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LINEAR ELASTIC STRESS ANALYSIS
for a
THICK SHELL OF REVOLUTION
SUBJECTED TO HYDROSTATIC PRESSURE

L.C. Mercer, BSc, CEng, MICE, MStructE
Submission for PhD degree

Department of Civil Engineering
Structures Division
The City University, London

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N.B. These are the Illustrations and Computer Plots for Part One - Theory. See the beginning of Part Two - Experiment for the continuation.

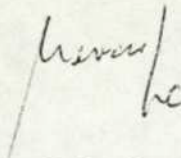
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██████████ adopted various ingenious devices to produce economic micro-concrete models, ██████████ adapted the pressure vessel, ██████████ assembled the pressure pumping system, and Mr ██████████ did the strain gauge assembly.

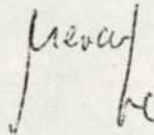
Furthermore the Computer Advisory helped me to start off in their department, and the Library staff helped me to trace some of the more obscure literature.

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L.C. Mercer.

Declaration

I hereby authorise the University Librarian to allow this thesis to be copied in whole or in part without further reference to myself at his or her own discretion with the provision that these be only single copies made for study purposes and subject to normal conditions of acknowledgement.

A handwritten signature in dark ink, appearing to read 'L.C. Mercer', with a stylized flourish at the end.

L.C. Mercer.

Abstract

The thesis is divided into two parts, theory and experiment.

Part One contains the theoretical analysis of the thick shell of revolution and is intended to be essentially practical in that the theories are developed for the sole purpose of producing numerical results which can be assessed in application.

Beginning with the finite element Displacement Method, some element types are gone into, and followed with computer programmes written in Fortran. A simple thin shell theory follows, first to assess a definition of a deep submergence shell, and then to set up equations for a particular geometry which can be used in comparison with the finite element method as a quick check that results obtained are of a similar order. Also a general method of plotting stresses in finite elements is carried out.

Part Two describes the experimental work which was carried out in the University Structures Laboratories for incorporation within the thesis. Results are analysed and discussed in comparison with the results of the theoretical analysis.

Symbols and Abbreviations

1. The Finite Element Method.

z = vertical coordinate axis corresponding to the axis of revolution.

r = horizontal coordinate axis.

θ = horizontal angular coordinate measured from the r - z plane about the axis of symmetry.

u & v = displacements in the r & z directions, respectively.

$\{f\}$ = displacement function.

$\{\delta\}$ = nodal displacement matrix.

Δ = area of triangle.

$[N]$ = shape function.

$\{\epsilon\}$ = nodal strain matrix.

$\{\sigma\}$ = nodal stress matrix.

$[D]$ = elasticity matrix.

E = Young's Modulus.

ν = Poisson's Ratio.

$[B]$ = strain displacement matrix.

$[k]$ = stiffness matrix.

p_s = uniform applied pressure.

l_{ij} = distance between nodes i and j .

P_{ju} = total integrated horizontal load component at j .

P_{jv} = total integrated vertical load component at j .

Introduction

Objectives

The research objective was to produce an efficient general method of computing and plotting linear elastic stresses in a thick shell of revolution subjected to hydrostatic pressure. A second objective was to assess the validity of the theoretical results by carrying out experiments on micro-concrete models in a pressure vessel. An advantage could then be found in providing the values and locations of stress concentrations, and the positions where reinforcement should be placed along the lines of principal tensile stress. This design information is a primary requirement which should be satisfied to enable further research to proceed into related questions.

Applications

As a mature student with industrial experience the author felt the need to consider the future applications of research in relation to the needs of mankind, and found support for this attitude in the utterances of senior members of the engineering professions. Past President of the Institution of Civil Engineers, Sir Norman Rowntree, is quoted from his Presidential address, on future engineering research ; " If it is allowed, that five years is a reasonable time for a major research study, then a research programme should be based on the type of works likely to be required 15 years later, and research started now should be aimed at needs after 1990."¹ His successor, John Baxter, also calls for " better planning and more research " in his

Presidential address. He says, " The need is for wise long-term foresight instead of short-term expediency related to the next election, or even to the next by-election, and for policies related to human well being regardless of the careers and reputations of individuals."²

Today North Sea gas and oil are being successfully extracted south of the Shetlands by working from above sea-level on floating or fixed platforms which can operate efficiently in water depths of up to 150 fathoms. However, the water depth increases to 400 fathoms west and north of the Shetlands towards Spitsbergen, and there are further possible oil fields on the vast Russian continental shelf up to 1000 fathoms deep.

With the recent growth of the offshore oil industry the technology for working underwater has begun to develop. Divers are meanwhile carrying out essential tasks, and robot or semi-robot systems are being developed to work from inside submersibles.

The concept of a submerged concrete shell structure containing the apparatus for manned drilling on the sea-bed at atmospheric pressure is the motivation behind the thesis. To carry out such a project in practice would involve some of the most advanced branches of science and engineering, including life-support systems, Earth resources and energy studies, and in civil engineering further research would be required into new areas of concrete technology.

Although submersibles at present in use are constructed in steel plate, concrete offers better properties where the need is for a large working space. It is economical, can be constructed in the form of a thick shell, and the weight of the concrete shell

can balance the buoyancy of the air inside the structure to give approximately zero buoyancy so that it can be submerged and floated on a caisson.

The possibility remains of alternative applications arising unexpectedly in the future. The structural type may be useful in space or deep ocean, and stress analysis itself when improved can bring advantages to other branches of engineering. Professor Ben C. Gerwick of the University of Berkeley, California, and President of the Federation Internationale de la Precontrainte, has spoken³ of ocean thermal energy plant in the deep sea using the gravity anchor principle. The floating nuclear power stations of which he speaks need not necessarily float at the surface, however, and his argument for the use of concrete applies equally well for deep-sea conditions in a variety of cases. He says, "Deep water, soft soils, more severe waves, colder temperatures, earthquakes, and ice certainly stretch our capabilities to the extreme."

In Nature itself, in human artifacts from the domestic wine glass to space rocketry, the shell of revolution is a proven form for innumerable applications. In the future centuries it may be utilised by mankind on other planets as a radiation shield, and to resist high pressures, insulate against high temperature ranges, and effectively contain life-support systems. It may then be that something like concrete could be the only applicable material. Therefore the author sees his work as both a step in the right direction for the distant future, and also as applicable within his own working life.

Existing Knowledge

The literature on algebraic analysis is enormous, since each geometry may give rise to a unique set of non-linear equations. The Finite Element Method offers a general but approximate representation which can be improved progressively, depending mainly upon the capacity of the computer which is being used.

Some experiment work with thick mass concrete shells has been carried out by Harvey H. Haynes, Jerry D. Stachiw and L.F. Khan, of the United States Naval Civil Engineering Laboratory, Port Hueneme, California. Spherical and cylindrical models have shown that concrete can be used to advantage up to a depth of 1000 fathoms. Empirical equations have been devised from the test results and are at present being applied in offshore structural designs.^{4,5,6}

The work of Harvey H. Haynes has provided valuable information about this unique type of structure. The models were tested to destruction and showed two types of collapse mechanism; rapid implosion of the thinner shells, and local fracture in thicker shells eventually leading to implosion. For spherical models the critical shell thickness at which both types of failure occurred simultaneously was about one sixteenth of the outer diameter. The sphere is an ideal shape to resist uniform pressure, and has been used for deep-sea diving-bells. But other, flat-based, types of geometry may be more practical for working on the sea-bed.

In the field of Finite Element analysis the theory is clearly expounded by O.C. Zienkiewicz⁷ and a large number of works on the subject have less rather than more to offer. However, the theory is further described most elegantly by

J. Robinson⁸ who in developing the application to aircraft structures has thoroughly gone into the underlying assumptions which give rise to many different methods. He suggests the stress element as a promising line of research for large structures, although this would involve the inversion of a larger matrix than the displacement method on account of the need to go into redundant constraints.

A further text on the various element types is given in the paper by Bruce M. Irons⁹, and Clough and Rashid deal specifically with the axi-symmetric case¹⁰. The detailed and practical paper by Doherty, Wilson and Taylor¹¹ was most informative in adding to the guidelines of Zienkiewicz. Some of the element types therein were investigated and compared for performance with others suggested by Zienkiewicz. The method of treating the thick shell as a solid of revolution by integrating two-dimensional elements for a revolution about the axis of symmetry was adopted for the analysis.

The work of Timoshenko¹² and Flugge¹³ are the recognised classics in shell theory. What Timoshenko refers to as the "shell of constant strength" was analysed for the thesis by postulating a thin shell structure under hydrostatic loading with both constant thickness and constant membrane stresses. This resulted in a spiral meridian geometry with a flat base similar to the shell of a sea-urchin, and as depth of submergence increases the shape tends towards a sphere. This spherical shape is the condition of uniform pressure, and to facilitate both computation and the model analysis uniform pressure is assumed to represent deep-sea pressures.

The publications by the author's research supervisor,

Professor J.E. Gibson,^{14,15} cover the fields of thin shell theory, applied computer programming, and structural model engineering. The simplified thin shell theory is modified and adapted for certain types of geometry. The superiority of his approach over the finite element Method is demonstrated in an analysis of box beams.¹⁶

Other authors such as E.E. Abdelmigid¹⁷ at Cairo University have emphasized the need to develop alternative methods and check them against each other, and warned about the numerous types of error and problems of computer time and core involved in the Finite Element Method.

There are commercially available universal Finite Element programmes on the market, but they are both expensive and their results open to doubt. The experimental method is thus regarded as the most reliable check on the validity of the theories and computer methods.

Finally a subroutine has been written in the thesis for plotting the principal planes by computer, applying the basic assumption of constant strain over each element. Even with a wide mesh the computer capacity can be easily exceeded, but the general flow of stresses and their approximate order can be discerned from the plots which have been carried out.

PART ONE - THEORY
CHAPTER ONE

THE FINITE ELEMENT DISPLACEMENT METHOD

i) The plane triangular element.

This type of element has been dealt with in some detail by Zienkiewicz⁷ is further detailed here as some short cuts are introduced to facilitate the computer programming in the following chapter. The same notation has been adopted for convenience.

In the case of a solid of revolution, of which the thick shell of revolution is a type, the consideration of symmetry enables all stresses and strains to be specified with respect to those in any radial plane.

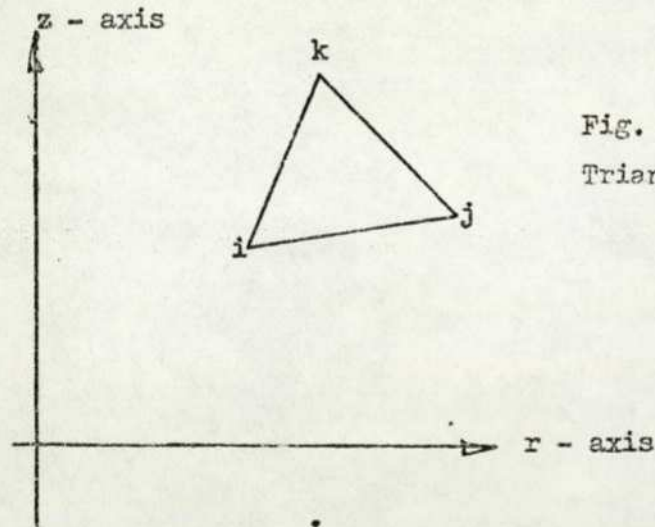


Fig. 1
Triangular Element

The z - axis is taken to be the axis of revolution and the r - axis is on the plane of the shell base where it is encastre. The origin is at the intersection of these axes. The triangular element nodes, i, j, k , are lettered anticlockwise. The nodal displacements are u and v in the r and z directions respectively.

Then the nodal displacements are written as

$$\{\delta\}^e = \begin{Bmatrix} \delta_i \\ \delta_j \\ \delta_k \end{Bmatrix} = \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_k \\ v_k \end{Bmatrix} \quad \text{--- (1)}$$

With six components of nodal displacement the simplest displacement function is a set of linear polynomials

$$\{f\} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} \alpha_1 + \alpha_2 r + \alpha_3 z \\ \alpha_4 + \alpha_5 r + \alpha_6 z \end{Bmatrix} \quad \text{--- (2)}$$

Then by substituting the nodal values we have six equations to solve for constants

$$u_i = \alpha_1 + \alpha_2 r_i + \alpha_3 z_i \quad \text{--- (3)}$$

$$u_j = \alpha_1 + \alpha_2 r_j + \alpha_3 z_j \quad \text{--- (4)}$$

$$u_k = \alpha_1 + \alpha_2 r_k + \alpha_3 z_k \quad \text{--- (5)}$$

$$v_i = \alpha_4 + \alpha_5 r_i + \alpha_6 z_i \quad \text{--- (6)}$$

$$v_j = \alpha_4 + \alpha_5 r_j + \alpha_6 z_j \quad \text{--- (7)}$$

$$v_k = \alpha_4 + \alpha_5 r_k + \alpha_6 z_k \quad \text{--- (8)}$$

Defining further constants

$$a_i = r_j z_k - r_k z_j$$

$$a_j = r_k z_i - r_i z_k$$

$$a_k = r_i z_j - r_j z_i$$

$$b_i = z_j z_k$$

$$b_j = z_k z_i$$

$$b_k = z_i z_j$$

$$c_i = r_k - r_j$$

$$c_j = r_i - r_k$$

$$c_k = r_j - r_i$$

Also $2\Delta = 2 \times (\text{Area of triangle } ijk)$

$$= \begin{vmatrix} 1 & r_i & z_i \\ 1 & r_j & z_j \\ 1 & r_k & z_k \end{vmatrix}$$

$$\begin{aligned}
 \therefore 2\Delta &= (r_j z_k - z_j r_k) - r_i (z_k - z_j) + z_i (r_k - r_j) \\
 &= (r_j z_k - z_j r_k) + (r_k z_i - r_i z_k) + (r_i z_j - r_j z_i) \\
 &= a_i + a_j + a_k
 \end{aligned}$$

$$(3) - (4)$$

$$(u_i - u_j) = -c_k \alpha_2 + b_k \alpha_3 \quad \text{--- (9)}$$

$$(3) - (5)$$

$$(u_i - u_k) = c_j \alpha_2 - b_j \alpha_3 \quad \text{--- (10)}$$

$$(9) \times c_j$$

$$(u_i - u_j) c_j = -c_k c_j \alpha_2 + b_k c_j \alpha_3 \quad \text{--- (11)}$$

$$(10) \times -c_k$$

$$(u_i - u_k)(-c_k) = -c_k c_j \alpha_2 + b_j c_k \alpha_3 \quad \text{--- (12)}$$

$$(11) - (12)$$

$$(u_i - u_j) c_j + (u_i - u_k) c_k = (b_k c_j - b_j c_k) \alpha_3$$

$$\therefore \alpha_3 = \frac{(c_j + c_k) u_i - c_j u_j - c_k u_k}{(b_k c_j - b_j c_k)} \quad \text{--- (13)}$$

$$(9) \times -b_j$$

$$(u_i - u_j)(-b_j) = b_j c_k \alpha_2 - b_j b_k \alpha_3 \quad \text{--- (14)}$$

$$(10) \times b_k$$

$$(u_i - u_k) b_k = b_k c_j \alpha_2 - b_j b_k \alpha_3 \quad \text{--- (15)}$$

$$(15) - (14)$$

$$(u_i - u_j) b_j + (u_i - u_k) b_k = (b_k c_j - b_j c_k) \alpha_2$$

$$\therefore \alpha_2 = \frac{(b_j + b_k) u_i - b_j u_j - b_k u_k}{(b_k c_j - b_j c_k)} \quad \text{--- (16)}$$

$$\text{But } (b_k c_j - b_j c_k)$$

$$= (z_i - z_j)(r_i - r_k) - (z_k - z_i)(r_j - r_i)$$

$$= \cancel{r_i z_i} - r_i z_j - r_k z_i + r_k z_j$$

$$- z_k r_j + r_i z_k + r_j z_i - \cancel{r_i z_i}$$

$$= (r_k z_j - z_k r_j) + (r_i z_k - r_k z_i) + (r_j z_i - r_i z_j)$$

$$= -(a_i + a_j + a_k) = -2\Delta$$

Also $b_j + b_k = -b_i$

And $c_j - c_k = -c_i$

∴ In (16)

$$\alpha_2 = \frac{b_i u_i + b_j u_j + b_k u_k}{2\Delta} \quad \text{--- (17)}$$

In (13)

$$\alpha_3 = \frac{c_i u_i + c_j u_j + c_k u_k}{2\Delta} \quad \text{--- (18)}$$

Substituting (17) and (18) into (3)

$$u_i = \alpha_1 + \frac{1}{2\Delta} [(b_i u_i + b_j u_j + b_k u_k) r_i + (c_i u_i + c_j u_j + c_k u_k) z_i] \quad \text{--- (19)}$$

$$\therefore \alpha_1 = \frac{1}{2\Delta} [(a_i + a_j + a_k - r_i b_i - z_i c_i) u_i - (r_i b_j + z_i c_j) u_j - (r_i b_k + z_i c_k) u_k] \quad \text{--- (20)}$$

Now $r_i b_i + z_i c_i$
 $= r_i z_j - r_i z_k + z_i r_k - z_i r_j$
 $= a_j + a_k$

Also $r_i b_j + z_i c_j$
 $= r_i z_k - \cancel{r_i z_i} + \cancel{r_j z_i} - r_k z_i$
 $= -a_j$

And $r_i b_k + z_i c_k$
 $= \cancel{r_i z_i} - r_i z_j + r_j z_i - \cancel{r_k z_i}$
 $= -a_k$

∴ In (20)

$$\alpha_1 = \frac{1}{2\Delta} (a_i u_i + a_j u_j + a_k u_k) \quad \text{--- (21)}$$

Substituting (17), (18) and (21) into (3)

$$u = \frac{1}{2\Delta} \left[(a_i u_i + a_j u_j + a_k u_k) \right. \\ \left. + (b_i u_i + b_j u_j + b_k u_k) r \right. \\ \left. + (c_i u_i + c_j u_j + c_k u_k) z \right]$$

Rearranging (22)

$$u = \frac{1}{2\Delta} \left[(a_i + b_i r + c_i z) u_i \right. \\ \left. + (a_j + b_j r + c_j z) u_j \right. \\ \left. + (a_k + b_k r + c_k z) u_k \right]$$

Similarly from equations (6), (7) and (8)

$$v = \frac{1}{2\Delta} \left[(a_i + b_i r + c_i z) v_i \right. \\ \left. + (a_j + b_j r + c_j z) v_j \right. \\ \left. + (a_k + b_k r + c_k z) v_k \right]$$

$$\therefore \{f\} = \begin{Bmatrix} u \\ v \end{Bmatrix} = [N] \{s\}^e \\ = [I N'_i, I N'_j, I N'_k] \{s\}^e$$

where $\{s\}^e$ is as in equation (1)

$[I]$ is a 2 X 2 identity matrix

and $N'_i = \frac{1}{2\Delta} (a_i + b_i r + c_i z)$

$N'_j = \frac{1}{2\Delta} (a_j + b_j r + c_j z)$

$N'_k = \frac{1}{2\Delta} (a_k + b_k r + c_k z)$

Definition of the shape function $[N]$ is given in Appendix I along with the general equations which follow.

Let θ be the angular coordinate about the axis of symmetry.

Due to symmetry there are four components of strain which are defined as

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_z \\ \epsilon_r \\ \epsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial r} \\ \frac{u}{r} \\ \frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \end{Bmatrix} \quad (27)$$

From (ii) in Appendix I

$$\begin{aligned} \{\epsilon\} &= [B] \{\delta\}^e \\ &= [B_i, B_j, B_k] \{\delta\}^e \end{aligned} \quad (28)$$

Substituting (25) and (26) into (27) and (28)

$$B_i = \frac{1}{2\Delta} \begin{bmatrix} 0 & \frac{\partial N_i}{\partial z} \\ \frac{\partial N_i}{\partial r} & 0 \\ \frac{N_i}{r} & 0 \\ \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial r} \end{bmatrix} = \frac{1}{2\Delta} \begin{bmatrix} 0 & c_i \\ b_i & 0 \\ \frac{a_i}{r} + b_i + \frac{c_i z}{r} & 0 \\ c_i & b_i \end{bmatrix} \quad (29)$$

and similarly for B_j and B_k

For the method which involves evaluating the $[B]$ matrix at the centroid of the element, row 3 in equation (29) can be further reduced by substituting the algebraic expansions of the terms therein

$$\begin{aligned} \text{e.g.} \quad & \frac{1}{2\Delta} (a_i/\bar{r} + b_i + c_i \bar{z}/\bar{r}) \\ & = \frac{1}{2\Delta} (a_i + b_i \bar{r} + c_i \bar{z})/\bar{r} \end{aligned}$$

cont.

$$\begin{aligned}
&= \frac{1}{2\Delta} \left[3a_i + (z_i - z_k)(v_i + v_j + v_k) \right. \\
&\quad \left. + (v_k - v_j)(z_i + z_j + z_k) \right] / 3\bar{r} \\
&= \frac{1}{2\Delta} \left[\cancel{3}(v_j z_k - v_k z_j) + (v_i z_j + \cancel{v_j z_j} + \cancel{v_k z_j} - v_i z_k \right. \\
&\quad \left. - \cancel{v_j z_k} - \cancel{v_k z_k} + v_k z_i + \cancel{v_k z_j} + \cancel{v_k z_k} - v_j z_i - \cancel{v_j z_j} \right. \\
&\quad \left. - \cancel{v_j z_k} \right) \right] / 3\bar{r} \\
&= \frac{1}{2\Delta} (a_i + a_j + a_k) / 3\bar{r} \\
&= 1/3\bar{r}
\end{aligned}$$

This must also follow for the 3rd row of B_j and B_k

$$\therefore \bar{B}_i = \frac{1}{2\Delta} \begin{bmatrix} 0 & c_i \\ b_i & 0 \\ 1/3\bar{r} & 0 \\ c_i & b_i \end{bmatrix} \quad \text{--- (30)}$$

We next require the Elasticity matrix.

There are four components of stress corresponding to the strain given in equation (iii) Appendix I

$$\{\sigma\} = \begin{Bmatrix} \sigma_z \\ \sigma_r \\ \sigma_\theta \\ \tau_{rz} \end{Bmatrix} = [D] \{\epsilon\} \quad \text{--- (31)}$$

where $[D]$ is the 4 X 4 Elasticity Matrix.

For an isotropic material

$$E \epsilon_z = \sigma_z - \nu \sigma_r - \nu \sigma_\theta \quad \text{--- (32)}$$

$$E \epsilon_r = -\nu \sigma_z + \sigma_r - \nu \sigma_\theta \quad \text{--- (33)}$$

$$E \epsilon_\theta = -\nu \sigma_z - \nu \sigma_r + \sigma_\theta \quad \text{--- (34)}$$

$$E \gamma_{rz} = 2(1 + \nu) \tau_{rz} \quad \text{--- (35)}$$

where E is Young's Modulus

ν is Poisson's Ratio

From (35)

$$\gamma_{rz} = \frac{E \gamma_{rz}}{2(1+\nu)} \quad \text{--- (36)}$$

From (32)

$$\sigma_z = E \epsilon_z + \nu(\sigma_r + \sigma_\theta) \quad \text{--- (37)}$$

Substituting (37) into (33)

$$E \epsilon_r = -\nu E \epsilon_z - \nu^2(\sigma_r + \sigma_\theta) + \sigma_r - \nu \sigma_\theta$$

$$\therefore E \left(\frac{\epsilon_r + \nu \epsilon_z}{(1+\nu)} \right) = (1-\nu) \sigma_r - \nu \sigma_\theta \quad \text{--- (38)}$$

$$(33) - (34) \quad E(\epsilon_r - \epsilon_\theta) = (1+\nu)(\sigma_r - \sigma_\theta)$$

$$\therefore \sigma_\theta = \sigma_r - E \frac{(\epsilon_r - \epsilon_\theta)}{(1+\nu)} \quad \text{--- (39)}$$

Substituting (39) into (38)

$$(1-\nu) \sigma_r - \nu \sigma_r = \frac{E}{(1-\nu)} [(\epsilon_r + \nu \epsilon_z) - \nu(\epsilon_r - \epsilon_\theta)]$$

$$\therefore \sigma_r = \frac{E}{(1+\nu)(1-2\nu)} [\nu \epsilon_z + (1-\nu) \epsilon_r + \nu \epsilon_\theta] \quad \text{--- (40)}$$

Similarly for σ_z and σ_θ

$$\therefore [D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & \nu & 0 \\ \nu & (1-\nu) & \nu & 0 \\ \nu & \nu & (1-\nu) & 0 \\ 0 & 0 & 0 & 0.5(1-2\nu) \end{bmatrix} \quad \text{--- (41)}$$

Finally with regard to the element characteristic matrices we require the element stiffness matrix which is formed from equation (vi) in Appendix I

$$\text{where } [K]^e = \int [B]^T [D] [B] d(\text{vol}).$$

Since the $[B]$ matrix depends upon the coordinates chosen, it is a convenient approximation to take them at the centroid of the triangle so that integration is carried out by applying Pappus' Theorem.

If $[\bar{B}]$ is the $[B]$ matrix at the centroid

$$\text{where } \bar{r} = \frac{1}{3}(r_i + r_j + r_k)$$

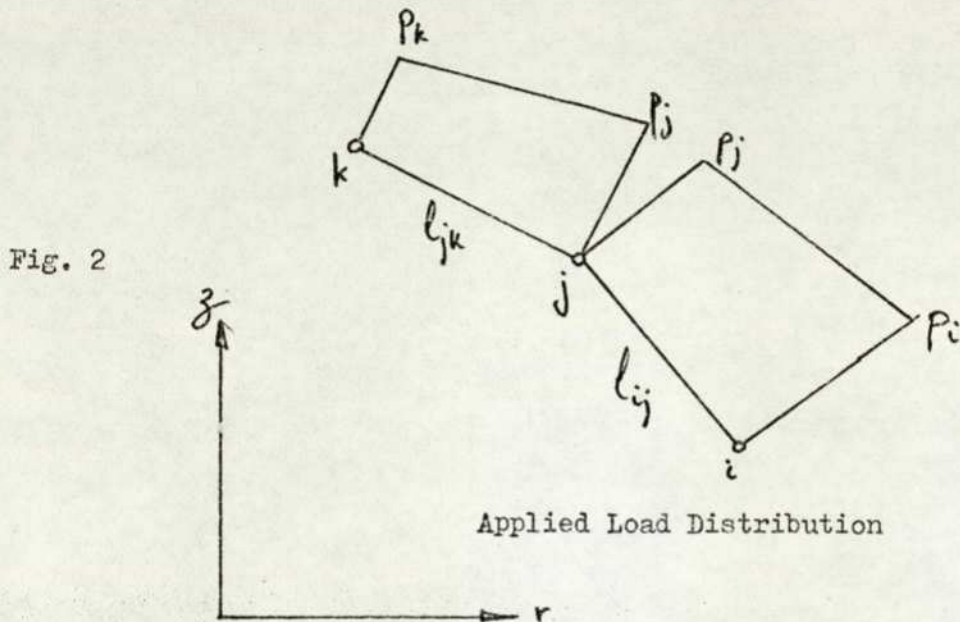
$$\text{and } \bar{z} = \frac{1}{3}(z_i + z_j + z_k)$$

are the coordinates at the centroid

then

$$[k]^e = [\bar{B}]^T [D] [\bar{B}] \cdot 2\pi \bar{r} \Delta \quad \text{--- (42)}$$

ii) Applied loading.



where p_s = uniform applied pressure

$$p_i = 2\pi p_s r_i$$

$$p_j = 2\pi p_s r_j$$

$$p_k = 2\pi p_s r_k$$

where p_i , p_j , and p_k are the integrated boundary loads at nodes i, j, and k respectively.

Reaction at j due to l_{ij}

$$\begin{aligned} &= p_{ij} \frac{l_{ij}}{2} + (p_i - p_j) \frac{l_{ij}}{6} \\ &= \frac{l_{ij}}{6} (p_i + 2p_j) \end{aligned}$$

cont.

$$= \frac{\pi p_s}{3} l_{ij} (r_i + 2r_j)$$

$$\text{Horizontal component} = \frac{\pi p_s}{3} (r_i + 2r_j) (z_j - z_i)$$

$$\text{Vertical component} = \frac{\pi p_s}{3} (r_i + 2r_j) (r_i - r_j)$$

Similarly, reaction at j due to l_{jk}

$$= \frac{\pi p_s}{3} l_{jk} (2r_j + r_k)$$

$$\text{Horizontal component} = \frac{\pi p_s}{3} (2r_j + r_k) (z_k - z_j)$$

$$\text{Vertical component} = \frac{\pi p_s}{3} (2r_j + r_k) (r_j - r_k)$$

$$\therefore P_{ju} = \frac{\pi p_s}{3} [(r_i + 2r_j) (z_j - z_i) + (2r_j + r_k) (z_k - z_j)]$$

$$P_{jv} = \frac{\pi p_s}{3} [(r_i + 2r_j) (r_i - r_j) + (2r_j + r_k) (r_j - r_k)]$$

where P_{ju} is the total horizontal reaction at j

and P_{jv} is the total vertical reaction at j.

$$\therefore P_{ju} = \frac{\pi p_s}{3} [r_i z_i - r_i z_i + 2r_j z_j - 2r_j z_i + 2r_j z_k - 2r_j z_j + r_k z_k - r_k z_j]$$

$$= \frac{\pi p_s}{3} [r_i (z_j - z_i) + 2r_j (z_k - z_i) + r_k (z_k - z_i)] \quad \text{--- (43)}$$

$$P_{jv} = \frac{\pi p_s}{3} [r_i^2 - r_i r_j + 2r_i r_j - 2r_j^2 + 2r_j^2 - 2r_j r_k + r_j r_k - r_k^2]$$

$$= \frac{\pi p_s}{3} [r_i^2 + r_i r_j - r_k^2 - r_j r_k]$$

$$= \frac{\pi p_s}{3} [(r_i + r_k) (r_i - r_k) + r_j (r_i - r_k)]$$

$$= \frac{\pi p_s}{3} (r_i - r_k) (r_i + r_j + r_k) \quad \text{--- (44)}$$

iii) Boundary conditions.

The stiffness matrix at this stage is singular because the support conditions have not been applied. It is convenient to keep the same number of equations in computer storage, and so, instead of substituting boundary conditions directly, the diagonal coefficient of matrix $[k]$, which corresponds to a known boundary condition in $\{s\}$, is multiplied by a large number, say 10^8 . The term on the right-hand side of the equation is replaced by the known displacement multiplied by the same large number. Thus the equation concerned is replaced by one which gives the specified boundary displacement condition without removing it from the overall matrix.

iv) Equation solving.

Many methods exist for the solution of simultaneous equations, either by matrix inversion, or by methods involving both sides of the equation.

In this case the matrix $[k]$ is symmetrical and banded. Since one of the points which limits the application of the Finite Element Method lies in the available computer storage capacity, there are ^{s in which} standard subroutines ~~which store~~ half the symmetrical band width only is stored as a column matrix.

The program efficiency depends very much on careful numbering and positioning of nodes in the first place to reduce the band width to a minimum, while maintaining a desirable finite element mesh layout.

v) Principal Stresses.

When displacements are known the stresses in each element at the centroid are found directly from equation (iv) in Appendix I.

$$\{\sigma\} = [D] [B] \{\delta\}^e$$

in which from equation (31)

$$\{\sigma\} = \begin{Bmatrix} \sigma_z \\ \sigma_r \\ \sigma_\theta \\ \tau_{rz} \end{Bmatrix}$$

Since there is no shear stress in the θ direction, the circumferential stress, σ_θ , is a principal stress. It is only necessary, therefore, to find the principal stresses in the $r - z$ plane.

If a principal plane is defined as that which has a zero shear stress, the principal stresses can be found by resolving forces in the r and z directions.

Consider a triangular prism of unit width with forces acting upon the two coordinate planes, and the principal stress, σ_p , at angle ϕ to the r -axis acting upon the hypotenuse of unit length.

Tensile stresses are taken as positive and ϕ is positive above the r -axis.

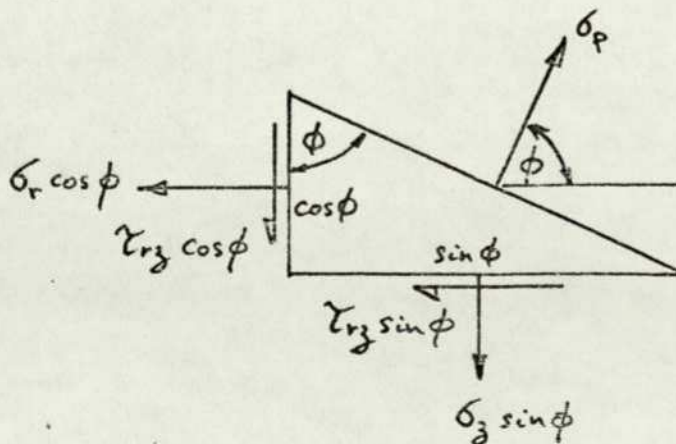


Fig. 3 Principal Stress Prism

For equilibrium

$$\begin{aligned}\sigma_p \sin \phi &= \sigma_r \sin \phi + \tau_{rz} \cos \phi \\ \therefore \sigma_p - \sigma_r &= \tau_{rz} \cot \phi \quad \text{--- (45)}\end{aligned}$$

and

$$\begin{aligned}\sigma_p \cos \phi &= \sigma_z \cos \phi + \tau_{rz} \sin \phi \\ \therefore \sigma_p - \sigma_z &= \tau_{rz} \tan \phi \quad \text{--- (46)}\end{aligned}$$

Multiplying (45) $\times$ (46)

$$\begin{aligned}(\sigma_p - \sigma_r)(\sigma_p - \sigma_z) &= \tau_{rz}^2 \\ \therefore \sigma_p^2 - (\sigma_r + \sigma_z)\sigma_p + (\sigma_r\sigma_z - \tau_{rz}^2) &= 0 \\ \therefore \sigma_p &= \frac{1}{2}(\sigma_r + \sigma_z) + \frac{1}{2}\sqrt{(\sigma_r + \sigma_z)^2 - 4(\sigma_r\sigma_z - \tau_{rz}^2)} \\ \therefore \sigma_p &= \frac{1}{2}(\sigma_r + \sigma_z) + \frac{1}{2}\sqrt{(\sigma_r - \sigma_z)^2 + 4\tau_{rz}^2} \quad \text{--- (47)}\end{aligned}$$

and a second solution

$$\sigma_q = \frac{1}{2}(\sigma_r + \sigma_z) - \frac{1}{2}\sqrt{(\sigma_r - \sigma_z)^2 + 4\tau_{rz}^2} \quad \text{--- (48)}$$

This gives two principal stresses, σ_p and σ_q , which are found by Mohr's Stress Circle Theory to act at right-angles to each other.

To draw Mohr's Circle on σ - τ axis the centre is at

$$\begin{aligned}\sigma &= \frac{1}{2}(\sigma_r + \sigma_z) \quad \text{--- (49)} \\ \text{and} \quad \tau &= 0\end{aligned}$$

while the radius

$$\tau_{\max} = \frac{1}{2}\sqrt{(\sigma_r - \sigma_z)^2 + 4\tau_{rz}^2} \quad \text{--- (50)}$$

gives the maximum shear stress acting at half a right-angle to the principal stresses.

vi) Quadrilateral elements.

So far the finite element method has been applied using only the constant strain triangle (CST) element in the displacement method, and the integral has been taken by application of Pappus' Theorem. Next a quadrilateral element may be formed at the end of the computation by averaging the centroidal stresses in adjacent triangles. With an arrangement of triangles which readily pair off into quadrilaterals some of the oscillating numerical results can be more evenly distributed to give consistent results. This type of quadrilateral may be abbreviated as the Q2CST in order to place it within the hierarchy of elements which follow. The simple method of taking averages is commonly used in the Finite Element Method in order to restore the principles of continuity and compatibility.

Further techniques aimed at closer representation include the use of nodeless variables formed by element substructuring, numerical integration, and local coordinates with skew axes. These are introduced in the elements which follow, progressing to the curved quadrilateral element.

vii) The quadrilateral element formed by four constant strain triangles. (Q4CST).

In addition to the four corner nodes which form the Q4CST element there is a fifth node within the element where the four CST subelements meet. This extra node could be anywhere within the element, and is taken for convenience at the centroid of the quadrilateral. In order to form a stiffness matrix it is necessary to reduce it by eliminating the matrix terms due to the centroidal node

in such a way that they are included in the terms for the corner nodes as nodeless variables. The assembled stiffness matrix may then be the same size as for the simpler case of CST elements.

The equation relating total potential energy to combined element properties is

$$\frac{\partial \chi^e}{\partial \{\delta\}^e} = [k]^e \{\delta\}^e + \{F\}^e \quad \text{--- (51)}$$

where χ^e is the total potential energy for the element.

Subdividing $\{\delta\}^e$ into those parts, $\{\bar{\delta}\}^e$, which are common to other elements or external constraints, and others, $\{\bar{\bar{\delta}}\}^e$, internal to the element, then

$$\frac{\partial \chi^e}{\partial \{\bar{\bar{\delta}}\}^e} = 0 \quad \text{--- (52)}$$

and equation (51) can be reformulated

$$\begin{aligned} \frac{\partial \chi^e}{\partial \{\delta\}^e} &= \begin{Bmatrix} \frac{\partial \chi^e}{\partial \{\bar{\delta}\}^e} \\ \frac{\partial \chi^e}{\partial \{\bar{\bar{\delta}}\}^e} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \chi^e}{\partial \{\bar{\delta}\}^e} \\ 0 \end{Bmatrix} \\ &= \begin{bmatrix} [\bar{k}]^e & [\hat{k}]^e \\ [\hat{k}]^{eT} & [\bar{k}]^e \end{bmatrix} \begin{Bmatrix} \{\bar{\delta}\}^e \\ \{\bar{\bar{\delta}}\}^e \end{Bmatrix} + \begin{Bmatrix} \{\bar{F}\}^e \\ \{\bar{\bar{F}}\}^e \end{Bmatrix} \quad \text{--- (53)} \end{aligned}$$

From the second set of equations in (53) the deflexions of the internal node can be expressed in terms of those for the corner nodes

$$\{\bar{\bar{\delta}}\}^e = -[\bar{k}]^{e-1} ([\hat{k}]^{eT} \{\bar{\delta}\}^e + \{\bar{\bar{F}}\}^e) \quad \text{--- (54)}$$

Substituting (54) into the first set of equations in (53)

$$\frac{\partial \chi^e}{\partial \{\bar{\delta}\}^e} = [k^*]^e \{\bar{\delta}\}^e + \{F^*\}^e \quad \text{--- (55)}$$

$$\text{where } [k^*]^e = [\bar{k}]^e - [\hat{k}]^e [\bar{k}]^{e-1} [\hat{k}]^{eT} \quad \text{--- (56)}$$

$$\{F^*\}^e = \{\bar{F}\}^e - [\hat{k}]^e [\bar{k}]^{e-1} \{\bar{\bar{F}}\}^e \quad \text{--- (57)}$$

are the reduced element properties and loadings.

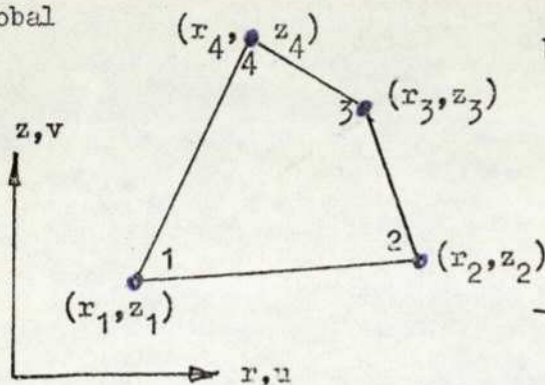
viii) Plane quadrilateral element. (Q4).

A simple mathematical model may be constructed conveniently by specifying the shape function in terms of local element coordinates on skew axes.

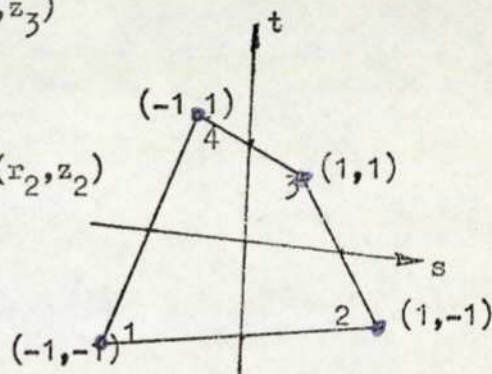
By numbering the element nodes 1 to 4 as shown the global and local coordinates are related by the interpolation functions.

Fig. 4 Plane Quadrilateral Element Co-ordinates

a) Global



b) Local



$$\begin{aligned} r(s, t) &= \sum_{i=1}^4 h_i r_i \\ z(s, t) &= \sum_{i=1}^4 h_i z_i \end{aligned} \quad \text{----- (58)}$$

$$\begin{aligned} \text{where } h_1 &= (1-s)(1-t)/4 \\ h_2 &= (1+s)(1-t)/4 \\ h_3 &= (1+s)(1+t)/4 \\ h_4 &= (1-s)(1+t)/4 \end{aligned} \quad \text{----- (59)}$$

It is next convenient to designate the Q4 as an isoparametric element and to use the Interpolation Functions also as Shape Functions of the same form.

$$\therefore u(s,t) = \sum_{i=1}^4 h_i u_i$$

$$v(s,t) = \sum_{i=1}^4 h_i v_i \quad \text{----- (60)}$$

where u, v are displacements along global axes, and h is defined as in the Interpolation Functions.

In order to derive the strain/displacement matrix from the Shape Functions it is necessary to transform partial differentials with respect to global coordinates into partial differentials with respect to local coordinates using a transformation matrix. Since the global coordinates are defined in terms of local coordinates, we can write :

$$\begin{Bmatrix} \frac{\partial}{\partial s} \\ \frac{\partial}{\partial t} \end{Bmatrix} = \begin{bmatrix} \frac{\partial r}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial r}{\partial t} & \frac{\partial z}{\partial t} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial z} \end{Bmatrix} \quad \text{----- (61)}$$

Solving (61)

$$\begin{Bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial z} \end{Bmatrix} = \frac{1}{J} \begin{bmatrix} \frac{\partial z}{\partial t} & -\frac{\partial z}{\partial s} \\ -\frac{\partial r}{\partial t} & \frac{\partial r}{\partial s} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial s} \\ \frac{\partial}{\partial t} \end{Bmatrix} \quad \text{----- (62)}$$

where $J = J(s,t) = \frac{\partial r}{\partial s} \frac{\partial z}{\partial t} - \frac{\partial r}{\partial t} \frac{\partial z}{\partial s} \quad \text{----- (63)}$

The Jacobian, J , is required in explicit form.

Substituting equations (58) and (59) into (63) :

$$J = \sum_{i=1}^4 \sum_{j=1}^4 r_i \left(\frac{\partial h_i}{\partial s} \frac{\partial h_j}{\partial t} - \frac{\partial h_i}{\partial t} \frac{\partial h_j}{\partial s} \right) z_j$$

$$= \sum_{i=1}^4 \sum_{j=1}^4 r_i p_{ij} z_j$$

$$\therefore J = \{r_i\}^T [P] \{z_j\} \quad \text{----- (64)}$$

where

$$\{r_i\}^T = [r_1 \ r_2 \ r_3 \ r_4]$$

$$\{z_j\}^T = [z_1 \ z_2 \ z_3 \ z_4]$$

and

$$[P] = \sum_{i=1}^4 \sum_{j=1}^4 \left(\frac{\partial h_i}{\partial s} \frac{\partial h_j}{\partial t} - \frac{\partial h_i}{\partial t} \frac{\partial h_j}{\partial s} \right)$$

$$\therefore [P] = \frac{1}{8} \begin{bmatrix} 0 & 1-t & -s+t & -1+s \\ & 0 & 1+s & -s-t \\ & & 0 & 1+t \\ \text{skew-symmetric} & & & 0 \end{bmatrix} \quad \text{-----}(65)$$

Multiplying out equation (64), first of all

$$\begin{aligned} [P] \{z'\} &= \frac{1}{8} \begin{bmatrix} 0 & 1-t & -s+t & -1+s \\ -1+t & 0 & 1+s & -s-t \\ s-t & -1-s & 0 & 1+t \\ 1-s & s+t & -1-t & 0 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{Bmatrix} \\ &= \frac{1}{8} \begin{Bmatrix} z_{24} - s z_{34} - t z_{23} \\ -z_{13} + s z_{34} + t z_{14} \\ -z_{24} + s z_{12} - t z_{14} \\ z_{13} - s z_{12} + t z_{23} \end{Bmatrix} \quad \text{-----}(66) \end{aligned}$$

$$\text{where } r_{ij} = r_i - r_j$$

$$z_{ij} = z_i - z_j$$

$$\begin{aligned} \therefore J &= \{r'\}^T [P] \{z'\} \\ &= \frac{1}{8} [r_1 \ r_2 \ r_3 \ r_4] \begin{bmatrix} z_{24} - s z_{34} - t z_{23} \\ -z_{13} + s z_{34} + t z_{14} \\ -z_{24} + s z_{12} - t z_{14} \\ z_{13} - s z_{12} + t z_{23} \end{bmatrix} \end{aligned}$$

$$\therefore J = \frac{1}{8} \left\{ (r_{13} z_{24} - r_{24} z_{13}) + s (r_{34} z_{12} - r_{12} z_{34}) + t (r_{23} z_{14} - r_{14} z_{23}) \right\} \quad \text{-----}(67)$$

The matrix $[B]$ in the equation, $\{E\} = [B] \{\delta\}$, is now found from the general equation for the axi-symmetric case :

$$\{E\} = \begin{Bmatrix} E_z \\ E_r \\ E_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial r} \\ \frac{u}{r} \\ \frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \end{Bmatrix} \quad \text{--- (68)}$$

We have

$$E_\theta = \frac{u}{r} = \sum_{i=1}^4 \frac{h_i u_i}{r} = \sum_{i=1}^4 G_i u_i \quad \text{--- (69)}$$

$$E_r = \frac{\partial u}{\partial r} = \frac{\partial u}{\partial s} \frac{\partial s}{\partial r} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial r} \quad \text{--- (70)}$$

Applying the transformation matrix of equation 62, then

$$E_r = \frac{1}{J} \sum_{i=1}^4 \sum_{j=1}^4 u_i \left(\frac{\partial h_i}{\partial s} \frac{\partial h_j}{\partial t} - \frac{\partial h_i}{\partial t} \frac{\partial h_j}{\partial s} \right) z_j$$

$$\therefore E_r = \frac{\{u\}^T [P] \{z'\}}{J} \quad \text{--- (71)}$$

Similarly

$$E_z = \frac{\partial v}{\partial z} = \frac{\partial v}{\partial s} \frac{\partial s}{\partial z} + \frac{\partial v}{\partial t} \frac{\partial t}{\partial z} \quad \text{--- (72)}$$

Whence again

$$E_z = - \frac{\{v'\}^T [P] \{r'\}}{J} \quad \text{--- (73)}$$

Finally

$$\begin{aligned} \gamma_{rz} &= \frac{\partial v}{\partial r} + \frac{\partial u}{\partial z} \\ &= \frac{\partial v}{\partial s} \frac{\partial s}{\partial r} + \frac{\partial v}{\partial t} \frac{\partial t}{\partial r} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial z} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial z} \end{aligned} \quad \text{--- (74)}$$

and

$$\gamma_{rz} = - \frac{\{u\}^T [P] \{r'\}}{J} + \frac{\{v'\}^T [P] \{z'\}}{J} \quad \text{--- (75)}$$

Now let

$$\frac{[P]\{z'\}}{J} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} = \{y\}$$

76

and

$$-\frac{[P]\{r'\}}{J} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \{x\}$$

Equations (76) are found explicitly from (66)

Thus $x_1 = -\frac{1}{8J} (r_{24} - 5r_{34} - tr_{23})$

$$y_1 = \frac{1}{8J} (z_{24} - 5z_{34} - tz_{23})$$

77

etc.

$$\therefore \begin{Bmatrix} \varepsilon_z \\ \varepsilon_r \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^4 (x_i v_i) \\ \sum_{i=1}^4 (y_i u_i) \\ \sum_{i=1}^4 (z_i w_i) \\ \sum_{i=1}^4 (x_i u_i + y_i v_i) \end{Bmatrix}$$

78

$$\therefore [B_i] = \begin{Bmatrix} 0 & x_i \\ y_i & 0 \\ z_i & 0 \\ x_i & y_i \end{Bmatrix}$$

79

The stiffness matrix is found from the equation

$$[K] = \int \int \int [B]^T [D] [B] r d\theta dr dz$$

We have

$$[K] = \int \int 2\pi [B]^T [D] [B] r dr dz \quad \text{--- (80)}$$

Transforming equation (80) into local coordinates :

$$[K] = 2\pi \int_{-1}^1 \int_{-1}^1 [B]^T [D] [B] r J ds dt \quad \text{--- (81)}$$

This may be integrated numerically by several possible methods using weighting factors.

Using an N-point scheme

$$[K] = 2\pi \sum_{i=1}^N [B(s_i, t_i)]^T [D] [B(s_i, t_i)]$$

$$r_i J(s_i, t_i) w_i \quad \text{--- (82)}$$

where w_i is the weighting factor.

The Gaussian quadrature with two points in each direction, giving four points altogether, has weighting factors = 1, and $N = 4$.

$$\begin{array}{ll} s_1 = a_1 & t_1 = a_1 \\ s_2 = a_1 & t_2 = -a_1 \\ s_3 = -a_1 & t_3 = a_1 \\ s_4 = -a_1 & t_4 = -a_1 \end{array} \quad \text{--- (83)}$$

$$\text{where } a_1 = 0.57735 \ 02691 \ 89626 \quad \text{--- (84)}$$

ix) The Q5 element.

The Q5 element is derived from the Q4 element by the addition of a nodeless variable to the Shape Function.

$$\begin{aligned} u &= \sum_{i=1}^4 h_i u_i + (1-s^2)(1-t^2) u_c \\ v &= \sum_{i=1}^4 h_i v_i + (1-s^2)(1-t^2) v_c \end{aligned} \quad \text{--- (85)}$$

From this the displacement matrix has two more columns

$$[B_c] = \begin{bmatrix} 0 & x_c \\ y_c & 0 \\ q_c & 0 \\ x_c & y_c \end{bmatrix} \quad \text{--- (86)}$$

Where

$$\begin{aligned} q_c &= \frac{(1-s^2)(1-t^2)}{r} \\ x_c &= \frac{\partial}{\partial z} \left\{ (1-s^2)(1-t^2) \right\} \\ y_c &= \frac{\partial}{\partial r} \left\{ (1-s^2)(1-t^2) \right\} \end{aligned} \quad \text{--- (87)}$$

u_c and v_c are displacements at the origin, C, of the local axes relative to that specified by the Shape Functions for the corner nodes.

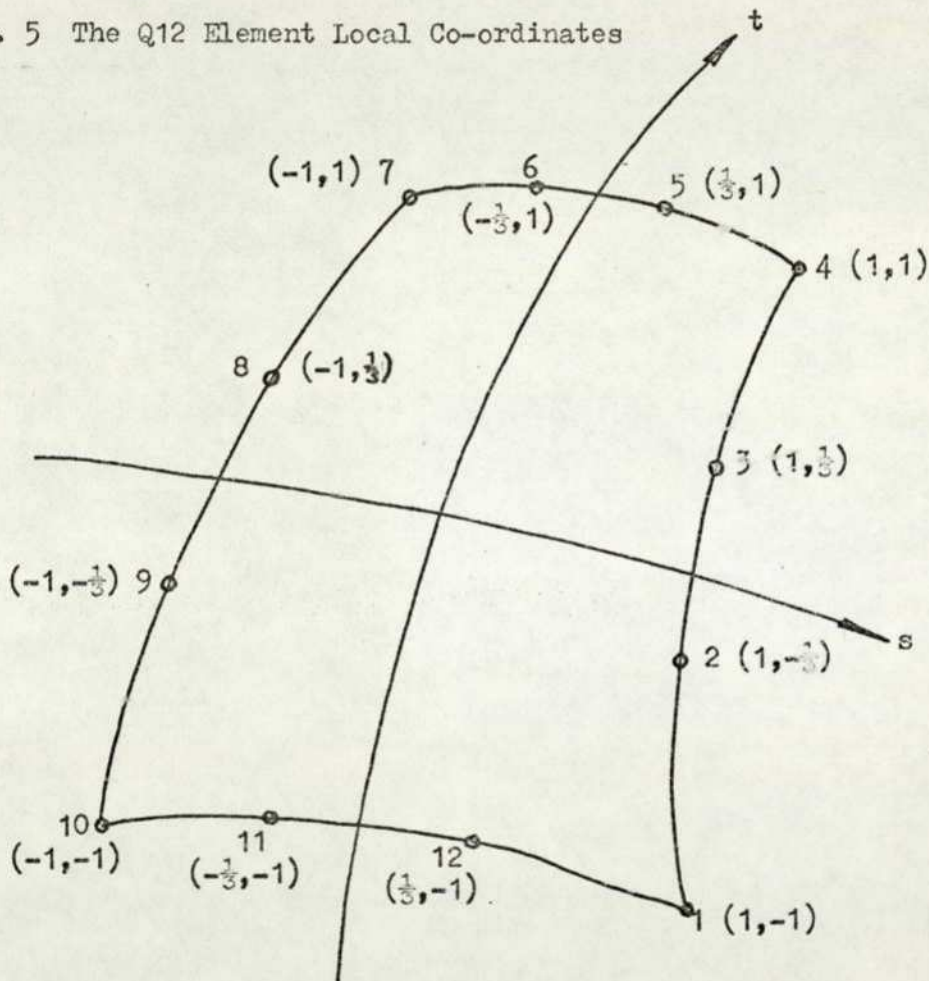
After the element stiffness matrix has been formed, the internal node is eliminated as for the Q4CST element, before assembly into the structural stiffness matrix.

x) The Cubic Serendipity Element. (Q12).

