_2)^2 \quad (6.10)$$

$$\tau = \frac{Na}{2A_i} (u_1 - u_2)^2 (\rho_2 + \rho_1) \quad (6.11)$$

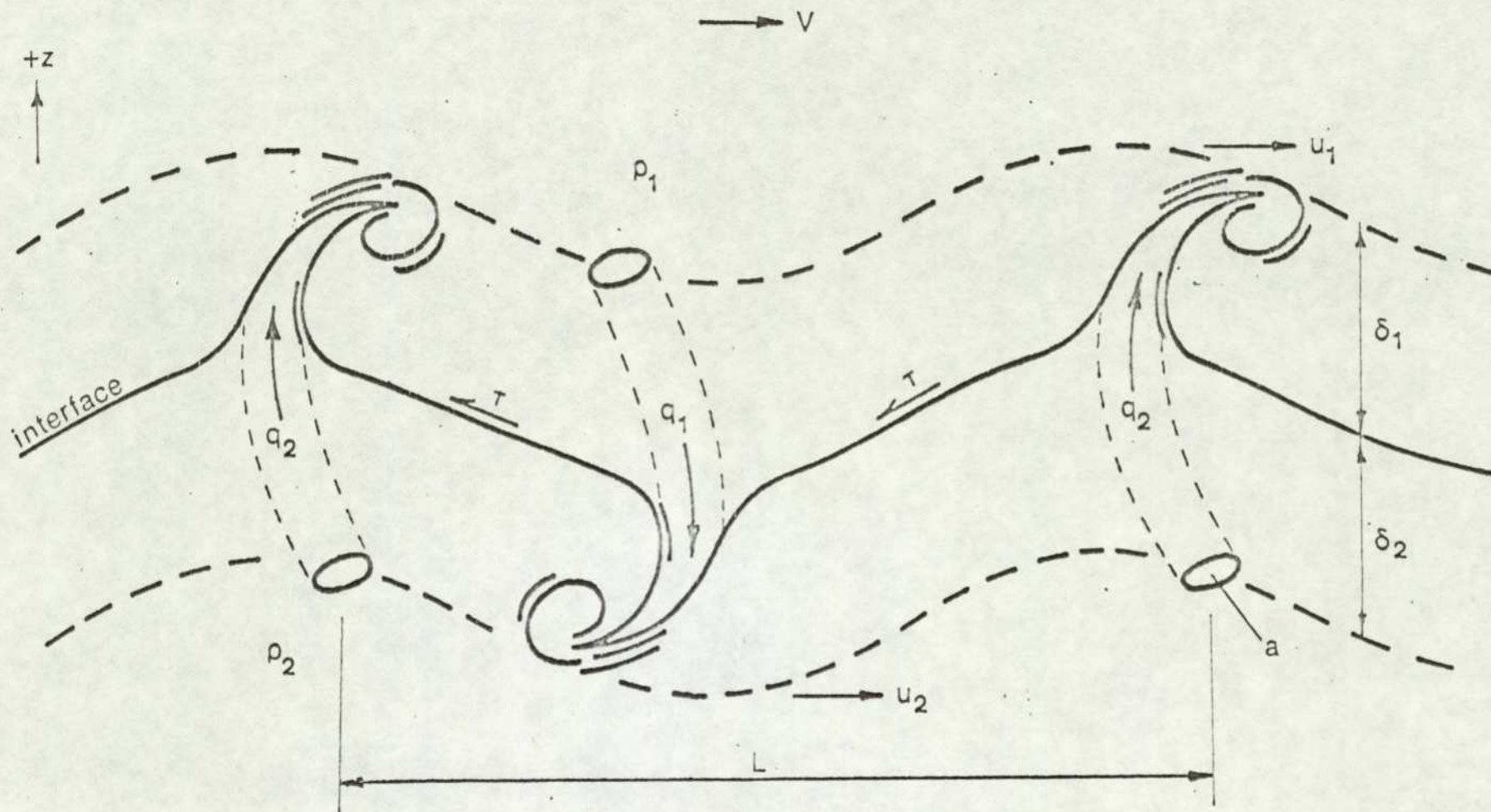


FIG. 6.4a DIAGRAM SHOWING BREAKING INTERFACIAL WAVES

$$\text{but} \quad \tau = \rho_1 q_1 u_1 - \rho_2 q_2 u_2 \quad (6.12)$$

hence since $\rho_2 \approx \rho_1$

$$\tau = \frac{Na}{A_i} (u_1 - u_2)^2 \rho_2 = \rho_2 (q_1 u_1 - q_2 u_2) \quad (6.13)$$

Because the mixing between layers is due to turbulent diffusion, $q_1 = q_2$.

$$\tau = \frac{Na}{A_i} (u_1 - u_2)^2 \rho_2 = \rho_2 q_2 (u_1 - u_2) \quad (6.14)$$

$$\tau = \frac{Na}{4A_i} \rho_2 V^2 = 0.5 \rho_2 q_2 V \quad (6.15)$$

$$q_2 = \frac{Na}{2A_i} V \quad (6.16)$$

Eq.(6.16) suggests that there is mixing at the interface for all velocities. Since this is not the case, and mixing only begins when the average velocity in the upper layer exceeds a certain critical value V_c ; V in Eq.(6.16) must be replaced by $(V - V_c)$.

$$q_2 = K_m (V - V_c) \quad (6.17)$$

where $K_m = Na/(2A_i)$, and $(V - V_c)/V$ is usually less than 0.5.

It will be seen that Eq.(6.11) is strictly only valid when interfacial mixing is taking place. It is however also used to calculate the stresses before mixing takes place of a laminar or quasi-laminar nature.

6.3 Experimental Results and Analysis for a Saline Wedge

Two sets of experiments were performed on arrested saline wedges. The first set was to ascertain the conditions at which the interface became unstable, and the second series of tests to measure the amount of mixing from the lower to the upper layer. In a laboratory channel free from wind effects the generation of short interfacial waves is caused by shearing stresses generated at the bed of the channel by the circulation current and possibly also by secondary currents. Eddies formed at the bed spread upwards towards the interface increasing in size as they go and on reaching the interface cause it to assume a wavy surface. Mortimer, McNaught and Stewart (98) were of the opinion that the generation of short internal waves in a natural lake was caused by the influence of eddies generated at the surface and spreading down to the interface. It follows that in saline wedge estuaries the interfacial waves must be generated both through the action of wind stresses at the surface and also shearing stresses at the bed resulting from the circulation current. It is probably not a coincidence that the average thickness of the interfacial mixing layer in a salt wedge estuary is about $1/5$ of the total depth when the average von Karman mixing length or eddy diameter either generated at the surface or the bed is also $1/5$ of the total depth, at mid-depth where the interface is.

When fresh water flows over a salt water wedge, the following conditions are observed to occur. If the velocity of the upper fluid is low no interfacial waves are evident and the wedge is perfectly stable. As the velocity increases interfacial waves are observed along the whole length of the wedge which increase in amplitude as the velocity increases. These waves are observed to break intermittently near the tip of the wedge where the depth is small and at the seaward end of the wedge where the velocity is high. As the velocity is further increased waves are observed to break intermittently along the whole length of the wedge and it is at this point when critical conditions are said to occur, (see Figs. 6.4b , 6.4c).

From the experiments to determine the critical conditions at which intermittent wave breaking begins the following results were obtained. Fig.6.5 shows the variation of the criterion of mixing Θ , with the Reynolds number of the upper fluid at the tip of the wedge. It can be seen that Θ is independent of the river Reynolds number and has a value of 0.140 . The hydraulic radius R was defined in its normal way as $H_0 b / (2H_0 + b)$, where b is the width of the channel. The results are also shown plotted in Fig.6.6 together with the data of Keulegan(76), Rumer(118) and Ippen and Harleman(67). It can be seen that over the range of Reynolds numbers tested the scatter of the points is large but suggests that Θ is a constant. It should however not be assumed that Θ is still the

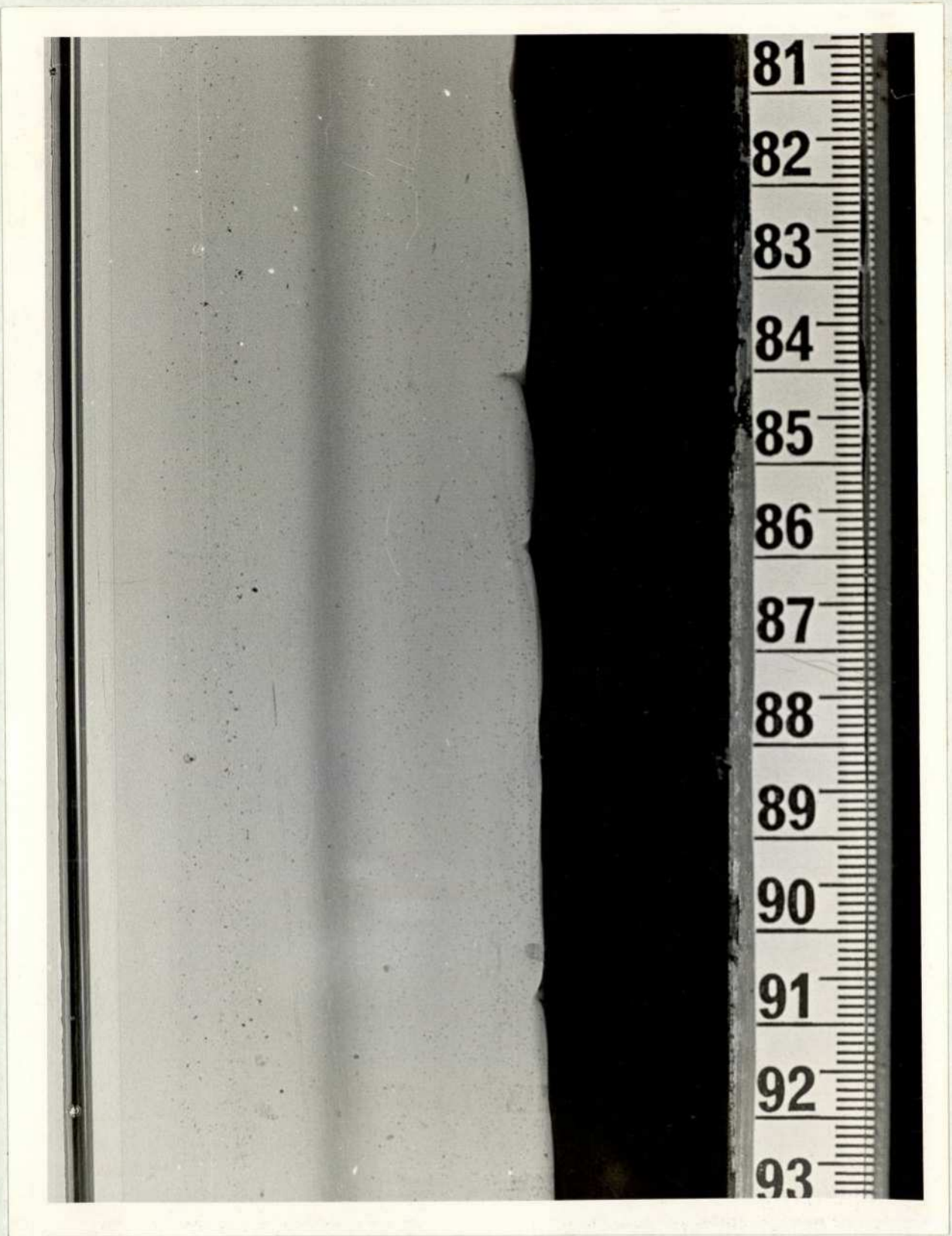


FIG. 6.1b WAVES BREAKING IN UPPER LAYER

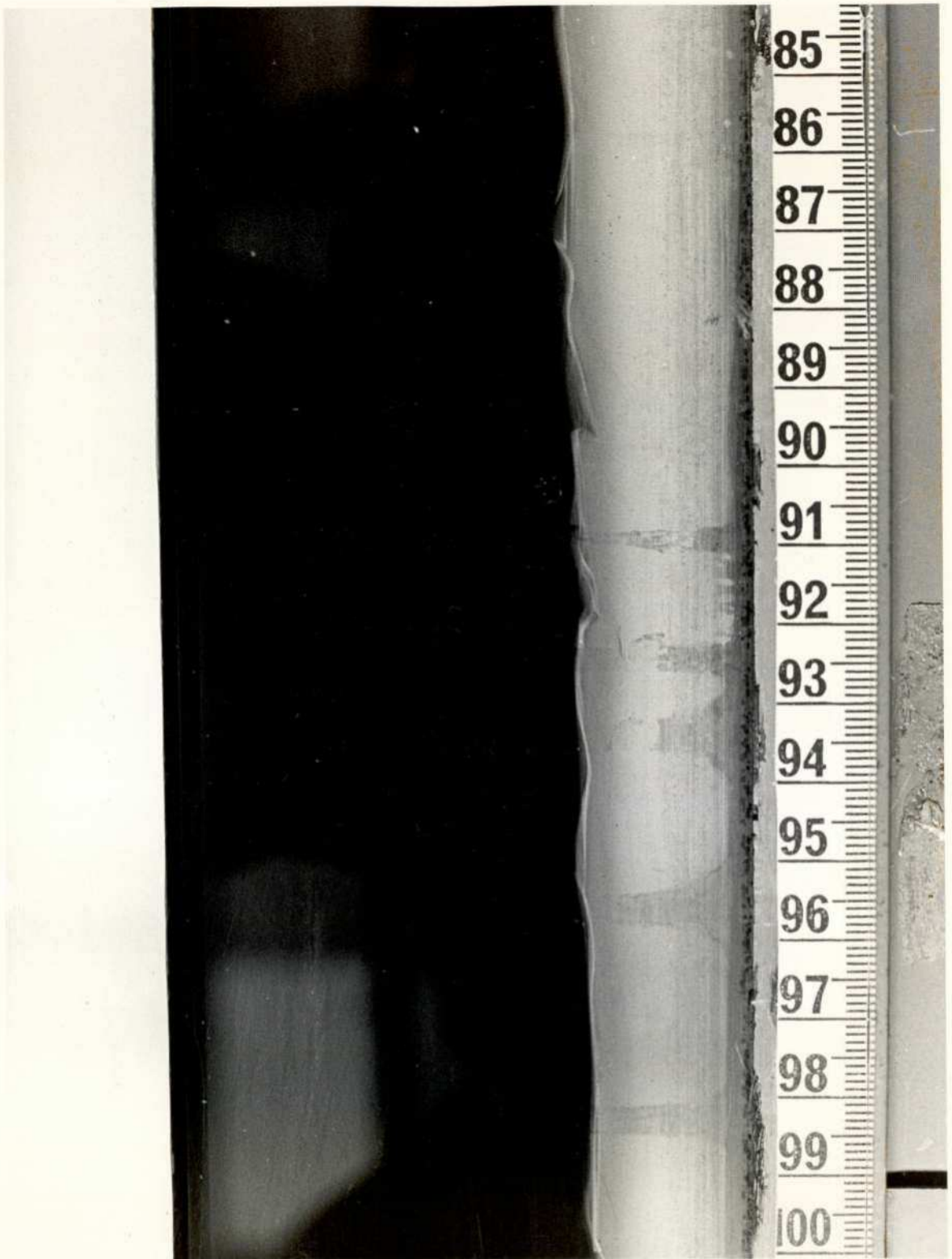


FIG. 6.4c WAVES BREAKING IN LOWER LAYER

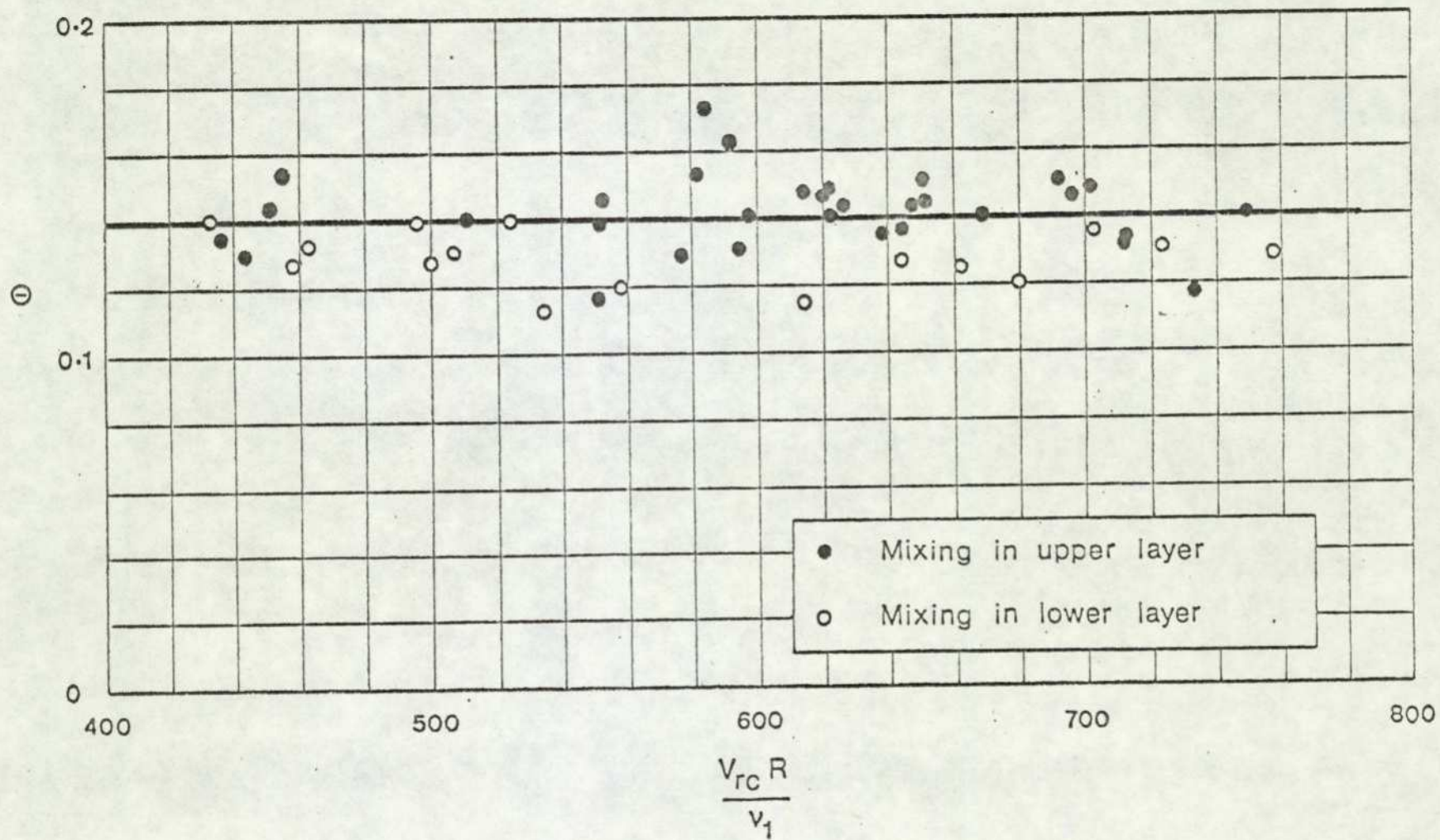


FIG. 6.5 CRITERION OF MIXING AGAINST REYNOLDS NUMBER

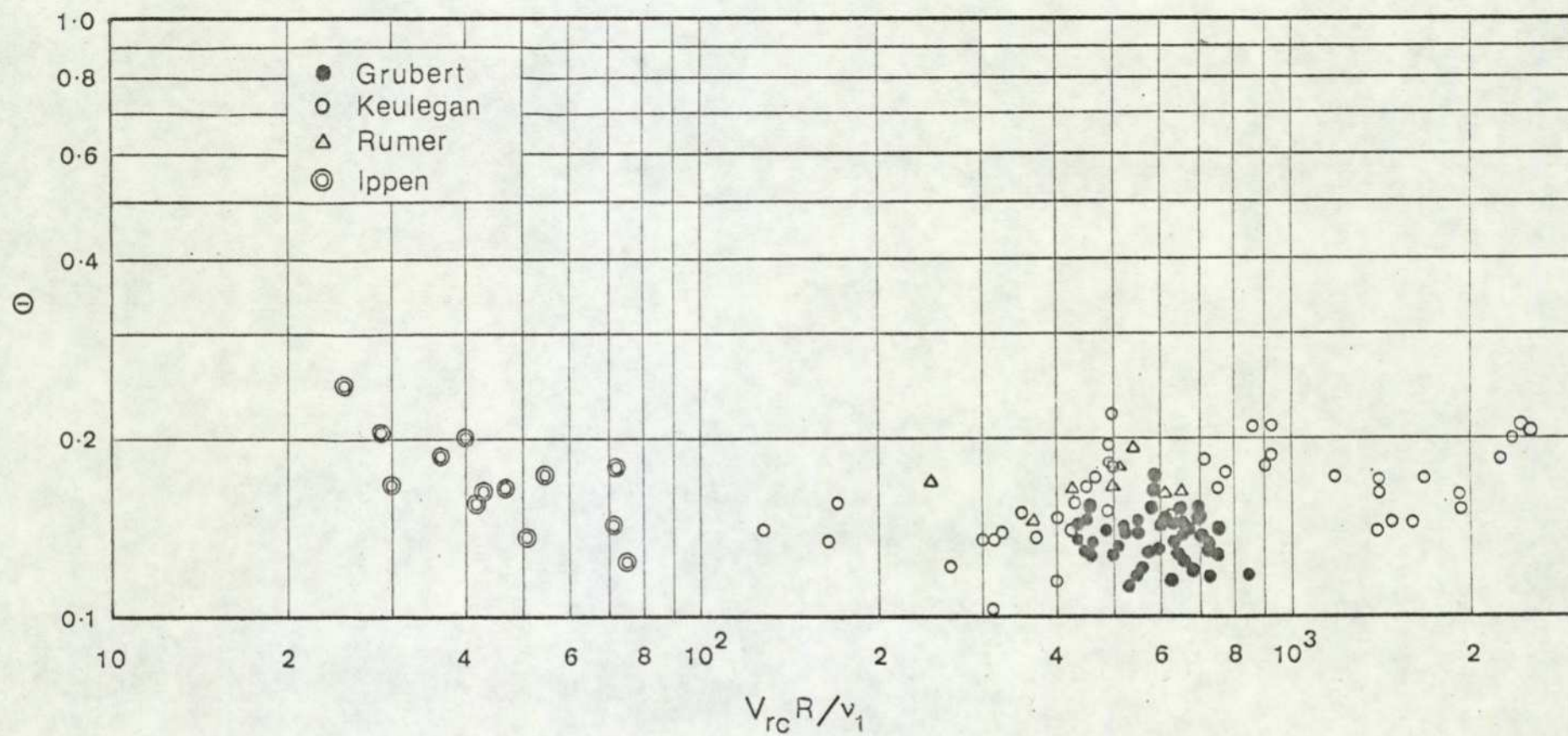


FIG. 6.6 EXPERIMENTAL VARIATIONS IN THE CRITERION OF MIXING

same constant for Reynolds numbers of the order of 10^7 as found in real estuaries. The results of Ippen and Harleman(67) in the laminar region ($V_{rc}R/\nu_1 < 400$), indicated to them that Θ is a function of Reynolds number. However, more tests at lower Reynolds numbers are necessary in view of the large scatter of the data, before this assumption can be accepted without question.

Fig.6.7 shows the variation of Θ with the relative densities of the two liquids. In this case Θ is a function of the relative density and can be expressed with the following equation.

$$\Theta = 0.526 (1 - \epsilon) + 0.127 \quad (6.18)$$

The results are again shown plotted in Fig.6.8 together with the data of Keulegan(76). It is evident from this figure also that Θ is a function of the relative density, and this alone strongly suggests that the rate of interfacial mixing should also be some function of the relative density.

Fig.6.9 shows the critical shear velocity for interfacial wave breaking in the upper fluid, and Fig.6.10 similarly for the lower fluid. Figs.6.11 and 6.12 show the relationship between wave length and thickness of the upper and lower parts of the interfacial mixing layer respectively. These results are combined in Fig.6.13 to give the variation of critical velocity V_c with interfacial densimetric velocity $(g'\delta)^{1/2}$. It can be concluded from these results that the theoretical

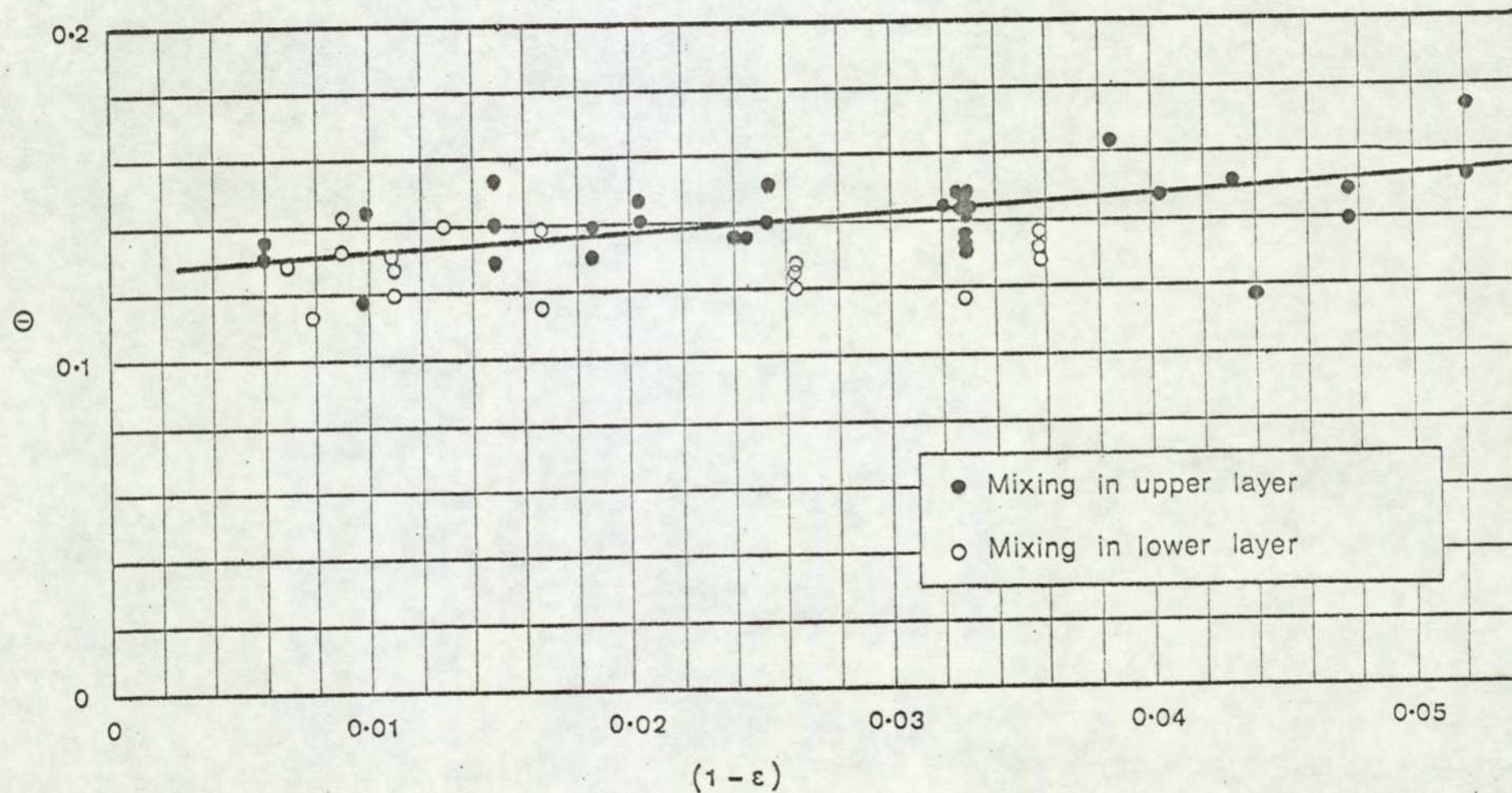
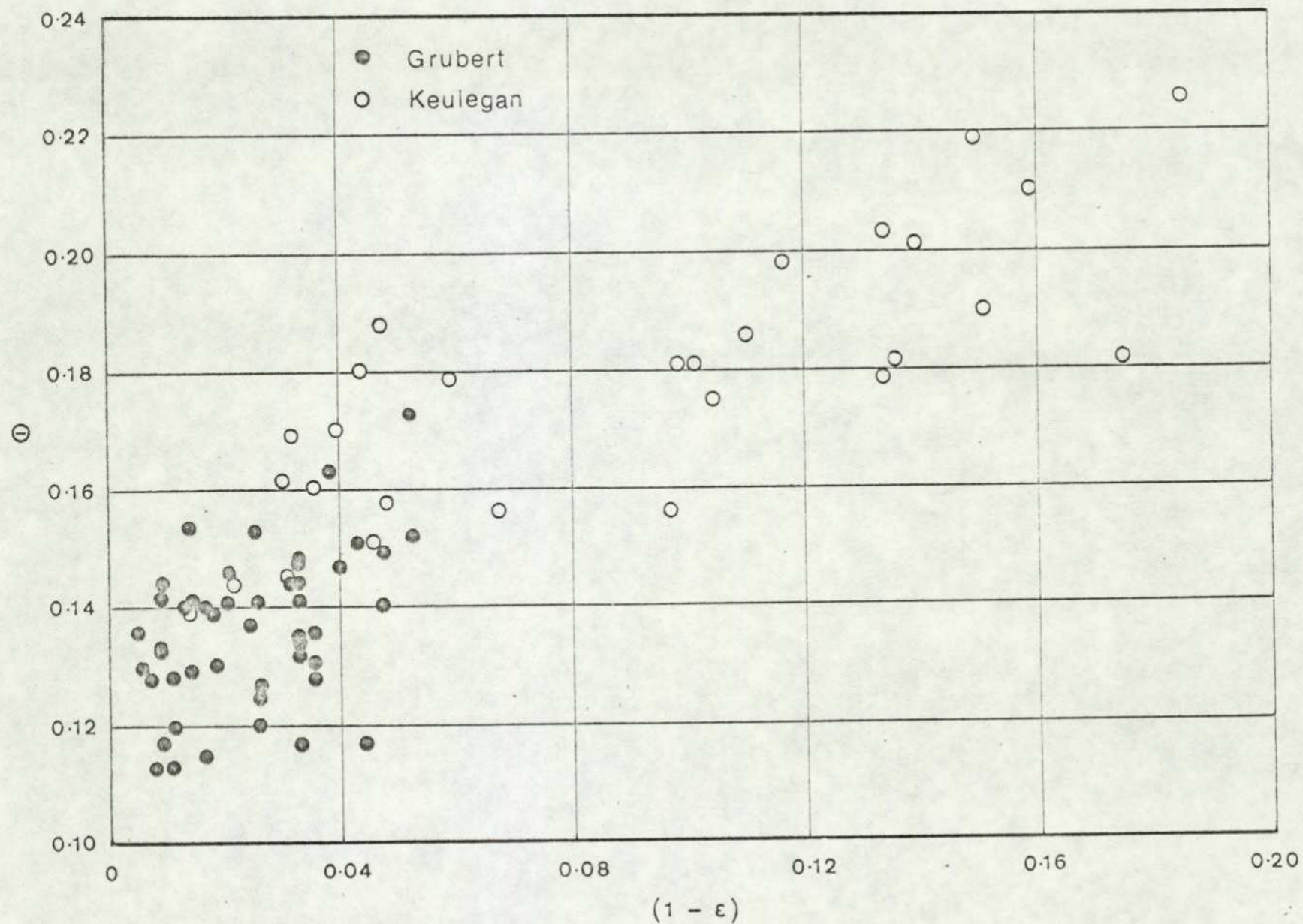
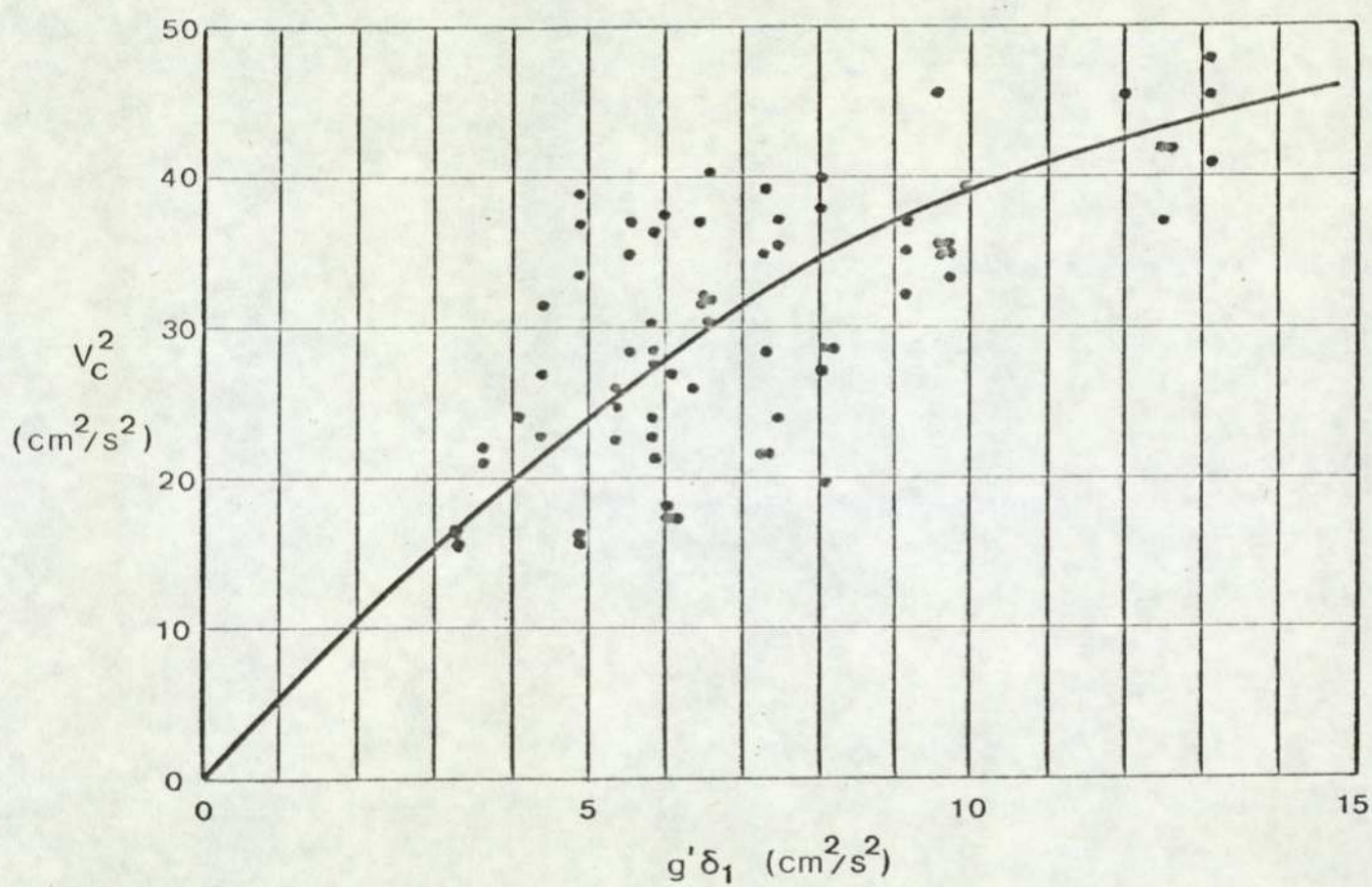


