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The local and global causal geometries of
Minkowski Space

by

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A thesis submitted for the degree of Doctor of
Philosophy in Mathematics.

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London.

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Table of Contents

	<u>Page No.</u>
Acknowledgements	3
Declaration	4
Abstract	5
Dedication	6
Introduction	7
<u>Part I.</u> <u>Foundations.</u>	
Chapter 1. The historical and philosophical background	11
Chapter 2. The Robb axiom system for Minkowski space	24
Chapter 3. The standard affine transformation groups	35
<u>Part II.</u> <u>Causal issues.</u>	
Chapter 4. The globally causal geometry	50
Chapter 5. The locally causal geometry	74
<u>Part III.</u> <u>Kinematics, inertial frames and the</u> <u>principle of relativity.</u>	
Chapter 6. Aspects of Space-Time kinematics	88
Chapter 7. Inertial frames	109
Concluding remarks	135

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Declaration

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Abstract

One of the most interesting aspects of the Special Theory of Relativity is its intimate connection with geometry by way of Minkowski's four dimensional *space-time world*. In this thesis we examine that connection from various points of view. On the one hand we consider *Minkowski space* as an axiomatic system built on certain postulates in the style of Peano and Hilbert, an avenue of development which began with Robb prior to 1914 and continues still with the work of Schutz in 1973 and 1979. On the other hand certain groups and group geometries are given prominence; this avenue of development began with Lorentz and Larmor before 1905, continued with Bateman and Cunningham in 1909 and received a boost recently with the work of Zeeman in 1964. The unifying theme of this part of our development concerns the way of making manifest, in the various group geometries associated with Minkowski space, key matters of physical experience and intuition, collectively known as the *principle of causality*. The procedures by which causal issues are treated in the Minkowski space model are subjected to critical review and the thesis contains original and/or simpler proofs of the theorems associated with the local and global causal groups. In addition, we present a systematic mathematical development of a procedure for dealing with the class of uniformly accelerated frames within the theory of Special Relativity, without the necessity of dealing with the curved space-time of General Relativity. The problems of physical interpretation that arise regarding the preservation of local and global causality and the restrictions on the use of the allowable transformations are also discussed and critically evaluated.

Dedication

This thesis is dedicated to [REDACTED]
[REDACTED]

Introduction

We shall be concerned with Minkowski space as a mathematical entity in its own right, and, separately, its use as a mathematical foundation for the physics of Special Relativity. Indeed, since the suitability of Minkowski space for that modelling rôle is well appreciated, it is clearly appropriate that its character as a mathematical structure should be examined.

In Part I, Chapter 1, we give brief historical accounts of the growth of Special Relativity and of the development of axiom systems for Minkowski space. There is also a discussion which gives some flavour of the interplay between geometry and physics, and a section devoted to causality. Lastly, there is an explanation that we shall be concerned with Minkowski space as a mathematical structure first, and with physics, second.

In Chapter 2, we look at the axiom of system of Robb (1914) in some detail ; here, Minkowski space, M_4 , is regarded as a four dimensional absolute geometry with the Euclidean property for parallels. The whole edifice is built on a single strict partial order relation which Robb refers to as *after* , this relation generates all the geometric entities, line, plane and hyperplane, as well as a "null cone" which separates each class of geometrical entity into three varieties.

Chapter 3 is concerned with notation and nomenclature, and the definitions and elementary properties of the standard affine groups associated with M_4 , viz. the *Lorentz* and *Poincaré* groups.

In Part II we examine certain bijective maps of parts, or of the whole, of M_4 which preserve aspects of its mathematical structure, aspects which reflect those physical concepts known as (local and global) "causality". Clearly these maps form groups, and the groups generate geometries ; Theorems A and B of Part II respectively *identify* the global and local causality groups with standard groups, both affine and non-affine, associated with M_4 . Theorem A, which identifies the global causality group with the "augmented" orthochronous Poincaré group, $P^{\uparrow*}$, is due in its explicit form to Zeeman (1964). However, our proof is new, and it invokes properties of the conformal group associated with M_4 ; indeed, this proof is designed to point the distinction between the affine subgroup $P^{\uparrow*}$ of the full conformal group and the group itself, regarding global causality preservation. Theorem B makes explicit ideas that were implicit in Wess (1960), and in Flato and Sternheimer (1966), concerning the local causality preservation properties of the *special conformal group*.

In Part III we make a more detailed examination of the properties of the special conformal group, S_4 . At this stage we are equipped with an abstract mathematical structure M_4 and a transformation group S_4 , a pair which characterizes the "locally causal geometry" for M_4 . We look at two classes of subsets of M_4 ; the homogeneous hyperplanes, and the homogeneous hyperbolic hypersurfaces, and the most important new result which emerges is Theorem C of Part III. This is a purely geometric (or group theoretic) theorem, in which the group generated by S_4 and the augmented orthochronous Lorentz group, $L^{\uparrow*}$, appears as the local invariance group for regions of those hyperplanes and hypersurfaces

mentioned above.

We now interpret Theorem C in terms of the kinematics associated with one-dimensional motion ; in particular, we discover the existence of a member of the collection of maps mentioned in Theorem C which associates (locally) a straight line through the origin with a given hyperbolic arc through the origin. Such a map is suggestive of a procedure for transforming from the coordinate system of a "non-accelerated observer" to that of a "uniformly accelerated observer". However, before that suggestion is fully pursued, we find a rather surprising fact, the map that would operate this procedure is not "physically" limited by the presence of the "singular cones" (where the transformations break down).

In Chapter 7, we proceed in a manner which may be described as "physically naive". We do not yet have the concept of "inertial frame", and we deliberately delay its introduction. That delay is well worth while since we may clearly establish the need for a class of privileged Minkowski coordinate systems even at this level of "naive physics". The privileged systems are then referred to as *inertial*. Finally, we make a full examination of the procedure for mapping from an inertial frame to a *uniformly accelerated frame* ; this is Theorem D, the third new result of the thesis. That completes Part III of the thesis.

The work ends with a brief collection of concluding remarks, together with some suggestions for further investigation.

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PART I. FOUNDATIONS

Chapter 1. The historical and philosophical background.

In common with many mathematical disciplines, the kinematics and dynamics associated with the motion of massive and massless particles since 1880 has developed in two ways. On the one hand, the equations of J. Clerk Maxwell (1873), which so profoundly categorize the behaviour of electromagnetic fields and massless particles were found not to match the equations of Newtonian mechanics regarding the motion of massive particles. This mismatch is immediately noticeable in the sense that, while the maximal linear group under which Maxwell's equations are invariant is the Poincaré group, that for the invariance of the Newtonian mechanical equations is the Galilean group. Clearly there would have to be some means of reconciling the two formulations, or one formulation would have to be rejected. This theoretical problem, thrown up by the mathematical physicists, was then echoed by the experimentalists in their search for evidence of absolutes in the fabric of the universe ; it led to Einstein's theory of Special Relativity (Einstein, 1905) and to Minkowski's formulation of kinematics in terms of a four dimensional space-time "world" (Minkowski, 1908 ; 1952). Quite apart from Einstein's theory of gravitation (Einstein, 1916), which would further widen the class of transformations which were to be allowable in the description of Physics, the Special Theory led to a plethora of results in mechanics, optics and electrodynamics . In addition, sustained efforts to absorb the theories of gravitation and of electromagnetic phenomena into a single "unified field theory" were undertaken.

There is another trend which balances that toward increasing theoretical complexity ; it is that toward economical axiomatization. On the face of it, this would seem to have started with the work of C. Carathéodory (1924) and that of H. Reichenbach (1924), one of Einstein's students in Berlin at the time of his first course of lectures on Relativity. Surprisingly, it was a British mathematician A. A. Robb (1913, 1914, 1921) who had a prior claim. An analysis of Robb's intellectual life around 1900 reveals an interesting fusion of the ideas of the axiomatists G. Peano (1889, 1894) and D. Hilbert (1899, 1902) with those of the Cambridge school of mathematical physicists, notably Larmor (1900). We examine both the historical development and the mathematical content of Robb's contribution in Chapter 2.

It is clear that Carathéodory and Reichenbach are little influenced by the first detailed expression of Robb's theory (1914), and, indeed, the continental school was inclined to dismiss it as being of little account.

Thus, Herman Weyl (1923 : 8) :

"For that reason, we can, if we are working with the entire infinite universe, construct the geometry of Minkowski [space], completely on the single concept of the null element ; the straight line becomes superfluous. This was developed a few years ago by the English (sic) mathematician Robb in a system of axioms which is admittedly very complicated and artificial ..."

Yet Robb's work was favourably acknowledged by Eddington (1924), Carnap (1958), Whitrow (1961) and Synge (1965), and begins to re-

assert its influence with the appearance of the paper "Causality implies the Lorentz group" of E. C. Zeeman (1964). Here, Minkowski space is treated as an essentially geometric entity, but without the confusion brought about by the terms "pseudo-Euclideanism" and the "indefinite metric" ; the treatment is carried to its logical end in Nanda (1976).

Further considerations of the axiomatics of Minkowski space had flared briefly with McVittie (1942) and Walker (1948, 1959) following Milne and Whitrow (1938) and Reid (1938). Recently, in a collection of papers under the editorship of P. Suppes (1973) several authors put forward axiom systems of their own, notably Domotor (Suppes, 1973 : 3 - 55) and Latzer (Suppes, 1973 : 321 - 365). Each pays due deference to Robb (1914) and each indicates a newly awakened interest in Minkowski space geometry as the result of Zeeman's paper (Zeeman, 1964). In addition, an Australian school, initiated by Szkeres (1968) has led to the development of the systems of Schutz (1973, 1979) ; these have a flavour of Hilbert's (1899) system as opposed to Robb's (1914), which is clearly fashioned in the Peano (1889) mould.

It is appropriate to give Robb's axiom system pride of place, in spite of the degree of artificiality associated with it. The system relies on the single primitive relation *after*, and is the archetype of every system in which there is an *a priori* acceptance of the primacy of "an observer's time order". Once that primacy is accepted we have a clearer appreciation of the subjective nature of "causality".

1.2 Geometry and Physics.

Greek geometry was clearly not initially an abstract invention of the human intellect; it must have grown out of, and developed along with, primitive physics. Indeed, human endeavour has always progressed by that procedure we know as "trial and error"; it is likely that it always will. In ancient times, as now, there existed a breed of man more inquisitive than his fellows ; in modern terms we would say that these primitive scientists "observed phenomena", "designed experiments" to verify those observations, and used that body of accumulated knowledge to make predictions. The very word "observation" indicates the importance of primitive optical science, and geometry in the ancient civilizations necessarily developed as a distillation from observed natural phenomena. No doubt experience soon taught the ancients the geodesic value of the "straight line" on simple conservationist and economical grounds. But how would one judge collinearity ? By optical alignment, as we do, for example, in surveying, today. Thus, there are no *hyperbolic* grids for irrigation systems, roads are straight, buildings often of rectangular plan. Is that only because straightness was a vogue of early civilization ? Hardly, for if light rays had non-straight paths, *always*, not only might we be inclined to ape that non-straightness, we would surely christen those paths "straight"! Of course, there might have been trouble if it were found that geodesic straightness was not of the same nature as optical straightness. To modern (post-renaissance) minds such a disparity would be evidence of a lack of the underlying unity that we hope for in Nature.

Even on the evidence of the parts of the geometry of Euclid

that have survived, Greek geometry was incredibly inventive and rich. It must be remembered, however, that it was abstracted from the properties of geodesy and optical alignment ; it was also, frequently, of practical contemporary use. Seventeen hundred years and more later, Galileo and Newton had the surviving body of Greek geometrical theory and intuition with which to build their models of mechanics. Without that theory they would scarcely have been able to start.

Nowadays we may examine Newton's description "uniform motion in a straight line" critically, and pose certain questions. What is a straight line ? What is uniform ? What reference system is implied ? But clearly Newton had no doubts : it was Euclidean geometry, the geometry that grew out of terrestrial geodesy and optical alignment that he knew, and was using.

In our era the two disciplines, physics and geometry, do not exist in isolation or in mutual antipathy, but are mutually supportive, as they were in ancient times. Whether or not, in the future, physics will be "geometrized away" in the sense that the universe will be modelled by a system without the concepts of particle, mass, charge, motion, rest, collapse, decay or rebirth, is speculation. The irony is that if such a "space time manifold" was designed so that its intrinsic differential geometry and topology in the form of knots, folds, cuts, holes, plateaux etc., manifested physical phenomena, we would still not use the model to build bridges. Of the interface between geometry and physics Poincaré (1952 : 50) says "One geometry cannot be more true than another ; it can only be more convenient". If, by that statement, he meant to imply the relation between geometry and the *uses* to which physics

is put in the world of our existence, he is undoubtedly correct. Whether or not there will be a system in which physics is completely geometrized is an unanswered, and possibly, unanswerable question. The claim of such a system would then be that for the purposes of what we take to be normal civil engineering projects, Euclidean geometry was appropriate but only as an approximation of very high order. The distinction might be too fine for engineers, designers and builders.

1.3 What is causality ?

In Special Relativity the physical universe is one of equal opportunity inasmuch as each observer (whether or not absolutely accelerated) is entitled to set up his own Minkowski space model, with its attendant system of coordinates - his "frame". Thus he is entitled to assign quadruples of numbers to events using (co-accelerated) measuring rods and clocks ; if he is either *unaccelerated* in some absolute sense or *uniformly accelerated* in some absolute sense, he will find that the history of a light pulse is a local straight line in his geometry - otherwise not. However, it is not our purpose to assume *any* physics at the outset but simply to contend that Minkowski space is an interesting mathematical structure and that certain groups and their geometries (abstractly available) are also of interest.

Now, although those groups are abstractly available, one of our aims is that the groups should give mathematical expression to a physically intuitive concept - "causality". It is instructive, therefore, to examine exactly what physical concepts might be *considered* to be built into an axiomatic construction of Minkowski

space, even though we shall initially deny such physicality. For example, according to Robb (1914), each observer is implicitly aware of a natural phenomenon which he characterizes as *his* "past", "present" and "future". He is also inclined to assign to remote pairs of events (i.e. those events that do not lie in *his* history) a "causal" connection ; thus, he will give his opinion that two events A, B are (i) "causally connected", with A "in the future of" B, (ii) "causally connected", with B "in the future of" A, (iii) not causally connected. Similarly, it is assumed, on good experimental grounds, that there is a finite maximum speed for signals by which information concerning remote events is brought to an observer, so that there is a second physical aspect to be enshrined in the axiomatic structure. This is achieved by the presence in Minkowski space of a null cone at each point, and the mathematical assertion that there are three "conical" regions, each with its distinctive nature, at each point. The null cone reflects the physical limitation on the signal speed and models the aspects of antecedence, futurity and simultaneity not only for the observer himself but for events remote from him.

Thus *causality* is essentially a subjective concept, one which is peculiar to a single observer. On the other hand, *the principle of causality* concerns the question of comparing information between (any) two observers. For example, if O, O' are the observers, and A, B are two happenings (events), then if O assigns a causal relation between A and B, with A "in the future of" B the requirement that O' should assign the same description to the pair A and B is called the (physical) *principle of causality*.

It is worth noting that such a principle is quite independent of the assumptions involving *constancy* of the signal speed, *finite maximum* for the signal speed or the truth or falsity of the statement that O, O' are in uniform relative motion ; it is simply a requirement that the subjective assessments of two observers concerning a causal relation, or lack of it, agree.

1.4 The treatment of causality in the mathematical model.

We have adopted the view that the mathematical edifice known as Minkowski space will always be a physically useful model. We have also accepted that causality is an intuitively important and physically necessary concept, and that its qualities should be reflected in the mathematical model.

Robb's effective advocacy of the importance of causality, and his strong conviction that the *principle of causality* should have a priori acceptance (Robb, 1936 : 20) ensures a place of importance for his views. However, the discussion of his axiomatic development in Chapter 2 is subsidiary to the main line of development of this thesis.