The general expressions which have been derived for the Q4 element may also be used in an analysis which employs a higher-order element.

The number of nodes and corresponding Shape Functions will differ, however, and it is proposed to apply the Q12 element to the shell of revolution.

Fig. 5 The Q12 Element Local Co-ordinates



The isoparametric element geometry is defined by the Interpolation Functions being used as Shape Functions. These are found by inspection as follows :

$$\begin{aligned}
h_1 &= \frac{1}{32} (1+s)(1-t) \{-10+9(s^2+t^2)\} \\
h_2 &= \frac{9}{32} (1+s)(1-t^2)(1-3t) \\
h_3 &= \frac{9}{32} (1+s)(1-t^2)(1+3t) \\
h_4 &= \frac{1}{32} (1+s)(1+t) \{-10+9(s^2+t^2)\} \\
h_5 &= \frac{9}{32} (1+t)(1-s^2)(1+3s) \\
h_6 &= \frac{9}{32} (1+t)(1-s^2)(1-3s) \\
h_7 &= \frac{1}{32} (1-s)(1+t) \{-10+9(s^2+t^2)\} \\
h_8 &= \frac{9}{32} (1-s)(1-t^2)(1+3t) \\
h_9 &= \frac{9}{32} (1-s)(1-t^2)(1-3t) \\
h_{10} &= \frac{1}{32} (1-s)(1-t) \{-10+9(s^2+t^2)\} \\
h_{11} &= \frac{9}{32} (1-t)(1-s^2)(1-3s) \\
h_{12} &= \frac{9}{32} (1-t)(1-s^2)(1+3s)
\end{aligned}$$

equations (88)

In order to form the strain/displacement matrix the first-order differentials with respect to s and t are required :

$$\begin{aligned}
\frac{\partial h_1}{\partial s} &= \frac{1}{32} (1-t)(-10+18s+27s^2+9t^2) \\
\frac{\partial h_2}{\partial s} &= \frac{9}{32} (1-t^2)(1-3t) \\
\frac{\partial h_3}{\partial s} &= \frac{9}{32} (1-t^2)(1+3t) \\
\frac{\partial h_4}{\partial s} &= \frac{1}{32} (1+t)(-10+18s+27s^2+9t^2) \\
\frac{\partial h_5}{\partial s} &= \frac{9}{32} (1+t)(3-2s-9s^2) \\
\frac{\partial h_6}{\partial s} &= \frac{9}{32} (1+t)(-3-2s+9s^2) \\
\frac{\partial h_7}{\partial s} &= \frac{1}{32} (1+t)(10+18s-27s^2-9t^2) \\
\frac{\partial h_8}{\partial s} &= -\frac{9}{32} (1-t^2)(1+3t) \\
\frac{\partial h_9}{\partial s} &= -\frac{9}{32} (1-t^2)(1-3t) \\
\frac{\partial h_{10}}{\partial s} &= \frac{1}{32} (1-t)(10+18s-27s^2-9t^2) \\
\frac{\partial h_{11}}{\partial s} &= \frac{9}{32} (1-t)(-3-2s+9s^2) \\
\frac{\partial h_{12}}{\partial s} &= \frac{9}{32} (1-t)(3-2s-9s^2)
\end{aligned}$$

equations (89)

$$\begin{aligned}
\frac{\partial h_1}{\partial t} &= \frac{1}{32} (1+s)(10-9s^2+18t-27t^2) \\
\frac{\partial h_2}{\partial t} &= \frac{9}{32} (1+s)(-3-2t+9t^2) \\
\frac{\partial h_3}{\partial t} &= \frac{9}{32} (1+s)(3-2t-9t^2) \\
\frac{\partial h_4}{\partial t} &= \frac{1}{32} (1+s)(-10+9s^2+18t+27t^2) \\
\frac{\partial h_5}{\partial t} &= \frac{9}{32} (1-s^2)(1+3s) \\
\frac{\partial h_6}{\partial t} &= \frac{9}{32} (1-s^2)(1-3s) \\
\frac{\partial h_7}{\partial t} &= \frac{1}{32} (1-s)(-10+9s^2+18t+27t^2) \\
\frac{\partial h_8}{\partial t} &= \frac{9}{32} (1-s)(3-2t-9t^2) \\
\frac{\partial h_9}{\partial t} &= \frac{9}{32} (1-s)(-3-2t+9t^2) \\
\frac{\partial h_{10}}{\partial t} &= \frac{1}{32} (1-s)(10-9s^2+18t-27t^2) \\
\frac{\partial h_{11}}{\partial t} &= -\frac{9}{32} (1-s^2)(1-3s) \\
\frac{\partial h_{12}}{\partial t} &= -\frac{9}{32} (1-s^2)(1+3s)
\end{aligned}$$

equations (90)

The matrix $[P]$ in which

$$p_{ij} = \frac{\partial h_i}{\partial s^i} \frac{\partial h_j}{\partial t^j} - \frac{\partial h_j}{\partial t^j} \frac{\partial h_i}{\partial s^i} \quad \text{---(91)}$$

has again

$$p_{ii} = 0 \quad \text{---(92)}$$

$$p_{ji} = -p_{ij} \quad \text{---(93)}$$

and, instead of forming it algebraically, it is more efficiently formed numerically by multiplying the values of the differentials out in the computer program.

The element stiffness matrix is formed by numerical integration with a 3 X 3 - point Gaussian Quadrature.

For the distribution of applied loading on the element boundary nodes we apply the equation which is derived by Zienkiewicz from virtual work principles :

$$\{F\}_b^e = - \int [N]^T \{g\} d(\text{area}) \quad \text{---(94)}$$

where $[N]$ is the Shape Function

$\{g\}$ is the distributed load

and $d(\text{area})$ the differential element upon which

$\{g\}$ is applied.

For the case in point the nodes are placed so that only the first four lie on the boundary under applied pressure, so :

$$s = 1$$

$$[N] = \begin{Bmatrix} h_1 & 0 & h_2 & 0 & h_3 & 0 & h_4 & 0 \\ 0 & h_1 & 0 & h_2 & 0 & h_3 & 0 & h_4 \end{Bmatrix}$$

where

$$\begin{aligned} h_1 &= \frac{1}{16} (1-t)(9t^2-1) \\ h_2 &= \frac{9}{16} (1-t^2)(1-3t) \\ h_3 &= \frac{9}{16} (1-t^2)(1+3t) \\ h_4 &= \frac{1}{16} (1+t)(9t^2-1) \end{aligned} \quad \text{equations (95)}$$

$$\therefore \{F\}_b^e = - \int [N]^T 2\pi r p_s \begin{Bmatrix} dz \\ -dr \end{Bmatrix} \quad \text{---(96)}$$

$$= 2\pi p_s \int_{-1}^{+1} [N]^T r \begin{Bmatrix} -\frac{dz}{dt} \\ \frac{dr}{dt} \end{Bmatrix} dt \quad \text{---(97)}$$

$$r = \sum_{i=1}^4 h_i r_i$$

$$\frac{dz}{dt} = \sum_{i=1}^4 \frac{dh_i}{dt} z_i \quad \text{equations (98)}$$

$$\frac{dr}{dt} = \sum_{i=1}^4 \frac{dh_i}{dt} r_i$$

and

$$\begin{aligned}\frac{\partial h}{\partial t}^1 &= \frac{1}{16} (1+18t-27t^2) \\ \frac{\partial h}{\partial t}^2 &= \frac{9}{16} (-3-2t+9t^2) \\ \frac{\partial h}{\partial t}^3 &= \frac{9}{16} (3-2t-9t^2) \\ \frac{\partial h}{\partial t}^4 &= \frac{1}{16} (-1+18t+27t^2)\end{aligned}$$

equations (99)

The integral is then evaluated by a three-point Gaussian Quadrature

$$\begin{aligned}\int_{-1}^{+1} f(t) dt &= \sum_{j=1}^3 H_j f(a_j) \\ &= \frac{2\pi p_s}{9} \left\{ 5f(a_1) + 8f(0) + 5f(-a_1) \right\} \\ &\quad \text{where } a_1 = 0.7745966692\end{aligned}\tag{100}$$

It is assumed that the applied pressure, p_s , is uniform in this case, and that the shell may be a general "cubic" meridional profile of revolution.

CHAPTER TWO

FINITE ELEMENT COMPUTER PROGRAMMING

i) Constant Strain Element. (CST).

The required sequence of operations is first specified by a flow chart :-

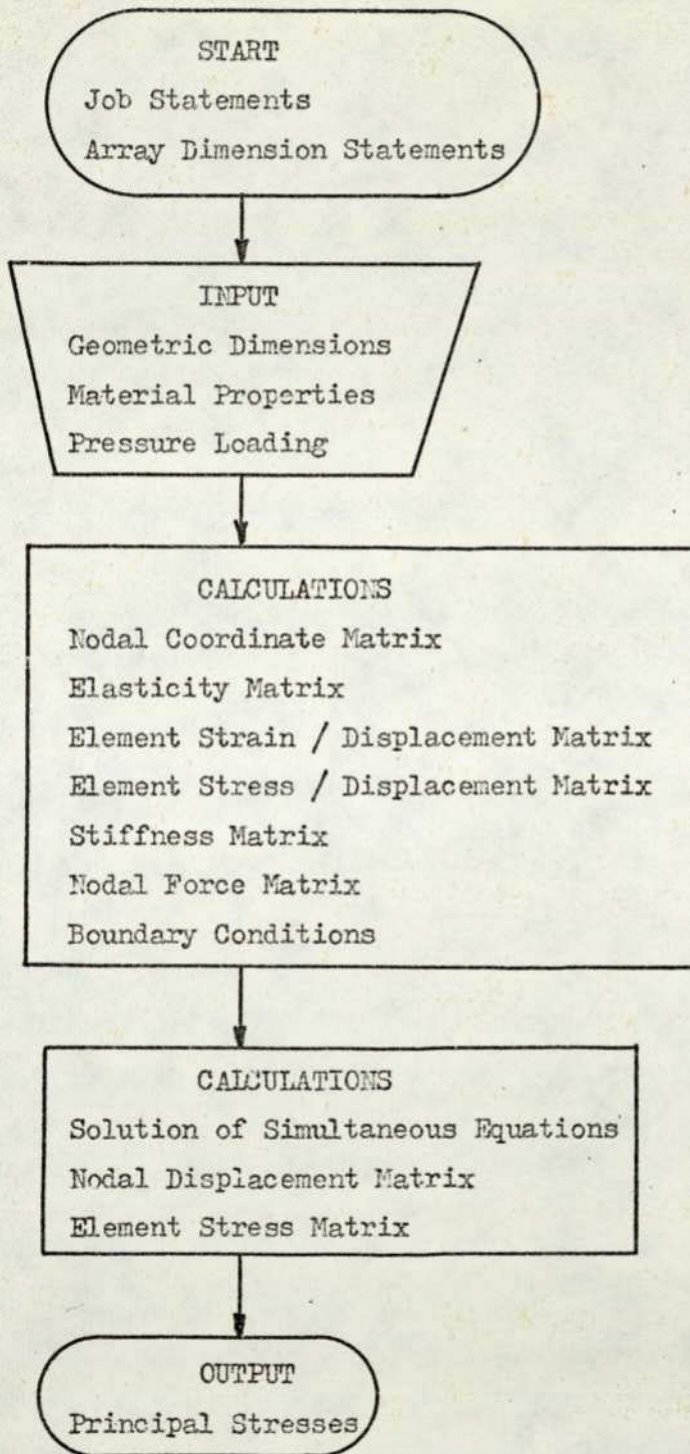


Fig. 6 Computer Programme - Sequence of Operations

The next step is to divide the sequence of operations into a master program and subroutines. In this case the output is being plotted.

Master

Input data	Data cards
Generate elasticity matrix	Subroutine
Generate nodal coordinates	Subroutine
Start plotting	Library subroutines
Generate stiffness matrix	Subroutines
Generate force matrix	Subroutine
Enter boundary conditions	
Solve simultaneous equations	Library subroutine
Plot element boundaries	Subroutines &
& principal stresses	library subroutines
Stop plotting	
End	

The program can now be written, beginning with the blue job cards for the George 2 system :-

```

JOB CST,CIR214,MERCER
SECNDS 600
LINES 1000
BFORTRAN ,590,OV,32000
****
DGC PRGC
.
.
.
****
DGC DATA
.
.
****

```

where CST is the job name
CIR214 a personal code number
MERCER the name of the writer
OV is the code for "overlay"

The program is written in Fortran with limits of 600 sec 1000 lines output and 32K core

With the overlay code entered it is necessary to specify the library subroutines, etc., by Fortran cards placed after the DGC PRGC card.

The library cards are as follows :--

```

LIBRARY(ED,SUBCRUPNAGF)
LIBRARY(ED,SUBCRUPSRF7)
LIBRARY(ED,SUBCRUPSRGP)
LIBRARY(ED,SUBCRUPS-RS)
LIST(LP)
PROGRAM(PPGCCIR214 )
COMPACT
INPUT 1,5=CRO
OUTPUT 2,6=LPO
TRACE 2
COMPRESS INTEGER AND LOGICAL
END

```

these cards start on number 7, as does the MASTER card which follows, whereas the blue job cards begin at statement number one

```

MASTER PLCT
DIMENSION F(160,1),E(4,4),D(4,6),A(160,12)
1,R(200),Z(200),X(160,12),W(160,1)
COMMON F,E,D,A

```

where F is the force matrix due to applied loading
 E is the elasticity matrix
 D is the element strain/displacement matrix
 A is the stiffness matrix
 R & Z are the coordinates
 X is a reserve matrix required by the subroutine for the solution of the simultaneous equations
 W is the displacement matrix

```

READ (1,100) H1, R1, R2, T1, T2, T3, YM,PR,PS

```

where H1,R1,R2,T1,T2,T3 are dimensions shown in the diagram below
 YM is Young's Modulus
 PR is Poisson's Ratio
 PS is the applied uniform pressure

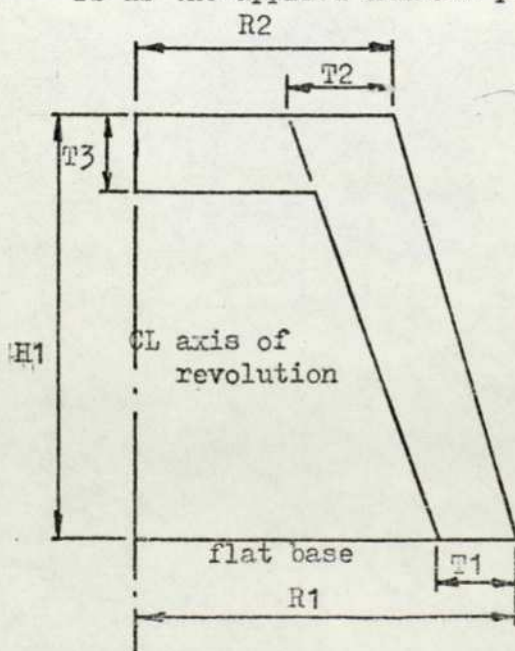


Fig. 7 Geometric Notation for Frustum of a Cone

An A4 size sheet is set out by the plotter :-

```
CALL HGPSTART (6HMERCE,6)
R(2)=21.
R(3)=21.
Z(3)=29.7
Z(4)=29.7
CALL HGPDASHLN (R(1),Z(1),R(2),Z(2),0.)
CALL HGPDASHLN (R(2),Z(2),R(3),Z(3),0.)
CALL HGPDASHLN (R(3),Z(3),R(4),Z(4),0.)
CALL HGPDASHLN (R(4),Z(4),R(1),Z(1),0.)
CALL HGPSYMBL (9.,6.,0.42,3HCST,0.,3)
CALL HGPORIGIN (6.,10.)
```

The element nodes may here be generated :-

```
DO 1 I=1,4
  T4=FL/AT(I-1)/3.
  H2=H1-T3*T4
  R3=R2-T2*T4+T3*T4*((R1-R2)-(T1-T2)*T4)/H1
  R4=R1-T1*T4
DO 2 J=1,45,4
  H3=FL/AT(J-1)/4
  K=I+J-1
  R(K)=R4-(R4-R3)*H3/12.
2  Z(K)=H2*H3/12.
DO 3 J=49,77,4
  H3=FL/AT(J-49)/4.
  K=I+J-1
  R(K)=R3-R3*H3/7.
3  Z(K)=H2
1  CONTINUE
```

Call in the elasticity matrix :-

```
CALL ELAST (YM,PR)
```

Plot the axis and base line of the shell structure :-

```
CALL HGPDASHLN (R(80),Z(80),R(81),Z(81),0.5)
CALL HGPDASHLN (R(81),Z(81),R(4),Z(4),0.5)
CALL HGPSYMBL (-0.21,4.,0.21,18HAXIS OF REVOLUTION,90.,
118)
CALL HGPSYMBL (6.,-0.42,0.21,9HFLAT BASE,0.,9)
```

We now call in the stiffness matrix.

Primarily the continuum has been divided into quadrilaterals.

The program loops through each of these in turn, finding the shortest diagonal with a conditional statement to form the pairs of constant strain triangular elements for which the subroutines generate element stiffness matrices and add them together to form the overall stiffness matrix.


```

DØ 4 M=1,73,4
DØ 4 N=1,3
I=M+N-1
J=I+4
K=I+5
L=I+1
IF ((R(I)-R(K))**2+(Z(I)-Z(K))**2
1-(R(J)-R(L))**2-(Z(J)-Z(L))**2)
1 11,12,12
11 CØNTINUE
CALL STIFF (R,Z,I,J,K)
CALL STIFF (R,Z,I,K,L)
GØ TØ 4
12 CØNTINUE
CALL STIFF (R,Z,I,J,L)
CALL STIFF (R,Z,J,K,L)
4 CØNTINUE

```

The nodal force matrix is formed by a loop on each pair of elements which form the boundary surface under applied pressure :-

```

DØ 5 J=5,73,4
I=J-4
K=J+4
CALL FØRCE (PS,R,Z,I,J,K)
5 CØNTINUE
CALL FØRCE (PS,R,Z,73,77,77)
F(153,1)=0.

```

This last statement is a boundary condition, that the shell does not deflect horizontally at the axis of revolution, and applying this condition and the encastre condition at the base of the shell to the stiffness matrix :-

```

DØ 6 I=1,4
J=2*I
K=J+151
L=J-1
A(J,12)=A(J,12)*100000000.
A(K,12)=A(K,12)*100000000.
6 A(L,12)=A(L,12)*100000000.

```

A library subroutine is used to solve the simultaneous equations with the stiffness matrix which has been stored efficiently as a column matrix, and the force matrix known, and the deflexion matrix to be found :-

```

K=1
CALL FO4ACF (A,160,F,160,160,11,1,W,160,X,160,12,K)
IF (K) 7,8,7
7 WRITE (2,101)
8 CØNTINUE

```

The program repeats the loop through the elements, this time to call the subroutines which generate and plot the element boundaries and principal stresses :-

```

      DØ 9 M=1,73,4
      DØ 9 N=1,3
      I=M+N-1
      J=I+4
      K=I+5
      L=I+1
      CALL HGPDASHLN (R(J),Z(J),R(I),Z(I),0.)
      CALL HGPDASHLN (R(I),Z(I),R(L),Z(L),0.)
      IF ((R(I)-R(K))**2+(Z(I)-Z(K))**2
1  - (R(J)-R(L))**2-(Z(J)-Z(L))**2)
1  13,14,14
13  CØNTINUE
      CALL HGPDASHLN (R(I),Z(I),R(K),Z(K),0.)
      CALL PRINC (W,R,Z,I,J,K)
      CALL PRINC (W,R,Z,I,K,L)
      GØ TØ 9
14  CØNTINUE
      CALL HGPDASHLN (R(L),Z(L),R(J),Z(J),0.)
      CALL PRINC (W,R,Z,I,J,L)
      CALL PRINC (W,R,Z,J,K,L)
9   CØNTINUE
      DØ 10 M=4,76,4
      N=M+4
      CALL HGPDASHLN (R(M),Z(M),R(N),Z(N),0.)
10  CØNTINUE
      CALL HGPDASHLN (R(80),Z(80),R(77),Z(77),0.)

```

The master program is brought to a close :-

```

      CALL HGPFINISH
100  FØRMAT (9F0.0)
101  FØRMAT (1H1,1X,28HMATRIX NØT PØSITIVE DEFINITE)
      STØP
      END

```

The subroutines follow, beginning with the elasticity matrix :-

```

SUBRØUTINE ELAST (YM,PR)
DIMENSION F(160,1),E(4,4),D(4,6),A(160,12)
CØMØN F,E,D,A
PN=YM/((1.+PR)*(1.-2.*PR))
DØ 1 I=1,3
  E(I,4)=0.
  E(4,I)=0.
DØ 1 J=1,3
  E(I,J)=PN*PR
1  E(I,I)=PN*(1.-PR)
  E(4,4)=0.5*PN*(1.-2.*PR)
RETURN
END

```


The element displacement matrix is generated :-

```

SUBROUTINE DISPL (R,Z,I,J,K)
DIMENSION F(160,1),E(4,4),D(4,6),A(160,12)
1,R(200),Z(200)
COMMON F,E,D,A
DO 1 M=1,3
DO 1 N=1,6
1 D(M,N)=0.
RB=1./(R(I)+R(J)+R(K))
ZB=(Z(I)+Z(J)+Z(K))*RB
RB=3.*RB
D(1,2)=R(K)-R(J)
D(4,1)=D(1,2)
D(1,4)=R(I)-R(K)
D(4,3)=D(1,4)
D(1,6)=R(J)-R(I)
D(4,5)=D(1,6)
D(2,1)=Z(J)-Z(K)
D(4,2)=D(2,1)
D(2,3)=Z(K)-Z(I)
D(4,4)=D(2,3)
D(2,5)=Z(I)-Z(J)
D(4,6)=D(2,5)
D(3,1)=RB*(R(J)*Z(K)-R(K)*Z(J))+D(2,1)+D(1,2)*ZB
D(3,3)=RB*(R(K)*Z(I)-R(I)*Z(K))+D(2,3)+D(1,4)*ZB
D(3,5)=RB*(R(I)*Z(J)-R(J)*Z(I))+D(2,5)+D(1,6)*ZB
RETURN
END

```

The overall stiffness matrix is formed as a column matrix :-

```

SUBROUTINE STIFF (R,Z,I,J,K)
DIMENSION F(160,1),E(4,4),D(4,6),A(160,12)
1,R(200),Z(200),C(4,6),G(6,6),B(6,4),N1(6)
COMMON F,E,D,A
RB=(R(I)+R(J)+R(K))/3.
PI=3.14159265
D2=RB*PI/(R(J)*Z(K)-Z(J)*R(K)-R(I)*
1(Z(K)-Z(J))+Z(I)*(R(K)-R(J)))
CALL DISPL (R,Z,I,J,K)
DO 1 L=1,4
DO 1 M=1,6
B(M,L)=D(L,M)*D2
C(L,M)=0.
DO 1 N=1,4
1 C(L,M)=C(L,M)+E(L,N)*D(N,M)
N1(2)=2*I
N1(4)=2*J
N1(6)=2*K
N1(1)=N1(2)-1
N1(3)=N1(4)-1
N1(5)=N1(6)-1
DO 2 L=1,6
DO 2 M=1,L
C(L,M)=0.
DO 3 N=1,4
3 G(L,M)=G(L,M)+B(L,N)*C(N,M)
N2=N1(L)-N1(N)

```

```

      IF (N2) 4,5,5
4     NR=N1(M)
      NC=12+N2
      GO TO 2
5     NR=N1(L)
      NC=12-N2
2     A(NR,NC)=A(NR,NC)+G(L,M)
      RETURN
      END

```

Since the stiffness matrix is symmetrical banded only the diagonal and lower half have been stored.

The right-hand side of the equation is formed from the force subroutine as follows :-

```

SUBROUTINE FORCE (PS,R,Z,I,J,K)
DIMENSION F(160,1),E(4,4),D(4,6),A(160,12)
1,R(200),Z(200)
COMMON F,E,D,A
PI= 3.14159265
PA=PI*PS/3.
NV=2*J
NU=NV-1
F(NU,1)=PA*(R(I)*(Z(J)-Z(I))+2.*R(J)*
1(Z(K)-Z(I))+R(K)*(Z(K)-Z(J)))
F(NV,1)=PA*(R(I)-R(K))*(R(I)+R(J)+R(K))
RETURN
END

```

Now, the nodal displacements having been found by solving the simultaneous equations with a library subroutine, the stress matrix and principal stresses are found and plotted for each element in turn.

```

SUBROUTINE PRINC (W,R,Z,I,J,K)
DIMENSION F(160,1),E(4,4),D(4,6),A(160,12)
1,W(160,1),R(200),Z(200),N(6),B(4,1),S(4,1)
COMMON F,E,D,A
D2=1./((R(J)*Z(K)-Z(J)*R(K)-R(I)*(Z(K)-Z(J))+Z(I)*(R(K)
1-R(J)))
CALL DISPL (R,Z,I,J,K)
N(2)=2*I
N(4)=2*J
N(6)=2*K
N(1)=N(2)-1
N(3)=N(4)-1
N(5)=N(6)-1
DO 1 L=1,4
B(L,1)=0.
DO 1 M=1,6
1 B(L,1)=B(L,1)+D(L,M)*D2*W(N(M),1)
DO 2 L=1,4
S(L,1)=0.
DO 2 M=1,4

```



```

2      S(L,1)=S(L,1)+E(L,M)*B(M,1)
      SA=SQRT(0.25*(S(1,1)-S(2,1))**2+S(4,1)**2)
      SB=0.5*(S(1,1)+S(2,1))
      SP=SB+SA
      SQ=SB-SA
      SR=SP-S(1,1)
      IF (SR) 6,7,6
7      TH=1.570796325
      GO TO 5
6      CONTINUE
      TH=S(4,1)/SR
      IF (TH) 3,4,4
3      TH=-TH
      TH=-ATAN(TH)
      GO TO 5
4      TH=ATAN(TH)
5      ST=S(3,1)
      CALL PLOTT (TH,SP,SQ,R,Z,I,J,K)
      RETURN
      END

```

The plotter subroutine which follows is somewhat lengthy, but the underlying idea is simple. If the stress flow in the radial plane is represented by ⁿ principal planes spaced at a distance between each other which is inversely proportional to the stress factor at some convenient scale, then for the constant strain element this will result in a mesh of orthogonal parallel straight lines.

The plot begins by rotating the coordinate axes to be parallel with the element principal planes, XR and YZ.

TH1 is the angle of rotation in radians, and SP1 and SQ1 are the principal stress factors.

```

SUBROUTINE PLOTT (TH1,SP1,SQ1,R,Z,I,J,K)
  DIMENSION F(160,1),E(4,4),D(4,6),A(160,12)
  1,R(200),Z(200),XR(200),YZ(200)
  COMMON F,E,D,A
  S=SIN(TH1)
  C=COS(TH1)
  XR(I)=Z(I)*S+R(I)*C
  YZ(I)=Z(I)*C-R(I)*S
  XR(J)=Z(J)*S+R(J)*C
  YZ(J)=Z(J)*C-R(J)*S
  XR(K)=Z(K)*S+R(K)*C
  YZ(K)=Z(K)*C-R(K)*S

```

The sign of the stress factor is next found, and the orientation of the element nodes established relative to the rotated axes :-

```

      IF (SP1) 1,2,3
1     SP1=-SP1
      M=-1
      GO TO 4
3     M=1
4     CONTINUE
      IF (YZ(I)-YZ(J)) 5,6,6
5     CONTINUE
      IF (YZ(I)-YZ(K)) 7,8,8
6     CONTINUE
      IF (YZ(J)-YZ(K)) 9,8,8
7     XR(101)=XR(I)
      XR(102)=XR(J)
      XR(103)=XR(K)
      YZ(101)=YZ(I)
      YZ(102)=YZ(J)
      YZ(103)=YZ(K)
      GO TO 10
9     XR(101)=XR(J)
      XR(102)=XR(K)
      XR(103)=XR(I)
      YZ(101)=YZ(J)
      YZ(102)=YZ(K)
      YZ(103)=YZ(I)
      GO TO 10
8     XR(101)=XR(K)
      XR(102)=XR(I)
      XR(103)=XR(J)
      YZ(101)=YZ(K)
      YZ(102)=YZ(I)
      YZ(103)=YZ(J)
10    CONTINUE
      N=1
11    PN=FL/AT(N)/SP1+YZ(101)

```

Label number 11 starts the loop to draw each principal plane across the element, where PN is the YZ coordinate of this line and the points where it cuts the element boundaries are XR(N1) and XR(N2) :-

```

      N2=2*N+103
      N1=N2-1
      IF (PN-YZ(103)) 12,12,13
12    XR(N1)=XR(101)+(XR(103)-XR(101))*(PN-YZ(101))/(YZ(103)-
      YZ(101))
      GO TO 14
13    CONTINUE
      IF (PN-YZ(102)) 15,2,2
15    XR(N1)=XR(103)+(XR(102)-XR(103))*(PN-YZ(103))/(YZ(102)-
      YZ(103))
14    CONTINUE
      IF (PN-YZ(102)) 17,17,18
17    XR(N2)=XR(101)+(XR(102)-XR(101))*(PN-YZ(101))/(YZ(102)-
      YZ(101))
      GO TO 20
18    CONTINUE
      IF (PN-YZ(103)) 19,2,2

```



```

19   XR(N2)=XR(102)+(XR(103)-XR(102))*(PN-YZ(102))/(YZ(103)-
    1YZ(102))
20   YZ(N1)=PN
    YZ(N2)=PN

```

The coordinates are transposed back into the R-Z axes and plotted with a dashed line if the stress is compressive or a continuous line if it is tensile, this being registered with the integer M.

```

    CALL TRANS (N1,R,Z,XR,YZ,S,C)
    CALL TRANS (N2,R,Z,XR,YZ,S,C)
    IF (M) 21,2,22
21   CONTINUE
    CALL HGPDASHLN (R(N1),Z(N1),R(N2),Z(N2),0.5)
    GO TO 16
22   CONTINUE
    CALL HGPDASHLN (R(N1),Z(N1),R(N2),Z(N2),0.0)

```

The loop is repeated until the element has been covered, and of course was bypassed in the first place if the stress was on or so close to zero that no line was required, and the same procedure followed to plot the orthogonal plane and complete the subroutine :-

```

16   N=N+1
    GO TO 11
2   CONTINUE
    IF (SQ1) 23,24,25
23   SQ1=-SQ1
    M=-1
    GO TO 26
25   M=1
26   CONTINUE
    IF (XR(I)-XR(J)) 27,28,28
27   CONTINUE
    IF (XR(I)-XR(K)) 29,30,30
28   CONTINUE
    IF (XR(J)-XR(K)) 31,30,30
29   YZ(101)=YZ(I)
    YZ(102)=YZ(J)
    YZ(103)=YZ(K)
    XR(101)=XR(I)
    XR(102)=XR(J)
    XR(103)=XR(K)
    GO TO 32
31   YZ(101)=YZ(J)
    YZ(102)=YZ(K)
    YZ(103)=YZ(I)
    XR(101)=XR(J)
    XR(102)=XR(K)
    XR(103)=XR(I)
    GO TO 32
30   YZ(101)=YZ(K)
    YZ(102)=YZ(I)

```

```

YZ(103)=YZ(J)
XR(101)=XR(K)
XR(102)=XR(I)
XR(103)=XR(J)
32  CONTINUE
   N=1
33  QN=FLOAT(N)/SQ1+XR(101)
   N2=2*N+103
   N1=N2-1
   IF (QN-XR(102)) 34,34,35
34  YZ(N1)=YZ(101)+(YZ(102)-YZ(101))*(QN-XR(101))/(XR(102)-
1XR(101))
   GO TO 36
35  CONTINUE
   IF (QN-XR(103)) 37,24,24
37  YZ(N1)=YZ(102)+(YZ(103)-YZ(102))*(QN-XR(102))/(XR(103)-
1XR(102))
36  CONTINUE
   IF (QN-XR(103)) 38,38,39
38  YZ(N2)=YZ(101)+(YZ(103)-YZ(101))*(QN-XR(101))/(XR(103)-
1XR(101))
   GO TO 40
39  CONTINUE
   IF (QN-XR(102)) 41,24,24
41  YZ(N2)=YZ(103)+(YZ(102)-YZ(103))*(QN-XR(103))/(XR(102)-
1XR(103))
40  XR(N1)=QN
   XR(N2)=QN
   CALL TRANS (N1,R,Z,XR,YZ,S,C)
   CALL TRANS (N2,R,Z,XR,YZ,S,C)
   IF (N) 42,24,43
42  CONTINUE
   CALL HGFDPASHIN (R(N1),Z(N1),R(N2),Z(N2),0.5)
   GO TO 44
43  CONTINUE
   CALL HGFDPASHIN (R(N1),Z(N1),R(N2),Z(N2),0.0)
44  N=N+1
   GO TO 33
24  CONTINUE
   RETURN
   END

SUBROUTINE TRANS (N,R,Z,X,Y,S,C)
  DIMENSION F(160,1),E(4,4),D(4,6),A(160,12)
  1,R(200),Z(200),X(200),Y(200)
  COMMON F,E,D,A
  R(N)=X(N)*C-Y(N)*S
  Z(N)=X(N)*S+Y(N)*C
  RETURN
  END

```

With the addition of the data cards the program is thus completed.

ii) The Q2CST Element.

This element is useful in that by averaging the stresses for adjacent CST elements an effective constant strain quadrilateral is obtained which has a more even flow of results compared with those obtained for the CST element. The computer program is identical to that of the CST element for the formation and solution of the simultaneous equations which give rise to numerical values for the nodal deflexions. There is a more extended procedure required for the plotting of the principal planes, and in addition the numerical results will be printed out.

After the library subroutine for solution of the simultaneous equations has been called as before, the master program now continues as follows :-

```

8      CONTINUE
      WRITE (2,102)
      WRITE (2,103)
      PK=ABS(PS)
      PK=1./PK
      PJ=180./3.14159265
      DO 9 M=1,73,4
      DO 9 N=1,3
      I=M+N-1
      J=I+4
      K=I+5
      L=I+1
      CALL HCPDASHLN (R(J),Z(J),R(I),Z(I),0.0)
      CALL HCPDASHLN (R(I),Z(I),R(L),Z(L),0.0)
      CALL PRINC (PK,PJ,W,R,Z,I,J,K,L)
9      CONTINUE
      DO 10 M=4,76,4
      N=M+4
      CALL HCPDASHLN (R(M),Z(M),R(N),Z(N),0.0)
10     CONTINUE
      CALL HCPDASHLN (R(80),Z(80),R(77),Z(77),0.0)
      CALL HCPFINISH
100    FORMAT (9F0.0)
101    FORMAT (1H1,1X,28HMATRIX NOT POSITIVE DEFINITE)
102    FORMAT (1H1,1X,13HELEMENT NODES,3X,15HPRINCIPAL PLANE,
19X,14HSTRESS FACTORS)
103    FORMAT (1H ,2X,1H1,3X,1HJ,3X,1HK,3X,1HL,5X,9H(DEGREES),
18X,5HMAJOR,4X,5HMINOR,5X,4HHOFP,4X,5HSHEAR/)
      STOP
      END

```

The subroutine to find the principal stresses takes a different form from the CST element in calling the global stress matrix and weighting the average of adjacent CSTs in proportion to their areas.

```

SUBROUTINE PRINC (PK,PJ,W,R,Z,I,J,K,L)
DIMENSION F(160,1),E(4,4),D(4,6),A(160,12)
1,W(160,1),R(200),Z(200),S(4,1),T(4,1)
COMMON F,E,D,A
DO 1 M=1,4
S(M,1)=0.
1 T(M,1)=0.
IF ((R(I)-R(K))**2+(Z(I)-Z(K))**2
1 -(R(J)-R(L))**2-(Z(J)-Z(L))**2
1 11,12,12
11 CONTINUE
CALL GLQBA (DS,S,W,R,Z,I,J,K)
CALL GLQBA (DT,T,W,R,Z,I,K,L)
GO TO 13
12 CONTINUE
CALL GLQBA (DS,S,W,R,Z,I,J,L)
CALL GLQBA (DT,T,W,R,Z,J,K,L)
13 CONTINUE
D2=1./(DS+DT)
DO 2 M=1,4
2 S(M,1)=(S(M,1)*DS+T(M,1)*DT)*D2
SA=SQRT(0.25*(S(1,1)-S(2,1))**2+S(4,1)**2)
SB=0.5*(S(1,1)+S(2,1))
ST=SB+SA
SQ=SB-SA
ST=S(3,1)
SR=SP-S(1,1)
IF (SR) 3,4,3
4 TH=1.570796325
GO TO 5
3 TH=S(4,1)/SR
IF (TH) 6,7,7
6 TH=-TH
TH=-ATAN(TH)
GO TO 5
7 TH=ATAN(TH)
5 TS=TH+0.7853981625
TT=TS-1.570796325
IF (TT) 8,9,10
10 TS=TT
GO TO 8
9 TS=0.
8 CONTINUE
CALL PLQTT (TH,SP,SQ,R,Z,I,J,K,L)
TH=TH*PJ
SP=SP*PK
SQ=SQ*PK
ST=ST*PK
SA=SA*PK
WRITE (2,104) I,J,K,L,TH,SP,SQ,ST,SA
104 FORMAT (1H ,1X,I2,2X,I2,2X,I2,2X,I2,6X,F7.2,7X,F7.2,2X
1,F7.2,2X,F7.2,2X,F7.2)

```



```

RETURN
END

```

The subroutine to find the global stress matrix for a CST follows :-

```

SUBROUTINE GLGBA (D1,S,W,R,Z,I,J,K)
DIMENSION F(160,1),E(4,4),D(4,6),A(160,12)
1,S(4,1),W(160,1),R(200),Z(200),B(4,1),N(6)
COMMON F,E,D,A
D1=R(J)*Z(K)-Z(J)*R(K)-R(I)*(Z(K)-Z(J))+Z(I)*(R(K)-R(J))
D2=1./D1
CALL DISPL (R,Z,I,J,K)
N(2)=2*I
N(4)=2*J
N(6)=2*K
N(1)=N(2)-1
N(3)=N(4)-1
N(5)=N(6)-1
DO 1 L=1,4
B(L,1)=0.
DO 1 M=1,6
1 B(L,1)=B(L,1)+D(L,M)*D2*W(N(M),1)
DO 2 L=1,4
S(L,1)=0.
DO 2 M=1,4
2 S(L,1)=S(L,1)+E(L,M)*B(M,1)
RETURN
END

```

The plotter output of principal planes follows the same routine as for the CST, but is more lengthy with the fourth boundary of the quadrilateral :-

```

SUBROUTINE PLOT (U,V,W,R,Z,I,J,K,L)
DIMENSION F(160,1),E(4,4),D(4,6),A(160,12)
1,R(200),Z(200),X(200),Y(200)
COMMON F,E,D,A
S=SIN(U)
C=COS(U)
X(I)=Z(I)*S+R(I)*C
X(J)=Z(J)*S+R(J)*C
X(K)=Z(K)*S+R(K)*C
X(L)=Z(L)*S+R(L)*C
Y(I)=Z(I)*C-R(I)*S
Y(J)=Z(J)*C-R(J)*S
Y(K)=Z(K)*C-R(K)*S
Y(L)=Z(L)*C-R(L)*S
IF (V) 1,2,3
1 V=-V
M=-1
GO TO 4
3 M=1
4 CONTINUE

```

```

5      IF (Y(I)-Y(J)) 5,6,6
      CCONTINUE
6      IF (Y(I)-Y(K)) 7,8,8
      CCONTINUE
7      IF (Y(J)-Y(K)) 9,8,8
      CCONTINUE
8      IF (Y(I)-Y(L)) 10,13,13
      CCONTINUE
9      IF (Y(K)-Y(L)) 12,13,13
      CCONTINUE
10     IF (Y(J)-Y(L)) 11,13,13
      CALL QUADS (X,Y,I,J,K,L)
      GOTO 14
11     CALL QUADS (X,Y,J,K,L,I)
      GOTO 14
12     CALL QUADS (X,Y,K,L,I,J)
      GOTO 14
13     CALL QUADS (X,Y,L,I,J,K)
14     CCONTINUE
      N=1
15     P=FL/AT(N)/V+Y(101)
      N2=2*N+104
      N1=N2-1
      IF (P-Y(104)) 16,17,17
16     X(N1)=X(101)+(X(104)-X(101))*(P-Y(101))/(Y(104)-Y(101))
      GOTO 18
17     CCONTINUE
      IF (P-Y(103)) 19,19,20
19     X(N1)=X(104)+(X(103)-X(104))*(P-Y(104))/(Y(103)-Y(104))
      GOTO 18
20     CCONTINUE
      IF (P-Y(102)) 21,2,2
21     X(N1)=X(103)+(X(102)-X(103))*(P-Y(103))/(Y(102)-Y(103))
18     CCONTINUE
      IF (P-Y(102)) 22,22,23
22     X(N2)=X(101)+(X(102)-X(101))*(P-Y(101))/(Y(102)-Y(101))
      GOTO 24
23     CCONTINUE
      IF (P-Y(103)) 25,25,26
25     X(N2)=X(102)+(X(103)-X(102))*(P-Y(102))/(Y(103)-Y(102))
      GOTO 24
26     CCONTINUE
      IF (P-Y(104)) 27,2,2
27     X(N2)=X(103)+(X(104)-X(103))*(P-Y(103))/(Y(104)-Y(103))
24     Y(N1)=P
      Y(N2)=P
      CALL TRANS (N1,R,Z,X,Y,S,C)
      CALL TRANS (N2,R,Z,X,Y,S,C)
      IF (M) 28,2,29
28     CCONTINUE
      CALL HCPDASHLN (R(N1),Z(N1),R(N2),Z(N2),0.5)
      GOTO 30
29     CCONTINUE
      CALL HCPDASHLN (R(N1),Z(N1),R(N2),Z(N2),0.0)
30     N=N+1
      GOTO 15
2     CCONTINUE
      IF (M) 61,62,62

```



```

61 V=-V
62 CCONTINUE
   IF (W) 31,32,33
31 W=-W
   M=1
   GOTO 34
33 M=1
34 CCONTINUE
   IF (X(I)-X(J)) 35,36,36
35 CCONTINUE
   IF (X(I)-X(K)) 37,38,38
36 CCONTINUE
   IF (X(J)-X(K)) 39,38,38
37 CCONTINUE
   IF (X(I)-X(L)) 40,43,43
38 CCONTINUE
   IF (X(K)-X(L)) 42,43,43
39 CCONTINUE
   IF (X(J)-X(L)) 41,43,43
40 CALL QUADS (X,Y,I,J,K,L)
   GOTO 44
41 CALL QUADS (X,Y,J,K,L,I)
   GOTO 44
42 CALL QUADS (X,Y,K,L,I,J)
   GOTO 44
43 CALL QUADS (X,Y,L,I,J,K)
44 CCONTINUE
   N=1
45 Q=FLD(1,N)/W+X(101)
   N2=2*N+104
   N1=N2-1
   IF (Q-X(102)) 46,46,47
46 Y(N1)=Y(101)+(Y(102)-Y(101))*(Q-X(101))/(X(102)-X(101))
   GOTO 48
47 CCONTINUE
   IF (Q-X(103)) 49,49,50
49 Y(N1)=Y(102)+(Y(103)-Y(102))*(Q-X(102))/(X(103)-X(102))
   GOTO 48
50 CCONTINUE
   IF (Q-X(104)) 51,32,32
51 Y(N1)=Y(103)+(Y(104)-Y(103))*(Q-X(103))/(X(104)-X(103))
48 CCONTINUE
   IF (Q-X(104)) 52,52,53
52 Y(N2)=Y(101)+(Y(104)-Y(101))*(Q-X(101))/(X(104)-X(101))
   GOTO 54
53 CCONTINUE
   IF (Q-X(103)) 55,55,56
55 Y(N2)=Y(104)+(Y(103)-Y(104))*(Q-X(104))/(X(103)-X(104))
   GOTO 54
56 CCONTINUE
   IF (Q-X(102)) 57,32,32
57 Y(N2)=Y(103)+(Y(102)-Y(103))*(Q-X(103))/(X(102)-X(103))
54 X(N1)=Q
   X(N2)=Q
   CALL TRANS (N1,R,Z,X,Y,S,C)
   CALL TRANS (N2,R,Z,X,Y,S,C)
   IF (M) 58,32,59
58 CCONTINUE

```

```

      CALL HCPDASHLN (R(N1),Z(N1),R(N2),Z(N2),0.5)
      GO TO 60
59  CONTINUE
      CALL HCPDASHLN (R(N1),Z(N1),R(N2),Z(N2),0.0)
60  N=N+1
      GO TO 45
32  CONTINUE
      IF (M) 63,64,64
63  W=-W
64  CONTINUE
      RETURN
      END

```

```

SUBROUTINE QUADS (X,Y,I,J,K,L)
DIMENSION F(160,1),E(4,4),D(4,6),A(160,12)
1,X(200),Y(200)
COMMON F,E,D,A
X(101)=X(I)
X(102)=X(J)
X(103)=X(K)
X(104)=X(L)
Y(101)=Y(I)
Y(102)=Y(J)
Y(103)=Y(K)
Y(104)=Y(L)
RETURN
END

```

```

SUBROUTINE TRANS (I,R,Z,XR,YZ,S,C)
DIMENSION F(160,1),E(4,4),D(4,6),A(160,12)
1,R(200),Z(200),XR(200),YZ(200)
COMMON F,E,D,A
R(I)=XR(I)*C-YZ(I)*S
Z(I)=XR(I)*S+YZ(I)*C
RETURN
END

```

Again the addition of data cards completes the program.

iii) The quadrilateral element formed by four CSTs. (Q4CST).

The introduction of a fifth node into the quadrilateral element gives a 10 X 10 element stiffness matrix called G in the program :-

```

MASTER PL0T
DIMENSION F(160,1),E(4,4),D(4,6),A(160,12),G(10,10)
1,R(190),Z(190),W(162,1),X(160,12)
COMMON F,E,D,A,G
READ (1,100) H1,R1,R2,T1,T2,T3,YM,PR,PS
CALL HGPSTART (6HMERCE,6)
R(2)=21.
R(3)=21.
Z(3)=29.7
Z(4)=29.7
CALL HGPDASHLN (R(1),Z(1),R(2),Z(2),0.)
CALL HGPDASHLN (R(2),Z(2),R(3),Z(3),0.)
CALL HGPDASHLN (R(3),Z(3),R(4),Z(4),0.)
CALL HGPDASHLN (R(4),Z(4),R(1),Z(1),0.)
CALL HGPSYMBL (9.,6.,0.42,5HQ4CST,0.,5)
CALL HGPORIGIN (6.,10.)
DO 1 I=1,4
T4=FL0AT(I-1)/3.
H2=H1-T3*T4
R3=R2-T2*T4+T3*T4*((R1-R2)-(T1-T2)*T4)/H1
R4=R1-T1*T4
DO 2 J=1,45,4
H3=FL0AT(J-1)/4.
K=I+J-1
R(K)=R4-(R4-R3)*H3/12.
2 Z(K)=H2*H3/12.
DO 3 J=49,77,4
H3=FL0AT(J-49)/4.
K=I+J-1
R(K)=R3-R3*H3/7.
3 Z(K)=H2
1 C0NTINUE
CALL HGPDASHLN (R(80),Z(80),R(81),Z(81),0.5)
CALL HGPDASHLN (R(81),Z(81),R(4),Z(4),0.5)
CALL HGPSYMBL (-0.21,4.,0.21,18HAXIS 0F REV0LUTION,90.,18)
CALL HGPSYMBL (6.,-0.42,0.21,9HFLAT BASE,0.,9)
CALL ELAST (YM,PR)

```

The program now loops through each element to call a subroutine which assembles the stiffness matrix :-

```

DO 4 M=1,73,4
DO 4 N=1,3
I=M+N-1
J=I+4
K=I+5
L=I+1
CALL ASSEM (R,Z,I,J,K,L)
4 C0NTINUE
DO 5 J=5,73,4
I=J-4

```

```

      K=J+4
      CALL FORCE (PS,R,Z,I,J,K)
5     CONTINUE
      CALL FORCE (PS,R,Z,73,77,77)
      F(153,1)=0.
      DO 6 I=1,4
      J=2*I
      K=J+151
      L=J-1
      A(J,12)=A(J,12)*1000000000.
      A(K,12)=A(K,12)*1000000000.
6     A(L,12)=A(L,12)*1000000000.
      K=1
      CALL FO4ACF (A,160,F,160,160,11,1,W,160,X,160,12,K)
      IF (K) 7,8,7
7     WRITE (2,101)
8     CONTINUE

```

Having the displacement matrix for the corner nodes, the program loops through each quadrilateral element again to call the subroutine which plots the principal planes :-

```

      DO 9 M=1,73,4
      DO 9 N=1,3
      I=M+N-1
      J=I+4
      K=I+5
      L=I+1
      CALL HCPDASHLN (R(J),Z(J),R(I),Z(I),0.)
      CALL HCPDASHLN (R(I),Z(I),R(L),Z(L),0.)
      CALL VARIA (W,R,Z,I,J,K,L)
9     CONTINUE
      DO 10 M=4,76,4
      N=M+4
      CALL HCPDASHLN (R(M),Z(M),R(N),Z(N),0.)
10    CONTINUE
      CALL HCPDASHLN (R(80),Z(80),R(77),Z(77),0.)
      CALL HCPFINISH
100   FORMAT (9F0.0)
101   FORMAT (1H1,1X,28HMATRIX NOT POSITIVE DEFINITE)
      STOP
      END

```

The subroutines for the elasticity, force, and displacement matrices are as for the CST element with matrix G included in their dimension and common statements. The element stiffness matrix is formed by assembling each CST subelement :-

```

      SUBROUTINE STIFF (R,Z,I,J,K)
      DIMENSION F(160,1),E(4,4),D(4,6),A(160,12),G(10,10)
1, R(190),Z(190),B(6,4),C(4,6),N1(6)
      COMMON F,E,D,A,G
      RB=(R(I)+R(J)+R(K))/3.