FIG. 6.7 VARIATION OF THE CRITERION OF MIXING WITH RELATIVE DENSITY





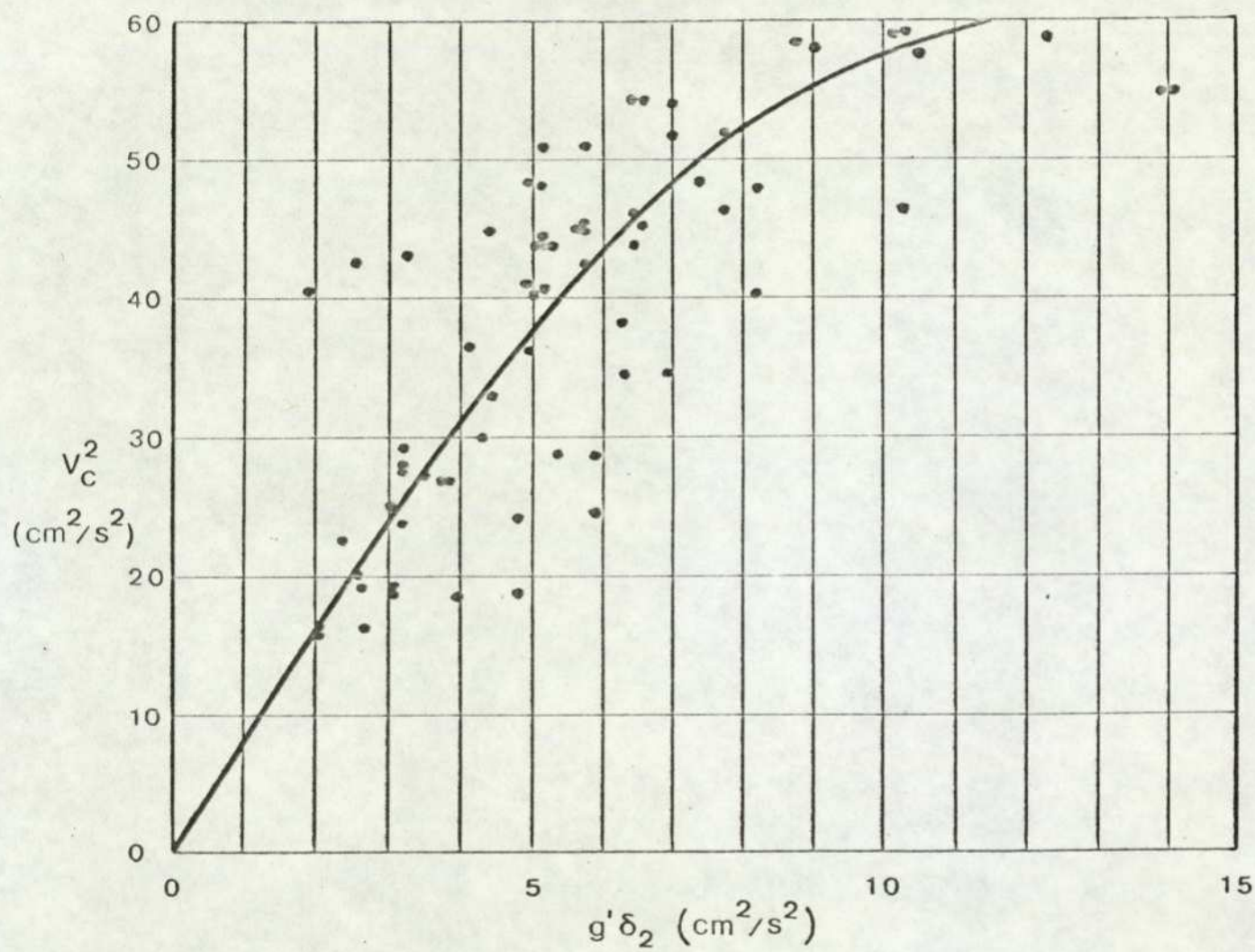


FIG. 6.10 CRITICAL SHEAR VELOCITY IN LOWER LAYER

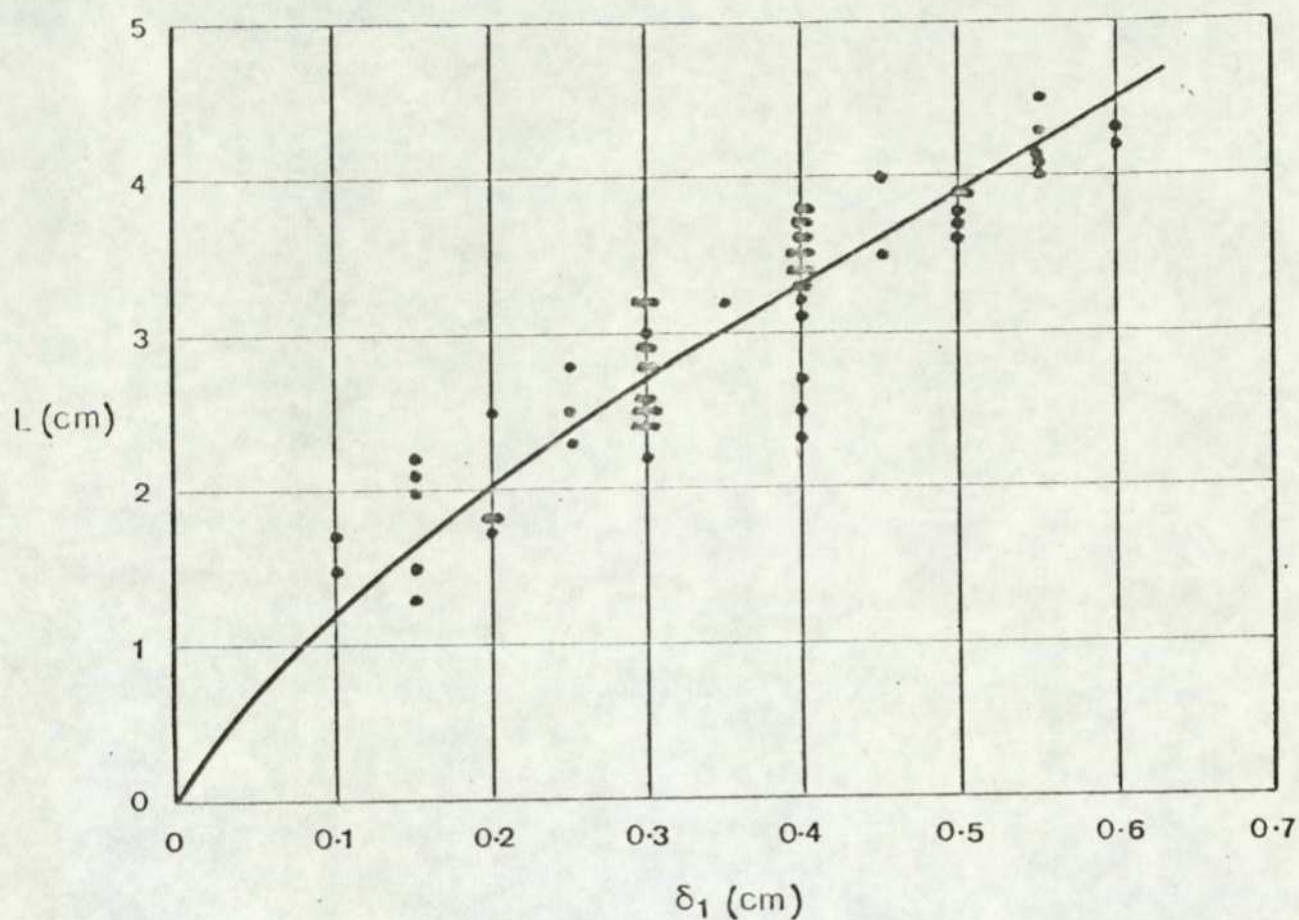


FIG. 6.11 WAVE LENGTH AGAINST THICKNESS OF UPPER PART OF MIXING LAYER

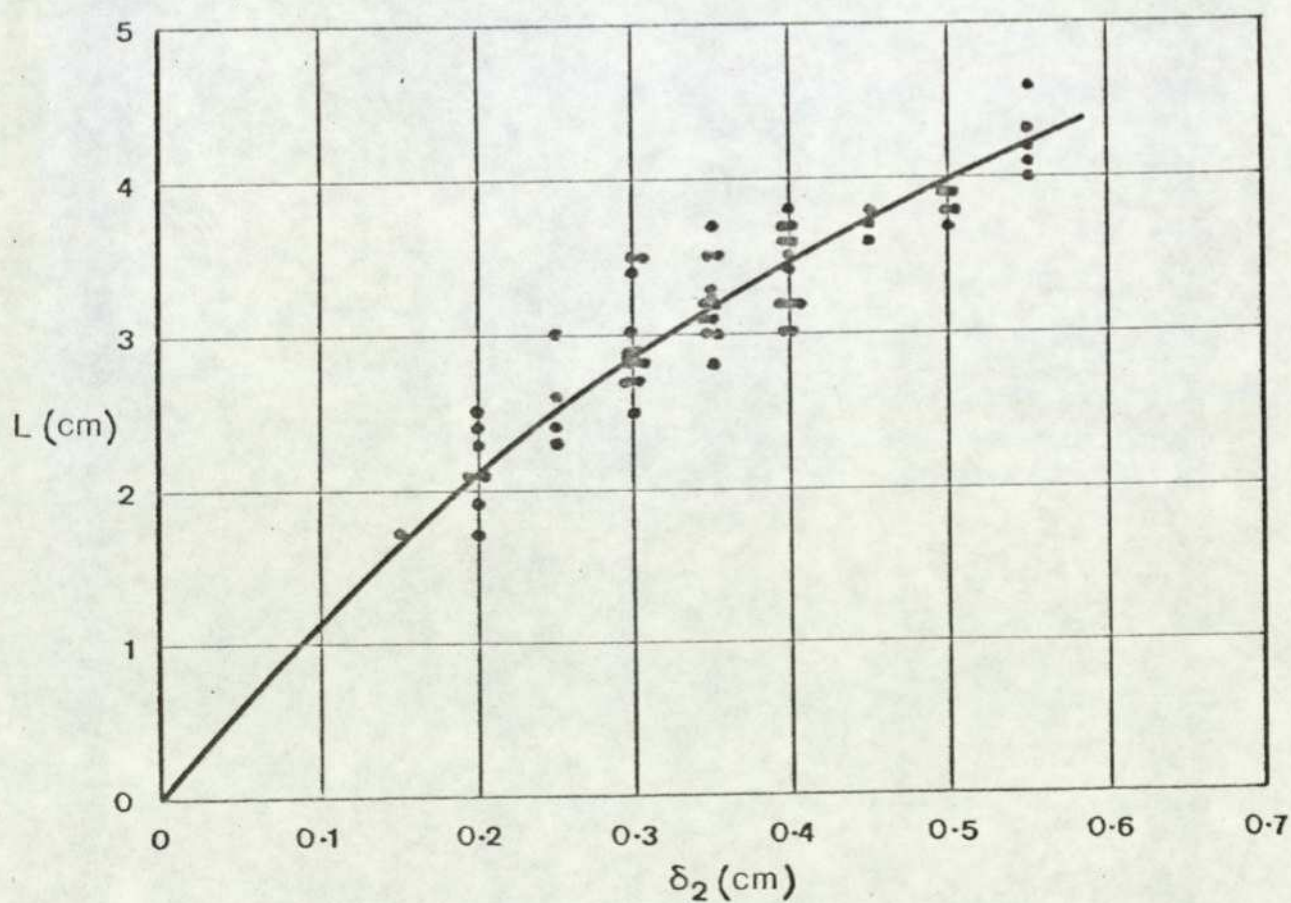


FIG. 6.12 WAVE LENGTH AGAINST THICKNESS OF LOWER PART OF MIXING LAYER

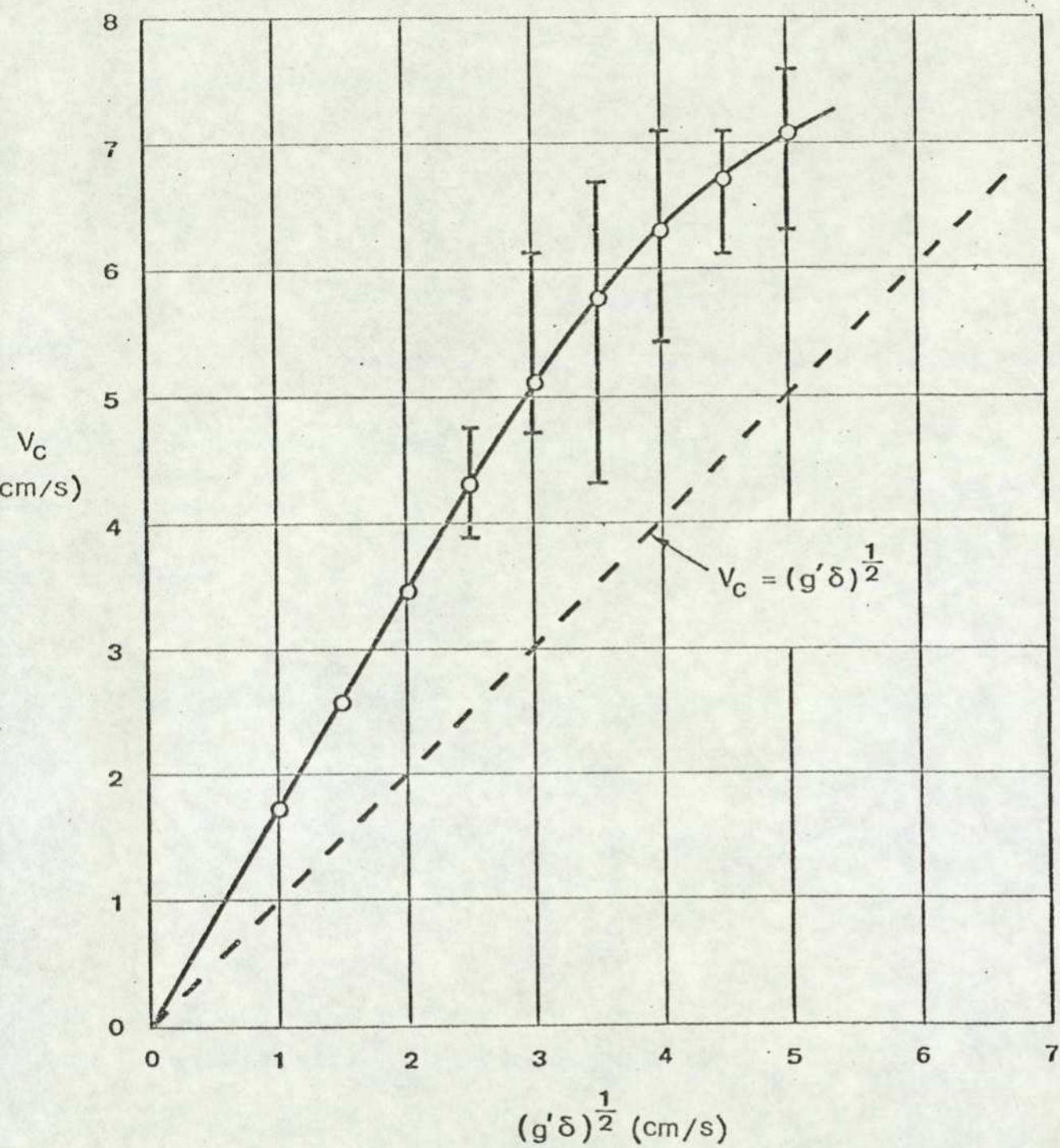


FIG. 6.13 VARIATION OF CRITICAL VELOCITY WITH INTERFACIAL DENSIMETRIC VELOCITY

stability condition expressed by $V_c = (g'\delta)^{1/2}$ is sufficient but not necessary for stability and that a better lower bound for critical velocities is that

$$F_\delta = V_c / (g'\delta)^{1/2} < 1.25 \quad (6.19)$$

The variation of wave length L with thickness of the interfacial mixing layer δ is shown in Fig.6.14 . The white circles with the limit lines represent the combined data given in Figs.6.11 and 6.12 . The black circles are data from tests for the measurement of interfacial mixing and therefore represent conditions where the velocities were greater than critical. The theoretical relationship $L = \pi\delta$, given by Eq.(6.4) which is obtained using Brunt - Väisälä's hypothesis for the shortest period of internal waves in a stratified fluid, provides a lower bound. This also agrees with the results of Thorpe (138) shown in Fig.6.15 and as Rumer (118) has noted these results seem to support the hypothesis that interfacial instability is intimately related to the natural frequency of oscillation of displaced fluid particles in a nonhomogeneous density field. Actual field measurements to support this theory are given by Mortimer, McNaught and Stewart (98), from Central Lake Michigan.

All the results obtained from the tests are plotted on the Taylor-Goldstein stability diagram shown in Fig.6.16 . Again the white circles with the limit lines are the results at critical conditions whilst the black

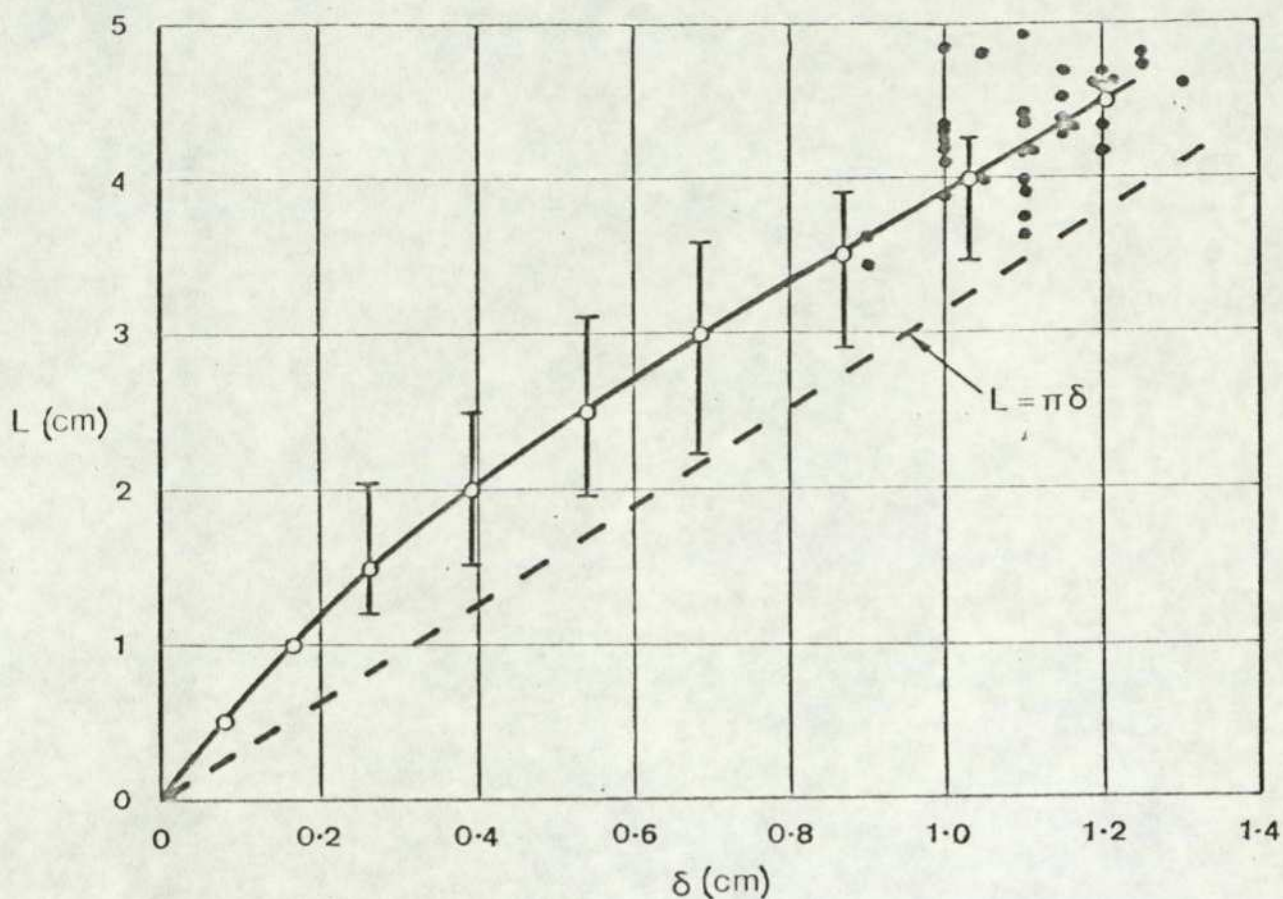


FIG. 6.14 WAVE LENGTH AGAINST THICKNESS OF INTERFACIAL MIXING LAYER

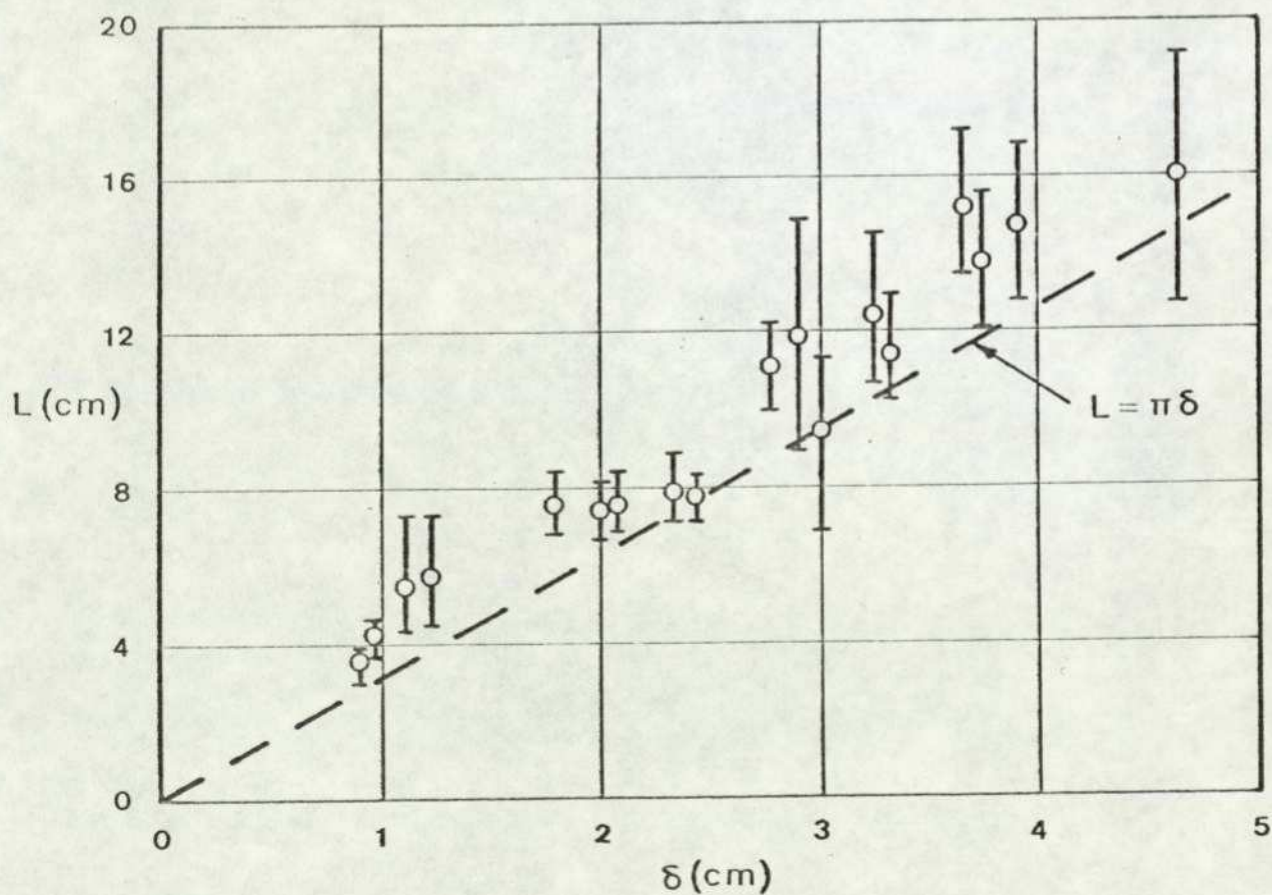


FIG. 6.15 EXPERIMENTAL RESULTS OBTAINED BY THORPE

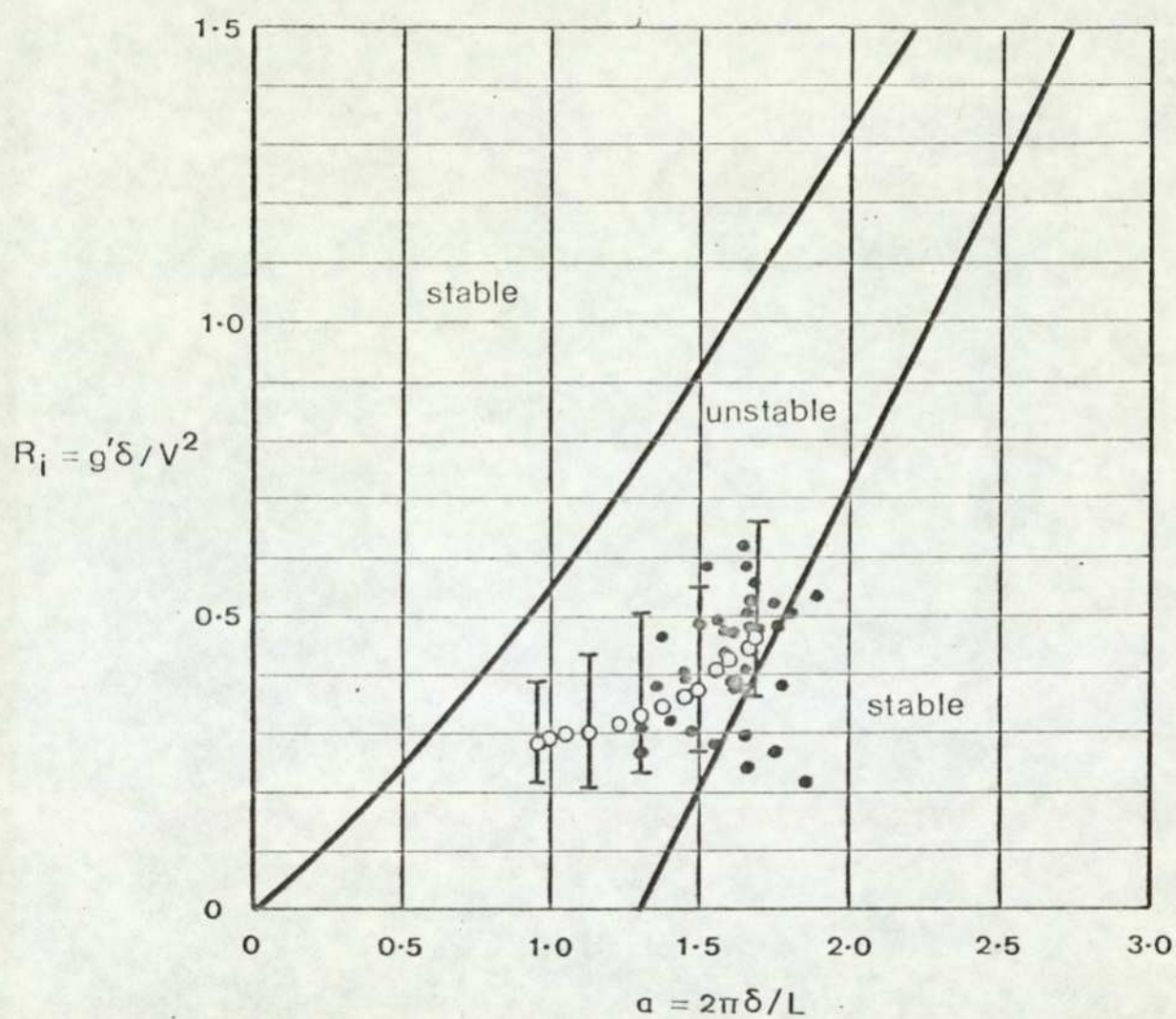


FIG. 6.16 INTERFACIAL INSTABILITY RESULTS

circles represent the data from the mixing experiments. This particular stability diagram represents a situation where the interfacial mixing layer has a constant mean density and a linear velocity distribution. It can be seen that the data obtained from the tests agrees reasonably well with the stability conditions represented by Fig.6.1 . The data however does not fit the stability conditions of Miles and Howard (96) shown in Fig.6.3, where the density distribution is assumed to be linear or the results of Goldstein (47) for an exponential density distribution. It could be construed from these results that the density distribution within the experimental interfacial mixing layers should be a constant mean value. It can be seen in Fig.6.17, from the average of the measured density distribution profiles that this interpretation is untrue in this case. Although the results agree reasonably well with the Taylor-Goldstein stability condition, the simple Helmholtz stability condition plus the Brunt-Väisälä hypothesis give even closer agreement, that is :-

$$R_1 / \alpha < 0.5 , \text{ and } \alpha < 2 \quad (6.20)$$

The results indicate that it is the waves of intermediate wave length that are unstable and that the short waves are stabilized according to Yih (155), due to the vorticity being confined to a finite region by viscous forces. The maximum wave number at which waves break appears to be about two, and the tests show that

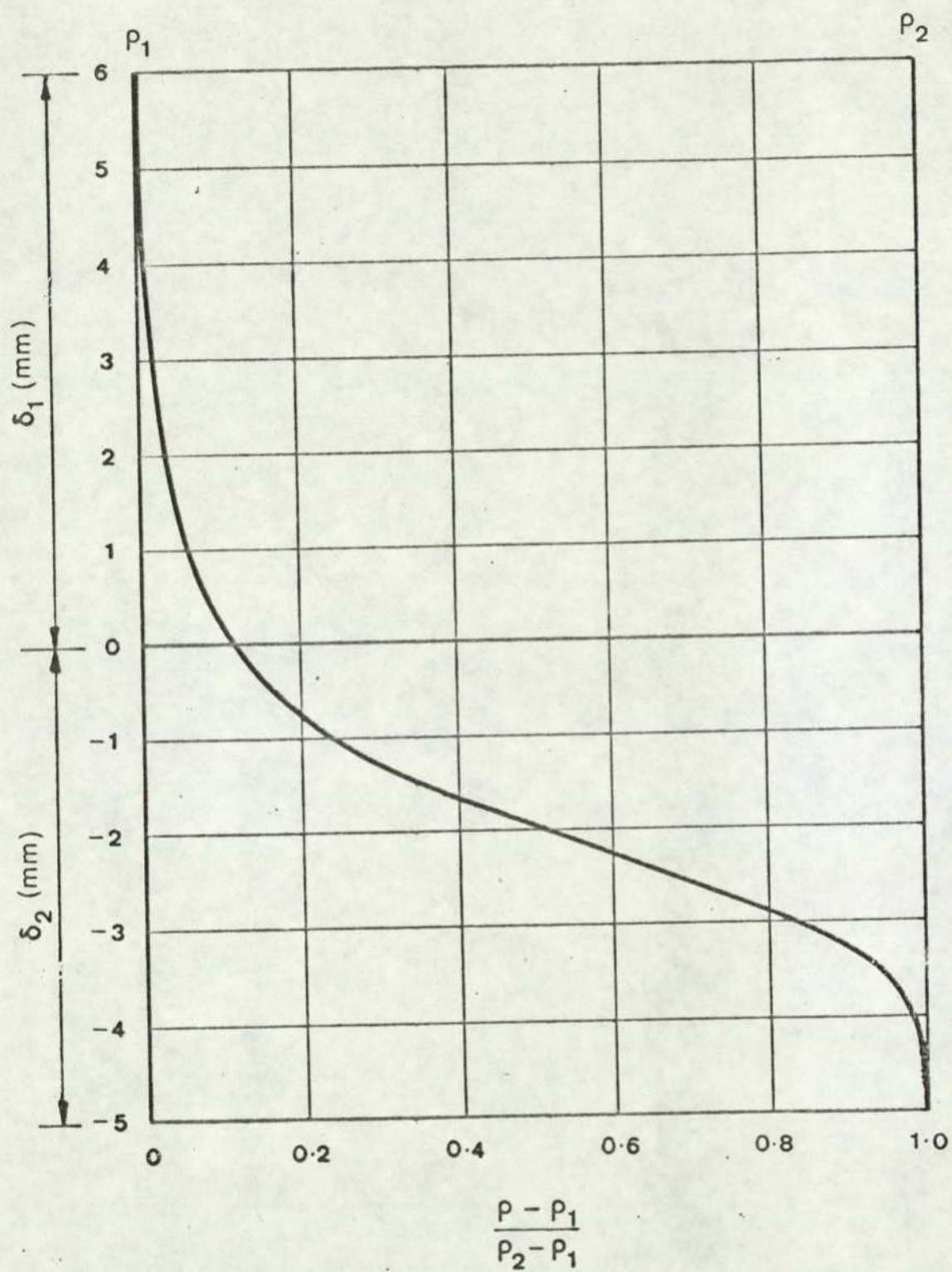


FIG. 6.17 INTERFACIAL DENSITY DISTRIBUTION PROFILE

for an experimental saline wedge a region of instability exists for

$$R_i / a < 0.4 , \text{ and } a < 2 \quad (6.21)$$

Also, Ellison and Turner (42) have shown that for $R_i > 0.8$ ($F_\delta < 1.1$) mixing at the interface of a stratified fluid is negligible, which agrees with the experimental results obtained.

The rate of fluid transfer from the lower to the upper layer was assumed to be given by the equation

$$q_2 = k_m (V - V_c) \quad (6.22)$$

where k_m is an unknown function of the relative density. This equation was summed along the whole length of the wedge using values of V_c calculated from Θ given by Eq.(6.18). From the density gradient of the upper mixing layer measured at the seaward end of the wedge, an average value for the coefficient of mixing k_m along the wedge was calculated for each test. The density gradient was measured using a wave monitor with a probe consisting of a pair of stainless steel wires 1.5mm diameter and 12.5mm apart. By insulating the probe along its whole length except for the tips a record of the density at any particular depth could be obtained on a chart recorder. The chart recorder trace was also used to calculate the average wave length and also the thickness of the upper and lower mixing layers for each test.

The wave monitor and UV chart recorder, which used 1600B galvanometers, are shown in Fig.5.2 . The calibration curve which was constantly checked, had to be calibrated within the channel where the profile was being taken since the readings altered significantly for the same solution of saline water when contained in either a test-tube, beaker or the channel. The calibration curve is shown in Fig.6.18 and a typical UV recorder trace (Test 1), in Fig.6.19 . The density profile for the upper mixing layer, obtained from this trace is shown in Fig.6.20, and the arrested saline wedge from which it was obtained is shown in Fig.6.21 .