From Chapter 3 onwards M_4 is a given mathematical structure, already coordinatized, and with the distinction between null, time-like and spacelike vectors already made. In Chapter 3 itself there is an account of the group theoretic and geometric qualities of M_4 associated with the Lorentz group and the Poincaré group. There is another group associated with Minkowski space, the non-affine conformal group, C_4 ; we examine aspects of its geometry in both Chapters 4 and 5. We notice how the affine group P^{+*} and the

special conformal group S_4 (also non-affine, and an abelian subgroup of C_4) absorb into their geometries mathematical qualities which model "global" and "local" principles of causality.

1.5 The principle of local causality and kinematical equivalence.

There is still no mention of other physical phenomena, whether rods, clocks or light signals - no concept of inertial frame, none of uniform motion. All the kinematics that is implied at this stage is as follows :- if an observer O , at the origin of a Minkowski coordinate system, assigns a straight line history to a particle P , then P and O are *relatively unaccelerated*. If the history of P is not straight P and O are *relatively accelerated*. We now select the class of hyperbolic world lines for special consideration (Briginshaw, 1980) and that leads us to a technique with which to conduct a simple thought experiment, that involved in the "clock paradox".

At this point, we ask the following question :

"Can the class of coordinate systems to which the principle of local causality applies be considered as *kinematically* equivalent ?". For the mathematical model to make physical sense the answer is "no" and we are forced to introduce the concept of "inertial frame". Only then do we make contact with the standard developments of Special Relativity (Pauli, 1958 ; Rindler, 1960 ; Synge, 1965 ; etc.).

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Chapter 2. The Robb axiom system for Minkowski space.

2.1 The historical perspective.

Alfred Arthur Robb (b. 1873 ; d. 1936) traced the origins of his theory as far back as 1902, when he attended a meeting of the British Association in Belfast to hear a paper from Lord Rayleigh (1902). At that time Robb, the son of a well-to-do farming family in Ulster, was twenty-nine ; after obtaining a B.A. at Queen's University, Belfast and an M.A. at St. John's College, Cambridge, he was awarded his Ph.D. at Göttingen in 1904 (Robb, 1904), and returned to St. John's as a Fellow. Robb was an associate of Sir Joseph Larmor, a fellow Ulsterman, who was himself soon (1903) to be appointed to the Lucasian Professorship of Mathematics in succession to Sir George Stokes and it was natural that Robb should be involved in the discussions that were provoked by the failure of experiment to "detect uniform motion through the aether". It was just such an experiment that Rayleigh (1902) was describing at the meeting in Belfast.

Now Robb studied at Göttingen between 1902 and 1905, and his doctorate was awarded for the thesis "On the theory of the Zeeman effect" (Robb, 1904). It was quite possible that at this stage of his career Robb became aware of Hilbert's approach to the axiomatization of three dimensional Euclidean geometry. He certainly became familiar with axiom systems propounded by Guiseppe Peano (1889, 1894) and the Italian school, for Robb published a paper (Robb, 1913a) and a short note (Robb, 1913b) in which he proved that two axioms of the Peano system for plane geometry were redundant. Incidentally the references to these

proofs given in Robb (1914, 1936) are incorrect ; they are correctly given in Briginshaw (1979), and in this thesis.

Clearly, at the same time that he was assimilating what was then the recent work of Peano (1889, 1894) and Pasch (1882), and the very recent work of Hilbert (1899) and Veblen (1904), Robb was working on his concept of "conical order". It is this concept, based on a single primitive relation *after*, from which he would eventually fabricate the full geometric structure that is Minkowski space (i.e. Minkowski's "space-time world"). Certainly, at the time of publication of "Optical Geometry of Motion" (Robb, 1911), the conically ordered structure had not fully crystallized as an axiomatic geometry, for Robb (1911 ; 1 - 2) writes :

"It is proposed in the following pages to refer briefly , in the first place, to certain Optical phenomena which occur in free space, upon which we might suppose some of the chief axioms to have their foundations ; and then to employ these to establish on a new basis some of the groundwork of the theory of 'Relativity'.

The writer does not propose, in the present paper, to go into the more minute Logical details of the foundations of Geometry ; as it seems to him that these would tend to obscure the general standpoint which he desires to emphasise. *For this reason he prefers to reserve them for a future occasion*". (My italics).

"Optical Geometry of Motion" was lodged with the London Mathematical Society in 1910 but not published in the Society's Proceedings, and Robb arranged for it to be published privately, as a pamphlet. The fact that it was not published in the Proceedings is not surprising, for the development is deficient in one key respect; it is based on the highly artificial concept "index of a particle" (Robb, 1911 ; 5). Although Robb himself abandoned the

idea of index, the pamphlet nevertheless had an important influence on his later work. In addition, the term "rapidity" is introduced for the first time [(Schutz, 1973 ; 233) wrongly assigns its first usage to Robb (1921)], and Robb's concept of "fundamental particle" recurs later, in the work of Walker (1959).

It is certain that the grand design for Robb's axiom system was complete before 1914, for another privately financed pamphlet (Robb, 1913c) was the forerunner, in *précis* form, to the comprehensive work "A Theory of Time and Space" (Robb, 1914). Throughout the remainder of his life he was to refine and reconstitute his proposals (Briginshaw, 1979) and the culmination of his work was to be the book "Geometry of Time and Space" (Robb, 1936).

2.2 A Peano axiom system for four dimensional absolute geometry .

The following set of fifteen axioms is an appropriate extension of Peano's (1889) system for a three dimensional absolute geometry.

Let $\mathcal{L}$ be a set of elements with a three element primitive relation *linearly between* ; we shall use the abbreviated form ABC for B is linearly between A and C.

- A.1 $\mathcal{L}$ has at least two elements.
- A.2 For each $A \in \mathcal{L}$, $\nexists X$, AXA.
- A.3 If A,B are distinct, $\exists C$, ACB.
- A.4 If ACB then BCA.

Definitions :

- (AB) = { C : ACB } is called the *linear segment* of A and B.
- A'B = { C : ABC } is called the *linear prolongation* of (AB).

- A.5 If A, B are distinct, $\exists C \in A'B$.
- A.6 If A, D are distinct, and if $C \in (AD)$, $B \in (AC)$ then $B \in (AD)$.
- A.7 If A, B are distinct and $C, D \in A'B$, then either $C = D$ or $C \in (BD)$ or $D \in (BC)$.
- A.8 If A, B, C, D $\in \ell$ and $B \in (AC)$, $C \in (BD)$, then $C \in (AD)$.

Definition :

$AB = A'B \cup (AB) \cup B'A$ is called the *straight line* through A and B.

- A.9 If ℓ is a straight line, $\exists A \notin \ell$.
- A.10 If A, B, C are three non-collinear points and $D \in (BC)$, $E \in (AD)$ then $\exists F \in B'E \cap (AC)$.
- A.11 If A, B, C are three non-collinear points and $D \in (BC)$, $F \in (AC)$ then $\exists E \in (AD) \cap (BF)$.

Definition :

If A, B, C are non-collinear points and $D \in AB$, then $\Pi = \{E : E \in CD, D \text{ arbitrary}\}$ is called the *plane* through A, B and C.

- A.12 If Π is a plane, $\exists P \notin \Pi$.

Definition :

If A, B, C, D are non-coplanar points and $E \in \Pi$, where Π is the plane through A, B and C, then

$$H_{D,ABC} = \{F : F \in DE, E \text{ arbitrary}\}$$

is called a *hyperplane*, cf Forsyth (1930 : 6).

- A.13 If H is a hyperplane, $\exists Q \notin H$.
- A.14 If $F \notin H_{D,ABC}$,
then $\ell = \{G : G \in FE, E \in H_{D,ABC}, E \text{ arbitrary}\}$.

A.15 If ℓ is a straight line and $A \notin \ell$, there is one and only one line ℓ' through A , and which lies in the plane of A and ℓ , such that ℓ, ℓ' have no common point.

2.2.1 Remark

Axiom 15 concerns the uniqueness of parallels in the system and makes the geometry Euclidean as opposed to Elliptic or Hyperbolic. There is no metric associated with the system.

2.3 The axiomatic geometry of A. A. Robb

The procedure of Robb (1936) is fourfold in operation. Firstly, he starts with an abstract *event set*, in which there is a primitive partial order, and constructs the *null cone* (abstractly ; there is no vector space structure and no coordinate system until the very last stage of the procedure) at each point appropriate to the partial order. It is the null cone that serves as a selector for the assemblage of geometric entities that contain the point. In this way, these entities, which are elements of a non-metric Euclidean geometry (Robb, 1936 : 121 - 124, 131 - 132, 148, 188, 192) are categorized into mutually exclusive subsets, which Robb calls *optical*, *inertia* and *separation* ; terms which are more usually known by the equivalent titles *null*, *timelike* and *spacelike*, respectively. Secondly, the geometric entities themselves are fabricated in two stages, beginning with *optical line*, and obtaining at one stage the assemblage of planes and lines, and then the assemblage of hyperplanes (Robb uses the term *threefold*) ; there is then an axiom by which the hierarchy of subspaces is limited to four varieties.

When the non-metric Euclidean geometry has been completed, Robb does not attempt to metrize the whole four dimensional edifice, instead, he metrizes sets of lines of the same variety (all inertia lines belong to one variety, as do all separation lines, but skew optical lines are not all members of the same variety for the purposes of metrization), and the method of congruence by which this is achieved leads to an implied correspondence between points of a *general line* and the set of rational points on the real metric Euclidean line (this is the line with a linear order induced by $d(x, y) = |x - y|$). The final axiom is designed to make each variety of line in the four-dimensional geometry a copy of the whole of the real metric Euclidean line, including the irrationals. In addition, having chosen a scale for metrization of inertia lines, Robb uses the scale induced on the *conjugate* (Robb, 1936 : 163) separation line for application to the whole variety of separation lines. There is no attempt to compare the scales for inertia or separation lines with those on optical lines. When this is achieved, Robb is able to distil from the abstract pure geometrical design, an appropriate coordinatization. Only one variety of plane and one variety of hyperplane, in each case, the *separation* variety, has a metric geometry associated with it in the sense that these are the only entities in which the analogues of Pythagoras' theorem and the triangle inequality are valid. Indeed, Robb shows that each separation plane can be identified with the metric Euclidean plane R^2 , and each separation hyperplane with the metric Euclidean variety R^3 . As for the other two varieties of plane and hyperplane, the optical and inertia types, the geometry is quite different, because, for certain regions of

each entity, the analogue of Pythagoras' theorem and the triangle inequality are both violated (for example, there are triples of points in an inertia plane for which the inequality is reversed).

2.4 The analytical geometry of the system

Finally, Robb achieves an analytic description of the whole geometry, and the quadratic form whose sign differentiates the three varieties of line, and, as a result, the three varieties of both plane and hyperplane. The coordinate description of the geometry is also one in which the three subspace entities, line, plane and hyperplane have affine (inhomogeneous linear) equations ; the full precision of the design is then realised by noticing that the vector space structure applies throughout the geometry, in other words, that the axiomatic geometry, when coordinatized , is Minkowski Space.

A complete catalogue of appropriate equations for the homogeneous varieties is as follows :-

The hyperplane

$$a_0x_0 - a_1x_1 - a_2x_2 - a_3x_3 = 0$$

is optical, inertia or separation according as $a_0^2 - a_1^2 - a_2^2 - a_3^2$ is zero, positive or negative.

A general plane, with equations

$$a_0x_0 - a_1x_1 - a_2x_2 - a_3x_3 = 0$$

$$b_0x_0 - b_1x_1 - b_2x_2 - b_3x_3 = 0$$

is optical, inertia or separation according as

$(a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3)^2 - (a_0^2 - a_1^2 - a_2^2 - a_3^2)(b_0^2 - b_1^2 - b_2^2 - b_3^2)$ is zero, positive or negative.

A general line, whose equations are

$$\frac{x_0}{l_0} = \frac{x_1}{l_1} = \frac{x_2}{l_2} = \frac{x_3}{l_3}$$

is optical, inertia or separation according as $l_0^2 - l_1^2 - l_2^2 - l_3^2$ is zero, positive or negative.

A variety which is the translate of one of these is tested using the same criteria.

2.5 Interpretation of the model

The geometry that Robb fabricated is a logical structure built from certain postulates ; it is self contained irrespective of any application . However, the motivation of the whole work was to obtain a logical and acceptable model for the description of space-time, and it is in this light that Robb sought to interpret his theory. In the world of space-time, according to Robb, two events may or may not be related, but if they *are* related then one event is after the other, and this information is absolute. As a result of this intrinsic quality of space-time, it is possible to set up a variety of coordinate systems at any point of the universal set in each of which the coordinate directions are mutually normal. For a material particle (and for a photon) we may say that the locus of its coincident event is its history. We may also define the absence of acceleration for a particle : a particle is said to be unaccelerated between the events A and B if its history between A and B lies in the inertia line AB.

Suppose now that we select any inertia line and consider the class of inertia lines parallel to it ; we must agree to interpret this class as containing the histories of unaccelerated particles which are at relative rest, and agree to refer to the class as

an inertial frame. Equally, any other inertia line and its parallel class is a second inertial frame. Now, in the first inertial frame, the constituents of the second inertial frame are interpreted as representing unaccelerated material particles, and vice versa, and it is taken as axiomatic that there are vectors associated with each pair of frames, called their relative velocities, which characterize the state of no acceleration. In addition, if the relative velocity of second with respect to first is $\underline{u}$ implies that the relative velocity of first with respect to second is $-\underline{u}$, this must be a matter of agreement, too. With these matters agreed, as well as a method of calculating the relative velocities associated with a pair of frames, it is now possible to obtain the equations of transformation between the event-coordinates according to two coordinate systems, one in each inertial frame. Clearly an individual's assessment of causality is endemic in the system; Robb (1936 : 20) accepts as axiomatic the principle of causality, too, so that the most natural procedure from which to obtain these Poincaré (inhomogeneous Lorentz) transformations is that of Theorem A in Chapter 4.

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Chapter 3. The standard affine transformation groups.

3.1 Perspective.

It is important to be clear exactly what mathematical entity is implied when we use the description *Minkowski space*. For example, from an algebraic viewpoint Minkowski space is a vector space, but not a linear algebra ; also, because of the vector space structure there are the elements of the usual (affine) geometry, these are entities for which the invariance group is the affine group. What, however, is the status of any topology associated with the space, is it a metric topology that we imply ? Certainly the vector space structure alone always generates a natural topology and that topology is semi-metric ; to what extent is that appropriate when the crucial quadratic form is not positive definite ? These are important questions and ones which a succession of authors, beginning with Zeeman (1967) have sought to answer. As far as *this* work is concerned, as the title indicates, it is concerned with causality and geometry and the interplay between the two ; when questions of continuity arise, as they do in Chapters 4 and 5, the topology is taken to be the R_4 metric topology. As for the construction of topologies finer than the metric topology, which are themselves non-metrizable, but which match the Lorentz geometry in a remarkable way, there is a brief survey in Chapter 7.

The present chapter, for its part, is concerned with nomenclature and notation, and the establishment of a number of standard results concerning the Lorentz group and allied groups, and their geometrical nature. We define Minkowski space to be a

four dimensional *orthogonal space* (Porteous, 1969 : 146-164).

3.2 The vector space, V_4

We shall refer to the collection of quadruples of real numbers of the type (x_0, x_1, x_2, x_3) with the usual vector space structure [i.e. $x + y \equiv (x_0 + y_0, x_1 + y_1, x_2 + y_2, x_3 + y_3)$; $\lambda x \equiv (\lambda x_0, \lambda x_1, \lambda x_2, \lambda x_3)$] as V_4 .