```



```

      PI=3.14159265
      D2=RB*PI/(R(J)*Z(K)-Z(J)*R(K)
1-R(I)*(Z(K)-Z(J))+Z(I)*(R(K)-R(J)))
      CALL DISPL (R,Z,I,J,K)
      DØ 1 L=1,4
      DØ 1 M=1,6
      B(M,L)=D(L,M)*D2
      C(L,M)=0.
      DØ 1 N=1,4
1      C(L,M)=C(L,M)+E(L,N)*D(N,M)
      N1(2)=2*I
      N1(4)=2*J
      N1(6)=2*K
      N1(1)=N1(2)-1
      N1(3)=N1(4)-1
      N1(5)=N1(6)-1
      DØ 2 L=1,6
      DØ 2 M=1,6
      DØ 2 N=1,4
2      G(N1(L),N1(M))=G(N1(L),N1(M))+B(L,N)*C(N,M)
      RETURN
      END

```

A separate subroutine forms the submatrices required to eliminate the parts of the stiffness matrix which are due to the internal node only, and include its effect upon the element stiffness in the remaining parts due to corner nodes only. The internal node is placed at the centroid of the quadrilateral element, and the stiffness matrices of each CST subelement first called in assembled form from the previous subroutine :-

```

SUBROUTINE CENTER (R,Z,I,J,K,L,II)
DIMENSION F(160,1),E(4,4),D(4,6),A(160,12),G(10,10)
1,X(190),Y(190),R(190),Z(190)
COMMON F,E,D,A,G
X(1)=R(I)
X(2)=R(J)
X(3)=R(K)
X(4)=R(L)
Y(1)=Z(I)
Y(2)=Z(J)
Y(3)=Z(K)
Y(4)=Z(L)
X1=R(I)+R(J)+R(K)
X2=R(I)+R(K)+R(L)
Y1=Z(I)+Z(J)+Z(K)
Y2=Z(I)+Z(K)+Z(L)
D1=R(J)*Z(K)-Z(J)*R(K)-
1R(I)*(Z(K)-Z(J))+Z(I)*(R(K)-R(J))
D2=R(K)*Z(L)-Z(K)*R(L)-
1R(1)*(Z(L)-Z(K))+Z(I)*(R(L)-R(K))
D3=1./(3.*(D1+D2))

```

```

      X(5)=(D1*X1+D2*X2)*D3
      Y(5)=(D1*Y1+D2*Y2)*D3
      R(II)=X(5)
      Z(II)=Y(5)
      DO 1 M=1,10
      DO 1 N=1,10
1      G(M,N)=0.
      CALL STIFF (X,Y,1,2,5)
      CALL STIFF (X,Y,2,3,5)
      CALL STIFF (X,Y,3,4,5)
      CALL STIFF (X,Y,4,1,5)
      GJ=1./(G(9,9)*G(10,10)-G(9,10)*G(10,9))
      G(9,10)=-G(9,10)*GJ
      G(10,9)=-G(10,9)*GJ
      GP=G(10,10)*GJ
      G(10,10)=G(9,9)*GJ
      G(9,9)=GP
      RETURN
      END

```

The subroutine to assemble the element into the overall stiffness matrix follows, eliminating the internal node within the operation of addition.

```

      SUBROUTINE ASSEM (R,Z,I,J,K,L)
      DIMENSION F(160,1),E(4,4),D(4,6),A(160,12),G(10,10)
1      R(190),Z(190),N1(8)
      COMMON F,E,D,A,G
      CALL CENTR (R,Z,I,J,K,L,81)
      N1(2)=2*I
      N1(4)=2*J
      N1(6)=2*K
      N1(8)=2*L
      N1(1)=N1(2)-1
      N1(3)=N1(4)-1
      N1(5)=N1(6)-1
      N1(7)=N1(8)-1
      DO 1 M=1,8
      DO 1 N=1,M
      N2=N1(M)-N1(N)
      IF (N2) 2,3,3
2      NR=N1(N)
      NC=12+N2
      GO TO 1
3      NR=N1(M)
      NC=12+N2
1      A(NR,NC)=A(NR,NC)+G(M,N)
      1-G(M,9)*(G(9,9)*G(9,N)+G(9,10)*G(10,N))-
      1-G(M,10)*(G(10,9)*G(9,N)+G(10,10)*G(10,N))
      RETURN
      END

```

After the solution of the simultaneous equations, the subroutine to find and plot principal planes first reforms the element stiffness

submatrices to find the deflexion at the centroidal node. The subroutines for plotting principal planes for each CST subelement are then called.

```

SUBROUTINE VARIA (W,R,Z,I,J,K,L)
DIMENSION F(160,1),E(4,4),D(4,6),A(160,12),G(10,10)
1,R(190),Z(190),N(8),W(162,1)
COMMON F,E,D,A,G
CALL CENTR (R,Z,I,J,K,L,81)
N(2)=2*I
N(4)=2*J
N(6)=2*K
N(8)=2*L
N(1)=N(2)-1
N(3)=N(4)-1
N(5)=N(6)-1
N(7)=N(8)-1
W(161,1)=0.
W(162,1)=0.
DO 1 M=1,8
W(161,1)=W(161,1)-W(N(M),1)*(G(9,9)*G(9,M)+G(9,10)*G(10,M)
1))
W(162,1)=W(162,1)-W(N(M),1)*(G(10,9)*G(9,M)+G(10,10)*G(10
1,M))
CALL PRINC (W,R,Z,I,J,81)
CALL PRINC (W,R,Z,J,K,81)
CALL PRINC (W,R,Z,K,L,81)
CALL PRINC (W,R,Z,L,I,81)
CALL HCPDASHLN (R(I),Z(I),R(81),Z(81),0.5)
CALL HCPDASHLN (R(J),Z(J),R(81),Z(81),0.5)
CALL HCPDASHLN (R(K),Z(K),R(81),Z(81),0.5)
CALL HCPDASHLN (R(L),Z(L),R(81),Z(81),0.5)
RETURN
END

```

The CST element subroutines and data cards complete the program.

iv) Plane quadrilateral element. (Q4).

This type of element has some scope for variation in the method of computation, and it was found that in practice that it had a negligible effect upon the effectiveness of the program whether the stiffness matrix was computed from the algebraically expanded terms or formed directly from the numerical values of the partial differentials involved. For the more complex types of element, however, the latter method saves many hours of mental labour for all but the exceptions.

A complete program follows which employs the algebraic method and a four-point Gaussian Quadrature in the Q4 element to output numerical values of principal planes and stresses for the case of a spherical shell. Although the program can be used for a general case, the sphere is one for which solutions are known by other theoretical approaches which have been also checked against experimental results.

In the job cards the NAG, dispenses with the need for library cards as it brings in the subroutine for solving simultaneous equations, and the plotter is not being used this time.

```
JOB Q4ALGEBRA,CIR214,MERCER
SECONDS 300
BFORTRAN ,290,NAG,25000
****
DGC PRG
    MASTER PLT
    DIMENSION F(200,1),E(4,4),D(4,8),A(200,12),G(8,8)
    1,R(190),Z(190),W(200,1),X(200,12)
    COMMON F,E,D,A,G,GJ
    READ (1,100) R1,T1,YM,PR,PS
    100  FORMAT (5F0.0)
    WRITE (2,101) R1,T1,YM,PR,PS
    101  FORMAT (/5F10.2/)
    CALL SPHERE (R1,T1,R,Z,4,100)
    CALL ELAST (YM,PR)
    DO 4 N=1,93,4
    DO 4 N=1,3
    I=M+N-1
    J=I+4
    K=I+5
    L=I+1
    CALL ASSEM (R,Z,I,J,K,L)
```



```

4      CONTINUE
      DO 5 J=5,93,4
      I=J-4
      K=J+4
      CALL FORCE (PS,R,Z,I,J,K)
5      CONTINUE
      CALL FORCE (PS,R,Z,93,97,97)
      CALL FORCE (PS,R,Z,1,1,5)
      F(193,1)=0.
      F(2,1)=0.
      DO 6 I=1,4
      J=2*I
      K=J+191
      A(J,12)=A(J,12)*100000000.
      A(K,12)=A(K,12)*100000000.
6      CONTINUE
      K=1
      CALL FORCE (A,200,F,200,200,11,1,W,200,X,200,12,K)
      IF (K) 7,8,7
7      WRITE (2,102)
102    FORMAT (1H1,1X,28HMATRIX NOT POSITIVE DEFINITE)
8      CONTINUE
      WRITE (2,103) ((F(I,1),W(I,1)),I=1,200)
103    FORMAT (1H ,10X,F6.1,10X,F6.1)
      DO 9 N=1,93,4
      DO 9 N=1,3
      I=M*N-1
      J=I+4
      K=I+5
      L=I+1
      CALL PRINC (W,R,Z,I,J,K,L)
9      CONTINUE
      STOP
      END

      SUBROUTINE SPHERE (RADIUS, THICKNESS, R,Z,M,N)
      DIMENSION F(200,1),E(4,4),D(4,8),A(200,12),G(8,8)
1,R(190),Z(190)
      COMMON F,E,D,A,G,GJ
      T=THICKNESS/FLØAT(M-1)
      L=N-M+1
      DO 1 I=1,L,M
      THETA=1.570796325*FLØAT(I-1)/FLØAT(L)
      CØ=CØS(THETA)
      SI=SIN(THETA)
      DO 1 J=1,M
      S=RADIUS-T*FLØAT(J-1)
      K=I+J-1
      R(K)=S*CØ
1      Z(K)=S*SI
      RETURN
      END

      SUBROUTINE ELAST (YM,PR)
      DIMENSION F(200,1),E(4,4),D(4,8),A(200,12),G(8,8)
      COMMON F,E,D,A,G,GJ
      PN=YM/((1.+PR)*(1.-2.*PR))
      DO 1 I=1,3

```

```

DØ 1 J=1,3
E(I,J)=PN*PR
1 E(I,I)=PN*(1.-PR)
E(4,4)=0.5*PN*(1.-2.*PR)
RETURN
END

SUBROUTINE DISPL (S,T,X,Y)
DIMENSION F(200,1),E(4,4),D(4,8),A(200,12),G(8,8)
1,X(4),Y(4)
COMMON F,E,D,A,G,GJ
GJ=0.125*((X(1)-X(3))*(Y(2)-Y(4))-(X(2)-X(4))*
1(Y(1)-Y(3))+S*((X(1)-X(4))*(Y(2)-Y(3))-(X(2)-X
1(3))*(Y(1)-Y(4)))+T*((X(3)-X(4))*(Y(1)-Y(2))-(X
1(1)-X(2))*(Y(3)-Y(4))))
CJ=0.125/GJ
D(1,2)=-CJ*(X(2)-X(4)+S*(X(2)-X(3))-T*(X(3)-X(4)))
D(1,4)= CJ*(X(1)-X(3)+S*(X(1)-X(4))-T*(X(3)-X(4)))
D(1,6)= CJ*(X(2)-X(4)-S*(X(1)-X(4))-T*(X(1)-X(2)))
D(1,8)=-CJ*(X(1)-X(3)-S*(X(2)-X(3))-T*(X(1)-X(2)))
D(2,1)= CJ*(Y(2)-Y(4)+S*(Y(2)-Y(3))-T*(Y(3)-Y(4)))
D(2,3)=-CJ*(Y(1)-Y(3)+S*(Y(1)-Y(4))-T*(Y(3)-Y(4)))
D(2,5)=-CJ*(Y(2)-Y(4)-S*(Y(1)-Y(4))-T*(Y(1)-Y(2)))
D(2,7)= CJ*(Y(1)-Y(3)-S*(Y(2)-Y(3))-T*(Y(1)-Y(2)))
DØ 1 I=1,7,2
J=I+1
D(4,J)=D(2,I)
1 D(4,I)=D(1,J)
H1=(1.+S)*(1.-T)*0.25
H2=(1.+S)*(1.+T)*0.25
H3=(1.-S)*(1.+T)*0.25
H4=(1.-S)*(1.-T)*0.25
R=1./(X(1)*H1+X(2)*H2+X(3)*H3+X(4)*H4)
D(3,1)=H1*R
D(3,3)=H2*R
D(3,5)=H3*R
D(3,7)=H4*R
RETURN
END

SUBROUTINE STIFF (X,Y)
DIMENSION F(200,1),E(4,4),D(4,8),A(200,12),G(8,8)
1,S(4),T(4),X(4),Y(4),R(4),C(4,8),B(8,4)
COMMON F,E,D,A,G,GJ
A1=0.577350269
PI=3.14159265
DØ 1 I=1,8
DØ 1 J=1,I
1 G(I,J)=0.
S(1)=A1
S(2)=A1
S(3)=-A1
S(4)=-A1
T(1)=A1
T(2)=-A1
T(3)=A1
T(4)=-A1
DØ 2 I=1,4

```



```

CALL DISPL (S(I),T(I),X,Y)
H1=(1.+S(I))*(1.-T(I))*0.25
H2=(1.+S(I))*(1.+T(I))*0.25
H3=(1.-S(I))*(1.+T(I))*0.25
H4=(1.-S(I))*(1.-T(I))*0.25
R(I)=H1*X(1)+H2*X(2)+H3*X(3)+H4*X(4)
GJ=GJ*R(I)*PI*2.
DØ 3 J=1,4
DØ 3 K=1,8
C(J,K)=0.
B(K,J)=D(J,K)*GJ
DØ 3 L=1,4
3 C(J,K)=C(J,K)+E(J,L)*D(L,K)
DØ 4 J=1,8
DØ 4 K=1,J
DØ 4 L=1,4
4 G(J,K)=G(J,K)+B(J,L)*C(L,K)
2 CØNTINUE
RETURN
END

```

```

SUBROUTINE ASSEM (R,Z,I,J,K,L)
DIMENSION F(200,1),E(4,4),D(4,8),A(200,12),G(8,8)
1,X(4),Y(4),R(190),Z(190),N1(8)
CØMMON F,E,D,A,G,GJ
X(1)=R(I)
X(2)=R(J)
X(3)=R(K)
X(4)=R(L)
Y(1)=Z(I)
Y(2)=Z(J)
Y(3)=Z(K)
Y(4)=Z(L)
CALL STIFF (X,Y)
N1(2)=2*I
N1(4)=2*J
N1(6)=2*K
N1(8)=2*L
N1(1)=N1(2)-1
N1(3)=N1(4)-1
N1(5)=N1(6)-1
N1(7)=N1(8)-1
DØ 1 M=1,8
DØ 1 N=1,M
N2=N1(M)-N1(N)
IF (N2) 2,3,3
2 NR=N1(N)
NC=12+N2
CØ TØ 1
3 NR=N1(M)
NC=12-N2
1 A(NR,NC)=A(NR,NC)+G(M,N)
RETURN
END

```

```

SUBROUTINE FORCE (PS,R,Z,I,J,K)
DIMENSION F(200,1),E(4,4),D(4,8),A(200,12),G(8,8)
1,R(190),Z(190)

```

```

COMMON F,E,D,A,G,GJ
PI=3.14159265
PA=PI*PS/3.
NV=2*J
NU=NV-1
F(NU,1)=PA*(R(I)*(Z(J)-Z(I))+
12.*R(J)*(Z(K)-Z(I))+R(K)*(Z(K)-Z(J)))
F(NV,1)=PA*(R(I)-R(K))*(R(I)+R(J)+R(K))
RETURN
END

```

```

SUBROUTINE PRINC (W,R,Z,I,J,K,L)
DIMENSION F(200,1),E(4,4),D(4,8),A(200,12),G(8,8)
1,X(4),Y(4),R(190),Z(190),N1(8),S1(4),W(200,1),S(4)
COMMON F,E,D,A,G,GJ
X(1)=R(I)
X(2)=R(J)
X(3)=R(K)
X(4)=R(L)
Y(1)=Z(I)
Y(2)=Z(J)
Y(3)=Z(K)
Y(4)=Z(L)
CALL DISPL (0.,0.,X,Y)
N1(2)=2*I
N1(4)=2*J
N1(6)=2*K
N1(8)=2*L
N1(1)=N1(2)-1
N1(3)=N1(4)-1
N1(5)=N1(6)-1
N1(7)=N1(8)-1
DO 1 M=1,4
S1(M)=0.
DO 1 N=1,8
1 S1(M)=S1(M)+D(M,N)*W(N1(N),1)
DO 2 M=1,4
S(M)=0.
DO 2 N=1,4
2 S(M)=S(M)+S1(N)*E(M,N)
SA=SQRT(0.25*(S(1)-S(2))**2+S(4)**2)
SB=0.5*(S(1)+S(2))
SP=SB+SA
SQ=SB-SA
SR=SP-S(1)
IF (SR) 6,7,6
7 TH=1.570796325
GO TO 5
6 CONTINUE
TH=S(4)/SR
IF (TH) 3,4,4
3 TH=-TH
TH=-ATAN(TH)
GO TO 5
4 TH=ATAN(TH)
5 ST=S(3)
WRITE (2,102) TH,SP,SQ,ST
102 FORMAT (4F10.2)

```


RETURN
END

FINISH

D/C DATA

11.00 1.10 1.00 0.20 -1.00

The input data shown is for a sphere of outer radius 11 cms, thickness 1.1 cms, with material properties, $E=1$ and $\nu=0.2$. The applied pressure of 1 on the outer surface of the shell is taken as negative for the sign convention used in the program, and gives the stress factors in the shell. In the linear elastic case the value of E cancels out in the analysis for stresses, and is only required if other information such as deflexions are included in the output.

v) The cubic quadrilateral element. (Q12).

```
MASTER PL0T
DIMENSION F(136,1),E(4,4),A(136,24),G(24,24)
1,R(68),Z(68),X(136,24),W(136,1),P(12),Q(12)
1,N(24),FF(8,1),S(4)
```

In this program which is for the case of a sphere as with the example given for the Q4 element, the arrangement of subroutines differs a little from those of the previous cases. It was not convenient to employ the repetitive "common" statement, and the dual system of nodal numbering is introduced as subroutines for the element and structural systems.

The job cards were similar to the previous case with allowance for extra core and time.

```
READ (1,100) R1,T1,YM,PR,PS
WRITE (2,102) R1,T1,YM,PR,PS
CALL ELAST (YM,PR,E)
R2=R1-T1
D0 1 I=1,57,8
TH=3.14159265*FL0AT(I-1)/128.
C0=C0S(TH)
SI=SIN(TH)
THD=3.14159265/48.
TH1=TH+THD
TH2=TH1+THD
C01=C0S(TH1)
SI1=SIN(TH1)
C02=C0S(TH2)
SI2=SIN(TH2)
J=I+4
K=I+6
R(J)=R1*C0
Z(1)=R1*SI
R(J)=R1*C01
Z(J)=R1*SI1
R(K)=R1*C02
Z(K)=R1*SI2
J=I+3
K=I+5
L=I+7
R(J)=R2*C0
Z(J)=R2*SI
R(K)=R2*C01
Z(K)=R2*SI1
R(L)=R2*C02
Z(L)=R2*SI2
K=I+2
L=I+1
CALL THIRD (R,Z,I,J,K,L)
```



```

1  CONTINUE
   R(65)=0.
   Z(65)=0.
   R(68)=0.
   Z(68)=0.
   CALL THIRD (R,Z,65,68,67,66)
   DO 6 I=1,57,8
   CALL REDUCE (P,R,I)
   CALL REDUCE (Q,Z,I)
   CALL STIFF (E,G,P,Q)
   CALL FORCE (PS,P,Q,FF)
   CALL EXPAND (N,I)
   DO 7 J=1,24
   DO 7 K=1,J
   M=N(J)-N(K)
   IF (M) 8,9,9
8  NR=N(K)
   NC=24+M
   GO TO 15
9  NR=N(J)
   NC=24-M
15 CONTINUE
   A(NR,NC)=A(NR,NC)+G(J,K)
7  G(J,K)=0.
   DO 10 J=1,3
   F(N(J),1)=F(N(J),1)+FF(J,1)
10 FF(J,1)=0.
6  CONTINUE
   F(2,1)=0.
   F(129,1)=0.
   DO 11 I=1,7,2
   J=I+1
   K=I+128
   A(J,24)=A(J,24)*1000000000.
   A(K,24)=A(K,24)*1000000000.
11 CONTINUE
   K=1
   CALL FO4ACF (A,136,F,136,136,23,1,W,136,X,136,24,K)
   IF (K) 12,13,12
12 WRITE (2,101)
13 CONTINUE
   WRITE (2,104) ((F(I,1),W(I,1)),I=1,136)
104 FORMAT (1H,10X,F6.1,10X,F6.1)
   DO 14 I=1,57,8
   CALL REDUCE (P,R,I)
   CALL REDUCE (Q,Z,I)
   CALL EXPAND (N,I)
   DO 14 J=1,12
   K=N(2*J)/2
   CALL PRINC (E,W,P,Q,J,N,S)
   WRITE (2,103) J,K,(S(L),L=1,4)
14 CONTINUE
100 FORMAT (5F0.0)
101 FORMAT (1H1,1X,28HMATRIX NOT POSITIVE DEFINITE)
102 FORMAT (/5F7.2/)
103 FORMAT (2I5,4F10.2)
   STOP
   END

```

The elasticity matrix is formed as before :-

```

SUBROUTINE ELAST (YM,PR,E)
DIMENSION E(4,4)
PN=YM/((1.+PR)*(1.-2.*PR))
DO 1 I=1,3
DO 1 J=1,3
E(I,J)=PN*PR
1 E(I,I)=PN*(1.-PR)
E(4,4)=0.5*PN*(1.-2.*PR)
RETURN
END

```

The midside nodes are automatically coordinated :-

```

SUBROUTINE THIRD (R,Z,I,J,K,L)
DIMENSION R(68),Z(68)
P=1./3.
R(K)=P*(R(I)+2.*R(J))
R(L)=P*(2.*R(I)+R(J))
Z(K)=P*(Z(I)+2.*Z(J))
Z(L)=P*(2.*Z(I)+Z(J))
RETURN
END

```

The structural nodal coordinate system which gives the minimum stiffness matrix band width is transferred into a local element numbering system which goes round the element boundary anticlockwise :-

```

SUBROUTINE REDUCE (A,B,I)
DIMENSION A(12),B(68)
A(1)=B(I)
A(2)=B(I+4)
A(3)=B(I+6)
A(4)=B(I+8)
A(5)=B(I+9)
A(6)=B(I+10)
A(7)=B(I+11)
A(8)=B(I+7)
A(9)=B(I+5)
A(10)=B(I+3)
A(11)=B(I+2)
A(12)=B(I+1)
RETURN
END

```

Similarly the element nodal numbering system is transferred back into the structural system :-

```

SUBROUTINE EXPAND (N,I)
DIMENSION N(24)
N(2)=2*I
N(4)=2*(I+4)

```



```

      N(6)=2*(I+6)
      N(8)=2*(I+8)
      N(10)=2*(I+9)
      N(12)=2*(I+10)
      N(14)=2*(I+11)
      N(16)=2*(I+7)
      N(18)=2*(I+5)
      N(20)=2*(I+3)
      N(22)=2*(I+2)
      N(24)=2*(I+1)
      DO 1 J=1,23,2
1      N(J)=N(J+1)-1
      RETURN
      END

```

The displacement matrix is formed numerically in the following subroutine from combinations of the shape functions and their partial derivatives :-

```

      SUBROUTINE DISPL (D,RJ,S,T,X,Y)
      DIMENSION D(4,24),X(12),Y(12),H(12),R(12),Z(12)
      1,P(12,12),Q(12)
      B=1./32.
      C=B*9.
      US=1.-S
      UT=1.-T
      VS=1.+S
      VT=1.+T
      WS=US*VS
      WT=UT*VT
      U3S=1.-3.*S
      U3T=1.-3.*T
      V3S=1.+3.*S
      V3T=1.+3.*T
      U=-10.+9.*(S*S+T*T)
      H(1)=B*VS*UT*U
      H(4)=B*VS*VT*U
      H(7)=B*US*VT*U
      H(10)=B*US*UT*U
      H(2)=C*VS*WT*U3T
      H(3)=C*VS*WT*V3T
      H(5)=C*VT*WS*V3S
      H(6)=C*VT*WS*U3S
      H(8)=C*US*WT*V3T
      H(9)=C*US*WT*U3T
      H(11)=C*UT*WS*U3S
      H(12)=C*UT*WS*V3S
      U=-10.+18.*S+27.*S*S+9.*T*T
      V=10.+18.*S-27.*S*S-9.*T*T
      R(1)=B*UT*U
      R(4)=B*VT*U
      R(7)=B*VT*V
      R(10)=B*UT*V
      R(2)=C*WT*U3T
      R(3)=C*WT*V3T
      R(8)=-R(3)

```

```

R(9)=-R(2)
U=-3.-2.*S+9.*S*S
V=3.-2.*S-9.*S*S
R(5)=C*VT*V
R(6)=C*VT*U
R(11)=C*UT*U
R(12)=C*UT*V
U=10.-9.*S*S+18.*T-27.*T*T
V=-10.+9.*S*S+18.*T+27.*T*T
Z(1)=B*VS*U
Z(4)=B*VS*V
Z(7)=B*US*V
Z(10)=B*US*U
U=-3.-2.*T+9.*T*T
V=3.-2.*T-9.*T*T
Z(2)=C*VS*U
Z(3)=C*VS*V
Z(8)=C*US*V
Z(9)=C*US*U
Z(5)=C*WS*V3S
Z(6)=C*WS*U3S
Z(11)=-Z(6)
Z(12)=-Z(5)
RI=0.
DØ 1 I=2,12
K=I-1
DØ 1 J=1,K
P(I,J)=R(I)*Z(J)-R(J)*Z(I)
1 P(J,I)=-P(I,J)
DØ 2 I=1,12
RI=RI+H(I)*X(I)
Q(I)=0.
DØ 2 J=1,12
2 Q(I)=Q(I)+P(I,J)*Y(J)
GJ=0.
DØ 3 I=1,12
3 GJ=GJ+Q(I)*X(I)
CJ=1./GJ
IF (RI) 6,7,6
7 RJ=0.
GØ TØ 8
6 CØNTINUE
RJ=1./RI
8 CØNTINUE
DØ 4 I=1,12
J=2*I
K=J-1
D(2,K)=Q(I)*CJ
D(1,J)=0.
DØ 5 I=1,12
5 D(1,J)=D(1,J)-P(I,L)*X(L)
D(1,J)=D(1,J)*CJ
D(4,K)=D(1,J)
D(4,J)=D(2,K)
4 D(3,K)=H(I)*RJ
RJ=RI*CJ*2.*3.14159265
RETURN
END

```


The stiffness matrix is formed with a 3 X 3 point Gauss Quadrature. This subroutine gives the 24 X 24 element stiffness matrix which is assembled into the structural stiffness matrix in the master program.

```

SUBROUTINE STIFF (E,G,X,Y)
DIMENSION E(4,4),G(24,24),X(12),Y(12),D(4,24)
1,S(9),T(9),W(9),C(4,24),B(24,4)
A1=0.7745966692
W1=1./9.
W2=W1*8.
W1=W1*5.
W(1)=W1*W1
W(2)=W1*W2
W(3)=W(1)
W(4)=W(2)
W(5)=W2*W2
W(6)=W(2)
W(7)=W(1)
W(8)=W(2)
W(9)=W(1)
S(1)=-A1
S(2)=-A1
S(3)=-A1
S(4)=0.
S(5)=0.
S(6)=0.
S(7)=A1
S(8)=A1
S(9)=A1
T(1)=-A1
T(2)=0.
T(3)=A1
T(4)=-A1
T(5)=0.
T(6)=A1
T(7)=-A1
T(8)=0.
T(9)=A1
DO 1 I1=1,9
CALL DISPL (D,RJ,S(I1),T(I1),X,Y)
DO 2 I=1,4
DO 2 J=1,24
C(I,J)=0.
B(J,I)=D(I,J)*RJ*W(I1)
DO 2 K=1,4
2 C(I,J)=C(I,J)+E(I,K)*D(K,J)
DO 3 I=1,24
DO 3 J=1,I
DO 3 K=1,4
3 C(I,J)=G(I,J)+B(I,K)*C(K,J)
1 CONTINUE
RETURN
END

```

The nodal force matrix is also formed in a subroutine which employs the 3 point Gauss Quadrature along the element boundaries which are subjected to applied pressure which in this case is assumed to be uniform pressure.

```

SUBROUTINE FORCE (PS,X,Y,F)
DIMENSION X(12),Y(12),F(8,1),G(8,1),H(4),S(4)
Z=9.*8.*16.*16.
Z=3.14159265*PS/Z
N=1
5   IF (N) 1,2,3
1   T=0.774596669
   GO TO 4
2   T=0.
   GO TO 4
3   T=-0.774596669
4   A=1.-T
   B=1.+T
   C=1.-3.*T
   D=2.-C
   E=-C*D
   H(1)=A*E
   H(2)=A*B*C*9.
   H(4)=B*E
   H(3)=A*B*D*9.
   S(1)=-E+18.*T*A
   S(2)=(E-2.*B)*9.
   S(3)=(-E+2.*A)*9.
   S(4)=E+18.*T*B
   R=0.
   RT=0.
   ZT=0.
   DO 6 I=1,4
   R=R+H(I)*X(I)
   RT=RT+S(I)*X(I)
6   ZT=ZT+S(I)*Y(I)
   DO 8 I=1,4
   J=2*I
   K=J-1
   G(K,1)=R*H(I)*ZT
8   G(J,1)=-R*H(I)*RT
   IF (N) 10,11,12,
10  CONTINUE
   DO 13 I=1,8
13  F(I,1)=5.*G(I,1)+F(I,1)
   GO TO 7
11  CONTINUE
   DO 14 I=1,8
14  F(I,1)=8.*G(I,1)+F(I,1)
   GO TO 16
12  CONTINUE
   DO 15 I=1,8
15  F(I,1)=5.*G(I,1)+F(I,1)
16  N=N+1
   GO TO 5

```



```

7      DØ 17 I=1,8
17     F(I,1)=F(I,1)*Z
      RETURN
      END

```

The equivalent applied nodal loading and nodal deflexion matrices are printed out and the final subroutine produces the stresses at the element nodes.... these are principal stress factors in this case and the angle of the principal planes is in radians. Also the nodal deflexions are based on a unit value for E, and must be divided by this true value of Young's Modulus which may be found from experiments or some constant value assumed for the purpose of establishing numerical results :-

```

      SUBROUTINE PRINC (E,W,X,Y,I,N,A)
      DIMENSION E(4,4),W(136,1),X(12),Y(12),N(24)
1      A(4),D(4,24),B(4),C(4),S(12),T(12)
      P=1./3.
      S(1)=1.
      S(2)=1.
      S(3)=1.
      S(4)=1.
      S(5)=P
      S(6)=-P
      S(7)=-1.
      S(8)=-1.
      S(9)=-1.
      S(10)=-1.
      S(11)=-P
      S(12)=P
      T(1)=-1.
      T(2)=-P
      T(3)=P
      T(4)=1.
      T(5)=1.
      T(6)=1.
      T(7)=1.
      T(8)=P
      T(9)=-P
      T(10)=-1.
      T(11)=-1.
      T(12)=-1.
      CALL DISPL (D,RJ,S(I),T(I),X,Y)
      DØ 1 J=1,4
      B(J)=0.
      DØ 1 K=1,24
1      B(J)=B(J)+D(J,K)*W(N(K),1)
      DØ 2 J=1,4
      C(J)=0.
      DØ 2 K=1,4
2      C(J)=C(J)+E(K)*E(J,K)

```

```

A(4)=C(3)
R=SQRT(0.25*(C(1)-C(2))**2+C(4)**2)
Z=0.5*(C(1)+C(2))
A(2)=R+Z
A(3)=-R+Z
F=A(2)-C(1)
IF (F) 3,4,3
4  A(1)=1.570796325
   GO TO 5
3  CONTINUE
   A(1)=C(4)/F
   IF (A(1)) 6,7,7
6  A(1)=-A(1)
   A(1)=ATAN(A(1))
   GO TO 5
7  A(1)=ATAN(A(1))
5  CONTINUE
   RETURN
   END

```


vi) Comparative assessment of element types.

There are two considerations required for comparison, the economy of the computer program arising from the use of an element type, and accuracy of results. An assessment of the efficiency of an element arises from a balance between these two factors. The matter of generality which arises with other methods is not a problem with the Finite Element displacement method, being its greatest attraction compared with other methods. However, efforts to capitalise on this aspect by the commercial development of "universal" finite element programs which are intended to be as general as possible result in unwieldy creations of doubtful validity which are more difficult to test than to write a new program for a specific problem. The examples carried out illustrate the variety of operations which can be incorporated into a standardised program which include variations for the method of integration and assembly of the stiffness matrix and in the form of the output required. The parts which do not vary are so small a part of the total program that they are a simple matter to reintroduce into each program.

On the question of economy it can be seen from the programs that they get longer and bigger with increased complexity of element type as would appear logical. The question of accuracy can be assessed in two ways, either by comparing results with cases for which solutions have been found by other methods, or by comparing the results given by different element types with each other and with experimental work on more general cases. Most finite elements have been applied to the case of the uniform sphere under uniform pressure, and this specific case is helpful in the first place to check whether the program could be working as well as the degree of nonuniformity in the results. But it is found that many elements which stand up

well in the case of the uniform sphere, show large variations in numerical values where stress concentrations occur in general cases.

The computer drawings which follow illustrate first the three types of constant strain elements and the stress factor flow pattern which arises in the frustum of a cone under uniform unit applied pressure. The continuous lines within the elements represent the direction of principal tensile planes and the broken lines are those in compression. The number of lines per centimetre is the stress factor. The great variation in numerical values can be clearly seen particularly in the areas of stress concentration. These values can be further varied by rearrangement of the mesh pattern.

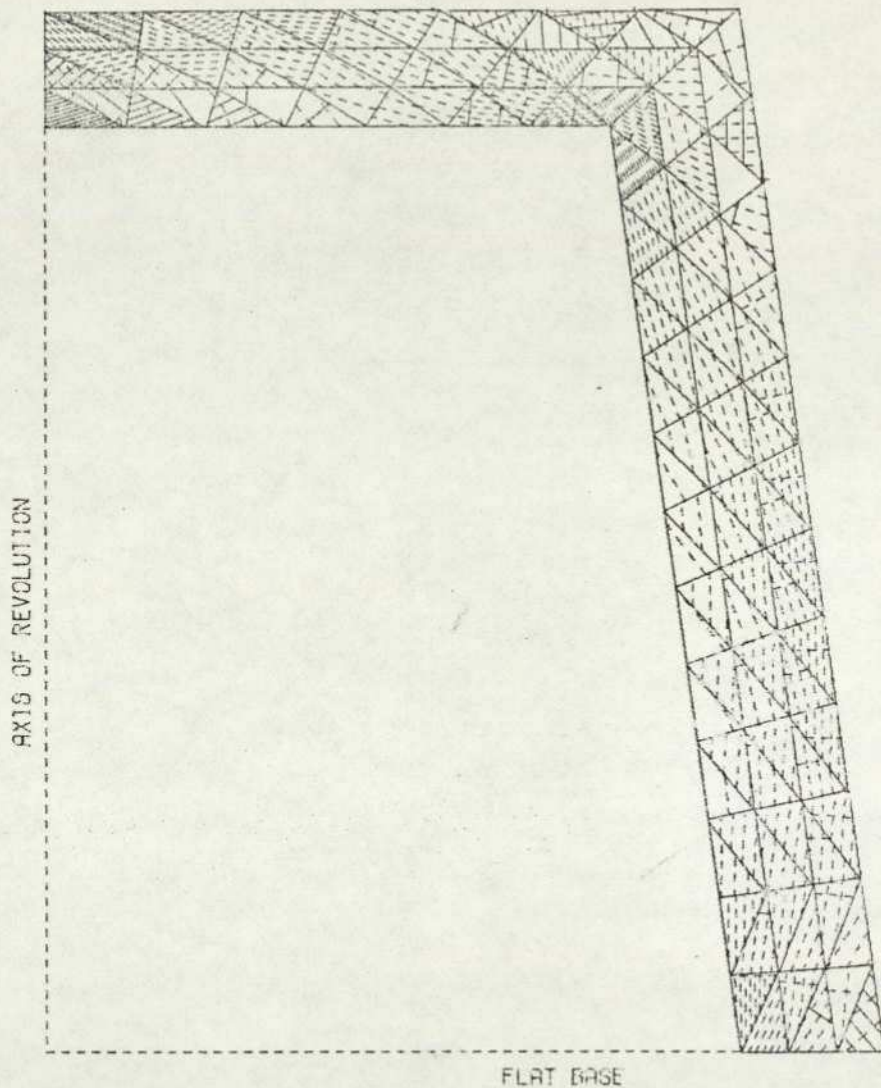
The main value in the method then is in the production of a picture of the way in which the stresses flow in the continuum. The Q2CST element gives smoother lines by dispensing with some of the local aberrations of the method. Experienced structural engineers can draw these lines approximately without the assistance of a computer and arrange the finite element mesh with a larger number of smaller elements in the areas of stress concentration in order to obtain results which are closer to convergence.

The three computer drawings which follow the CST element output and its combinations are some of the nodal numbering systems which were used in the Q4 and Q12 element programs. Three elements are taken through the thickness of the shell for the Q4 element as was done for the CST and its combinations. One Q12 element is taken across the thickness of the shell which is equivalent to the four nodes of the previous cases and the program, though larger, is still something of the same order.

Finally the numerical output of stress factors as obtained for the case of the sphere in the last two programs are shown.

Frustum of a Cone - General Case.

Computer graphics for principal stress factors.



CST Element Type.

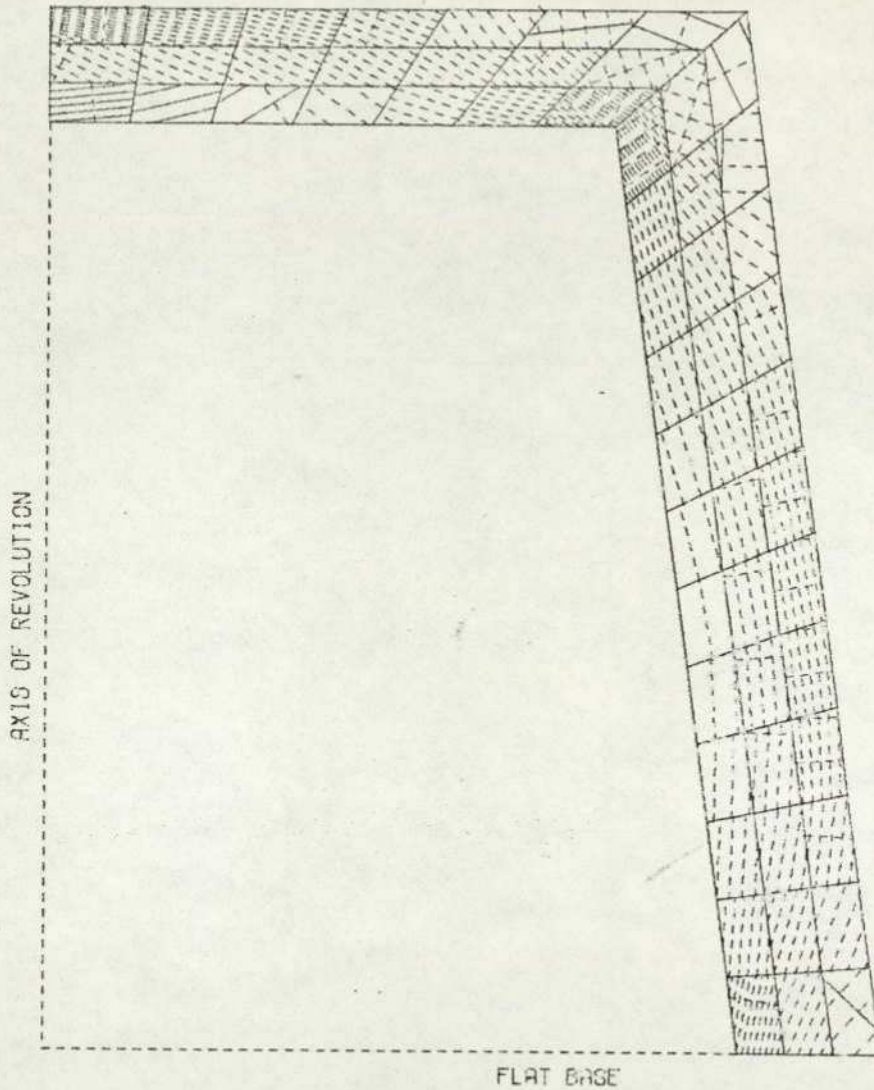
Note: Stress Factor = Number of lines per centimetre within each
element.

Continuous lines = tension

Broken lines = compression

Frustum of a Cone - General Case.

Computer graphics for principal stress factors.



Q2CST Element type.

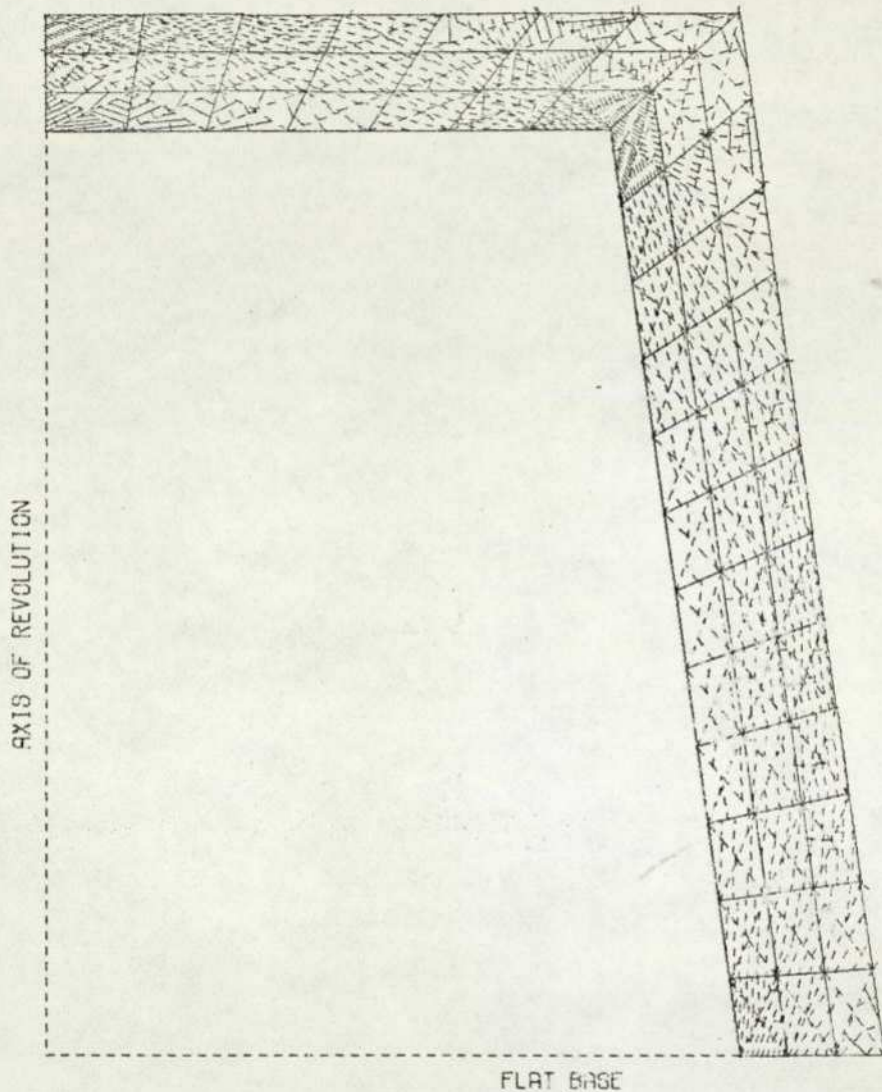
Note: Stress factor = Number of lines per centimetre within each element.

Continuous lines = tension

Broken lines = Compression

Frustum of a Cone - General Case.

Computer Graphics for principal stress factors.



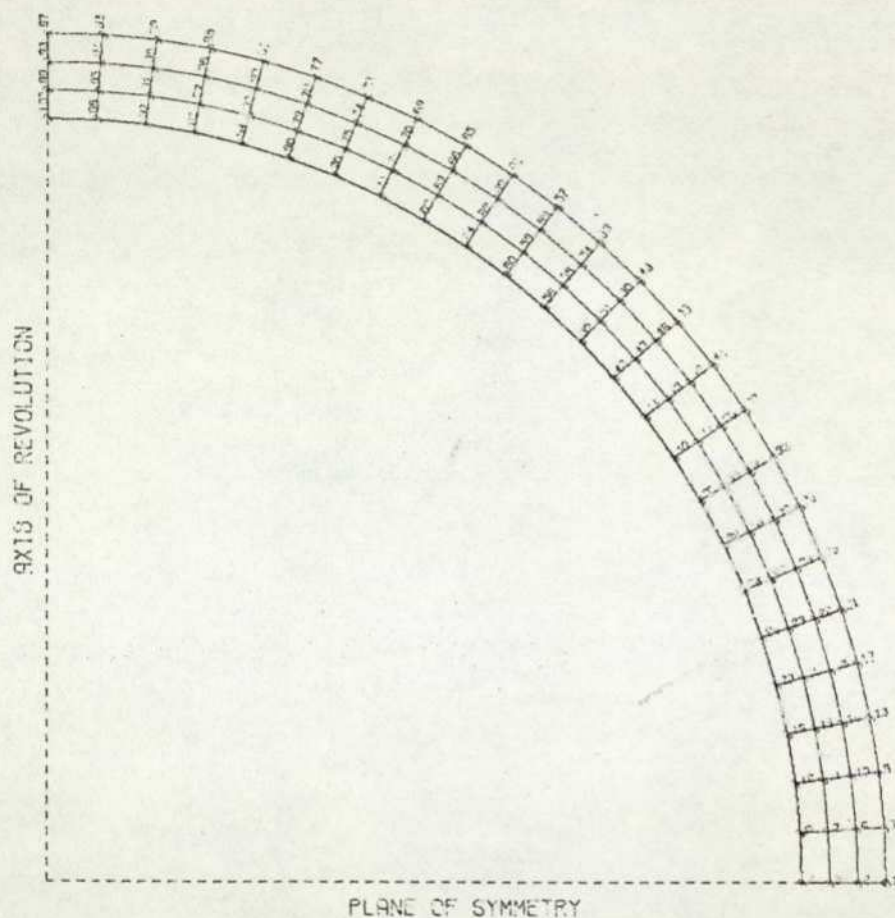
Q4CST Element Type.

Note : Stress Factor = Number of lines per centimetre within each element.

Continuous lines = tension

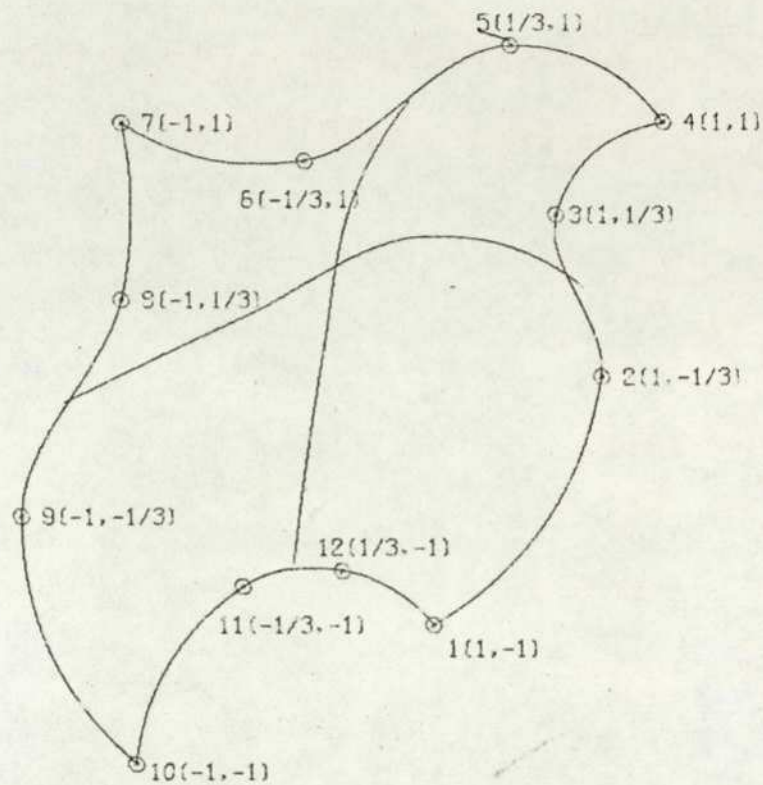
Broken lines = Compression

Hemispherical shell - computer graphics to illustrate general case of global nodal numbering system.



Q4 element (also applicable to CST, Q2CST, and Q4CST type elements).

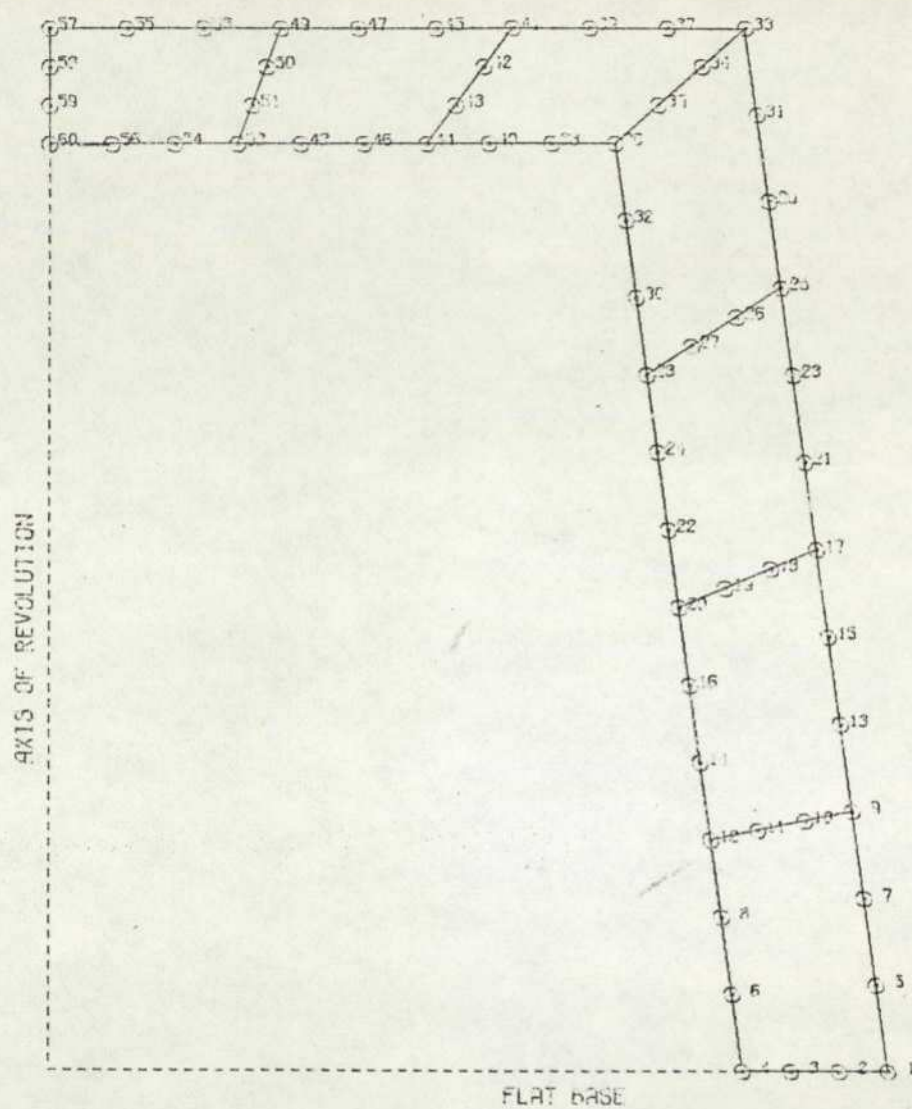
Computer graphics to illustrate a general Q12 element
local nodal numbering system.



Q12 SERENDIPITY CUBIC QUADRILATERAL
ISOPARAMETRIC ELEMENT LOCAL COORDINATES

Frustum of a Cone - General case

Computer graphics to illustrate global nodal numbering system for the Q12 element.



Q12 Element.

Numerical output for the Q4 element applied to a thick uniform sphere subjected to uniform external pressure.

Outer diameter = 22 cm thickness = 1.1 cm

Assumed $E = 1$ and $\nu = 0.2$ applied pressure = -1

nodal coordinates		principal plane radians	principal stresses		
local	structural		radial plane major	minor	hoop
1	2	0.02	-0.91	-5.23	-5.23
2	1	0.02	-0.81	-4.99	-4.99
3	5	0.05	-0.81	-4.99	-4.99
4	6	0.05	-0.91	-5.23	-5.23
1	3	0.02	-0.61	-5.40	-5.40
2	2	0.02	-0.50	-5.12	-5.12
3	6	0.05	-0.50	-5.12	-5.12
4	7	0.05	-0.61	-5.40	-5.40
1	4	0.02	-0.26	-5.60	-5.60
2	3	0.02	-0.14	-5.28	-5.28
3	7	0.05	-0.14	-5.28	-5.28
4	8	0.05	-0.26	-5.60	-5.60

These results repeat for each trio of elements and only show any significant error approaching the singularity which arises in shell theory at the axis of revolution where the stresses become indeterminate. In the program this turns up in the formation of the displacement matrix where a conditional statement is inserted to the effect that when the radius $RI=0$, then $RJ=1./RI=0$, so that the boundary condition that the shear stress must be zero at the axis of revolution is satisfied.

In the radial plane the term "major" principal stress refers to that solution which is found from the quadratic equation assuming the positive term, and the "minor" is that with the negative. The major stress factor thus turns out to be the numerically smaller of the two because all the stresses in this case are negative. Due to symmetry these coincide with the shell stresses so that the major principal stress is in the radial direction towards the centre of the sphere, and the minor and hoop principal stresses are in the plane of

the shell and due to symmetry again are equal to each other at any point in the continuum.