In all, thirty nine different unstable arrested saline wedges were tested with the relative density $(1 - \epsilon)$ varying from 0.012 - 0.052 , and from these an average value for the coefficient of mixing k_m along the wedge was calculated for each test. A typical calculation for k_m along the wedge shown in Fig.6.21 is as follows.

Test 1

For the saline wedge shown in Fig.6.21, the following data was obtained:-

$$\delta_1 = 0.60\text{cm}; L_0 = 155\text{cm}; \bar{d}_1 = 6.50\text{cm}; b = 3.5\text{cm};$$

$$Q = 129.1\text{cm}^3/\text{s}; v_2 = 0.99 \times 10^{-2}\text{cm}^2/\text{s}; g' = 17.35\text{cm}/\text{s}^2;$$

Since $(1 - \epsilon) = 0.0177$, from Eq.(6.18), $\Theta = 0.136$.

$$\Theta = (g' v_2)^{1/3} / V_c = 0.136 \text{ hence } V_c = 4.09\text{cm}/\text{s} .$$

From Fig.6.20 the average density in the upper interfacial

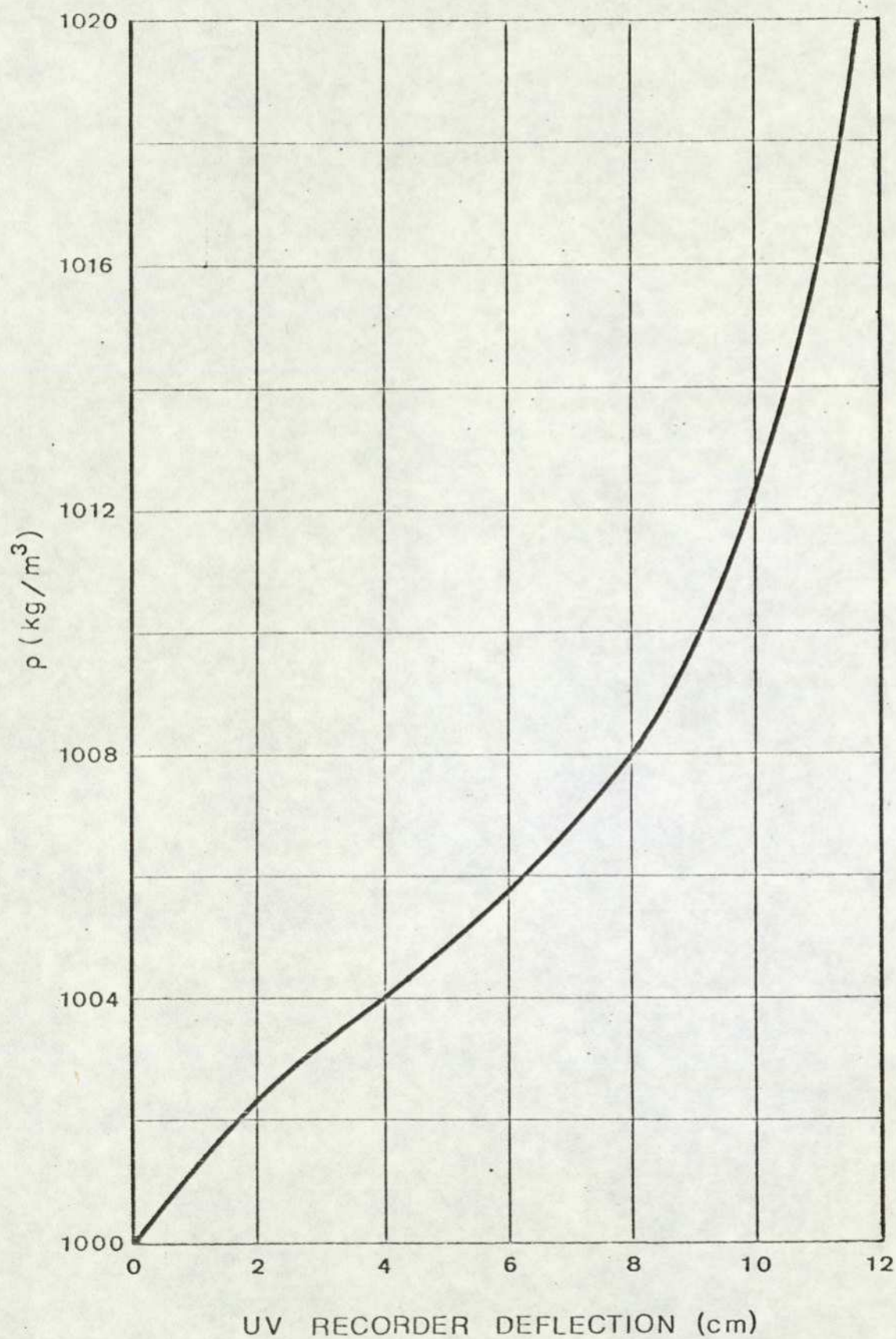


FIG. 6.18 CALIBRATION CURVE FOR DENSITY

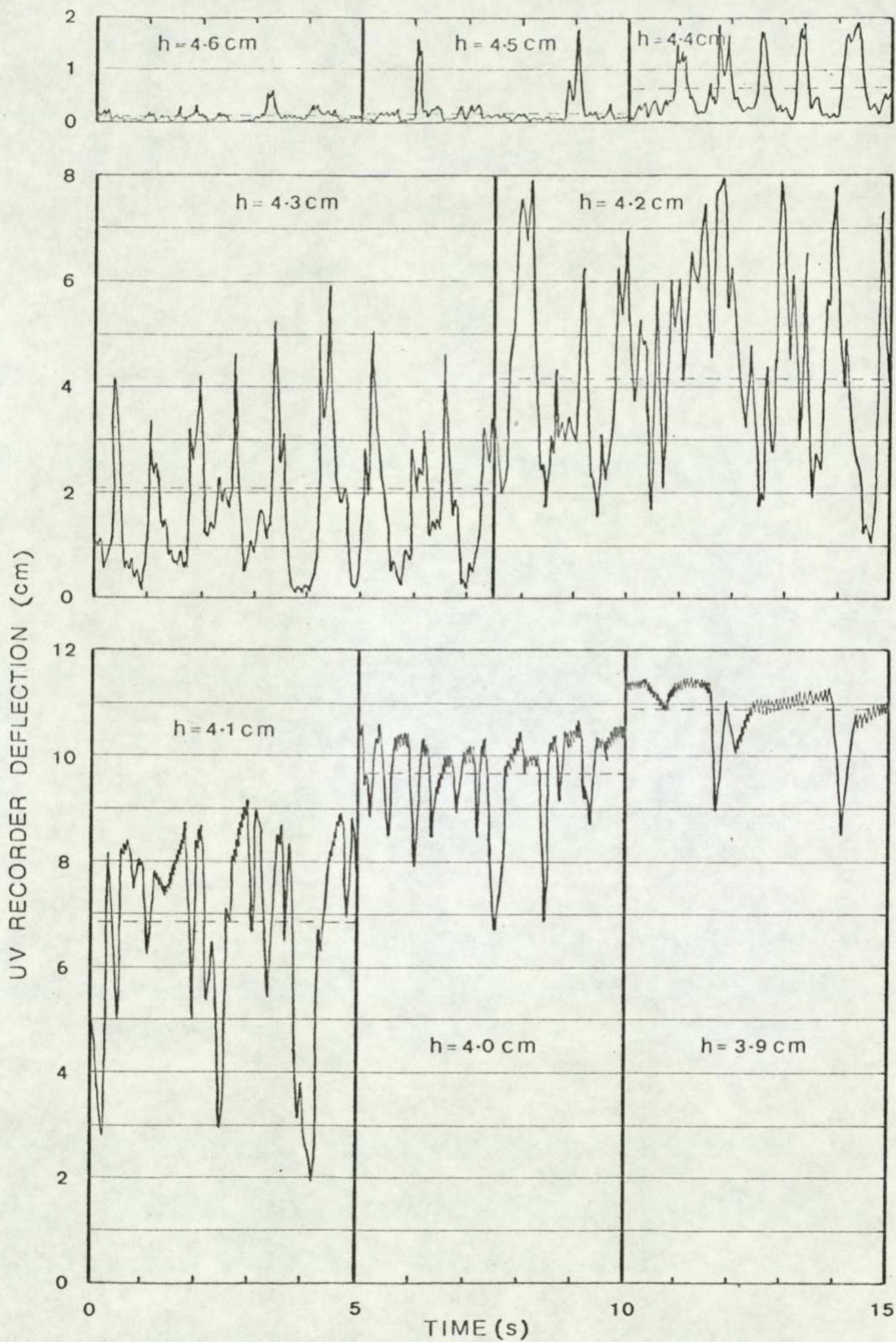


FIG. 6.19 TRACES OBTAINED AT DIFFERENT LEVELS IN UPPER MIXING LAYER

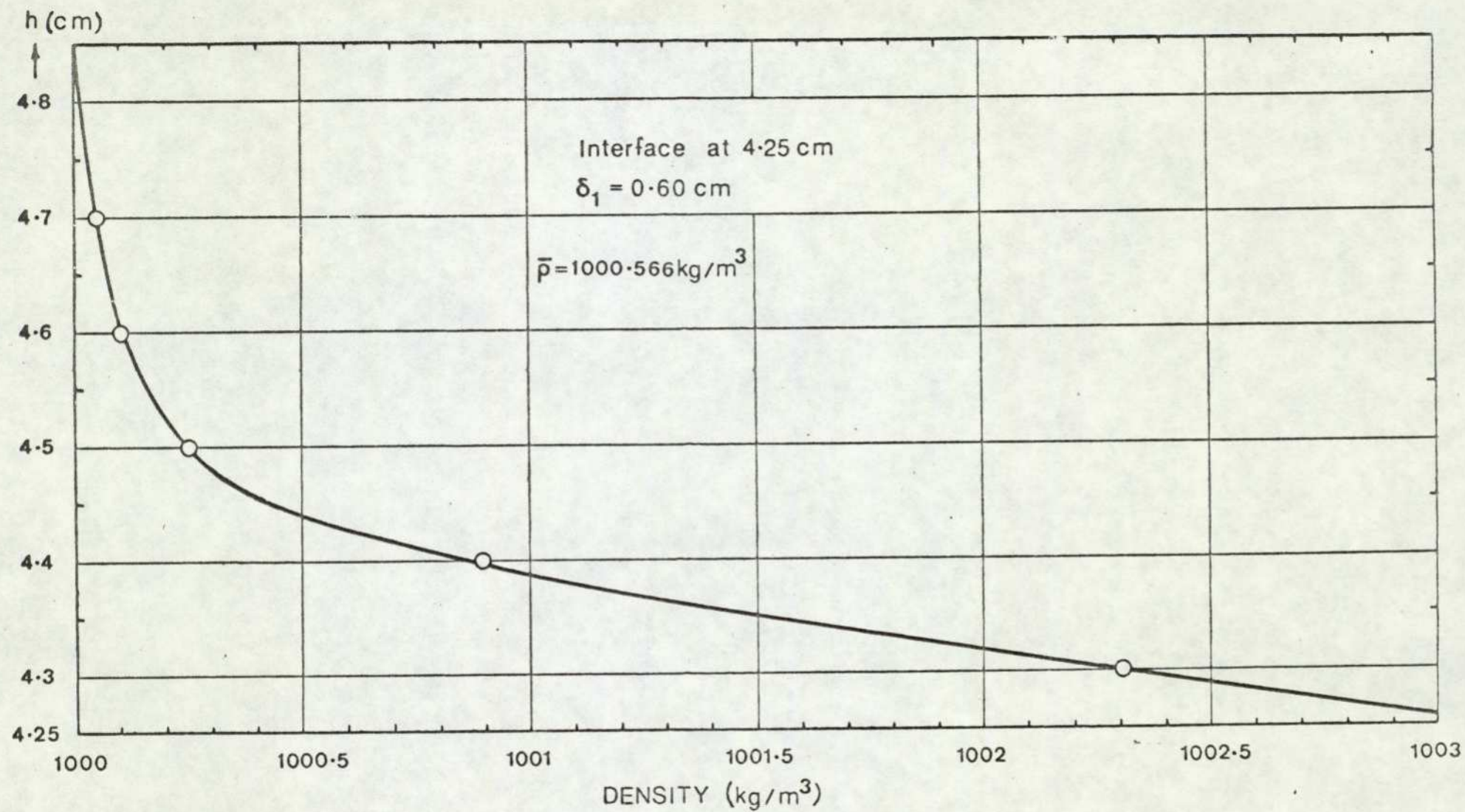


FIG. 6.20 DENSITY PROFILE OF UPPER MIXING LAYER

FIG. 6.21 TYPICAL ARRESTED SALINE WEDGE

mixing layer at the seaward end of the wedge was calculated to be 1000.566 kg/m^3 . Since $q_2 = k_m (V - V_c)$ then equating salinities.

$$0.566/1000 = \int_0^{L_0} \frac{q_2 (1 - \epsilon) dx}{V \delta_1} \quad (6.23)$$

$$0.566/1000 = \frac{k_m (1 - \epsilon)}{\delta_1} \int_0^{L_0} \left(1 - \frac{V_c b d_1}{Q}\right) dx \quad (6.24)$$

$$0.566/1000 = k_m \times 0.0177 \times 155/0.60 (1 - 4.09 \times 3.5 \times 6.50/129.1)$$

$$0.566/1000 = 1.28 k_m$$

$$\text{hence } k_m = 4.42 \times 10^{-4} \text{ when } (1 - \epsilon) = 0.0177$$

The results for k_m are plotted against the relative density $(1 - \epsilon)$ in Fig.6.22. Since the slope of the line is equal to minus one, the resulting equation for mixing is :-

$$q_2 = k'_m (V - V_c) / (1 - \epsilon) \quad (6.25)$$

$$\text{where } k'_m = 0.94 \times 10^{-5}.$$

From the experiments the average diameter of an entrainment plume was about 1mm. The surface area on which the shear stress acts is $L \times b$. It was observed that in a channel of 3.5cm width, two plumes developed across the width of the channel. Over the whole length of the wedge an average of only about one half of all

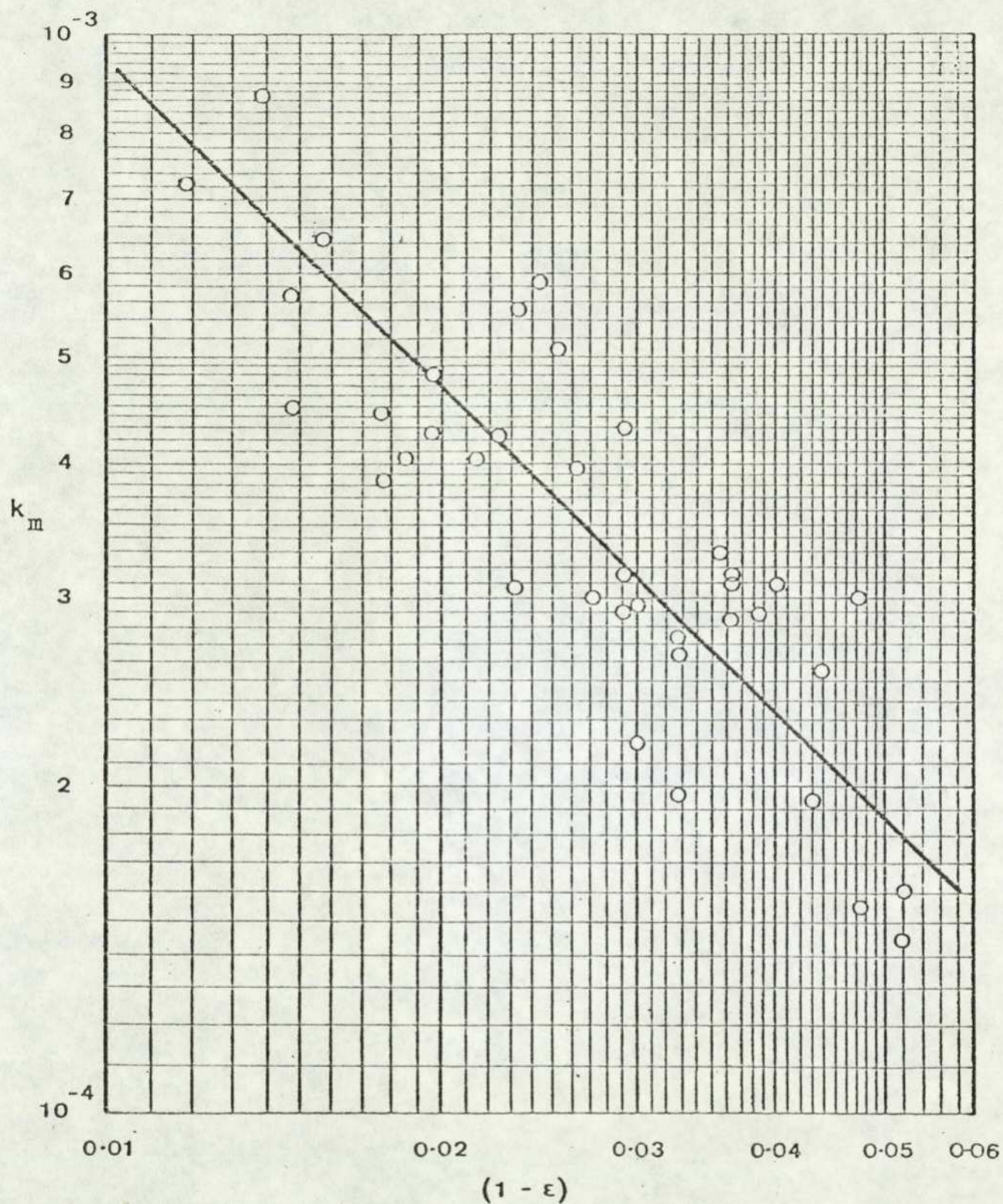


FIG. 6.22 COEFFICIENT OF MIXING AGAINST RELATIVE DENSITY

waves are breaking at any particular instant, therefore $N = 1$. The average wavelength $L = 4.5\text{cm}$. The theoretical coefficient of mixing in Eq.6.17 is therefore :-

$$K_m = 0.5 Na/A_1 = 2.5 \times 10^{-4} \quad (6.26)$$

The measured rate of mixing in the upper layer for $(1 - \epsilon) = 0.02$ gave a coefficient k_m shown in Fig.6.22 of 4.7×10^{-4} . The mixing coefficients obtained from theory and experiment are very similar indicating that the assumed mixing process due to momentum exchange of turbulent motion is valid.

6.4 Application of Results to Mathematical Models

As stated in the previous chapter, for the successful modelling of a stratified flow it is necessary that the densimetric Froude numbers should be the same in both model and prototype. The second dimensionless term in Eq.(5.1) is the densimetric Reynolds number, which has only a minor influence on the flow conditions. Since the density in the model was the same as that in the prototype, one can better refer to the scaling simply in terms of Froude and Reynolds numbers. The experimental work of Righer (115) indicated that the Reynolds number of the flow is a minor but relevant term for a stable arrested saline wedge, but decreased in importance as turbulent mixing at the interface increased. The unimportance of the Reynolds number for a salt wedge where both layers are in a turbulent state is suggested by Fig.6.5 where it

is seen to be independent of the stability parameter Θ . This is in contrast to the findings of Ippen and Harleman (67) who conducted tests on laminar stratified flows and found Θ to be a function of Reynolds number.

Since the wavelength L is not a geometrical element of the channel but proportional to the wave celerity.

$$L(\text{prototype}) = L(\text{model}) \times Y \quad (6.27)$$

where $Y = \text{prototype/model vertical scale ratio}$.

In the prototype the ratio of the interfacial mixing layer thickness δ to the total depth should from Eq.(6.4) be the same as in the model. Extra proof is given by Long(90), and Simmons (126) where the interfacial layer thickness for highly stratified estuaries, such as the South-West Pass of the River Mississippi was found to be about one fifth of the total depth, which agrees with the model tests, hence.

$$\delta(\text{prototype}) = \delta(\text{model}) \times Y \quad (6.28)$$

Also, since the interfacial densimetric Froude number in model and prototype are equal, V_c in Eq.(6.5) must increase in the prototype by the same ratio as all the other prototype velocities. Because the viscosity is a function of the relative density and the variation of Θ with relative density is given in Eq.(6.18), then from Eq.(6.7), $\Theta \propto 1/V_c$. The value of the stability parameter Θ in the prototype becomes.

$$\Theta(\text{prototype}) = \Theta(\text{model}) / Y^{0.5} \quad (6.29)$$

Eq.(6.25) remains unchanged for the prototype since the rate of mixing has the dimensions of a velocity and the densities in model and prototype are equal.

It can be seen from Eq.(6.15) that the interfacial shear stress can be expressed as :-

$$\tau = 0.5 \rho_2 q_2 V \quad (6.30)$$

but τ can also be written as :-

$$\tau = \frac{1}{8} f \rho_2 V^2 \quad (6.31)$$

where f is the interfacial friction coefficient.

From Eqs.(6.30) and (6.31)

$$f = 4.0 q_2 / V \quad (6.32)$$

substituting into Eq.(6.25) gives

$$f = 3.8 \times 10^{-5} (V - V_c) / [V(1 - \epsilon)] \quad (6.33)$$

For $(1 - \epsilon) = 0.02$, $f < 1.0 \times 10^{-3}$, which can be compared with the values used in various mathematical models of saline wedges (79,25,146,63) shown in Table 6.1 . It can be seen that the value for f obtained in the physical model is less than that used in the mathematical models. This is due to the fact that conditions in the physical model produced a more highly stratified flow than can be expected to occur in a real estuary. Table 6.1 clearly shows that as the level of stratification decreases so the interfacial friction coefficient increases. This is consistent with the increased level of turbulent mixing at the interface, due mainly to tidal action. Also, it

Estuary (1)	Type of estuary (2)	(1 - ϵ) (3)	Tidal range in metres (4)	$f \times 10^3$ (5)
South Pass Mississippi	Highly stratified	0.020	0	2
Grand Rhône	Highly stratified	0.025	0.13-0.40	3 - 5
Rotterdam Waterway	Partly mixed	0.024	1.6	42
Fraser River	Partly mixed	0.022	3.3	60
Note : 1 m = 3.28 ft				

TABLE 6.1 INTERFACIAL FRICTION COEFFICIENTS USED IN ESTUARINE MATHEMATICAL MODELS

should not be forgotten that friction coefficients in mathematical models are not truly representative of the shear stresses acting. This is because these coefficients have to absorb all the errors in the mathematical model, to enable a reasonable comparison with the prototype to be made. Therefore when a set of equations for a highly stratified estuary are used for a partly mixed estuary the value of the interfacial friction coefficient used will not reflect the true level of mixing at the interface. However, this does not mean that the model should not be used for partly mixed estuaries because an alternative multi-layer stratified model could prove to be very expensive and still give no better results.

Computer tests were carried out for the salt wedge in the South Pass of the River Mississippi in order to compute the interfacial mixing coefficient for this estuary. The program used was the writers extended implicit finite difference scheme for fully stratified flows described in Section 4.4 . The data used is that given by Keulegan (79), with a bed slope of 2×10^{-5} (27) and a Chezy bed friction coefficient of $60 \text{ m}^{1/2}/\text{s}$. It was assumed that conditions were critical in the upper layer at the seaward end of the estuary and that the wedge at its landward limit assumed a wave front form at a densimetric Froude number of unity. These conditions gave an upper layer depth at the sea of 5.81m and a lower layer depth of 7.91m, plus a front height of 1.26m for the salt wedge. These depths

were used as boundary data for the model plus the river discharge of $2831.7 \text{ m}^3/\text{s}$ and the upper and lower layer densities of 1000 kg/m^3 and 1020 kg/m^3 respectively.

It is known that the length of the arrested saline wedge in the South Pass is 14 miles (22.5 km), and that the average discharge in the salt wedge should be zero. By running the program for different wedge lengths and interfacial mixing coefficients k'_m given in Eq.(6.25), the corresponding discharges in the wedge at the seaward end were obtained. Figure 6.23 shows the relationship between wedge discharge and wedge length for different values of the interfacial mixing coefficient. It can be seen that for zero wedge discharge and a wedge length of 22.5 km, $k'_m = 4 \times 10^{-5}$. For a moving salt wedge in a fully stratified estuary it is reasonable to conclude from the results that the interfacial mixing coefficient k'_m can vary from 5×10^{-5} on the flood flow to 3×10^{-5} on the ebb flow. The average value of $(V - V_c)/V$ where mixing is taking place is 0.3 and since $(1 - \epsilon) = 0.02$, then from Eqs.(6.25) and (6.32)

$$f = 4.0 k'_m (V - V_c) / [V (1 - \epsilon)] = 60 k'_m \quad (6.34)$$

Therefore the average value of the interfacial friction coefficient for the South Pass (Mississippi River) saline wedge is 2.4×10^{-3} . This compares very favourably with the analytically derived value of 2.1×10^{-3} obtained by Poggi from Eq.(5.9). For moving fully stratified saline wedges the value of f will vary from 3×10^{-3} on the flood

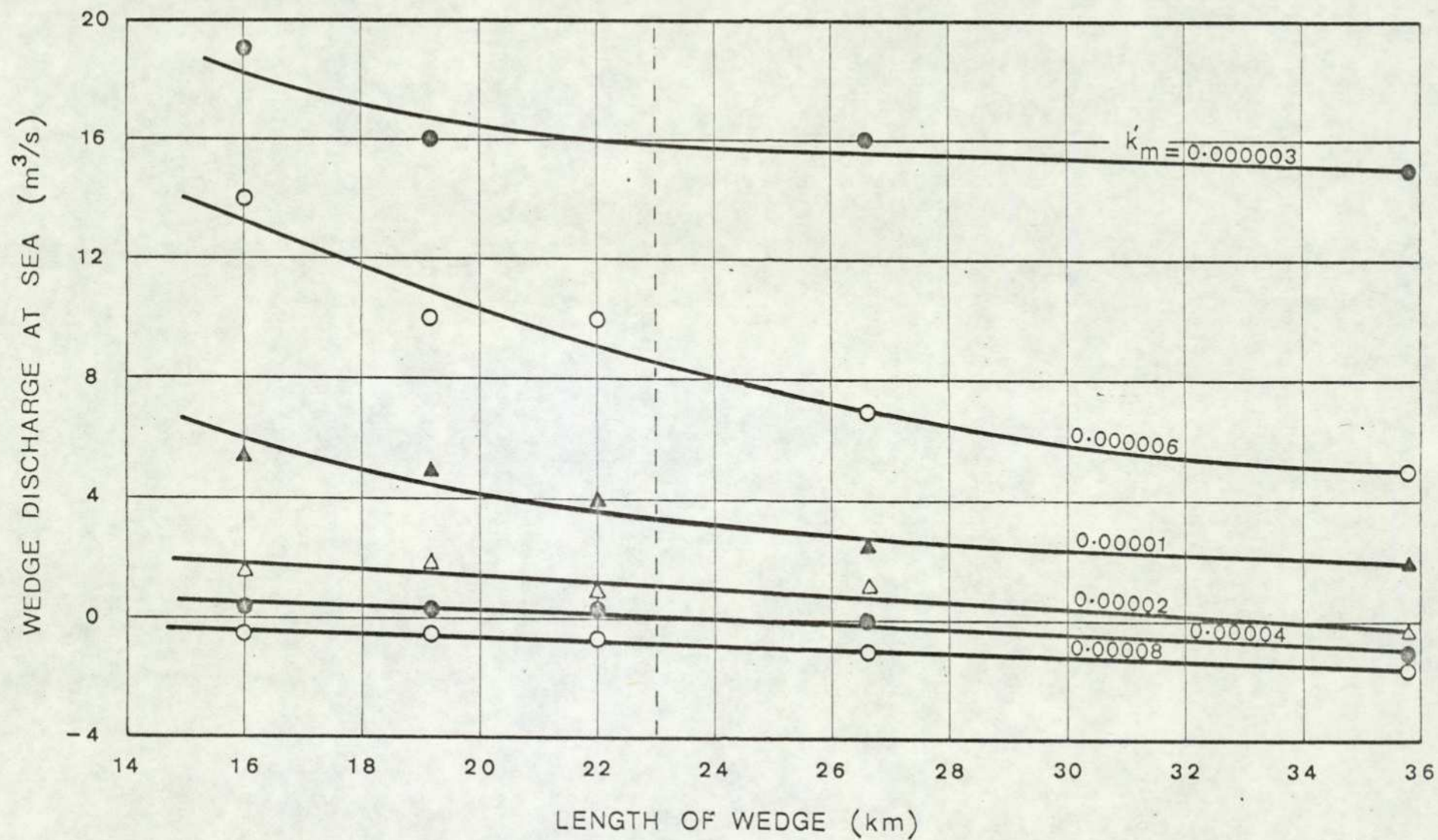


FIG. 6.23 WEDGE DISCHARGE AGAINST LENGTH FOR DIFFERENT INTERFACIAL MIXING COEFFICIENTS

flow to 2×10^{-3} on the ebb flow, whilst a good value for the interfacial friction coefficient during quasi-laminar flow conditions before mixing takes place is $f = 1 \times 10^{-3}$.

Other types of stratified flows give different values for the interfacial friction coefficient. Pedersen (103) indicated that for fjord flows where brackish water flows over a stagnant salt water pool, so that the entrainment is from the lower to the upper layer only, f is about 1×10^{-3} . Lofquist (89) studied the stability of the interface for a forced turbulent underflow beneath a stagnant upper layer and found that for densimetric Froude numbers between 0.32 and 0.71, f varied from 2.4×10^{-4} to 4×10^{-3} . For comparison the densimetric Froude number in the upper layer over most of the model salt wedge varied from 0.45 to 0.75. For lock exchange flows the results of the Delft Hydraulics Laboratory, Project M 772, 1968, reported by Abraham and Eysink (15) showed that for high Reynolds numbers at a densimetric Froude number of 0.8, f tends to a constant value of about 6×10^{-3} . These results suggest together with energy considerations that the interfacial friction coefficient is proportional to the square of the densimetric Froude number.

The results obtained indicate that in order to model mathematically a highly stratified estuary, the shear stress at the interface, when mixing is taking place can be represented in the momentum equation of the upper

layer by Eq.(6.12) as shown in (51). When mixing is not taking place Eq.(6.31) can be used with $f = 1 \times 10^{-3}$. The rates of mixing q_1 and q_2 are calculated from Eq.(6.25) using values of V_c obtained from Eqs.(6.18, 6.29 and 6.7) . The coefficient of mixing k'_m in Eq.(6.25) needs to be estimated depending on the level of stratification of the estuary. For a highly stratified estuary, k'_m will vary from 2×10^{-5} to 6×10^{-5} depending on the phase of the tide. If the average velocity in the salt wedge is not zero, as it will not be for a moving salt wedge, then V becomes the average relative velocity between layers. If an estuary of the partly mixed type is to be modelled, it will probably be found necessary in order to obtain a high interfacial shear stress without too much entrainment, to use two terms in each momentum equation. The first term will be Eq.(6.12) which will provide the necessary mass transfer together with its associated interfacial shear stress, plus a term such as Eq.(6.31). The friction coefficient f in Eq.(6.31) will need to be estimated by trial and error until a good correspondence between model and prototype is obtained.

CHAPTER 7

APPLICATIONS OF EXTENDED IMPLICIT MODEL

7.1 The River Hooghly

An estuary is classified highly stratified, partly mixed, or well-mixed by determining the ratio of mean upland discharge flowing into the estuary during a tidal cycle to its mean tidal prism. The estuary is highly stratified if the ratio is > 1 , partly mixed if the ratio is > 0.25 , and well-mixed if the ratio is < 0.1 . Applying this criterion to the Hooghly, computations show that this ratio varies between 0.07 and 0.064 at Saugar for an average tide. The estuary is thus well-mixed.

The Hooghly is formed by the confluence of three rivers: the Bhagirathi, the Jalengi and the Churni as shown in Fig.7.1 . The Bhagirathi and the Jalengi meet at Swarupganj, the river then being known as the Hooghly. The Churni meets the Hooghly about 68km below Swarupganj. Calcutta is situated about 140km below Swarupganj, on the left bank of the Hooghly. The estuary is funnel-shaped, meeting the Bay of Bengal at Saugar at a distance of about 140km below Calcutta.

The Bhagirathi, once the main branch of the Ganges, has silted up to such an extent in the course of time, that the flow from the Ganges to the Bhagirathi now takes place only when the level in the former rises by about 4.6m over the general dry season level. The offtake is

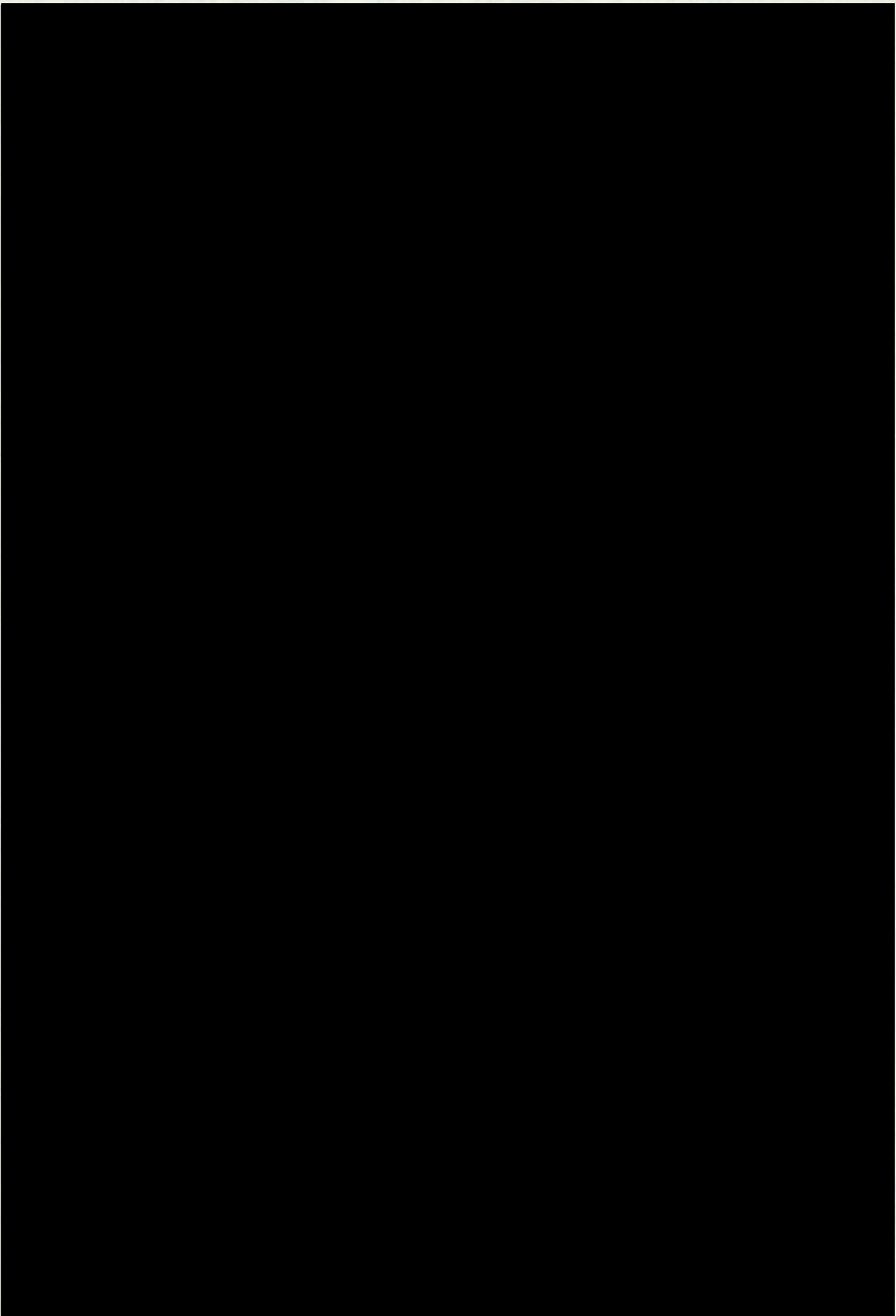


FIG. 7.1 HOOGHLY RIVER SYSTEM

unstable and has shifted by about 38km longitudinally and 16km laterally in the last few decades. However the Bhagirathi remains almost the only source of freshwater flow into the Hooghly, as the Jalengi flows are small and of short duration, while the Churni is practically a moribund channel. Owing to the nature of the offtake, there is almost no freshwater flow into the Hooghly in the dry season from November to June, the discharge rises steeply around mid-July to a magnitude of about $3000\text{m}^3/\text{s}$, falling suddenly to very low values at the end of the monsoon season around mid-October.