3.3 Nomenclature

We shall say that an homogeneous linear transformation of the components of the quadruple,

$$x'_i = \sum a_{ij} x_j , \text{ is linear } ,$$

and that an inhomogeneous linear transformation,

$$x'_i = b_i + \sum a_{ij} x_j , \text{ is affine } .$$

Transformations which are not inhomogeneous linear are often called non-linear, but we shall use the term *non-affine*.

3.4 The usual geometry for V_4

The sets $\{a\}$, $\{a + \lambda b : a, b \in V_4, \lambda \in R\}$,
 $\{a + \lambda_1 b_1 + \lambda_2 b_2 : a, b_1, b_2 \in V_4 ; \lambda_1, \lambda_2 \in R\}$,
 $\{a + \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 : a, b_1, b_2, b_3 \in V_4 ; \lambda_1, \lambda_2, \lambda_3 \in R\}$,
are the elements of the *usual geometry* associated with V_4 ;
they are respectively points, lines, planes and hyperplanes.

It is clear that the group of non-singular affine transformations is the invariance group for this geometry, i.e. the entities point, line, plane and hyperplane are invariant figures

for the group.

3.5 Definition : orthogonal space.

We say that V_4 is assigned an *inner product* $\cdot$ if there is a mapping $p : V_4 \times V_4 \rightarrow \mathbb{R}$ given by $p(u,v) = u \cdot v$ with the properties

$$(i) \quad u \cdot v = v \cdot u \quad \forall u, v : \text{symmetry}$$

$$(ii) \quad u \cdot (\lambda v + \mu w) = \lambda u \cdot v + \mu u \cdot w$$

$$\forall u, v, w \in V_4; \forall \lambda, \mu \in \mathbb{R} : \text{linearity}.$$

The pair $(V_4, \cdot)$ is said to be an *orthogonal space* (Porteous, 1969 : 148) .

3.6 Definition : Minkowski space.

It is clear that $p : V_4 \times V_4 \rightarrow \mathbb{R}$ given by $p(u,v) = u \cdot v = u_0 v_0 - u_1 v_1 - u_2 v_2 - u_3 v_3$ is an inner product; in this case, the pair $(V_4, \cdot)$ is said to be *Minkowski 4-space* or *four dimensional Minkowski space*, and denoted by M_4 .

3.7 Remark.

The inner product as introduced above does *not* have the property that $u \cdot u \geq 0 \quad \forall u$, and does *not* therefore generate a semi-norm in V_4 ; M_4 is *not* a pre-Hilbert space.

3.8 Definition : Lorentz transformation .

A linear transformation of V_4 which preserves $\cdot$ is said to be a *Lorentz transformation* of M_4 . Thus $\Lambda : M_4 \rightarrow M_4$ is a Lorentz transformation if Λ is linear and if $(\Lambda u) \cdot (\Lambda v) = u \cdot v \quad \forall u, v \in M_4$ (See Chapter 4, if f is a bijection, which preserves $\cdot$

then it is necessarily linear).

3.9 Theorem 1. (Trautman, 1965)

Λ is a Lorentz transformation iff

$$(\Lambda u) \cdot (\Lambda u) = u \cdot u, \forall u \in M_4.$$

Proof :

Sufficiency : suppose that $(\Lambda u) \cdot (\Lambda u) = u \cdot u, \forall u \in M_4$;

$$\text{then } \Lambda(u + v) \cdot \Lambda(u + v) = (u + v) \cdot (u + v)$$

$$\therefore \Lambda u \cdot \Lambda u + 2\Lambda u \cdot \Lambda v + \Lambda v \cdot \Lambda v$$

$$= u \cdot u + 2u \cdot v + v \cdot v$$

$$\therefore \Lambda u \cdot \Lambda v = u \cdot v, \forall u, v \in M_4$$

Λ is a Lorentz transformation.

3.10 Matrix representation of Λ .

Consider an appropriate basis for V_4 , viz.

$$e_0 = (1, 0, 0, 0) ; e_1 = (0, 1, 0, 0) ; e_2 = (0, 0, 1, 0) ; e_3 = (0, 0, 0, 1).$$

Now $e_0 \cdot e_0 = 1$, $e_i \cdot e_i = -1$ for $i \neq 0$, and $e_i \cdot e_j = 0$ for $i \neq j$.

Thus, if Λ is a Lorentz transformation

$$\Lambda e_0 = \lambda_{00}e_0 + \lambda_{01}e_1 + \lambda_{02}e_2 + \lambda_{03}e_3$$

$$\text{where } \Lambda e_0 \cdot \Lambda e_0 = \lambda_{00}^2 - \lambda_{01}^2 - \lambda_{02}^2 - \lambda_{03}^2 = 1,$$

$$\Lambda e_i = \lambda_{i0}e_0 + \lambda_{i1}e_1 + \lambda_{i2}e_2 + \lambda_{i3}e_3, \quad i = 1, 2, 3,$$

$$\text{where } \Lambda e_i \cdot \Lambda e_i = \lambda_{i0}^2 - \lambda_{i1}^2 - \lambda_{i2}^2 - \lambda_{i3}^2 = -1.$$

$$\text{Also, } \Lambda e_i \cdot \Lambda e_j = 0 \quad \text{for } i \neq j.$$

The matrix representation of Λ is then

$$A = \begin{pmatrix} \lambda_{00} & \lambda_{01} & \lambda_{02} & \lambda_{03} \\ \lambda_{10} & \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{20} & \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{30} & \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix}$$

3.11 Theorem 2.

A is non-singular.

Proof:-

Consider scalars $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ such that

$$\alpha_0 \lambda_{00} + \alpha_1 \lambda_{10} + \alpha_2 \lambda_{20} + \alpha_3 \lambda_{30} = 0$$

$$\alpha_0 \lambda_{01} + \alpha_1 \lambda_{11} + \alpha_2 \lambda_{21} + \alpha_3 \lambda_{31} = 0$$

$$\alpha_0 \lambda_{02} + \alpha_1 \lambda_{12} + \alpha_2 \lambda_{22} + \alpha_3 \lambda_{32} = 0$$

$$\alpha_0 \lambda_{03} + \alpha_1 \lambda_{13} + \alpha_2 \lambda_{23} + \alpha_3 \lambda_{33} = 0$$

Now

$$\begin{aligned} \alpha_0^2 &= (\alpha_1 \lambda_{10} + \alpha_2 \lambda_{20} + \alpha_3 \lambda_{30})^2 \\ &\quad - (\alpha_1 \lambda_{11} + \alpha_2 \lambda_{21} + \alpha_3 \lambda_{31})^2 \\ &\quad - (\alpha_1 \lambda_{12} + \alpha_2 \lambda_{22} + \alpha_3 \lambda_{32})^2 \\ &\quad - (\alpha_1 \lambda_{13} + \alpha_2 \lambda_{23} + \alpha_3 \lambda_{33})^2 \\ &= -\alpha_1^2 - \alpha_2^2 - \alpha_3^2 + 2\alpha_1\alpha_2[\lambda_{10}\lambda_{20} - \lambda_{11}\lambda_{21} - \lambda_{12}\lambda_{22} - \lambda_{13}\lambda_{23}] \\ &\quad + 2\alpha_1\alpha_3[\lambda_{10}\lambda_{30} - \lambda_{11}\lambda_{31} - \lambda_{12}\lambda_{32} - \lambda_{13}\lambda_{33}] \end{aligned}$$

$$+ 2\alpha_2\alpha_3[\lambda_{20}\lambda_{30} - \lambda_{21}\lambda_{31} - \lambda_{22}\lambda_{32} - \lambda_{23}\lambda_{33}] \rightarrow 0$$

$$\therefore \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 0$$

$$\therefore \alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = 0$$

Thus the column vectors of A are independent, and $\det A \neq 0$.

3.12 Definition : the Lorentz group.

Consider the collection of Lorentz transformations of M_4 ; if Λ is any member of the collection, then the inverse transformation exists, from theorem 2. Consider arbitrary elements $\Lambda^{-1}u$, $\Lambda^{-1}v$, then

$$(\Lambda\Lambda^{-1}u) \cdot (\Lambda\Lambda^{-1}v) = (\Lambda^{-1}u) \cdot (\Lambda^{-1}v) ,$$

since Λ is Lorentz ; it is clear therefore that Λ^{-1} is also Lorentz, as is 1_{M_4} . Thus, with composition as the group operation, it is clear that the collection is a group, which we denote by L, and call the *Lorentz group*.

3.13 Types of vector.

The vectors of M_4 may be classified into three types as follows :-

- (i) u is *timelike* if $u \cdot u > 0$,
- (ii) u is *null* if $u \cdot u = 0$,
- (iii) u is *spacelike* if $u \cdot u < 0$.

3.14 Theorem 3.

If u, v are timelike vectors, with $u_0 > 0$, $v_0 > 0$, then $u \cdot v > 0$.

Proof:-

$$u.v = u_0 v_0 - u_1 v_1 - u_2 v_2 - u_3 v_3$$

$$\text{Now } \left| \sum_{i=1}^3 u_i v_i \right| \leq \sqrt{\sum_{i=1}^3 u_i^2} \cdot \sqrt{\sum_{i=1}^3 v_i^2}$$

$$\text{Thus } \left| \sum_{i=1}^3 u_i v_i \right| < |u_0| |v_0| = u_0 v_0$$

$$\therefore u_1 v_1 + u_2 v_2 + u_3 v_3 < u_0 v_0$$

$$\text{i.e. } 0 < u.v$$

Corollary.

If u, v are timelike vectors, with $u_0 < 0, v_0 < 0$, then $u.v > 0$.

3.15 Time orientation.

Let $\mathcal{J}$ be the set of all timelike vectors in M_4 , then $\mathcal{J}^+ = \{u \in \mathcal{J} : u_0 > 0\}$ is said to be the *positive time direction* or *future*, and $\mathcal{J}^- = \{u \in \mathcal{J} : u_0 < 0\}$ is called the *negative time direction* or *past*.

3.16 Definition : orthochronous Lorentz transformation.

A Lorentz transformation Λ is said to be *orthochronous* if, for each $u \in M_4$, $(\Lambda u)_0$ has the same sign as u_0 . The collection of orthochronous Lorentz transformations is a subgroup of L ; we shall refer to it as the group $L^\uparrow$.

3.17 Inner orientation (Trautman, 1965)

Let V_N be a real vector space and S_M be a vector subspace

of V_N . Let $\mathcal{B}$ be the set of all bases of S_M ; now, if $\{e_i\}$, $\{f_i\}$, $i = 1, 2, \dots, M$ are two such bases, then

$$e_j = A_{ji} f_i, \quad \text{where } |A_{ji}| \neq 0,$$

and we may define the equivalence relation O on $\mathcal{B}$ as follows :-

$$(\{e_i\}, \{f_i\}) \in O \quad \text{iff } |A_{ji}| > 0.$$

Thus, there are two equivalence classes in S_M , $[\{e_i\}]$, $[\{-e_i\}]$, and we may single out one of them, and call it the *positive inner orientation*, the other equivalence class then characterizes the negative orientation. The same procedure applies to the whole space V_N , in which case the inner orientation is called the *orientation* of V_N . In the case of the subspace S_M , there is also the possibility of an outer orientation, which does not exist for the whole space.

3.18 Outer orientation of a subspace (Trautman, 1965)

Suppose that V_N has an orientation, and S_M has an inner orientation; let $\{e_i\}$, $i = 1, 2, \dots, M$ be a basis of S_M with positive inner orientation, we may extend the collection of basis vectors to $\{e_i\}$, $i = 1, 2, \dots, N$ in such a way that the basis for V_N has positive orientation. Now the collection $e_{M+1}, e_{M+2}, \dots, e_N$ is a basis for $V_N - S_M$, and we stipulate that it defines a positive inner orientation for $V_N - S_M$, which we refer to as the *outer orientation* of S_M .

3.19 Definition : orientation preserving Lorentz transformation.

A Lorentz transformation Λ is said to be *orientation preserving* if, for every basis $\{e_i\}$ of M_4 , $\{\Lambda e_i\}$ and $\{e_i\}$

belong to the same equivalence class of orientations i.e. they are both positive or both negative orientations of $M_4 (V_4)$.

3.20 Definition : proper Lorentz transformation.

If Λ is both orthochronous and orientation-preserving, we say that Λ is a *proper Lorentz transformation*. The collection of all proper Lorentz transformations is a subgroup of L and is called the *proper Lorentz group* ; it is denoted by $L_+^\uparrow$.

3.21 Remark.

A proper Lorentz transformation of M_4 is one which preserves the time orientation and also the inner orientation of a triad of spacelike vectors.

3.22 Remark.

We may improve on the result that the matrix of a Lorentz transformation is non singular ; in fact, its determinant is ± 1 .

3.23 Adjoint of a Lorentz transformation.

Let $\Lambda \in L$, then there is a unique linear map

$$\Lambda^\# : M_4 \rightarrow M_4 \quad \text{such that}$$

$$x \cdot \Lambda^\# y = y \cdot \Lambda x \quad \forall x, y \in M_4.$$

3.24 Theorem 4. (Porteous, 1969 : 154).

$$\Lambda \in L \text{ iff } \Lambda^\# \Lambda = 1_{M_4}.$$

Proof :-

$$\text{Let } \Lambda^\# \Lambda = 1_{M_4}$$

$$\text{now } \Lambda^\# x.y = x.\Lambda y \quad \forall x,y \in M_4$$

$$\text{in particular } \Lambda^\# \Lambda x.y = \Lambda x.\Lambda y$$

$$\text{i.e. } x.y = \Lambda x.\Lambda y$$

so that $\Lambda \in L$.

$$\text{Conversely, from } \Lambda^\# \Lambda x.y = \Lambda x.\Lambda y$$

with $\Lambda \in L$, we obtain

$$\Lambda^\# \Lambda x.y = x.y$$

$$\text{i.e. } (\Lambda^\# \Lambda - 1_{M_4})x.y = 0, \quad \forall y \in M_4$$

$$\text{Now, let } (\Lambda^\# \Lambda - 1_{M_4})x = (\alpha_0, \alpha_1, \alpha_2, \alpha_3),$$

Choose $y = (1, 0, 0, 0)$ etc.,

$$\text{then } \alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = 0,$$

$$\text{thus } \Lambda^\# \Lambda = 1_{M_4}.$$

3.25 Theorem 5.

$$\text{If } \Lambda \in L \text{ then } (\det \Lambda)^2 = 1.$$

Proof:-

Consider the isomorphism f from the general linear group in 4 dimensions to the set of all non-singular matrices.

$$\text{Let } f\Lambda = A \text{ and } f\Lambda^\# = A^\#;$$

$$\text{it is clear that } A^\# = A' \text{ (transpose)}$$

$$\text{for, let } A = (A_{ij}) \text{ and } A^\# = (A_{ij}^\#),$$

$$\text{then } \Lambda e_i = \sum_{j=0}^3 A_{ij} e_j$$

$$\therefore e_k \cdot \Lambda e_i = A_{ik}$$

$$\text{Similarly } \Lambda^\# e_k \cdot e_i = A_{ki}^\#.$$

Also $A^\# A = I$, from Theorem 4,

$$\therefore A^\# = A' = A^{-1} \text{ i.e. } A'A = I$$

$$\therefore (\det A)^2 = 1,$$

3.26 Remark.

The sign of the determinant decides whether or not orientations are preserved, so that $\det \Lambda = +1$ corresponds to $\Lambda \in L_+$ and $\det \Lambda = -1$ means that Λ is an improper Lorentz map (Porteous, 1969 : 158).

3.27 Definition : dilatation.

If V_N is a real vector space, a map $d_\lambda : V_N \rightarrow V_N$ given by $d_\lambda(x) = \lambda x$, where $\lambda \in \mathbb{R}^+$, is said to be a *dilatation* of V_N ; the collection of dilatations is an abelian group, since $\lambda = 0$ is excluded.