The structural node numbers 1 and 5, 2 and 6, 3 and 7, and 4 and 8, are symmetrically identical and give identical results within each element, but not between elements where discontinuity is exhibited across the element boundaries. The usual practice in this case is to find the nodal averages which may be satisfactory for the uniform sphere, but not necessarily for a general case involving stress concentrations. Another method is to take the principal stresses at the element centroid which gives results as follows with the Q4 element :-

outer element	0.03	-0.86	-5.10	-5.10
middle "	0.03	-0.55	-5.26	-5.26
inner "	0.03	-0.20	-5.43	-5.44

The results for the Q12 element are as follows :-

1	1	0.00	-1.00	-5.03	-5.03
2	5	0.07	-1.00	-5.03	-5.03
3	7	0.13	-1.00	-5.03	-5.03
4	9	0.20	-1.00	-5.03	-5.03
5	10	0.20	-0.71	-5.18	-5.18
6	11	0.20	-0.38	-5.34	-5.34
7	12	0.20	-0.00	-5.54	-5.54
8	8	0.13	-0.00	-5.54	-5.54
9	6	0.07	-0.00	-5.54	-5.54
10	4	0.00	-0.00	-5.54	-5.54
11	3	-0.00	-0.38	-5.34	-5.34
12	2	-0.00	-0.71	-5.18	-5.18

These results then repeat over six more elements and in the final eighth element which was taken to the axis of revolution there is a different result as with the previous program. Ignoring the final boundary results, the Q12 element gives numerical values which agree with those obtained by other analytical methods, as do the results for the Q4 element taken at the centroid. The nodal values for the Q4 element is not in agreement which suggests an error in the program.

The Q12 program with eight elements occupied 26 K core and was programmed with a time limit of 600 seconds.

The identical node groups

1, 5, 7, 9

2, 10

3, 11

4, 6, 8, 12

in the structural numbering system all give identical results and there is no necessity to take nodal averages. Although this program is less economical than the simpler elements, it does open the possibility of efficient plotting of cubic curves for the principal planes and stress contours.

However, although the Q12 program was successful for the sphere, the problem of continuity of stresses at the element boundary arises with a more general case. For instance, by adding a boundary condition to the master program :-

$$F(1,1)=0.$$

and in the loop for the stiffness matrix :-

$$A(I,24)=A(I,24)*100000000.$$

the program becomes the case of a hemisphere which is encastre at the equator. The principal planes and stress factors are then as follows :

nodes		principal		stress factor	
local	structural	plane radians	major	minor	hoop
1	1	-1.17	1.99	0.16	0.43
2	5	-0.22	0.16	-1.45	-0.57
3	7	0.05	-0.80	-3.42	-1.74
4	9	0.33	-1.21	-4.98	-2.80
5	10	0.12	-0.81	-5.23	-2.77
6	11	0.10	-0.13	-5.36	-2.69
7	12	0.20	0.61	-5.59	-2.64
8	8	0.10	-0.39	-7.18	-2.47
9	6	-0.01	-1.33	-9.70	-2.59
10	4	-0.16	-2.96	-13.21	-3.23
11	3	-0.21	-1.64	-7.87	-1.90
12	2	-0.35	-0.38	-3.19	-0.71
1	9	0.25	-1.28	-5.01	-2.82
2	13	0.31	-1.23	-5.88	-3.63
3	15	0.34	-1.09	-6.33	-4.24
4	17	0.38	-0.96	-6.51	-4.69

5	18	0.40	-0.60	-5.61	-4.50
6	19	0.41	-0.40	-4.93	-4.39
7	20	0.40	-0.25	-4.05	-4.26
8	16	0.34	0.22	-4.03	-3.66
9	14	0.30	0.50	-4.51	-3.10
10	12	0.23	0.69	-5.34	-2.57
11	11	0.09	-0.11	-5.35	-2.69
12	10	0.07	-0.72	-5.12	-2.73
1	17	0.37	-0.93	-6.41	-4.66
2	21	0.44	-0.97	-6.33	-4.98
3	23	0.52	-1.00	-6.19	-5.18
4	25	0.60	-0.99	-5.94	-5.27
5	26	0.61	-0.69	-5.54	-5.26
6	27	0.61	-0.34	-5.06	-5.23
7	28	0.59	0.06	-4.54	-5.22
8	24	0.52	-0.02	-4.31	-4.97
9	22	0.45	-0.11	-4.13	-4.65
10	20	0.37	-0.25	-4.05	-4.26
11	19	0.40	-0.38	-4.87	-4.37
12	18	0.40	-0.61	-5.66	-4.51
1	25	0.60	-1.00	-6.01	-5.29
2	29	0.66	-1.00	-5.78	-5.31
3	31	0.72	-1.00	-5.59	-5.30
4	33	0.78	-1.00	-5.40	-5.26
5	34	0.80	-0.70	-5.33	-5.37
6	35	0.80	-0.37	-5.27	-5.49
7	36	0.79	-0.00	-5.21	-5.64
8	32	0.72	0.00	-5.04	-5.56
9	30	0.66	0.02	-4.81	-5.42
10	28	0.59	0.05	-4.57	-5.22
11	27	0.61	-0.34	-5.08	-5.24
12	26	0.61	-0.70	-5.56	-5.26
1	33	0.78	-1.00	-5.41	-5.26
2	37	0.85	-0.99	-5.26	-5.21
3	39	0.92	-0.99	-5.15	-5.16
4	41	0.98	-1.00	-5.07	-5.11
5	42	0.99	-0.72	-5.21	-5.29
6	43	0.99	-0.38	-5.36	-5.48
7	44	0.98	0.01	-5.54	-5.70
8	40	0.92	-0.01	-5.48	-5.71
9	38	0.85	-0.02	-5.37	-5.69
10	36	0.79	-0.01	-5.23	-5.65
11	35	0.80	-0.37	-5.28	-5.49
12	34	0.80	-0.71	-5.35	-5.37
1	41	0.98	-1.00	-5.08	-5.11
2	45	1.05	-0.99	-5.02	-5.07
3	47	1.11	-0.99	-4.99	-5.04
4	49	1.18	-1.00	-4.97	-5.01
5	50	1.18	-0.72	-5.16	-5.20
6	51	1.18	-0.38	-5.37	-5.41
7	52	1.18	0.00	-5.62	-5.65
8	48	1.11	-0.01	-5.61	-5.68
9	46	1.05	-0.01	-5.59	-5.69
10	44	0.98	0.01	-5.55	-5.70
11	43	0.99	-0.38	-5.37	-5.48
12	42	0.99	-0.72	-5.21	-5.29
1	49	1.18	-1.00	-4.97	-5.01
2	53	1.24	-1.00	-4.96	-4.99

3	55	1.31	-1.00	-4.96	-4.98
4	57	1.37	-1.00	-4.96	-4.97
5	58	1.37	-0.72	-5.15	-5.16
6	59	1.37	-0.38	-5.36	-5.37
7	60	1.37	-0.00	-5.60	-5.60
8	56	1.31	-0.00	-5.61	-5.62
9	54	1.24	-0.00	-5.62	-5.64
10	52	1.18	0.00	-5.62	-5.65
11	51	1.18	-0.38	-5.37	-5.41
12	50	1.18	-0.72	-5.16	-5.20
1	57	1.37	-1.00	-4.96	-4.97
2	61	1.44	-1.00	-4.96	-4.97
3	63	1.51	-1.00	-4.96	-4.97
4	65	-1.57	0.05	-3.92	-0.77
5	66	-1.57	0.39	-4.04	-0.73
6	67	-1.57	0.79	-4.18	-0.68
7	68	-1.57	1.24	-4.34	-0.62
8	64	1.51	-0.00	-5.59	-5.59
9	62	1.44	-0.00	-5.59	-5.59
10	60	1.37	-0.00	-5.60	-5.60
11	59	1.37	-0.38	-5.36	-5.37
12	58	1.37	-0.72	-5.15	-5.16

It can be seen that there is stress turbulence at the encastre boundary, and this gives rise to local tensile stress and a high compressive stress factor which falls off towards a more even distribution over most of the shell.

The numerical values of stress factors at nodes which lie on a common boundary between elements are not unique at any given node in the region of perturbation, but the values for adjacent elements tends towards equality over the rest of the shell. This is due to limitations on the element in the representation of shear forces and bending moments. This limitation is seen to be a common factor with simple elements. Its effect is reduced by increasing the program core either with more complex elements or by increasing the number of simpler elements.

The imposition of partial contiguity renders full numerical convergence unattainable in practice when the finite element method is applied to a continuum. The degree of accuracy attainable ultimately depends upon the capacity of the computer, and compared with shell theory the method has poor economy and low efficiency.

CHAPTER THREE

SHELL THEORY

i) The shell of constant strength.

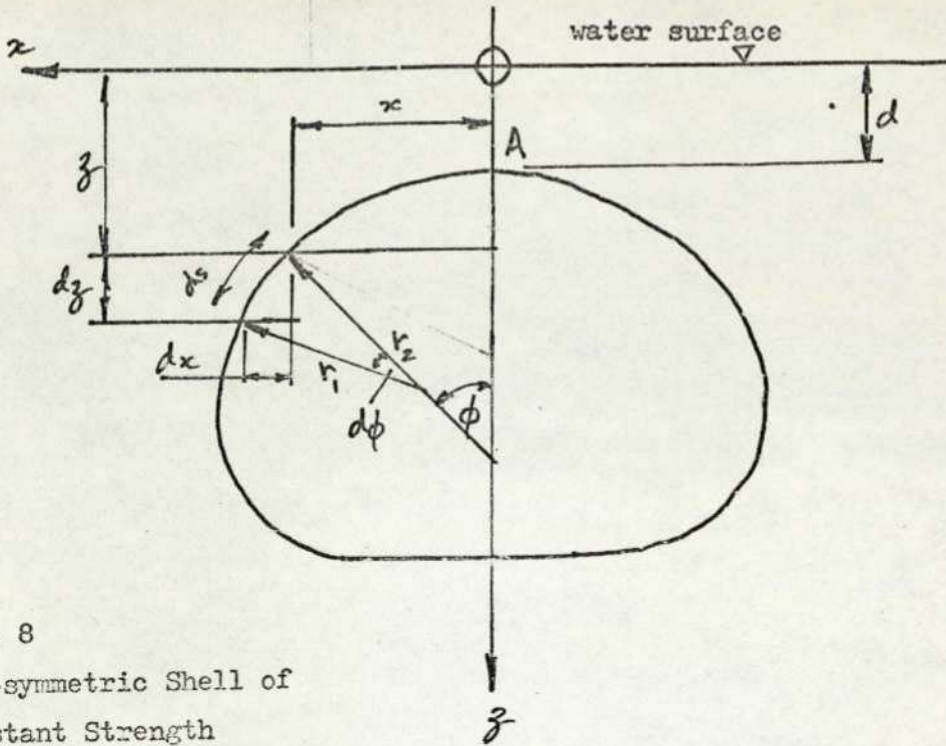


Fig. 8
Axi-symmetric Shell of
Constant Strength

A starting point in the understanding of thin shell theory for the submerged shell of revolution is the shell of constant strength analysed by membrane theory.

It is assumed that the shell is of constant thickness and the membrane stress resultants are constant throughout.

$$\text{ie, } N_\theta = N_\phi = \text{constant}$$

The applied pressure on the outer surface of the shell is

$$Z = \gamma_z$$

The basic equations of membrane theory are :-

$$\frac{dN_\phi}{d\phi} - N_\theta r_1 \cos\phi + \gamma r_1 r = 0, \quad N_\phi/r_1 + N_\theta/r_2 + Z = 0$$

Which become the single equation :-

$$N_\theta \left(\frac{1}{r_2} + \frac{1}{r_1} \right) + \gamma_z = 0 \quad \text{--- (1)}$$

We have

$$r_2 = \frac{x}{\sin \phi}$$

and

$$ds = r_1 d\phi = \frac{dx}{\cos \phi}$$

substituting into equation (1)

$$N_\theta \left(\frac{\sin \phi}{x} + \frac{d \sin \phi}{dx} \right) + \gamma z = 0 \quad \text{--- (2)}$$

At the crown of the shell, A, we also have $z = d$

$$\text{and } r_1 = r_2 = r_c$$

again substituting into equation (1)

$$\frac{2N_\theta}{r_c} = -\gamma d$$

and putting

$$a^2 = \frac{r_c d}{2}$$

then

$$\frac{N_\theta}{\gamma} = -a^2 \quad \text{--- (3)}$$

substituting (2) into (3)

$$\frac{\sin \phi}{x} + \frac{d(\sin \phi)}{dx} = \frac{z}{a^2} \quad \text{--- (4)}$$

also by definition of the meridional geometry

$$\frac{dz}{dx} = \tan \phi \quad \text{--- (5)}$$

putting

$$u = \sin \phi \quad \text{--- (6)}$$

$$\text{differentiating (6)} \quad du = \cos \phi d\phi \quad \text{--- (7)}$$

and substituting (6) into (4)

$$\frac{du}{dx} = \frac{z}{a^2} - \frac{u}{x} \quad \text{--- (8)}$$

The meridian can then be plotted using finite differences with ϕ as the independent variable.

In addition to equation (6), the difference equations are written :-

$$\text{from (7)} \quad \Delta u = \cos \phi \Delta \phi \quad \text{--- (9)}$$

$$\text{from (8)} \quad \frac{\Delta u}{\Delta x} = \frac{3}{a^2} - \frac{u}{x} \quad \text{--- (10)}$$

$$\text{and from (5)} \quad \frac{\Delta z}{\Delta x} = \tan \left(\phi + \frac{\Delta \phi}{2} \right) \quad \text{--- (11)}$$

The values for $\Delta \phi$ are then chosen to avoid the case

$$\phi + \frac{\Delta \phi}{2} = \frac{\pi}{2}$$

which in (11) gives

$$\frac{\Delta z}{\Delta x} = \infty$$

initially

$$\left. \begin{aligned} \Delta u &= \Delta \phi \\ \Delta x &= \frac{2a^2}{d} \Delta \phi \\ \Delta z &= \Delta x \tan \frac{\Delta \phi}{2} \end{aligned} \right\} \text{--- (12)}$$

The following computer program illustrates the procedure :-

JOB SUBMEMBRANE, CIR214, MERCER

SECONDS 200

REFRTRAN , , 190, PLOT

DOC PRG

{ where PLOT opens the plotter

{ library subroutines

```

MASTER PLOT
DIMENSION X(151),Y(151)
READ (1,10) G,DF,DN,CN
10  FORMAT (4F0.0)
A=-CN/G
I=1
U=0.
Z=DN
F=0.
X(I)=0.
Y(I)=0.
I=I+1
DU=DF
DDX=2.*A*DF/DN
DZ=DDX*TAN(DF/2.)
U=DU
Z=Z+DZ
F=DF
X(I)=DDX
Y(I)=Z-DN
1  J=I

```

{ X and Y are coordinates
{ G is the specific weight
{ of fluid, DF the increment
{ of ϕ , the angle, DN the
{ submerged depth at the crown
{ and CN the value of the
{ stress resultant.


```

I=I+1
DUX=Z/A-U/X(J)
DU=DF*COS(F+DF/2.)
DDX=DU/DUX
DZX=TAN(F+DF/2.)
DZ=DDX*DZX
U=U+DU
Z=Z+DZ
F=F+DF
X(I)=X(J)+DDX
Y(I)=Z-DN
IF (I-150) 1,1,2
2 CALL HGPSTART (6HMERGER,6)
CALL HGPCURVE (X,Y,151,3,0.0,4.712389)
CALL HGPFINISH
STOP
END
FINISH

```

DGC DATA

1. 0.0314159265 10. -30.

The curve may be plotted with an alternative which makes no assumptions for the slope, or with the assumption made at the start or finish only, by replacing the number 3 in the plotter subroutine by 0, 1, or 2, respectively. The meridian is plotted with the crown of the shell at the origin of the axes which have not been included in the above program.

By variation of the data it can be shown how the meridian is a spiral curve with the crown of the shell near the surface, and tends towards a sphere as depth increases.

The relative influence of the variable component of pressure is a function of the ratio

$$d' = \frac{\text{depth of submergence}}{\text{height of shell structure}}$$

Then the percentage of the total applied pressure which is due to the variable component may be expressed as

$$R = \frac{100}{2d' + 1} \%$$

Thus when $d' = 0$, $R = 100 \%$, and when $d' > 10$, $R < 5 \%$, and it may be considered that this implies that, if the structure is submerged

more than ten times its own height, it can be classified as a deep submergence structure and analysed with the assumption of uniform applied pressure. In the experimental work which was carried out on shells about 10 or 15 cm high in an oil pressure tank, the variable component is approximately

$$= \frac{0.9 \times 62.4 \times 5}{1728 \times 2} = 0.08 \text{ psi}$$

and with readings being taken at increments of 100 psi

$$R = \frac{100 \times 0.08}{100.08} = 0.08 \%$$

which can be ignored in the analysis.

At 100 psi pressure the consideration of the geometry of pressure distribution gives $d' = 600$, which is equivalent to a 10 ft high model placed at a depth of 1000 fathoms. Thus the equivalent depth in sea water under the assumption of uniform pressure and a constant value of specific weight of sea water will be the applied pressure divided by this specific weight to give the stress level encountered in practice.

Thus for a pressure increment of 100 psi in sea water weighing 64 lb/cu ft, the increment in depth of submergence will be approximately

$$d = \frac{100 \times 144}{64 \times 6} = 37.5 \text{ fathoms}$$

and this is the equivalent depth which is represented by the stress analysis.

The self weight of a shell may require consideration in the analysis of a concrete dam, for instance, but with the deep submergence shell structure it is found to be a very small variable compared with the applied pressure and is ignored in the stress analysis.

ii) Thin hemispherical shell.

We can proceed by picking up the general equations for shells of revolution under axisymmetric loading which are laid down in the work of Prof. J.E. Gibson¹⁵, using the notation of Timoshenko¹².

They are as follows :-

$$\frac{d}{d\phi}(N_{\phi}r) - N_{\theta}r_1 \cos\phi - Q_{\phi}r_2 \sin\phi + Yr_1r_2 \sin\phi = 0$$

—————(13)

$$\frac{d}{d\phi}(Q_{\phi}r) + N_{\theta}r_1 \sin\phi + N_{\phi}r_2 \sin\phi + Zr_1r_2 \sin\phi = 0$$

—————(14)

$$\frac{d}{d\phi}(M_{\phi}r) - M_{\theta}r_1 \cos\phi - Q_{\phi}r_1r_2 \sin\phi = 0$$

—————(15)

$$N_{\theta} = Et \left[(\nu \cot\phi - \omega)/r_2 + \nu \left(\frac{d\nu}{d\phi} - \omega \right)/r_1 \right] / (1-\nu^2)$$

—————(16)

$$N_{\phi} = Et \left[\left(\frac{d\nu}{d\phi} - \omega \right)/r_1 + \nu (\nu \cot\phi - \omega)/r_2 \right] / (1-\nu^2)$$

—————(17)

$$M_{\phi} = -D \left[\frac{1}{r_1} \frac{d\alpha}{d\phi} + \frac{\nu}{r_2} \alpha \cot\phi \right] \quad \text{—————(18)}$$

$$M_{\theta} = -D \left[\frac{1}{r_2} \alpha \cot\phi + \frac{\nu}{r_1} \frac{d\alpha}{d\phi} \right] \quad \text{—————(19)}$$

$$\text{where } D = Et^3/12(1-\nu^2) \quad \text{—————(20)}$$

$$\text{and } \alpha = \frac{\nu}{r_1} + \frac{1}{r_1} \frac{d\nu}{d\phi} \quad \text{—————(21)}$$

Into the general equations we substitute as follows for the spherical shell :-

$$\gamma = 0 \quad z = p$$

$$r_1 = r_2 = a \quad r = a \sin \phi$$

$$\therefore \frac{d}{d\phi} (N\phi \sin \phi) - N_0 \cos \phi - Q\phi \sin \phi = 0 \quad \text{--- (22)}$$

$$\frac{d}{d\phi} (Q\phi \sin \phi) + N_0 \sin \phi + N\phi \sin \phi + pa \sin \phi = 0 \quad \text{--- (23)}$$

$$\frac{d}{d\phi} (M\phi \sin \phi) - M_0 \cos \phi - Q\phi a \sin \phi = 0 \quad \text{--- (24)}$$

$$N\phi = \frac{Et}{a} \left\{ \left(\frac{dv}{d\phi} - w \right) + v(v \cot \phi - w) \right\} / (1-v^2) \quad \text{--- (25)}$$

$$N_0 = \frac{Et}{a} \left\{ (v \cot \phi - w) + v \left(\frac{dv}{d\phi} - w \right) \right\} / (1-v^2) \quad \text{--- (26)}$$

$$M\phi = -D \left\{ \frac{d\alpha}{d\phi} + v\alpha \cot \phi \right\} / a \quad \text{--- (27)}$$

$$M_0 = -D \left\{ \alpha \cot \phi + v \frac{d\alpha}{d\phi} \right\} / a \quad \text{--- (28)}$$

$$\alpha = (v + \frac{dw}{d\phi}) / a \quad \text{--- (29)}$$

$$D = Et^3 / (12(1-v^2)) \quad \text{--- (30)}$$

By Gibson's simplified theory, we can proceed by retaining higher order derivatives only.

$$\therefore \alpha = \frac{dw}{d\phi} / a \quad \text{--- (31)}$$

In equations (27) and (28)

$$M_\phi = -D \frac{d^2 w}{d\phi^2} / a^2 \quad \text{--- (32)}$$

$$M_\theta = -D \nu \frac{d^2 w}{d\phi^2} / a^2 = \nu M_\phi \quad \text{--- (33)}$$

We now reduce equation (24) to

$$\frac{dM_\phi}{d\phi} = Q_\phi a \quad \text{--- (34)}$$

Differentiating equation (32)

$$\frac{dM_\phi}{d\phi} = -D \frac{d^3 w}{d\phi^3} / a^2$$

and substituting into equation (34)

$$\frac{d^3 w}{d\phi^3} = -\frac{Q_\phi}{D} a^3 \quad \text{--- (35)}$$

By eliminating N_θ between equations (22) and (23) we obtain :-

$$\begin{aligned} \frac{d}{d\phi} (N_\phi \sin\phi) \sin\phi - Q_\phi \sin^2\phi + \frac{d}{d\phi} (Q_\phi \sin\phi) \cos\phi \\ + N_\phi \sin\phi \cos\phi + pa \sin\phi \cos\phi = 0 \end{aligned}$$

Putting this in the form of differentials :-

$$\frac{d}{d\phi} (N_\phi \sin^2\phi) + \frac{d}{d\phi} (Q_\phi \sin\phi \cos\phi) + \frac{d}{d\phi} (\frac{1}{2} pa \sin^2\phi) = 0$$

Integrating, we obtain

$$N_\phi \sin\phi + Q_\phi \cos\phi + \frac{1}{2} pa \sin\phi = C$$

$$\text{When } \phi = 0 \quad \sin\phi = 0 \quad Q_\phi = 0$$

$$\therefore C = 0$$

$$\therefore N_\phi = -Q_\phi \cot\phi - \frac{1}{2} pa \quad \text{--- (36)}$$

Rearranging equation (23)

$$N_\theta = -N_\phi - pa - \frac{d}{d\phi} (Q_\phi \sin\phi) / \sin\phi$$

Substituting equation (36) for N_ϕ and expanding the differential

$$N_\theta = Q_\phi \cot \phi - \frac{1}{2} p a - \frac{dQ_\phi}{d\phi} - Q_\phi \cot \phi$$

$$\therefore N_\theta = -\frac{dQ_\phi}{d\phi} - \frac{1}{2} p a \quad \text{--- (37)}$$

By mutual elimination between equations (25) and (26) we have

$$N_\theta - \nu N_\phi = \frac{Et}{a} (\nu \cot \phi - w) \quad \text{--- (38)}$$

$$N_\theta - \nu N_\theta = \frac{Et}{a} \left(\frac{d\nu}{d\phi} - w \right) \quad \text{--- (39)}$$

Differentiating equation (38)

$$\frac{dN_\theta}{d\phi} - \nu \frac{dN_\phi}{d\phi} = \frac{Et}{a} \left\{ \frac{d\nu}{d\phi} \cot \phi - \nu / \sin^2 \phi - \frac{dw}{d\phi} \right\} \quad \text{--- (40)}$$

Eliminating $\frac{d\nu}{d\phi}$ between equations (39) and (40)

$$\frac{dN_\theta}{d\phi} - \nu \frac{dN_\phi}{d\phi} - (N_\phi - \nu N_\theta) \cot \phi$$

$$= \frac{Et}{a} \left(w \cot \phi - \nu / \sin^2 \phi - \frac{dw}{d\phi} \right)$$

Now by dispensing with N_ϕ , N_θ , w and ν , and again retaining only the highest order derivatives for $\frac{dN_\phi}{d\phi}$ and $\frac{dN_\theta}{d\phi}$ from equations (36) and (37), we obtain :-

$$\frac{d^2 Q_\phi}{d\phi^2} = \frac{Et}{a} \frac{dw}{d\phi} \quad \text{--- (41)}$$

Differentiating (41) twice, and substituting into (35)

$$\frac{d^4 Q_\phi}{d\phi^4} + \frac{Et a^2}{D} Q_\phi = 0 \quad \text{--- (42)}$$

which is normally written

$$\frac{d^4 Q_\phi}{d\phi^4} + 4\beta^4 Q_\phi = 0 \quad \text{--- (43)}$$

where

$$\beta^4 = \frac{Et a^2}{4D} = 3(1-\nu^2) \left(\frac{a}{t} \right)^2 \quad \text{--- (44)}$$

A standard form of solution to equation (43) will be :-

$$Q_\phi = C_1 a_1 + C_2 a_2 + C_3 a_3 + C_4 a_4 \quad \text{--- (45)}$$

where C_1 to C_4 are constants

and

$$\left. \begin{aligned} a_1 &= \sin \beta \phi \sinh \beta \phi \\ a_2 &= \sin \beta \phi \cosh \beta \phi \\ a_3 &= \cos \beta \phi \sinh \beta \phi \\ a_4 &= \cos \beta \phi \cosh \beta \phi \end{aligned} \right\} \text{--- (46)}$$

Differentiating equation (45)

$$\frac{1}{\beta} \frac{dQ_\phi}{d\phi} = C_1 (a_3 + a_1) + C_2 (a_4 + a_1) + C_3 (a_4 - a_1) + C_4 (a_3 - a_2)$$

$$\frac{1}{\beta^2} \frac{d^2 Q_\phi}{d\phi^2} = C_1 a_4 + C_2 a_3 - C_3 a_2 - C_4 a_1$$

Now at the crown, where $\phi = 0$, we have $Q_\phi = 0$

and

$$a_1 = a_2 = a_3 = 0, \quad a_4 = 1$$

$$\therefore C_4 = 0$$

Also $\frac{dw}{d\phi} = 0$ and $\therefore$ from equation (41)

$$\frac{d^2 Q_\phi}{d\phi^2} = 0$$

$$\therefore C_1 = 0$$

$\therefore$ Equation (45) becomes

$$Q_\phi = C_2 a_2 + C_3 a_3 \quad \text{--- (47)}$$

With Q_ϕ now expressed explicitly, all the other shell variables can be expressed in terms of its derivatives.

We already have

$$N_\phi = -Q_\phi \cot \phi - \frac{1}{2} p a \quad (36)$$

$$N_\theta = -\frac{dQ_\phi}{d\phi} - \frac{1}{2} p a \quad (37)$$

From equation (41)

$$\frac{dw}{d\phi} = \frac{a}{Et} \frac{d^2 Q_\phi}{d\phi^2} \quad (48)$$

From equations (32) and (48)

$$\begin{aligned} M_\phi &= -\frac{D}{Et a} \frac{d^3 Q_\phi}{d\phi^3} \\ &= -\frac{a}{4\beta^4} \frac{d^3 Q_\phi}{d\phi^3} \quad (49) \end{aligned}$$

And as before,

$$r_{i\theta} = \nu h_\phi \quad (50)$$

Rearranging equation (38), we can obtain horizontal deflexion,

$$\begin{aligned} H &= \nu \cos \phi - w \sin \phi \\ \therefore H &= a \sin \phi (N_\theta - \nu N_\phi) / Et \\ &= \frac{a \sin \phi}{Et} \left\{ -\frac{dQ_\phi}{d\phi} - \frac{1}{2} p a (1-\nu) + \nu Q_\phi \cot \phi \right\} \\ &= \left\{ \nu a Q_\phi \cot \phi - a \sin \phi \left(\frac{dQ_\phi}{d\phi} + \frac{1}{2} p a (1-\nu) \right) \right\} / Et \end{aligned}$$

In this expression for H we can identify the components of shell deflexion,

$$\nu = \nu a Q_\phi / Et \quad (51)$$

$$w = a \left\{ \frac{dQ_\phi}{d\phi} + \frac{1}{2} p a (1-\nu) \right\} / Et \quad (52)$$

and we can neglect ν to obtain

$$H = -a \sin \phi \left\{ \frac{dQ_\phi}{d\phi} + \frac{1}{2} p a (1-\nu) \right\} / Et \quad (53)$$

Similarly for the vertical deflexion, V , we may obtain

$$\begin{aligned} V &= w \sin \phi + v \cos \phi \\ &= a \cos \phi \left\{ \frac{dQ_\phi}{d\phi} + \frac{1}{2} pa (1-\nu) \right\} / Et \end{aligned} \quad (53)$$

Equation (53) as applied to a hemisphere gives $V = 0$ at the support. If, however, $V = V_b$ at the boundary, we require this term to give more general application.

$$\therefore V = V_b + a \cos \phi \left\{ \frac{dQ_\phi}{d\phi} + \frac{1}{2} pa (1-\nu) \right\} / Et \quad (54)$$

Finally, the explicit forms of equation (47) differentiated :

$$Q_\phi = C_2 a_2 + C_3 a_3 \quad (47)$$

$$\frac{dQ_\phi}{d\phi} = \beta [C_2 (a_4 + a_1) + C_3 (a_4 - a_1)] \quad (55)$$

$$\frac{d^2 Q_\phi}{d\phi^2} = 2\beta^2 [C_2 a_3 - C_3 a_2] \quad (56)$$

$$\frac{d^3 Q_\phi}{d\phi^3} = 2\beta^3 [C_2 (a_4 - a_1) - C_3 (a_4 + a_1)] \quad (57)$$

Again summarising the shell equations :

$$N_\phi = -Q_\phi \cot \phi - \frac{1}{2} pa \quad (36)$$

$$N_\theta = -\frac{dQ_\phi}{d\phi} - \frac{1}{2} pa \quad (37)$$

$$\frac{dw}{d\phi} = \frac{a}{Et} \frac{d^2 Q_\phi}{d\phi^2} \quad (48)$$

$$M_\phi = -\frac{a}{4\beta^2} \frac{d^3 Q_\phi}{d\phi^3} \quad (49)$$

$$M_\theta = \nu M_\phi \quad (33)$$

$$H = -a \sin \phi \left\{ \frac{dQ_\phi}{d\phi} + \frac{1}{2} pa (1-\nu) \right\} / Et \quad (52)$$

$$V = a \cos \phi \left\{ \frac{dQ_\phi}{d\phi} + \frac{1}{2} p a (1-\nu) \right\} / Et + V_3 \quad \text{--- (54)}$$

The difficulty of equation (36) when $\phi = 0$ can be overcome as follows :

If $\phi = 0$ in equation (37), we obtain

$$N_\theta = -\beta (C_2 + C_3) - \frac{1}{2} p a \quad \text{--- (58)}$$

If ϕ is small in equation (36), we have

$$a_2 = a_3 = \beta \phi$$

$$\text{and } \cot \phi = 1/\phi$$

$$\begin{aligned} \therefore Q_\phi \cot \phi &= \beta \phi (C_2 + C_3) / \phi \\ &= \beta (C_2 + C_3) \end{aligned}$$

$$\therefore N_\phi = -\beta (C_2 + C_3) - \frac{1}{2} p a = N_\theta$$

$$\text{ie, when } \phi = 0 \quad N_\phi = N_\theta \quad \text{--- (59)}$$

Thus, although $Q_\phi = 0$ at the crown, $Q_\phi \cot \phi \neq 0$, and we do not violate boundary conditions.

When the hemisphere is encastre at the equator, the constants are found as follows :

$$\phi = \frac{\pi}{2}$$

$$\frac{dw}{d\phi} = 0$$

$$H = 0$$

$$a_n = A_n$$

Substituting the above into equations (48) and (56)

$$\frac{dw}{d\phi} = \frac{a}{Et} \frac{d^2 Q_\phi}{d\phi^2} = \frac{2\beta^2 a}{Et} (C_2 a_3 - C_3 a_2) = 0$$

$$\therefore C_3 = C_2 A_3/A_2 \text{ ————— (60)}$$

And in equations (52) and (55)

$$\begin{aligned} H &= -a \sin\phi \left\{ \frac{dQ_\phi}{d\phi} + \frac{1}{2} pa(1-\nu) \right\} / Et \\ &= -\frac{a}{Et} \left\{ \beta [C_2 (A_4 + A_1) + C_3 (A_4 - A_1)] + \frac{1}{2} pa(1-\nu) \right\} \\ &= 0 \\ \therefore C_2 (A_4 + A_1) + C_3 (A_4 - A_1) &= \frac{-pa(1-\nu)}{2\beta} \end{aligned}$$

Eliminating C_3

$$\begin{aligned} A_2 C_2 (A_4 + A_1) + C_2 A_3 (A_4 - A_1) \\ = -\frac{A_2 pa(1-\nu)}{2\beta} \end{aligned}$$

$$\therefore C_2 = \frac{-A_2 pa(1-\nu)}{2\beta [A_2 (A_4 + A_1) + A_3 (A_4 - A_1)]} \text{ ————— (62)}$$

This solution does not apply when the shell thickness varies.

APPENDIX I

The Finite Element Displacement Method - Basic Equations

It is assumed that a continuum is divided into a finite number of imaginary elements for which equilibrium is established by transfer of forces between elements at certain nodal points on the element boundaries. Distributed body forces within the continuum and initial stress and strain are neglected.

We begin by defining what are known as the Finite Element Characteristic Functions.

The displacement of any point within an element is defined by the relation

$$\{f\} = [N] \{\delta\}^e \quad \text{--- (i)}$$

where $\{f\}$ is the displacement function

$[N]$ is the shape function

and $\{\delta\}^e$ the nodal displacements for element e .

The generalised strains within the element are given by

$$\{\epsilon\} = [B] \{\delta\}^e \quad \text{--- (ii)}$$

where $\{\epsilon\}$ is the strain

and $[B]$ can be derived directly from $[N]$.

The generalised stresses are given by

$$\{\sigma\} = [D] \{\epsilon\} \quad \text{--- (iii)}$$

where $\{\sigma\}$ is the stress

and $[D]$ the elasticity matrix.

Substituting (ii) into (iii)

$$\{\sigma\} = [D][B]\{\delta\}^e \text{ ————— (iv)}$$

where $[D][B]$ forms the stress matrix.

Applying the principle of virtual work, work done by nodal forces, $\{F\}^e = (d\{\delta\}^e)^T \{F\}^e$

$$\text{Work done by stresses} = (d\{\epsilon\})^T \{\sigma\}$$

$$\text{From (ii)} \quad (d\{\epsilon\})^T = (d\{\delta\}^e)^T [B]^T$$

$$\text{Work done by stresses} = (d\{\delta\}^e)^T [B]^T \{\sigma\}$$

Equating external and internal work done over the whole element, and cancelling out the common multiplier

$$\{F\}^e = \int [B]^T \{\sigma\} d(vol) \text{ ————— (v)}$$

Substituting (iv) into (v)

$$\{F\}^e = \int [B]^T [D][B] d(vol) \{\delta\}^e \text{ ————— (vi)}$$

where $\int [B]^T [D][B] d(vol)$ forms the stiffness matrix.

It can be shown that, provided the virtual work principle is maintained by the continuity of displacements across element boundaries, the same functions apply throughout the continuum.

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CHAPTER ONE

Pressure Test on Perspex Cylinders - Mark I and Mark II

Manufacture of Models

The construction materials were ready-made half-inch thick flat perspex plate, and quarter-inch thick tubing, all of which are available from laboratory materials suppliers. The end plates were cut accurately by a technician and drilled through where required to carry out the internal strain gauge wiring. The tubes were cut to length and in the case of the Mark I Model which was formed from two tubes fitted one inside the other it was necessary to heat the outer tube to expand it. The inner tube was then inserted and the space between them was filled with a suitable adhesive.

Before the end plates could be glued into position, the internal strain gauges and wiring were fixed and numbered. And finally, after completion of the model construction, the external strain gauges were placed and their wiring and numbering. At this stage the models were ready for testing. In the case of the perspex models, which were very buoyant, it was necessary to attach a weight to the bases.

Testing Equipment

The central component is the pressure vessel in which the model to be tested is placed. A mild steel cylindrical tank was made available utilising oil as a pressure fluid. It was adapted to the present application by an experienced technician who made an end plate sealed by O-rings, with 120 brass valves which he copied from a bicycle tube valve, having a conical seating enclosed in rubber tubing, and passing through the

end plate. The internal and external strain gauge wiring were then soldered to the respective ends of these brass studs by myself, as were all the data logger and dummy strain gauges and their connections so that I could check that all strain gauge numbering was correctly ordered.

Another component of equipment was an electric oil pressure pump capable of applying up to 2000psi pressure to the tank. This required some practise in its use prior to any model testing, as it had both fine and coarse control. The fine control only was required for the testing, but it is possible when the upper limit setting of the fine control is passed for the coarse control to take over automatically and blow up the pressure vessel, which was originally designed to go to 400psi, in a matter of seconds.

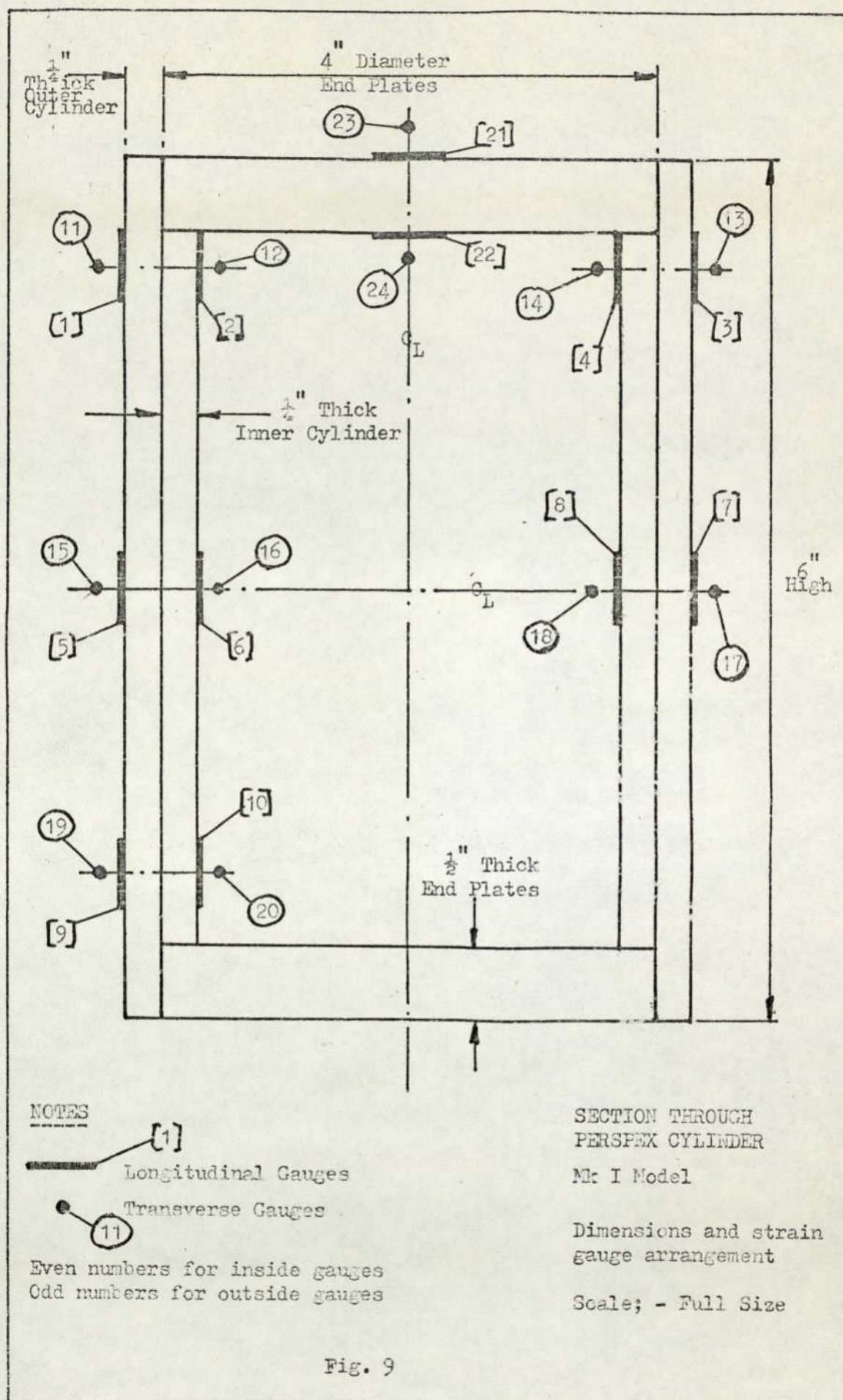
The tank itself was tested for soundness up to 1200psi on a single overload run, as the ultimate design criteria to have a maximum working load of 400psi must also be based upon fatigue considerations as to how often it is estimated that the pressure in the tank would be raised from zero, as well as how much and/or for how long. In the end the only way to be certain that everything would be alright was by trying it out with a substantial working margin for error.

Test Procedure

The model to be tested was prepared with all strain gauge wiring attached before being placed inside the pressure vessel on a stand. It was necessary to drain some of the oil to avoid excess displacement by the volume of the model, and it was topped up almost level with the underside of the end plate before that too was positioned.



Photograph Number 1.
Laboratory Equipment.



PERSPEX CYLINDER

Mk I Model

Strain Gauge Readings

Strain Gauge Numbers	11th Aug 1980 p = 200 psi	8th Sept 1980 p = 200 psi
1	-	+2.0
2	+744.0	+738 .0
3	-253.5	-222.0
4	+554.0	+540.0
5	+95.0	+108.0
6	+25.3	+35.0
7	+75.5	+87.0
8	-	+20.0
9	-	-248.0
10	+748.5	+741.0
11	-	-12.0
12	-	+11.0
13	+6.5	-4.0
14	-	-
15	+262.5	+263.0
16	-	-
17	+243.5	+252.0
18	+394.0	+400.0
19	-	-3.0
20	+27.3	+26.0
21	+629.0	+626.0
22	-405.5	-406.0
23	+619.5	+511.0
24	-	-

1 DLU = 1.94 psi

Compression positive

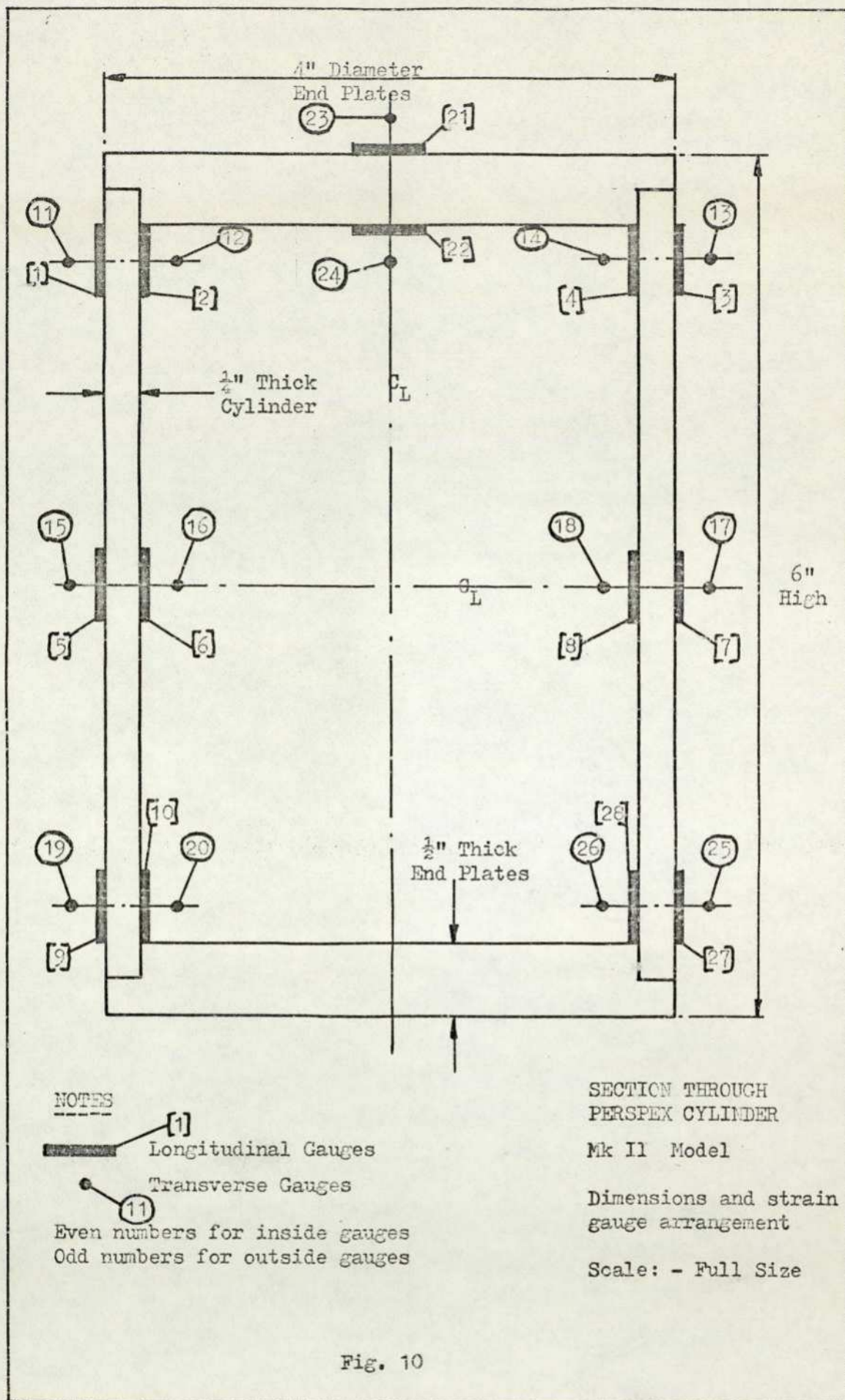
Tension negative

PERSPEX CYLINDER

Mk I Model

Analysis of Experimental Results

Gauge Numbers	Average Reading	Stress Factors (Compression Negative)
1	2.0	-0.02
2	741.0	-7.19
3	-237.75	2.31
4	547.0	-5.31
5	101.5	-0.98
6	30.15	-0.29
7	81.25	-0.79
8	20.0	-0.19
9	-248.0	2.41
10	744.75	-7.22
11	-12.0	0.12
12	11.0	-0.11
13	1.25	-0.01
14	-	-
15	262.75	-2.55
16	-	-
17	247.75	-2.40
18	397.0	-3.85
19	-3.0	0.03
20	26.65	-0.26
21	627.5	-6.09
22	-405.75	3.94
23	565.25	-5.48
24	-	-



PERSPEX CYLINDER

Mk II Model

Experimental Results

Strain Gauge Number	40 psi DLU's (27th July 1981)	60 psi DLU's	Average Stress Factor
2	-143	-229	-8.21
3	24	38	1.37
4	-12	-20	-0.72
6	-21	-29	-1.12
7	-7	-27	-0.70
8	-16	-31	-1.02
9	5	11	0.35
10	-118	-176	-6.53
11	16	20	0.82
12	15	24	0.86
13	17	20	0.84
14	15	20	0.79
15	-88	-148	-5.18
16	-122	-194	-6.98
17	-100	-156	-5.66
18	-106	-163	-5.96
19	23	35	1.29
20	13	24	0.81
23	-130	-204	-7.38
24	99	154	5.60
25	60	93	3.39
26	-2	15	0.22
27	22	31	1.19
28	-134	-211	-7.63

1 DLU = 2.22 psi
 Compression negative
 Tension Positive

CHAPTER TWO

Pressure Test on a Hemispherical Closed Shell

Twelve microconcrete hemispherical shells were produced by the University concrete laboratory staff and left to cure in water. From these those closest to a shell of revolution were allocated for testing and those which were not symmetrical or had other faults such as cracks or air bubbles were rejected. The shell for which strain gauge readings were first obtained was cast on 6th July 1976 and, although the inner form had moved upwards to reduce the crown thickness during casting, it is a shell of revolution well within one millimetre of error both in thickness of shell and out of roundness.

The thickness measurements were taken by setting up a dial gauge with a roller attachment on a flat steel plate which was designed for laboratory measurement work. The results are as follows.

Along the hemispherical equator the shell thickness at quarter points was

A	B	C	D	
16.08	15.94	15.57	17.14 mm	average 16.18 mm

A measurement taken halfway between the equator and the crown was 11.15 mm thick and the crown itself measured 9.12 mm thick. For the purposes of analysis the equator and crown thicknesses are taken to be 16 mm and 9 mm respectively.

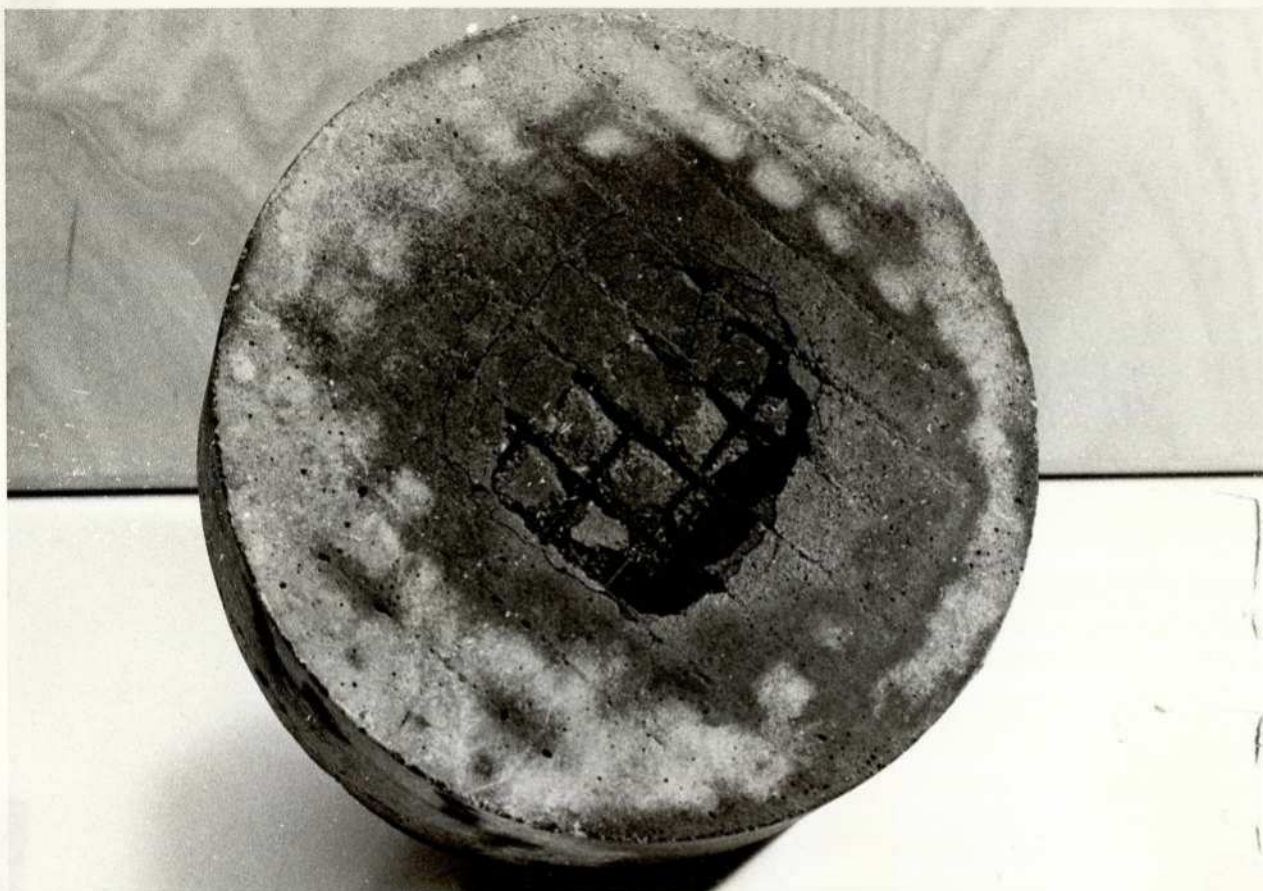
The diameter measured at the equator : -

AC	inner	201 mm	outer	234 mm	diff.	33 mm
ED	"	201 mm	"	233 mm	"	32 mm

From the thickness measurements

$$A + C = 31.65 \text{ mm}$$

$$B + D = 33.08 \text{ mm}$$



Photograph Number 2.