The tidal ranges at the mouth of the Hooghly exhibit large spring-neap variations (1.25 - 5.5m) and in the dry season the tidal limit extends to Swarupganj. The main component of the tide is semi-diurnal. During the monsoon the major part of the river is practically free from salinity, at Saugar the salinity drops to only 7 ppt, but during the dry season the salinity at Saugar reaches that of the ocean, about 34 ppt. The maximum dry season salinity at Palta, where the river is a source of drinking water supplies to the City of Calcutta, may reach 3 ppt. This is well above the potable limit. In fact, supply of drinking water to Calcutta beyond March is a serious problem, and therefore salinity variations in the river for different flow conditions is of great importance.

A program incorporating the generalized algorithm described in chapter 4.2 was coded in ALGOL 60 and run on an IBM 360/65 computer in order to compute water

level, discharge and density variations for the Hooghly. The program was run using the data shown in Table 7.1 obtained from Ref.(144). The friction coefficient was estimated by running the program with various roughness heights, until the known maximum velocities on the ebb and flood flows for zero upland discharge were obtained at Calcutta. It is known from studies by the author (50) and by Basu (22) that the roughness coefficient varies with the phase of the tide. This variation in friction coefficient with tidal phase is partly due to bed roughness, depth and density changes, but primarily due to the errors caused by using a normal flow friction parameter for unsteady flow conditions. In this model however, because of its research nature and therefore inadequate data input for a commercial study, only one roughness height was used for the river. Constant distance steps of 5km were used with time steps of 900s giving a Courant number of 2, although higher Courant numbers are possible (52). The longitudinal diffusion coefficient was taken as $200\text{m}^2/\text{s}$, since this is the value recommended by Haag and Bedford (56) for estuaries when only one constant value is to be used.

The computation grid used is shown in Fig.7.2 . At the left hand side of the grid the fresh water upland discharge and density at Swarupganj are used as boundary conditions at $j=0$ and $j=1$ respectively. At the right hand side of the grid a known vertical tide at Saugar Light House with a range of 5.5m is used at section jj for the dynamic boundary data. Since at the seaward end of the grid the

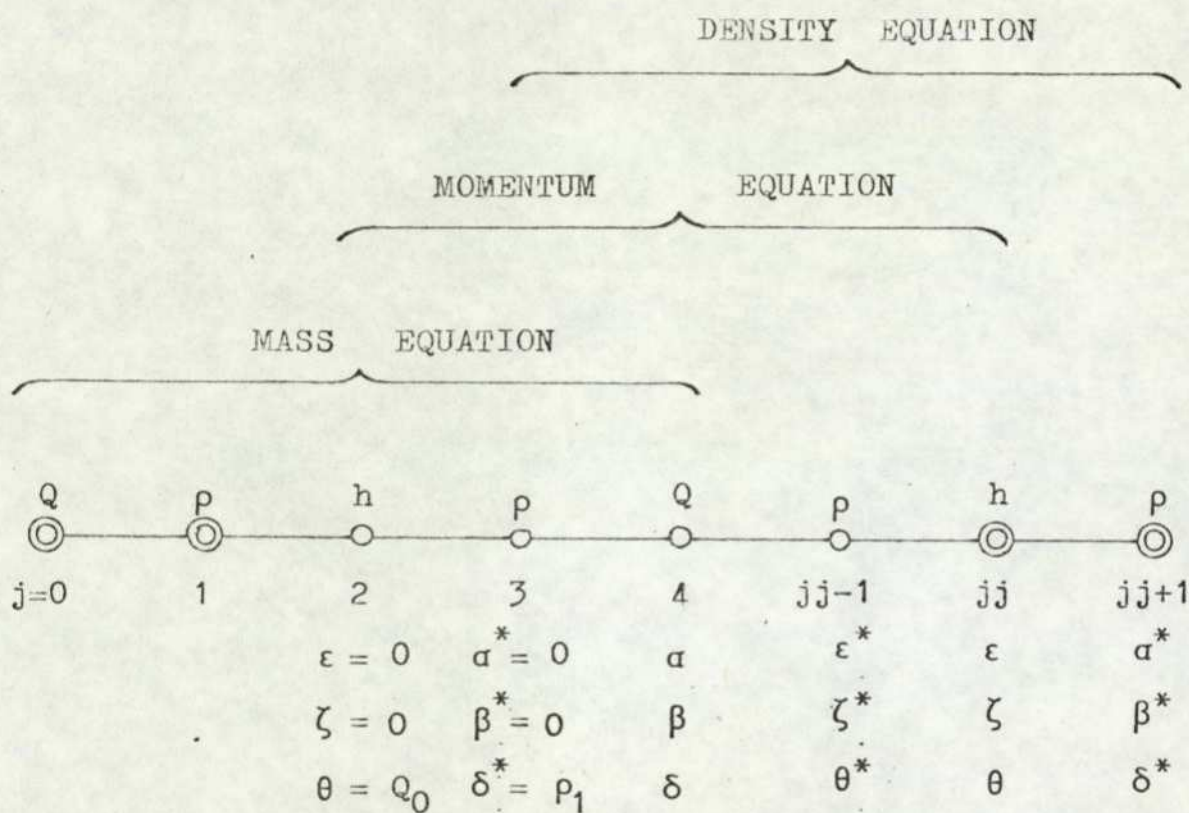


FIG. 7.2 COMPUTATION GRID

Distance from Saugar Light House, in kilometres	Schematized breadth of channel, in metres	Schematized bed level with zero datum at lowest water level, in metres
(1)	(2)	(3)
0	13500	-12.0
10	10800	-12.0
20	8400	-11.9
30	5900	-11.9
40	4260	-11.8
50	3230	-11.7
60	2440	-11.5
70	1920	-11.4
80	1340	-11.2
90	1100	-11.0
100	940	-10.7
110	790	-10.4
120	670	-10.1
130	570	-9.8
140	480	-9.4
150	420	-9.1
160	370	-8.7
170	320	-8.4
180	300	-8.0
190	280	-7.6
200	240	-7.2
210	210	-6.8
220	180	-6.4
230	150	-5.9
240	130	-5.5
250	120	-5.1
260	110	-4.7
270	110	-4.3
280	100	-3.9
290	100	-3.4

TABLE 7.1 RIVER HOOGHLY DATA

density is a function of the upland discharge it cannot be used directly as a boundary condition. However, the density gradient at the mouth of the Hooghly remains almost constant at 0.00016kg/m^4 and this is used in Eq.(4.36) to compute the density at section $jj+1$.

The simultaneous tidal curves obtained by running the program with low and high upland discharge are shown in Figs. 7.3 and 7.4, respectively, together with the observed prototype curves obtained from Ref.(144). The salinity variations in the Hooghly are shown in Fig. 7.5 for the low and high upland discharges and are compared with the prototype data given in Refs.(24) and (48).

Salinity variations in an estuary take much longer to settle down from guessed initial conditions than do the water levels and discharges. It was therefore found necessary to run the program over twelve tidal cycles in order to achieve a constant cyclic state for the salinity. The CPU time on the IBM 360/65 for calculating 150 prototype hours for the Hooghly at a Courant number of two is less than five minutes, and the storage requirement was 120kB.

The results indicate that although the computed simultaneous tidal curves agree closely with the measured curves, the computed longitudinal distribution of salinity at low flow does not. The reason for this is partly due to using a constant longitudinal diffusion coefficient for the whole estuary, but mainly due to the fact that the channel data plus the simultaneous tidal curves are for the Hooghly in 1900 whereas the salinity profiles are for the Hooghly

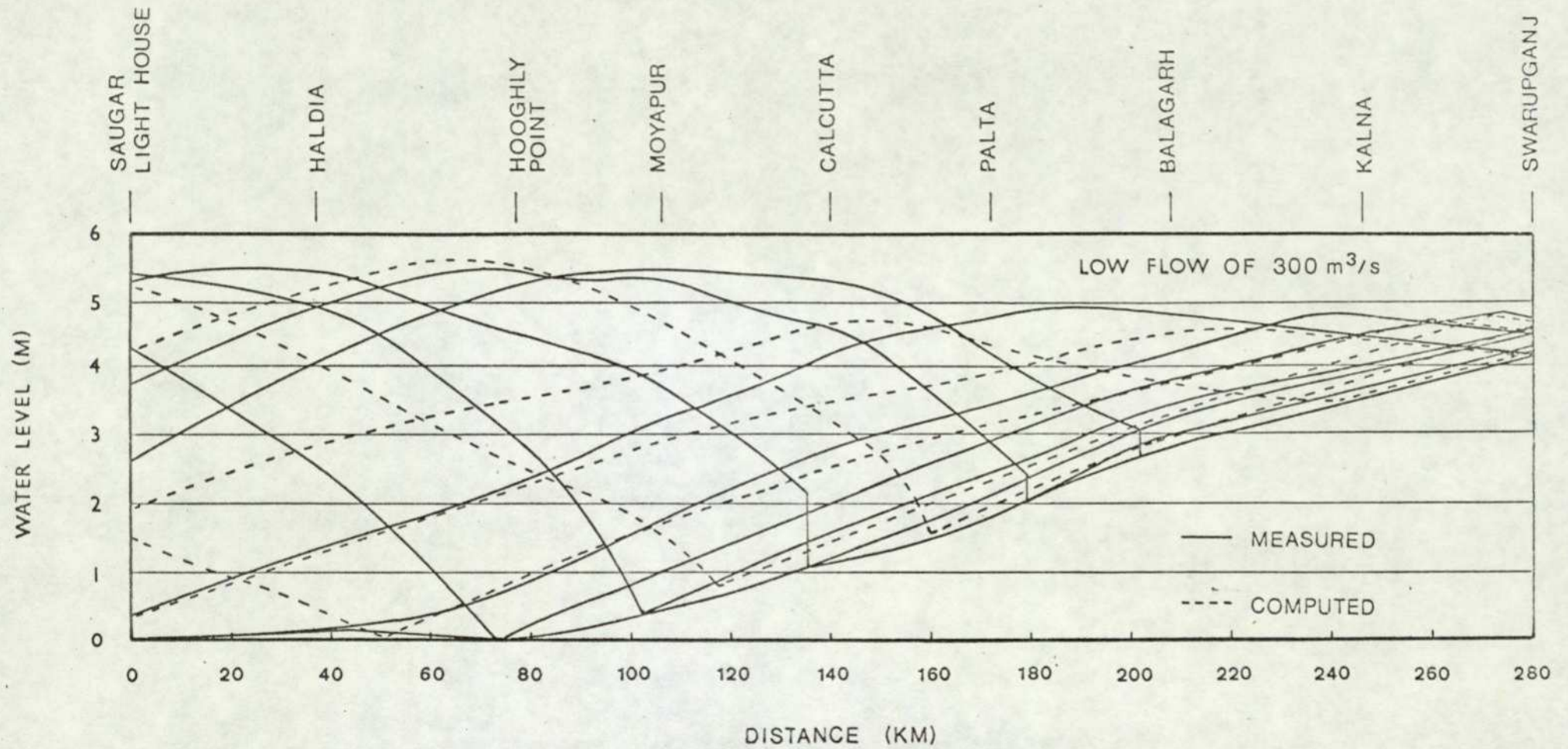


FIG. 7.3 DRY SEASONAL SIMULTANEOUS TIDAL CURVES (RIVER HOOGHLY)

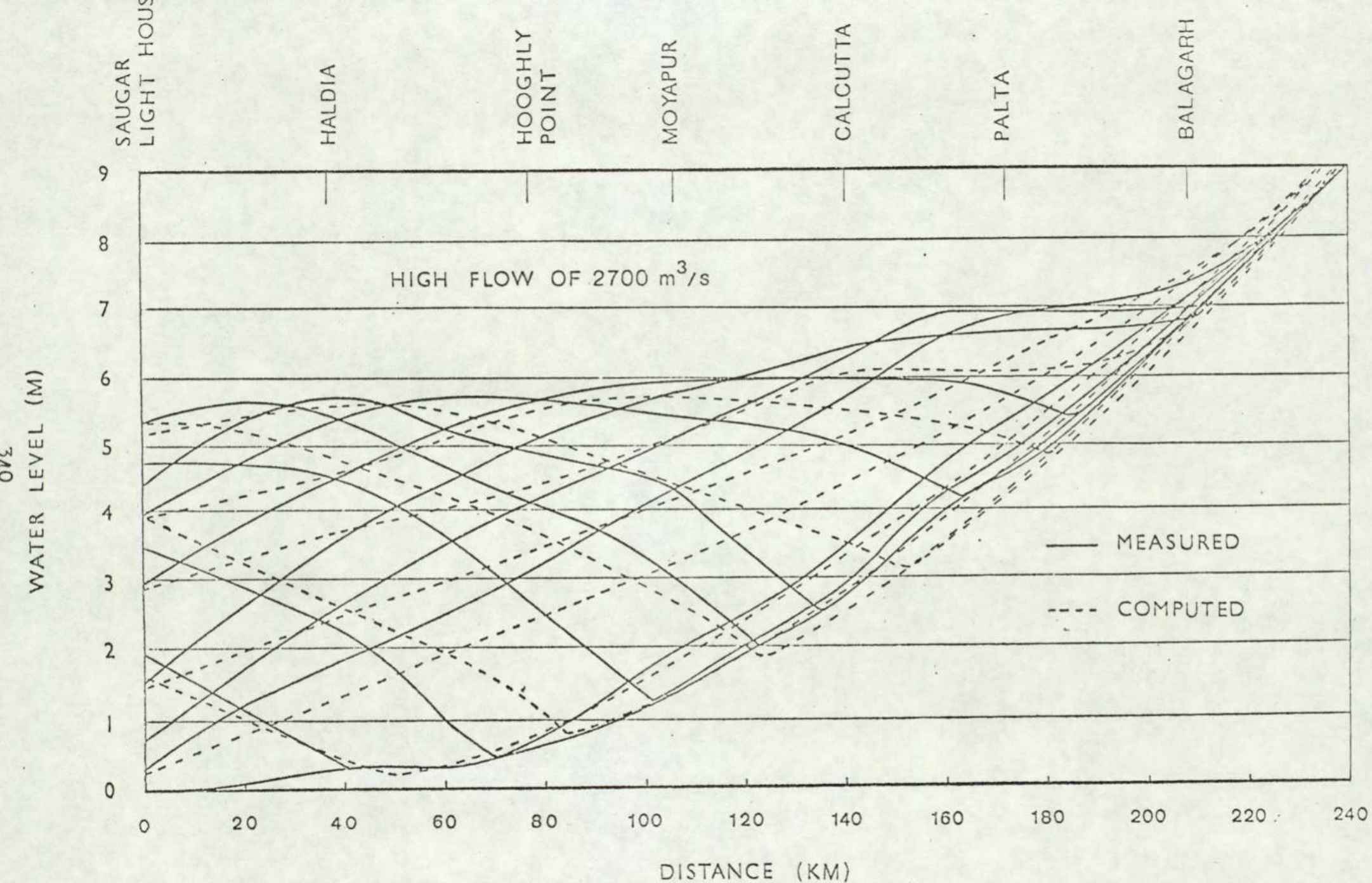


FIG. 7.4 MONSOON SEASONAL SIMULTANEOUS TIDAL CURVES (RIVER HOOGHLY)

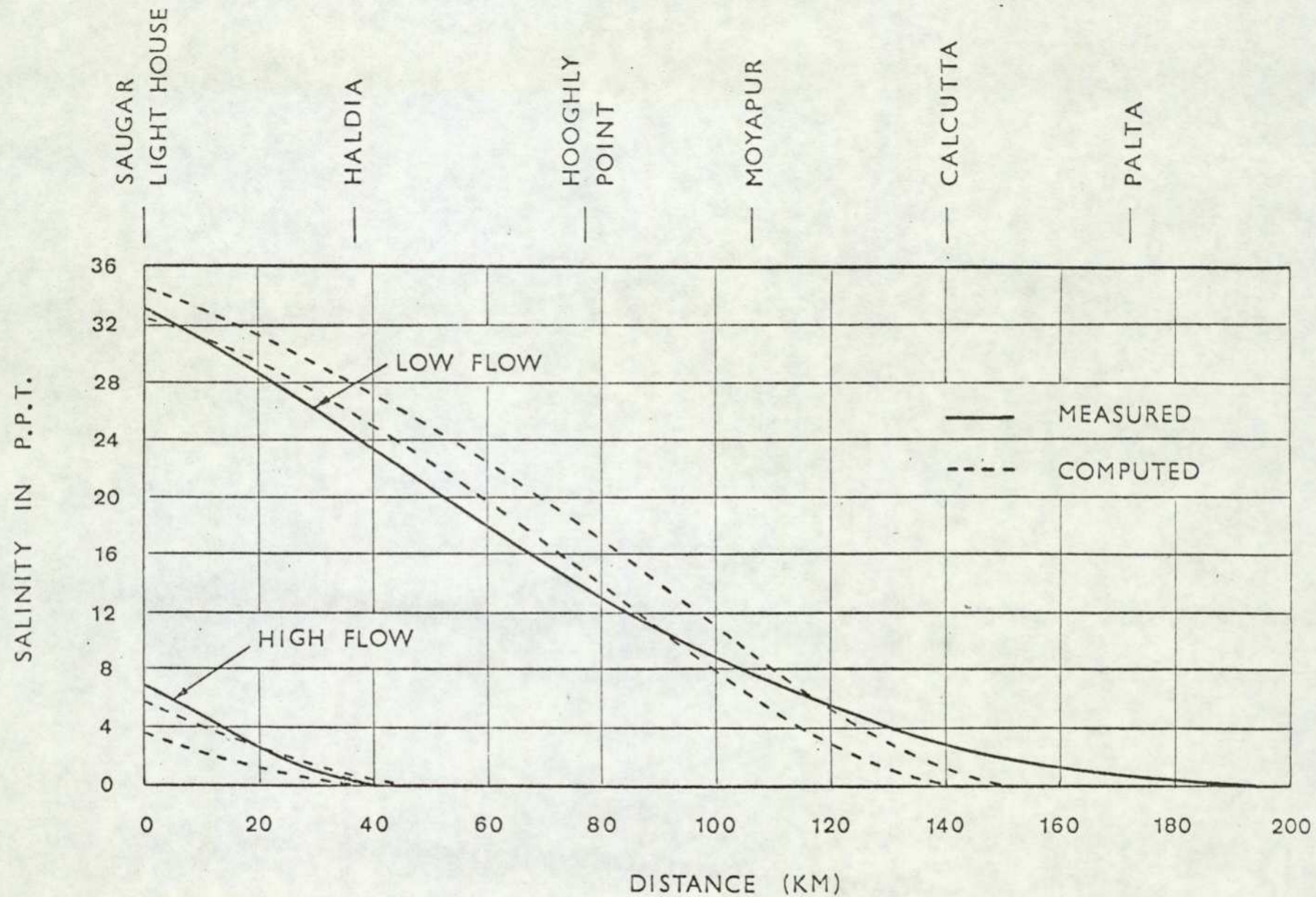


FIG. 7.5 LONGITUDINAL DISTRIBUTION OF SALINITY (RIVER HOOGHLY)

in 1963. It should be noted that there are no salinity measurements for the Hooghly around 1900 and that the only comprehensive published tidal data is that for the year 1900.

Tidal ranges in the Hooghly have increased steadily from 1911 to 1958 by half a metre at Calcutta but are now being controlled to a limited extent by dredging. This indicates that the channel has been and still is becoming wider and shallower and is in fact chocking up from the upstream end. Because of the increased tidal range and zero upland discharge, salinity intrusion is rapidly progressing landwards affecting the industrial centres located along the river.

During the dry season a tidal bore develops in the Hooghly on the spring tide and reaches a height of 1.5m at Calcutta. These tidal bores seriously hinder the day to day activities of people in the vicinity of Calcutta and reflect the deterioration of the river caused by too little upstream discharge entering the river system. This lack of upstream discharge has caused silting up of the river, resulting in higher tidal ranges and steeper upstream bed gradients all of which contribute to the formation of the tidal bore. Usually tidal bores in numerical computations cause stability problems, but the iteration of equation coefficients in the centred implicit scheme causes enough dissipation to accommodate the bore.

7.2 The River Gambia

This river flows through the West African country The Gambia from which the country takes its name. The Gambia is a long narrow country surrounded by Senegal and as shown in Fig.7.6 follows the length of the river over its entire 500km tidal reach. The tidal range at the sea varies between 0.6m and 1.5m and the estuary is classified as well-mixed. Because of the size of the estuary and its very small upland discharge over nine months of the year, tidal variations at the sea have a considerable effect on the movement of water within the estuary. The tidal limit extends beyond Gouloumbo, during the dry season, which is 520km from Banjul, whilst the maximum extent of saline intrusion is 250km from Banjul.

A mathematical model of the Gambia estuary was built by Sanmuganathan (119) using a very limited amount of prototype data which was subsequently used in this thesis. Sanmuganathan's model which was designed to predict salinity variations in the estuary is an analytical solution of the salt transport-diffusion equation using Laplace transformations.

The extended implicit model for well-mixed flows was similarly used to predict salinity variations in the Gambia, given a freshwater hydrograph, tidal conditions at the sea, and an initial salinity profile. The seaward boundary of the model was taken at Banjul and the landward boundary at the tidal limit 510km from Banjul. The variation in cross-sectional area of the estuary is shown in Fig.7.7 . The width of the estuary had to be estimated from a map and from this data it was decided to use a mean depth of water at Banjul

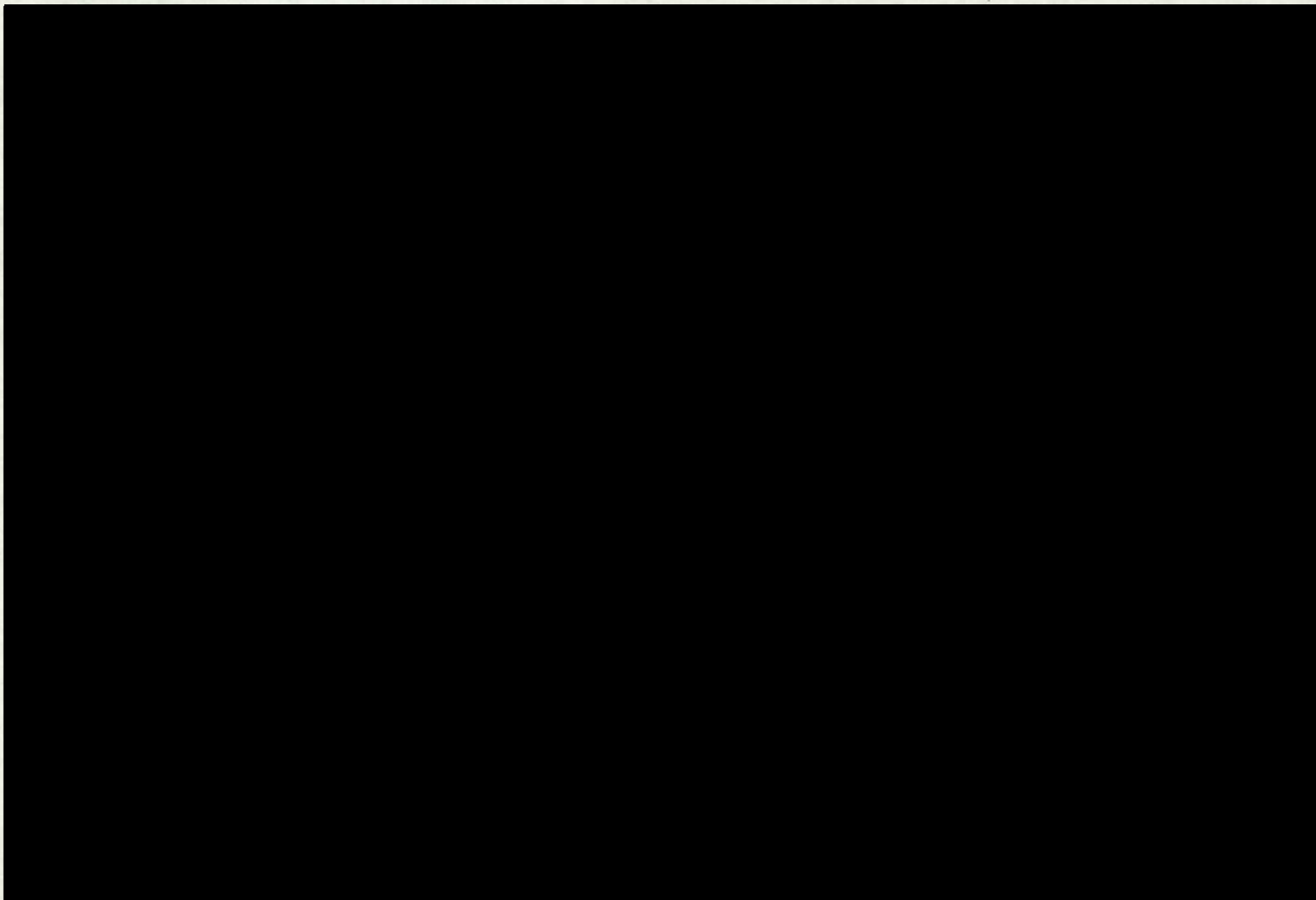


FIG. 7.6 THE GAMBIA ESTUARY

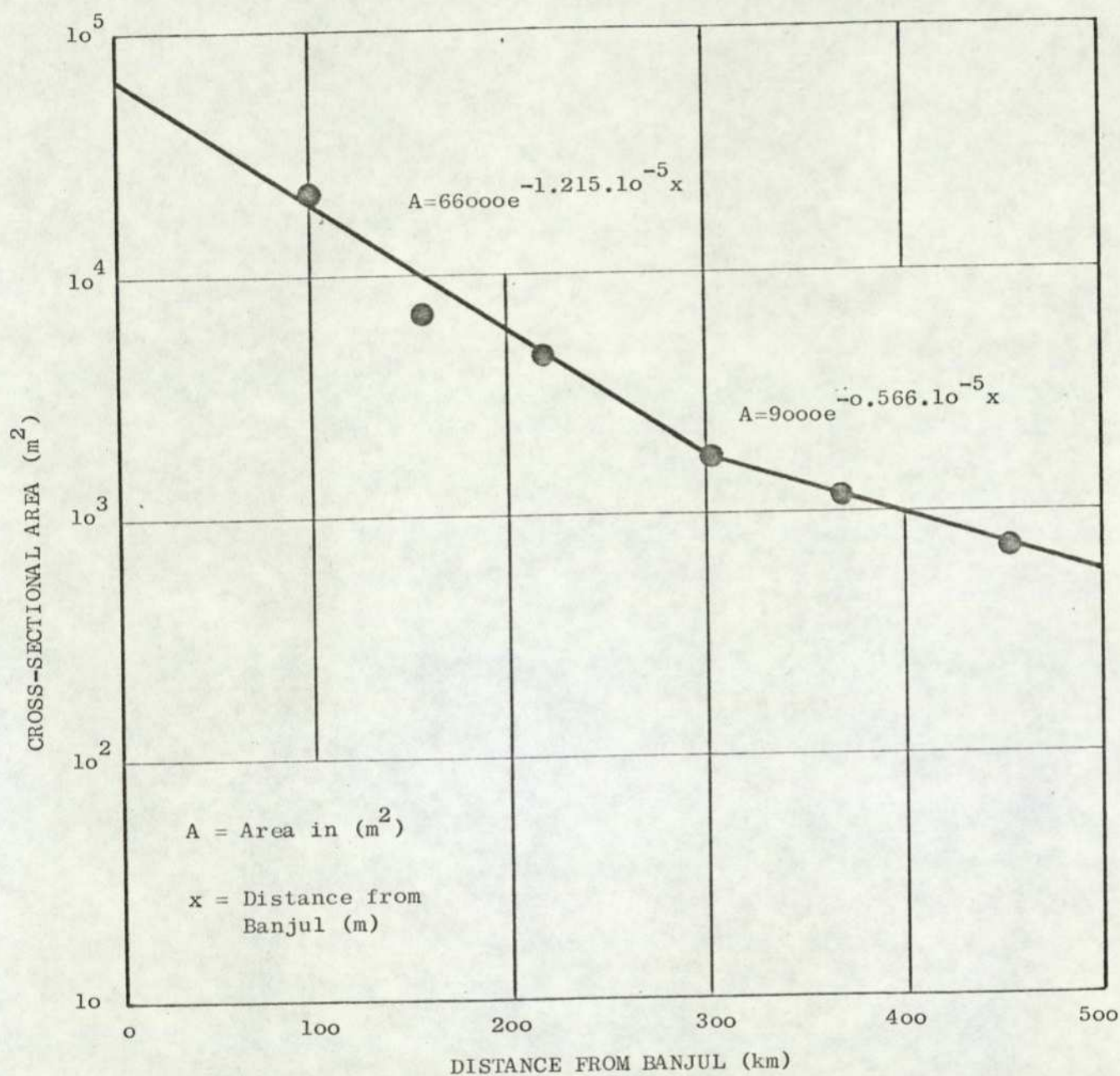


FIG. 7.7 VARIATION OF CROSS-SECTIONAL AREA FOR THE GAMBIA ESTUARY

of 12m . Since no information was available on bed levels, a logarithmic bed profile was assumed given by the equation:-
 Bed level = $0.01 e^{1.1 \times 10^{-5} L}$, where L is the distance from Banjul. The landward boundary of the model was the fresh water hydrograph shown in Fig.7.8, whilst at Banjul a constant sinusoidal vertical tide of 12.5hr period and 1m range was used.

Since known prototype values for the maximum tidal velocities along the Gambia were available, the Chézy bed friction coefficient was estimated by running the model with guessed values of Chézy coefficient and zero upland discharge until the maximum tidal velocities agreed with the prototype. It was found that a linear variation in Chézy coefficient decreasing from $100\text{m}^{1/2}/\text{s}$ at Banjul to $40\text{m}^{1/2}/\text{s}$ at 300km from Banjul and then remaining constant gave the best results which are shown in Fig.7.9 .