3.28 Remark.

For M_4 , each d_λ is a linear transformation and has orthochronous and orientation preserving properties; however, d_λ does not preserve inner product (except $\lambda = 1$), and is not, therefore, a Lorentz transformation. The dilatation map does share some of the qualities associated with L in that it preserves the null cone and timelike and spacelike qualities.

3.29 Definition : augmented Lorentz transformation, group.

We shall denote the group generated by the elements of $\mathbb{R}^+$ and L as L^* , and refer to it as the *augmented Lorentz group*; the elements of L^* are called augmented Lorentz transformations.

Similarly, there is an *augmented orthochronous Lorentz group*, $L^{\dagger*}$.

3.30 Definition : translation.

A *translation* of a real vector space V_N is a map $t_a : V_N \rightarrow V_N$ given by $t_a(x) = x + a$, where $a \in V_N$. Under composition, the collection $\{t_a\}$ is an abelian group, known as the *translation group*.

3.31 Remark.

In the case of M_4 , it is clear that translations are not Lorentz maps, but they do preserve time and spatial orientation and also the null, timelike and spacelike qualities of vectors.

3.32 Definition : Poincaré transformation, group.

A map $\Pi : M_4 \rightarrow M_4$ given by $\Pi x = \Lambda x + a$, where $\Lambda \in L$, $a \in M_4$ is said to be a *Poincaré* (inhomogeneous Lorentz) *transformation*; the collection of such maps is the *Poincaré group*, which we denote by P .

3.33 Theorem 6.

P is a group.

Proof:-

It is clear that composition is associative.

For each Π , define Π^{-1} by $\Pi^{-1}x = \Lambda^{-1}(x - a)$,

then $\Pi\Pi^{-1}x = x = \Pi^{-1}\Pi x$

thus $\Pi\Pi^{-1} = 1_{M_4} = \Pi^{-1}\Pi$.

3.34 Nomenclature.

If $\Lambda \in L^\uparrow$, then the Poincaré transformation generated by Λ and any $a, \Pi = (\Lambda, a) \in P^\uparrow$, the *orthochronous* Poincaré group; similarly if $\Lambda^* \in L^*$ (respectively $L^{\uparrow*}$) then $\Pi^* = (\Lambda, \lambda, a) \in P^*$ (respectively $P^{\uparrow*}$), the *augmented* (respectively *orthochronous*) Poincaré group.

3.35 Remark.

In semi-direct product formulation (Roman, 1975 : 101) P can be looked upon as the semi-direct sum $R \ltimes L$ (Michel : 1968), P^* as the semi-direct product $R^+ \ltimes P$, and so on.

3.36 Preservation of the null cone.

Clearly, if $\Pi^* = (\Lambda, \lambda, a) \in P^*$,
then

$$\begin{aligned} & (\Pi^*x - \Pi^*b) \cdot (\Pi^*y - \Pi^*c) \\ &= (\lambda\{\Lambda x + a\} - \lambda\{\Lambda b + a\}) \cdot (\lambda\{\Lambda y + a\} - \lambda\{\Lambda c + a\}) \\ &= \lambda^2 (\Lambda x - \Lambda b) \cdot (\Lambda y - \Lambda c) \\ &= \lambda^2 (x - b) \cdot (y - c). \end{aligned}$$

In particular, the elements of P^* preserve sets of the type $\{x : (x - b) \cdot (x - b) = 0\}$, and it has been shown by Frank (1911) that P^* is the largest *affine* group with this property.

3.37 The Lorentz and Poincaré geometries.

In geometric parlance we can say that each of the groups, $L, L^\uparrow, L^{\uparrow*}; P, P^\uparrow, P^{\uparrow*}$ has an affine geometry associated with it, in which lines, planes and hyperplanes are *a priori* invariant

figures. In addition, what is common to the geometries is the preservation of the qualities null, timelike, spacelike associated with the invariant figures ; in particular, we look upon these groups as preserving the "surface" and the "interior" and "exterior" of the null cones $\{x : (x - b) \cdot (x - b) = 0\}$.

3.38 Remark.

The purpose of Chapter 3, to make explicit the structure and the group theoretic and geometric terminology associated with M_4 , is now achieved ; we are now in a position to discuss the recent work concerning local and global causal automorphisms of M_4 .

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Chapter 4 The globally causal geometry4.1 Perspective.

In 1964, the paper of E.C. Zeeman entitled "Causality implies the Lorentz group." (Zeeman, 1964) was published. In it, Zeeman proved that without an a priori assumption of linearity, or even continuity, the requirement of global causality preservation was enough to ensure that the allowable mappings to be associated with the model of Special Relativity - Minkowski space, were members of the group $P^{\uparrow*}$. The result was challenging and stimulating, but, on reflection, not as surprising as it first seemed.

Thus, it had always been assumed that the linearity of transformations of Special Relativity was axiomatic (Pauli, 1958:9), but that, to accord with an *independent physical principle of causality*, the maps were of the orthochronous variety. Now, it was clear that the linearity was consequent on the requirement of causality preservation. Clearly the fact of an individual's causality is ~~woven~~ into the fabric of his model of the universe in any viable axiomatic system (e.g. Robb, 1936; Schutz, 1973; Schutz, 1979; Carnap, 1958: 212), so that the acceptance of the principle of causality *first* and its consequence of linearity seemed to be an important achievement. However, as Flato and Sternheimer (1966) were quick to point out the mappings that Zeeman (1964) refers to as "causal automorphisms" are everywhere bijective; that fact alone, together with the preservation of null cones was, as Weyl (1923) and Fock (1955) had noticed, enough to guarantee affine transformations. This fact was reinforced by Alexandrov (1967), and reiterated by Dutta, Mukherjee and Sen (1970), Hegerfeldt (1972), and Borchers

and Hegerfeldt (1972). More recently still, Nanda (1976) has given a geometrical proof of the theorem, using a method reminiscent of Alexandrov (1967); and Lester and McKiernan (1977) have proved that null cone preserving mappings are linear under considerably weaker conditions than heretofore. In every contribution, it is made clear that the theorem is not valid for M_2 ; for M_2 there are certainly non-affine causal automorphisms.

There were other responses to the Zeeman (1964) paper; Alonso and Yndurain (1966) proved that for a space which is locally homeomorphic (in the Euclidean metric topology) to M_n ($n \geq 3$), every causal automorphism is continuous. Barucchi and Teppati (1967) and, separately, Barucchi (1968) and Teppati (1968), using a mixture of group theoretic and geometric methods, provided alternative proofs of the causality theorem; their work covered similar ground to that of Alexandrov (1967).

In this chapter of the thesis, we present a new proof of the theorem of Zeeman that proceeds from the continuity of the causal maps as proved by Nanda (1976) to their membership of the conformal group. We then use an easily established result concerning the causal violation property of the non-affine elements of the conformal group to remove them from consideration.

4.2 Properties of the conformal group.

4.2.1 Null line.

If x, y are null related i.e. $(x-y) \cdot (x-y) = 0$, we say that the set $N_{x,y} = \{z: z = x + \mu b, \mu \in \mathbb{R}, b = x - y\}$ is the *null line* through x and y .

4.2.2 Conformal map, group.

Let $f: \overline{M}_4 \rightarrow \overline{M}_4$ (M_4 compactified by the addition of an ideal "hyperplane at infinity") be such that f is continuous at every non singular point and $\forall$ appropriate x, y , if $x' = fx$, we have

$$\frac{dx'.dy'}{\sqrt{dx'.dx'}\sqrt{dy'.dy'}} = \frac{dx.dy}{\sqrt{dx.dx}\sqrt{dy.dy}},$$

then f is said to be *conformal*. Under composition, the collection of such maps is a group, called the *conformal group*, C .

4.2.3 Generators of C .

Following the method of Lie and Engel (1888), it was established by Bateman (1909), in conjunction with Cunningham (to whom Bateman assigns the discovery of the inversion map) that the conformal group is that generated by dilatations, Lorentz transformations, translations and inversions (see the next paragraph). Cunningham and Bateman (1909) refer to the inversion as a mapping "by reciprocal radii" or "an inversion with respect to the psuedosphere" because of the close analogy between maps which preserve null cones in M_4 and those that preserve hyperspheres in R_4 .

4.2.4 The inversion map.

The conformal map $I: \overline{M}_4 \rightarrow \overline{M}_4$ given by $Ix = \frac{x}{x.x}$ is said to be an *inversion* of M_4 .

4.2.5 Remark.

Wess (1960) has drawn attention to a collection of conformal maps which constitutes an abelian subgroup of the full group; the collection is called the *special conformal group*, and its elements are of particular importance in the consideration of "local causality".

4.2.6 Special conformal maps, group.

If $t_a: M_4 \rightarrow M_4$ given by $t_a x = x + a$ is an arbitrary translation of M_4 , then the composite $I \circ t_a \circ I$ is said to be a *special conformal map* of M_4 .

The collection of special conformal maps with composition is an abelian subgroup of C ; it is denoted by S and called the *special conformal group*. We shall consider the group S in more detail in Chapter 5.

4.2.7 Commutation properties of the generators of C .

If λ, t_a, Λ are respectively arbitrary dilatation, translation and Lorentz maps, and I is the inversion map, for M_4 , then the commutation properties for pairs of such maps is described in the following theorem.

Theorem 2.

- (i) $\lambda \circ I = I \circ \lambda^{-1}$
- (ii) $\Lambda \circ I = I \circ \Lambda$
- (iii) $\lambda \circ \Lambda = \Lambda \circ \lambda$
- (iv) $\Lambda \circ t_a = t_{\Lambda a} \circ \Lambda$

$$(v) \quad \lambda \circ t_a = t_{\lambda a} \circ \lambda,$$

there is no similar commutation relation for t_a and I .

Proof:-

$$\begin{aligned} (i) \quad I \circ \lambda x &= \frac{\lambda x}{\lambda x \cdot \lambda x} = \lambda^{-1} \frac{x}{x \cdot x} \\ &= \lambda^{-1} \circ I x \end{aligned}$$

$$\begin{aligned} (ii) \quad \Lambda \circ I x &= \Lambda \left(\frac{x}{x \cdot x} \right) \\ &= \frac{1}{x \cdot x} \Lambda x \\ &= \frac{1}{\Lambda x \cdot \Lambda x} \Lambda x \\ &= I \circ \Lambda x \end{aligned}$$

$$(iii) \quad \text{Clearly } \lambda \circ \Lambda = \Lambda \circ \lambda$$

$$(iv) \quad \Lambda \circ t_a x = \Lambda(x+a) = \Lambda x + \Lambda a$$

$$t_{\Lambda a} \circ \Lambda x = t_{\Lambda a}(\Lambda x) = \Lambda x + \Lambda a$$

$$(v) \quad \lambda \circ t_a x = \lambda(x+a) = \lambda x + \lambda a$$

$$t_{\lambda a} \circ \lambda x = t_{\lambda a}(\lambda x) = \lambda x + \lambda a.$$

4.3 Causal relations.

There are three relations in M_4 , induced by the inner product

and the time orientation, as follows:-

we write

$$(i) \quad y < x \text{ if } (x-y).(x-y) > 0 \text{ and } y_0 < x_0 \quad (\text{Zeeman, 1964}),$$

$$(ii) \quad y < .x \text{ if } (x-y).(x-y) = 0 \text{ and } y_0 < x_0 \quad (\text{Zeeman, 1964}),$$

$$(iii) \quad y \ll x \text{ if } (x-y).(x-y) \geq 0 \text{ and } y_0 < x_0 \quad (\text{Nanda, 1976}).$$

4.3.1 Theorem 3

$<$ is a partial order.

Proof:-

Suppose that $z < y$ and $y < x$

$$(x-z).(x-z) = (x-y+y-z).(x-y+y-z)$$

$$= (x-y).(x-y) + 2(x-y).(y-z) + (y-z).(y-z)$$

Now, if $u = x-y$ and $v = y-z$

then $u > 0$ and $v > 0$

so that from §3.14 $u.v > 0$;

thus $(x-z).(x-z) > 0$.

Also if $z_0 < y_0$ and $y_0 < x_0$ then $z_0 < x_0$,

so that $z < y$ and $y < x \Rightarrow z < x$.

4.3.2 Remark

It is clear that $\ll$ is also a partial order.

However, $< .$ is not a partial order; for, consider

$$z = (0,0,0,0); \quad y = \left(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right); \quad x = \left(2, \frac{3}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right)$$

$$(y-z) \cdot (y-z) = 1 - \frac{1}{4} - \frac{1}{4} - \frac{1}{2} = 0 \quad \text{and } z_0 < y_0 \quad \text{i.e. } z < y$$

$$(x-y) \cdot (x-y) = 1 - 1 - 0 - 0 = 0 \quad \text{and } y_0 < x_0 \quad \text{i.e. } y < x$$

$$(x-z) \cdot (x-z) = 4 - \frac{9}{4} - \frac{1}{4} - \frac{1}{2} = 1 \quad \text{so that } z \not< x.$$

4.4 Definition: the causal space $\hat{M}_4$.

We refer to the pair $(M_4, <)$ as a *causal space*, and denote it by $\hat{M}_4$.

4.4.1 Definition: causal automorphism.

If $f: \hat{M}_4 \rightarrow \hat{M}_4$ is one: one and onto, and has the property that $y < x \iff fy < fx$, then f is said to be a *causal automorphism* of $\hat{M}_4$.

4.4.2 Remark.

The description "morphism" in the definition given above implies that some aspect of the *structure* of $\hat{M}_4$ is being preserved; it is the partial order, of course, that is being preserved, not the linearity.

4.4.3 Definition: causality group

It is clear that, with composition as the group operation, the collection of causal automorphisms is a group, which we call the *causality group* and denote by $\hat{G}$.

4.4.4 Remark.

We might further distinguish the structures $(M_4, \ll)$, $(M_4, <.)$, refer to them as $\hat{M}_4$ and $\hat{\hat{M}}_4$, and define two other types of causal automorphism according to whether $\ll$ or $<.$ is preserved; then the collections of such automorphisms would be respectively $\hat{\hat{G}}$ and $\hat{G}$. In fact, we shall show that $\hat{G}$, $\hat{\hat{G}}$ and $\hat{G}$ are identical.

4.4.5 Theorem 4.

$$\hat{G} = \hat{\hat{G}} = \hat{G}$$

Proof:

We show first that $\hat{G} = \hat{\hat{G}}$

Let $f \in \hat{G}$;

if $x \not\prec y$ then $fx \not\prec fy$

for, suppose that $x \not\prec y \Rightarrow fx < fy$

$$\Rightarrow f^{-1}fx < f^{-1}fy$$

i.e. $x < y$, which is a contradiction.

Now $x < . y \Leftrightarrow \text{both } x \not\prec y$

and $y < z \Rightarrow x < z$;

so that, since $f \in \hat{G}$

$x < . y \Rightarrow \text{both } fx \not\prec fy$

and $fy < fz \Rightarrow fx < fz$

$$\Rightarrow fx < . fy.$$

There is a similar property for f^{-1}

$$\therefore f \in \hat{\hat{G}}.$$

Conversely, let $F \in \hat{G}$,

if $x \not\prec y$ then $Fx \not\prec Fy$,

for, suppose that, $x \not\prec y \Rightarrow Fx < Fy$

$$\Rightarrow F^{-1}Fx < F^{-1}Fy$$

i.e. $x < y$, which is a contradiction.

Now $x < y \Leftrightarrow$ both $x \not\prec y$

and for some z , $x < z < y$;

so that, since $F \in \hat{G}$

$$x < y \Rightarrow \text{both } Fx \not\prec Fy$$

and for some z , $Fx < Fz < Fy$

$$\Rightarrow Fx < Fy.$$

There is a similar property for F^{-1}

$$\therefore F \in \hat{G}.$$

Thus $\hat{G} = \hat{G}$.