First trial model base - failure by punching shear.

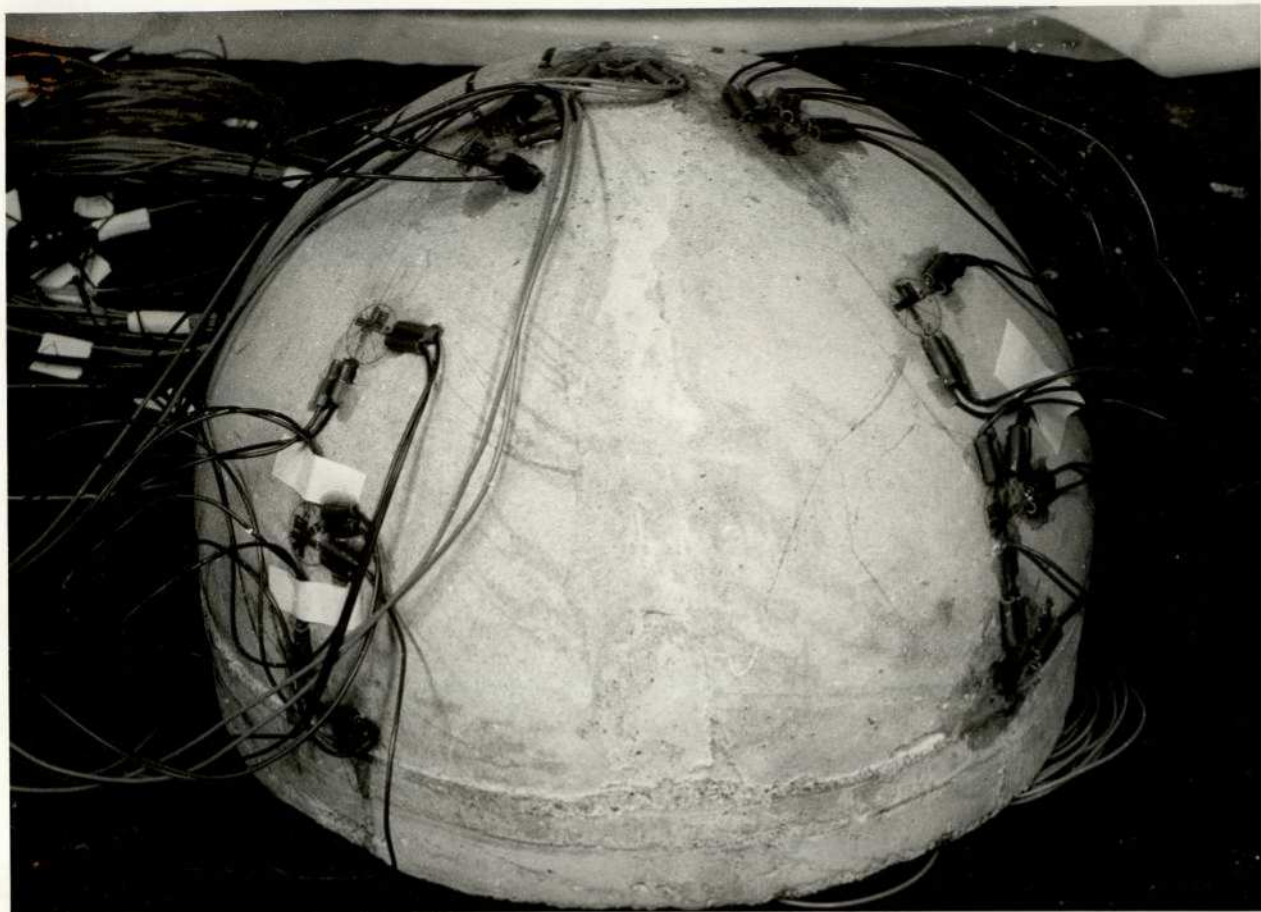


Photograph Number 3(a)



Photograph Number 3(b)

Second trial model base - failure by bending shear.



Photograph Number 4(a). Hemispherical Model. External Surface.



Photograph Number 4(b). Hemispherical Model. Internal Surface.

All these measurements are found to be represented within an error of 1 mm by taking the inner diameter as 201 mm and the outer diameter as 233 mm with the inner form displaced 7 mm towards the crown of the hemisphere.

The base of the shell was cast on 27th October 1976, after two unsuccessful attempts with other shells in which the base failed too early to give useful results. Testing began on 26th November 1976 and strain gauge readings were first taken up to 400 psi applied pressure until the roll of print output paper ran out. This stage is marked by a * in the result sheets which follow.

In order to verify the consistency of the results they were taken progressively as follows : -

0 psi - 100 psi - 0 psi - 100 psi - 200 psi - 100 psi - 0 psi - 100 psi - 200 psi - 300 psi - 200 psi - etc, so that they oscillated from zero each time 100 psi higher in 100 psi steps. The readings were taken without delay as soon as each pressure was registering on the gauge mounted upon the pressure vessel.

The pressure reading shown on the pressure pump gauge was lower as shown on the next page.

An hour later, when another roll of output paper had been obtained, further reading were taken until that also ran out after 800 psi had been reached, and, since there was none immediately available, the testing was abandoned for the day, a Friday.

The next week testing continued. The previous stoppage is marked as a ** on the results sheets. The final run was taken in 200 psi steps until the model failed at precisely 1000 psi before there was time to press the data logger button for a set of readings. Unlike the flat base failures which sounded like a series of feeble cracking noises until the pressure gauge needle slowly flopped down

TEST EQUIPMENT PRESSURE READINGS

Let p_1 = reading on pressure gauge attached directly to test tank

Let p_2 = reading on pressure gauge attached directly to oil pump

p_1 psi	p_2 psi
300	287
200	189
100	87
0	6
100	95
200	197
300	286
400	391
300	287
200	189
0	3
100	90
200	196
300	291
400	394
500	490
400	391
300	288
200	190
100	90
0	6
100	90
200	195
300	290
400	394
500	500
600	599
500	491
400	391
300	287
200	188
100	87
0	5
100	90
200	191
300	292
400	394
500	498
600	595
700	693
600	587
500	493
400	394
300	286
200	192
100	87
0	5
100	96
200	193
300	292

p_1 is raised from zero in 100 psi increments and returned to zero again in 100 psi increments.

The maximum reading is progressively increased with each load oscillation, up to a maximum of $p_1 = 800$ psi.

readings (cont.)	
p_1 psi	p_2 psi
400	389
500	499
600	594
700	690
800	792
700	683

at final failure, the shell failed with a loud report similar to a rifle shot and the pressure gauge needle moved too fast for the eye until it was down to about 400 psi. Air bubbled out from the vessel lid for over an hour until the needle stopped at about 50 psi.

When the model $\sqrt{s}$ removed from the pressure vessel the following day, the pressure gauge remained slightly above zero, and it is assumed that the shock wave at failure caused some alteration in its calibration, although no actual damage was found when it was opened up for inspection.

The shell had failed in a circle about 10 cm diameter placed about 3 cm off centre from the crown. The fragments were up to 1 cm long, irregular shapes. There were few of these, however, to be found, the rest of the failed area having been transformed into concrete dust which could be seen suspended in the oil which was siphoned out of the shell.

The final reading on the strain gauges, taken after failure is shown on the results sheets with the mark thus, ***. The lessons learned in this first test with full strain gauging, were first of all to have equipment fully loaded to go the full test run.

The results showed that many strain gauges became useless after standing in the oil of the pressure vessel for four days. The model had been placed in the vessel, topped up and enclosed the day before testing, 25th November 1976. A pressure of about 400 psi was applied in error that day when checking the pump which was found to have a delayed action from zero before picking up pressure. This was the second lesson, but fortunately the model appears to be elastic up to 400 psi in this case.

Therefore future tests should be carried out with a more reliable procedure, although it involves a long day of testing when more care is taken.



Photograph Number 5(a). Third trial model base.



Photograph Number 5(b). Hemispherical model after destruction.

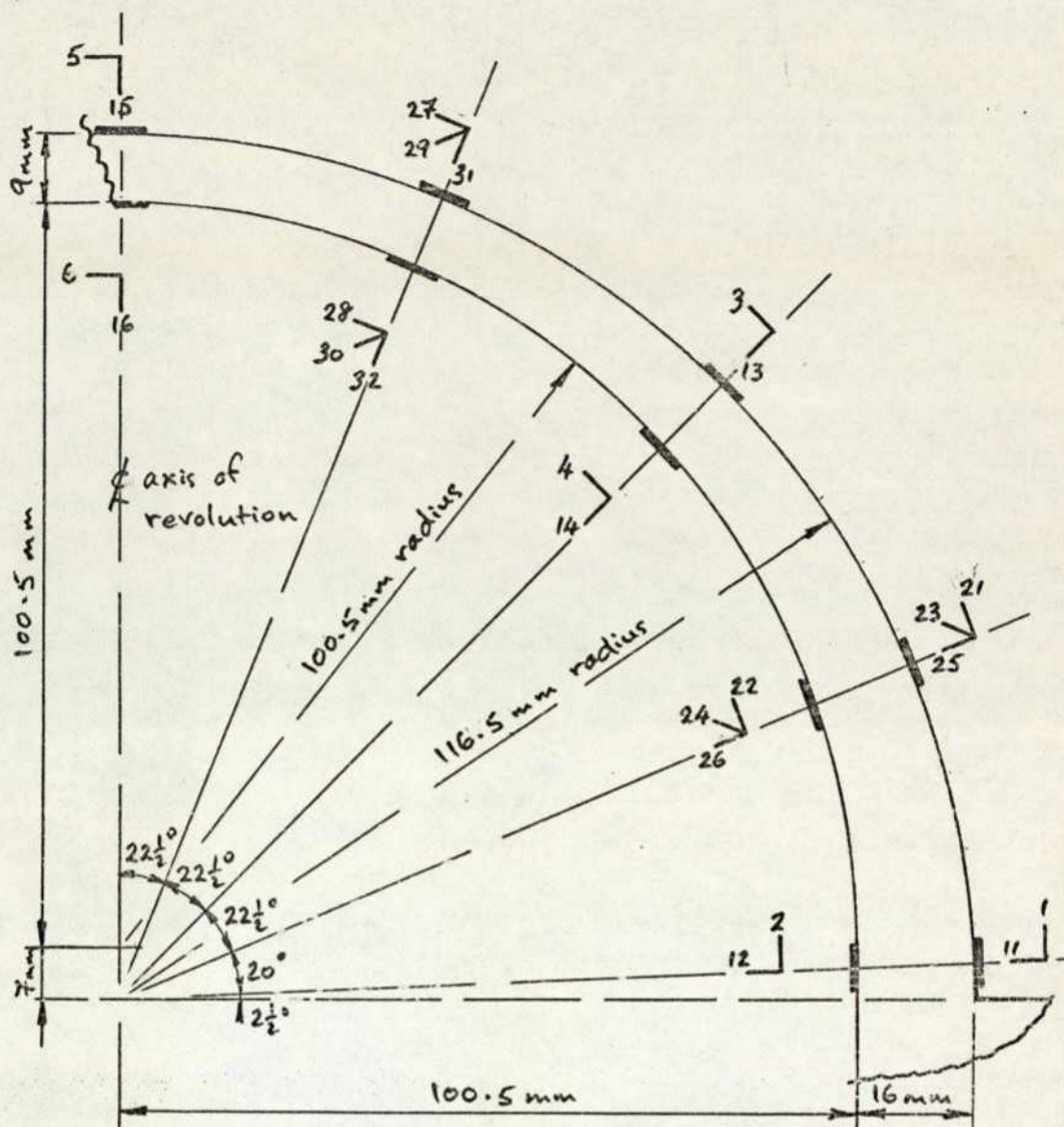


Fig. 11 a) Half - section through hemisphere

cast 6/1/76

tested 26 to 29/11/76

Cubes cast 6/7/76, tested 2/12/76 to failure : -

53.7 N/mm^2

43.9

35.1

55.7

Average $47.1 \times 1000 / 6.895 = \underline{6831}$ psi

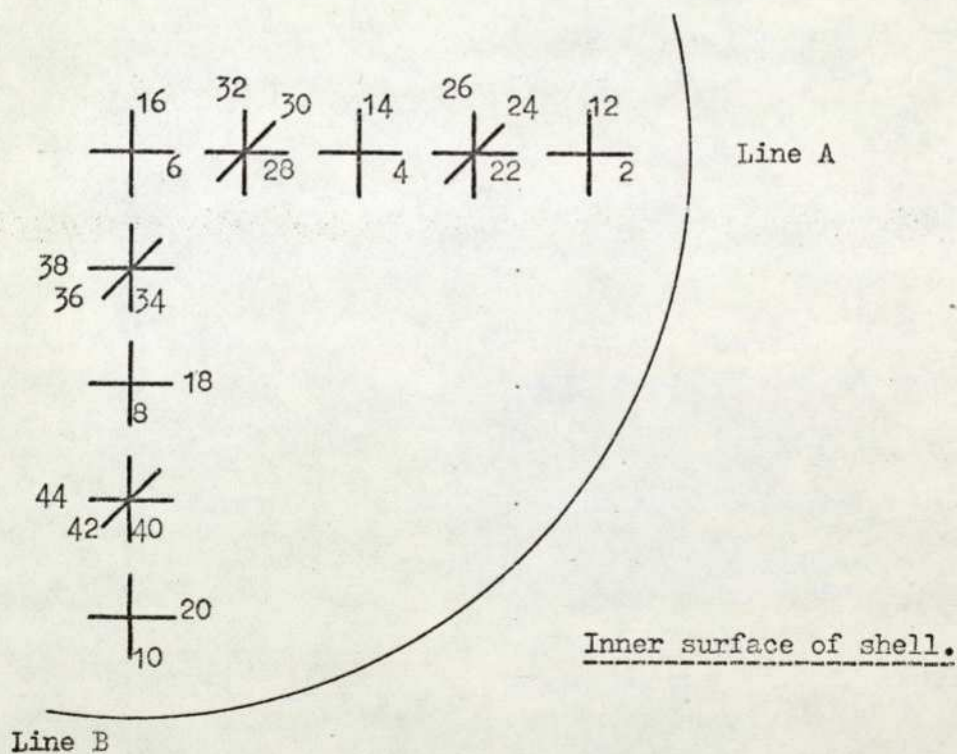
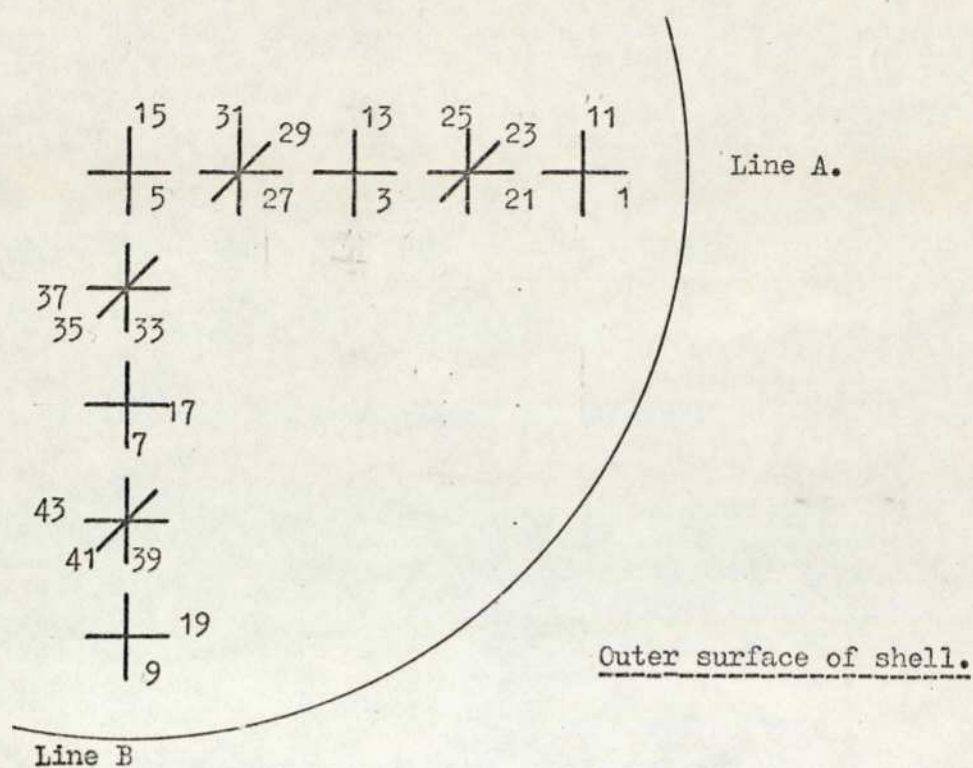


Fig. 11 b) Strain gauge numbering.

(Hemisphere cast 6/7/76, tested 26 to 1/12/76)

Pressure	Opsi	100	200	300	400
01	56	39	20	6	-7
02	630	615	592	577	560
03	290	274	255	239	221
04	591	568	542	519	493
05	463	456	447	438	428
06	360	355	350	345	340
07	-	-	-	-	-
08	547	531	513	495	474
09	707	694	678	660	637
10	435	432	429	426	422
11	105	98	91	82	72
12	667	646	622	600	574
13	343	333	323	311	296
14	554	540	525	510	492
15	351	337	326	317	308
16	762	756	750	745	738
17	349	338	324	312	299
18	479	484*	441	425	406
19	669	660	649	636	621
20	554	537	513	490	466
21	363	354	347	338	325
22	537	524	511	497	480
23	290	276	263	250	236
24	511	495	475	456	435
25	364	349	331	317	300
26	494	471	442	419	394
27	342	343	349	357	366
28	508	504	500	497	492
29	391	387	382	374	364
30	-28	-32	-34	-37	-39
31	1225	1220	1207	1194	1181
32	1481	1468	1456	1446	1436
33	924	914	904	893	879
34	548	530	510	491	468
35	796	788	779	771	760
36	1381	1362	1338	1318	1296
37	233	217	198	182	162
38	962	942	918	898	876
39	-	-	-	-	-
40	1035	1023	1009	993	975
41	357	344	329	315	299
42	356	340	320	303	284
43	88	72	52	35	16
44	387	368	347	329	308

* high, assume 462.

Selected strain gauge readings.

Pressure	0-100psi	100-200	200-300	300-400	av. diff.
01	-17	-19	-14	-13	-15.75
02	-15	-23	-15	-17	-17.50
03	-16	-19	-16	-18	-17.25
04	-23	-26	-23	-26	-24.50
05	-7	-9	-9	-10	-8.75
06	-5	-5	-5	-5	-5.00
07	-	-	-	-	-
08	-16	-18	-18	-21	-18.25
09	-13	-16	-18	-23	-17.50
10	-3	-3	-3	-4	-3.25
11	-7	-7	-9	-10	-8.25
12	-21	-24	-22	-26	-23.25
13	-10	-10	-12	-15	-11.75
14	-14	-15	-15	-18	-15.50
15	-14	-11	-9	-9	-10.75
16	-6	-6	-5	-7	-6.00
17	-11	-14	-12	-13	-12.50
18	-17	-21	-16	-19	-18.25
19	-9	-11	-13	-15	-12.00
20	-17	-24	-23	-24	-22.00
21	-9	-7	-9	-13	-9.50
22	-13	-13	-14	-17	-14.25
23	-14	-13	-13	-14	-13.50
24	-16	-20	-19	-21	-19.00
25	-15	-18	-14	-17	-16.00
26	-23	-29	-23	-25	-25.00
27	+1	+6	+8	+9	+6.00
28	-4	-4	-3	-5	-4.00
29	-4	-5	-8	-10	-6.75
30	-4	-2	-3	-2	-2.75
31	-5	-13	-13	-13	-11.00
32	-13	-12	-10	-10	-11.25
33	-10	-10	-11	-14	-11.25
34	-18	-20	-19	-23	-20.00
35	-8	-9	-8	-11	-9.00
36	-19	-24	-20	-22	-21.25
37	-16	-19	-16	-20	-17.75
38	-20	-24	-20	-22	-21.50
39	-	-	-	-	-
40	-12	-14	-16	-18	-15.00
41	-13	-15	-14	-16	-14.50
42	-16	-20	-17	-19	-18.00
43	-16	-20	-17	-19	-18.00
44	-19	-21	-18	-21	-19.75

Strain gauge reading differences.

	Pressure 0-100psi	100-200	200-300	300-400	Check sum
01	-07.94	-20.63	+11.11	+17.46	-0.00
02	+14.29	-31.43	+14.29	+02.86	+0.01
03	+07.25	-10.14	+07.25	-04.35	+0.01
04	+06.12	-06.12	+06.12	-06.12	+0.00
05	+20.00	-02.86	-02.86	-14.29	-0.01
06	0.00	0.00	0.00	0.00	0.00
07	-	-	-	-	-
08	+12.33	+01.37	+01.37	-15.07	+0.00
09	+25.71	+08.57	-02.86	-31.43	-0.01
10	+07.69	+07.69	+07.69	-23.08	-0.01
11	+15.15	+15.15	-09.09	-21.21	+0.00
12	+09.68	-03.23	+05.38	-11.83	+0.00
13	+14.89	+14.89	-02.13	-27.66	-0.01
14	+09.68	+03.23	+03.23	-16.13	+0.01
15	-30.23	-02.33	+16.28	+16.28	-0.00
16	0.00	0.00	+16.67	-16.67	0.00
17	+12.00	-12.00	+04.00	-04.00	0.00
18	+06.85	-15.07	+12.33	-04.11	+0.00
19	+25.00	+08.33	-08.33	-25.00	0.00
20	+22.73	-09.09	-04.55	-09.09	+0.00
21	+05.26	+26.32	+05.26	-36.84	+0.00
22	+08.77	+08.77	+01.75	-19.30	-0.01
23	-03.70	+03.70	+03.70	-03.70	0.00
24	+15.79	-05.26	0.00	-10.53	+0.00
25	+06.25	-12.50	+12.50	-06.25	0.00
26	+08.00	-16.00	+08.00	0.00	0.00
27	-83.33	0.00	+33.33	+50.00	-0.00
28	0.00	0.00	+25.00	-25.00	0.00
29	+40.74	+25.93	-18.52	-48.15	+0.00
30	-45.45	+27.27	-09.09	+27.27	-0.00
31	+54.55	-18.18	-18.18	-18.18	+0.01
32	-15.56	-06.67	+11.11	+11.11	-0.01
33	+11.11	+11.11	+02.22	-24.44	+0.00
34	+10.00	0.00	+05.00	-15.00	+0.00
35	+11.11	0.00	+11.11	-22.22	0.00
36	+10.59	-12.94	+05.88	-03.53	+0.00
37	+09.86	-07.04	+09.86	-12.68	+0.00
38	+06.98	-11.63	+06.98	-02.33	+0.00
39	-	-	-	-	-
40	+20.00	+06.67	-06.67	-20.00	0.00
41	+10.34	-03.45	+03.45	-10.34	0.00
42	+11.11	-11.11	+05.56	-05.56	0.00
43	+11.11	-11.11	+05.56	-05.56	0.00
44	+03.80	-06.33	+08.86	-06.33	+0.00

Percentage difference from the mean.

<u>Range (%)</u>	<u>Frequency (actual)</u>	<u>Frequency (%)</u>
95.00 - 100.00	0.00	0.00
85.00 - 94.99	0	0.00
75.00 - 84.99	0	0.00
65.00 - 74.99	0	0.00
55.00 - 64.99	0	0.00
45.00 - 54.99	2	1.19
35.00 - 44.99	1	0.60
25.00 - 34.99	8	4.76
15.00 - 24.99	10	5.95
5.00 - 14.99	50	29.76
(-4.99) - (4.99)	40	23.81
(-5.00) - (-14.99)	29	17.26
(-15.00) - (-24.99)	18	10.71
(-25.00) - (-34.99)	6	3.57
(-35.00) - (-44.99)	1	0.60
(-45.00) - (-54.99)	2	1.19
(-55.00) - (-64.99)	0	0.00
(-65.00) - (-74.99)	0	0.00
(-75.00) - (-84.99)	1	0.60
(-85.00) - (-94.99)	0	0.00
(-95.00) - (-100.00)	0	0.00
total	<u>168</u>	<u>100.00</u>

Frequency distribution of percentage
error from the mean.

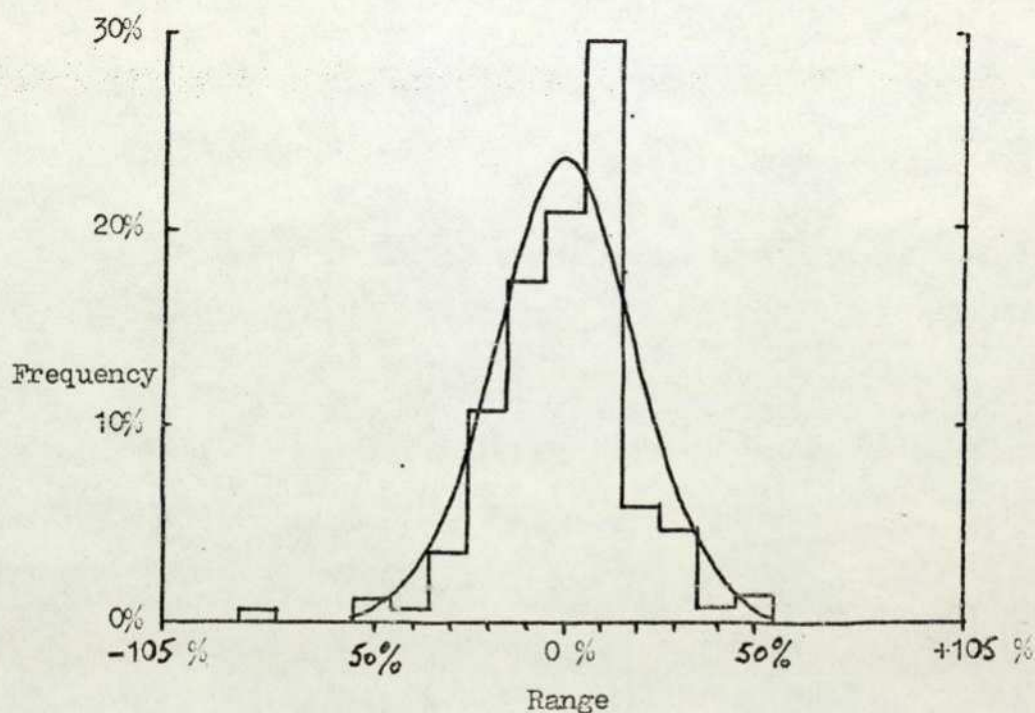
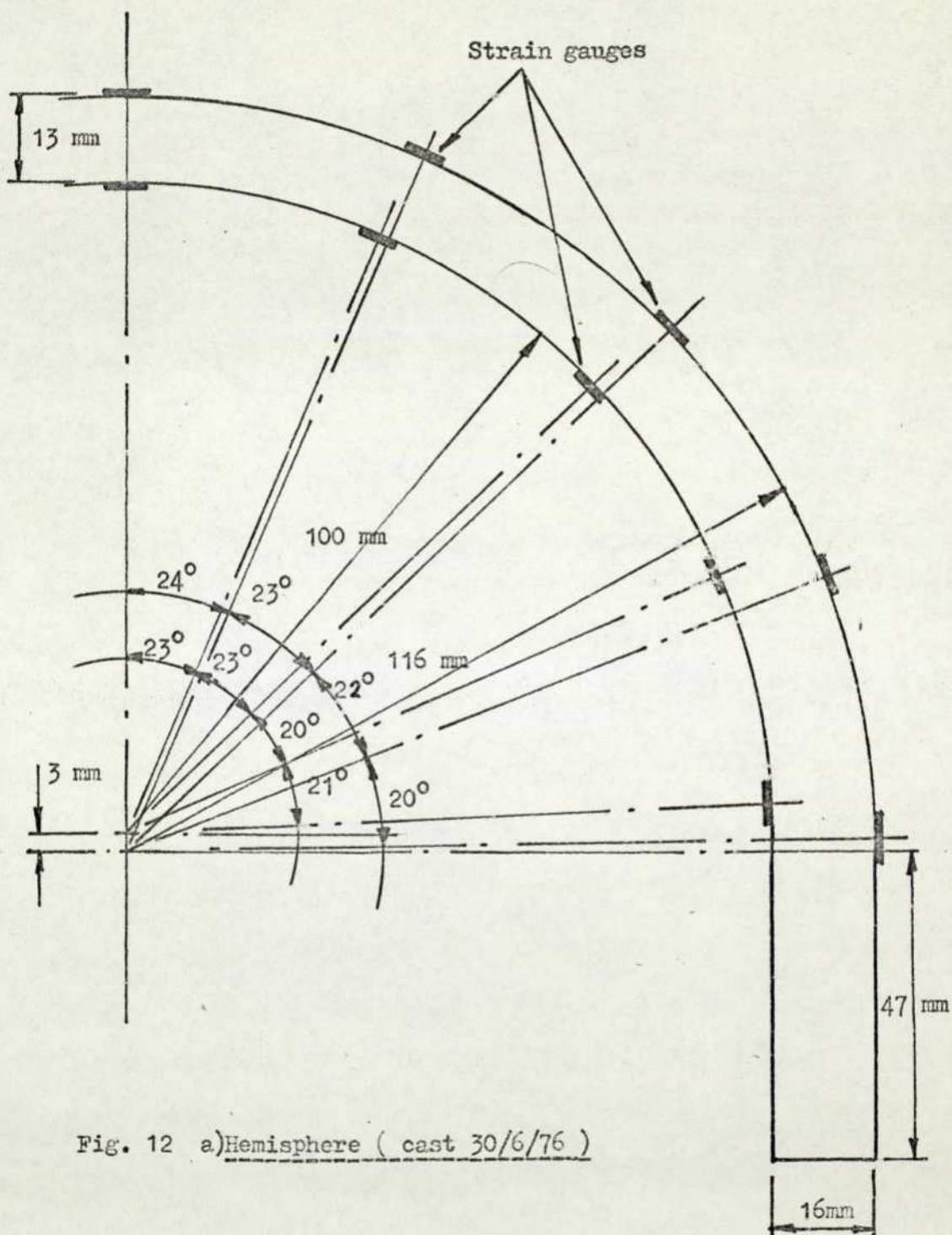


Fig. 11 c)

Strain Gauge.Stress factor.

01	-3.59
02	-3.99
03	-3.93
04	-5.59
05	-2.00
06	-1.14
07	-
08	-4.16
09	-3.99
10	-0.74
11	-1.88
12	-5.30
13	-2.68
14	-3.53
15	-2.45
16	-0.91
17	-2.85
18	-4.16
19	-2.74
20	-5.02
21	-2.17
22	-3.25
23	-3.08
24	-4.33
25	-3.65
26	-5.70
27	+0.95
28	-0.91
29	-1.54
30	-0.63
31	-2.51
32	-2.57
33	-2.57
34	-4.56
35	-2.05
36	-4.85
37	-4.05
38	-4.90
39	-
40	-3.42
41	-3.31
42	-4.10
43	-4.10
44	-4.50

Experimental stress factors.



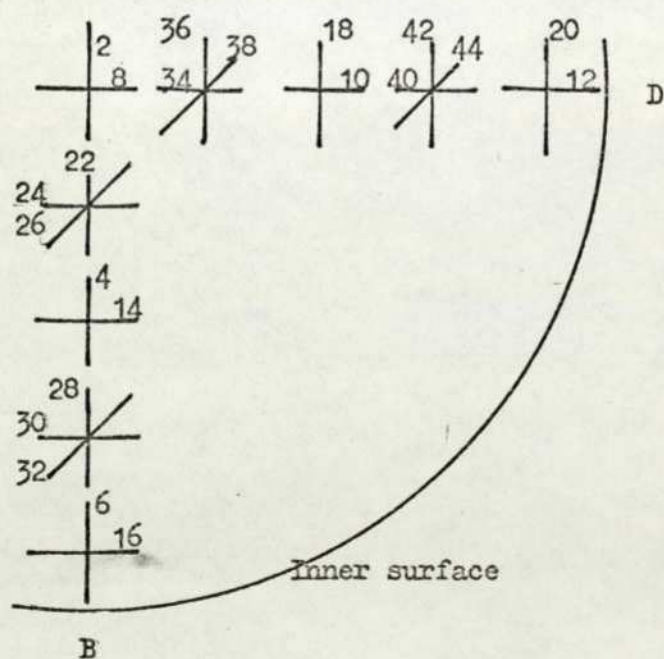
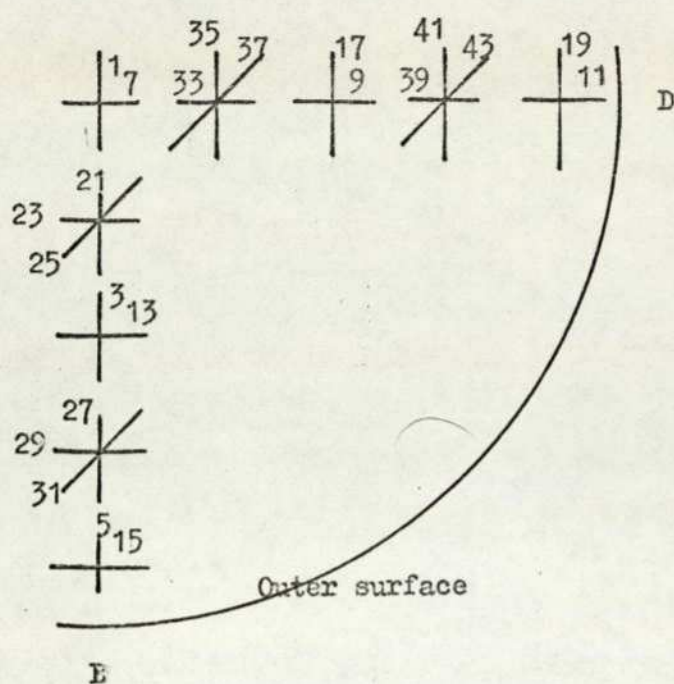


Fig. 12 b) Hemisphere (cast 30/6/76)

Pressure Opsi		100	200	300	400
01	erratic				
02	407	393	377	360	344
03	240	229	216	203	189
04	465	448	429	410	390
05	114	94	80	67	55
06	erratic				
07	erratic				
08	413	396	377	359	339
09	not working				
10	229	212	193	174	155
11	392	371	350	328	306
12	erratic				
13	510	498	484	469	455
14	711	696	679	662	645
15	erratic				
16	905	884	863	845	830
17	614	606	595	583	570
18	620	608	594	579	564
19	674	654	635	616	599
20	569	547	523	500	479
21	112	109	102	92	80
22	306	295	283	271	257
23	-19	-30	-45	-61	-76
24	307	294	279	265	250
25	15	11	0	-15	-30
26	242	228	212	197	182
27	366	378	375	368	361
28	323	306	287	269	251
29	577	566	553	541	528
30	194	179	162	147	129
31	1177	1162	1148	1134	1119
32	1508	1490	1472	1454	1436
33	erratic				
34	242	225	206	187	166
35	erratic				
36	1204	1189	1173	1156	1137
37	erratic				
38	860	849	834	820	805
39	1120	1111	1101	1090	1080
40	1021	1001	982	964	947
41	erratic				
42	158	142	124	106	87
43	erratic				
44	346	330	311	293	272

Selected strain gauge readings.

	Pressure 0-100psi	100-200	200-300	300-400	av. diff.
01	-	-	-	-	-
02	-14	-16	-17	-16	-15.75
03	-11	-13	-13	-14	-12.75
04	-17	-19	-19	-20	-18.75
05	-20	-14	-13	-12	-14.75
06	-	-	-	-	-
07	-	-	-	-	-
08	-17	-19	-18	-20	-18.50
09	-	-	-	-	-
10	-17	-19	-19	-19	-18.50
11	-21	-21	-22	-22	-21.50
12	-	-	-	-	-
13	-12	-14	-15	-14	-13.75
14	-15	-17	-17	-17	-16.50
15	-	-	-	-	-
16	-21	-21	-18	-15	-18.75
17	-8	-11	-12	-13	-11.00
18	-12	-14	-15	-15	-14.00
19	-20	-19	-19	-17	-18.75
20	-22	-24	-23	-21	-22.50
21	-3	-7	-10	-12	-8.00
22	-11	-12	-12	-14	-12.25
23	-11	-15	-16	-15	-14.25
24	-13	-15	-14	-15	-14.25
25	-4	-11	-15	-15	-11.25
26	-14	-16	-15	-15	-15.00
27	12	-3	-7	-7	+1.25
28	-17	-19	-18	-18	-18.00
29	-11	-13	-12	-13	-12.25
30	-15	-17	-15	-18	-16.25
31	-15	-14	-14	-15	-14.50
32	-18	-18	-18	-18	-18.00
33	-	-	-	-	-
34	-17	-19	-19	-21	-19.00
35	-	-	-	-	-
36	-15	-16	-17	-19	-16.75
37	-	-	-	-	-
38	-11	-15	-14	-15	-13.75
39	-9	-10	-11	-10	-10.00
40	-20	-19	-18	-17	-18.50
41	-	-	-	-	-
42	-16	-18	-18	-19	-17.75
43	-	-	-	-	-
44	-16	-19	-18	-21	-18.50

Strain gauge reading differences.

	<u>Pressure 0-100psi</u>	<u>100-200</u>	<u>200-300</u>	<u>300-400</u>	<u>Check Sum</u>
02	+11.11	-1.59	-7.94	-1.59	-0.01
03	+13.73	-1.96	-1.96	-9.80	+0.01
04	+9.33	-1.33	-1.33	-6.67	+0.00
05	-35.59	+5.08	+11.86	+18.64	-0.01
08	+8.11	-2.70	+2.70	-8.11	+0.00
10	+8.11	-2.70	-2.70	-2.70	+0.01
11	+2.33	+2.33	-2.33	-2.33	+0.00
13	+12.73	-1.82	-9.09	-1.82	+0.00
14	+9.09	-3.03	-3.03	-3.03	+0.00
16	-12.00	-12.00	+4.00	+20.00	+0.00
17	+27.27	0.00	-9.09	-18.18	+0.00
18	+14.29	0.00	-7.14	-7.14	+0.01
19	-6.67	-1.33	-1.33	+9.33	+0.00
20	+2.22	-6.67	-2.22	+6.67	+0.00
21	+62.50	+12.50	-25.00	-50.00	+0.00
22	+10.20	+2.04	+2.04	-14.29	-0.01
23	+22.81	-5.26	-12.28	-5.26	+0.01
24	+8.77	-5.26	+1.75	-5.26	+0.00
25	+64.44	+2.22	-33.33	-33.33	+0.00
26	+6.67	-6.67	0.00	0.00	+0.00
27	erratic				
28	+5.56	-5.56	0.00	0.00	+0.00
29	+10.20	-6.12	+2.04	-6.12	+0.00
30	+7.69	-4.62	+7.69	-10.77	-0.01
31	-3.45	+3.45	+3.45	-3.45	+0.00
32	0.00	0.00	0.00	0.00	+0.00
34	+10.53	0.00	0.00	-10.53	+0.00
36	+10.45	+4.48	-1.49	-13.43	+0.01
38	+20.00	-9.09	-1.82	-9.09	+0.00
39	+10.00	0.00	-10.00	0.00	+0.00
40	-8.11	-2.70	+2.70	+8.11	+0.00
42	+9.86	-1.41	-1.41	-7.04	+0.00
44	+13.51	-2.70	+2.70	-13.51	+0.00

Percentage difference from the mean.

<u>Range (%)</u>	<u>Frequency (actual)</u>	<u>Frequency (%)</u>
55.00 - 64.99	2	1.56
45.00 - 54.99	0	0.00
35.00 - 44.99	0	0.00
25.00 - 34.99	1	0.78
15.00 - 24.99	3	2.34
5.00 - 14.99	26	20.31
(-4.99) - (4.99)	58	45.31
(-5.00) - (-14.99)	31	24.22
(-15.00) - (-24.99)	2	1.56
(-25.00) - (-34.99)	3	2.34
(-35.00) - (-44.99)	1	0.78
(-45.00) - (-54.99)	1	0.78
total	128	99.98

Frequency distribution of percentage
error from the mean.

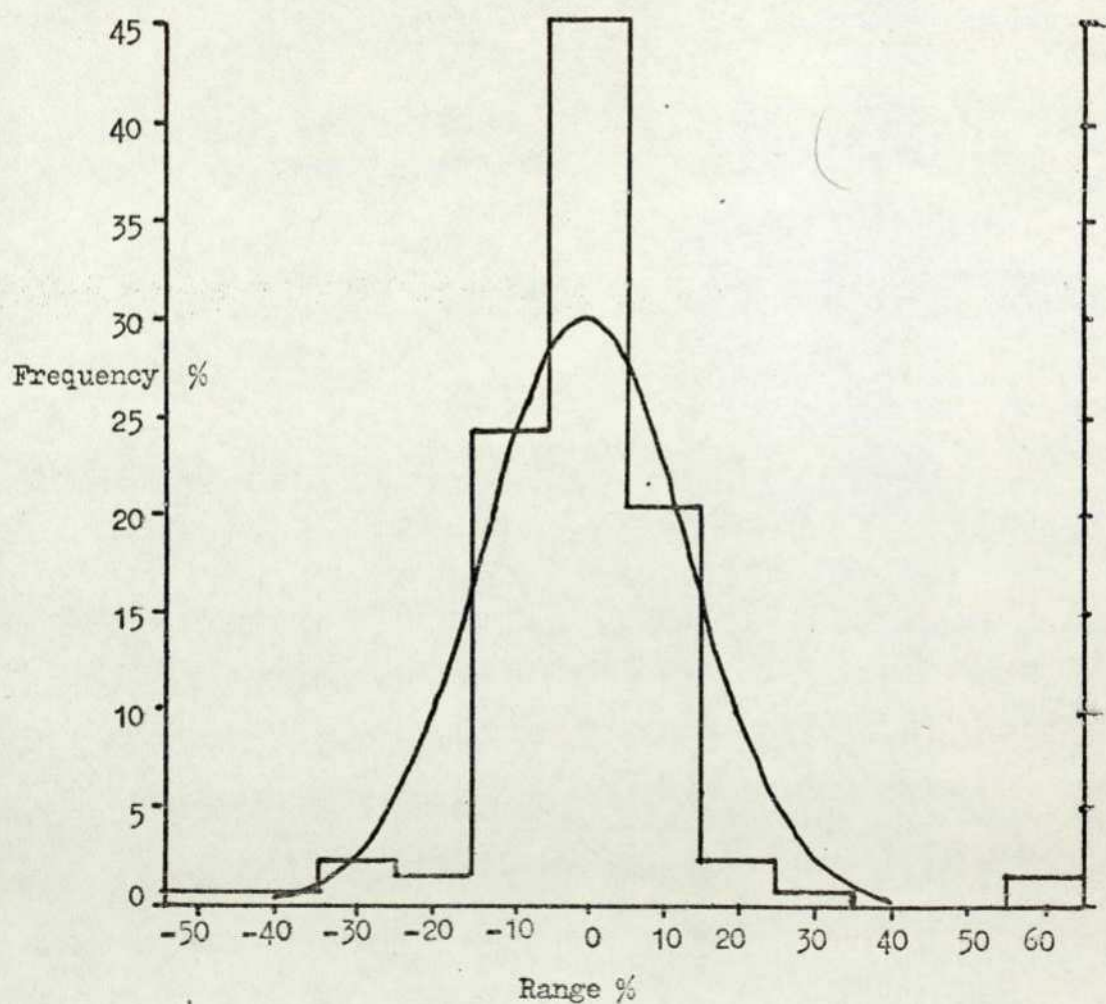


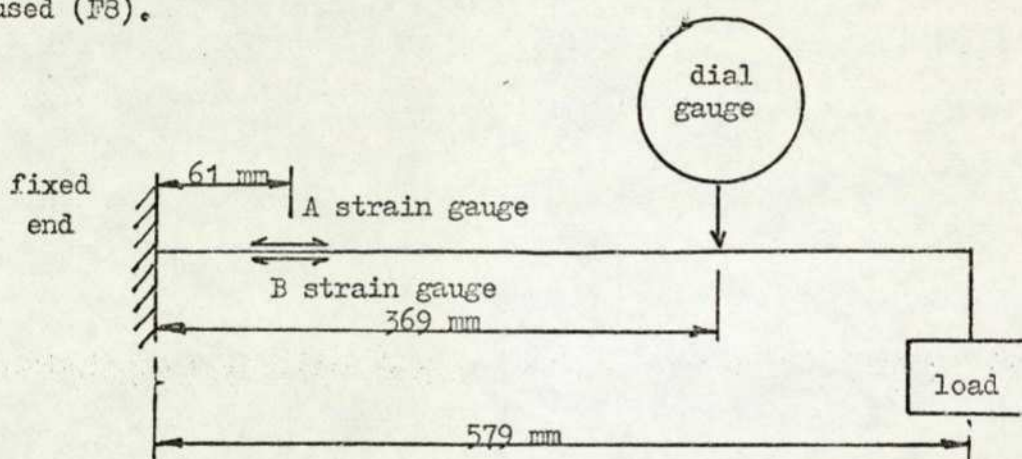
Fig. 12 c)

Strain Gauge NumberStress Factor

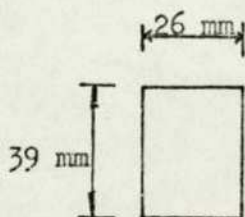
02	-2.96
03	-2.40
04	-3.53
05	-2.77
08	-3.48
10	-3.48
11	-4.04
13	-2.59
14	-3.10
16	-3.53
17	-2.07
18	-2.63
19	-3.53
20	-4.23
21	-1.50
22	-2.30
23	-2.68
24	-2.68
25	-2.12
26	-2.82
28	-3.38
29	-2.30
30	-3.06
31	-2.73
32	-3.38
34	-3.57
36	-3.15
38	-2.59
39	-1.88
40	-3.48
42	-3.34
44	-3.48

Analysis of Experimental Results

In order to make some sense out of the results of strain gauge readings under varying applied pressure, it is necessary to establish the relationship between the strain gauge readings and the actual stresses and strains. This is done by carrying out loading tests on a simple structure for which relationships are known between the load and stress/strain relationships. The simple cantilever beam was used in the following case made from a simple plain microconcrete specimen, cast 12th May 1975, and tested 2nd December 1976. It had a pair of strain gauges already attached of the same type as those used (F8).



a) line diagram of test beam



b) section through beam

Fig. 13

Test results on calibration beam.

<u>Load (Newtons)</u>	<u>Deflexion (mm)</u>
0	1.83
1	1.84
2	1.84
3	1.82
4	1.80
5	1.80
6	1.785
7	1.775
8	1.755
8*	1.745
7	1.75
6	1.76
5	1.775
4	1.791
3	1.805
2	1.822
1	1.835
0	1.85

* from the reading thus marked onwards the dial gauge was lightly tapped as it seemed to stick slightly with the addition or subtraction of each increment of loading.

readings were continued with higher loading:-

<u>Load (Newtons)</u>	<u>Deflexion (mm)</u>
10	1.72
12	1.693
14	1.67
16	1.632
18	1.601
16	1.624
14	1.652
12	1.677
10	1.70
0	1.837

with the corresponding data logger output:-

00 + 490
01 + 265
00 + 490
01 + 264
00 + 491
01 + 264
00 + 492
01 + 263
00 + 492
01 + 263
00 + 493
01 + 263
00 + 494
01 + 262
00 + 495
01 + 262
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01 + 264
00 + 490
01 + 264
00 + 489
01 + 264

and

00 + 496
01 + 260
00 + 497
01 + 259
00 + 499
01 + 258
00 + 500
01 + 257
00 + 502
01 + 256
00 + 501
01 + 258
00 + 499
01 + 258
00 + 497
01 + 259
00 + 496
01 + 260
00 + 489
01 + 264

where 00 is A strain gauge
01 is B " "

The results are seen to be linear and can be smoothed out:-

<u>Load (Newtons)</u>	<u>Deflexion (mm)</u>	<u>Gauge A</u>	<u>Gauge B</u>
0	1.84	489	265
2	1.82	491	264
4	1.79	492	263
6	1.76	494	262
8	1.74	495	261
10	1.71	496	260
12	1.68	497	259
14	1.65	499	258
16	1.63	501	257
18	1.60	502	256

They reduce to:-

0	0	0	0
2	-0.02	2	-1
4	-0.05	3	-2
6	-0.08	5	-3
8	-0.10	6	-4
10	-0.13	7	-5
12	-0.16	8	-6
14	-0.19	10	-7
16	-0.21	12	-8
18	-0.24	13	-9

∴ the ratio of load/deflexion = $\frac{18}{-0.24} = -75 \text{ N/mm}$

The strain gauge readings show different properties in tension and compression.

we have $f = M/Z$

where f is the stress

M the moment

Z the section modulus

$$Z = \frac{26 \times 39^2}{6} = 6,591 \text{ mm}^3$$

$$M = 518 \times \text{load} = 518 W$$

$$\therefore f = 0.07859 W \text{ N/mm}^2 = 11.4 W \text{ psi (load } W \text{ in Newtons)}$$

when $W = 18$

$$f = 11.4 \times 18 = 205.2 \text{ psi}$$

$\therefore$ one strain gauge data logger unit in tension corresponds to

$$\frac{205.2}{13} = \underline{15.8 \text{ psi}}$$

and in ~~tension~~ compression

$$- \frac{205.2}{9} = \underline{-22.8 \text{ psi}}$$

e.g. Strain gauge number 12 on the last run from zero pressure to 800 psi before failure at 1000 psi gave readings of 655 and 435 respectively, a difference of 220, or more precisely -220 where compression is negative.

The corresponding stress is $-220 \times 22.8 = 5016 \text{ psi}$

at 1000 psi the stress is thus $5016 \times 1000/800$

$$= \underline{6270 \text{ psi}}$$

this failure stress is comparable with the average cube strength of 6831 psi. In fact just before failure at 1000 psi the pressure gauge overshoot the 1000 psi mark by about 50 psi due to the delayed action of the pump mentioned earlier. This cuts the remaining difference by half.

Young's Modulus, E, is also found from the calibration beam : -

Integrating the relation $EIy'' = M = -Wx$

$$EIy' = -\frac{1}{2} W x^2 + C_1$$

$$\text{when } x = L, y' = 0 \therefore C_1 = \frac{1}{2} W L^2$$

$$\therefore EIy' = -\frac{1}{2} W x^2 + \frac{1}{2} W L^2$$

$$\text{Integrating again} \quad EIy = -\frac{1}{6} W x^3 + \frac{1}{2} W L^2 x + C_2$$

$$\text{when } x = L, y = 0 \therefore C_2 = -\frac{1}{2} W L^3 + \frac{1}{6} W L^3 = -\frac{1}{3} W L^3$$

$$\therefore EIy = -\frac{1}{6} W x^3 + \frac{1}{2} W L^2 x - \frac{1}{3} W L^3$$

$$= -\frac{1}{6} W (x^3 - 3 L^2 x + 2 L^3)$$

where $I = b d^3 / 12$, the moment of area

y = deflexion, distance x from the end of the
cantilever,

and L = length of cantilever.

Now we have established from the experiment that $W/y = 75 \text{ N/mm}$

also $x = 210 \text{ mm}$

$$L = 579 \text{ mm}$$

$$I = 26 \times 39^3 / 12 = 128\,524.5 \text{ mm}^4$$

$$x^3 - 3 L^2 x + 2 L^3 = (9.26 - 211.2 + 388.2) \times 10^6 \\ = 186.26 \times 10^6 \text{ mm}^3$$

$$\therefore E = \frac{7.5}{6} \times \frac{186.26 \times 10^6}{128\,524.5} \times \frac{1000}{7}$$

$$\therefore \underline{E = 2.6 \times 10^6 \text{ psi}}$$

Through the preceding pages the data logger output is given in full, followed by a breakdown of the readings to give a single average value for each strain gauge which represents the difference for an increment of 100 psi in applied loading. Inspection of the output shows first of all that strain gauges 7 and 39 were not in operation. Since this was not discovered until the whole apparatus had been set up and the cause was not found in the external equipment, it could be that the strain gauges were faulty, or the soldered connections were inadequate. The former case appears most likely as both these gauges are on the outer surface of the shell in close proximity to each other where they may have been damaged at the same time. Inspection of the soldered connections showed them to be all firmly in place when the apparatus was dismantled to remove the model from the pressure vessel after completion of testing.