In order to estimate the longitudinal diffusion coefficient, several tests were made initially with the coefficient constant and the resulting salinity advance for salinities of 1.5g/l, 5.0g/l, and 10.0g/l recorded. The salinity advance curves for a constant diffusion coefficient of $200\text{m}^2/\text{s}$ and $500\text{m}^2/\text{s}$ are shown in Figs.7.10 and 7.11 respectively. These curves indicate that the diffusion coefficient should be between $200\text{--}500\text{m}^2/\text{s}$ at 150km from Banjul but that a value of $500\text{m}^2/\text{s}$ is the right order between 50-100km from Banjul. Since most estuaries have diffusion coefficients of about $1000\text{m}^2/\text{s}$ at the sea reducing to about $100\text{m}^2/\text{s}$ in their middle reaches, a diffusion coefficient varying with distance as shown in Fig.7.12 was

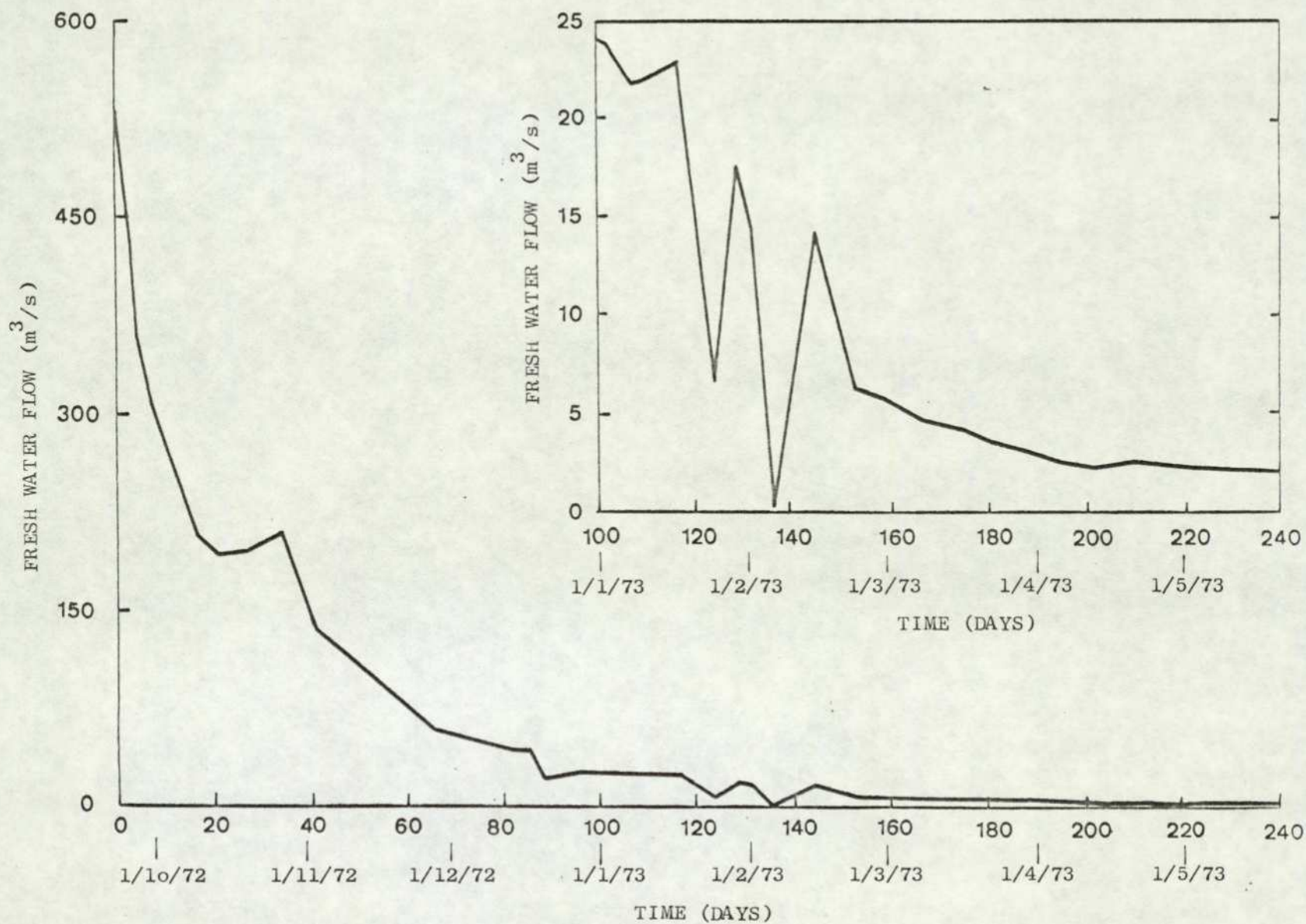


FIG. 7.8 HYDROGRAPH FOR THE RIVER GAMBIA

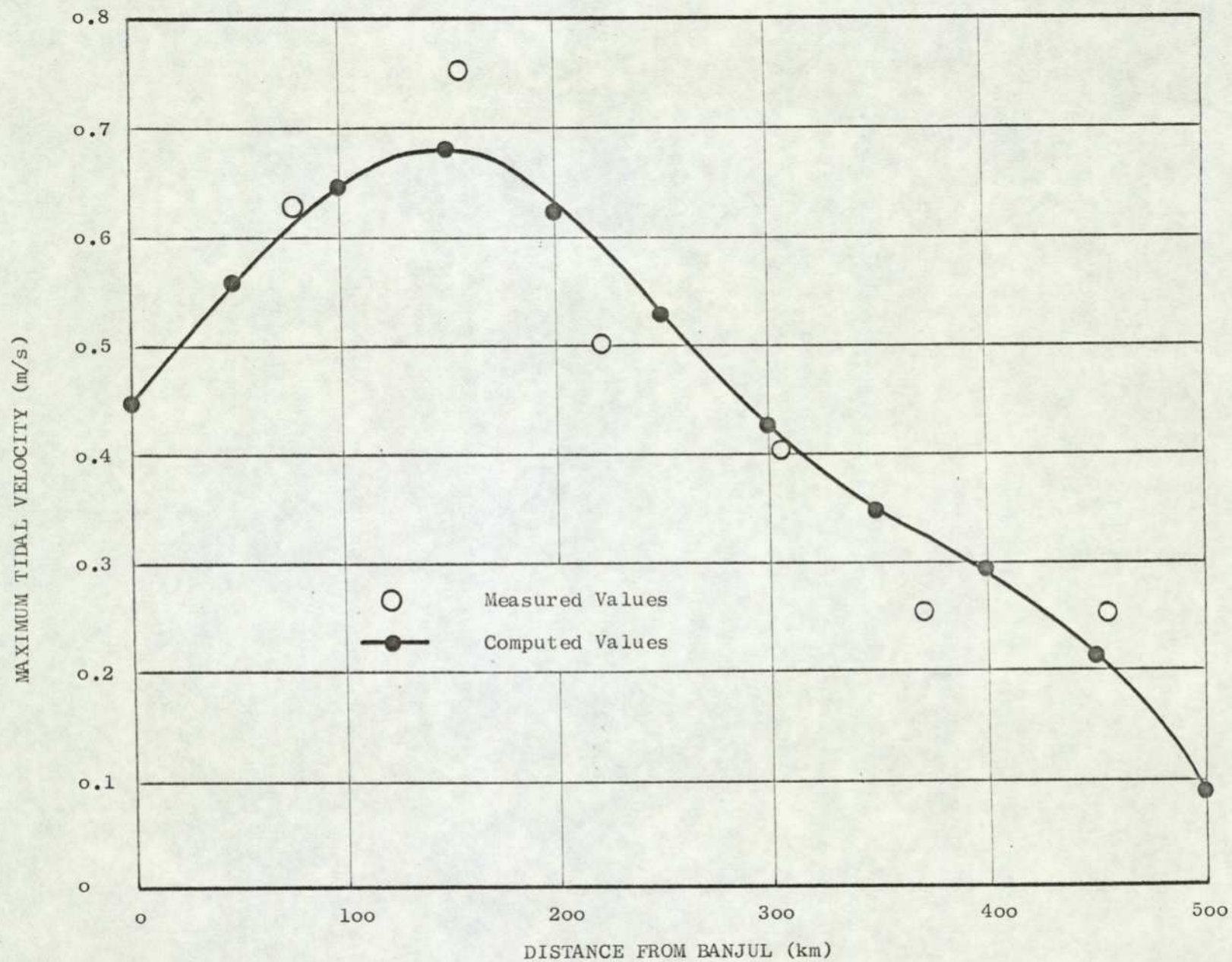


FIG. 7.9 VARIATION OF MAXIMUM TIDAL VELOCITY WITH DISTANCE FROM BANJUL

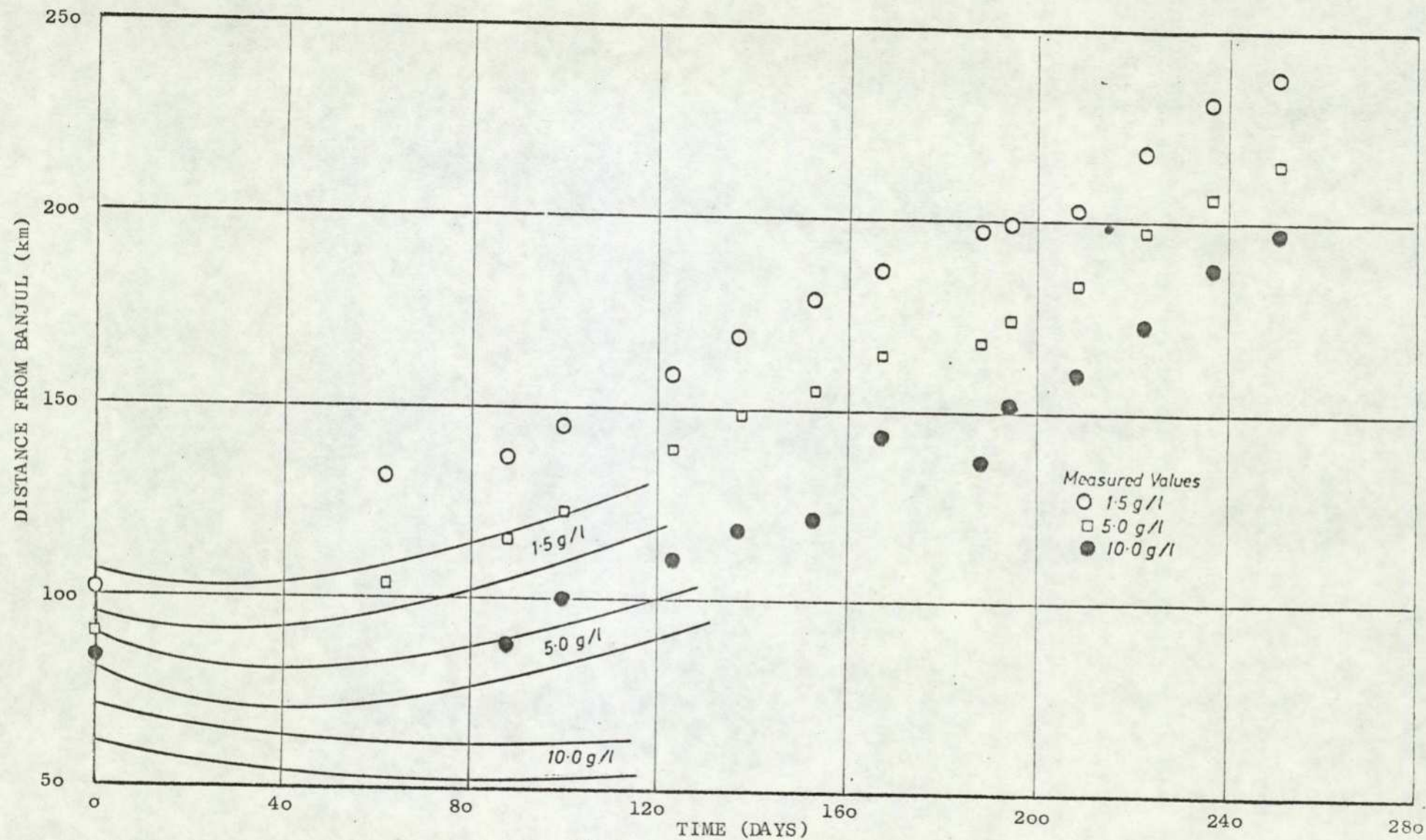


FIG. 7.10 COMPUTED SALINITY ADVANCE FOR A DIFFUSION COEFFICIENT OF $200 \text{ m}^2/\text{s}$ (GAMBIA ESTUARY)

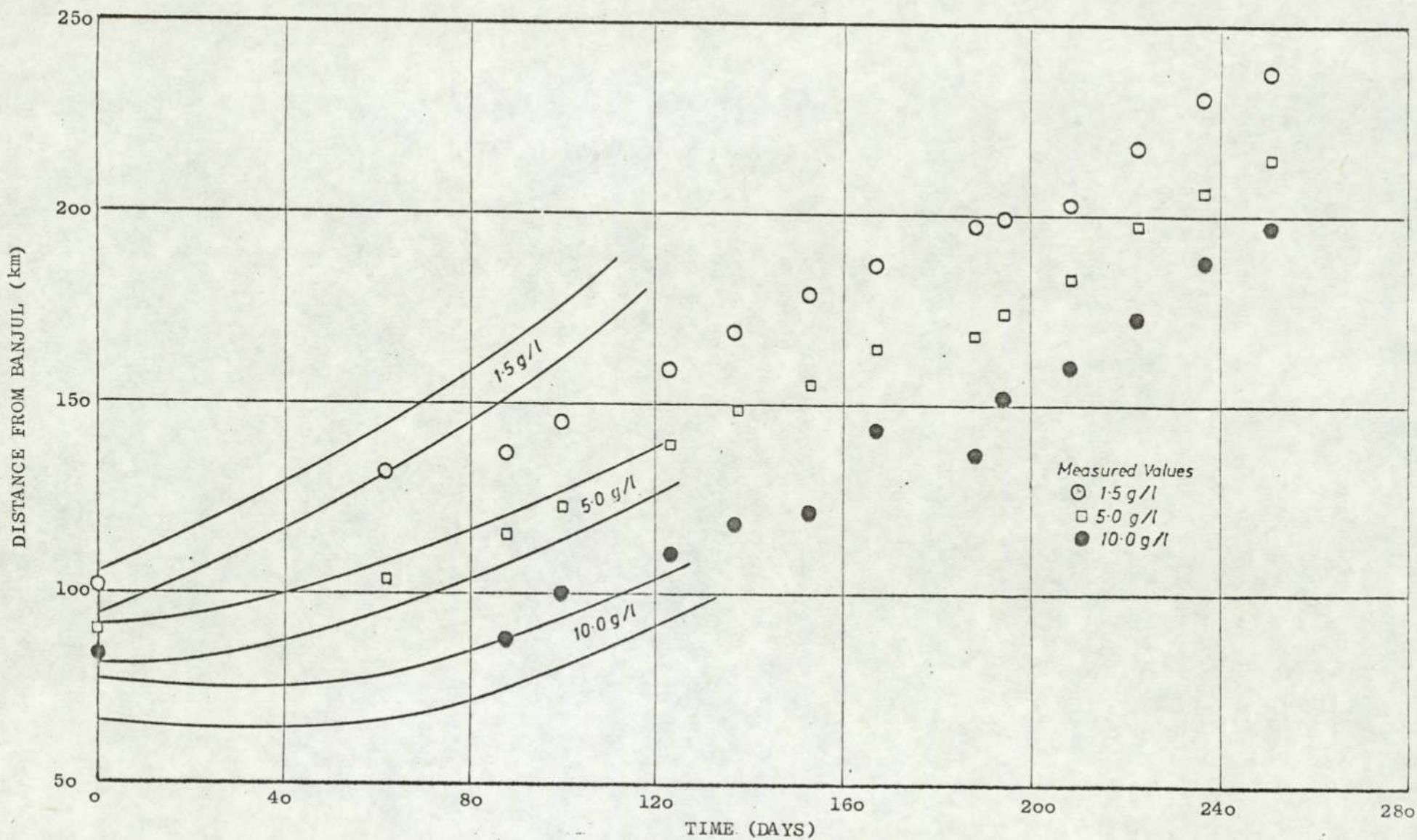


FIG. 7.11 COMPUTED SALINITY ADVANCE FOR A DIFFUSION COEFFICIENT OF $500 \text{ m}^2/\text{s}$ (GAMBIA ESTUARY)

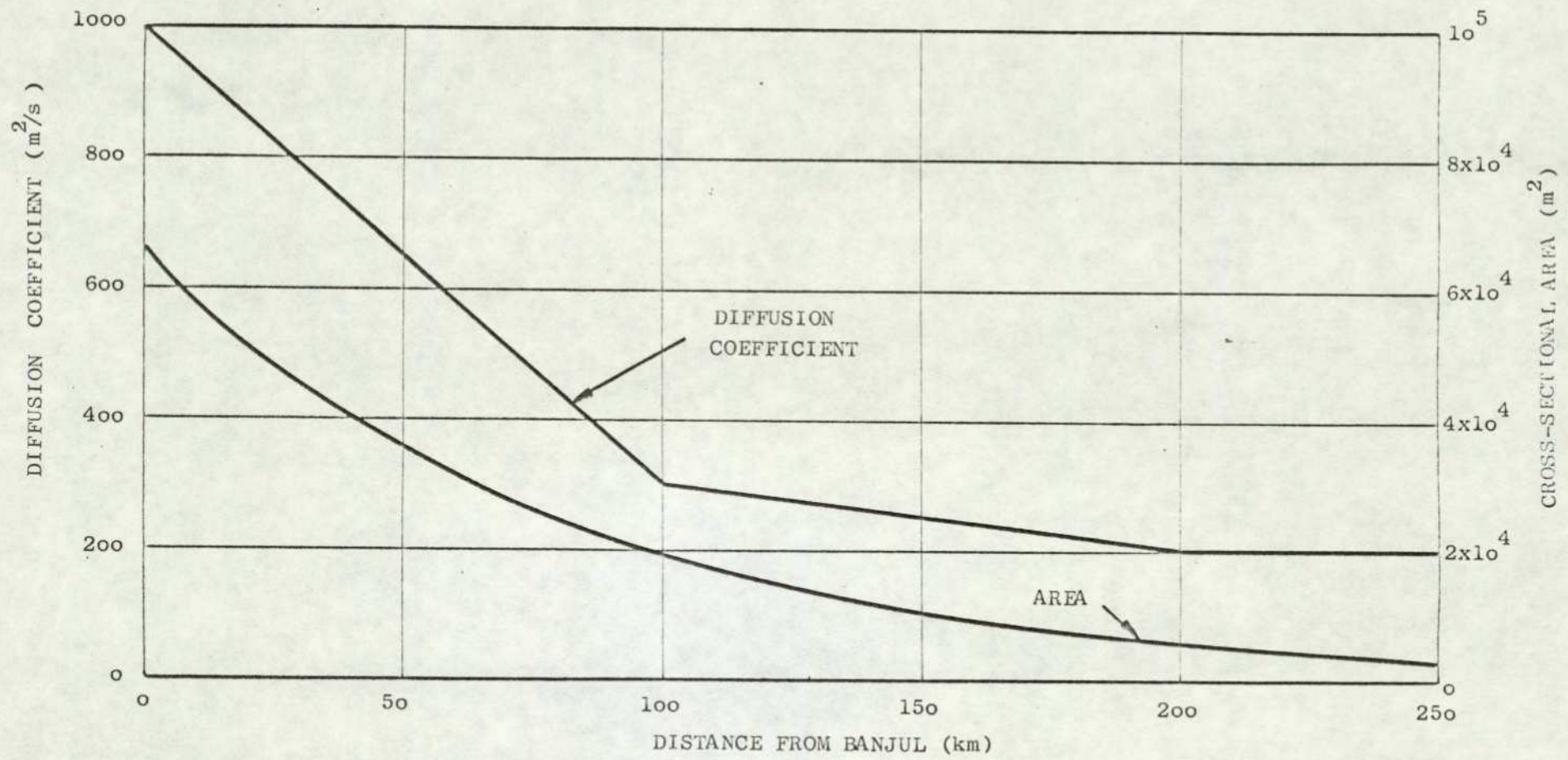


FIG. 7.12 VARIATION OF DIFFUSION COEFFICIENT AND CROSS-SECTIONAL AREA WITH DISTANCE FROM BANJUL

used. These values take no account of the considerable variations in upland discharge shown in Fig.7.8, therefore the final results for salinity advance will be very approximate.

It should be made quite clear at this point that the longitudinal diffusion coefficient is perhaps more correctly called a longitudinal dispersion coefficient. This is because its value includes not only a contribution from molecular and turbulent diffusion but also a multitude of dispersion effects. The two dominant influences on the longitudinal dispersion coefficient come from the lateral circulation caused by depth variations in the cross-section and the width to depth ratio of the estuary. Clearly the value of the longitudinal diffusion (dispersion) coefficient will vary both with distance and time depending on the state of the tide, the volume of fresh water entering the estuary, the cross-sectional shape of the bed and the width to depth ratio of the estuary.

The model was run for 250 prototype days with distance and time steps of 5km and 30minutes respectively, and took 100minutes of CPU time on an IBM 360/65 computer. Using the new generation of IBM 370 series computers this CPU time can be reduced considerably. A typical example of water surface profiles over a tidal cycle is shown in Fig.7.13 for day 62. Computed and observed longitudinal salinity profiles are shown in Fig.7.14, whilst Fig.7.15 shows how the salinity varies over a tidal cycle; in this case for day 62 at 100km from Banjul. It will be seen from

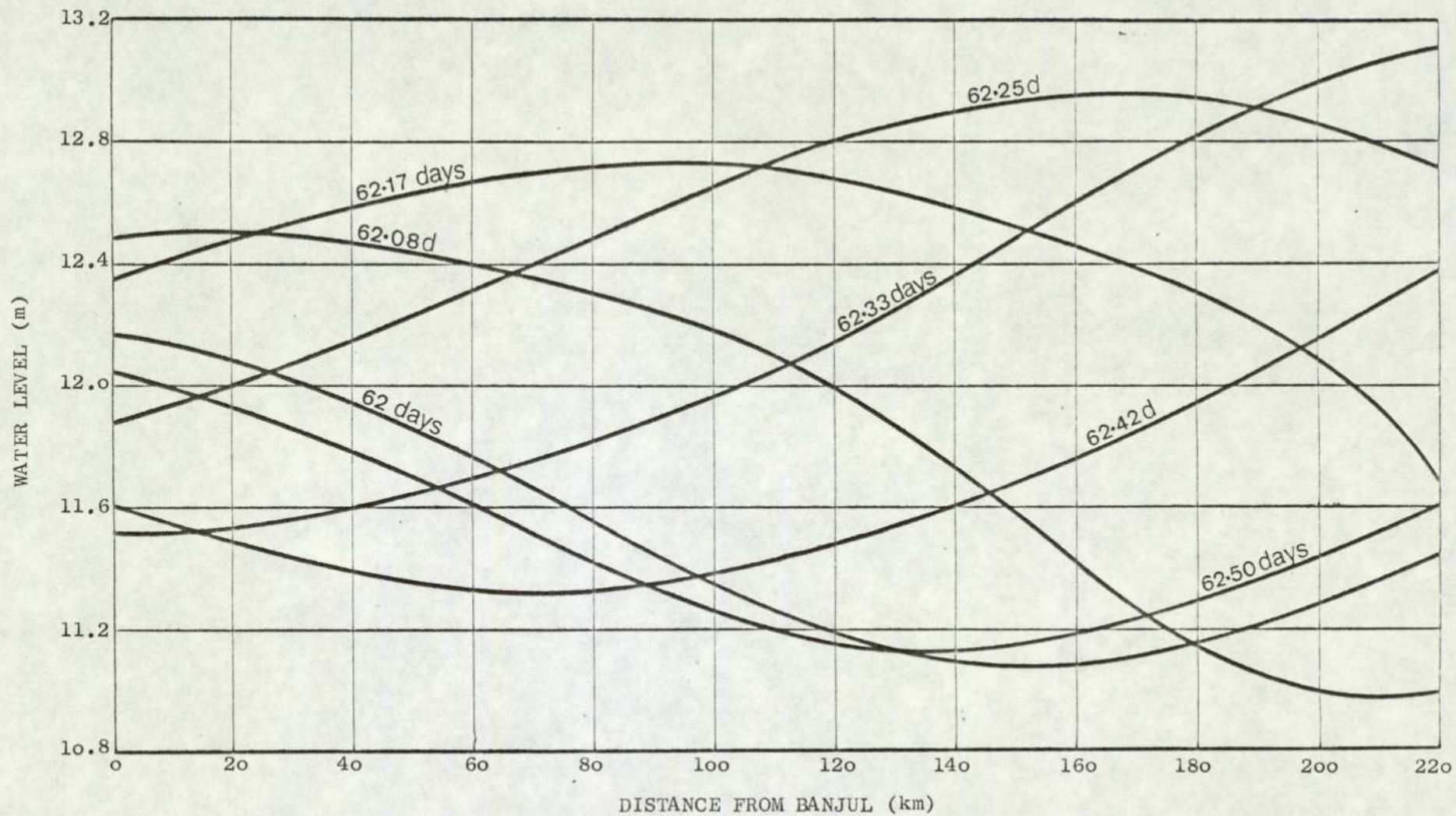


FIG. 7.13 SIMULTANEOUS TIDAL CURVES FOR THE GAMBIA

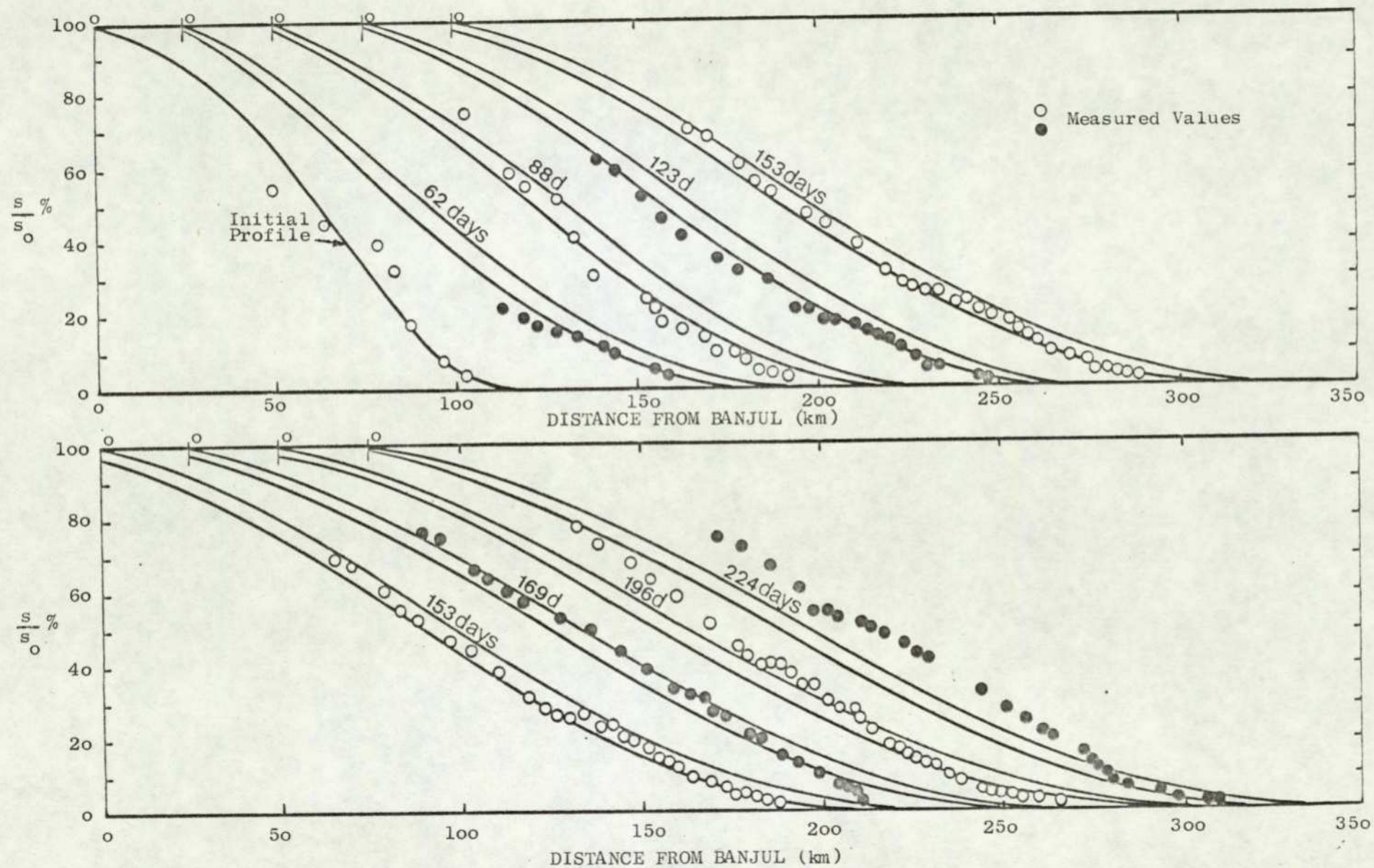


FIG. 7.14 COMPUTED AND OBSERVED LONGITUDINAL SALINITY PROFILES FOR THE GAMBIA ESTUARY

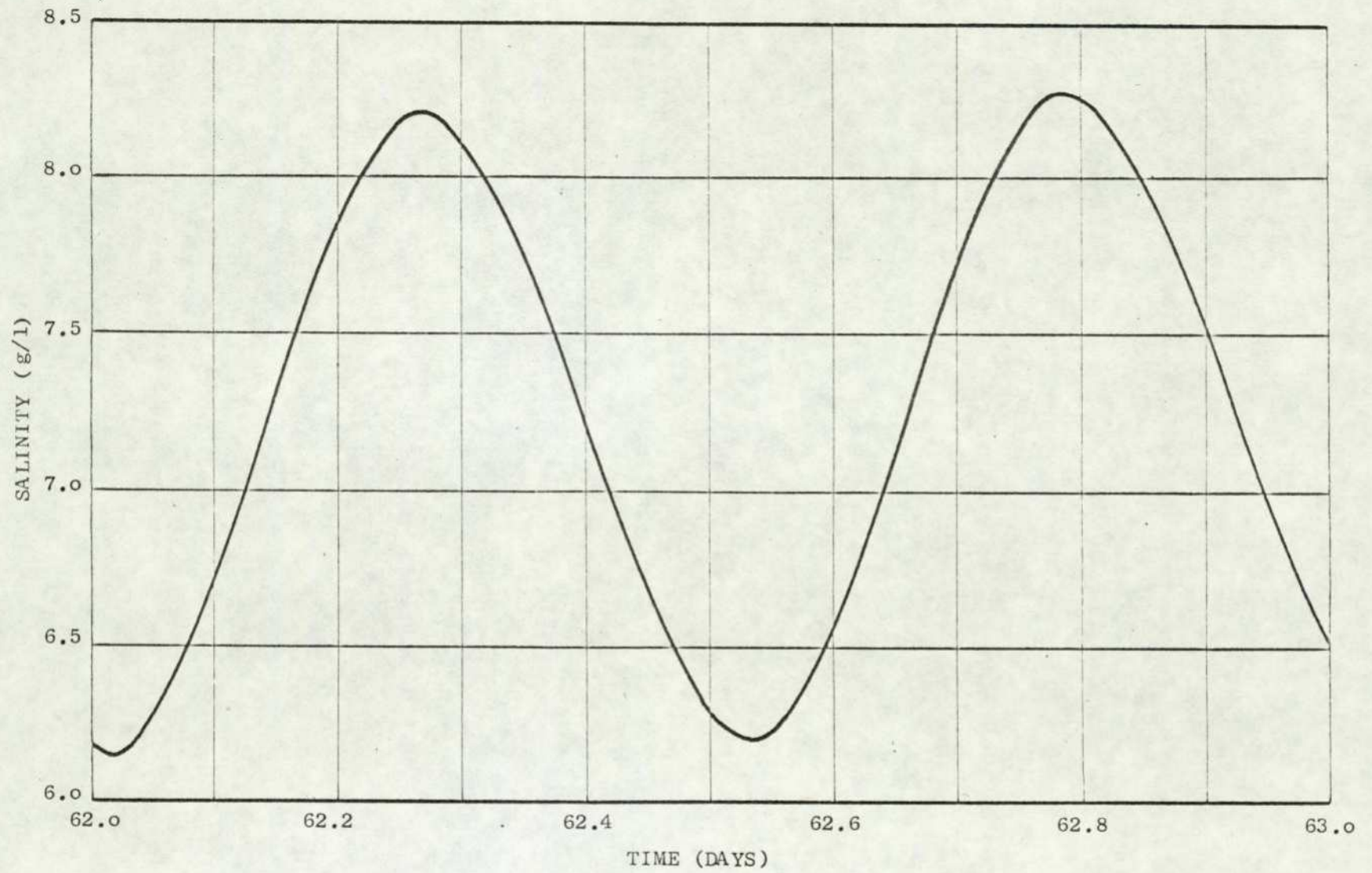


FIG. 7.15 TIDAL VARIATION OF SALINITY AT 100 km FROM BANJUL

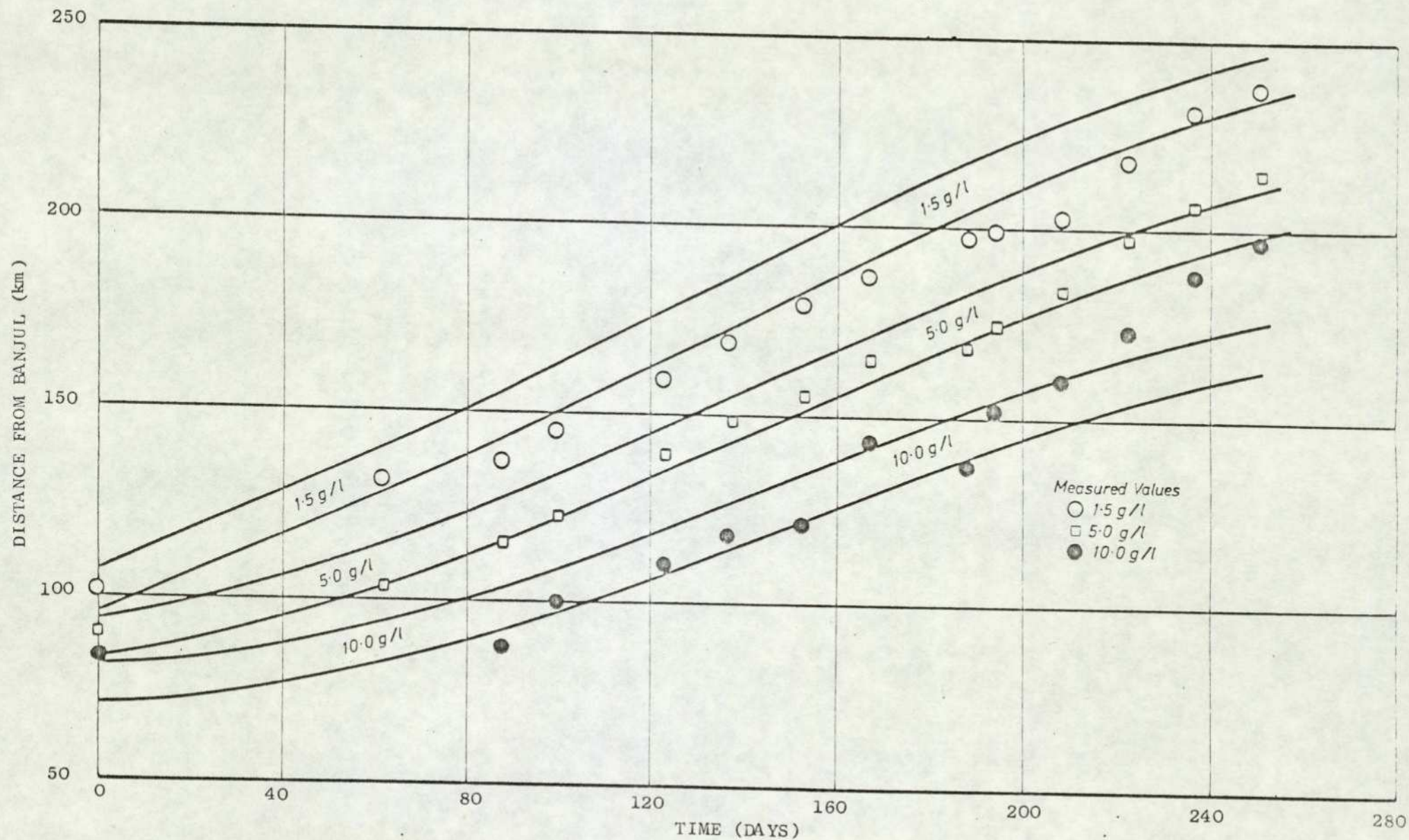


FIG. 7.16 COMPUTED AND OBSERVED SALINITY ADVANCE FOR THE GAMBIA ESTUARY

Fig.7.14 that starting with a given initial salinity profile the computed salinity profiles compare well with the measured values for 169 days, that is until the middle of March 1973, but then lag behind the observed salinities. It is difficult to see why this should happen unless the tidal range at the sea increased significantly at the spring equinox causing greater dispersion of salt in the estuary. The salinity advance curves for salinities of 1.5g/l, 5.0g/l and 10.0g/l are shown in Fig.7.16 together with the measured prototype values. It can be seen that the computed salinities match the measured values reasonably well for 190 days but then there is a rapid increase in the prototype dispersion rate, which reduces again after 240 days. As mentioned before it is most likely that this surge in dispersion of salt is due to increased maximum tidal velocities in the estuary caused by higher tides at the spring equinox.

7.3 The South Pass of the Mississippi River

The Mississippi forms a delta shown in Fig. 7.17 of the multilobate type with relatively large river branches, indicating that the action of the river predominates over that of the sea. Several of these branches have been artificially widened and deepened by dredging so as to improve navigation from the Gulf of Mexico to New Orleans 150km inland.

The South Pass is one of these waterways and is 14m deep, 450m wide and extends from the Gulf of Mexico to meet the confluence of the delta rivers about 30km inland. Because of the large fresh-water discharge into the South Pass ($2800\text{m}^3/\text{s}$) and low spring tidal range (0.6m), salt-water enters the Pass from the sea along the bed and intrudes landwards for about 23km. Since the tidal action at the sea is slight, the salt-wedge has a maximum tidal excursion of less than 2km. Also, because the waterway is deep and straight there is relatively little mixing between the salt-wedge and the fresh-water flowing above it, resulting in a highly stable wedge. These conditions produce a fully stratified two-layer flow which terminates landwards with the homogeneous fresh-water flow through a discontinuity called a saline front.