Now $x \ll y \Rightarrow$ either $x < y$

or $x < y$.

If $f \in \hat{G}$

$$x \ll y \Rightarrow \text{either } fx < fy$$

or $fx < fy$

$$\Rightarrow fx \ll fy$$

with a similar result for f^{-1} .

$$\text{i.e. } f \in \hat{\hat{G}}$$

Finally $x < y \Leftrightarrow \text{both } x \ll y$

$$\text{and } \forall z, x < z$$

then either $y < z$ or $z < y$.

If $F \in \hat{\hat{G}}$

$$x < y \Rightarrow \text{both } Fx \ll Fy$$

$$\text{and } \forall Fz, Fx < Fz$$

then either $Fy < Fz$ or $Fz < Fy$

$$\Rightarrow Fx < Fy$$

There is a similar procedure for f^{-1}

$$\therefore F \in \hat{\hat{G}} \subset \hat{\hat{G}}$$

thus $\hat{\hat{G}} = \hat{\hat{G}} = \hat{\hat{G}}$.

4.5 The principle of causality.

It is clear that the physical "principle of global causality" as outlined in §1.3 has its mathematical formulation in the restriction of the allowable mappings to the collection of causal automorphisms.

4.6 Causal properties for $L^{\uparrow*}$ and $P^{\uparrow*}$.

It is clear that the elements of $L^{\uparrow*}$ and $P^{\uparrow*}$ preserve the partial order $<$, so that $L^{\uparrow*} \subseteq P^{\uparrow*} \subseteq G$. The question that poses itself is whether there are non-affine maps of M_4 which preserve $<$. As Zeeman (1964) has pointed out, this matter is crucially dependent on the dimension of the space; the theorem of Zeeman (1964) is that for M_4 (and also for M_N , with $N \geq 3$, but not for M_2) there are no non-affine causal automorphisms of the causal space.

4.7 Non-affine causal automorphisms of M_2 .

In M_2 , we have the inner product $\cdot$ defined by $x \cdot x = x_0^2 - x_1^2$, where $x = (x_0, x_1) \in M_2$, and this, in turn, induces the three causal relations and the single causal automorphism group, as in §4.4.5. We now write $\xi_1 = x_0 + x_1$; $\xi_2 = x_0 - x_1$, so that

$$x \cdot x = \xi_1 \xi_2.$$

Thus $x < y$ corresponds to

$$(\xi_1 - \eta_1)(\xi_2 - \eta_2) > 0 \quad \text{and} \quad \xi_1 + \xi_2 < \eta_1 + \eta_2.$$

Let $x < y$ and suppose that $f: M_2 \rightarrow M_2$ is given by

$$f(x_0, x_1) = (X_0, X_1) = X,$$

$$\text{where } X_0 + X_1 = (x_0 + x_1)^3, \quad X_0 - X_1 = (x_0 - x_1)^3.$$

Now, since $x < y$, we must have $\xi_1 < \eta_1$ and $\xi_2 < \eta_2$

together with $\xi_1 + \xi_2 < \eta_1 + \eta_2$.

Then $\xi_1^3 \leq \eta_1^3$ and $\xi_2^3 < \eta_2^3$, from which $\xi_1^3 + \xi_2^3 < \eta_1^3 + \eta_2^3$;

i.e. $X < Y$.

There is a similar procedure for f^{-1} , and it is clear that f is a non-affine causal automorphism of M_2 .

4.8 The interval topology.

Define $O_{x,y} = \{z: x < z < y\}$, then the collection $\{O_{x,y} : x, y \in M_4\}$ is a base of neighbourhoods for a topology on M_4 , that we call the *interval topology* (Nanda, 1976). We denote the interval topology by $\mathcal{I}$ and the Euclidean metric topology by $\mathcal{E}$; the following two results are those of Nanda (1976).

4.9 Lemma 1.

$$\mathcal{I} = \mathcal{E}$$

Proof:-

(i) We prove first that $\mathcal{I} \subset \mathcal{E}$.

Without loss of generality, we may consider two points a, b such that $a = (T, 0, 0, 0)$ and $b = (-T, 0, 0, 0)$.

Consider any $x \in O_{b,a}$ and the Euclidean ball, centre x , radius ϵ ;

if $||z-x||^2 = \sum_{i=0}^3 (z_i - x_i)^2$, then $B = \{z: ||z-x|| < \epsilon\}$ is the ball.

Now if $y \in B$, $|y_i - x_i| < \epsilon$ for each $i=0,1,2,3$;

thus $y_i = x_i + \lambda_i \epsilon$, where $-1 < \lambda_i < 1$.

We have

$$\begin{aligned} & (y-b) \cdot (y-b) \\ &= (x_0 + T + \lambda_0 \epsilon)^2 - (x_1 + \lambda_1 \epsilon)^2 - (x_2 + \lambda_2 \epsilon)^2 - (x_3 + \lambda_3 \epsilon)^2 \\ &= (x-b) \cdot (x-b) + 2\epsilon [\lambda_0 (x_0 + T) - \lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 x_3] \\ &\quad + (\lambda \cdot \lambda) \epsilon^2. \end{aligned}$$

Also

$$\begin{aligned} & (y-a) \cdot (y-a) \\ &= (x_0 - T + \lambda_0 \epsilon)^2 - (x_1 + \lambda_1 \epsilon)^2 - (x_2 + \lambda_2 \epsilon)^2 - (x_3 + \lambda_3 \epsilon)^2 \\ &= (x-a) \cdot (x-a) + 2\epsilon [\lambda_0 (x_0 - T) - \lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 x_3] + (\lambda \cdot \lambda) \epsilon^2. \end{aligned}$$

We are given that $(x-b) \cdot (x-b) > 0$, $(x-a) \cdot (x-a) > 0$

and $-T < x_0 < +T$; it is clear that, for sufficiently small ϵ , we have

$$(y-b) \cdot (y-b) > 0, (y-a) \cdot (y-a) > 0 \text{ and } -T < y_0 < +T.$$

It follows that $\mathcal{Y} \subset \Xi$.

(ii) To prove that $\Xi \subset \mathcal{Y}$, consider the ball

$$B = \{ z: ||z|| < \delta \}; \text{ let } y \in B \text{ and construct}$$

$0_{b,a}$ for which $b = (y_0 - \epsilon, y_1, y_2, y_3)$; $a = (y_0 + \epsilon, y_1, y_2, y_3)$.

We ask whether, for sufficiently small ϵ , $0_{b,a} \subset B$.
 Let $x \in 0_{b,a}$;

clearly $y_0 - \epsilon < x_0 < y_0 + \epsilon$

$$\therefore x_0 = y_0 + \lambda_0 \epsilon, \text{ where } -1 < \lambda_0 < 1.$$

$$\text{i.e. } |x_0 - y_0| = \mu_0 \epsilon, \text{ where } 0 < \mu_0 < 1.$$

Also

$$\begin{aligned} \sqrt{\sum_{i=1}^3 (x_i - y_i)^2} &< \sqrt{\sum_{i=1}^3 (x_i - a_i)^2} + \sqrt{\sum_{i=1}^3 (a_i - y_i)^2} \\ &< |x_0 - a_0| + |y_0 - a_0| \\ &< 3\epsilon \end{aligned}$$

Now $x = y + (x-y)$

$$\text{so that } ||x|| \leq ||y|| + ||x-y||$$

$$< ||y|| + \sqrt{10}\epsilon.$$

Clearly, for sufficiently small ϵ , $||x|| < \delta$, so that $x \in B$.

We have the result $\mathcal{Y} = \mathcal{X}$.

4.10 Lemma 2.

If $f \in \hat{G}$, then f is a homeomorphism of (M_4, Σ) .

Proof:- Let $k(x) = \{z: (z-x) \cdot (z-x) > 0\}$, and define

$$k^+(x) = \{z \in k(x): x_0 < z_0\},$$

$$k^-(x) = \{z \in k(x): z_0 < x_0\}.$$

Clearly $0_{x,y} = k^+(x) \wedge k^-(y)$, so that

$$f(0_{x,y}) = k^+(fx) \wedge k^-(fy) = 0_{fx,fy};$$

there is a similar result for f^{-1} . Thus f, f^{-1} are both open mappings i.e. f is a homeomorphism.

4.10.1 Corollary

If $f \in \hat{G}$ then f is a homeomorphism of (M_4, Σ) .

Proof:- $f \in \hat{G} \Rightarrow f \in \hat{G}$.

4.11 Lemma 3.

There are no null line triangles in M_4 .

Proof:-

Without loss of generality we may prove that if $0 < \alpha < \beta$, and if $0 < \beta$ then $0, \alpha, \beta$ are points of the same null line. Write $\alpha - \beta = p$, so that $p.p = 0$; now $\beta + p$ is a null vector,

$$\therefore \beta.\beta + 2\beta.p + p.p = 0$$

$$\text{i.e. } \beta.p = 0$$

If β, p are not in the same direction, then

$$\begin{aligned} & |\beta_1 p_1 + \beta_2 p_2 + \beta_3 p_3| \\ & < \sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2} \sqrt{p_1^2 + p_2^2 + p_3^2} \\ & = |\beta_0| |p_0| \end{aligned}$$

$$= \beta_0 p_0$$

$$\therefore \beta \cdot p > 0.$$

Thus $0, \alpha, \beta$ are collinear.

4.12 Lemma 4.

If $f \in \hat{G}$, then f maps every null line onto a null line.

Proof:-

Let N_p be an arbitrary null line through p , and let $q \in N_p$ be such that $p < q$.

Suppose that p, q are fixed and that $\xi \in N_p$ is such that $p < \xi < q$. Since $f \in \hat{G}$ it follows that $fp < f\xi < fq$, and that $fp, f\xi, fq$ are collinear.

4.13 Lemma 5.

If I is the inversion map for $\bar{M}_4$, then $I \notin \hat{G}$.

Proof:-

Consider the null line $N = \{x: x = p + \mu b, \mu \in \mathbb{R}, b \cdot b = 0, b \neq 0\}$.

Now, if $x \in N$,

$$Ix = \frac{p + \mu b}{p \cdot p + 2\mu b} = \frac{p}{p \cdot p} + \frac{\mu [(p \cdot p)b - 2(p \cdot b)p]}{(p \cdot p)[p \cdot p + 2\mu(p \cdot b)]},$$

and there is clearly a singular point on N , where the value of

$$\mu \text{ is } \mu^* = -\frac{p \cdot p}{2p \cdot b}.$$

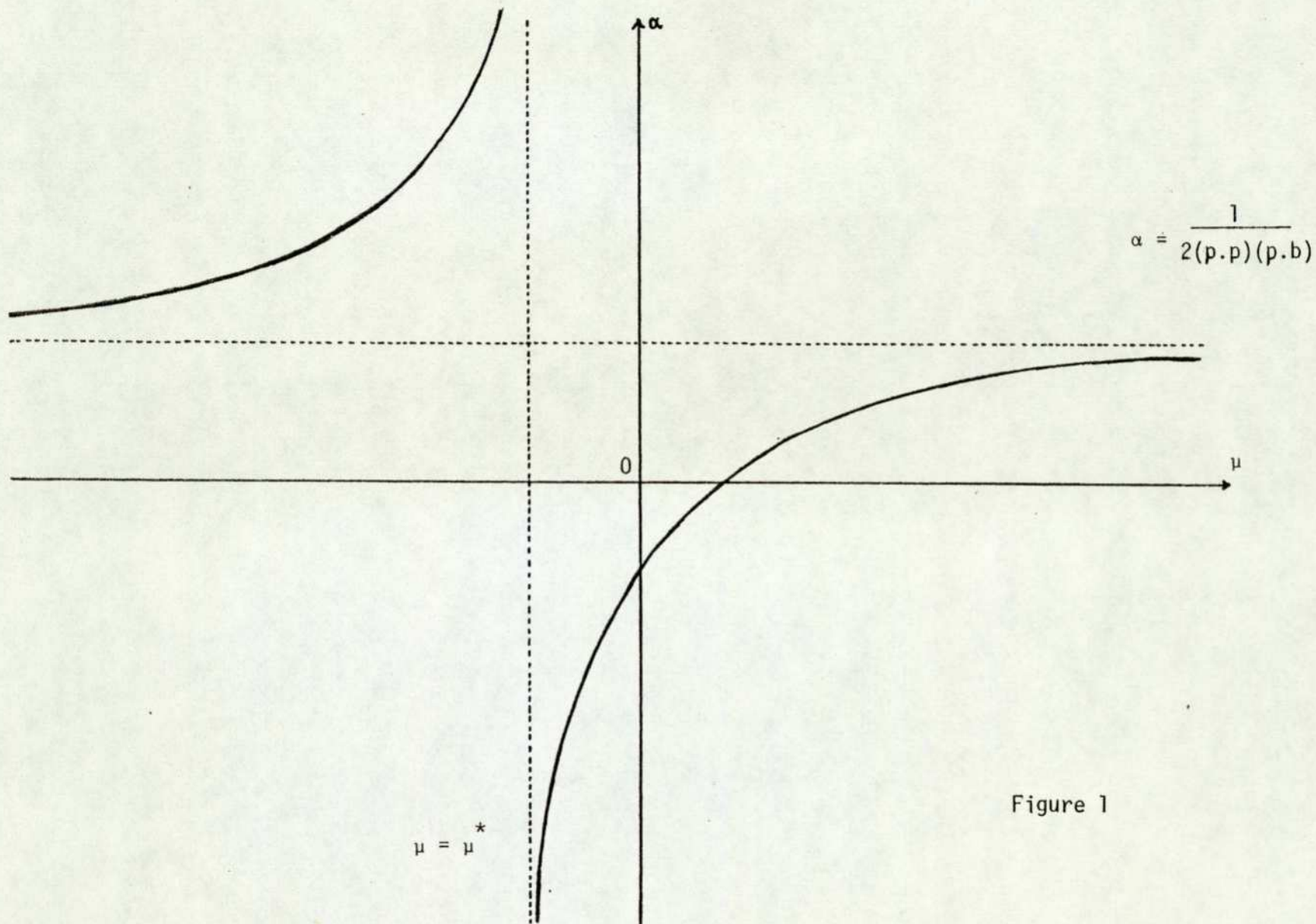


Figure 1

In addition, the expression for Ix has the form

$$Ix = \frac{p}{p.p} + \alpha X$$

where $X = (p.p)b - 2(p.b)p$ is null.

Thus the map of a null line is a null line;

$$\text{in addition } \alpha = \frac{1}{2(p.p)(p.b)} \left[1 - \frac{p.p}{p.p + 2\mu(p.b)} \right].$$

Now p is an arbitrary point on N and we may choose it such that $0 < p$ i.e. $p.p > 0$ and $p_0 > 0$; equally well, without loss of generality, we may consider $0 < b$ i.e. $b.b = 0$ and $b_0 > 0$, in which case $p.b > 0$.

Thus, if $\mu_1 < \mu_2 < \mu^*$, or if $\mu^* < \mu_1 < \mu_2$, then

$\alpha_1 < \alpha_2$, and the relation $<$ is preserved.

On the other hand, if $\mu_1 < \mu^* < \mu_2$, then $\alpha_2 < \alpha_1$, and the relation $<$ is violated.

The behaviour of α as a function of μ is illustrated in diagram 1.

4.14 Theorem A

$$p^{\dagger*} = \hat{G}$$

Proof:

It is clear that $p^{\dagger*} \subseteq \hat{G}$.

Also, if $f \in \hat{G}$, then f is continuous, from lemma 2;
and f preserves null lines, from lemma 4;
from Weyl (1923), it follows that $f \in C$.

Now, if $f = \alpha_1^0 \alpha_2^0 \alpha_3^0 \alpha_4^0 \dots \alpha_N$,

where each α_i is an element of the generating subgroups of dilatation, translation, Lorentz and inversion maps, then we may arrange f thus:

$$f = \lambda \circ \Lambda \circ t_a \circ I \circ t_b \circ I \circ t_c \circ I \dots t_p \circ I \circ t_q$$

where λ, Λ are respectively a dilatation and a Lorentz transformation and $t_a, t_b, \dots, t_q$ are translations.