In the last phase of testing a batch of about half a dozen gauges were not working, and at first this was thought to be due to deterioration from leaving the model immersed in the oil pressure vessel over the weekend. However, while dismantling the apparatus it was noticed that one of the junction leads into the data logger was halfway out of place, and it corresponded with the inoperative gauges. Therefore there is no conclusive evidence that several days of immersion in the oil causes any deleterious effects upon the strain gauges.

The general impression from the data logger output is of elastic behaviour up to 400 psi applied pressure, and above this the signs of creep can be detected as the readings fall progressively as the applied pressures were brought down again, suggesting a compressive strain setting in. Only strain gauge 27 reads tensile, all the rest show compressive strain. The readings were selected for

the range zero to 400 psi of applied loading, generally from those found for the first complete run in this range.

However, where some erratic readings appeared such as in strain gauge 20, then the results were inspected further and other more representative ones chosen. Assuming the strain to be zero at zero applied pressure, then the strain increments for the corresponding increments in applied pressure are seen to be constant for each gauge throughout the tests. The variation is found to follow a normal distribution about the mean, as shown in the histogram. There the curve was fitted with a calculated standard deviation of 16.9 %.

Therefore the assumption of a linear relation between load and strain gives a maximum of 50 % error, anything higher being accountable as a low strain value for which only small variations in readings give a high percentage error.

The average differences for each strain gauge can now be multiplied by the factors found from the calibration beam to convert them into stresses, and dividing by 100 psi increment of applied pressure, the stress factors for the shell are found. Finally the significance of the results can be assessed in their relation to the behaviour of the shell structure as predicted both by theoretical considerations and the assumptions and conditions attendant upon the experimental testing.

CHAPTER THREE

Pressure Test on a Conical Closed Shell.

A very accurate frustum of a cone was cast on 20th July, 1976, and the strain gauge readings from the tests made on 27th January, 1977 are in the pages which follow. No strain gauge was placed on the inner face of the end closure as this small part at the apex end of the model was too badly cast to find a reasonably plane area, reducing the number of gauges to 42. Of these gauges 8, 10, 20, 24 and 35, were either completely out of action or gave obviously erratic readings which could be discounted as false. No visible evidence was found to explain the cause of any faults.

On 31st January, 1977, four cubes which had been cast at the same time as the shell were destroyed on the Avery machine at a speed of 10 % per minute of the 1250 kN range. The microconcrete appears consistently stronger than the results for the four cubes tested for the hemispherical shell, although they were about the same age. The model itself was tested up to 1000 psi applied pressure and neither failed nor revealed any signs of cracking when inspected afterwards. The strain gauge readings, however, indicate that they were nonlinear in some parts of the shell and the results were selected for the 0 to 400 psi range of applied pressure where they remained linear and suitable for the purpose of this analysis.

Readings were taken at 50 psi intervals of applied pressure, following a similar pattern as for the previous test on the hemisphere, but no advantage arose from the extra information and the 100 psi interval results only have been recorded here. Accuracy in the linear range is seen to be limited by a time lag effect. At one point in the test, going from 200 to 250 psi for the first time, the pressure pump went out of control because the coarse control had been

inadvertently set to work automatically, and the pressure rose rapidly to 500 psi before it could be released. Fortunately the model was not damaged and the strain gauge readings exhibit elastic continuity as the experiment proceeded. Another minor delay occurred after 600 psi was reached, when the data logger output roll finished.

The following results were obtained for the tests cubes

62.1 N/mm²

61.4

60.6

65.0

Average $62.275 \times 1000 / 6.895 = \underline{9032}$ psi

On 28th January, 1977, a plane microconcrete beam was tested to determine elastic properties. It had been cast 5/1/77 with the following dimensions.

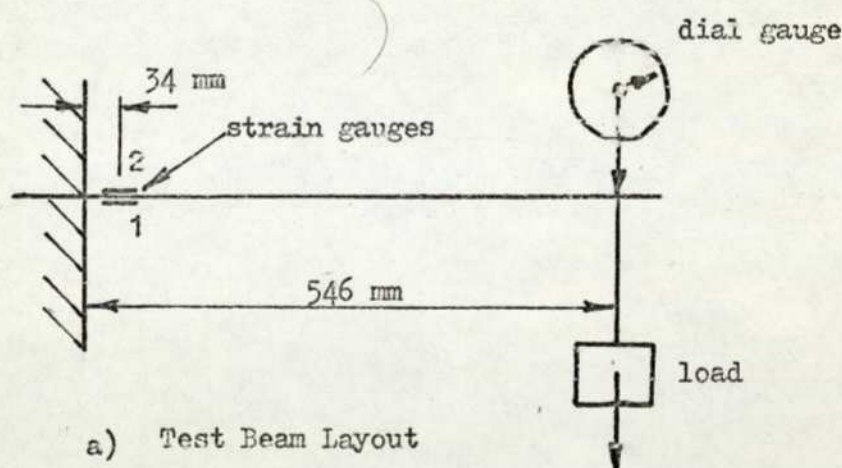
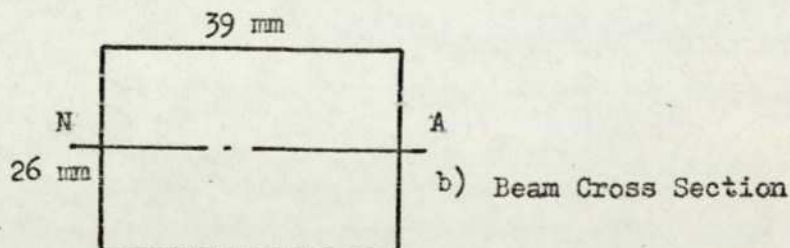


Fig. 14



The beam test results follow: -

Dial gauge mm	diff.	Load lb	Data logger 1	diff.	2	diff.
3.00	-0.37	0	-625	8	672	-8
2.63	-0.41	2	-617	8	664	-9
2.22	-0.50	4	-609	11	655	-11
1.72		6	-598		644	
		8 collapse				

Working in lb in. units,

$$\text{section modulus} = \frac{bd^2}{6} = 0.268 \text{ in}^3$$

bending moment for a 2 lb load = $2 \times 20.157 = 40.315 \text{ lb-in.}$

$$\text{maximum stress} = \frac{40.315}{0.268} = 150.429 \text{ psi}$$

which is represented by 8 units on the data logger

therefore one unit on the data logger is 18.8 psi in the linear elastic range both in tension and compression.

The above result suggests that the first test beam which was cast 12/5/75 and tested 2/12/76 was reinforced to cause different properties in tension and compression, and the above result will be adopted for both the hemispherical and conical shells.

In the equation

$$EIy = -\frac{1}{6} W (x^3 - 3L^2x + L^3)$$

we have $x = 0$

$$L = 21.5 \text{ in}$$

$$I = 0.137 \text{ in}^4$$

$$W = 2 \text{ lb}$$

$$y = -0.0146 \text{ in.}$$

$$\therefore E = 3,300,000 \text{ psi.}$$

The cube strengths for the beam material were found for three cubes on 2/2/77 to be :-

$$40.6, 52.0, \text{ and } 52.3 \text{ N/mm}^2$$

and a cylinder failed at 31.8 N/mm^2

Also, the microconcrete weighed 135 lb/ft^3

Three further cubes were destroyed on 16/2/77, cast 5/1/77, with the following results :-

48.1 N/mm²

42.7

53.3

Average 48.0

A test was also carried out on the same day on a cylinder of the same material to obtain further strain gauge readings. One gauge was accidentally damaged.

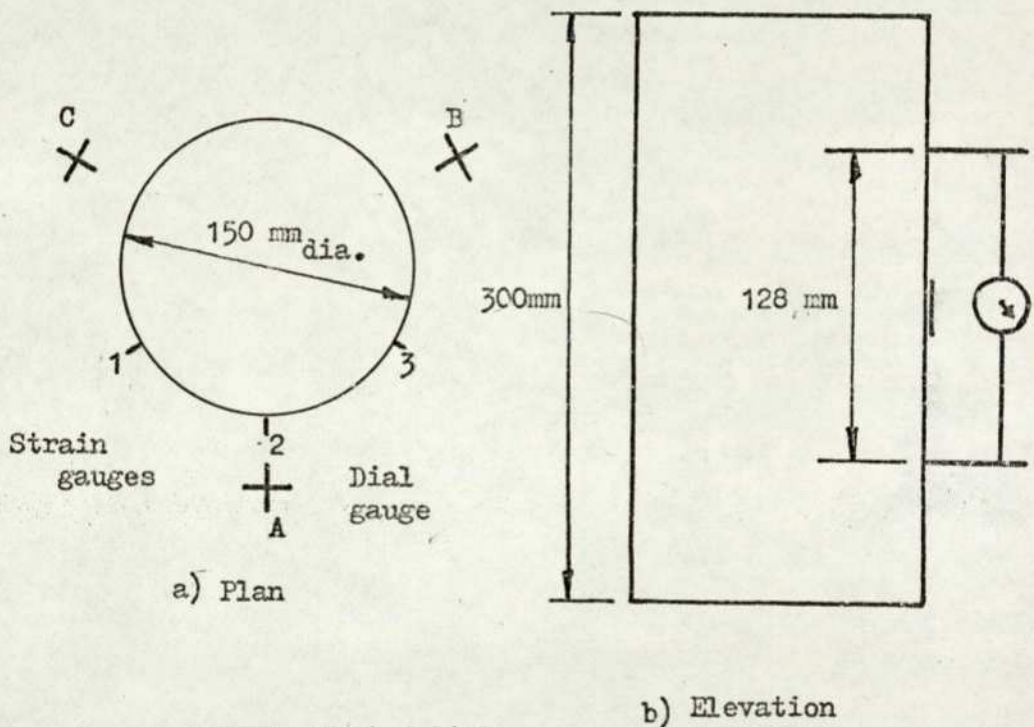
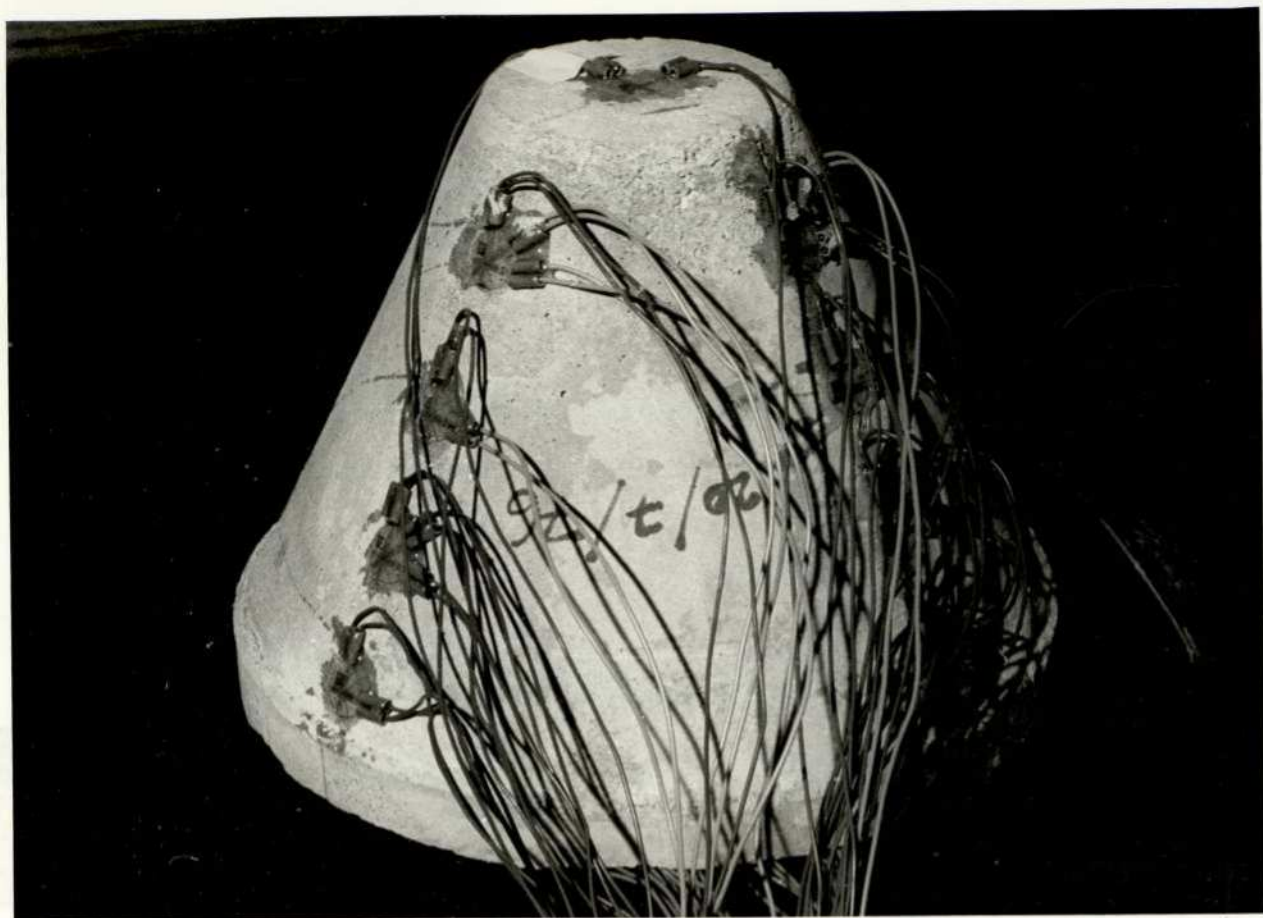
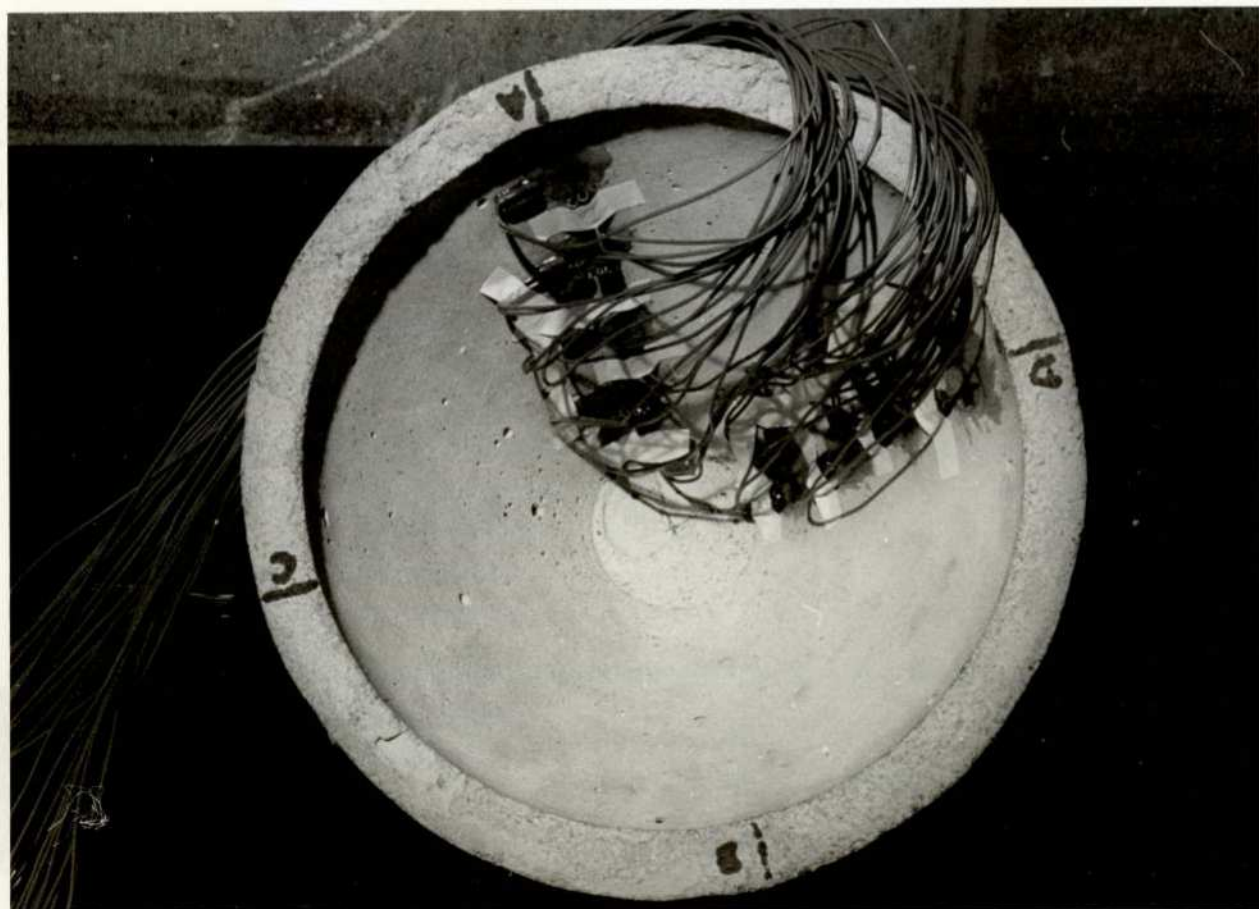


Fig.15 Test Cylinder

Load		Dial gauge .0001"				Strain gauge					
kN	diff.	A	diff.	B	diff.	C	diff.	2	diff.	3	diff.
0		11.18	0	14.40	.08	5.73	.04	-1532	-7	-1519	-25
50	50	11.18	-.02	14.48	.10	5.77	.06	-1539	-7	-1544	-29
100	50	11.16	-.01	14.58	.09	5.83	.06	-1546	-8	-1573	-27
150	50	11.15	0	14.67	.10	5.89	.06	-1554	-7	-1600	-24
200	50	11.15	0	14.77	.08	5.95	.05	-1561	-10	-1624	-22
250	50	11.15	0	14.85	.08	6.00	.05	-1571	-10	-1646	-22
0	-250	11.15	0	14.58	-.27	5.70	-.30	-1531	40	-1526	120
300	300	11.15	0	14.98	.40	6.03	.33	-1585	-54	-1678	-152
350	50	11.15	0	15.06	.08	6.07	.04	-1596	-11	-1694	-16
400	50	11.18	.03	15.15	.09	6.11	.04	-1609	-13	-1692	2
0	-400	11.16	-.02	14.62	-.53	5.71	-.40	-1533	76	-1538	154
450	450	11.22	.06	15.27	.65	6.13	.42	-1629	-96	-1712	-174
500	50	11.26	.04	15.36	.09	6.15	.02	-1647	-18	-1730	-18
550	50	11.30	.04	15.46	.10	6.18	.03	-1666	-19	-1748	-18
600	50	11.35	.05	15.62	.16	6.21	.03	-1690	-24	-1770	-22
650	50	11.38	.03	15.71	.09	6.24	.03	-1712	-22	-1788	-16
0	-650	11.20	-.18	14.75	-.96	5.74	-.50	-1543	169	-1568	220
700	700	11.44	.24	15.91	1.16	6.31	.57	-1745	-202	-1814	-246
750	50	11.49	.09	16.12	.21	6.38	.07	-1787	-42	-1846	-32
800	50	11.58	.09	16.48	.36	6.44	.06	-1823	-36	-1873	-27
failure at 800 kN											
0		10.78		15.79		5.74		-1611		-1638	



Photograph Number 6(a). Frustum of a Cone. External Surface.



Photograph Number 6(b). Frustum of a Cone. Internal Surface.

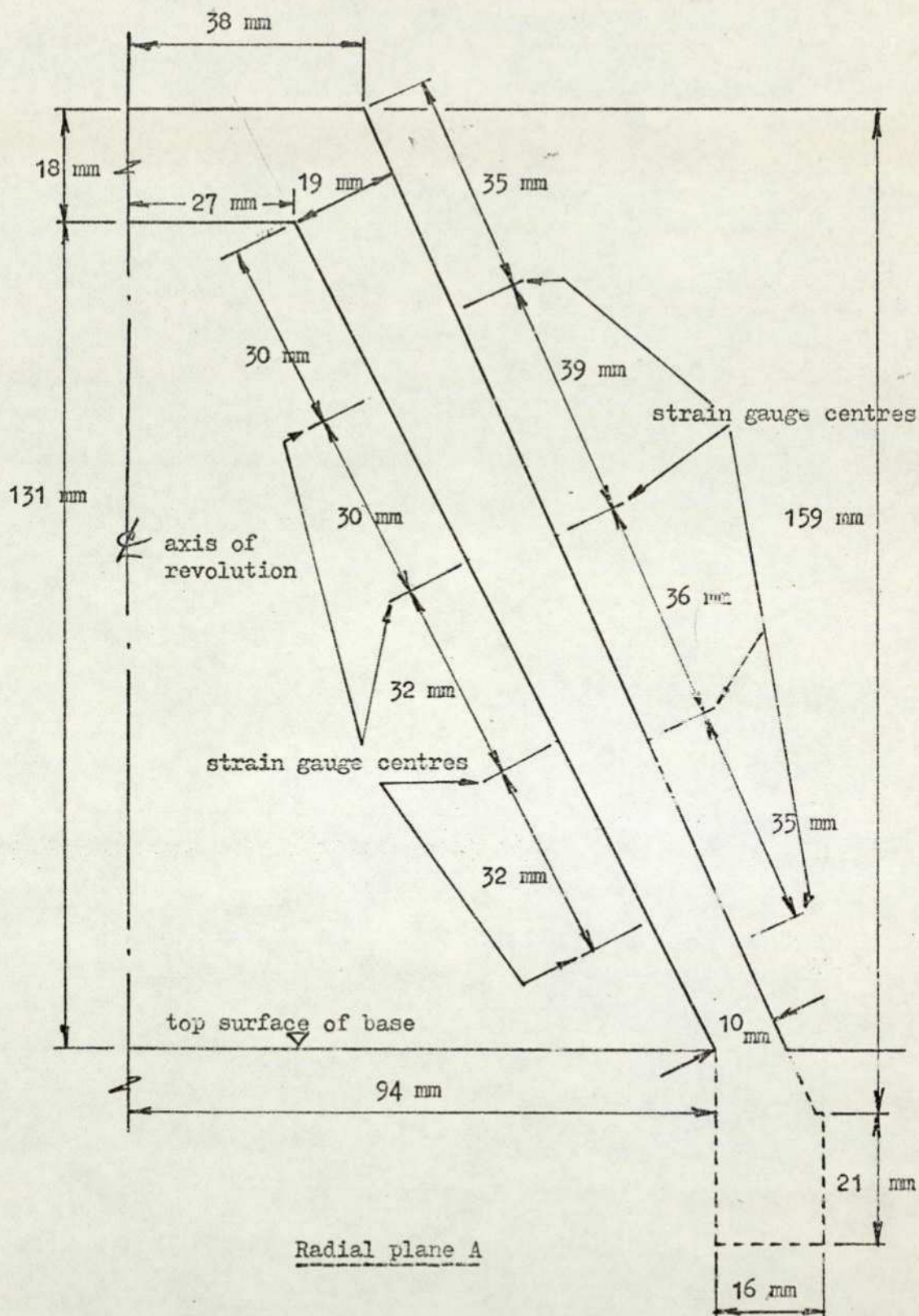


Fig. 16 a) Frustum of a cone.

(cast 20/7/76, tested 27/1/77)

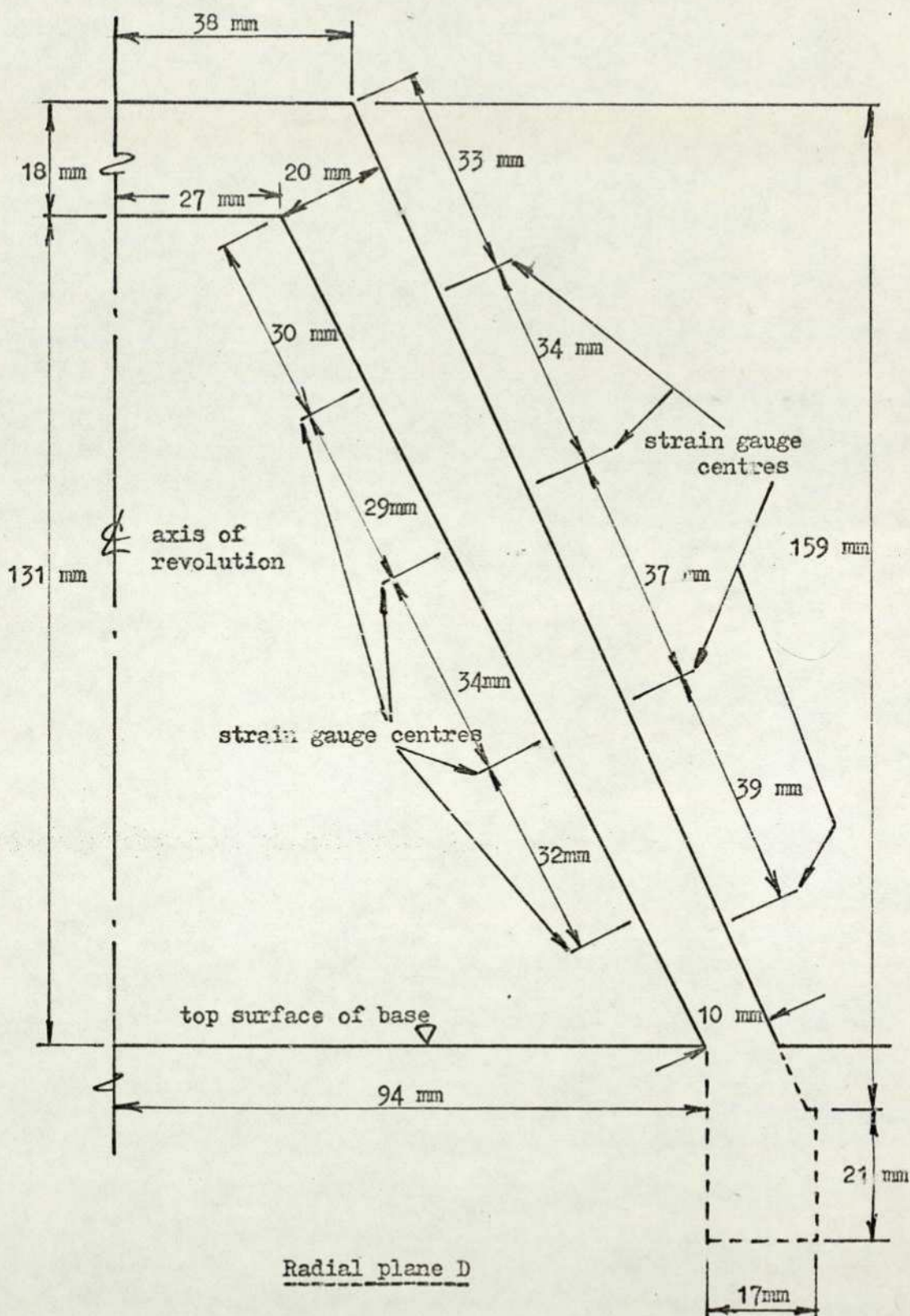
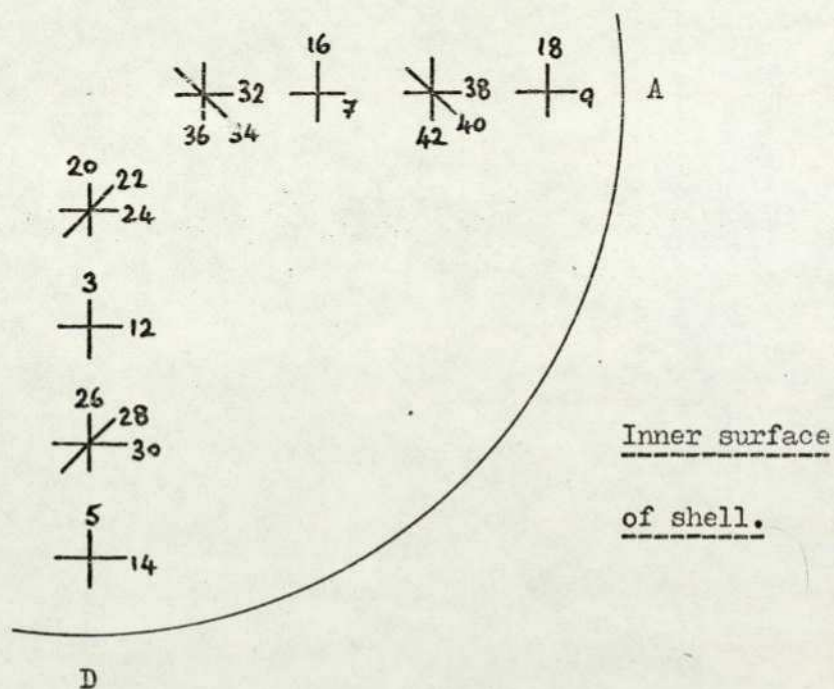
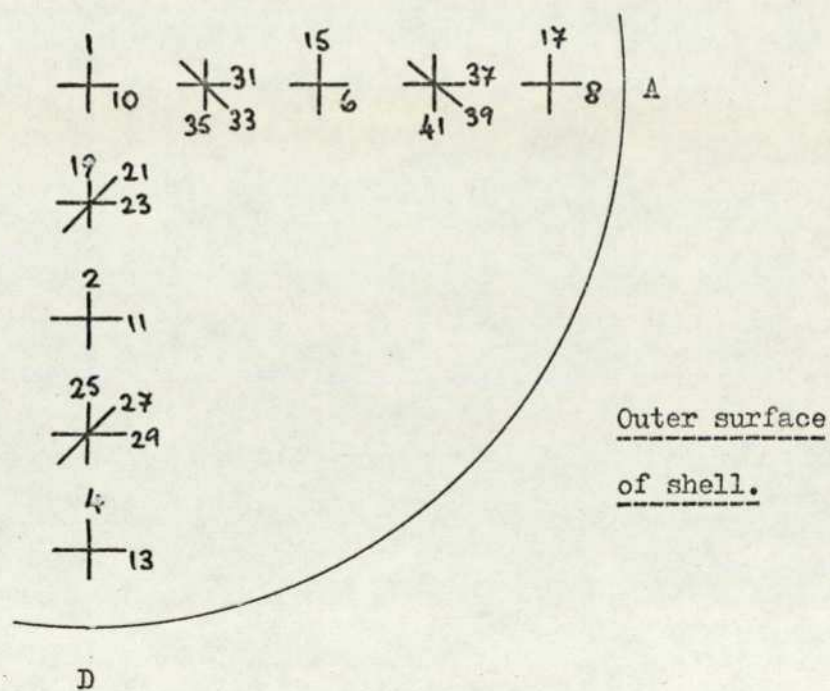


Fig. 16 b) Frustum of a cone.
(cast 20/7/76, tested 27/1/77)



Strain gauge numbering.

(Frustum of cone cast 20/7/76)

Fig. 16 c)

Strain Gauge Number	Pressure psi					
	0	100	200	300	400	
1	235	228	220	212	203	
2	330	325	319	313	306	
3	530	523	515	507	499	
4	436	438	441	444	447	
5	532	527	521	514	506	
6	-44	-51	-60	-70	-79	
7	390	382	373	365	356	
8	-	-	-	-	-	erratic
9	509	499	487	473	459	
10	-	-	-	-	-	erratic
11	480	467	459	447	432	
12	917	897	874	851	827	
13	617	609	600	592	585	
14	713	699	685	672	658	
15	354	341	326	310	295	
16	632	609	582	555	528	
17	651	641	631	622	615	
18	618	603	588	575	561	
19	693	691	688	685	683	
20	-	-	-	-	-	gauge not working
21	381	377	372	368	363	
22	128	118	107	96	84	
23	198	192	186	179	172	
24	-	-	-	-	-	gauge not working
25	70	51	29	6	-16	
26	366	365	365	365	365	
27	191	171	150	128	106	
28	327	316	307	297	286	
29	315	295	273	251	228	
30	158	134	110	86	61	
31	741	738	735	732	729	
32	1045	1036	1030	1022	1014	
33	340	336	332	328	323	
34	-173	-182	-192	-203	-213	
35	-	-	-	-	-	erratic
36	769	755	741	727	712	
37	-73	-97	-122	-148	-176	
38	489	487	486	485	485	
39	627	607	584	561	536	
40	660	645	630	616	600	
41	65	46	26	7	-11	
42	269	241	211	181	150	

Frustum of a Cone

Selected strain gauge readings.

Strain Gauge Number	Pressure psi				av.
	0 - 100	100 - 200	200 - 300	300 - 400	
1	-7	-8	-8	-9	-9
2	-5	-6	-6	-7	-6
3	-7	-8	-8	-8	-7.75
4	2	3	3	3	2.75
5	-5	-6	-7	-8	-6.5
6	-7	-9	-10	-9	-8.75
7	-8	-9	-8	-9	-8.5
8	-	-	-	-	-
9	-10	-12	-14	-14	-12.5
10	-	-	-	-	-
11	-13	-8	-12	-15	-12
12	-20	-23	-23	-24	-22.5
13	-8	-9	-8	-7	-8
14	-14	-14	-13	-14	-13.75
15	-13	-15	-16	-15	-14.75
16	-23	-27	-27	-27	-26
17	-10	-10	-9	-7	-9
18	-15	-15	-13	-14	-14.25
19	-2	-3	-3	-2	-2.5
20	-	-	-	-	-
21	-4	-5	-4	-5	-4.5
22	-10	-11	-11	-12	-11
23	-6	-6	-7	-7	-6.5
24	-	-	-	-	-
25	-19	-22	-23	-22	-21.5
26	-1	0	0	0	-0.25
27	-20	-21	-22	-22	-21.25
28	-11	-9	-10	-11	-10.25
29	-20	-22	-22	-23	-21.75
30	-24	-24	-24	-25	-24.25
31	-3	-3	-3	-3	-3
32	-9	-6	-8	-8	-7.75
33	-4	-4	-4	-5	-4.25
34	-9	-10	-11	-10	-10
35	-	-	-	-	-
36	-14	-14	-14	-15	-14.25
37	-24	-25	-26	-28	-25.75
38	-2	-1	-1	0	-1
39	-20	-23	-23	-25	-22.75
40	-15	-15	-14	-16	-15
41	-19	-20	-19	-18	-19
42	-28	-30	-30	-31	-29.75

Strain gauge reading differences.

Strain Gauge Number	Hemisphere	Cone
1	-2.96	-1.69
2	-3.29	-1.13
3	-3.24	-1.46
4	-4.61	0.52
5	-1.65	-1.22
6	-0.94	-1.65
7	-	-1.60
8	-3.43	-
9	-3.29	-2.35
10	-0.61	-
11	-1.55	-2.26
12	-4.37	-4.23
13	-2.21	-1.50
14	-2.91	-2.59
15	-2.02	-2.77
16	-1.13	-4.89
17	-2.35	-1.69
18	-3.43	-2.63
19	-2.26	-0.47
20	-4.14	-
21	-1.79	-0.85
22	-2.68	-2.07
23	-2.54	-1.22
24	-3.57	-
25	-3.01	-4.04
26	-4.70	-0.05
27	1.13	4.00
28	-0.75	-1.93
29	-1.27	-4.09
30	-0.52	-4.56
31	-2.07	-0.56
32	-2.12	-1.46
33	-2.12	-0.80
34	-3.76	-1.88
35	-1.69	-
36	-4.00	-2.68
37	-3.34	-4.84
38	-4.04	-0.19
39	-	-4.28
40	-2.82	-2.82
41	-2.73	-3.57
42	-3.38	-5.59
43	-3.38	-
44	-3.71	-

Stress factors

CHAPTER FOUR

Comparison Between Theory and Experiment.

The following pages give the computer output in the form of principal planes and stress factors for the two cases which have been described in the experimental work, the hemisphere and the frustum of a cone. The analysis by the finite element displacement method employs the Q2CST element and cannot be directly compared with the experimental work in which surface strain gauges measure movement in the meridional and circumferential directions.

The chosen finite element program gives output values at the element centroid, and the surface stress factors can be found by assuming an even distribution of stress across the shell thickness or by extrapolation. Shell values are found from principal values by rearranging the equations which relate them to global or any other axes.

ie.

$$\sigma_p - \sigma_r = \tau_{rz} \cot \phi$$

$$\sigma_p - \sigma_z = \tau_{rz} \tan \phi$$

eliminating τ_{rz}

$$(\sigma_p - \sigma_r) \tan \phi = \tau_{rz} = (\sigma_p - \sigma_z) \cot \phi$$

$$\therefore \sigma_p \tan \phi - \sigma_r \tan \phi = \sigma_p \cot \phi - \sigma_z \cot \phi$$

also

$$\begin{aligned} \sigma_r + \sigma_z &= 2\sigma_p - 2\tau_{\max} \\ &= 2\sigma_p + 2\tau_{\max} \end{aligned}$$

$$\therefore \sigma_r + \sigma_z = \sigma_p + \sigma_q$$

combining these two relationships

$$\sigma_p \tan \phi + \sigma_q \cot \phi = \sigma_r (\tan \phi + \cot \phi)$$

$$\therefore \sigma_r = \frac{\sigma_p \tan \phi + \sigma_q \cot \phi}{\tan \phi + \cot \phi}$$

$$\therefore \sigma_r = \sigma_p \sin^2 \phi + \sigma_q \cos^2 \phi$$

and

$$\sigma_z = \sigma_p \cos^2 \phi + \sigma_q \sin^2 \phi$$

For convenience the finite element results are plotted as continuous curves of stress factor against the angle between the outer radius and axis of revolution of the hemisphere. For the frustum of a cone the stress factors are plotted against the distance measured along the surface meridian from the axis of revolution at the crown. The experimental results are superimposed as isolated values for comparison.

Of the two experiments the frustum shows more similarity between theory and experiment which may be put down to improved experience in carrying out a reliable testing procedure. The hemisphere equator may not have been fully encastre, while the frustum was embedded much more efficiently in the base by the use of a small pipe to release the air-lock inside the shell. That was during casting, and then the theoretical analysis suggests minor tensile stresses which could initiate rotation of the hemisphere equator.

The hemisphere shows wide variation of the experimental results at similar positions along the meridians which are at right-angles to each other. The corresponding pairs of results for the frustum appear close together and impart the sense of authenticity as an axisymmetric phenomenon.

Computer Programme Results

Element Type Q2CST

Cylindrical Perspex Model

Mk I

Element Nodes				Principal Plane (Degrees)	Stress Factors			
I	J	K	L		Major	Minor	Hoop	Shear
1	5	6	2	-8.05	-0.31	-2.38	-3.40	1.03
2	6	7	3	-6.43	-0.31	-2.65	-3.87	1.17
3	7	8	4	-3.56	-0.14	-2.71	-4.29	1.28
5	9	10	6	-3.69	-0.93	-2.83	-3.90	0.95
6	10	11	7	-7.18	-0.47	-2.48	-3.94	1.01
7	11	12	8	-4.41	-0.19	-2.38	-4.24	1.09
9	13	14	10	-2.48	-0.89	-3.01	-3.98	1.06
10	14	15	11	-5.06	-0.59	-2.53	-4.04	0.97
11	15	16	12	-3.99	-0.20	-2.10	-4.16	0.95
13	17	18	14	-1.26	-0.88	-3.14	-3.96	1.13
14	18	19	15	-2.79	-0.56	-2.52	-3.98	0.98
15	19	20	16	-2.46	-0.20	-1.93	-4.07	0.87
17	21	22	18	0.36	-0.86	-3.17	-3.81	1.15
18	22	23	19	0.05	-0.54	-2.50	-3.83	0.98
19	23	24	20	-0.50	-0.18	-1.83	-3.90	0.82
21	25	26	22	2.67	-0.85	-3.07	-3.50	1.11
22	26	27	23	3.85	-0.52	-2.49	-3.56	0.98
23	27	28	24	1.98	-0.17	-1.82	-3.66	0.83
25	29	30	26	6.59	-0.83	-2.79	-3.00	0.98
26	30	31	27	8.85	-0.49	-2.51	-3.19	1.01
27	31	32	28	4.90	-0.16	-1.94	-3.36	0.89
29	33	34	30	14.81	-0.77	-2.29	-2.32	0.76
30	34	35	31	14.93	-0.42	-2.58	-2.73	1.08
31	35	36	32	7.80	-0.14	-2.25	-3.03	1.06
33	37	38	34	35.50	-0.49	-1.67	-1.50	0.59
34	38	39	35	21.39	-0.30	-2.70	-2.21	1.20
35	39	40	36	10.33	-0.12	-2.80	-2.71	1.34
37	41	42	38	72.12	-0.08	-1.54	-0.96	0.73
38	42	43	39	25.71	-0.21	-2.88	-1.75	1.33
39	43	44	40	12.03	-0.08	-3.66	-2.49	1.79
41	45	46	42	-83.40	0.54	-1.15	-0.64	0.85
42	46	47	43	41.26	-1.75	-3.34	-2.25	0.80
43	47	48	44	15.80	-0.00	-4.68	-2.40	2.34
45	49	50	46	-63.76	-0.06	-1.07	-1.28	0.50
46	50	51	47	-58.68	-0.81	-1.46	-1.39	0.32
47	51	52	48	27.13	-3.00	-6.90	-4.17	1.95
49	53	54	50	-19.87	0.07	-0.86	-1.57	0.46
50	54	55	51	-1.01	-0.64	-1.44	-1.68	0.40
51	55	56	52	55.29	-3.17	-6.54	-4.09	1.68
53	57	58	54	7.02	0.80	-1.14	-1.91	0.97
54	58	59	55	45.47	-1.16	-3.99	-3.10	1.41
55	59	60	56	70.00	0.55	-3.37	-0.96	1.96
57	61	62	58	42.34	-0.33	-2.26	-3.49	0.97
58	62	63	59	58.50	0.64	-2.95	-2.02	1.79
59	63	64	60	61.84	0.44	-1.88	0.07	1.16
61	65	66	62	72.27	-0.41	-3.58	-4.60	1.58
62	66	67	63	59.77	0.43	-2.85	-2.06	1.64
63	67	68	64	41.62	0.87	-0.75	1.13	0.81
65	69	70	66	81.81	-0.63	-5.09	-5.70	2.23

				Mk I (continued)				
66	70	71	67	65.38	0.19	-2.60	-2.00	1.39
67	71	72	68	20.66	1.48	-0.20	2.00	0.84
69	73	74	70	87.04	-0.77	-6.25	-6.49	2.74
70	74	75	71	69.93	-0.31	-2.52	-1.98	1.11
71	75	76	72	5.16	2.20	-0.17	2.78	1.18
73	77	78	74	85.36	-2.45	-8.01	-7.33	2.78
74	78	79	75	67.49	-1.07	-2.16	-1.49	0.54
75	79	80	76	-2.80	3.31	-0.89	3.90	2.10

Input Data

7.62 5.715 5.715 1.270 1.270 1.270 1.0 0.39 -1.0

PERSPEX CYLINDER

Mk I Model

Analysis of Computer Results

Element Nodes				Meridional Stress Factor	Element Nodes				Meridional Stress Factor
I	J	K	L		I	J	K	L	
1	5	6	2	-2.34	43	47	48	44	-4.33
2	6	7	3	-2.62	45	49	50	46	-0.26
3	7	8	4	-2.70	46	50	51	47	-0.99
5	9	10	6	-2.82	47	51	52	48	-6.09
6	10	11	7	-2.45	49	53	54	50	-0.04
7	11	12	8	-2.37	50	54	55	51	-0.64
9	13	14	10	-3.01	51	55	56	52	-5.45
10	14	15	11	-2.51	53	57	58	54	0.77
11	15	16	12	-2.09	54	58	59	55	-2.60
13	17	18	14	-3.14	55	59	60	56	-2.91
14	18	19	15	-2.52	57	61	62	58	-1.21
15	19	20	16	-1.93	58	62	63	59	-1.97
17	21	22	18	-3.17	59	63	64	60	-1.36
18	22	23	19	-2.50	61	65	66	62	-3.29
19	23	24	20	-1.83	62	66	67	63	-2.02
21	25	26	22	-3.07	63	67	68	64	0.16
22	26	27	23	-2.48	65	69	70	66	-5.00
23	27	28	24	-1.82	66	70	71	67	-2.12
25	29	30	26	-2.76	67	71	72	68	1.27
26	30	31	27	-2.46	69	73	74	70	-6.24
27	31	32	28	-1.93	70	74	75	71	-2.26
29	33	34	30	-2.19	71	75	76	72	2.18
30	34	35	31	-2.44	73	77	78	74	-7.97
31	35	36	32	-2.21	74	78	79	75	-2.00
33	37	38	34	-1.27	75	79	80	76	3.30
34	38	39	35	-2.38					
35	39	40	36	-2.71					
37	41	42	38	-0.22					
38	42	43	39	-2.38					
39	43	44	40	-3.50					
41	45	46	42	0.52					
42	46	47	43	-2.65					

PERSPEX CYLINDER

Mk I Model

Analysis of Computer Results

Method of nodal averaging with assumption of constant stress across
each element.