A salt-wedge estuary such as the Mississippi is a highly complex flow system which appears deceptively simple. Even when there is no tidal action so that the wedge is stationary there is still a circulation of salt-water within the wedge as shown in Fig. 1.1 . Also, because the circulation velocity causes the flow in the upper part of

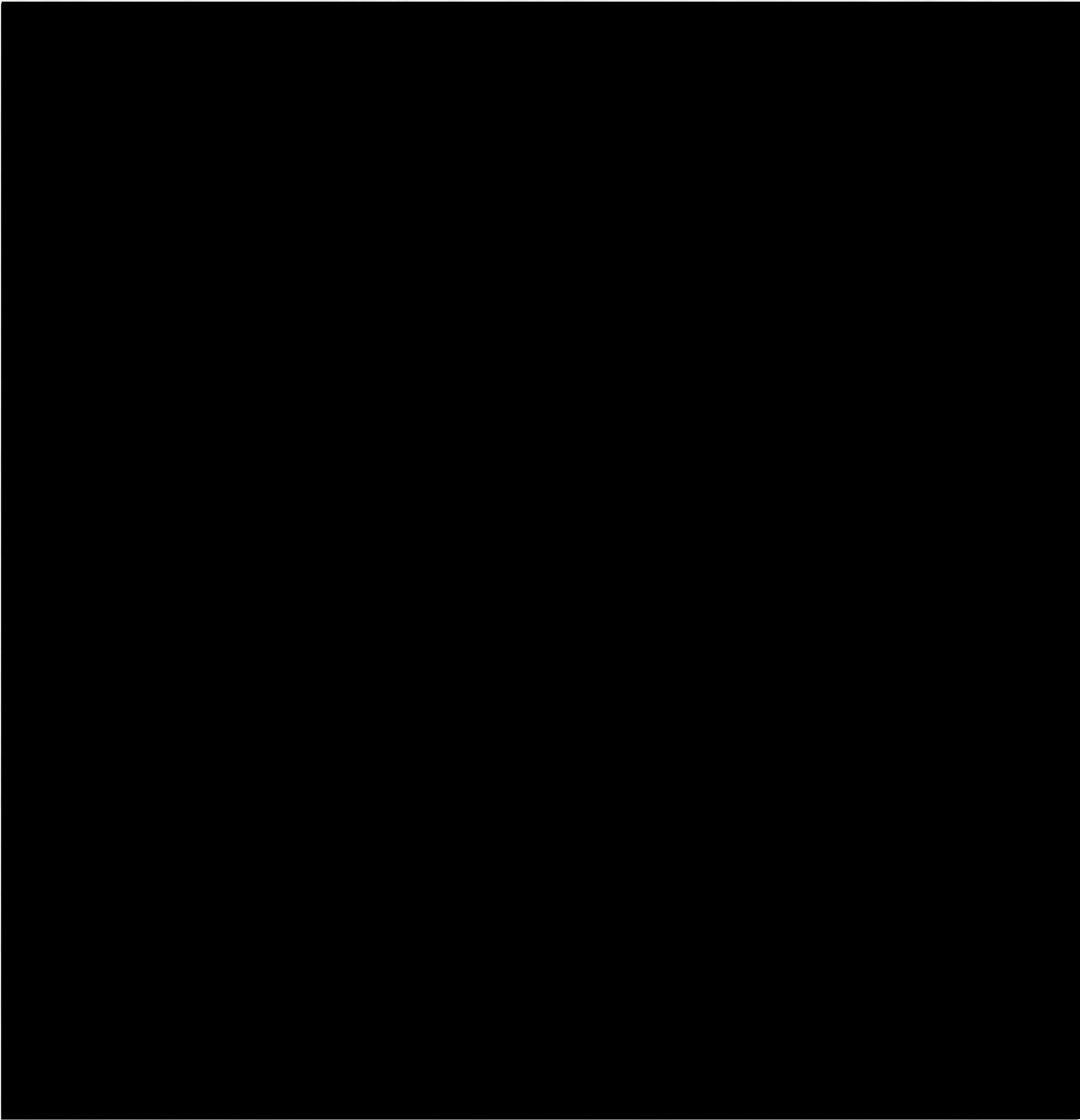


FIG. 7.17 MISSISSIPPI DELTA

the wedge to become turbulent, entrainment - diffusion mixing takes place which are two-way processes with equal quantities of water being transferred between layers. This mixing creates an interfacial mixing layer which has a thickness of about one fifth of the total depth. It is within this layer that most of the change in density from fresh to salt-water takes place. Also, after the interfacial mixing layer has been formed it acts as a barrier between the two fluids, regulating mixing between the layers. If this layer did not exist, mixing across the interface would increase continuously until the flow became well-mixed, since the rate of mixing is inversely proportional to the density difference between layers.

It is this wedge circulation which ensures that the salt front remains of high density. Without this circulation the front would become less and less dense due to entrainment mixing with the adjacent fresh water resulting in the rapid erosion of the front and consequently the wedge. Sea-water enters the wedge along the bed and flows towards the salt-front. Some of this sea-water convects itself vertically into the interior of the salt-wedge, replenishing the salinity which is constantly reduced due to entrainment-diffusion mixing across the interface. Salt-water on reaching the wedge-front is then swept back out to sea either in the upper half of the wedge due to the seaward velocities imparted into the wedge by the fresh-water flowing above it or in the upper layer after being entrained or diffused through the interface.

It is still popularly believed that in an arrested saline wedge salt-water enters the wedge from the sea to be progressively entrained through the interface into the upper layer, resulting in a zero discharge at the tip of the salt-wedge. However, as already explained this situation does not arise, and can be shown to be wrong by considering the solution of Eq.(5.9) for the length of an arrested saline wedge. In this calculation so long as the discharge in the wedge is taken everywhere to be zero, the length of the wedge is predicted very accurately when the known prototype value for the interfacial mixing coefficient is used in the equation. This analytical solution first derived by Schijf/Schönfeld (120) is based on the integration of the combined upper and lower layer dynamical differential equations. Results indicate that the average velocity in the salt-wedge at any position must be zero, implying that a circulatory flow is present in the wedge.

Under tidal conditions the salt-wedge moves landwards as water flows into the wedge on the flood flow and retreats seawards as water flows out of the wedge on the ebb flow. Because the relative velocity between layers increases on the flood flow and decreases on the ebb, more interfacial mixing takes place on the flood than on the ebb. Whilst this flooding and ebbing takes place in the lower layer the upper layer discharge remains seawards, although it varies depending on the phase of the tide. In a partially mixed estuary however, even the flow direction in the upper layer changes, and it is this that largely distinguishes it from a fully stratified estuary.

Any mathematical model capable of simulating the range of estuary flow types from fully stratified salt-wedge to well-mixed homogeneous flow needs to be based on a two-layer flow model. The well-mixed model is initiated when the lower layer thickness becomes zero, whilst partially-mixed flows can be approximated to saline wedges with considerable interfacial mixing. Clearly it is not feasible to model the interfacial mixing layer, and also not necessary. This is because good results are obtained from analytical solutions of the dynamical arrested saline wedge equations when average layer velocities are used.

It is necessary to build into a salt-wedge model realistic front conditions so that the saline wedge can move correctly up and down the estuary with the tide. Also, the amount of water exchanged between layers needs to be computed since this not only changes the densities in the layers but considerably alters the interfacial friction. In previous finite difference and characteristic models the exchange of water across the interface has always been neglected. Also, many modellers have either approximated the frontal depth to zero on the flood flows (146) or failed to understand that a wave-front cannot exist on the ebb flows (63). Clearly a knowledge of mathematics and computing is not enough to build realistic estuary models, that is why considerable effort has been directed towards investigating frontal conditions, interfacial mixing and instability in this thesis.

Using the extended implicit scheme for two-layer stratified flows described in chapter 4.4, the program shown

in appendix 1 was built to compute flow conditions in any salt-wedge estuary. Because a four-level scheme has been used, each layer can be computed separately without uncoupling the layers. The computational domain of the upper layer is therefore from the sea to the tidal limit whereas that for the lower layer is from the sea to the tip of the salt-wedge. If the salt-wedge length is reduced to zero the program should then calculate for a well-mixed estuary. It will be seen from the stratified equations for the upper layer in chapter 4.4 that when the lower layer thickness is reduced to zero these equations are identical to those in chapter 4.2 for a well-mixed flow. For economic reasons however, since the salt-wedge program has thirty times as many computational steps as the well-mixed program, it should not be used for well-mixed estuaries.

The complexity and size of the salt-wedge program is due not only to the complicated difference schemes but also to the special procedures required to scan and monitor the flow and the movement of the salt-wedge. Initially the four-level scheme was programmed to compute the flow conditions in an arrested saline wedge. This program was tested on the salt-wedge in the South Pass of the Mississippi for various interfacial mixing coefficients. Because this wedge was stationary the landward boundary could be taken at the wedge front, enabling the entire computation to embrace the two-layer fluid only. To extend this stable stratified program to one that could compute across the salt-wedge front into the homogeneous flow beyond, and also monitor the movement and changing frontal conditions without

the program going unstable took considerably longer. In fact the time required to work out the mathematics of the multi-level difference scheme plus the writing and testing of the stratified flow program was less than that required to extend the program so that it could calculate the tidal excursions of the wedge.

The salt-wedge program shown in appendix 1 consists of procedures each of which performs a specific task in the computation. Each procedure can be readily isolated from the program enabling improvements or modifications to be made without interfering with the remainder of the program. Since the program is designed to be used for any type of estuarine flow, many of the procedures are concerned with the organisation of the computation. A brief description of the role of each procedure is given below.

INITIAL: This contains the initial guessed conditions plus data concerning the channel such as bed widths and bed levels.

BOUND: Time dependent boundary conditions are given here such as the upland discharge and vertical tide at the sea.

BSTART: Because the position and type of boundary relative to the grid needs to be specified, integer numbers representing these data are read in with this procedure.

ITER: Since the equation coefficients in this type of implicit scheme need to be computed with centred dependent variables, this procedure centres these values before the computation is entered for the second time.

RIVSEC: This procedure calculates flow cross-sectional areas, hydraulic radii, mathematical widths and parameters required by the interfacial mixing equations.

EIGEN: This procedure calculates the number of positive and negative eigen values in each layer, at every grid point.

ABCL1, ABCU2, ABCL2: These three procedures compute coefficients for the lower layer at the second time level, the upper layer at the third time level and the lower layer at the fourth time level respectively.

FRONTCO: This procedure assigns starting recurrence coefficients to the grid points in the vicinity of the salt-wedge front.

PRECO: Depending on the number of positive and negative characteristics (eigen values) at a grid point, this procedure directs the computation into the correct procedure for computing the recurrence coefficients. The calculation is started at the left hand side of the grid using the given boundary data there and ends at the right hand side of the grid.

COLRSB, CORLSB, COLRSP, CORLSP: Using the above 'ABC...' coefficients these four procedures compute $\alpha, \beta, \gamma \dots$ recurrence coefficients in each layer depending on the flow direction and whether it is subcritical or supercritical. For example, if the flow is subcritical from left to right then procedure COLRSB is called.

SEADENSITY: Since the density at the sea is a function of the tidal discharges, it cannot be used directly as a boundary condition. However, this procedure computes the density at the sea in both the upper and lower layers from given density gradients depending on whether the flow is subcritical or supercritical.

CALQRH: With the recurrence coefficients known, the dependent variables namely discharge, level and density are computed alternately starting with the given boundary data at the right hand side of the grid, back to the left hand side.

CALFRONT: This procedure computes either the depth or discharge at the salt-wedge front depending on whether the front is adjacent to a level or flow grid section.

SHIFT: Since the salt-wedge front is a discontinuity or boundary in the flow, it cannot move by itself. Therefore at every time step the front must be shifted either landwards or seawards, depending on the computed frontal velocity. This procedure monitors the position of the front and also calculates the new grid variables as the front crosses over from one grid point to the next.

CONTROL: Because the flow does not usually change its eigen values with every time step, it would be uneconomical to use the eigen value procedure more than is necessary. Therefore the procedure CONTROL is used which scans the grid at each time step and detects using a simple control function (Eq. 4.66) whether any of the eigen values have changed sign. Only if a change is detected does the computation enter the EIGEN procedure.

OUTL1, OUTU2, OUTL2: These three procedures output all the computed values for the lower layer at time level two, the upper layer at time level three and the lower layer at time level four respectively.

Before the salt-wedge system just described could be applied to the Mississippi river it was necessary to compute the upstream fresh water discharge at 34km from

the sea, using a well-mixed model. The reason for this is that the constant upland discharge of $2800\text{m}^3/\text{s}$ in the South Pass for an arrested saline wedge is not constant at 34km from the sea when there is a tide. Also, it is far too expensive and unnecessary to run the salt-wedge program for say 150km of estuary when the salt-wedge only occupies the first 24km. The well-mixed model used a boundary 150km from the sea for the constant fresh water discharge of $2800\text{m}^3/\text{s}$, and a sinusoidal vertical tide at the sea of 0.5m range. Also, the downstream bed level was taken along the top of the salt-wedge using a Chezy coefficient of $200\text{m}^{1/2}/\text{s}$, whilst upstream of the wedge a bed friction coefficient of $50\text{m}^{1/2}/\text{s}$ was used. In order to ensure that any phase errors in the well-mixed model were the same as those in the salt-wedge model, the same distance and time steps were used in both models. The computed fresh water discharge at 34km from the sea is shown in Fig. 7.18, together with the corresponding vertical tide at the sea.

The Mississippi model was run for five tidal cycles with distance and time steps of 500m and 120s respectively. The CPU time required on an IBM 360/65 was 60 minutes, and the storage requirement was 840kB. An interfacial mixing coefficient $k_m' = 0.00004$ was used together with a quasi-laminar interfacial friction coefficient $f = 0.0012$; whilst longitudinal diffusion coefficients of $1000\text{m}^2/\text{s}$ were used in each layer.

Critical flow conditions were assumed to occur at the seaward end of the wedge giving variations in the

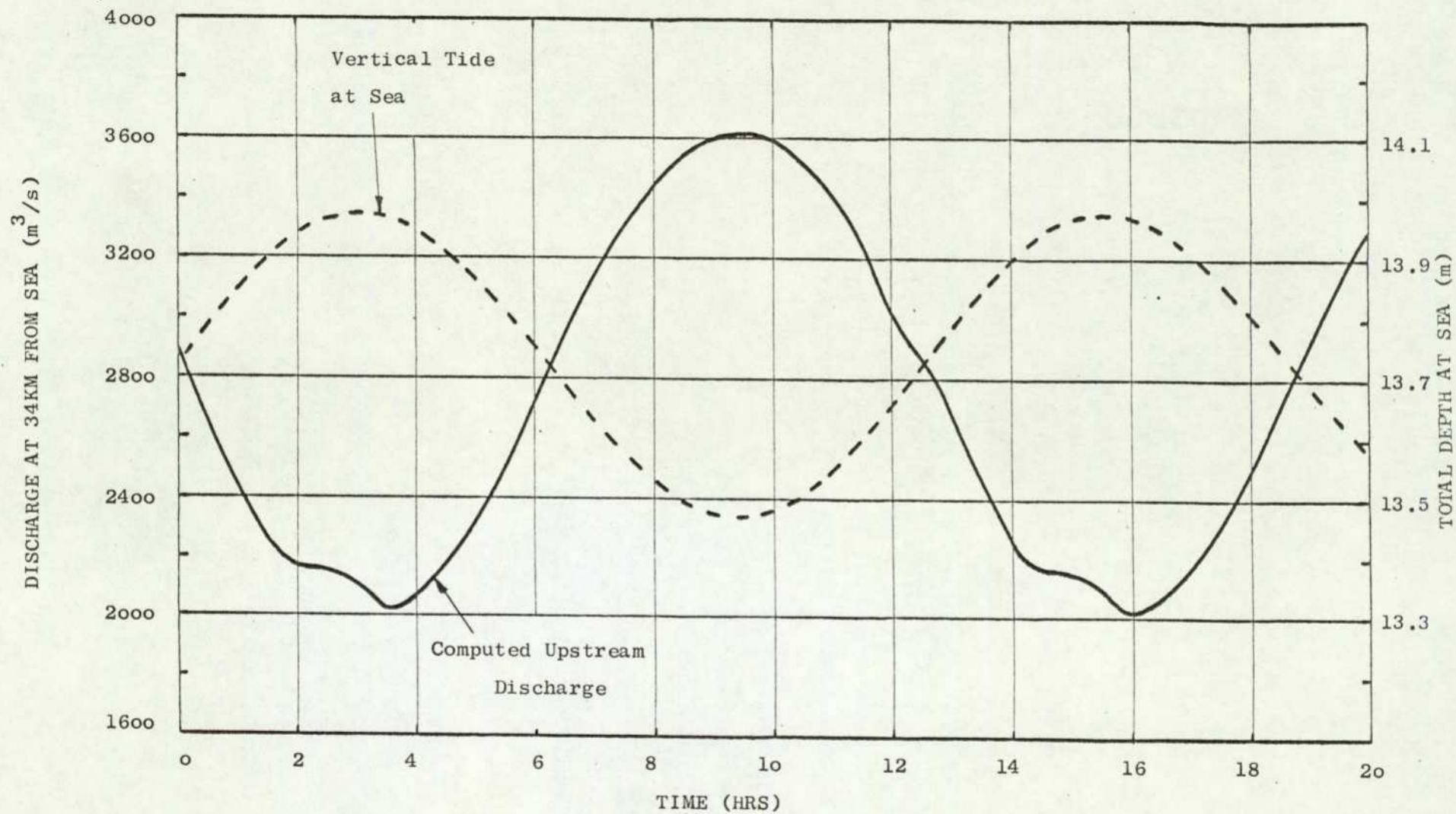


FIG. 7.18 BOUNDARY CONDITIONS IN MISSISSIPPI MODEL

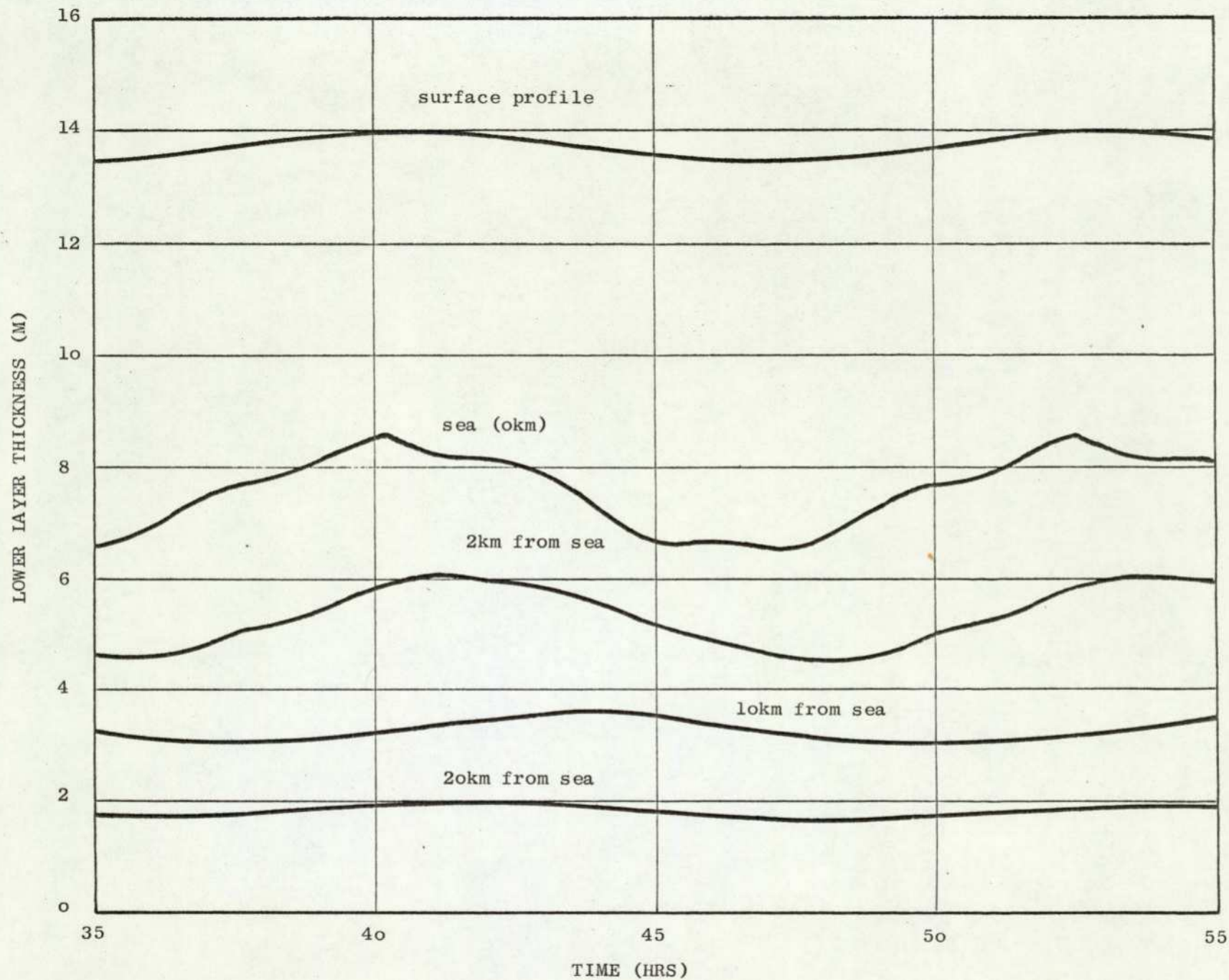


FIG. 7.19 COMPUTED SALT WEDGE THICKNESS FOR THE RIVER MISSISSIPPI

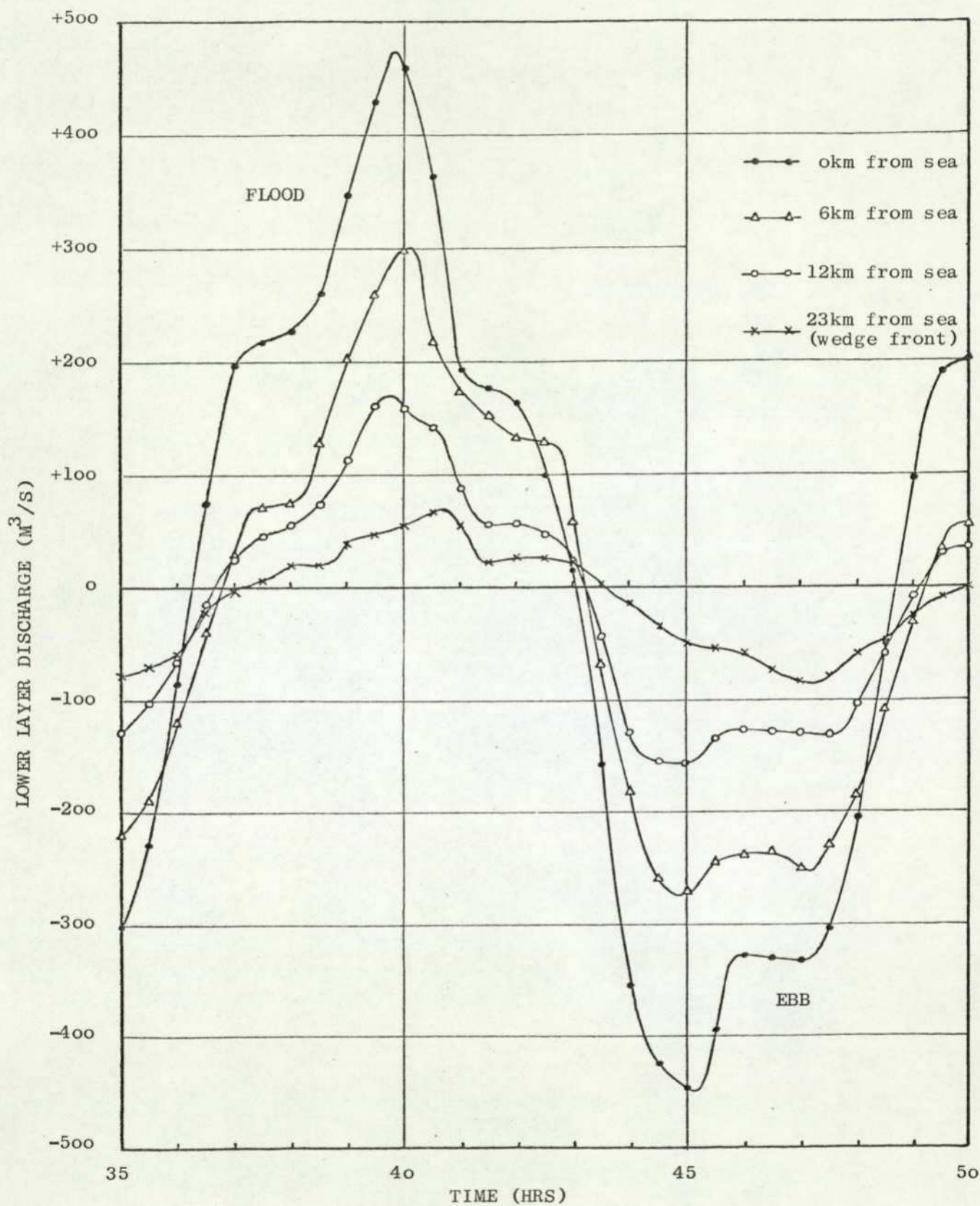


FIG. 7.20 COMPUTED DISCHARGE VARIATIONS IN THE MISSISSIPPI SALT WEDGE

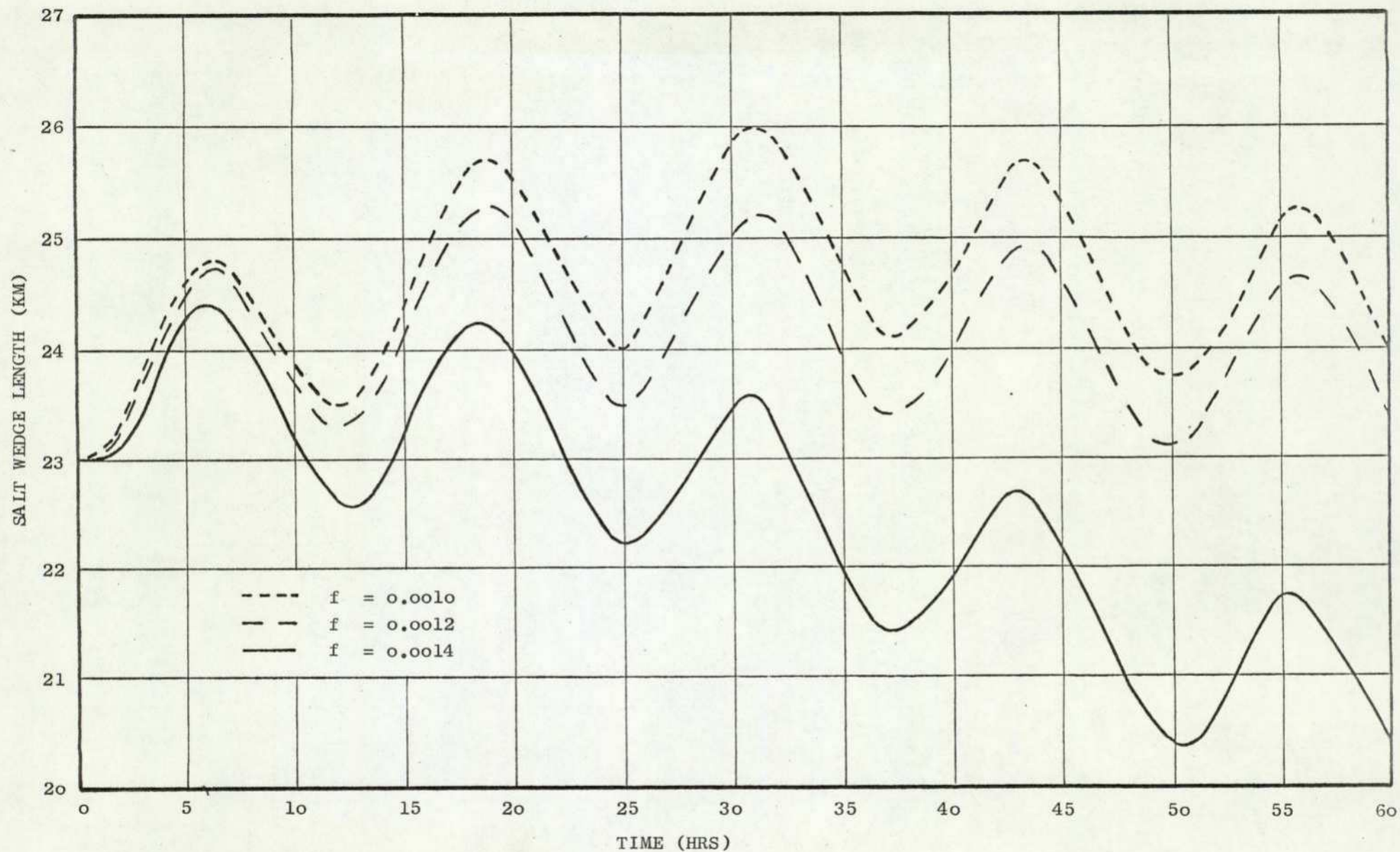


FIG. 7.21 COMPUTED TIDAL EXCURSIONS FOR THE MISSISSIPPI SALT WEDGE FRONT

lower layer thickness shown in Fig. 7.19. It will be seen that at the sea a tidal wave of 0.5m range at the surface, produces an interfacial tidal wave of about 2m range. Also, the phase shift of the wave within the wedge indicates that the internal wave celerity is initially about 1m/s compared with a surface wave celerity of at least 10m/s. It is this slow response of the interface to surface waves that occasionally causes supercritical flows to occur in the prototype where normally critical flow conditions would exist. Computed discharges in the lower layer at various distances from the sea are shown in Fig. 7.20. It can be seen from the steep gradients of the lower layer discharge curve at the sea, why in nature the lower layer flow has difficulty in adjusting to critical conditions at every stage of the tide. In Fig. 7.21 the computed tidal excursions of the wedge for various quasi-laminar interfacial friction factors are shown. Since the length of the salt-wedge in the South Pass is about 23km, the friction factor is seen to be about 0.0012 when a mixing coefficient $k'_m = 0.00004$ is used. With a tidal range at the sea of 0.5m, the salt-wedge is seen to move about 2km as the water ebbs and floods.

The computed average longitudinal density variations in the upper and lower layers of the South Pass are shown in Fig. 7.22. It must be noted that the only boundary conditions for density specified at the sea were those of density gradients. At 34km from the sea the water in the upper layer was assumed to be fresh whilst in the lower layer at the wedge-front the prototype wedge density of 1022kg/m^3 was used. It can be seen that only in the seaward

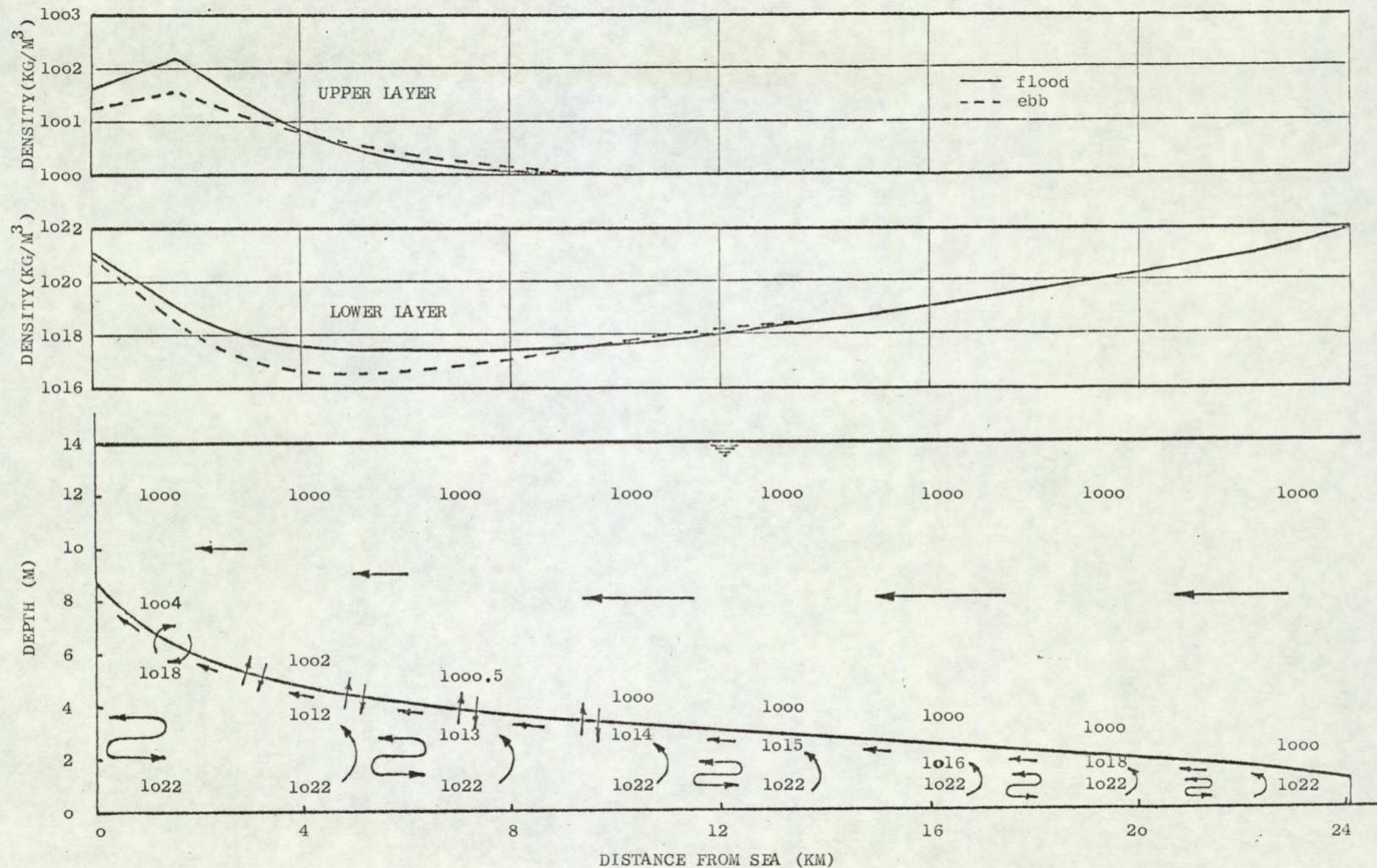


FIG. 7.22 COMPUTED LONGITUDINAL DENSITY VARIATIONS IN THE RIVER MISSISSIPPI

half of the wedge are velocities high enough for interfacial mixing to take place. This mixing increases the density in the upper layer until it reaches a maximum value of about 1002kg/m^3 at 2km from the sea and then decreases. The increase in interfacial mixing on the flood flows over the ebb is shown by the higher densities in the upper layer. Since the density is not specified at the sea in the lower layer, it can be seen that mixing through the interface decreases the density in the wedge far more than propagation of salt from the sea replenishes it. One might have assumed that the density in the wedge would decrease continuously from the sea to the wedge-front, but this is not the case. However, one must remember that these are average densities and that virtually all of the density changes for a fully stratified flow take place in the interfacial mixing layer. The higher density in the wedge on the flood flows indicates that the increased rate of interfacial mixing on the flood flow is not as significant as the increase in propagation of salt-water from the sea on the flood flows. The salt-wedge shown in Fig.7.22 indicates how the computed average densities may be distributed, bearing in mind the existence of an interfacial mixing layer. The arrows show the flow direction and indicate that most of the wedge flow changes direction with the tide.