Consider the image of an arbitrary null line under f ; without loss of generality we may take $q = 0$, and consider the line $N = \{x: x = \xi + \mu\eta, \text{ where } \xi \cdot \xi \neq 0, \eta \cdot \eta = 0, \eta \neq 0\}$. It is clear that, for each $f \in C$, with $b, \dots, p$ non-zero, there is a corresponding network of finitely many singular surfaces.

Thus, I is singular on $K(0)$, (as is $t_p \circ I$),

$I \circ t_p \circ I$ is singular on $K(0) \cup K(-p/p.p)$, (as is $t_j \circ I \circ t_p \circ I$);

if we now write $\Delta_{p,j} = 1 + 2(p.j) + (p.p)(j.j)$, then

$I \circ t_j \circ I \circ t_p \circ I$ is singular on

$K(0) \cup K(-p/p.p) \cup K(-p(j.j)/\Delta_{p,j} - j/\Delta_{p,j})$, and so on.

Now consider the point x^* at which N intersects the cone $K(0)$, and let μ^* be the parameter associated with x^* as in 4.13. Let μ_1, μ_2, μ_3 be close enough to μ^* to ensure that $x_1 < x^* < x_2 < x_3$ but that if S is the union of all singular surfaces for f ,

$$u, v \in S - K(0), u < x_1 < x_3 < v.$$

Thus $I x_2 < I x_3 < I x_1$; also, for every other map of the collection comprising the *non-affine* part of f these relations are preserved.

Case (i): suppose at least one of $b, c, \dots, p \neq 0$ and that Λ is non-orthochronous, then $<$ is violated for the pair x_2, x_3 .

Case (ii): suppose at least one of $b, \dots, p \neq 0$ and that Λ is orthochronous, then $\langle \cdot \rangle$ is violated for the pair x_1, x_2 .

Case (iii): suppose that every $b, \dots, p = 0$ and that Λ is non-orthochronous, then $\langle \cdot \rangle$ is violated for every pair on N . We conclude that every $b, c, \dots, p = 0$ and that Λ is orthochronous, in which case $f \in P^{\uparrow*}$.

4.15 Remark.

We recall that, in §3.8, a Lorentz map f of M_4 was defined to be one which was linear and such that $fx.fy = x.y$, $x, y \in M_4$. It is clear that the requirement of linearity can be weakened, and we have the following result.

4.15.1 Corollary.

If $f: M_4 \rightarrow M_4$ is a bijection for which $fx.fy = x.y$, $x, y \in M_4$, then f is linear and therefore a Lorentz map.

Proof:-

Clearly $f \in C$, now if $f = \alpha_1 \circ \alpha_2 \dots \circ \alpha_N$, and we write

$$f = \lambda \circ \Lambda \circ t_a \circ I \dots t_p \circ I \circ t_q,$$

then, as before, if at least one of $b, \dots, p$ is non-zero f is not bijective.

Thus $f = \lambda \circ \Lambda \circ t_a$, and if $a \neq 0$ or $\lambda \neq 1$, the condition $fx.fy = x.y$ is violated. We are left with $f = \Lambda$.

4.16 Remark.

For $n \geq 3$, the causal automorphisms of M_n are *always* elements of the conformal group. However, for $n = 2$, that is not true; the non-affine *global* causal automorphisms mentioned in §4.7 are not conformal.

For, let $\xi_1 = x_0 + x_1$, $\xi_2 = x_0 - x_1$,

then $x \cdot y = \frac{1}{2}(\xi_1 \eta_2 + \xi_2 \eta_1)$,

$$\text{and } \frac{dx \cdot dy}{\sqrt{dx \cdot dx} \sqrt{dy \cdot dy}} = \frac{\frac{1}{2} [d\xi_1 d\eta_2 + d\xi_2 d\eta_1]}{\sqrt{d\xi_1 d\xi_2} \sqrt{d\eta_1 d\eta_2}}.$$

Thus, if $f: M_2 \rightarrow M_2$ is given by

$$f(x_0, x_1) = (X_0, X_1) = X, \text{ where } X_0 + X_1 = (x_0 + x_1)^3,$$

$$X_0 - X_1 = (x_0 - x_1)^3, \text{ then}$$

$$\frac{dX \cdot dY}{\sqrt{dX \cdot dX} \sqrt{dY \cdot dY}} = \frac{\frac{1}{2} [9\xi_1^2 \eta_2^2 d\xi_1 d\eta_2 + \xi_2^2 \eta_1^2 d\xi_2 d\eta_1]}{3\sqrt{\xi_1^2 \xi_2^2 d\xi_1 d\xi_2} \sqrt{\eta_1^2 \eta_2^2 d\eta_1 d\eta_2}}.$$

4.17 Remark.

It is clear that the globally causal geometry for M_4 is the *affine* geometry associated with P^{t*} , but there are still questions to be asked. The physical relevance of the larger group C is clear, for it is the group for which Maxwell's equations are invariant (Cunningham, 1910; Bateman, 1910.) Yet C is not appropriate when global causality is involved.

Is the requirement of global causality a reasonable one, from a physical point of view? Certainly our observations of the universe are markedly local, however we may choose to extrapolate them; can we envisage, then, a *local* causality?

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Chapter 5. The locally causal geometry.

5.1 Perspective.

In this chapter we confine our attention to the conformal geometry of Minkowski space and investigate the behaviour of certain hyperbolic hypersurfaces under conformal maps ; as far as interpretation in terms of kinematical concepts is concerned, we pursue that matter in Chapter 6. We shall also explain exactly in what sense the special conformal maps are "locally causal preservers", again, as a geometrical concept, rather than as a physical one.

Gamba and Luzzatto (1964) in a rapid addendum to Zeeman's paper (1964), were quick to point out that for the map of M_4 known as an *inversion*, causality is violated ; their conclusion was then that the *non-affine* maps which were elements of the conformal group were not allowable in physics because they did not even preserve *causality* locally. However, in response, Flato and Sternheimer (1966) warned that the conclusion was hasty, and, indeed, invalid, and pointed out (as did Kastrup (1966), referring to Wess (1960)) that for special conformal maps (see below) there is a region of local causality preservation. We shall define the concept "locally causal at the origin" in such a way that it admits a composition group and show that the group is that generated by dilatations, orthochronous Lorentz maps and special conformal maps.

5.2 The special conformal group

We recall that the collection of maps of the type $I \circ t_a \circ I$ comprises the *special conformal group*, S .

5.2.1 The singular cones.

It is clear that $f \in S$ iff $fx = \frac{x + (x.x)a}{1 + 2(a.x) + (x.x)(a.a)}$,

provided that $x.x \neq 0$ and $1 + 2(a.x) + (x.x)(a.a) \neq 0$.

Now, if $a.a \neq 0$

$$1 + 2(a.x) + (x.x)(a.a) = (a.a) \left[x + \frac{a}{a.a} \right] \cdot \left[x + \frac{a}{a.a} \right],$$

so that f is singular on $K(0) \cup K\left(-\frac{a}{a.a}\right)$,

where $K(x) = \{y : (y - x).(y - x) = 0\}$ is the *null cone* at x .

5.2.2 Theorem 1.

S is an abelian group.

Proof:-

Let $\phi_a, \phi_b \in S$, and consider the collection of points of M_4 for which $\phi_a \circ \phi_b$ is non-singular,

$$\text{then } \phi_a = I \circ t_a \circ I ; \quad \phi_b = I \circ t_b \circ I$$

$$\begin{aligned} \therefore \phi_a \circ \phi_b &= I \circ t_a \circ I \circ I \circ t_b \circ I \\ &= I \circ t_{a+b} \circ I \\ &= \phi_b \circ \phi_a \end{aligned}$$

Also ϕ_{-a} has the properties

$$\phi_a \circ \phi_{-a} = \phi_{-a} \circ \phi_a = 1_{M_4}.$$

5.3 Definition : locally causal maps.

If $\phi : \hat{M}_4 \rightarrow \hat{M}_4$ is such that for $A \subset \bar{M}_4$ and $\forall x, y \in A$ for which $x < y$, then $\phi x < \phi y$, ϕ is said to be a *locally causal map* of $\bar{M}_4$ with *local causality region* A .

5.3.1 Remark.

It is clear from § 4.13 that each special conformal map is a global causality violator ; we now show that there is a local causality region associated with each $f \in S$.

5.4 Theorem 2.

If $f = I \circ t_a \circ I \in S$ then f is a locally causal map with local causality region $O_{0,-a/a,a}$.

Proof:-

Consider $x, y \in M_4$ and let $x' = fx, y' = fy$;
write $\Delta_x = 1 + 2(a.x) + (x.x)(a.a)$.

$$\text{Then } y' = \frac{y + (y.y)a}{\Delta_y}, \quad x' = \frac{x + (x.x)a}{\Delta_x}$$

$$\text{Now } (y' - x') \cdot (y' - x')$$

$$= \Delta_x^{-2} \Delta_y^{-2} \left[\Delta_x^2 (y.y) \{1 + 2(y.a) + (y.y)(a.a)\} + \Delta_y^2 (x.x) \{1 + 2(x.a) + (x.x)(a.a)\} \right. \\ \left. - 2\Delta_x \Delta_y \{(y.x) + (x.x)(a.y) + (y.y)(a.x) + (x.x)(y.y)(a.a)\} \right]$$

$$= \frac{y.y}{\Delta_y} + \frac{x.x}{\Delta_x} - 2 \frac{\{(y.x) + (x.x)(a.y) + (y.y)(a.x) + (x.x)(y.y)(a.a)\}}{\Delta_x \Delta_y}$$

$$= \frac{y.y - 2y.x + x.x}{\Delta_x \Delta_y}$$

$$= \frac{(y-x) \cdot (y-x)}{\Delta_x \Delta_y}$$

Let $y = x + \theta$, and suppose that $x < y$, so that $0 < \theta$.

It is clear that $\Delta_y = \Delta_x [1 + R_1]$,

where $|R_1| < p\theta_0$ and p depends on a, x ; $p > 0$.

Similarly $\Delta_y^{-1} = \Delta_x^{-1} [1+R_2]$,

where $|R_2| < q\theta_0$ and q depends on a, x ; $q > 0$

Thus $(y'-x').(y'-x') = \Delta_x^{-2} [1+R_2] (y-x).(y-x)$, where $|R_2| < q\theta_0$.

In addition,

$$\begin{aligned} y'_0 - x'_0 &= [y_0 + (y.y)a_0]\Delta_y^{-1} - [x_0 + (x.x)a_0]\Delta_x^{-1} \\ &= [y_0 + (y.y)a_0]\Delta_x^{-1} [1+R_2] - [x_0 + (x.x)a_0]\Delta_x^{-1} \\ &= (y_0 - x_0)\Delta_x^{-1} + a_0[2(\theta.x) + (\theta.\theta) + (y.y)R_2]\Delta_x^{-1} \\ &= \Delta_x^{-1}(y_0 - x_0) + R_3 , \quad \text{where } |R_3| < r\theta_0 \end{aligned}$$

and r depends on a, x ; $r > 0$.

Thus, for sufficiently small $\theta_0 > 0$, timelike displacements map to timelike displacements ; however, if Δ_x changes sign $y'_0 - x'_0$ has the opposite sign to $y_0 - x_0$, so that causality is violated.

We now look at the boundary $K(0)$; consider the inversion map I , and let $x' = Ix$,

then $(y'-x').(y'-x')$

$$\begin{aligned} &= \frac{y.y}{(y.y)^2} - \frac{2x.y}{(x.x)(y.y)} + \frac{x.x}{(x.x)^2} \\ &= \frac{1}{y.y} - \frac{2x.y}{(x.x)(y.y)} + \frac{1}{x.x} \\ &= \frac{(x-y).(x-y)}{(x.x)(y.y)} \end{aligned}$$

Thus, if $x.x > 0$ and $y.y < 0$ but $(y-x).(y-x) > 0$, then

$$(y'-x').(y'-x') < 0$$

so that a timelike segment is mapped to a spacelike one, and causality is violated.

Now consider $t_a \circ I$, with the same configuration for x, y causality is similarly violated ; suppose that

$$x'' = t_a \circ I x \quad \text{and} \quad y'' = t_a \circ I y$$

$$\text{then } (Iy'' - Ix'').(Iy'' - Ix'') = \frac{(y'' - x'').(y'' - x'')}{(x'' \cdot x'')(y'' \cdot y'')} .$$

If the segment $y'' - x''$ does not intersect $K(0)$, the spacelike quality is retained for $Iy'' - Ix''$. Thus, it is clear that $K(0)$ is the other element of the causality violation boundary, and following Wess (1960), the singular cones $K(0)$ and $K(-a/a.a)$ partition M_4 into nine regions, each of which is a local causality region for f .

In particular $A = {}^0_{0,-a/a.a}$ is a local causality region for f .

5.5 The singular cone causality grid.

The accompanying figures , 2 and 3, are from Wess (1960); in figure 2, the broken lines indicate the cone $K(-a/a.a)$, in figure 3, the broken lines indicate $K(+a/a.a)$; the unbroken lines indicate $K(0)$ in both diagrams. The numerals indicate corresponding regions of M_4 under the map $I \circ t_a \circ I$; each region is a local causality region either for $f = I \circ t_a \circ I$ or for $f^{-1} = I \circ t_{-a} \circ I$. Bateman (1910) concluded that, if Maxwell's equations are to be invariant, there is a certain class of bijections of a certain domain of M_4 for which the invariance holds, the bijections being elements of C ; the domain in question is the complement of