Outer Surface Node	Meridional Stress Factor	Hoop Stress Factor	Inner Surface Node	Meridional Stress Factor	Hoop Stress Factor
1	-2.34	-3.40	4	-2.70	-4.29
5	-2.58	-3.65	8	-2.54	-4.27
9	-2.92	-3.94	12	-2.23	-4.20
13	-3.08	-3.97	16	-2.01	-4.12
17	-3.16	-3.89	20	-1.88	-3.99
21	-3.12	-3.66	24	-1.83	-3.78
25	-2.92	-3.25	28	-1.88	-3.51
29	-2.48	-2.66	32	-2.07	-3.20
33	-1.73	-1.91	36	-2.46	-2.87
37	-0.75	-1.23	40	-3.11	-2.60
41	0.15	-0.80	44	-3.92	-2.45
45	0.13	-0.96	48	-5.21	-3.29
53	0.37	-1.74	56	-4.18	-2.53
57	-0.22	-2.70	60	-2.14	-0.45
61	-2.25	-4.05	64	-0.60	0.60
65	-4.15	-5.15	68	0.72	1.57
69	-5.62	-6.10	72	1.73	2.39
73	-7.11	-6.91	76	2.74	3.34
77	-7.97	-7.33	80	3.30	3.90

PERSPEX CYLINDER

Mk I Model

Outer Surface Meridional Stress Factor

Compressive
Stress Factor
Negative

— Theoretical Stress Factor

⊗ Experimental Stress Factor

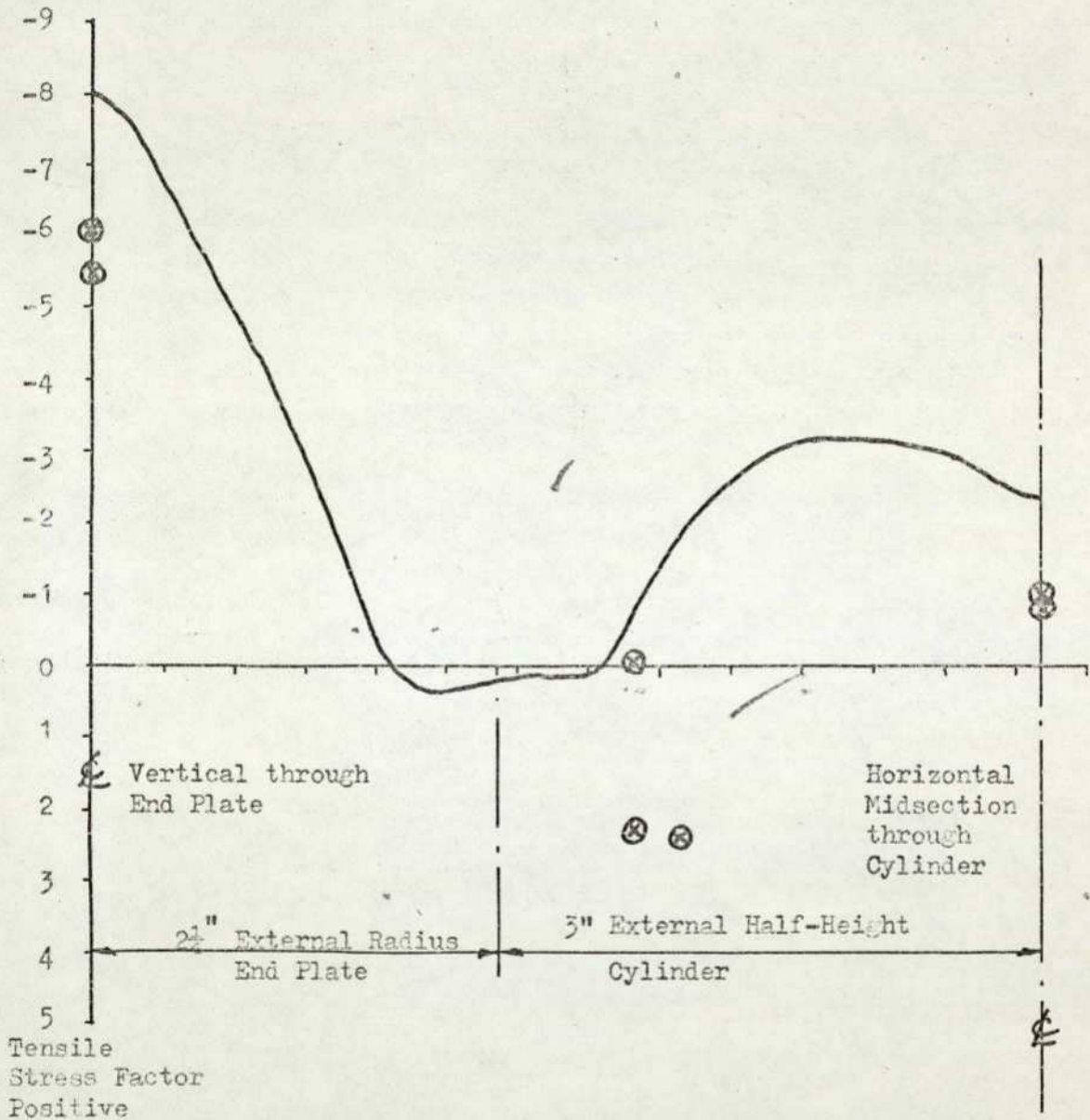


Fig. 17 a)

PERSPEX CYLINDER

Mk I Model

Inner Surface Meridional Stress Factor

Compressive
Stress Factor
Negative

..... Theoretical Stress Factor
⊗ Experimental Stress Factor

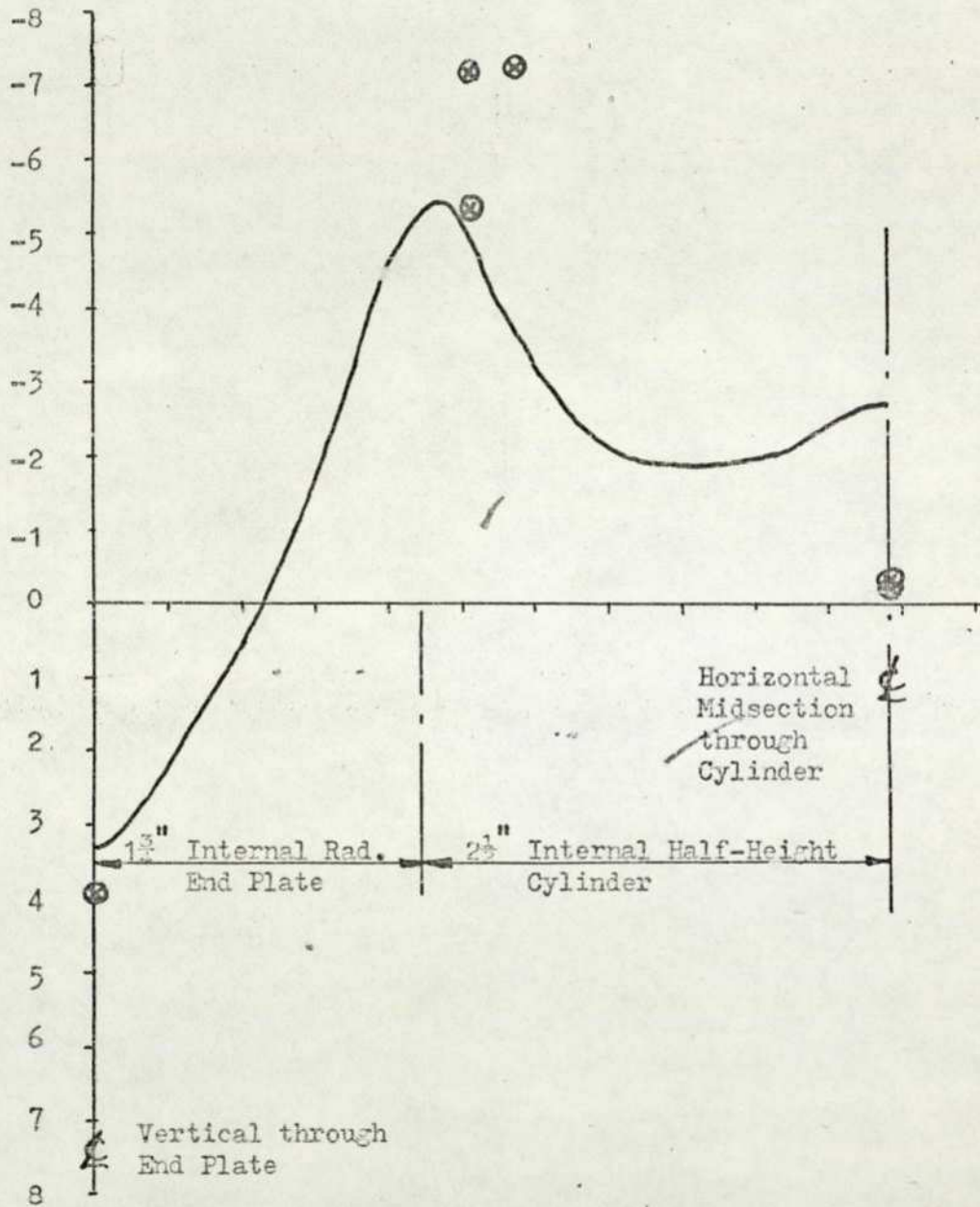


Fig. 17 b)

PERSPEX CYLINDER

Mk I Model

Outer Surface Hoop Stress Factor

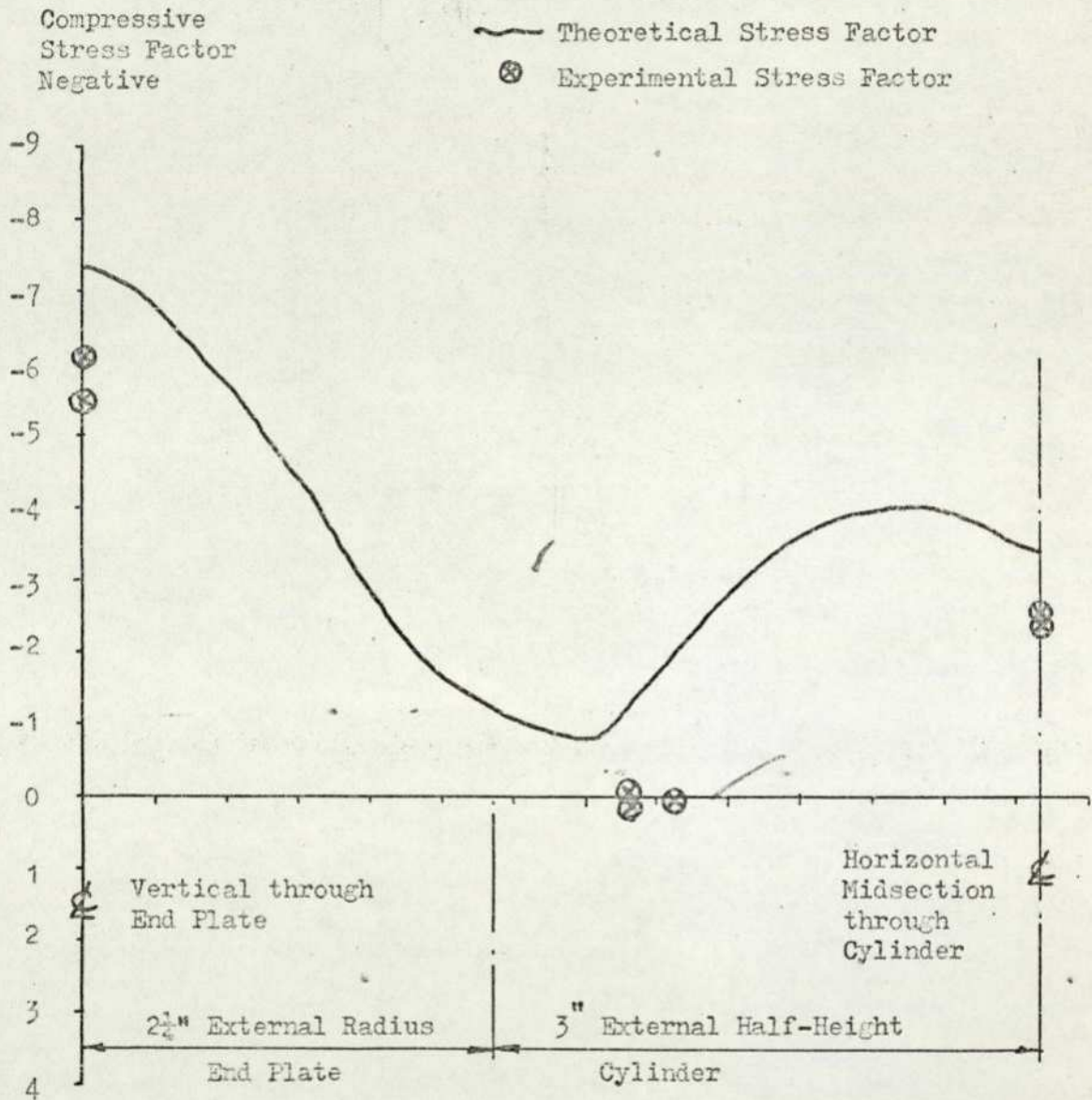


Fig. 17 c)

PERSPEX CYLINDER

Mk I Model

Inner Surface Hoop Stress Factor

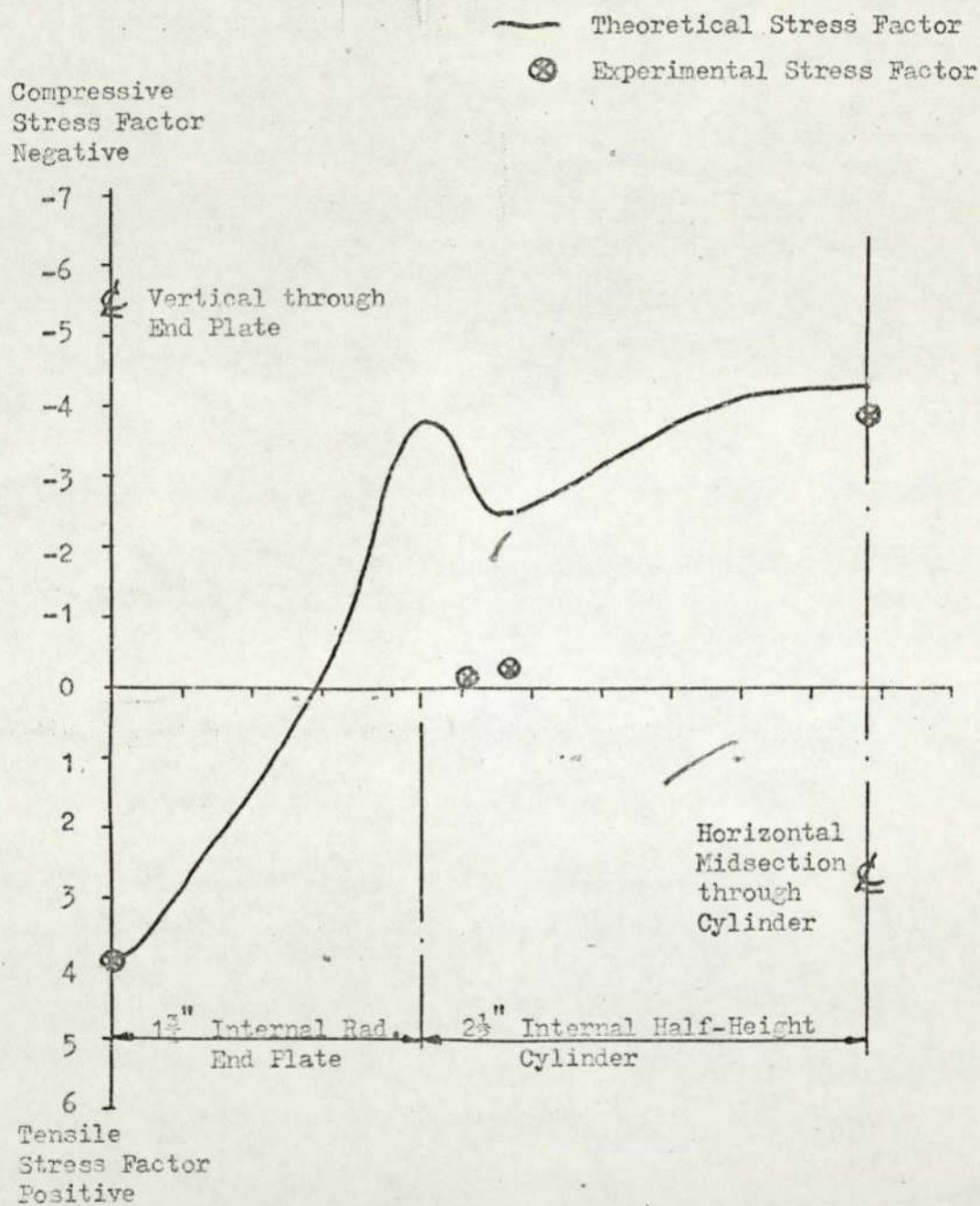


Fig. 17 d)

Computer Programme Results

Element Type Q2CST
Cylindrical Perspex Model
Mk II

Element Nodes				Principal Plane		Stress Factors			
I	J	K	L	(Degrees)	Major	Minor	Hoop	Shear	
1	5	6	2	-8.37	-0.05	-3.31	-5.87	1.63	
2	6	7	3	-6.47	-0.13	-4.38	-6.69	2.13	
3	7	8	4	-3.62	-0.09	-5.36	-7.48	2.63	
5	9	10	6	-3.93	-0.94	-4.30	-6.95	1.68	
6	10	11	7	-5.99	-0.46	-4.27	-7.10	1.91	
7	11	12	8	-3.73	-0.14	-4.51	-7.47	2.18	
9	13	14	10	-2.22	-0.81	-4.83	-7.43	2.01	
10	14	15	11	-3.68	-0.49	-4.32	-7.47	1.91	
11	15	16	12	-2.51	-0.10	-3.90	-7.57	1.90	
13	17	18	14	-0.84	-0.77	-5.21	-7.71	2.22	
14	18	19	15	-2.01	-0.44	-4.27	-7.63	1.92	
15	19	20	16	-1.26	-0.09	-3.45	-7.62	1.68	
17	21	22	18	-0.02	-0.78	-5.44	-7.65	2.33	
18	22	23	19	-0.77	-0.45	-4.23	-7.55	1.89	
19	23	24	20	-0.41	-0.10	-3.12	-7.49	1.51	
21	25	26	22	0.55	-0.85	-5.51	-7.18	2.33	
22	26	27	23	0.32	-0.50	-4.20	-7.14	1.85	
23	27	28	24	0.27	-0.15	-2.92	-7.12	1.39	
25	29	30	26	1.26	-0.97	-5.35	-6.24	2.19	
26	30	31	27	1.71	-0.57	-4.19	-6.36	1.81	
27	31	32	28	0.93	-0.22	-2.89	-6.47	1.34	
29	33	34	30	2.68	-1.14	-4.76	-4.81	1.81	
30	34	35	31	4.05	-0.68	-4.22	-5.27	1.77	
31	35	36	32	1.73	-0.28	-3.13	-5.58	1.42	
33	37	38	34	7.28	-1.24	-3.46	-2.94	1.11	
34	38	39	35	8.27	-0.79	-4.29	-3.95	1.75	
35	39	40	36	2.82	-0.41	-3.81	-4.61	1.70	
37	41	42	38	73.07	-0.77	-2.19	-1.17	0.71	
38	42	43	39	12.82	-0.30	-4.45	-2.47	2.07	
39	43	44	40	4.23	-0.33	-5.27	-3.75	2.47	
41	45	46	42	89.89	1.78	-0.58	0.49	1.18	
42	46	47	43	29.89	-2.56	-5.21	-2.84	1.33	
43	47	48	44	7.34	0.10	-7.08	-3.09	3.59	
45	49	50	46	-79.35	0.47	-0.41	-0.94	0.44	
46	50	51	47	-2.47	-0.50	-0.62	-0.63	0.06	
47	51	52	48	22.39	-3.15	-9.96	-4.93	3.41	
49	53	54	50	1.43	0.06	-0.69	-2.11	0.37	
50	54	55	51	24.50	0.05	-1.86	-1.54	0.96	
51	55	56	52	50.02	-2.32	-6.92	-3.44	2.30	
53	57	58	54	32.20	0.02	-1.54	-3.02	0.78	
54	58	59	55	48.16	-0.50	-4.12	-2.85	1.81	
55	59	60	56	65.55	0.94	-2.57	0.01	1.75	
57	61	62	58	63.93	-0.52	-3.24	-4.51	1.36	
58	62	63	59	59.15	0.75	-2.89	-1.88	1.82	
59	63	64	60	52.93	0.89	-1.21	1.02	1.05	
61	65	66	62	78.25	-0.47	-4.59	-5.46	2.06	
62	66	67	63	61.38	0.46	-2.69	-1.86	1.58	
63	67	68	64	31.63	1.39	-0.38	2.01	0.89	
65	69	70	66	84.26	-0.73	-5.83	-6.39	2.55	

				Mk II (continued)				
66	70	71	67	65.26	0.12	-2.48	-1.81	1.30
67	71	72	68	13.84	2.11	-0.14	2.87	1.13
69	73	74	70	87.93	-1.01	-6.90	-7.20	2.95
70	74	75	71	65.82	-0.51	-2.49	-1.81	0.99
71	75	76	72	3.25	3.02	-0.33	3.75	1.68
73	77	78	74	84.47	-3.14	-9.12	-8.27	2.99
74	78	79	75	50.78	-1.26	-2.29	-1.29	0.51
75	79	80	76	-1.91	4.41	-1.22	5.12	2.62

Input Data

7.62 5.080 5.080 0.635 0.635 1.270 1.0 0.39 -1.0

PERSPEX CYLINDER

Mk II Model

Analysis of Computer Results

Method of nodal averaging with assumption of constant stress
across each element

Outer Surface Node	Meridional Stress Factor	Hoop Stress Factor	Inner Surface Node	Meridional Stress Factor	Hoop Stress Factor
1	-3.24	-5.87	4	-5.34	-7.48
5	-3.76	-6.41	8	-4.92	-7.48
9	-4.55	-7.19	12	-4.19	-7.52
13	-5.02	-7.57	16	-3.67	-7.60
17	-5.33	-7.68	20	-3.29	-7.56
21	-5.48	-7.42	24	-3.02	-7.31
25	-5.43	-6.71	28	-2.91	-6.80
29	-5.05	-5.53	32	-3.01	-6.03
33	-4.09	-3.88	36	-3.47	-5.10
37	-2.16	-2.06	40	-4.52	-4.18
41	0.45	-0.34	44	-6.10	-3.42
45	1.11	-0.23	48	-7.97	-4.01
49	-0.13	-1.53	52	-6.60	-4.19
	-0.16			-4.58	
53	-0.18	-2.57	56	-3.50	-1.72
57	-1.57	-3.77	60	-1.21	0.52
61	-3.57	-4.99	64	0.23	1.52
65	-5.10	-5.93	68	1.44	2.44
69	-6.34	-6.80	72	2.50	3.31
73	-7.98	-7.74	76	3.71	4.44
77	-9.06	-8.27	80	4.40	5.12

PERSPEX CYLINDER

Mk II Model

Outer Surface Meridional Stress Factor

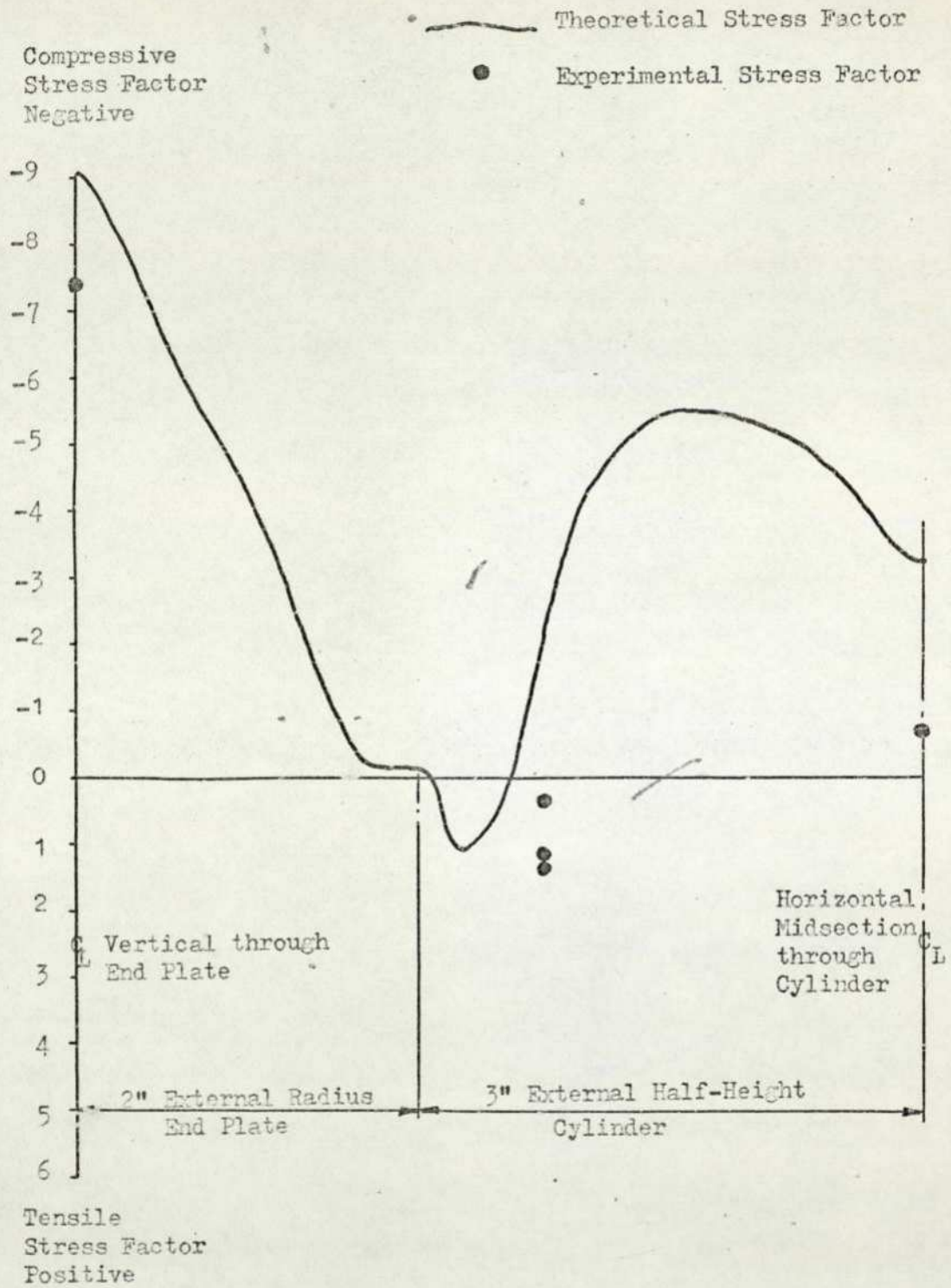


Fig. 18 a)

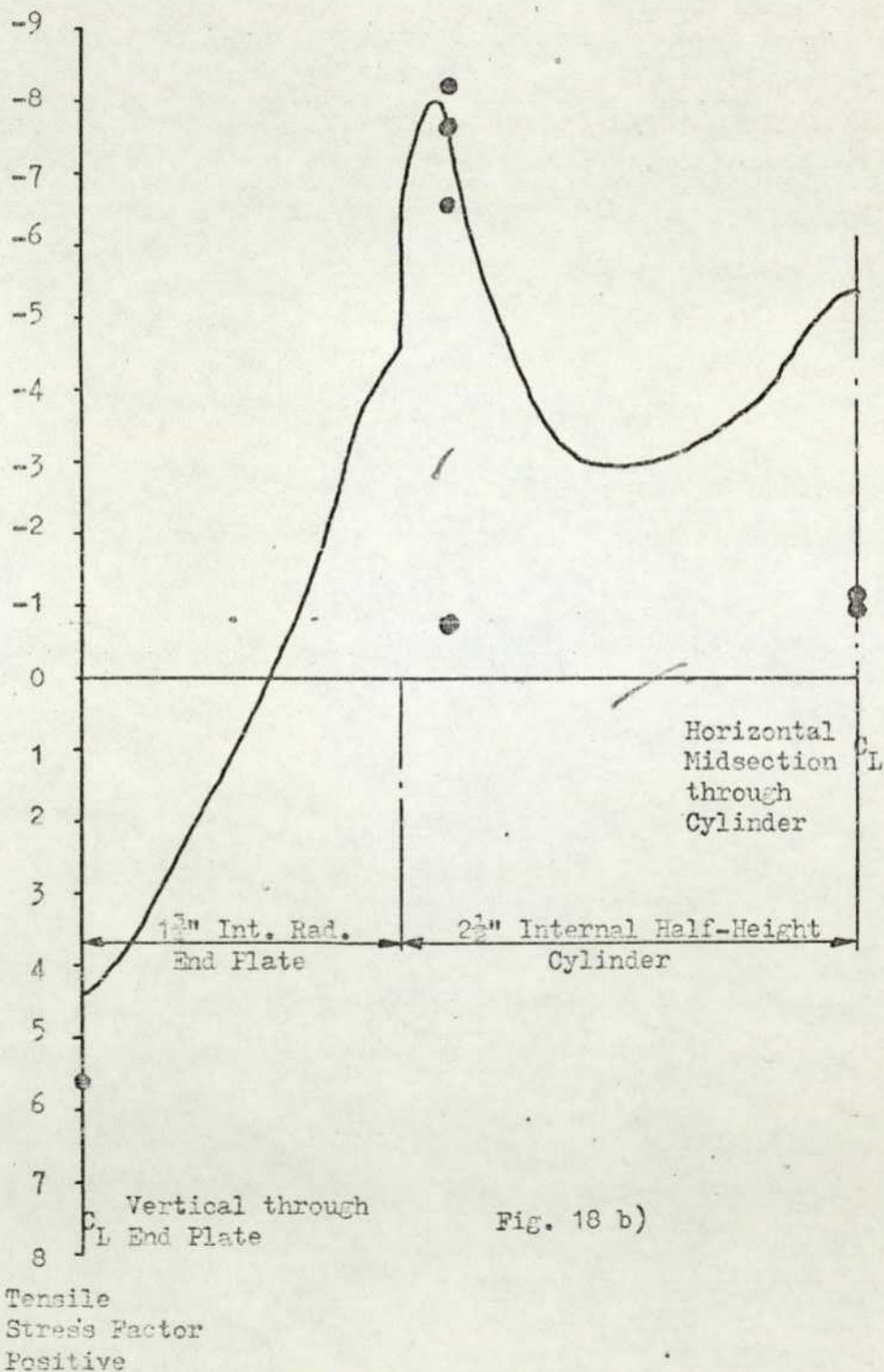
PERSPEX CYLINDER

Mk II Model

Inner Surface Meridional Stress Factor

Compressive Stress
Factor Negative

— Theoretical Stress Factor
● Experimental Stress Factor

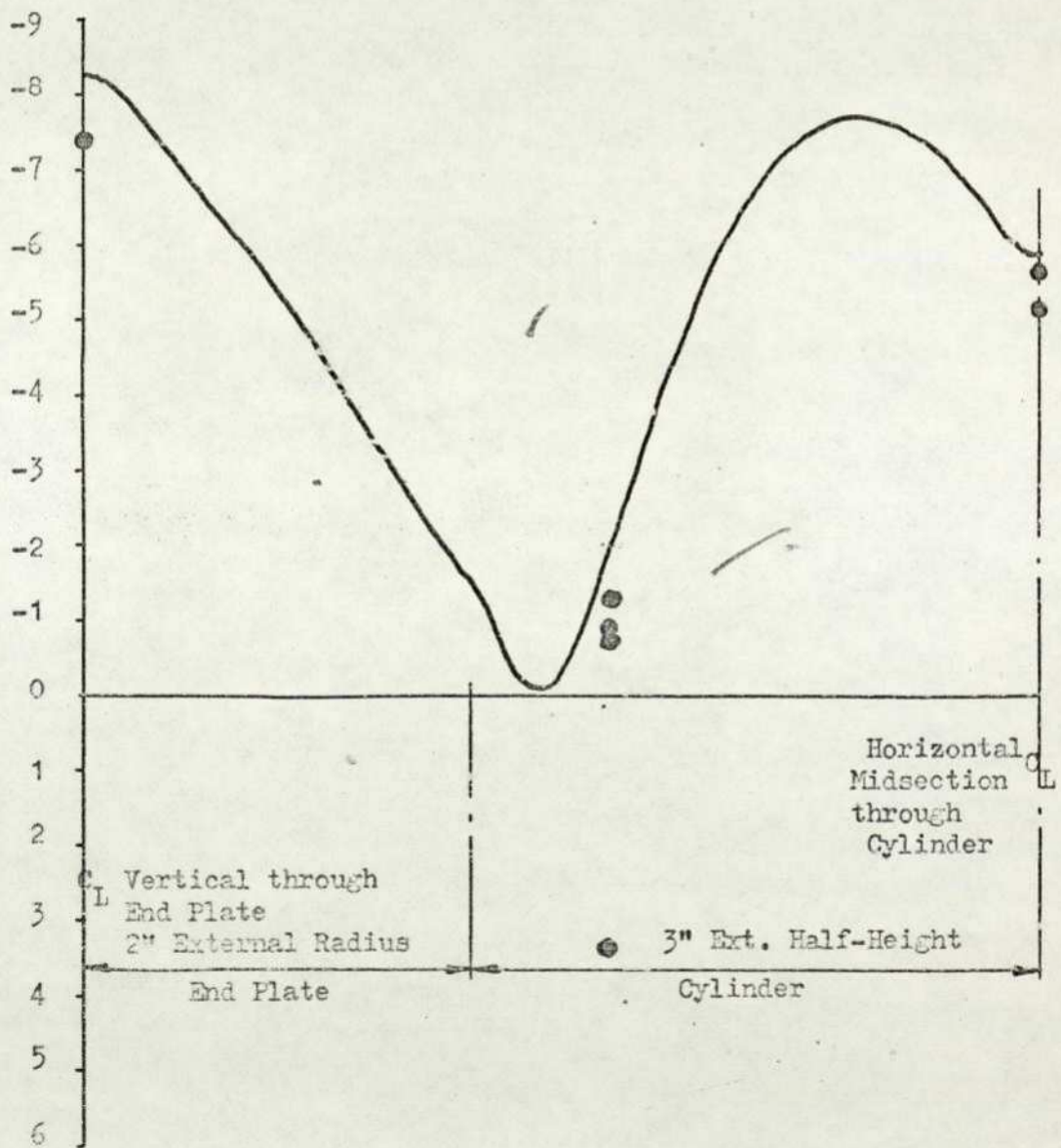


PERSPEX CYLINDER

Mk II Model

Outer Surface Hoop Stress Factor

— Theoretical Stress Factor
 • Experimental Stress Factor
 Compressive Stress Factor Negative



Tensile
 Stress Factor
 Positive

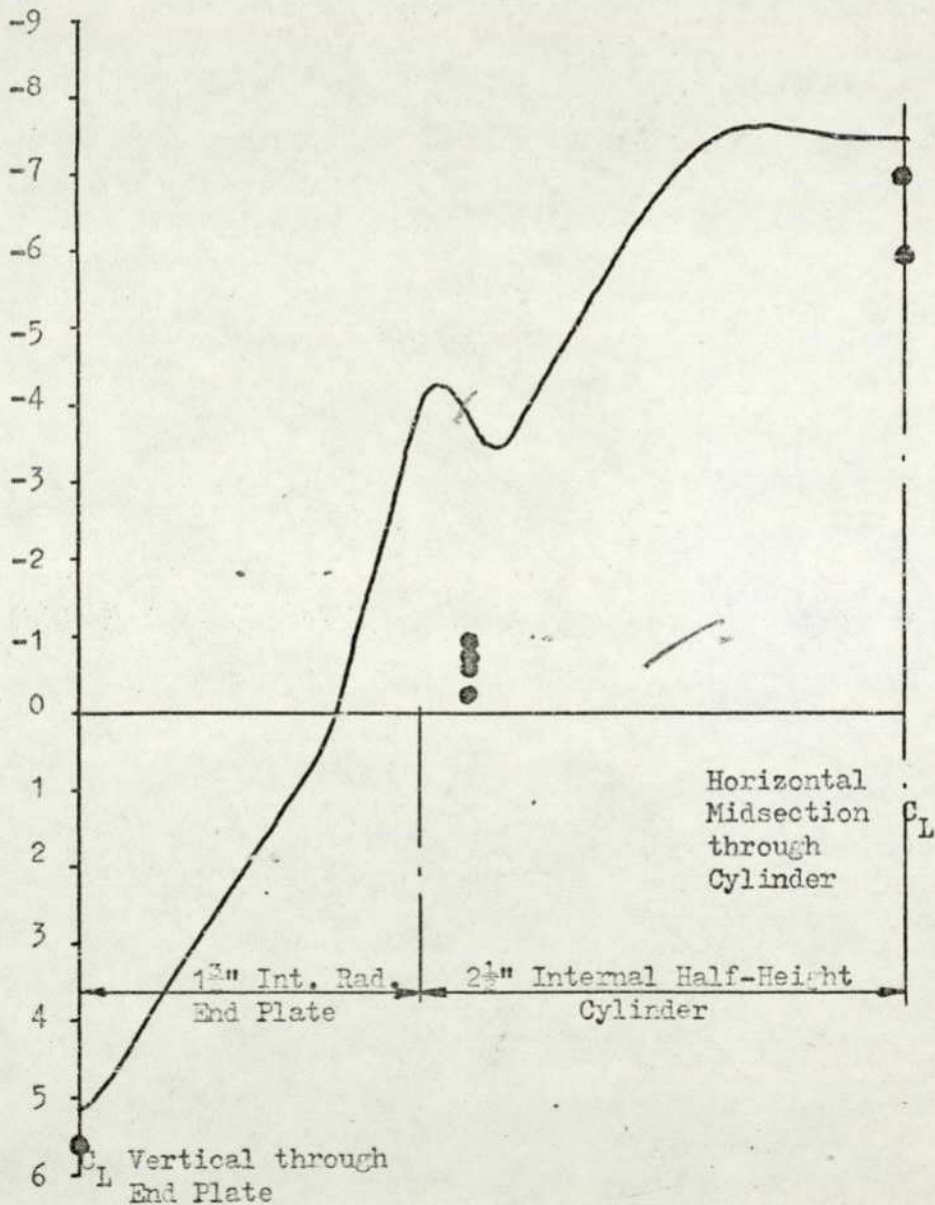
Fig. 18 c)

PERSPEX CYLINDER

Mk II Model

Inner Surface Hoop Stress Factor

— Theoretical Stress Factor
 • Experimental Stress Factor
 Compressive Stress Factor Negative



Tensile
 Stress Factor
 Positive

Fig. 18 d)

74	78	79	75	87.51	-0.68	-4.63	-4.58	1.98
75	79	80	76	87.56	-0.36	-4.95	-4.88	2.29

Q2CST computer programme results

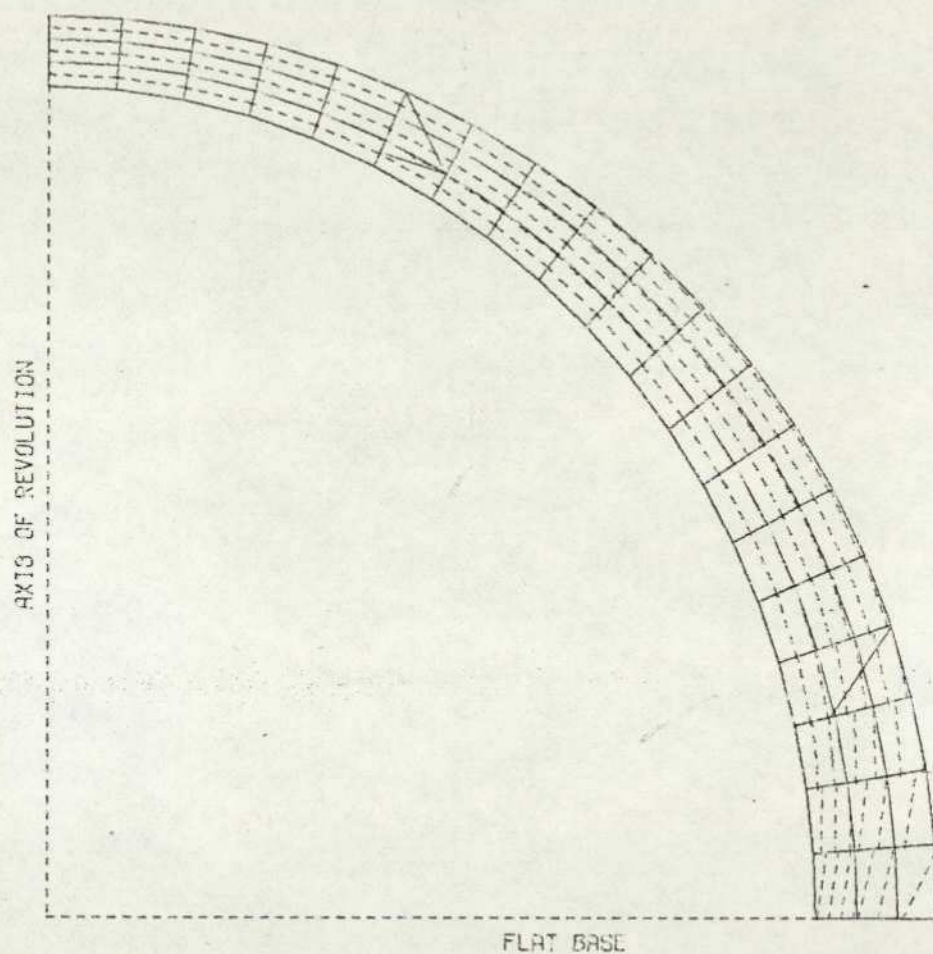
Hemisphere Mk I

Input data 11.6 11.6 1.6 1.3 1. .2 -1.

Element nodes				Principal planes (degrees)	Stress factors			
I	J	K	L		major	minor	hoop	shear
1	5	6	2	-25.43	-0.39	-2.05	-0.63	0.83
2	6	7	3	-12.66	-0.71	-4.04	-1.09	1.66
3	7	8	4	-7.71	-1.01	-6.29	-1.62	2.64
5	9	10	6	-7.41	-0.97	-2.75	-1.24	0.89
6	10	11	7	-6.42	-0.62	-3.94	-1.41	1.66
7	11	12	8	-0.37	-0.40	-5.19	-1.64	2.40
9	13	14	10	4.37	-0.97	-3.34	-1.82	1.19
10	14	15	11	2.31	-0.64	-3.89	-1.86	1.63
11	15	16	12	5.86	-0.30	-4.44	-1.93	2.07
13	17	18	14	12.56	-0.93	-3.79	-2.38	1.43
14	18	19	15	10.45	-0.61	-3.89	-2.34	1.64
15	19	20	16	12.17	-0.25	-3.98	-2.32	1.87
17	21	22	18	19.25	-0.90	-4.09	-2.88	1.60
18	22	23	19	17.75	-0.58	-3.93	-2.81	1.67
19	23	24	20	18.33	-0.21	-3.74	-2.73	1.76
21	25	26	22	25.16	-0.87	-4.29	-3.31	1.71
22	26	27	23	24.26	-0.55	-3.99	-3.22	1.72
23	27	28	24	24.20	-0.19	-3.66	-3.13	1.73
25	29	30	26	30.60	-0.86	-4.40	-3.66	1.77
26	30	31	27	30.13	-0.53	-4.06	-3.58	1.76
27	31	32	28	29.69	-0.18	-3.68	-3.50	1.75
29	33	34	30	35.75	-0.85	-4.46	-3.93	1.81
30	34	35	31	35.52	-0.53	-4.13	-3.87	1.80
31	35	36	32	34.84	-0.17	-3.78	-3.82	1.80
33	37	38	34	40.69	-0.84	-4.48	-4.12	1.82
34	38	39	35	40.57	-0.52	-4.20	-4.09	1.84
35	39	40	36	39.75	-0.17	-3.91	-4.08	1.87
37	41	42	38	45.49	-0.84	-4.43	-4.26	1.82
38	42	43	39	45.39	-0.53	-4.27	-4.27	1.87
39	43	44	40	44.50	-0.18	-4.05	-4.29	1.94
41	45	46	42	50.21	-0.85	-4.47	-4.35	1.81
42	46	47	43	50.07	-0.54	-4.33	-4.39	1.90
43	47	48	44	49.18	-0.19	-4.20	-4.46	2.01
45	49	50	46	54.87	-0.85	-4.46	-4.41	1.80
46	50	51	47	54.69	-0.55	-4.39	-4.48	1.92
47	51	52	48	53.84	-0.19	-4.33	-4.58	2.07
49	53	54	50	59.50	-0.86	-4.45	-4.44	1.79
50	54	55	51	59.29	-0.55	-4.44	-4.54	1.95
51	55	56	52	58.51	-0.20	-4.46	-4.68	2.13
53	57	58	54	64.13	-0.86	-4.44	-4.45	1.79
54	58	59	55	63.91	-0.56	-4.49	-4.58	1.96
55	59	60	56	63.22	-0.21	-4.56	-4.74	2.17
57	61	62	58	68.78	-0.87	-4.44	-4.45	1.79
58	62	63	59	68.57	-0.57	-4.53	-4.61	1.98
59	63	64	60	67.98	-0.22	-4.64	-4.79	2.21
61	65	66	62	73.44	-0.88	-4.44	-4.44	1.78
62	66	67	63	73.28	-0.58	-4.56	-4.62	1.99
63	67	68	64	72.77	-0.22	-4.71	-4.82	2.24
65	69	70	66	78.13	-0.89	-4.44	-4.42	1.78
66	70	71	67	78.05	-0.59	-4.59	-4.62	2.00
67	71	72	68	77.60	-0.23	-4.77	-4.85	2.27
69	73	74	70	82.80	-0.90	-4.42	-4.39	1.76
70	74	75	71	82.87	-0.61	-4.61	-4.61	2.00
71	75	76	72	82.43	-0.26	-4.83	-4.88	2.28
73	77	78	74	87.12	-0.96	-4.38	-4.33	1.71

Microconcrete closed hemisphere - Mk I

Computer plot of principal stress factors in radial plane.



Q2CST Element

Stress factor = Number of lines per centimetre

Tension = continuous lines within elements

Compression = broken lines within elements

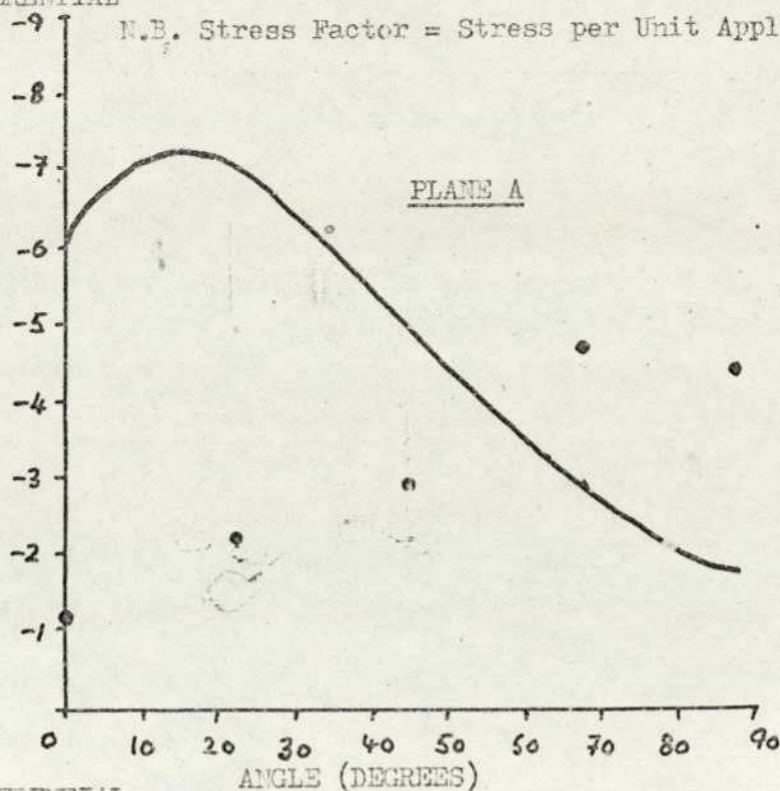
Fig. 19

CIRCUMFERENTIAL

STRESS

FACTOR

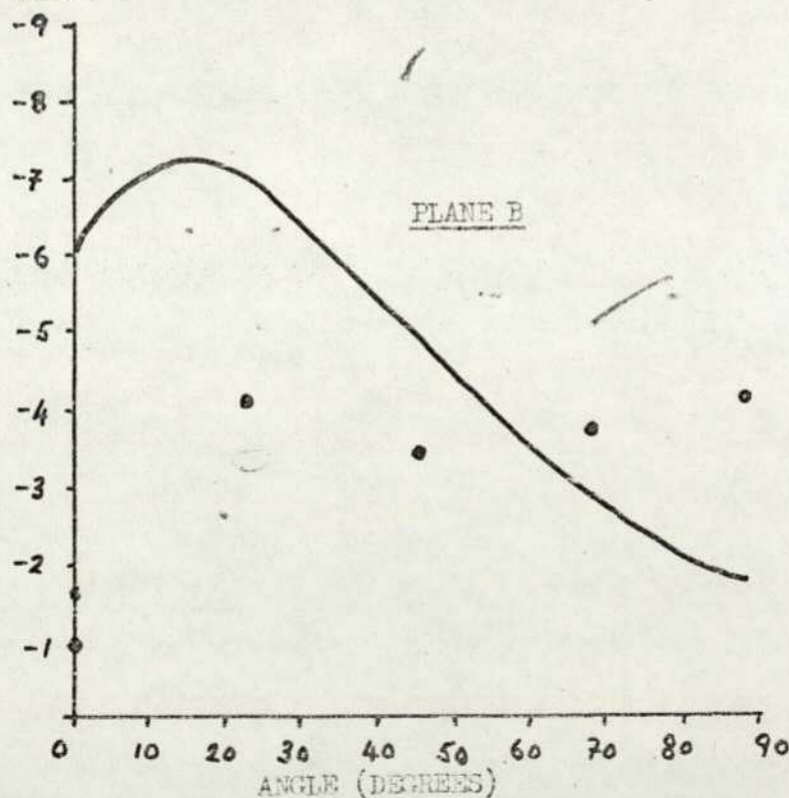
N.B. Stress Factor = Stress per Unit Applied Pressure



CIRCUMFERENTIAL

STRESS

FACTOR



MARK I HEMISPHERE (INTERNAL SURFACE)

CORRELATION BETWEEN THEORY AND EXPERIMENT

— THEORY

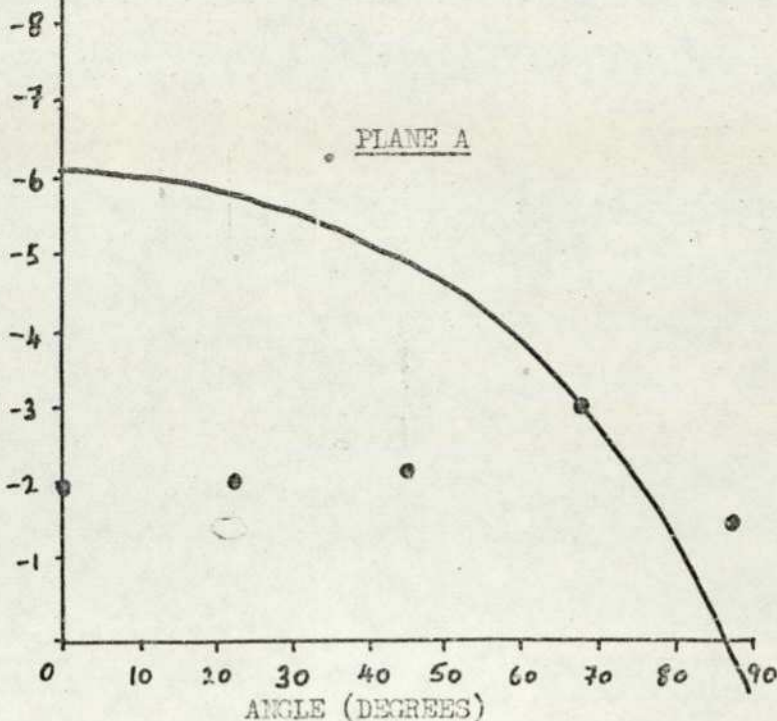
• EXPERIMENT

Fig. 20 (a)

CIRCUMFERENTIAL

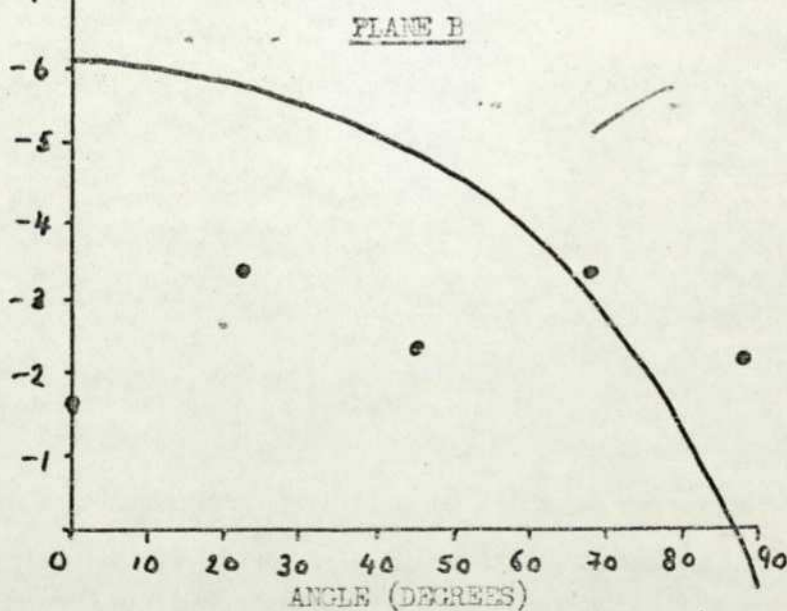
STRESS
FACTOR

N.B. Stress Factor = Stress per Unit Applied Pressure



CIRCUMFERENTIAL

STRESS
FACTOR



MARK I HEMISPHERE (EXTERNAL SURFACE)

CORRELATION BETWEEN THEORY AND EXPERIMENT

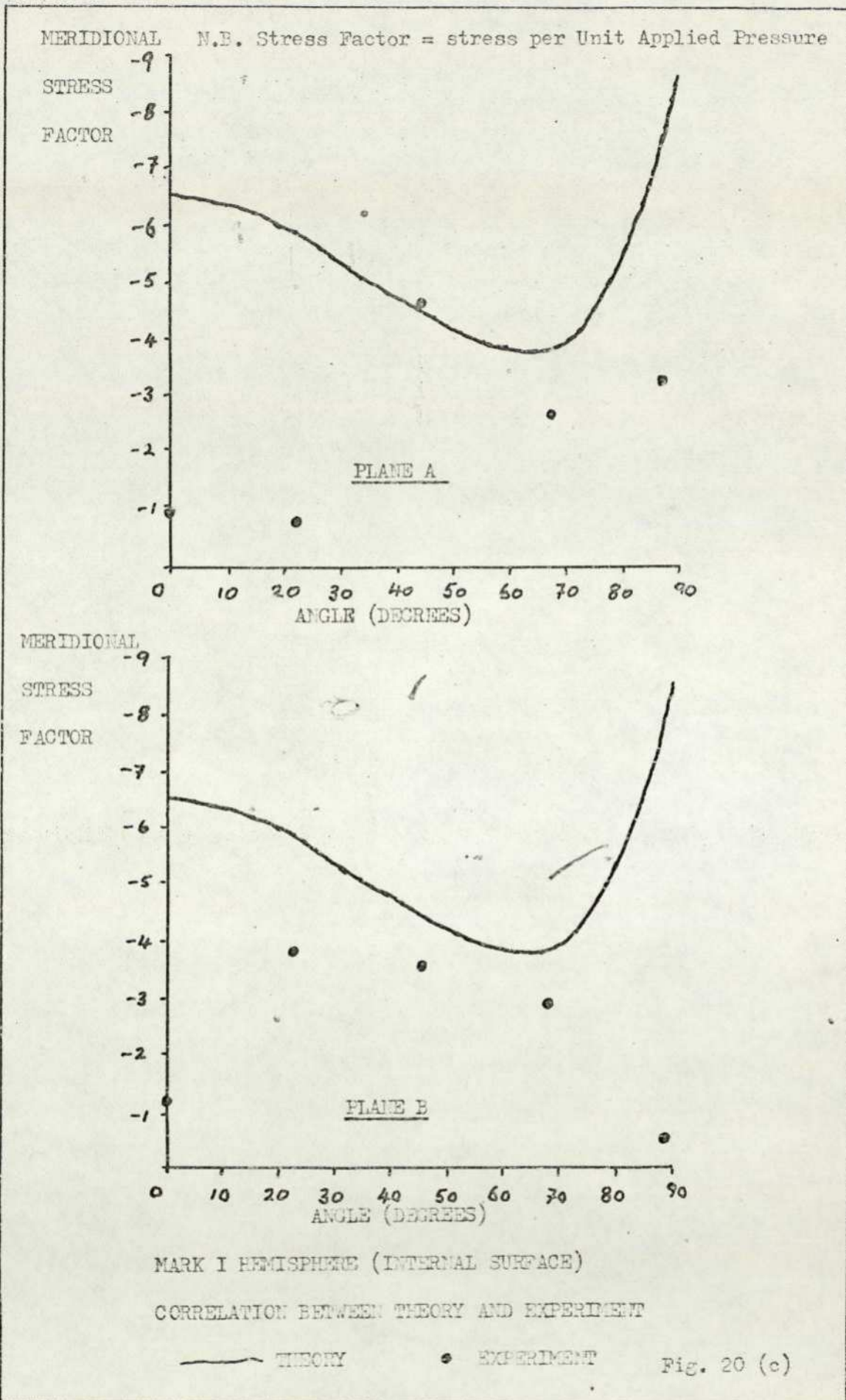
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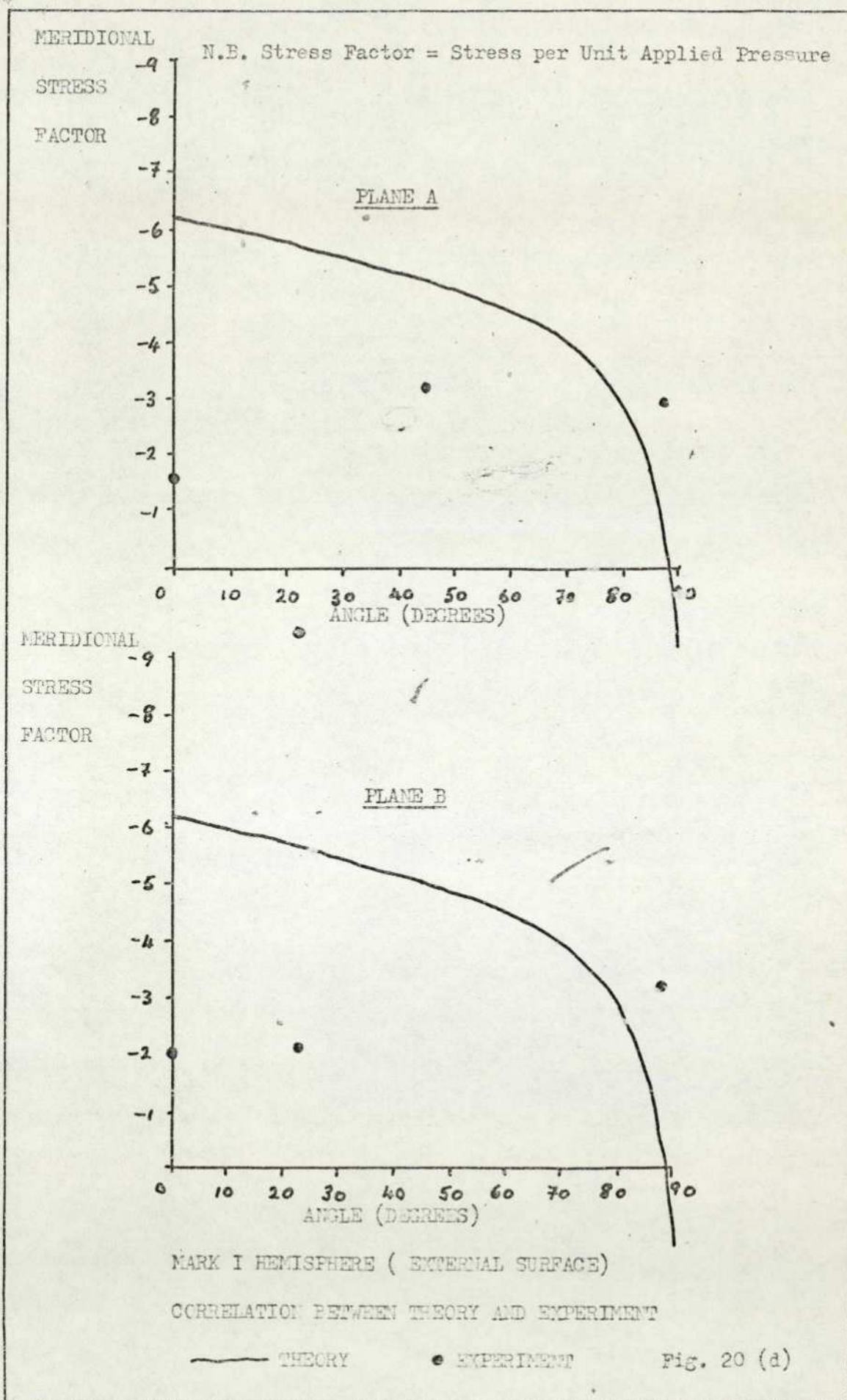
THEORY

•

EXPERIMENT

Fig. 20 (b)





74	78	79	75	87.01	-0.65	-6.27	-6.23	2.81
75	79	80	76	86.96	-0.32	-6.42	-6.37	3.05

Q2CST computer programme results

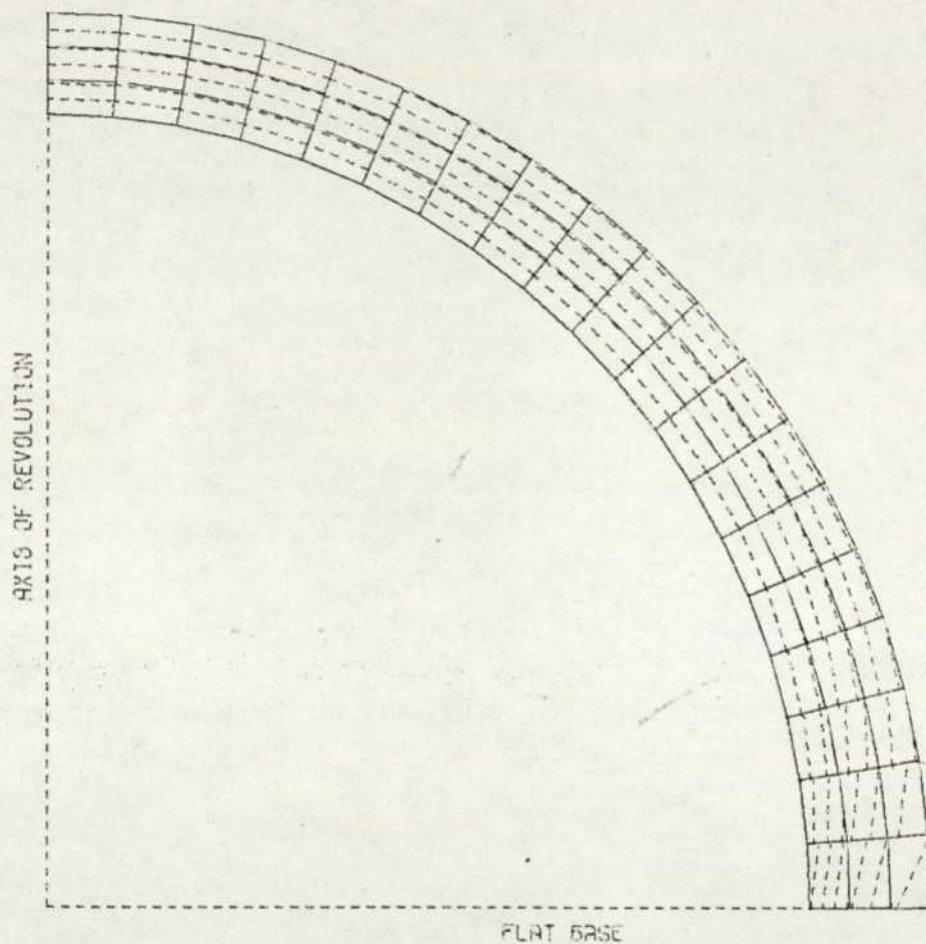
Hemisphere Mk II

Input data 11.65 11.65 1.6 0.9 1. .2 -1.