In a partly mixed estuary the increased tidal activity results in more waves developing at the interface from eddies originating at the bed than would occur in a fully stratified estuary. These waves partially destroy the interfacial mixing layer, allowing a greater level of mixing to take place between layers for the same relative velocity.

CHAPTER 8

CONCLUSIONS

8.1 Discussion of the Practical Results

The numerical results obtained in this thesis for the rivers Hooghly and Gambia indicate that a new completely stable, commercially viable, numerical scheme has been developed. This scheme computes simultaneously not only water level and discharge variations in any well-mixed estuary but also variations in flow density. Its numerical algorithm is a natural extension of the much used and well-proven central implicit difference scheme using two dependent variables namely water level and discharge. All the knowledge learned over the past decade by engineers and mathematicians engaged in using the basic central implicit scheme for river and estuarine flow computations applies equally well to this new scheme.

Because of the successful extension of the central implicit algorithm to three dependent variables (53), it was decided to incorporate this technique in a second generation implicit stratified flow model. Knowledge about generalized algorithms acquired from the development of the first implicit stratified flow model (51), plus the successful use of the 'fractional steps' technique to a new two-dimensional flow model (52) were brought together in order to develop the stratified flow model described in this thesis. The model has been tested for a salt-wedge in the Mississippi delta and the results indicate that the

algorithm works and remains stable over long periods of computation. There are however problems when using central implicit schemes for stratified flow computations, these problems resulting from the fact that such schemes have virtually no dissipative properties.

Since the interface in a stratified flow is considerably more unstable than the surface, stratified flow computations are much more likely to become unstable than well-mixed flow computations. This is especially true when starting the model with guessed initial conditions. In a well-mixed model virtually any realistic initial condition can be used, and the scheme will remain stable, but for stratified flows and especially salt-wedge flows this is not the case. If velocities and depths throughout the estuary in both layers do not correspond closely with the initial boundary data, large interfacial waves develop which rapidly cause instabilities in the model. Without detailed prototype data of velocities and depths at any instant in salt-wedge flows, considerable time can be spent trying various initial conditions just to start the computation. Once the calculation has begun it should remain stable for fully stratified flow conditions. However, the interface is unstable by nature especially for partly-mixed flows and unlike a real flow situation which develops an interfacial mixing layer to attenuate interfacial instabilities, the central implicit scheme has no equivalent built-in dissipative mechanism. Since the central implicit scheme needs to be retained for stratified flow computations because of its economy and

second order accuracy, non-linear instabilities if they develop should be suppressed by artificially smoothing the numerical solution.

The physical experiments with arrested saline wedges proved that interfacial mixing is a two-way process with equal amounts of water being transferred between layers. This transfer of fluid must therefore be due to turbulent diffusion mixing although intermittent entrainment mixing in both directions will be present to a greater or lesser extent. The rate of interfacial mixing was shown to exhibit a straight line relationship with the relative velocity between layers, and to be inversely proportional to the relative density between layers. The point at which intermittent interfacial mixing is initiated was shown to depend on the interfacial Richardson number. Also, the ratio of the thickness of the interfacial mixing layer to the total depth was found to be the same in both model and prototype salt wedges. These two facts indicate that the value of Keulegan's interfacial stability parameter as measured in the laboratory cannot be a constant for all salt wedge flows but must be scaled up to prototype flow conditions.

Physical experiments with moving salt wedges proved that basically two different frontal types can exist in nature. When the salt wedge is moving into the estuary on the flood flow the front is characterised by having a wall or 'head' of fluid, and is called a wave front. As the tide reverses and the salt wedge is drawn out of the estuary

the front of the salt wedge becomes wedge shaped with zero frontal depth and is called a St.Venant front. It was shown that the relationship between frontal velocity and depth for the wave front was not the same as predicted by theory, this being due to the large energy losses in the frontal zone caused by interfacial mixing. Similarly the relationship between frontal velocity and critical depth for the St.Venant front was different from that predicted by theory, but in this case the reason was due to energy losses caused by the presence of a laminar sub-layer in the vicinity of the front. It was deduced from the experimental results that for mathematical modelling the physical frontal conditions can be best represented by ending the salt-wedge at the critical flow section.

8.2 Recommendations for Further Study

Computationally the only deficiency in the extended central implicit scheme is its sensitivity to non-linear instabilities in stratified flow models. This could be eliminated if a second order implicit dissipative difference scheme was used similar in performance to the Lax-Wendroff explicit dissipative difference scheme. However, at the moment no such scheme exists, but remains a challenge to anyone wishing to work in this field of applied mathematics. Even with such a scheme which dissipates energy only when the waves start to steepen there is no guarantee that the small scale turbulent energy will not be diffused as well.

The scheme proposed above must not be confused with the normal implicit scheme which when weighted on the forward time level can generate as much dissipation as one requires. The disadvantage with a non-centred scheme is that it is not of

second order accuracy and can in addition produce information losses. For the salt-wedge flows the scheme was tested with the interfacial shear stress terms mounted on the implicit side of the algorithm. The result was to improve the stability of the scheme but at the cost of damping out most of the flow in the salt-wedge. The same results occurred when artificially smoothing the solution and it was found that the degree of smoothing necessary to damp out non-linear instabilities seriously affected the flow conditions in the salt-wedge. For these reasons a fully centred scheme was used without smoothing and more care taken with the initial conditions when starting the computation.

The application of the mathematical model to the Gambia estuary illustrated the extent to which the value of the dispersion/diffusion coefficient affects the results. Although previous attempts have been made to relate dispersion coefficients to simple estuarine parameters, these have only given 'order of magnitude' estimates and it is thought that this area still provides scope for further study, especially as field measurements are constantly being recorded and published.

Experiments to obtain values for the interfacial stability parameter suggest that this parameter varies depending on whether the flow is in a laminar or turbulent state and that it is a function of the flow Reynolds number. Since all the laboratory experiments have taken place in 'smooth' channels, further studies should be made to ascertain (a) the relationship between the interfacial stability parameter and the flow Reynolds number for laminar flow conditions and (b) what effect the relative roughness of the channel has on the value of this parameter for different flow Reynolds numbers under turbulent flow conditions.


```

'BEGIN' 'REAL' CX,EK,D1,D2,CF,CK,CQ1,CQ2,C+1,CH2,CR1,CR2,BB,Z,FI,
XF,DXF,LENGTH,AMP,AMPQ;
'INTEGER' N,NN,M,MM,J,JJ,JF,X,Y,DT,JFS1,JFS2,N1,JSP;
SYSACT(1,6,132); SYSACT(1,8,60); SYSACT(1,12,1);
IN INTEGER(C,JJ);
'BEGIN' 'REAL' 'ARRAY' A,B,C,D,E,F,AL,ALS,BE,BES,GA,GAS,DE,DES,
EP,EPS,ZE,ZES,ET,ETS,TH,THS,EE,FF,A1,A2,B1,B2,TA,UC,MX,
CA1,CA2,F+1,F+2,QQ1,QQ2,U1,U2,RR1,RR2,LAM,HR1,PHI,LQ1,HBB,V1,
BMD1,BMD2,BM1,BM2,TAU,IB,LC2,Q1,R1,HR2(/O:JJ+2/),
HY,QY,RY(/C:JJ+2,C:2,1:2/),BQ(/C:25/),
I,F1,F2,Q1,C2,R1,R2,U1,U2,Y1,Y2(/O:JJ+2,C:2/),P(/1:2/);
'INTEGER' 'ARRAY' LBS,RBS,LE,RB(/1:2/),PCS,NEG(/O:JJ+2,1:2/);
'PROCEDURE' INITIAL;
'BEGIN'
'FOR' J:=C 'STEP' 1 'UNTIL' JJ+2 'DO'
'BEGIN'
HBB(/J/):=(JJ+2-J)*DX/500000;
Q1(/J,0/):=Q1(/J,2/):=Q1(/J,1/):=CQ1-J*390/(JJ+2);
'IF' J<JF 'THEN'
Q2(/J,0/):=Q2(/J,2/):=Q2(/J,1/):=0 'ELSE'
Q2(/J,0/):=Q2(/J,2/):=Q2(/J,1/):=CQ2-400*(J-JF)/(JJ+2-JF);
R1(/J,0/):=R1(/J,2/):=R1(/J,1/):=CR1;
R2(/J,0/):=R2(/J,2/):=R2(/J,1/):=CR2;
H1(/J,0/):=H1(/J,2/):=H1(/J,1/):=CH1+(JJ+2-J)*DX/250000;
'IF' J>JJ-8 'THEN'
H2(/J,0/):=H2(/J,2/):=H2(/J,1/):=HBB(/J/)+5.0+3.5*(J-JJ+8)/10 'ELSE'
H2(/J,0/):=H2(/J,2/):=H2(/J,1/):=HBB(/J/)+5.0-(5.0-1.26)*
{JJ-8-J}/{JJ-30};
'IF' J<JF 'THEN'
H2(/J,0/):=H2(/J,2/):=H2(/J,1/):=0;
B1(/J/):=BB; B2(/J/):=BB;
LAM(/J/):=R1(/J,0/)/R2(/J,0/);
'IF' J<JF 'THEN'
U1(/J,0/):=U1(/J,2/):=Q1(/J,C/)/(B1(/J/)*(H1(/J,0/)-HBB(/J/))) 'ELSE'
U1(/J,0/):=U1(/J,2/):=Q1(/J,0/)/(B1(/J/)*(H1(/J,0/)-H2(/J,0/)));
U1(/J,1/):=U1(/J,0/);
'IF' J<JF 'THEN'
U2(/J,0/):=U2(/J,2/):=0 'ELSE'
U2(/J,0/):=U2(/J,2/):=Q2(/J,C/)/(B2(/J/)*(H2(/J,0/)-HBB(/J/)));
U2(/J,1/):=U2(/J,0/);
'IF' J<JF 'THEN'
Y1(/J,0/):=Y1(/J,2/):=H1(/J,C/)-HBB(/J/); 'ELSE'
Y1(/J,0/):=Y1(/J,2/):=H1(/J,0/)-H2(/J,0/);
'IF' J<JF 'THEN'
Y2(/J,0/):=Y2(/J,2/):=0 'ELSE'
Y2(/J,0/):=Y2(/J,2/):=H2(/J,0/)-HBB(/J/);
'IF' J<JF 'THEN' I(/J,0/):=0 'ELSE'
I(/J,0/):=U1(/J,0/)*U2(/J,C/)*((U1(/J,0/)**2/(GR*Y1(/J,0/))-1)*
{U2(/J,0/)**2/(GR*Y2(/J,0/))-1}-LAM(/J/));
V1(/J/):=C; PHI(/J/):=3.142/2;

```

```

RI (/J/) := 1000; QI (/J/) := 0;
LQ1 (/J/) := 0; LQ2 (/J/) := 0;
'END';
XF := 0;
'END' OF PROCEDURE INITIAL;
'PROCEDURE' BSTART;
'BEGIN' 'FOR' Y := 1, 2 'DO'
'BEGIN'
'IF' POS (/JJ-2, Y/) = 2 | POS (/JJ-2, Y/) = 1 'THEN'
'BEGIN' RES (/Y/) := 20; RB (/Y/) := 1;
'END';
'IF' POS (/JJ-2, Y/) = 0 'THEN'
'BEGIN' RES (/Y/) := 30; RB (/Y/) := 2;
'END';
'IF' POS (/JJ-2, Y/) = 3 'THEN'
'BEGIN' RES (/Y/) := 11; RB (/Y/) := 1;
'END';
'END';
'END' OF BSTART;
'PROCEDURE' BOUND;
'BEGIN'
AMP := 0.25;
H1 (/JJ, 2/) := CH1 + SIN (N * DT / (6.25 * 3600) * 3.142) * AMP;
H2 (/JJ, 2/) := H1 (/JJ, 2/) - (G1 (/JJ, 2/) ** 2 / (GR * B1 (/JJ/) ** 2 * 0.9 ** 2
* (1 - LAM (/JJ/)))) * (1/3);
X := ENTIER (N * CT / 1800);
'IF' X > 24 'AND' X < 50 'THEN' X := X - 25;
'IF' X > 49 'AND' X < 75 'THEN' X := X - 50;
'IF' X > 74 'AND' X < 100 'THEN' X := X - 75;
'IF' X > 99 'AND' X < 125 'THEN' X := X - 100;
'IF' X > 124 'AND' X < 150 'THEN' X := X - 125;
'IF' X > 149 'AND' X < 175 'THEN' X := X - 150;
'IF' X > 174 'AND' X < 200 'THEN' X := X - 175;
'IF' X > 199 'AND' X < 225 'THEN' X := X - 200;
'IF' X > 224 'AND' X < 250 'THEN' X := X - 225;
'IF' X > 249 'AND' X < 275 'THEN' X := X - 250;
'IF' X > 274 'AND' X < 300 'THEN' X := X - 275;
'IF' X > 299 'AND' X < 325 'THEN' X := X - 300;
'IF' X > 324 'AND' X < 350 'THEN' X := X - 325;
'IF' N * DT / 1800 = N * DT / 1800 'THEN' Q1 (/0, 2/) := BQ (/X/) 'ELSE'
Q1 (/0, 2/) := BQ (/X/) + DT / 1800 * (BQ (/X+1/) - BQ (/X/));
R1 (/1, 2/) := CR1;
Q1 (/0, 1/) := 0.5 * (Q1 (/0, 2/) + G1 (/0, 0/));
R1 (/1, 1/) := 0.5 * (R1 (/1, 2/) + F1 (/1, 0/));
H1 (/JJ, 1/) := 0.5 * (H1 (/JJ, 2/) + H1 (/JJ, 0/));
H2 (/JJ, 1/) := 0.5 * (H2 (/JJ, 2/) + H2 (/JJ, 0/));
'IF' JF / 4 = JF / 4 'THEN'
'BEGIN'
A2 (/JF/) := B2 (/JF/) * (H2 (/JF, 2/) - HBB (/JF/));
Q2 (/JF, 2/) := A2 (/JF/) * (U1 (/JF, 2/) - SQRT (GR * Y2 (/JF, 2/)
* (1 - LAM (/JF/))));
R2 (/JF+1, 2/) := CR2;
Q2 (/JF, 1/) := 0.5 * (Q2 (/JF, 2/) + Q2 (/JF, 0/));
R2 (/JF+1, 1/) := 0.5 * (R2 (/JF+1, 2/) + R2 (/JF+1, 0/));
JFS1 := JF + 2; JFS2 := JF + 4;

```



```

'END' 'ELSE'
'BEGIN'
Y2 (/JF,2/) := (U1 (/JF,2/) - U2 (/JF,2/)) ** 2 / (GR * (1 - LAM (/JF/)));
H2 (/JF,2/) := Y2 (/JF,2/) + HBE (/JF/);
R2 (/JF-1,2/) := CR2;
H2 (/JF,1/) := 0.5 * (H2 (/JF,2/) + H2 (/JF,0/));
R2 (/JF-1,1/) := 0.5 * (R2 (/JF-1,2/) + R2 (/JF-1,0/));
JFS1 := JF + 4; JFS2 := JF + 2;
'END';
'END' OF PROCEDURE BOUND;
'PROCEDURE' ITER;
'BEGIN'
'FOR' J := 0 'STEP' 1 'UNTIL' JJ 'DO'
'BEGIN'
QQ1 (/J/) := 0.5 * (Q1 (/J,0/) + Q1 (/J,2/));
QQ2 (/J/) := 0.5 * (Q2 (/J,0/) + Q2 (/J,2/));
HH1 (/J/) := 0.5 * (H1 (/J,0/) + H1 (/J,2/));
HH2 (/J/) := 0.5 * (H2 (/J,0/) + H2 (/J,2/));
RR1 (/J/) := 0.5 * (R1 (/J,0/) + R1 (/J,2/));
RR2 (/J/) := 0.5 * (R2 (/J,0/) + R2 (/J,2/));
'END';
'END' OF PROCEDURE ITER;
'PROCEDURE' RIVSEC;
'BEGIN'
'FOR' J := 0 'STEP' 1 'UNTIL' JJ 'DO'
'BEGIN'
'IF' J < JF 'THEN'
A1 (/J/) := B1 (/J/) * (HH1 (/J/) - HBE (/J/)) 'ELSE'
A1 (/J/) := B1 (/J/) * (HH1 (/J/) - HH2 (/J/));
A2 (/J/) := B2 (/J/) * (HH2 (/J/) - HBE (/J/));
UU1 (/J/) := QQ1 (/J/) / A1 (/J/);
UU2 (/J/) := QQ2 (/J/) / A2 (/J/);
LAM (/J/) := RR1 (/J/) / RR2 (/J/);
'IF' J < JF 'THEN'
HR1 (/J/) := A1 (/J/) / (B1 (/J/) + 2 * (HH1 (/J/) - HBE (/J/))) 'ELSE'
HR1 (/J/) := A1 (/J/) / (2 * (HH1 (/J/) - HH2 (/J/)));
HR2 (/J/) := A2 (/J/) / (B2 (/J/) + 2 * (HH2 (/J/) - HBE (/J/)));
'IF' J < JF 'THEN'
TAU (/J/) := LQ1 (/J/) := LQ2 (/J/) := 0 'ELSE'
'BEGIN'
TA (/J/) := (0.526 * (1 - LAM (/J/)) + 0.127) / 15.8;
UC (/J/) := (0.00001 * 9.81 * (1 - LAM (/J/))) ** 0.23333 / TA (/J/);
TAU (/J/) := FI * RI (/J/) * (UU2 (/J/) - UU1 (/J/)) * ABS (UU2 (/J/) - UU1 (/J/)) / 8;
'IF' ABS (UU1 (/J/) - UU2 (/J/)) > UC (/J/) 'THEN'
LQ2 (/J/) := EK * (ABS (UU1 (/J/) - UU2 (/J/)) - UC (/J/)) / (1 - LAM (/J/)) 'ELSE'
LQ2 (/J/) := 0;
LQ1 (/J/) := LQ2 (/J/);
'END';
'END';
'FOR' J := 0 'STEP' 1 'UNTIL' JJ 'DO'
'BEGIN'
BMD1 (/J/) := (A1 (/J/) + A2 (/J/)) / HH1 (/J/);
'IF' J < JF 'THEN' BMD2 (/J/) := B2 (/J/) 'ELSE'
BMD2 (/J/) := A2 (/J/) / HH2 (/J/);
'END';

```

```

'FOR' J:=1'STEP'1'UNTIL'JJ-1'DO'
'BEGIN'
DA1(/J/):=(A1(/J+1/)-A1(/J-1/))/2;
DA2(/J/):=(A2(/J+1/)-A2(/J-1/))/2;
BM1(/J/):=HF1(/J/)*(BMD1(/J+1/)-BMD1(/J-1/))/(HH1(/J+1/)-HH1(/J-1/))+
BMD1(/J/);
'IF' J<JF+1'THEN' BM2(/J/):=B2(/J/)'ELSE'
BM2(/J/):=HH2(/J/)*(BMD2(/J+1/)-BMD2(/J-1/))/(HH2(/J+1/)-HH2(/J-1/))+
BMD2(/J/);
'END';
BM2(/JF/):=2*BM2(/JF+1/)-BM2(/JF+2/);
'FOR' J:=1'STEP'1'UNTIL'JJ-1'DO'
'BEGIN'
FF(/J/):=GR*A1(/J/)-QQ1(/J/)**2*BM1(/J/)/A1(/J/)**2;
EE(/J/):=GR*(1-LAM(/J/))*A2(/J/)-QQ2(/J/)**2*BM2(/J/)/A2(/J/)**2;
'END';
'END' OF RIVSEC;
'PROCEDURE' ABCL1;
'BEGIN'
'FOR' J:=JFS1'STEP'4'UNTIL'JJ'DO'
'BEGIN'
MX(/J/):=B2(/J/)*(LAM(/J/)*LG1(/J/)-LG2(/J/))/(4*HF2(/J/));
A(/J/):=1/(16*DX);
B(/J/):=B2(/J/)*HH2(/J/)/(2*RR2(/J/)*DT)+QQ2(/J/)/(8*RR2(/J/)*DX);
C(/J/):=B2(/J/)/DT;
D(/J/):=E2(/J/)*HF2(/J/)/(2*RR2(/J/)*DT)-QQ2(/J/)/(8*RR2(/J/)*DX);
E(/J/):=-1/(16*DX);
F(/J/):=E(/J/)*Q2(/J+2,0/)+D(/J/)*R2(/J+1,C/)+C(/J/)*H2(/J,0/)+
B(/J/)*R2(/J-1,0/)+A(/J/)*G2(/J-2,0/)+
B2(/J/)*(LAM(/J/)*LG1(/J/)-LG2(/J/))/2;
'END';
'FOR' J:=JFS2'STEP'4'UNTIL'JJ-2'DO'
'BEGIN'
A(/J/):=EE(/J/)/(16*DX)-UU2(/J/)*B2(/J/)/DT;
B(/J/):=QQ2(/J/)*(1-2*B2(/J/)*HH2(/J/)/A2(/J/))/(2*RR2(/J/)*DT)
-QQ2(/J/)**2/(8*RR2(/J/)*A2(/J/)*DX);
C(/J/):=1/DT-(B2(/J/)*LG2(/J/)-2*B2(/J/)*LAM(/J/)*LG1(/J/))/(4*A2(/J/))
+GR*ABS(Q2(/J,0/))*CK**2/(2*A2(/J/)*HR2(/J/));
D(/J/):=QQ2(/J/)*(1-2*B2(/J/)*HF2(/J/)/A2(/J/))/(2*RR2(/J/)*DT)
+QQ2(/J/)**2/(8*RR2(/J/)*A2(/J/)*DX);
E(/J/):=-(EE(/J/)/(16*DX)+UU2(/J/)*B2(/J/)/DT);
F(/J/):=E(/J/)*H2(/J+2,0/)+D(/J/)*R2(/J+1,C/)+(1/DT+(B2(/J/)*LG2(/J/)-
2*B2(/J/)*LAM(/J/)*LG1(/J/))/(4*A2(/J/)))*Q2(/J,0/)+
(QQ2(/J/)*(1-2*B2(/J/)*
HH2(/J/)/A2(/J/))/(2*RR2(/J/)*DT)-QQ2(/J/)**2/(8*RR2(/J/)*A2(/J/)*DX))
*R2(/J-1,C/)+(EE(/J/)/(16*DX)-UU2(/J/)*B2(/J/)/DT)*H2(/J-2,0/)+
B2(/J/)*LG1(/J/)*LAM(/J/)/(2*A1(/J/))*Q1(/J,C/)-GR*LAM(/J/)*A2(/J/)*
(H1(/J+2,0/)-H1(/J-2,0/))/(8*DX)-TAU(/J/)/(2*RR2(/J/))*B2(/J/);
'END';
'FOR' J:=JF+1'STEP'2'UNTIL'JJ-1'DO'
'BEGIN'
A(/J/):=1/(4*DT)+(UU2(/J/)-D2*DA2(/J/)/(A2(/J/)*DX))/(16*DX)
-D2/(16*DX**2);
B(/J/):=1/(2*DT)+D2/(8*DX**2)+B2(/J/)*LG1(/J/)/(4*A2(/J/));
C(/J/):=1/(4*DT)-(UU2(/J/)-D2*DA2(/J/)/(A2(/J/)*DX))/(16*DX)