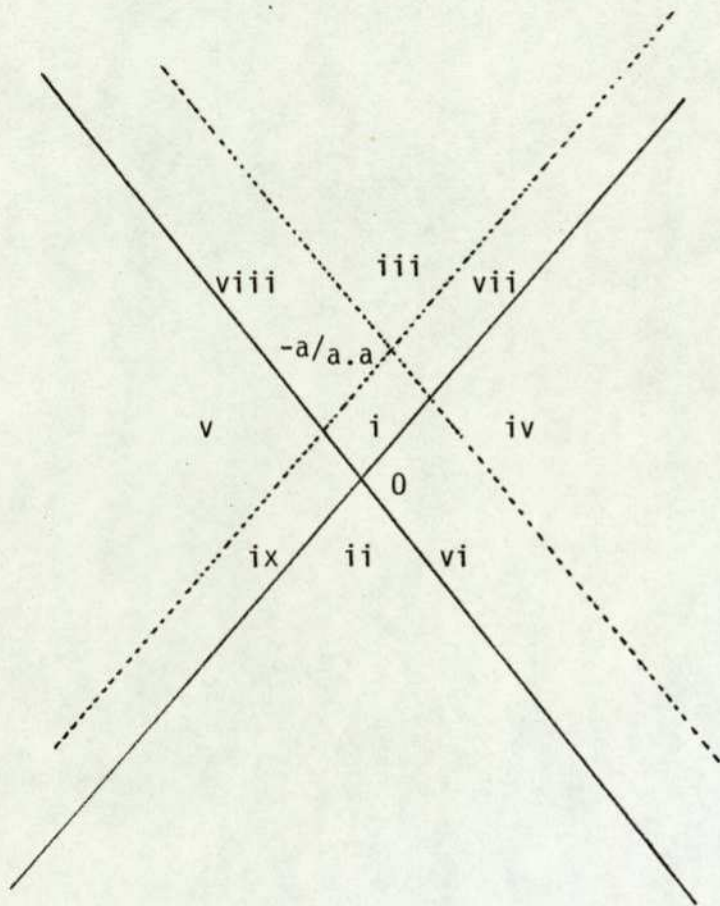


Figure 2

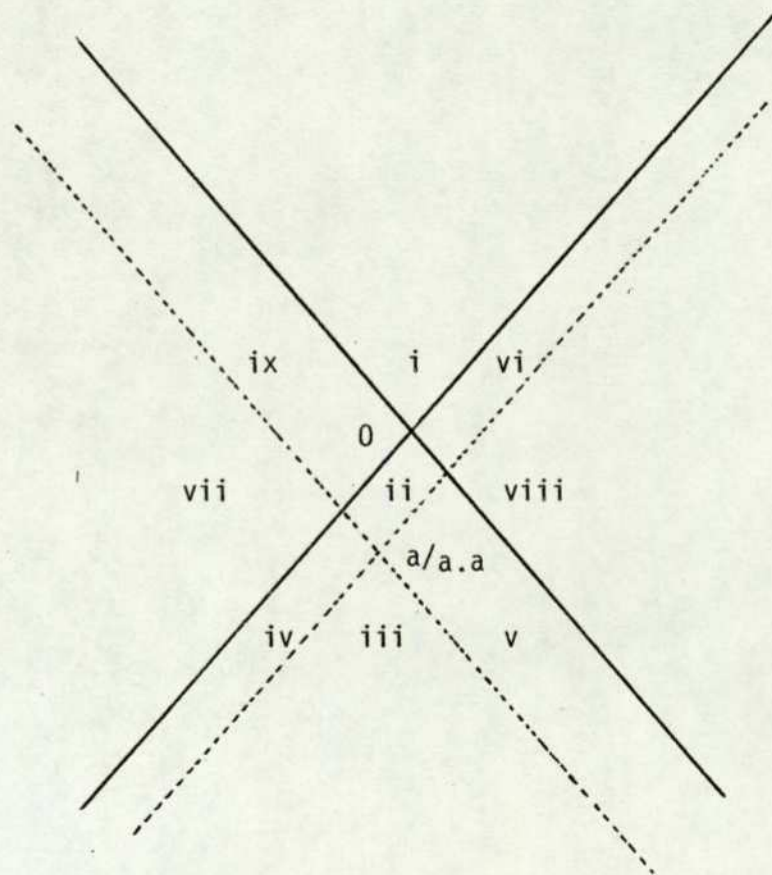


Figure 3

$$K(0) \cup K(-a/a.a) \cup K(+a/a.a).$$

5.6 Local bijections of M_4 and local causality preservation.

In Chapter 4, we were concerned with maps of M_4 that were globally bijective (i.e. maps for which both f, f^{-1} were defined at each point of M_4) and globally causal (i.e. both f, f^{-1} preserved $<$ or $<..$).

Even at that stage, a wider class of maps, the conformal maps, became involved in the discussion. As far as the bijective quality for conformal maps is concerned there are now *domains* of M_4 naturally associated with special conformal maps ; equally, those domains are associated with the preservation of $<$ and $<..$. We may properly refer to these maps as *locally causal automorphisms* in the following manner.

5.7 Definition : locally causal automorphism at 0.

If $\phi : M_4 \rightarrow M_4$ is such that

(i) $\phi 0 = 0$,

(ii) if $A = M_4 - \bigcup_1 X_i$, where $X_i = K^*(0) \cup K(-a_i) \cup K(a_i)$,

$i = 1, 2, \dots, n$, with, for each i , $0 < a_i$;

and $K^*(0) = \{x : x.x = 0, x \neq 0\}$, then

$\phi : A \rightarrow A$ is a *bijection*,

(iii) for each i , $K(0) \cup K(-a_i); K(0) \cup K(a_i)$ partition

M_4 into nine regions $C_1, C_2, \dots, C_9$; $C'_1, C'_2, \dots, C'_9$ in the manner of §5.5 such that

$x, y \in C_j$ with $x < y \iff \phi x, \phi y \in C'_j$ with $\phi x < \phi y$,

then ϕ is said to be a *locally causal automorphism* at 0.

5.7.1 Remark.

It is clear that $A = [k^-(0) \cap \bigcap_i k^+(+a_i)] \cup [k^+(0) \cap \bigcap_i k^-(-a_i)]$ is a local causality region for both ϕ, ϕ^{-1} in the sense of §5.3.

5.8 Remark.

It would be possible to define, as in §5.3, three varieties of locally causal map according to whether $<, <.,$ or $\ll$ was to be preserved. Whatever variety of causal map is under consideration, Theorem 2 of §5.4 is still true, and the singular cones still partition M_4 into regions where each one of the relations $<, <.,$ and $\ll$ is preserved. Thus a locally causal automorphism at 0 regarding $<, <.,$ which we defined in §5.7 will clearly be a locally causal automorphism at 0 regarding both $<.,$ and $\ll$. (definition understood).

5.9 Definition : locally causal automorphism group.

If ϕ, θ are both locally causal automorphisms, with $A_\phi = M_4 - [K^*(0) \cup K(a_1) \cup K(b_1)]$ and $A_\theta = M_4 - [K^*(0) \cup K(a_2) \cup K(b_2)]$, then $\phi \circ \theta$ and $\theta \circ \phi$ are defined on $A_\phi \cap A_\theta$, and each is locally causal at 0 with local causality region $A_\phi \cap A_\theta$.

Clearly, under composition, the collection is a group, for $\phi \circ (\theta \circ \psi)$ and $(\phi \circ \theta) \circ \psi$ are defined on the same subset of M_4 , the local causality region for each is the "smallest" of A_ϕ, A_θ, A_ψ , and $\phi \circ (\theta \circ \psi) = (\phi \circ \theta) \circ \psi$.

Also $\phi \circ 1_{M_4} = 1_{M_4} \circ \phi = \phi$, $\forall \phi$

and $\phi \circ \phi^{-1} = \phi^{-1} \circ \phi = 1_{M_4}$,

where 1_{M_4} is the restriction of the identity map to A_ϕ .

5.10 Remark.

It is natural to ask whether we can establish the nature of the locally causal automorphism group in a similar way to that of Zeeman (1964) for the globally causal automorphism group. It is clear that the locally causal automorphism group is exactly that generated by special conformal transformations, orthochronous Lorentz maps and dilatations.

5.11 Theorem B.

If G_0 is the group of locally causal automorphisms at 0, and C_0 is the group generated by special conformal maps, orthochronous Lorentz maps and dilatations, then

$$G_0 = C_0.$$

Proof :-

It is clear from §§5.4 and 5.5 that $C_0 \subseteq G_0$. Suppose now that $\phi \in G_0$ and that A is the bijective region for ϕ .

If $x, y \in A$ are any two points such that $x < y$, then the segment of the null line through x and y is mapped onto the null line segment through ϕx and ϕy .

From Weyl (1923), it follows that

$$\phi \in C.$$

We pursue the method of Theorem A, §4.14; write

$$\phi = \lambda \circ \Lambda \circ t_a \circ I \circ t_b \circ I \circ \dots \circ t_k \circ I \circ t_p \circ I \circ t_q$$

and assume that none of $a, b, c, \dots, p, q$ is zero.

There are two possibilities; we may group the generators to obtain

$$(i) \quad \phi = \lambda \circ \Lambda \circ t_a \circ \theta_b \circ t_c \circ \theta_d \dots \circ t_k \circ \theta_p \circ t_q$$

where $\theta_i = I \circ t_i \circ I$

or we may obtain

$$(ii) \quad \phi = \lambda \circ \Lambda \circ t_a \circ I \circ t_b \circ \theta_c \circ t_d \dots \circ t_k \circ \theta_p \circ t_q.$$

We may exclude (ii) because, here, $\phi 0 = \omega$ (a point on the hyperplane at infinity). Now, in (i), if Λ is non-orthochronous, there is no local causality region for ϕ and ϕ^{-1} in the vicinity of 0. Further, since $\phi 0 = 0$, we must have $a + c + \dots + k + q = 0$. Thus ϕ is a member of an equivalence class associated with a map generated by S and $P^{\uparrow*}$. In that sense, we may write $G_0 = C_0$.

5.12 The locally causal geometry.

It is now possible to speak of the geometry associated with the local causality group ; it is clear that, since the group is non-affine, lines, planes and hyperplanes in M_4 are not, in general, invariant figures. We shall find, however, that there is a class of hypersurfaces and hyperplanes which is invariant under the local causality group.

5.13 Definition : hyperbolic hypersurface.

A hypersurface with equation $x.x = x.b$, where b is a constant vector, is said to be *hyperbolic*. We shall confine our attention to those hypersurfaces for which $b.b = -k^2$, $k \in \mathbb{R}$.

5.14 Invariance of the hyperbolic hypersurfaces under linear transformation.

It is clear that hypersurfaces $\{x : x.x = -x.b, b.b = -k^2, k \in \mathbb{R}\}$ are mapped onto each other by elements of the group L^* , generated by the Lorentz transformations and dilatations. We notice also that if Λ is a non-orthochronous Lorentz map then the part of the hypersurface for which $x_0 > 0$ is mapped to the part for which $(\Lambda x)_0 < 0$.

It follows that the maximal *linear* group which preserves *both* the form of the hypersurface and the sign of x_0 is that generated by the orthochronous Lorentz maps and dilatations, $L^{\uparrow*}$.

5.15 Invariance under special conformal transformation.

Theorem 3

The special conformal map $f_a = I \circ t_a \circ I$ maps the set $\{x : x.x = x.b, \Delta_{x,a} \neq 0, b.b = -k^2\}$ onto $\{x' : x'.x' = x'.\beta, \text{ where } \beta = \frac{b}{1+a.b}\}$, provided that $a.b \neq -1$.

Proof:

We write $\Delta_{x,a} = 1 + 2(x.a) + (x.x)(a.a)$,

$$\text{then } \Delta_{x,a} x'.x' = x.x$$

$$\text{and } \Delta_{x,a} x'.b = x.b + (x.x)(a.b).$$

Thus, given that $a.b \neq -1$, the points of $x.x = x.b$ for which $\Delta_{x,a} \neq 0$ are mapped onto

$$x'.x' = x'.\beta$$

$$\text{where } \beta = \frac{b}{1+a.b}, \text{ as required.}$$

In addition, $\beta \cdot \beta = \frac{b \cdot b}{(1+a \cdot b)^2} < 0$.

5.16 Remark.

If we choose a such that $a \cdot b = -1$, then the set $\{x : x \cdot x = x \cdot b, \Delta_{x,a} \neq 0\}$ is mapped onto the hyperplane $\{x' : x' \cdot b = 0\}$. It is clear, conversely, that every hyperplane $\{x : x \cdot b = 0, b \cdot b = -k^2\}$ is mapped onto $\{x' : x' \cdot b' = 0, b' \cdot b' = -k^2\}$ under a Lorentz transformation, with a similar preservation for dilatations.

It is also easy to see that $\{x : x \cdot b = 0, b \cdot b = -k^2\}$ is mapped onto $\{x' : x' \cdot x' = \frac{x' \cdot b}{a \cdot b}\}$ under special conformal transformation, provided that $a \cdot b \neq 0$ and to $\{x' : x' \cdot b = 0, b \cdot b = -k^2\}$ under special conformal transformation for which $a \cdot b = 0$.

5.17 Notation.

We shall refer to the collection of all hyperbolic hypersurfaces and all homogeneous hyperplanes as H , and to those parts of the ingredients of H for which $x_0 \geq 0$ as $H^\uparrow$. We now prove that the maximal invariance group for the collection $H^\uparrow$ which affords *local* preservation for $x_0 \geq 0$ is that generated by the orthochronous Lorentz maps, the special conformal maps and dilatations.

5.18 Theorem C.

The maximal invariance group for local preservation of the collection $H^\uparrow$ is that generated by S_4 and $L^{\uparrow*}$, i.e. C_0

Proof:

Let W be the maximal invariance group for local preservation of $H^\uparrow$.

Clearly $C_0 \subseteq W$.

In addition, every element of W preserves the asymptotic hypersurface $x \cdot x = 0$, thus, from Bateman (1909), we have $W \subseteq C_0$.

5.19 Remark.

In the next chapter we shall associate hyperbolic lines in M_2 with the histories of uniformly accelerated material particles in relativistic kinematics ; it is inevitable that we shall lose the exact identity between the invariance group for the hyperbolic lines (the conformal group) and the local causality group (which, for M_2 , contains non-conformal maps).

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Chapter 6 Aspects of Space-Time kinematics.6.1 Perspective.

Since the work of Bateman and Cunningham (1909) (1910a) (1910b) it has been known that the conformal group is the largest covariance group for the comparison of observations concerning electromagnetic phenomena. At about the same time Born (1909) and Sommerfeld (1910) made a detailed study of relativistic uniformly accelerated motion, a type of motion which is also referred to as "hyperbolic" because of the hyperbolic world lines associated with it in Minkowski space. Both Born (1909) and, soon after, Noether (1910) and Hassé (1912) were concerned with phenomena concerning uniformly accelerated electrons, and Hassé (1912) used conformal maps in his treatment of the problem. Later, Page (1936) attempted to establish a theory based on a more general "principle of relativity" than the special principle, admitting as equivalent the class of mutually uniformly accelerated observers ; he, too, found that C_4 was naturally invoked. In Chapter 7 we shall investigate just such an augmented principle of relativity and comment on its usefulness. However, to return to the historical thread, Hill (1945) (1947) reiterated the association of uniform acceleration with the special conformal group by finding the generators of the group for which a certain characteristic differential equation is invariant. More recently a paper of Fulton, Rohrlich and Witten (1962) contains a discussion of the means of comparison of observations from pairs of special, conformal and general relativistic observers ; their paper starts in the realms of dynamics, which is beyond the scope of this thesis, and invokes the "principle of equivalence" which is also beyond our scope.

In Chapter 5, we have seen how the special conformal transformations and $L^{\dagger*}$ generate the locally causal automorphism group. We have also considered the same group's appropriateness for dealing with hyperbolic hypersurfaces. In particular, for M_2 , we know of the divergence between the local causality group which contains non-linear, non-conformal maps and the line/hyperbola invariance group, which is precisely that generated by S_2 and $L^{\dagger*}$. In this chapter we are concerned with the kinematics to be associated with the collection of straight lines and hyperbolic arcs through a point of M_2 and a discussion of the relative priority of the principles of local causality and special relativity will then follow in Chapter 7.

6.2 Kinematics and Minkowski space.

From the time of Minkowski's address to the Assembly of German Natural Scientists at Cologne on 21 September 1908 (Minkowski, 1952; see also Minkowski : 1908, 1909), at which he proposed a new structure "the space-time world" to model relativistic physics, it has been accepted that what we now call *Minkowski space* is indeed tailored to the needs of special relativity. Now, if each observer, whether or not "absolutely accelerated", is entitled to his own Minkowski space, based on *his* causality, and if we have a principle of local causality to provide an invariance group, then what is the status of an "inertial" observer, a matter which has so far not been discussed in our geometrical system ?

So far, if, in the estimation of any observer, a set of points in his Minkowski space constitutes a continuous path (in the R^4 topology) then he refers to the set as a *world line* or *history*.

Thus, if the observer registers a *straight* world line in his M_4 , he associates it with either an unaccelerated material particle or a photon, thus Robb (1936 : 19) " we are able to say in terms of these relations what we mean by a particle being unaccelerated". At this stage of the development, it is not clear why he could not have as well written *relatively* unaccelerated, and the question turns on the concepts of *inertial frame* and *observer*, and why they are essential parts of the theory.

Perhaps it should be remarked that the concepts *are* endemic in Reichenbach, (1969 : 29-54) but not clearly apparent in Robb (1936), Carnap (1958) or Schutz (1973,1979). We choose to avoid defining *inertial frame* until it is forced on us, which indeed it is, by the "clock paradox" and we reserve further discussion until that point is reached. For the moment, we contend that if an observer sees a world line in his M_4 which is *not straight*, he will register a (relative) acceleration for that history.