Element nodes				Principal planes (degrees)	Stress factors			
I	J	K	L		major	minor	hoop	shear
1	5	6	2	-27.03	-0.35	-1.99	-0.61	0.82
2	6	7	3	-12.98	-0.71	-4.08	-1.10	1.68
3	7	8	4	-7.85	-1.03	-6.43	-1.65	2.70
5	9	10	6	-8.28	-0.97	-2.69	-1.25	0.86
6	10	11	7	-6.79	-0.62	-3.98	-1.44	1.68
7	11	12	8	-0.71	-0.40	-5.32	-1.69	2.46
9	13	14	10	3.99	-0.97	-3.31	-1.87	1.17
10	14	15	11	1.91	-0.64	-3.95	-1.93	1.65
11	15	16	12	5.31	-0.29	-4.57	-2.01	2.14
13	17	18	14	12.32	-0.94	-3.80	-2.49	1.43
14	18	19	15	9.96	-0.61	-3.97	-2.47	1.68
15	19	20	16	11.38	-0.25	-4.13	-2.45	1.94
17	21	22	18	19.03	-0.90	-4.15	-3.06	1.62
18	22	23	19	17.14	-0.58	-4.05	-3.00	1.73
19	23	24	20	17.30	-0.21	-3.92	-2.93	1.86
21	25	26	22	24.91	-0.88	-4.41	-3.57	1.77
22	26	27	23	23.51	-0.55	-4.16	-3.49	1.80
23	27	28	24	22.91	-0.19	-3.88	-3.41	1.85
25	29	30	26	30.30	-0.86	-4.59	-4.01	1.87
26	30	31	27	29.22	-0.54	-4.29	-3.93	1.88
27	31	32	28	28.17	-0.18	-3.97	-3.86	1.89
29	33	34	30	35.38	-0.85	-4.73	-4.38	1.94
30	34	35	31	34.45	-0.53	-4.44	-4.32	1.96
31	35	36	32	33.13	-0.18	-4.13	-4.27	1.98
33	37	38	34	40.26	-0.84	-4.84	-4.69	2.00
34	38	39	35	39.35	-0.53	-4.61	-4.66	2.04
35	39	40	36	37.88	-0.18	-4.35	-4.64	2.09
37	41	42	38	45.00	-0.95	-4.95	-4.94	2.05
38	42	43	39	44.06	-0.53	-4.78	-4.94	2.12
39	43	44	40	42.54	-0.18	-4.60	-4.96	2.21
41	45	46	42	49.67	-0.85	-5.07	-5.16	2.11
42	46	47	43	48.68	-0.54	-4.97	-5.19	2.21
43	47	48	44	47.17	-0.19	-4.86	-5.24	2.34
45	49	50	46	54.30	-0.86	-5.20	-5.35	2.17
46	50	51	47	53.27	-0.55	-5.16	-5.40	2.30
47	51	52	48	51.85	-0.20	-5.13	-5.48	2.46
49	53	54	50	58.92	-0.86	-5.35	-5.52	2.24
50	54	55	51	57.90	-0.56	-5.36	-5.59	2.40
51	55	56	52	56.61	-0.21	-5.38	-5.69	2.59
53	57	58	54	63.58	-0.87	-5.51	-5.68	2.32
54	58	59	55	62.60	-0.56	-5.56	-5.76	2.50
55	59	60	56	61.47	-0.21	-5.62	-5.87	2.70
57	61	62	58	68.27	-0.87	-5.68	-5.82	2.41
58	62	63	59	67.40	-0.57	-5.75	-5.90	2.59
59	63	64	60	66.44	-0.22	-5.84	-6.02	2.81
61	65	66	62	73.01	-0.88	-5.85	-5.94	2.49
62	66	67	63	72.28	-0.58	-5.93	-6.03	2.67
63	67	68	64	71.51	-0.23	-6.03	-6.15	2.90
65	69	70	66	77.79	-0.88	-5.99	-6.04	2.55
66	70	71	67	77.25	-0.58	-6.07	-6.13	2.75
67	71	72	68	76.66	-0.23	-6.18	-6.26	2.98
69	73	74	70	82.59	-0.90	-6.09	-6.10	2.60
70	74	75	71	82.25	-0.60	-6.19	-6.20	2.79
71	75	76	72	81.83	-0.25	-6.30	-6.34	3.03
73	77	78	74	87.02	-0.94	-6.16	-6.12	2.61

Microconcrete closed hemisphere - Mk II

Computer plot of principal stress factors in radial plane.



Q2CST Element

Stress factor = Number of lines per centimetre

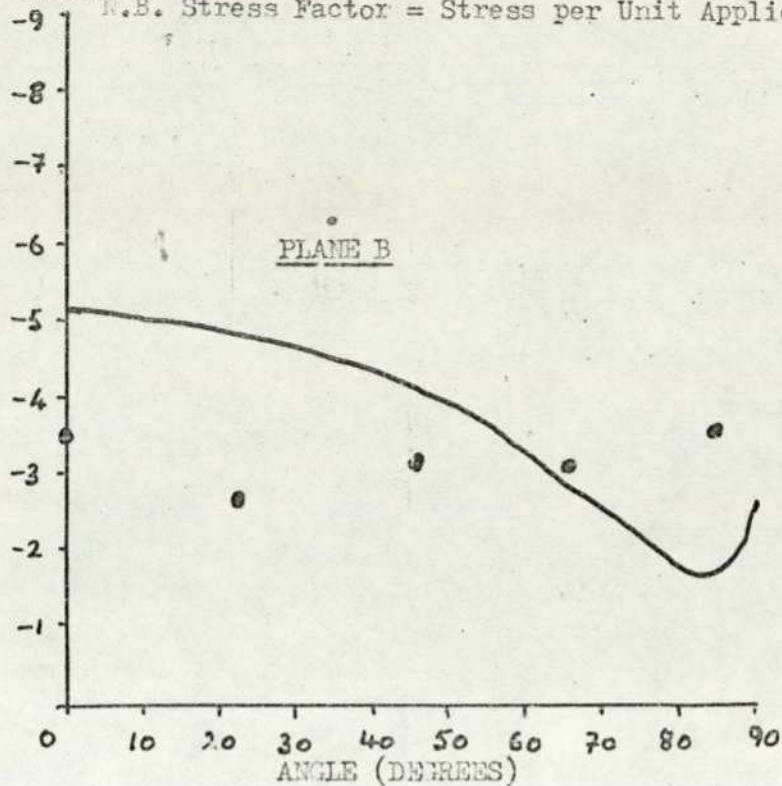
Tension = continuous lines within elements

Compression = broken lines within elements

Fig. 21

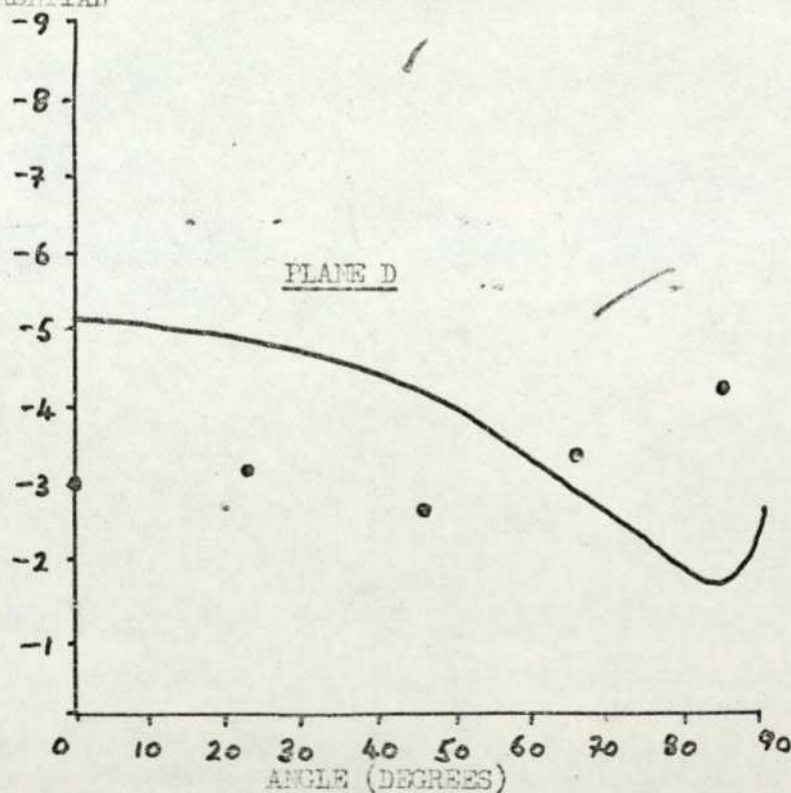
CIRCUMFERENTIAL N.B. Stress Factor = Stress per Unit Applied Pressure

STRESS
FACTOR



CIRCUMFERENTIAL

STRESS
FACTOR



MARK II HEMISPHERE (INTERNAL SURFACE)

CORRELATION BETWEEN THEORY AND EXPERIMENT

— THEORY • EXPERIMENT

Fig. 22 (a)

CIRCUMFERENTIAL

STRESS

FACTOR

-9

-8

-7

-6

-5

-4

-3

-2

-1

N.B. Stress Factor = Stress per Unit Applied Pressure

PLANE B

0

10

20

30

40

50

60

70

80

90

ANGLE (DEGREES)

CIRCUMFERENTIAL

STRESS

FACTOR

-9

-8

-7

-6

-5

-4

-3

-2

-1

PLANE D

0

10

20

30

40

50

60

70

80

90

ANGLE (DEGREES)

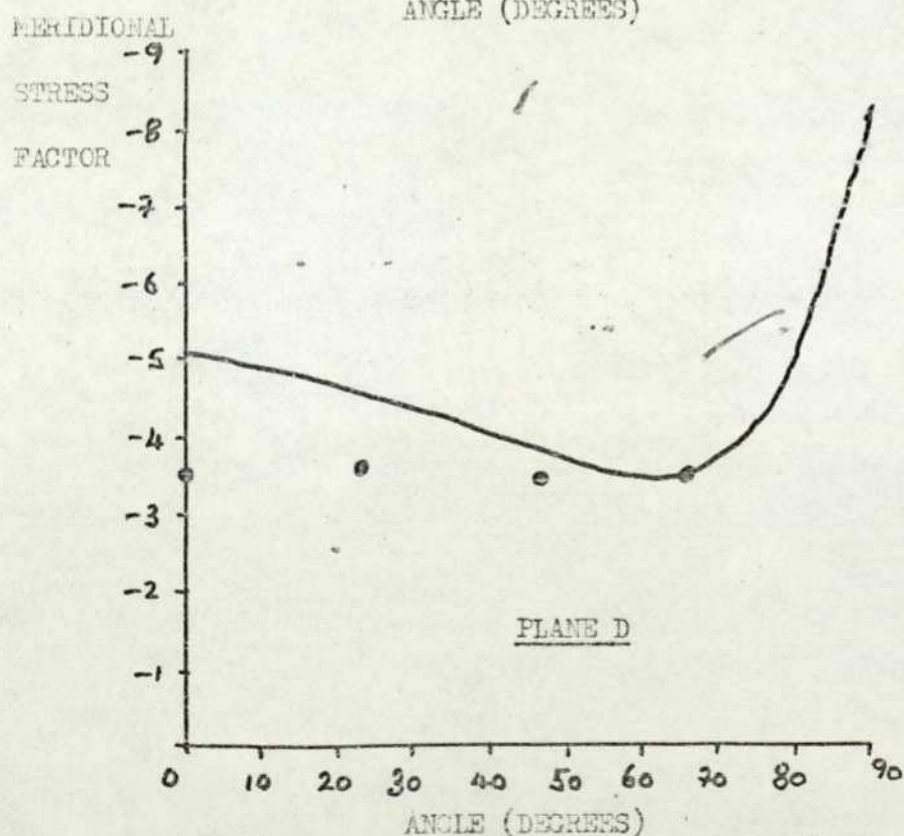
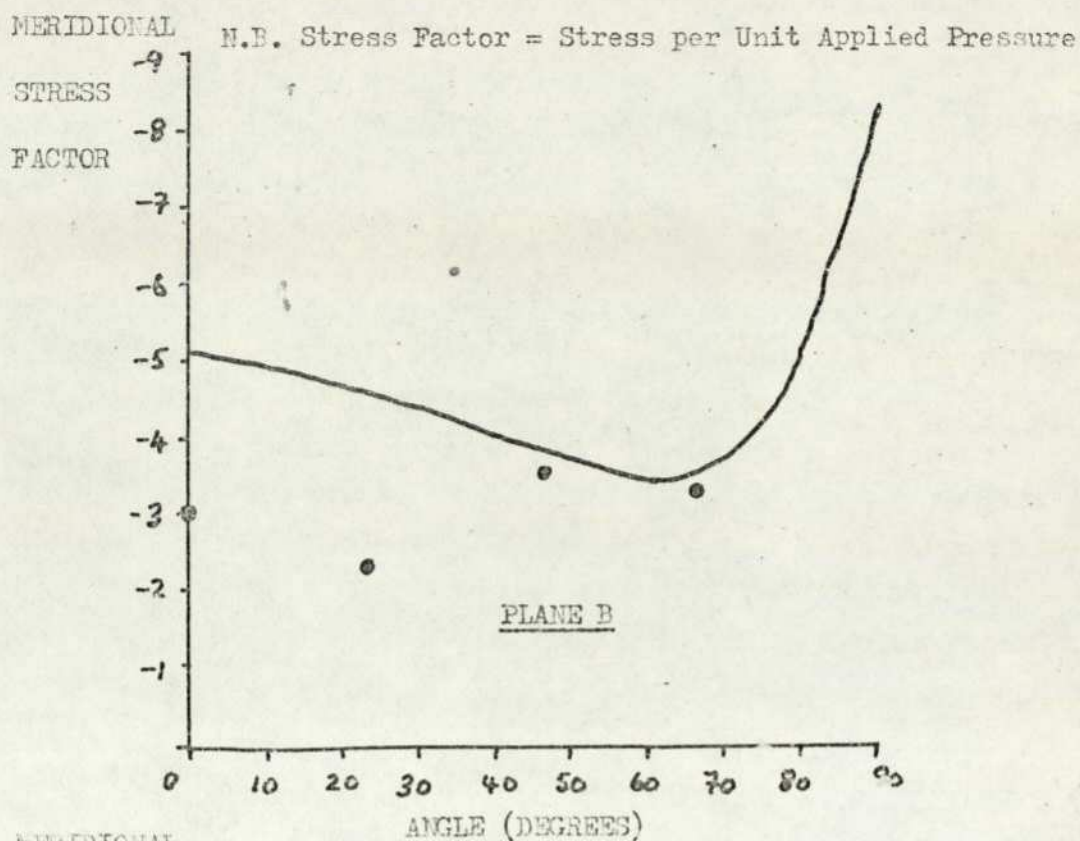
MARK II HEMISPHERE (EXTERNAL SURFACE)

CORRELATION BETWEEN THEORY AND EXPERIMENT

— THEORY

• EXPERIMENT

Fig. 22 (b)



MARK II HEMISPHERE (INTERNAL SURFACE)

CORRELATION BETWEEN THEORY AND EXPERIMENT

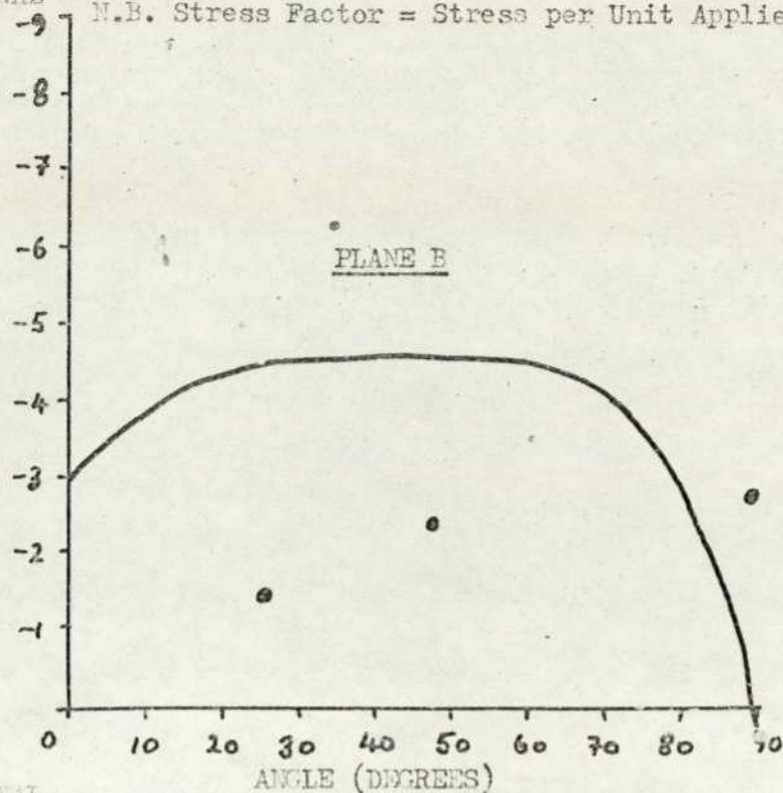
— THEORY

• EXPERIMENT

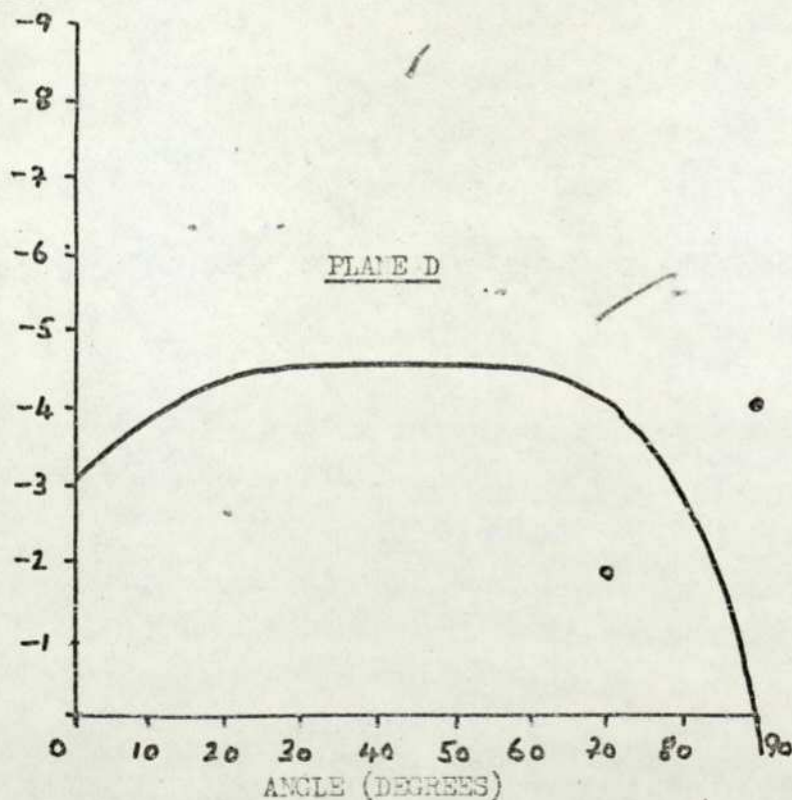
Fig. 22 (c)

MERIDIONAL
STRESS
FACTOR

N.B. Stress Factor = Stress per Unit Applied Pressure



MERIDIONAL
STRESS
FACTOR



MARK II HEMISPHERE (EXTERNAL SURFACE)

CORRELATION BETWEEN THEORY AND EXPERIMENT

— THEORY • EXPERIMENT

Fig. 22 (d)

74	78	79	75	84.11	-0.44	-1.71	-1.70	0.63
75	79	80	76	87.29	-0.13	-1.41	-1.33	0.64

Q2CST computer programme results

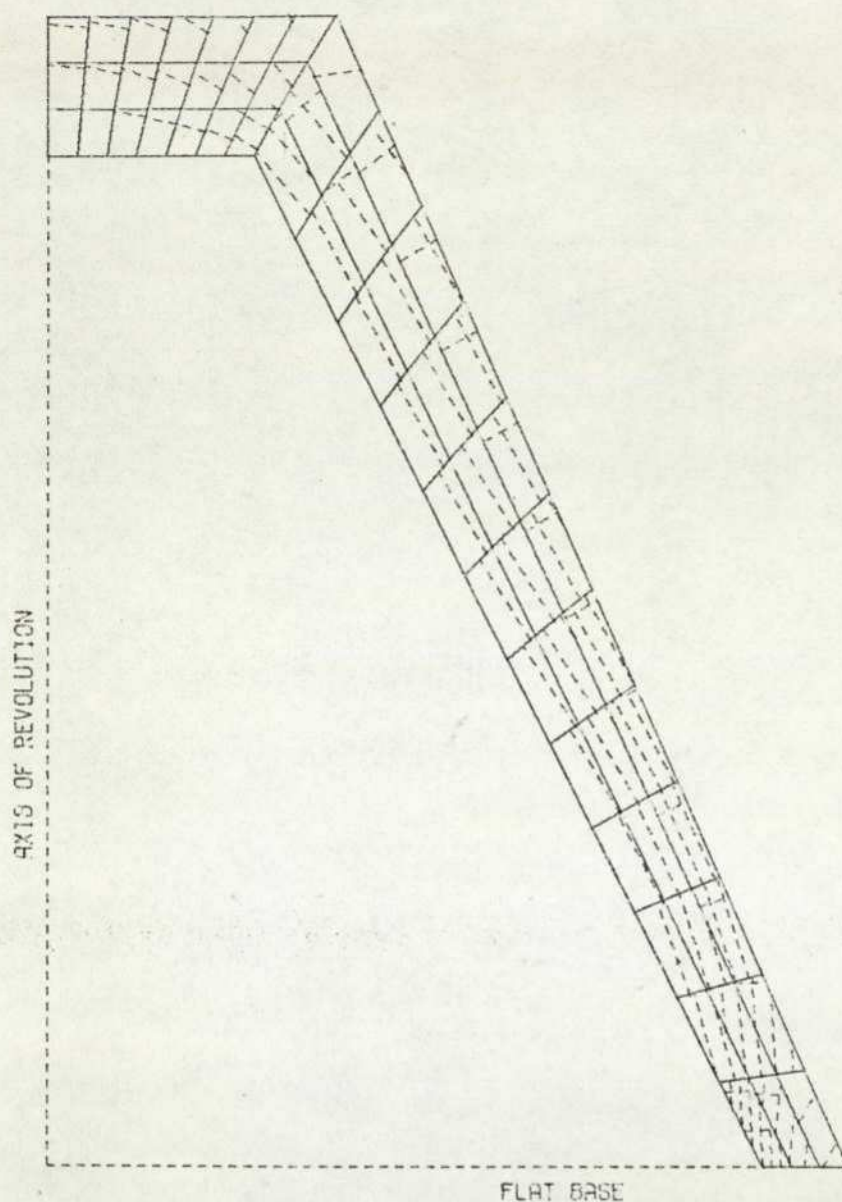
Frustum of a Cone

Input data 14.9 10.5 3.8 1.1 2.0 1.8 1. .2 -1.

Element nodes				Principal plane (degrees)	Stress factors			
I	J	K	L		major	minor	hoop	shear
1	5	6	2	-28.88	-0.26	-1.82	-0.82	0.78
2	6	7	3	-5.61	-1.10	-4.74	-1.57	1.82
3	7	8	4	3.52	-1.94	-8.35	-2.46	3.21
5	9	10	6	3.27	-0.82	-4.23	-2.62	1.71
6	10	11	7	5.91	-0.47	-4.96	-2.70	2.25
7	11	12	8	15.68	-0.03	-4.75	-2.58	2.36
9	13	14	10	18.11	-0.90	-5.55	-4.45	2.32
10	14	15	11	17.79	-0.22	-4.43	-4.13	2.10
11	15	16	12	19.68	0.09	-3.25	-3.87	1.67
13	17	18	14	24.54	-0.91	-5.67	-5.68	2.38
14	18	19	15	25.40	-0.54	-4.15	-5.42	1.81
15	19	20	16	27.36	-0.20	-2.67	-5.18	1.24
17	21	22	18	28.10	-0.76	-5.19	-6.15	2.22
18	22	23	19	30.65	-0.60	-3.88	-6.13	1.64
19	23	24	20	31.14	-0.24	-2.48	-6.06	1.12
21	25	26	22	29.36	-0.80	-4.39	-6.03	1.79
22	26	27	23	32.74	-0.47	-3.51	-6.22	1.52
23	27	28	24	33.58	-0.12	-2.52	-6.41	1.20
25	29	30	26	29.75	-0.85	-3.53	-5.49	1.34
26	30	31	27	33.06	-0.53	-3.14	-5.90	1.30
27	31	32	28	32.43	-0.19	-2.61	-6.34	1.21
29	33	34	30	29.11	-0.88	-2.81	-4.71	0.96
30	34	35	31	31.34	-0.58	-2.79	-5.28	1.10
31	35	36	32	30.63	-0.23	-2.66	-5.92	1.21
33	37	38	34	28.25	-0.89	-2.26	-3.84	0.69
34	38	39	35	30.37	-0.61	-2.48	-4.50	0.94
35	39	40	36	29.14	-0.25	-2.62	-5.27	1.19
37	41	42	38	30.15	-0.91	-1.83	-2.95	0.46
38	42	43	39	30.19	-0.61	-2.22	-3.62	0.81
39	43	44	40	28.44	-0.25	-2.53	-4.46	1.14
41	45	46	42	37.19	-0.98	-1.40	-2.09	0.21
42	46	47	43	35.32	-0.77	-2.08	-2.75	0.65
43	47	48	44	28.95	-0.23	-2.52	-3.57	1.15
45	49	50	46	8.02	-0.94	-1.06	-1.41	0.06
46	50	51	47	40.36	-0.94	-1.75	-1.94	0.40
47	51	52	48	40.96	-0.73	-2.92	-2.73	1.09
49	53	54	50	-1.68	-0.72	-1.05	-1.17	0.17
50	54	55	51	40.05	-0.88	-1.53	-1.58	0.33
51	55	56	52	57.96	-0.81	-3.38	-2.33	1.29
53	57	58	54	17.58	-0.69	-1.12	-1.29	0.21
54	58	59	55	49.77	-0.71	-1.98	-1.72	0.63
55	59	60	56	68.27	-0.28	-2.67	-2.00	1.20
57	61	62	58	40.07	-0.71	-1.29	-1.45	0.29
58	62	63	59	59.21	-0.47	-1.99	-1.72	0.76
59	63	64	60	73.88	-0.15	-2.24	-1.80	1.04
61	65	66	62	57.97	-0.73	-1.48	-1.62	0.37
62	66	67	63	65.38	-0.38	-1.92	-1.71	0.77
63	67	68	64	77.58	-0.11	-1.92	-1.65	0.91
65	69	70	66	70.67	-0.78	-1.67	-1.77	0.44
66	70	71	67	70.94	-0.39	-1.83	-1.70	0.72
67	71	72	68	80.79	-0.10	-1.69	-1.54	0.79
69	73	74	70	79.72	-0.83	-1.84	-1.88	0.50
70	74	75	71	76.99	-0.43	-1.75	-1.70	0.66
71	75	76	72	84.25	-0.11	-1.52	-1.45	0.71
73	77	78	74	85.41	-0.89	-1.96	-1.90	0.54

Microconcrete closed frustum of a cone.

Computer plot of principal stress factors in radial plane.



Q2CST Element

Stress factor = Number of lines per centimetre

Tension = continuous lines within elements

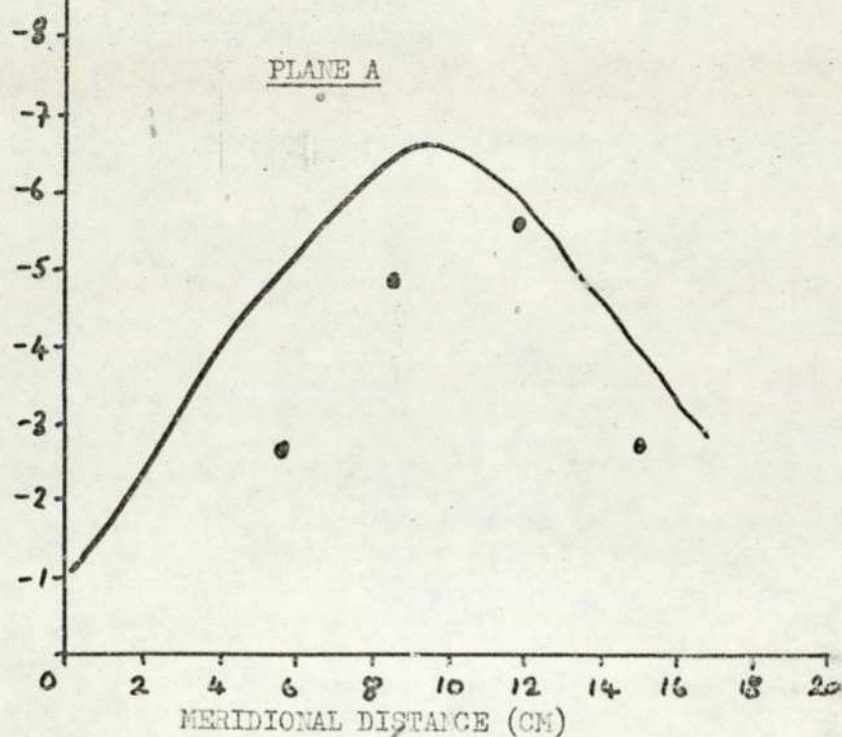
Compression = broken lines within elements

Fig. 23

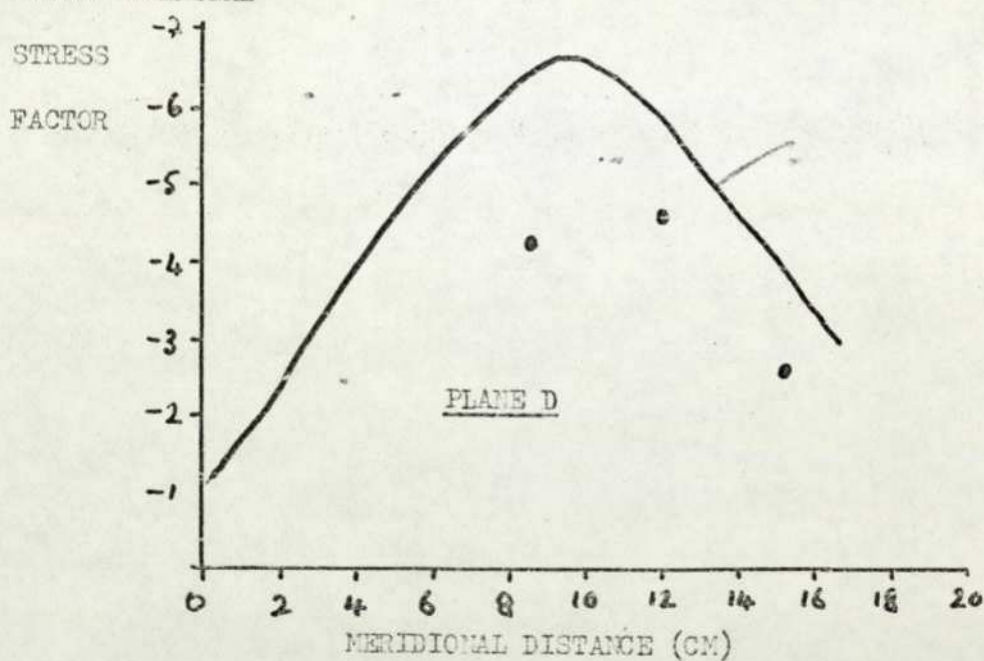
CIRCUMFERENTIAL

STRESS
FACTOR

N.B. Stress Factor = Stress per Unit Applied Pressure



CIRCUMFERENTIAL



FRUSTUM OF A CONE (INTERNAL SURFACE)

CORRELATION BETWEEN THEORY AND EXPERIMENT

— THEORY • EXPERIMENT

Fig. 24 (a)

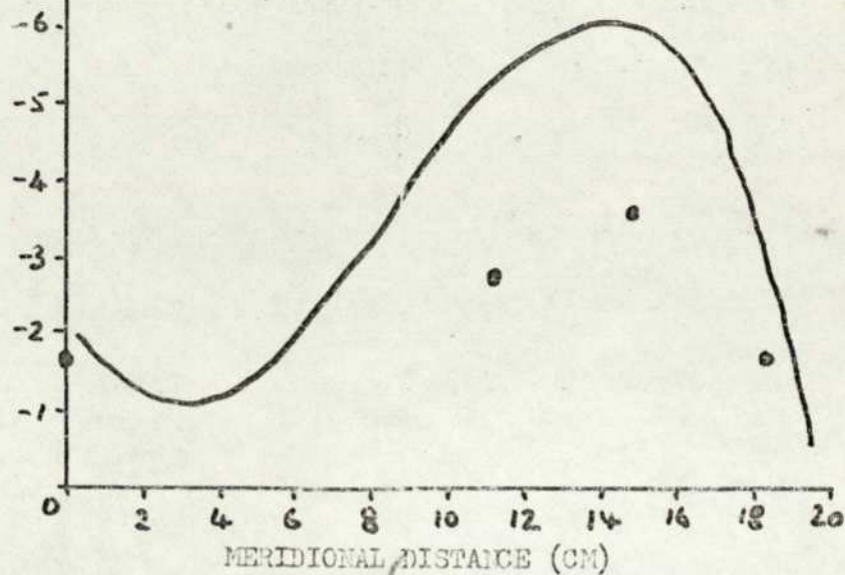
CIRCUMFERENTIAL

STRESS

FACTOR

N.B. Stress Factor = Stress per Unit Applied Pressure

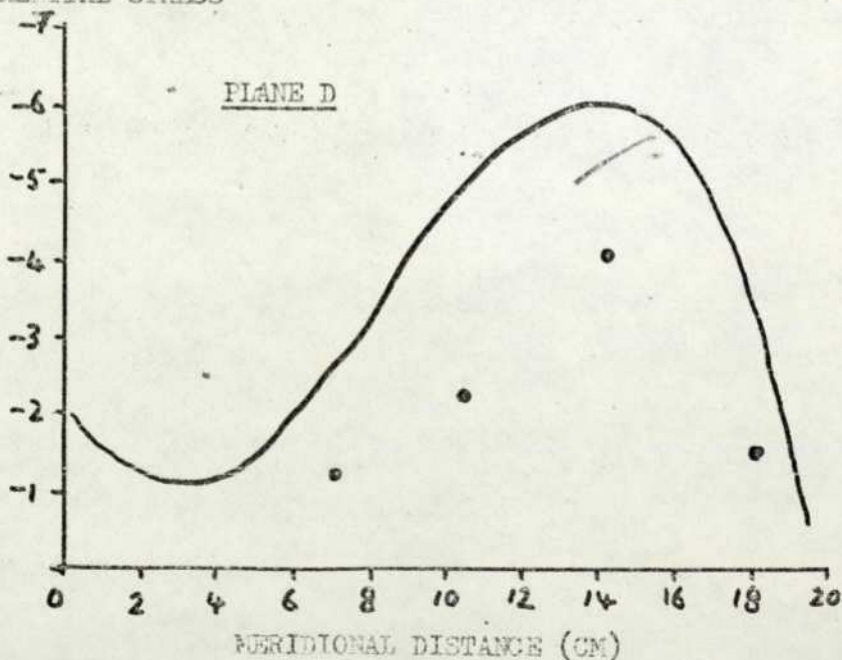
PLANE A



CIRCUMFERENTIAL STRESS

FACTOR

PLANE D



FRUSTUM OF A CONE (EXTERNAL DISTANCE)

CORRELATION BETWEEN THEORY AND EXPERIMENT

— THEORY

• EXPERIMENT

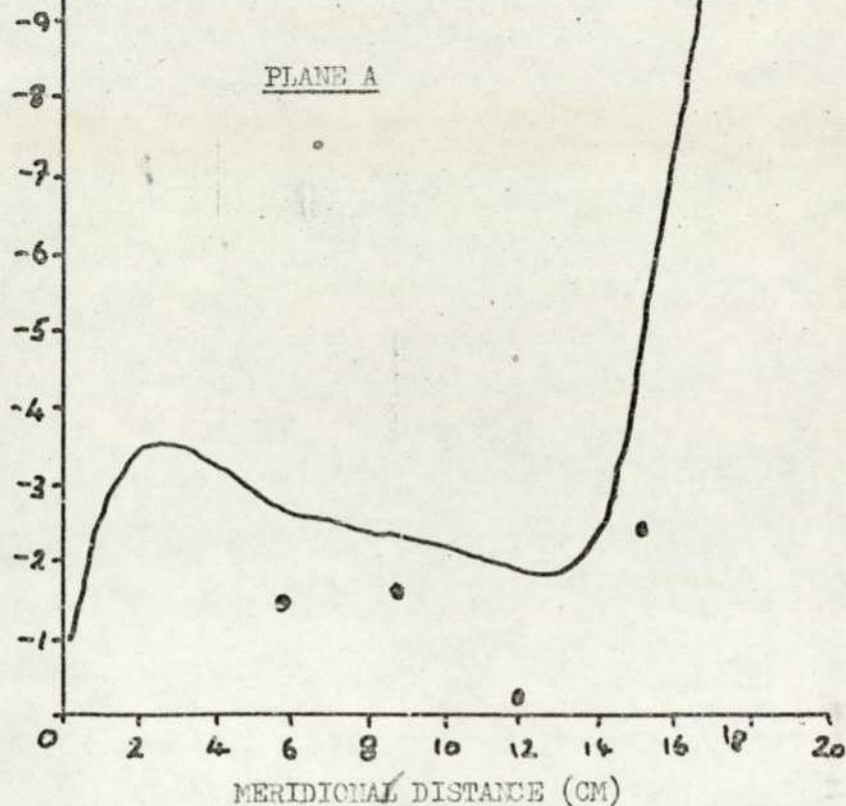
Fig. 24 (b)

MERIDIONAL

STRESS

FACTOR

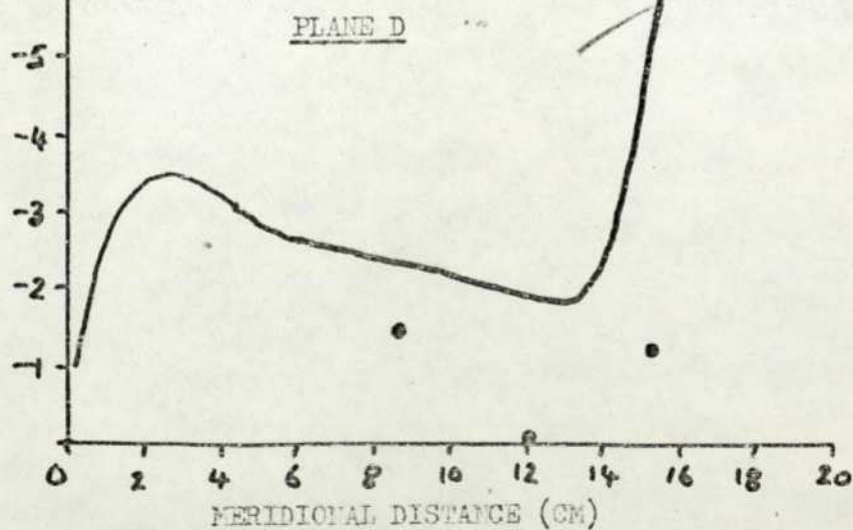
N.B. Stress Factor = Stress per Unit Applied Pressure



MERIDIONAL

STRESS

FACTOR



FRUSTUM OF A CONE (INTERNAL SURFACE)

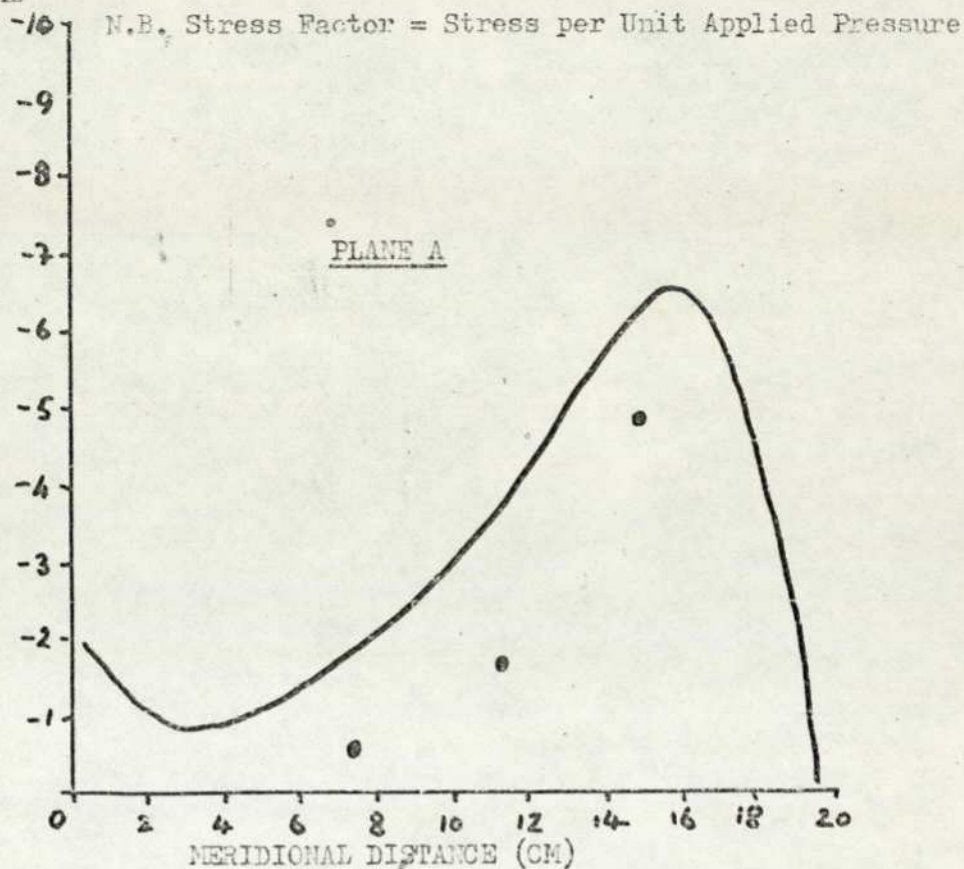
CORRELATION BETWEEN THEORY AND EXPERIMENT

— THEORY

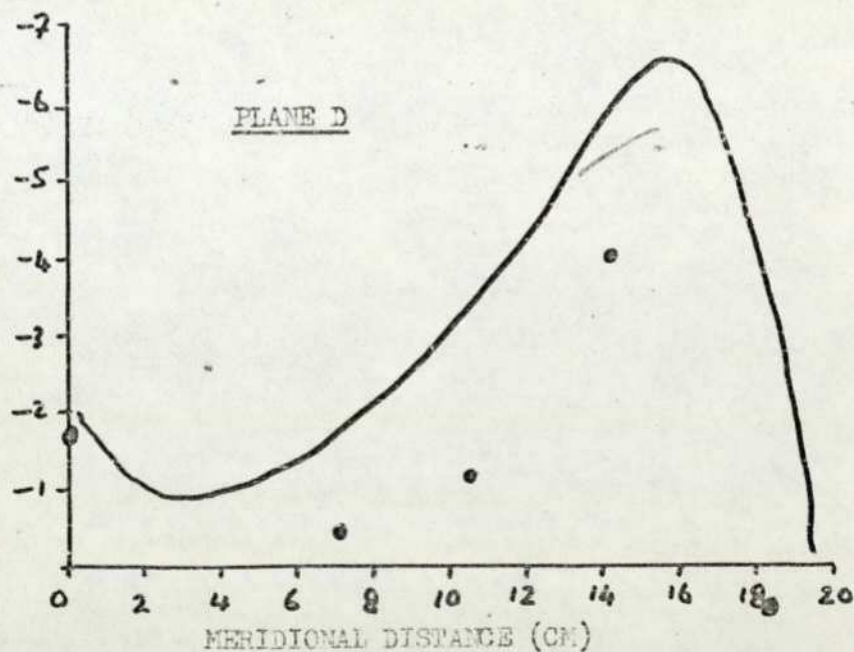
• EXPERIMENT

FIG. 24 (c)

MERIDIONAL
STRESS
FACTOR



MERIDIONAL
STRESS
FACTOR



FRUSTUM OF A CONE (EXTERNAL SURFACE)

CORRELATION BETWEEN THEORY AND EXPERIMENT

— THEORY

• EXPERIMENT

Fig. 24 (d)

Conclusions

The best results for correlation between the Finite Element analysis and the uniform pressure test are found to be for the linear elastic perspex model Mark II. It was the most uniform in construction, having been made out of single thicknesses of manufactured plate and tubing. The crazing which occurred in part of the Mark I perspex model appears to have caused some local stress redistribution. This type of locked-in initial stresses are also probably responsible to some extent for the poorer correlation of results obtained for the microconcrete models.

There are locked-in stresses which can be produced both during the casting of the shell component, and during its embedding in the flat base of the model. The values and distribution of initial stresses will vary with both geometric types, thicknesses, scale factors, casting procedures, and ambient temperatures. None of these stresses are measured in the experiments, because the surface strain gauges can only be attached to quantify relative movement under applied loading subsequent to completion of the model construction.

If the material is continuous with a constant value of Young's Modulus and Poisson's Ratio, as with the perspex models, the locked-in stresses can have less effect upon strain behaviour than for a microconcrete model in which linear elasticity is an approximation only over a small range of stress. It is quite possible in principle for locked-in casting stresses in a microconcrete shell and the subsequent stresses due to applied loading to cancel each other out to some extent in some regions of the continuum while being additive in other regions.

This is only one possible source of disagreement between theory and experiment. The work of Harvey Haynes of the U.S. Navy (referred to in the introduction to this thesis) has indicated that concrete shells of revolution are also unstable in that they do not necessarily have symmetrical radial deflection about the axis of revolution under uniform applied pressure. Long thin cylindrical geometry is particularly prone to this type of behaviour, as he has demonstrated. He has also shown experimentally that for spherical geometry minor surface defects can substantially reduce the value of applied uniform pressure required to cause the destruction of a shell.

The problem we have encountered in this thesis, where a low level of correlation is found between theory and experiment for the microconcrete hemispherical shells, may, in addition to the possibility of locked-in stresses, also be due to the fact that the strain gauges are placed on the shell surfaces where the relative movements they measure are most sensitive to local aberrations.

From the analytical point of view also, the Finite Element method cannot cope economically and accurately with high stress gradients and these are most likely to occur at the shell surface. On the other hand, while Haynes was able to apply the results of ready-worked thick shell theories to the standard spherical case, the general theory of thick shells of revolution results in the solution of complex differential equations.

In his experimental work, Haynes has succeeded in identifying two modes of thick shell failure in his spherical microconcrete models, thinner shells which sheared across the shell thickness with rapid implosion of the shell, and thicker shells which split up slowly by stages internally within the plane of the shell in a laminar fashion. He found the borderline case occurred when the shell thickness was about one sixteenth of its outer diameter.

If this is taken as a way of defining a thin or thick concrete shell in practice, then the mode of failure of the hemisphere as shown in the photographs at the end of this thesis was that of a thin shell.

Turning to the practical application of this research, it has first of all some resemblance to mass concrete arch dams which have been designed by shell principles, but under triangular pressure loading. Beyond this there are reinforced concrete shells in offshore structures, some of which have been designed partly utilising the results of Haynes' research on cylinders with hemispherical end closures. The knowledge established from all sources has been sufficient to design with confidence to a depth of about 500 metres. The exact edge of knowledge in this field is not clearly defined.

This research has taken the matter further by giving a detailed programming of the finite element method to move from the specific types to the general shell of revolution, so that a shell can be designed for buoyancy, floatation stability, and practicality of construction, for use at greater ocean depths and on a larger scale as will be required for the development of deep ocean resources.

The experimental result show the effects of a change of material upon the degree of confidence which can be placed in numerical results obtained from Finite Element computer programming in this application.

The direction of future research which is indicated would be the numerical solution by computer of differential equations of the general thick shell of revolution. The results obtained need to be matched by an extensive program of model testing with variation of material type and geometry, and accurate means of measurement throughout the shell thickness.