```



```

-C2/(16*DX**2);
D(/J/):=(1/(4*DT)-(UU2(/J/)-C2*CA2(/J/)/(A2(/J/)*DX))/(16*DX)
+D2/(16*DX**2))*R2(/J+2,0/)+(1/(2*DT)-D2/(8*DX**2)-B2(/J/)*LQ1(/J/)/
(4*A2(/J/)))*R2(/J,0/)+(1/(4*DT)+(UU2(/J/)-D2*CA2(/J/)/(A2(/J/
*DX))/(16*DX)+D2/(16*DX**2))*R2(/J-2,0/)+(E2(/J/)*LQ1(/J/)/(2*A2(/J/))
*R1(/J,C/);
*END*;
*FOR J:=C*STEP 1*UNTIL JJ+2*DO*
AL(/J/):=ALS(/J/):=BE(/J/):=BES(/J/):=GA(/J/):=GAS(/J/):=DE(/J/):=
DES(/J/):=EP(/J/):=EPS(/J/):=ZE(/J/):=ZES(/J/):=ET(/J/):=ETS(/J/):=
TH(/J/):=THS(/J/):=C;
*END* OF PROCEDURE ABCL1;
*PROCEDURE ABCU2;
*BEGIN*
*FOR J:=2*STEP 4*UNTIL JJ*DO*
*BEGIN*
MX(/J/):=(B2(/J/)*(LQ2(/J/)/LAM(/J/)-LQ1(/J/))*(1-LAM(/J/))+
QI(/J/)*RI(/J/)/RR1(/J/))/(2*HH1(/J/));
A(/J/):=1/(8*DX);
B(/J/):=QG1(/J/)/(4*RR1(/J/)*DX)-(B2(/J/)*HH2(/J/)-B1(/J/)*HH1(/J/))/(
(2*RR1(/J/)*DT);
C(/J/):=B1(/J/)/DT;
D(/J/):=-QG1(/J/)/(4*RR1(/J/)*DX)-(B2(/J/)*HF2(/J/)-B1(/J/)*HH1(/J/))/(
(2*RR1(/J/)*DT);
E(/J/):=-1/(8*DX);
F(/J/):=E(/J/)*G1(/J+2,0/)+C(/J/)*R1(/J+1,0/)+C(/J/)*H1(/J,C/)+
B(/J/)*R1(/J-1,0/)+A(/J/)*Q1(/J-2,0/)-(G2(/J+2,1/)-Q2(/J-2,1/))/(
(4*DX)-2*E2(/J/)*HH2(/J/)*(R2(/J,1/)-R2(/J,C/))/(RR2(/J/)*DT)
-QQ2(/J/)*(R2(/J+1,1/)-R2(/J-1,1/))/(2*RR2(/J/)*DX)
+B2(/J/)*(LQ2(/J/)/LAM(/J/)-LQ1(/J/))*(1-LAM(/J/))
+QI(/J/)*RI(/J/)/RR1(/J/);
*END*;
*FOR J:=4*STEP 4*UNTIL JJ-2*DO*
*BEGIN*
MX(/J/):=V1(/J/)*COS(PHI(/J/))*QI(/J/)*RI(/J/)/(2*RR1(/J/)*QG1(/J/));
A(/J/):=FF(/J/)/(8*DX)-UU1(/J/)*B1(/J/)/DT;
B(/J/):=(QG1(/J/)+2*UU1(/J/)*(B2(/J/)*HH2(/J/)-B1(/J/)*HF1(/J/)))/(
(2*RR1(/J/)*DT)-QQ1(/J/)**2/(4*RR1(/J/)*A1(/J/)*DX);
C(/J/):=1/DT-B2(/J/)*(LQ1(/J/)*(1-2*LAM(/J/))+2*LQ2(/J/)*
(1-1/LAM(/J/))-2*QI(/J/)*RI(/J/)/(B2(/J/)*RR1(/J/)))/(2*A1(/J/))
+GR*ABS(G1(/J,0/))*CK**2/(2*A1(/J/)*HR1(/J/));
D(/J/):=(QG1(/J/)+2*UU1(/J/)*(B2(/J/)*HF2(/J/)-B1(/J/)*HF1(/J/)))/(
(2*RR1(/J/)*DT)+QQ1(/J/)**2/(4*RR1(/J/)*A1(/J/)*DX);
E(/J/):=-FF(/J/)/(8*DX)-UU1(/J/)*B1(/J/)/DT;
F(/J/):=E(/J/)*H1(/J+2,0/)+C(/J/)*R1(/J+1,0/)+(C(/J/)-
GR*ABS(Q1(/J,0/))*CK**2/(2*A1(/J/)*HR1(/J/))*Q1(/J,0/)+
B(/J/)*R1(/J-1,0/)+A(/J/)*H1(/J-2,0/)+2*UU1(/J/)*(Q2(/J+2,1/)-
Q2(/J-2,1/))/(4*DX)+4*UU1(/J/)*B2(/J/)*HH2(/J/)*(R2(/J,1/)-R2(/J,0/))/(
(RR2(/J/)*DT)+UU1(/J/)*QQ2(/J/)*(R2(/J+1,1/)-R2(/J-1,1/))/(RR2(/J/)*DX)
+B2(/J/)*LQ2(/J/)*Q2(/J,1/)/(LAM(/J/)*A2(/J/))-UU1(/J/)**2*BM2(/J/))
*(F2(/J+2,1/)-H2(/J-2,1/))/(4*DX)+V1(/J/)*COS(PHI(/J/))
*G1(/J/)*RI(/J/)/RR1(/J/)+TAU(/J/)/RR1(/J/)*B2(/J/);
*END*;
*FOR J:=3*STEP 2*UNTIL JJ-1*DO*
*BEGIN*

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A(/J/):=1/(4*DT)+(UU1(/J/)-D1*DA1(/J/)/(A1(/J/)*DX))/(8*DX)
-D1/(8*DX**2);
B(/J/):=1/(2*DT)+D1/(4*DX**2)+(B2(/J/)*LQ2(/J/)-Q1(/J/))/(2*A1(/J/));
C(/J/):=1/(4*DT)-(UU1(/J/)-D1*DA1(/J/)/(A1(/J/)*DX))/(8*DX)
-D1/(8*DX**2);
D(/J/):=(1/(4*DT)-(UU1(/J/)-D1*DA1(/J/)/(A1(/J/)*DX))/(8*DX)
+D1/(8*DX**2))*R1(/J+2,0/)+(1/(2*DT)-D1/(4*DX**2)-(E2(/J/)*LQ2(/J/)-
Q1(/J/))/(2*A1(/J/))*R1(/J,C/)+(1/(4*DT)+(UU1(/J/)-D1*DA1(/J/)/
(A1(/J/)*DX))/(8*DX)+D1/(8*DX**2))*R1(/J-2,0/)+B2(/J/)*LQ2(/J/)/
A1(/J/)*R2(/J,1/)+Q1(/J/)*R1(/J/)/A1(/J/);
'END';
'FOR' J:=0 'STEP' 1 'UNTIL' JJ+2 'DO'
AL(/J/):=ALS(/J/):=BE(/J/):=BES(/J/):=GA(/J/):=GAS(/J/):=DE(/J/):=
DES(/J/):=EP(/J/):=EPS(/J/):=ZE(/J/):=ZES(/J/):=ET(/J/):=ETS(/J/):=
TH(/J/):=THS(/J/):=C;
'END' OF PROCEDURE ABCU2;
'PROCEDURE' ABCL2;
'BEGIN'
'FOR' J:=JFS1 'STEP' 4 'UNTIL' JJ 'DO'
'BEGIN'
A(/J/):=1/(16*DX);
B(/J/):=B2(/J/)*HH2(/J/)/(2*RR2(/J/)*DT)+Q2(/J/)/(8*RR2(/J/)*DX);
C(/J/):=B2(/J/)/DT;
D(/J/):=B2(/J/)*HH2(/J/)/(2*RR2(/J/)*DT)-Q2(/J/)/(8*RR2(/J/)*DX);
E(/J/):=-1/(16*DX);
F(/J/):=E(/J/)*Q2(/J+2,1/)+D(/J/)*R2(/J+1,1/)+C(/J/)*H2(/J,1/)+
B(/J/)*R2(/J-1,1/)+A(/J/)*Q2(/J-2,1/)+B2(/J/)*(LAM(/J/)*LQ1(/J/)-
LQ2(/J/))/2;
'END';
'FOR' J:=JFS2 'STEP' 4 'UNTIL' JJ-2 'DO'
'BEGIN'
A(/J/):=EE(/J/)/(16*DX)-UU2(/J/)*B2(/J/)/DT;
B(/J/):=Q2(/J/)*(1-2*B2(/J/)*HH2(/J/)/A2(/J/))/(2*RR2(/J/)*DT)-
QQ2(/J/)**2/(8*RR2(/J/)*A2(/J/)*DX);
C(/J/):=1/DT+CR*ABS(Q2(/J,1/))*CK**2/(2*A2(/J/)*HR2(/J/))-
(B2(/J/)*LQ2(/J/)-2*B2(/J/)*LAM(/J/)*LQ1(/J/))/(4*A2(/J/));
D(/J/):=Q2(/J/)*(1-2*B2(/J/)*HH2(/J/)/A2(/J/))/(2*RR2(/J/)*DT)+
QQ2(/J/)**2/(8*RR2(/J/)*A2(/J/)*DX);
E(/J/):=-EE(/J/)/(16*DX)-UU2(/J/)*B2(/J/)/DT;
F(/J/):=E(/J/)*H2(/J+2,1/)+(Q2(/J/)*(1-2*B2(/J/)*HH2(/J/)/A2(/J/))/
(2*RR2(/J/)*DT)+QQ2(/J/)**2/(8*RR2(/J/)*A2(/J/)*DX))*R2(/J+1,1/)+
(1/DT+(B2(/J/)*LQ2(/J/)-2*B2(/J/)*LAM(/J/)*LQ1(/J/))/(4*A2(/J/)))*
Q2(/J,1/)+
(QQ2(/J/)*(1-2*B2(/J/)*HH2(/J/)/A2(/J/))/(2*RR2(/J/)*DT)-
QQ2(/J/)**2/(8*RR2(/J/)*A2(/J/)*DX))*R2(/J-1,1/)+(EE(/J/)/(16*DX)-
UU2(/J/)*B2(/J/)/DT)*H2(/J-2,1/)+B2(/J/)*LQ1(/J/)*LAM(/J/)/(2*A1(/J/))
*Q1(/J,2/)-GR*LAM(/J/)*A2(/J/)/(8*DX)*(H1(/J+2,2/)-H1(/J-2,2/))-
TAU(/J/)/(2*RR2(/J/))*B2(/J/);
'END';
'FOR' J:=JF+1 'STEP' 2 'UNTIL' JJ-1 'DO'
'BEGIN'
A(/J/):=1/(4*DT)+(UU2(/J/)-D2*DA2(/J/)/(A2(/J/)*DX))/(16*DX)
-D2/(16*DX**2);
B(/J/):=1/(2*DT)+D2/(8*DX**2)+B2(/J/)*LQ1(/J/)/(4*A2(/J/));
C(/J/):=1/(4*DT)-(UU2(/J/)-D2*DA2(/J/)/(A2(/J/)*DX))/(16*DX)

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-D2/(16*DX**2);
D(/J/):=(1/(4*DT))-(UU2(/J/)-D2*DA2(/J/)/(A2(/J/)*DX))/(16*DX)
+D2/(16*DX**2))*R2(/J+2,1/)+(1/(2*DT))-D2/(8*CX**2)-B2(/J/)*LQ1(/J/)/
(4*A2(/J/))*R2(/J,1/)+(1/(4*DT))+(UU2(/J/)-D2*DA2(/J/)/(A2(/J/)*CX))/(
(16*CX))+D2/(16*DX**2))*R2(/J-2,1/)+B2(/J/)*LQ1(/J/)/(2*A2(/J/))
*R1(/J,2/);
'END';
'FOR' J:=0 'STEP' 1 'UNTIL' JJ+2 'DO'
AL(/J/):=ALS(/J/):=BE(/J/):=BES(/J/):=GA(/J/):=GAS(/J/):=DE(/J/):=
DES(/J/):=EP(/J/):=EPS(/J/):=ZE(/J/):=ZES(/J/):=ET(/J/):=ETS(/J/):=
TH(/J/):=THS(/J/):=0;
'END' OF PROCEDURE ABCL2;
'PROCEDURE' COLRSB;
'BEGIN'
'IF' (J+2)/4=(J+2)'/4' THEN
'BEGIN'
Z:=C(/J/)+E(/J/)*ZE(/J/);
AL(/J+2/):=(-B(/J/)-(E(/J/)*EP(/J/)+D(/J/))*BES(/J+1/))/Z;
BE(/J+2/):=-A(/J/)/Z;
DE(/J+2/):=(F(/J/)-D(/J/)*DES(/J+1/)-(TH(/J/)+EP(/J/)*DES(/J+1/))
+E(/J/))/Z;
'END';
'IF' (J+1)/4=(J+1)'/4' THEN
'BEGIN'
Z:=B(/J/)+C(/J/)*BES(/J/);
ZES(/J+2/):=-A(/J/)/Z;
THS(/J+2/):=(D(/J/)-C(/J/)*DES(/J/))/Z;
'END';
'IF' J/4=J'/4' THEN
'BEGIN'
Z:=C(/J/)+E(/J/)*BE(/J/);
EP(/J+2/):=(-B(/J/)-ZES(/J+1/))*{D(/J/)+E(/J/)*AL(/J/)} /Z;
ZE(/J+2/):=-A(/J/)/Z;
TH(/J+2/):=(F(/J/)-E(/J/)*DE(/J/)-THS(/J+1/))*{D(/J/)+E(/J/)*AL(/J/)} /Z;
'END';
'IF' (J-1)/4=(J-1)'/4' THEN
'BEGIN'
Z:=B(/J/)+C(/J/)*ZES(/J/);
BES(/J+2/):=-A(/J/)/Z;
DES(/J+2/):=(D(/J/)-C(/J/)*THS(/J/))/Z;
'END';
'END' OF PROCEDURE COLRSB;
'PROCEDURE' CORLSB;
'BEGIN'
'IF' (J+2)/4=(J+2)'/4' THEN
'BEGIN'
Z:=C(/J/)+E(/J/)*ZE(/J/)+ALS(/J+1/)*{D(/J/)+E(/J/)*EP(/J/)};
AL(/J+2/):=(-B(/J/)-(E(/J/)*EP(/J/)+D(/J/))*BES(/J+1/))/Z;
BE(/J+2/):=-A(/J/)/Z;
DE(/J+2/):=(F(/J/)-D(/J/)*DES(/J+1/)-(TH(/J/)+EP(/J/)*DES(/J+1/))
+E(/J/))/Z;
'END';
'IF' (J+1)/4=(J+1)'/4' THEN
'BEGIN'
Z:=-E(/J/)/C(/J/)-ALS(/J/)*AL(/J+1/)-BES(/J/);

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EPS(/J+2/):=ALS(/J/)*BE(/J+1/)/Z;
ZES(/J+2/):=A(/J/)/(C(/J/)*Z);
THS(/J+2/):=(ALS(/J/)*DE(/J+1/)+DES(/J/)-C(/J/)/C(/J/))/Z;
'END';
'IF' J/4=J'/'4' THEN
'BEGIN'
Z:=C(/J/)+E(/J/)*BE(/J/)+EPS(/J+1/)*(D(/J/)+E(/J/)*AL(/J/));
EP(/J+2/):=(-B(/J/)-ZES(/J+1/)*(D(/J/)+E(/J/)*AL(/J/)))/Z;
ZE(/J+2/):=-A(/J/)/Z;
TH(/J+2/):=(F(/J/)-E(/J/)*DE(/J/)-THS(/J+1/)*(D(/J/)+E(/J/)*AL(/J/)))/Z;
'END';
'IF' (J-1)/4=(J-1)'/4' THEN
'BEGIN'
Z:=-B(/J/)/C(/J/)-EPS(/J/)*EP(/J+1/)-ZES(/J/);
ALS(/J+2/):=EPS(/J/)*ZE(/J+1/)/Z;
BES(/J+2/):=A(/J/)/(C(/J/)*Z);
DES(/J+2/):=(EPS(/J/)*TH(/J+1/)+THS(/J/)-D(/J/)/C(/J/))/Z;
'END';
'END' OF CCRLSB;
'PROCEDURE' CORLSP;
'BEGIN'
'IF' (J+1)/4=(J+1)'/4' THEN
'BEGIN'
Z:=B(/J+1/)*C(/J/)*ALS(/J/)-E(/J+1/)*A(/J/);
AL(/J+1/):=(D(/J+1/)*A(/J/)-B(/J+1/)*(B(/J/)+C(/J/)*BES(/J/)))/Z;
BE(/J+1/):=(C(/J+1/)*A(/J/)-B(/J+1/)*C(/J/)*GAS(/J/))/Z;
GA(/J+1/):=A(/J+1/)*A(/J/)/Z;
DE(/J+1/):=(B(/J+1/)*(D(/J/)-C(/J/)*DES(/J/))-F(/J+1/)*A(/J/))/Z;
'END';
'IF' J/4=J'/'4' THEN
'BEGIN'
Z:=D(/J/)+E(/J/)*AL(/J/);
EPS(/J+1/):=-(C(/J/)+E(/J/)*BE(/J/))/Z;
ZES(/J+1/):=-B(/J/)/Z;
ETS(/J+1/):=-(A(/J/)+E(/J/)*GA(/J/))/Z;
THS(/J+1/):=(F(/J/)-E(/J/)*DE(/J/))/Z;
'END';
'IF' (J-1)/4=(J-1)'/4' THEN
'BEGIN'
Z:=B(/J+1/)*C(/J/)*EPS(/J/)-E(/J+1/)*A(/J/);
EP(/J+1/):=(D(/J+1/)*A(/J/)-B(/J+1/)*(B(/J/)+C(/J/)*ZES(/J/)))/Z;
ZE(/J+1/):=(C(/J+1/)*A(/J/)-B(/J+1/)*C(/J/)*ETS(/J/))/Z;
ET(/J+1/):=A(/J+1/)*A(/J/)/Z;
TH(/J+1/):=(B(/J+1/)*(D(/J/)-C(/J/)*THS(/J/))-F(/J+1/)*A(/J/))/Z;
'END';
'IF' (J+2)/4=(J+2)'/4' THEN
'BEGIN'
Z:=D(/J/)+E(/J/)*EP(/J/);
ALS(/J+1/):=-(C(/J/)+E(/J/)*ZE(/J/))/Z;
BES(/J+1/):=-B(/J/)/Z;
GAS(/J+1/):=-(A(/J/)+E(/J/)*ET(/J/))/Z;
DES(/J+1/):=(F(/J/)-E(/J/)*TH(/J/))/Z;
'END';
'END' OF CORLSP;
'PROCEDURE' COLRSP;

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'BEGIN'
'IF' (J+1)/4 = (J+1) / '4' THEN
'BEGIN'
Z := B(/J/) + C(/J/) * BES(/J/);
ZES(/J+2/) := -A(/J/) / Z;
THS(/J+2/) := (D(/J/) - C(/J/) * DES(/J/)) / Z;
'END';
'IF' J/4 = J / '4' THEN
'BEGIN'
EP(/J+2/) := -ZES(/J+1/) * (B(/J-2/) + C(/J-2/) * AL(/J/) + D(/J-2/) * BES(/J-1/))
+ E(/J-2/) * EP(/J-2/) * BES(/J-1/)) / A(/J-2/);
TH(/J+2/) := - (THS(/J+1/) * (B(/J-2/) + C(/J-2/) * AL(/J/) + D(/J-2/) * BES(/J-1/))
+ E(/J-2/) * EP(/J-2/) * BES(/J-1/)) + C(/J-2/) * DE(/J/) + D(/J-2/) * DES(/J-1/))
+ E(/J-2/) * (EP(/J-2/) * DES(/J-1/) + TH(/J-2/)) - F(/J-2/)) / A(/J-2/);
'END';
'IF' (J-1)/4 = (J-1) / '4' THEN
'BEGIN'
Z := E(/J/) + C(/J/) * ZES(/J/);
BES(/J+2/) := -A(/J/) / Z;
DES(/J+2/) := (D(/J/) - C(/J/) * THS(/J/)) / Z;
'END';
'IF' (J+2)/4 = (J+2) / '4' THEN
'BEGIN'
AL(/J+2/) := -BES(/J+1/) * (B(/J-2/) + C(/J-2/) * EP(/J/) + D(/J-2/) * ZES(/J-1/))
+ E(/J-2/) * AL(/J-2/) * ZES(/J-1/)) / A(/J-2/);
DE(/J+2/) := - (DES(/J+1/) * (B(/J-2/) + C(/J-2/) * EP(/J/) + D(/J-2/) * ZES(/J-1/))
+ E(/J-2/) * AL(/J-2/) * ZES(/J-1/)) + C(/J-2/) * TH(/J/) + D(/J-2/) * THS(/J-1/))
+ E(/J-2/) * (AL(/J-2/) * THS(/J-1/) + DE(/J-2/)) - F(/J-2/)) / A(/J-2/);
'END';
'END' OF COLRSP;
'PROCEDURE' FRONTCO;
'BEGIN'
'IF' JF/4 = JF / '4' THEN
'BEGIN'
TH(/JF+2/) := Q2(/JF, X/); DES(/JF+3/) := R2(/JF+1, X/);
'END' ELSE
'BEGIN'
DE(/JF+2/) := F2(/JF, X/); DES(/JF+1/) := R2(/JF-1, X/);
'END';
HY(/JJ, X, Y/) := H2(/JJ, X/);
RY(/JJ-1, X, Y/) := R2(/JJ-1, X/);
QY(/JJ-2, X, Y/) := Q2(/JJ-2, X/);
QY(/JF, X, Y/) := Q2(/JF, X/); RY(/JF+1, X, Y/) := R2(/JF+1, X/);
HY(/JF, X, Y/) := H2(/JF, X/);
RY(/JF-1, X, Y/) := R2(/JF-1, X/);
'END' OF FRONTCO;
'PROCEDURE' CALGRF;
'BEGIN'
'FOR' J := JJ STEP -4 UNTIL LB(/Y/) + 4 DO
'BEGIN'
'IF' RB(/Y/) = 2 'AND' J = JJ 'THEN' 'GOTO' SKIP;
'IF' RES(/Y/) = 20 'AND' J = JJ 'THEN'
RY(/J-1, X, Y/) := ALS(/J+1/) * HY(/J, X, Y/) + BES(/J+1/) * RY(/J+1, X, Y/)
+ DES(/J+1/);
QY(/J-2, X, Y/) := EP(/J/) * RY(/J-1, X, Y/) + ZE(/J/) * HY(/J, X, Y/) + ET(/J/)

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*QY(/J+2,X,Y/)+TH(/J/);
SKIP:
RY(/J-3,X,Y/):=EPS(/J-1/)*QY(/J-2,X,Y/)+ZES(/J-1/)*
RY(/J-1,X,Y/)+ETS(/J-1/)*HY(/J,X,Y/)+THS(/J-1/);
HY(/J-4,X,Y/):=AL(/J-2/)*RY(/J-3,X,Y/)+BE(/J-2/)*
QY(/J-2,X,Y/)+GA(/J-2/)*HY(/J,X,Y/)+DE(/J-2/);
RY(/J-5,X,Y/):=ALS(/J-3/)*HY(/J-4,X,Y/)+BES(/J-3/)*
RY(/J-3,X,Y/)+GAS(/J-3/)*QY(/J-2,X,Y/)+DES(/J-3/);
'END';
'FOR' J:=2 'STEP' 4 'UNTIL' JJ-4 'DO'
QY(/J,X,Y/):=(QY(/J+2,X,Y/)+QY(/J-2,X,Y/))/2;
QY(/JJ,X,Y/):=2*QY(/JJ-2,X,Y/)-QY(/JJ-4,X,Y/);
'FOR' J:=1 'STEP' 2 'UNTIL' JJ-1 'DO'
QY(/J,X,Y/):=(QY(/J+1,X,Y/)+QY(/J-1,X,Y/))/2;
'FOR' J:=2 'STEP' 4 'UNTIL' JJ-4 'DO'
HY(/J+2,X,Y/):=(HY(/J+4,X,Y/)+HY(/J,X,Y/))/2;
HY(/0,X,Y/):=2*HY(/2,X,Y/)-HY(/4,X,Y/);
'FOR' J:=1 'STEP' 2 'UNTIL' JJ-1 'DO'
HY(/J,X,Y/):=(HY(/J+1,X,Y/)+HY(/J-1,X,Y/))/2;
'FOR' J:=2 'STEP' 2 'UNTIL' JJ-2 'DO'
RY(/J,X,Y/):=(RY(/J+1,X,Y/)+RY(/J-1,X,Y/))/2;
RY(/0,X,Y/):=2*RY(/1,X,Y/)-RY(/2,X,Y/);
RY(/JJ,X,Y/):=0.5*(RY(/JJ+1,X,Y/)+RY(/JJ-1,X,Y/));
'IF' Y=1 'THEN' JSP:=1 'ELSE' JSP:=JF+1;
'FOR' J:=JSP 'STEP' 1 'UNTIL' JJ-1 'DO'
QY(/J,X,Y/):=(1-0.01)*QY(/J,X,Y/)+0.01*(QY(/J+1,X,Y/)+QY(/J-1,X,Y/))/2;
'END' OF PROCEDURE CALQRH;
'PROCEDURE' CALFRONT;
'BEGIN'
'IF' JF>1 'THEN'
'BEGIN'
'FOR' J:=0 'STEP' 1 'UNTIL' JF-1 'DO'
'BEGIN'
HY(/J,X,Y/):=QY(/J,X,Y/):=C; RY(/J,X,Y/):=CR2;
'END';
'IF' JF/4=JF1/4 'THEN'
'BEGIN'
HY(/JF,X,Y/):=2*HY(/JF+2,X,Y/)-HY(/JF+4,X,Y/);
HY(/JF+1,X,Y/):=0.5*(HY(/JF,X,Y/)+HY(/JF+2,X,Y/));
RY(/JF,X,Y/):=2*RY(/JF+1,X,Y/)-RY(/JF+2,X,Y/);
'END' 'ELSE'
'BEGIN'
QY(/JF,X,Y/):=2*QY(/JF+2,X,Y/)-QY(/JF+4,X,Y/);
QY(/JF+1,X,Y/):=0.5*(QY(/JF,X,Y/)+QY(/JF+2,X,Y/));
RY(/JF,X,Y/):=2*RY(/JF+1,X,Y/)-RY(/JF+2,X,Y/);
'END';
'END';
'END' OF CALFRONT;
'PROCEDURE' CUTL1;
'BEGIN'
OUTSTRING(1,('SALT WEDGE IN THE SOUTH PASS OF THE MISSISSIPPI'));
LINE(1,1);
OUTSTRING(1,('TIME(HRS) ='))'; AFIX(1,2,3,N*DT/3600);
OUTSTRING(1,('N='))'; AFIX(1,4,0,N);
OUTSTRING(1,('TIME STEP (SECS) ='))'; AFIX(1,4,0,DT);

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OUTSTRING(1,('DISTANCE STEP (M) =')));AFIX(1,4,0,EX);
OUTSTRING(1,('SALT WEDGE (KM) ='))); AFIX(1,3,3,LENGTH/1000 );
OUTSTRING(1,('P1 ='))); FIX(1,0,5,P1/(1/));
OUTSTRING(1,('P2 ='))); FIX(1,0,5,P1/(2/));
OUTSTRING(1,('CK ='))); AFIX(1,0,4,CK);
OUTSTRING(1,('FI ='))); AFIX(1,0,4,FI);
OUTSTRING(1,('D1 ='))); AFIX(1,4,0,D1);
OUTSTRING(1,('D2 ='))); AFIX(1,4,0,D2);
OUTSTRING(1,('EK ='))); AFIX(1,0,6,EK); LINE(1,2);
OUTSTRING(1,('LOWER LEVEL (M) AT (N+0.5)DT'))); LINE(1,1);
'FOR'J:=0'STEP'1'UNTIL'JJ'DO' FIX(1,2,3,H2/(J,1/));LINE(1,1);
OUTSTRING(1,('LOWER DISCHARGE (M**3/S) AT (N+0.5)DT')));LINE(1,1);
'FOR'J:=0'STEP'1'UNTIL'JJ'DO' FIX(1,5,0,Q2/(J,1/));LINE(1,1);
OUTSTRING(1,('LOWER DENSITY (KG/M**3) AT (N+0.5)DT')));LINE(1,1);
'FOR'J:=0'STEP'1'UNTIL'JJ'DO' FIX(1,4,2,R2/(J,1/));LINE(1,3);
'END' OF PROCEDURE OUTL1;
'PROCEDURE' CUTU2;
'BEGIN'
OUTSTRING(1,('UPPER LAYER POSITIVE CHARACTERISTICS'))); LINE(1,1);
OUTSTRING(1,('UPPER LAYER NEGATIVE CHARACTERISTICS'))); LINE(1,1);
'FOR'J:=0'STEP'2'UNTIL'JJ'DO' AFIX(1,1,0,POS/(J,1/)); LINE(1,1);
'FOR'J:=0'STEP'2'UNTIL'JJ'DO' AFIX(1,1,0,NEG/(J,1/)); LINE(1,1);
OUTSTRING(1,('UPPER LEVEL (M) AT (N+1)DT')));LINE(1,1);
'FOR'J:=0'STEP'1'UNTIL'JJ'DO'FIX(1,2,3,H1/(J,2/));LINE(1,1);
OUTSTRING(1,('UPPER VELOCITY (M/S) AT (N+1)DT')));LINE(1,1);
'FOR'J:=0'STEP'1'UNTIL'JJ'DO' FIX(1,2,3,U1/(J,2/)); LINE(1,1);
OUTSTRING(1,('UPPER DISCHARGE (M**3/S) AT (N+1)DT')));LINE(1,1);
'FOR'J:=0'STEP'1'UNTIL'JJ'DO' FIX(1,5,0,Q1/(J,2/));LINE(1,1);
OUTSTRING(1,('UPPER DENSITY (KG/M**3) AT (N+1)DT')));LINE(1,1);
'FOR'J:=0'STEP'1'UNTIL'JJ'DO'FIX(1,4,2,R1/(J,2/));LINE(1,3);
'END' OF PROCEDURE OUTU2;
'PROCEDURE' CUTL2;
'BEGIN'
OUTSTRING(1,('LOWER LAYER POSITIVE CHARACTERISTICS'))); LINE(1,1);
OUTSTRING(1,('LOWER LAYER NEGATIVE CHARACTERISTICS'))); LINE(1,1);
'FOR'J:=0'STEP'2'UNTIL'JJ'DO' AFIX(1,1,0,FCS/(J,2/)); LINE(1,1);
'FOR'J:=0'STEP'2'UNTIL'JJ'DO' AFIX(1,1,0,NEG/(J,2/)); LINE(1,1);
OUTSTRING(1,('LOWER LEVEL (M) AT (N+1)DT')));LINE(1,1);
'FOR'J:=0'STEP'1'UNTIL'JJ'DO'FIX(1,2,3,H2/(J,2/));LINE(1,1);
OUTSTRING(1,('LOWER VELOCITY (M/S) AT (N+1)DT')));LINE(1,1);
'FOR'J:=0'STEP'1'UNTIL'JJ'DO' FIX(1,2,3,U2/(J,2/)); LINE(1,1);
OUTSTRING(1,('LOWER DISCHARGE (M**3/S) AT (N+1)DT')));LINE(1,1);
'FOR'J:=0'STEP'1'UNTIL'JJ'DO'FIX(1,5,0,Q2/(J,2/));LINE(1,1);
OUTSTRING(1,('INTERFACIAL DISCHARGE PER DISTANCE STEP (M**3/S)')));
LINE(1,1); 'FOR'J:=0'STEP'1'UNTIL'JJ'DO'
FIX(1,5,0,LQ2/(J/)*DX*B2/(J/)); LINE(1,1);
OUTSTRING(1,('LOWER DENSITY (KG/M**3) AT (N+1)DT')));LINE(1,1);
'FOR'J:=0'STEP'1'UNTIL'JJ'DO'FIX(1,4,2,R2/(J,2/));LINE(1,3);
'END' OF PROCEDURE OUTL2;
'PROCEDURE' PRECO;
'BEGIN'
'FOR'J:=LB(/Y/)+3'STEP'1'UNTIL'JJ-RB(/Y/)+3'DO'
'BEGIN'
'IF' POS(/J-2,Y/)=3'AND' NEG(/J-2,Y/)=0 'THEN'
'BEGIN' J:=J-2; COLRSP; J:=J+2;

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'END';
'IF' POS(/J-2,Y/)=0 'AND' NEG(/J-2,Y/)=3 'THEN'
'BEGIN' J:=J-1; CORLSP; J:=J+1;
'END';
'IF' POS(/J-2,Y/)=2 'AND' NEG(/J-2,Y/)=1 'THEN'
'BEGIN' J:=J-2; COLRSB; J:=J+2;
'END';
'IF' POS(/J-2,Y/)=1 'AND' NEG(/J-2,Y/)=2 'THEN'
'BEGIN' J:=J-2; CORLSB; J:=J+2;
'END';
'END';
'END' OF PRECO;
'PROCEDURE' EIGEN;
'BEGIN'
'REAL' X1,X2,X3,X4,XXX;
'REAL' 'ARRAY' XDOT(/1:6/);
'FOR' J:=C'STEP'1 'UNTIL' JJ'DC'
'BEGIN'
X1:=SQRT(GR*(Y1(/J,2/)+Y2(/J,2/)));
X2:=SQRT(ABS((GR*(1-LAM(/J/))*Y1(/J,2/)*Y2(/J,2/))/(Y1(/J,2/)+Y2(/J,2/))
-((U1(/J,2/)-U2(/J,2/))*2*Y1(/J,2/)*Y2(/J,2/))
/(Y1(/J,2/)+Y2(/J,2/))*2));
X3:=(Y1(/J,2/)*U1(/J,2/)+Y2(/J,2/)*U2(/J,2/))/(Y1(/J,2/)+Y2(/J,2/));
X4:=(Y1(/J,2/)*U2(/J,2/)+Y2(/J,2/)*U1(/J,2/))/(Y1(/J,2/)+Y2(/J,2/));
XDOT(/1/):=X3+X1; XDOT(/2/):=X3-X1; XDOT(/3/):=U1(/J,2/);
XDOT(/4/):=X4+X2; XDOT(/5/):=X4-X2; XDOT(/6/):=U2(/J,2/);
POS(/J,1/):=POS(/J,2/):=NEG(/J,1/):=NEG(/J,2/):=0;
'IF' XDOT(/4/)>0 'AND' XDOT(/5/)>0 'THEN'
'BEGIN' 'IF' J>{(JF+JJ)/2} 'THEN'
'BEGIN' POS(/J,1/):=3; NEG(/J,1/):=0;
'IF' XDOT(/6/)>0 'THEN'
'BEGIN' POS(/J,2/):=2; NEG(/J,2/):=1;
'END' 'ELSE'
'BEGIN' PCS(/J,2/):=1; NEG(/J,2/):=2;
'END';
'END' 'ELSE'
'BEGIN' PCS(/J,2/):=3; NEG(/J,2/):=0;
'IF' XDOT(/3/)>0 'THEN'
'BEGIN' PCS(/J,1/):=2; NEG(/J,1/):=1;
'END' 'ELSE'
'BEGIN' POS(/J,1/):=1; NEG(/J,1/):=2;
'END';
'END';
'END';
'IF' XDOT(/4/)<0 'AND' XDOT(/5/)<0 'THEN'
'BEGIN' POS(/J,2/):=0; NEG(/J,2/):=3;
'IF' XDOT(/3/)>0 'THEN'
'BEGIN' PCS(/J,1/):=2; NEG(/J,1/):=1;
'END' 'ELSE'
'BEGIN' PCS(/J,1/):=1; NEG(/J,1/):=2;
'END';
'END';
'IF' XDOT(/4/)*XDOT(/5/)<0 'THEN'
'BEGIN' 'IF' XDOT(/3/)>0 'THEN'
'BEGIN' PCS(/J,1/):=2; NEG(/J,1/):=1;

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'END''ELSE'
'BEGIN' POS(/J,1/):=1; NEG(/J,1/):=2;
'END';
'IF' XDCT(/6/)>0 'THEN'
'BEGIN' POS(/J,2/):=2; NEG(/J,2/):=1;
'END''ELSE'
'BEGIN' PCS(/J,2/):=1; NEG(/J,2/):=2;
'END';
'END';
'IF' Y2(/J,2/)<0.001 'THEN' PCS(/J,2/):=NEG(/J,2/):=0;
'IF' J<JF 'THEN'
'BEGIN' 'IF' U1(/J,2/)>0 'THEN'
'BEGIN' POS(/J,1/):=2; NEG(/J,1/):=1;
'END''ELSE'
'BEGIN' PCS(/J,1/):=1; NEG(/J,1/):=2;
'END';
'END';
'END';
'END' OF EIGEN;
'PROCEDURE' CONTROL;
'BEGIN'
'FOR' J:=JF 'STEP' 1 'UNTIL' JJ 'DO'
I(/J,2/):=U1(/J,2/)*U2(/J,2/)*{(U1(/J,2/)**2/(GR*Y1(/J,2/))-1)*
(U2(/J,2/)**2/(GR*Y2(/J,2/))-1)-LAM(/J/)};
'FOR' J:=0 'STEP' 1 'UNTIL' JF-1 'DO'
I(/J,2/):=U1(/J,2/)*(U1(/J,2/)**2/(GR*Y1(/J,2/))-1);
'FOR' J:=0 'STEP' 1 'UNTIL' JJ 'DO'
'IF' I(/J,2/)*I(/J,0/)<0 'THEN'
'BEGIN' EIGEN; J:=JJ;
'END';
'END' OF CONTROL;
'PROCEDURE' SEADENSITY;
'BEGIN'
'IF' POS(/JJ-2,Y/)=1 | PCS(/JJ-2,Y/)=2 'THEN'
RY(/JJ+1,X,Y/):=(ALS(/JJ+1/)*HY(/JJ,X,Y/)+2*DX*P(/Y/)+
DES(/JJ+1/))/(1-BES(/JJ+1/));
'IF' POS(/JJ-2,Y/)=0 | POS(/JJ-2,Y/)=3 'THEN'
RY(/JJ-1,X,Y/):=(EPS(/JJ-1/)*QY(/JJ-2,X,Y/)+ETS(/JJ-1/)*HY(/JJ,X,Y/)+
THS(/JJ-1/)+2*DX*P(/Y/))/(1-ZES(/JJ-1/));
'END' OF SEADENSITY;
'PROCEDURE' NEGFRONT;
'BEGIN' 'FOR' J:=JF+1,JF 'DO'
'FOR' X:=0,1,2 'DO'
'BEGIN'
Q1(/J,X/):=2*Q1(/J+1,X/)-Q1(/J+2,X/);
Q2(/J,X/):=2*Q2(/J+1,X/)-Q2(/J+2,X/);
H1(/J,X/):=2*H1(/J+1,X/)-H1(/J+2,X/);
H2(/J,X/):=2*H2(/J+1,X/)-H2(/J+2,X/);
R1(/J,X/):=2*R1(/J+1,X/)-R1(/J+2,X/);
R2(/J,X/):=2*R2(/J+1,X/)-R2(/J+2,X/);
Y1(/J,X/):=H1(/J,X/)-H2(/J,X/);
Y2(/J,X/):=H2(/J,X/)-HBB(/J/);
U1(/J,X/):=Q1(/J,X/)/(Y1(/J,X/)*B1(/J/));
U2(/J,X/):=Q2(/J,X/)/(Y2(/J,X/)*B2(/J/));
A2(/J/):=B2(/J/)*Y2(/J,2/);

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LAM(/J/):=2*LAM(/J+1/)-LAM(/J+2/);
R2(/JF-1,X/):=CR2;
I(/J,0/):=U1(/J,0/)*U2(/J,0/)*{(U1(/J,0/)**2/(GR*Y1(/J,C/))-1)*
(U2(/J,C/)**2/(GR*Y2(/J,0/))-1)-LAM(/J/)};
'END';
EIGEN;
'END' OF NEGFRONT;
'PROCEDURE' POSFRONT;
'BEGIN' 'FOR' J:=JF-2,JF-1'DO'
'FOR' X:=C,1,2'DO'
'BEGIN'
Q1(/J,X/):=2*Q1(/J-1,X/)-Q1(/J-2,X/);
H1(/J,X/):=2*H1(/J-1,X/)-H1(/J-2,X/);
R1(/J,X/):=2*R1(/J-1,X/)-R1(/J-2,X/);
Y1(/J,X/):=H1(/J,X/)-HBB(/J/);
U1(/J,X/):=Q1(/J,X/)/(Y1(/J,X/)*B1(/J/));
LAM(/J/):=2*LAM(/J-1/)-LAM(/J-2/);
R2(/J,X/):=R2(/JF,X/):=R2(/JF+1,X/):=CR2;
I(/J,X/):=Y2(/J,X/):=H2(/J,X/):=U2(/J,X/):=Q2(/J,X/):=C;
'END';
EIGEN;
'END' OF PCSFRONT;
'PROCEDURE' SHIFT;
'BEGIN' DXF:=UU2(/JF/)*DT; XF:=DXF+XF;
'IF' ABS(XF)>DX 'THEN'
'BEGIN' 'IF' XF<0'THEN' JF:=JF-2'ELSE' JF:=JF+2;
'IF' JF/4=JF/'4' 'THEN'
'BEGIN' LBS(/2/):=20; LB(/2/):=JF+1;
'END' 'ELSE'
'BEGIN' LBS(/2/):=21; LB(/2/):=JF;
'END';
'IF' XF<0'THEN' NEGFRONT 'ELSE' POSFRONT;
'IF' XF<C'THEN' XF:=XF+2*CX 'ELSE' XF:=XF-2*CX;
'END';
LENGTH:=(JJ-JF)*CX-XF;
'END' OF SHIFT;
INREAL(0,CX); ININTEGER(0,DT);
INREAL(0,D1); INREAL(0,D2); INREAL(0,GR);
INREAL(0,CK); INREAL(0,CQ1); INREAL(0,CQ2); INREAL(0,CH1); INREAL(0,CH2);
INREAL(0,CR1); INREAL(0,CR2); INREAL(0,BB); ININTEGER(0,MM);
ININTEGER(0,NN); INREAL(0,EK); INREAL(0,F);
ININTEGER(0,LBS(/1/)); ININTEGER(0,LBS(/2/)); ININTEGER(0,LB(/1/));
ININTEGER(0,LB(/2/)); ININTEGER(0,RES(/1/)); ININTEGER(0,RBS(/2/));
ININTEGER(0,RB(/1/)); ININTEGER(0,RB(/2/));
INARRAY(0,P); ININTEGER(0,JF);
INARRAY(0,BQ);
INITIAL;
EIGEN;
N:=-1;
HAGUE: N:=N+1;
BOUND;
BSTART;
M:=0;
DELF1: N:=N+1;
ITER;

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RIVSEC;
ABCL1;
X:=1; Y:=2;
FRONTCD;
PRECC;
SEADENSITY;
CALQRH;
CALFRONT;
'FOR' J:=0 'STEP' 1 'UNTIL' JJ'DO'
'BEGIN'
H2(/J,X/):=HY(/J,X,2/); Q2(/J,X/):=QY(/J,X,2/); R2(/J,X/):=RY(/J,X,2/);
U2(/J,1/):=Q2(/J,1/)/(B2(/J/)*(H2(/J,1/)-FEB(/J/)));
'END';
'IF' N<1000 'AND' N*DT/7200=N*DT/'7200' THEN'
OUTL1;
'IF' N>999 'AND' N*DT/1800=N*DT/'1800' THEN'
OUTL1;
ABCU2;
'IF' LBS(/1/)=30 | LBS(/1/)=21 'THEN' DE(/4/):=H1(/2,2/);
'IF' LBS(/1/)=30 | LBS(/1/)=20 'THEN' TH(/2/):=G1(/0,2/);
DES(/3/):=R1(/1,2/);
HY(/JJ,2,1/):=H1(/JJ,2/); FY(/JJ-1,2,1/):=F1(/JJ-1,2/);
QY(/JJ-2,2,1/):=Q1(/JJ-2,2/); QY(/0,2,1/):=G1(/0,2/);
RY(/1,2,1/):=R1(/1,2/); HY(/2,2,1/):=H1(/2,2/);
X:=2; Y:=1;
PRECC;
SEADENSITY;
CALQRH;
'FOR' J:=0 'STEP' 1 'UNTIL' JJ'DO'
'BEGIN'
H1(/J,X/):=HY(/J,X,1/); Q1(/J,X/):=QY(/J,X,1/); R1(/J,X/):=RY(/J,X,1/);
Q1(/J,1/):=0.5*(Q1(/J,2/)+Q1(/J,0/));
'END';
'IF' N<1000 'AND' N*DT/7200=N*DT/'7200' THEN'
OUTU2;
'IF' N>999 'AND' N*DT/1800=N*DT/'1800' THEN'
OUTU2;
ABCL2;
X:=2; Y:=2;
FRONTCD;
PRECC;
SEADENSITY;
CALQRH;
CALFRONT;
'FOR' J:=0 'STEP' 1 'UNTIL' JJ'DO'
'BEGIN'
H2(/J,X/):=HY(/J,X,2/); Q2(/J,X/):=QY(/J,X,2/); R2(/J,X/):=RY(/J,X,2/);
U2(/J,2/):=Q2(/J,2/)/(B2(/J/)*(H2(/J,2/)-FBB(/J/)));
U1(/J,2/):=Q1(/J,2/)/(B1(/J/)*(H1(/J,2/)-F2(/J,2/)));
Y1(/J,2/):=H1(/J,2/)-H2(/J,2/);
Y2(/J,2/):=F2(/J,2/)-FBB(/J/);
U1(/J,1/):=0.5*(U1(/J,2/)+U1(/J,0/));
'END';
'IF' N<1000 'AND' N*DT/7200=N*DT/'7200' THEN'
OUTL2;

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'IF'N>999'AND' N*DT/1800=N*DT/'1800 'THEN'
OUTL2;
'IF' N<NN 'THEN' 'GOTO' DELFT;
CONTROL;
SHIFT;
'IF' N<NN 'THEN'
'BEGIN'
  'FOR'J:=0'STEP'1'UNTIL'JJ'DC'
  'BEGIN'
    H1(/J,0/):=H1(/J,2/);H2(/J,0/):=H2(/J,2/);
    Q1(/J,0/):=Q1(/J,2/);Q2(/J,0/):=Q2(/J,2/);
    R1(/J,0/):=R1(/J,2/);R2(/J,0/):=R2(/J,2/);
    I(/J,0/):=I(/J,2/);
  'END';
  'GOTO' FAGUE;
'END';
'END';
SYSACT(1,12,0);
'END'

```


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