In this chapter we are wholly concerned with M_2 , and its role in modelling one-dimensional motion, particularly regarding *uniform* (relative) acceleration. Later, in Chapter 7, still concerned with M_2 , the newly developed method of Chapter 5 for mapping to a zero (relative) acceleration system is used to calculate the discrepancy in clock readings due to two mutually uniformly accelerated observers. It is at this stage that we are forced to select a category of reference systems called *inertial frames*.

6.3 Definitions.

For a particle in one dimensional motion we shall identify three systems of coordinates :-

- (i) a system in which the particle is instantaneously at rest is called a *rest system* (Hill, 1945),
- (ii) a system in which the particle has zero acceleration instantaneously is called a *zero acceleration system* (Briginshaw, 1980),
- (iii) a system in which, instantaneously, the particle has zero acceleration and is also at rest is called a *proper system* (Hill, 1945).

At this stage we make no claim of privilege for any of the three coordinate systems ; indeed none is to be regarded as an "inertial system" for the time being.

6.4 The class of hyperbolic world lines in M_2 .

In M_2 , the world line $x.x = x.b$, with $b.b = -k^2$ and $x_0 \geq 0$ is the history of a particle which is accelerated relative to the coordinate system observer. In addition, the particle passes through 0 with speed $u = \frac{b_0}{b_1}$.

Thus, if $b_0 \neq 0$, we may consider the Lorentz transformation $X = \Lambda x$, given by

$$\begin{aligned} X_0 &= \frac{x_0}{\sqrt{1-u^2}} - \frac{u}{\sqrt{1-u^2}} x_1, \\ X_1 &= \frac{x_1}{\sqrt{1-u^2}} - \frac{u}{\sqrt{1-u^2}} x_0; \end{aligned}$$

for which the map of $x.x = x.b$ is $X.X = X.B$, where $B_0 = 0$ and $B_1 = \sqrt{1-u^2} b_1 \neq 0$. This is exactly the form of the world line in Pauli (1958 : 74 - 76) appropriate to *uniformly accelerated motion in special relativistic kinematics*. We shall assume that, whatever the status of the Minkowski coordinate system and observer, such a world line is associated with an important type of relatively

accelerated motion. By analogy with Pauli (1958 : 74 - 76) we shall refer to this type of motion as *uniformly accelerated* ; it is understood, however, that the motion is relative to a given Minkowski coordinate system whose status ("inertial" or "non-inertial") is not specified.

Using the same analogy we shall introduce a number K , given by

$$\frac{1}{K^2} = \frac{1}{4} B_1^2 \doteq \frac{1}{4} k^2, \text{ and speak of it as the}$$

magnitude of the relative uniform acceleration.

In the following paragraphs, prior to § 7.3.2, these conventions operate, and we do not imply (until then) uniform acceleration in the special relativistic sense.

However, two special conformally related observers will agree that a particle is uniformly accelerated, but they will not agree on the magnitude of the uniform acceleration.

For, if $x' = I \circ t_a \circ I x$, then the map of $x.x = x.b$ is

$$(\text{locally}) \quad x'.x' = x' \cdot \frac{b}{1+a.b}, \text{ in general. In the frame of } 0',$$

the particle passes $0'$ with speed $\frac{b_0}{b_1}$, as before, but the

magnitude of the uniform acceleration is now

$$K' = \frac{2}{k} (1+a.b) = K(1+a.b).$$

6.4.1 Alternative form for the special conformal map

It is often convenient to write the special conformal map $x' = fx = I \circ t_a \circ I x$ in a varied form, replacing a by $\frac{\alpha}{\alpha \cdot \alpha}$. If $a \cdot a \neq 0$ this is always possible, for the requirements that $a = \frac{\alpha}{\alpha \cdot \alpha}$ is satisfied by the choice $\alpha = \frac{a}{a \cdot a}$.

Thus, we may always consider a special conformal map in the form

$$f = I \circ t_{\alpha/\alpha \cdot \alpha} \circ I$$

in which case

$$fx = \frac{(\alpha \cdot \alpha)x + (x \cdot x)\alpha}{\alpha \cdot \alpha + 2x \cdot \alpha + x \cdot x}$$

6.5 The zero acceleration and proper systems.

If $x \cdot x = x \cdot b$ is a given world line we may obtain a zero-acceleration system by using the special conformal transformation

$$x = \frac{(b \cdot b)x - (x \cdot x)b}{b \cdot b - 2x \cdot b + x \cdot x} \quad (\text{subject to the usual non-singularity restrictions})$$

which gives a straight line history in the new system,

$$X.b = 0.$$

Thus, in the new system, the particle has constant speed $\frac{b_0}{b_1}$, and zero acceleration.

In addition, the proper system is obtained by applying the appropriate orthochronous Lorentz transformation to the zero acceleration system.

6.6 Remarks.

6.6.1 Unbounded uniform acceleration

It is clear from § 6.4 that super light speeds are not attained for an unbounded uniform acceleration, since the limiting situation as $K \rightarrow \infty$ is the light cone $N_{\frac{1}{2}b} = \{x : (x - \frac{1}{2}b) \cdot (x - \frac{1}{2}b) = 0\}$ as a degenerate hyperbola.

6.6.2 Zero uniform acceleration

We now examine the limiting case $K \rightarrow 0$; let $x.x = x.b$ be the usual hyperbolic world line, and let $u = \frac{b_0}{b_1}$ be the speed with which the particle passes 0.

From $x_0^2 - x_1^2 = b_0 x_0 - b_1 x_1$, we obtain

$$\frac{x_0^2 - x_1^2}{b.b} = \frac{b_0 x_0}{b.b} - \frac{b_1 x_1}{b.b},$$

and if we let $k^2 = -b.b \rightarrow \infty$ in such a way that $b_0/b.b / b_1/b.b \rightarrow u$, then we obtain $x_1 = u x_0$ as the limiting equation of the world line, corresponding to the case of zero uniform acceleration.

6.6.3 Degenerate form of the zero acceleration map for vanishing relative acceleration.

The system in which the particle appears to be uniformly accelerated is traditionally referred to as an "inertial system" or "frame", neither concept of which we have so far defined. However, looking forward to such a definition, and the selection of a certain class of systems for the purposes of reference, we show that as the relative acceleration associated with the world line $x.x = x.b$ approaches zero, so the map to the zero acceleration system approaches the identity map.

This is clear, for consider

$$x' = \frac{(b.b)x - (x.x)b}{b.b - 2x.b + x.x}$$

where $k^2 = -b.b$ measures the reciprocal of the magnitude of the uniform acceleration relative to the reference system. Thus $K \rightarrow 0$ is effected by letting $b.b \rightarrow -\infty$, and the fact of the hyperbolic arc "straightening out" implies that both $b_0, b_1 \rightarrow -\infty$, too.

Now

$$x' = \frac{x - (x.x) \frac{b}{b.b}}{1 - \frac{2x.b}{b.b} + \frac{x.x}{b.b}}$$

and, by hypothesis $u = \frac{b_0}{b_1} < 1$.

Clearly

$$\frac{b_0}{b_0^2 - b_1^2} = \frac{b_0}{\left(1 - \frac{1}{u^2}\right)b_0^2} = \frac{u^2}{(u^2 - 1)b_0} \rightarrow 0$$

$$\frac{b_1}{b_0^2 - b_1^2} = \frac{b_1}{(u^2 - 1)b_1^2} = \frac{1}{(u^2 - 1)b_1} \rightarrow 0.$$

Thus, in the limit that $b \cdot b \rightarrow -\infty$, the map degenerates to $x' = x$.

Since the invariance group for the line/hyperbola collection is, from § 5.18 Theorem C, that generated by positive dilatations, the orthochronous Lorentz group and the special conformal group, we conclude that, in kinematical terms, the degenerate form of the map from one system to another which has zero relative uniform acceleration is an augmented orthochronous Lorentz transformation.

6.7 Remark.

In the kinematics of one dimensional motion, modelled by M_2 , it is the nature of the group C_0 , that generated by $S_2, L^\dagger$ and dilatations to preserve the local *uniformity* of acceleration, the whole group does not itself preserve *zerness* and *non-zerness*. To preserve zerness and uniformity, we are confined to the subgroup $L^{\dagger*}$, and the progress to the zero uniform acceleration state is entirely consistent with that fact, as we have seen in § 6.6.3.

We also know that, whereas C_0 is included in the local causality group for M_2 , it is now a *proper* subgroup of that *local causality group* (since there are *globally* causal maps of M_2 which are not conformal maps). However, this is a matter of little importance ; what is more important is the extent to which physical limitations are imposed by the presence of the singular cones ; we investigate this matter in § 6.9.

6.8 Interpretation of a.

6.8.1 Theorem 1.

If (x_0, ξ_1) , with ξ_1 fixed, is the representative point of a

given straight world line in the coordinate system associated with 0,
then the map of the line under

$$x' = \frac{x + (x.x)a}{1+2(x.a)+(x.x)(a.a)}$$

is the hyperbola $(x'_0 - \beta_0)^2 - (x'_1 - \beta_1)^2 = -\mu^2$

$$\text{where } \beta_0 = 2\mu\xi_1 a_0 ; \beta_1 = \mu(2a_1\xi_1 - 1)$$

$$\text{and } \mu^{-1} = 2[(a.a)\xi_1 + a_1].$$

Proof :

We have $x' = fx$, where $f = I \circ t_a \circ I$

First inversion : write $y = Ix = \frac{x}{x.x}$

$$\text{i.e. } y_0 = \frac{x_0}{x.x} ; y_1 = \frac{\xi_1}{x.x}$$

$$\text{also } y.y = \frac{1}{x.x} = \frac{y_1}{\xi_1}$$

locus is $y_0^2 - y_1^2 - \frac{y_1}{\xi_1} = 0$, which is an hyperbola.

Translation : write $y' = t_a \circ Ix = t_a y = y+a$

$$\text{locus is } (y'_0 - a_0)^2 - (y'_1 - a_1)^2 - \frac{(y'_1 - a_1)}{\xi_1} = 0$$

$$\text{i.e. } (y'_0)^2 - (y'_1)^2 - 2a.y' + a.a - \frac{y'_1}{\xi_1} + \frac{a_1}{\xi_1} = 0$$

Second inversion : write $x' = Iy' = \frac{y'}{y'.y'}$

$$\text{so that } x'.x' = \frac{1}{y'.y'}$$

$$\text{and } a.y' = (y'.y')a.x' = \frac{a.x'}{x'.x'}$$

Thus the locus is

$$\frac{1}{x'.x'} - \frac{2a.x'}{x'.x'} + a.a - \frac{x'_1}{\xi_1(x'.x')} + \frac{a_1}{\xi_1} = 0$$

$$\text{i.e. } (x'.x') \left[a.a + \frac{a_1}{\xi_1} \right] - 2a.x' - \frac{x'_1}{\xi_1} + 1 = 0$$

which is an hyperbola.

Write $\mu^{-1} = 2[(a.a)\xi_1 + a_1]$, then we have

$$\frac{x'.x'}{2\xi_1\mu} - 2a_0x'_0 + 2a_1x'_1 - \frac{x'_1}{\xi_1} + 1 = 0$$

$$\text{i.e. } x'.x' - 4\mu\xi_1a_0x'_0 + 2\mu x'_1[2a_1\xi_1 - 1] + 2\xi_1\mu = 0$$

$$\begin{aligned} \text{i.e. } [x'_0 - 2\mu\xi_1a_0]^2 - [x'_1 - \mu(2\xi_1a_1 - 1)]^2 \\ - 4\mu^2\xi_1^2a_0^2 + \mu^2(2\xi_1a_1 - 1)^2 + 2\xi_1\mu = 0 \end{aligned}$$

$$\begin{aligned} \text{i.e. } [x'_0 - \beta_0]^2 - [x'_1 - \beta_1]^2 \\ = 4\mu^2 \left[\xi_1^2a_0^2 - \frac{1}{4}(4\xi_1^2a_1^2 - 4\xi_1a_1 + 1) - \frac{\xi_1}{2\mu} \right] = -\mu^2 \end{aligned}$$

6.8.2 Corollary.

If (x_0, ξ_1) , with ξ_1 fixed, is the representative point of a given straight world-line in the coordinate system associated with 0, then the map of the line under

$$x' = \frac{(\alpha.\alpha)x + (x.x)\alpha}{\alpha.\alpha + 2x.\alpha + x.x}$$

is the hyperbola

$$(x'_0 - \beta_0)^2 - (x'_1 - \beta_1)^2 = -\mu^2$$

$$\text{where } \mu = \frac{\alpha.\alpha}{2(\xi_1 + \alpha_1)},$$

$$\beta_0 = \frac{\xi_1 \alpha_0}{\xi_1 + \alpha_1} \quad \text{and} \quad \beta_1 = \frac{2\alpha_1 \xi_1 - \alpha \cdot \alpha}{2(\xi_1 + \alpha_1)}$$

6.8.3 Remark.

In the coordinate system associated with $0'$ we may identify the world line of §6.8.1 as being appropriate to a uniform acceleration $\pm 2[(a \cdot a)\xi_1 + a_1]$. Furthermore, if we let $\xi_1 \rightarrow 0$, we obtain the uniform acceleration of 0 relative to $0'$, viz $\pm 2a_1$.

In the notation of §6.8.2 these quantities are respectively

$$\pm \frac{2[\xi_1 + \alpha_1]}{\alpha \cdot \alpha} \quad \text{and} \quad \pm \frac{2\alpha_1}{\alpha \cdot \alpha}.$$

6.9 Zero acceleration systems and singularities of the conformal group

Since the group C_0 is known to be associated with M_2 , as the local invariance group for uniformly accelerated (relative) motion, it is natural to ask to what extent the singular cones represent physically significant barriers. We therefore investigate the properties of the region for which $f \in S_2$ represents a bijective map of M_2 , corresponding to the kinematical situation of a transformation from one system in which a particle appears to be uniformly accelerated to another in which it has zero acceleration.

The most natural estimate of the limitation imposed by the singular cones (lines) in this kinematical situation is the speed with which the particle approaches the lines of singularity.

6.9.1 Nature of the singular region.

It is clear that the map $F : M_2 \rightarrow M_2$ defined by

$$X = Fx = \frac{(b \cdot b)x - (x \cdot x)b}{b \cdot b - 2(x \cdot b) + x \cdot x}$$

is singular on the cones $K(0)$ and $K(b)$. We must now stipulate that the world line $x \cdot x = x \cdot b$ is to represent the history of a particle which passes through 0 with speed $u = \frac{b_0}{b_1} > 0$; for an appropriate world line

we must also have $b_0^2 - b_1^2 = -k^2$.

In that case, with the usual geometrical representation of M_2 , we look upon the world line as the hyperbola

$$(x_0 - \frac{1}{2} b_0)^2 - (x_1 - \frac{1}{2} b_1)^2 = \frac{1}{4}(b_0^2 - b_1^2)$$

for which the *centre* is at $[\frac{1}{2} b_1, \frac{1}{2} b_0]$ and

the appropriate *vertex* at $[\frac{1}{2} b_1 + \frac{1}{2} k, \frac{1}{2} b_0]$;

square brackets are used to indicate the cartesian *representation* of M_2 which is to be distinguished from (x_0, x_1) , the number pair description of an element of M_2 (see figure 4).

The implication of the coordinate representation is that b_0, b_1 are negative numbers, with $u = \frac{b_0}{b_1} > 0$, as required.

It is evident that $x.x = x.b$ is *asymptotic* to

$$(x_0 - \frac{1}{2} b_0)^2 - (x_1 - \frac{1}{2} b_1)^2 = 0$$

i.e. to the cone $K(\frac{1}{2} b)$; it is also clear that $x.x = x.b$ does not intersect $K(b)$ in the future of 0.

Thus the map $x \rightarrow X$ imposes no physical limitations in the future of 0.

Now consider the inverse map $F^{-1} : M^2 \rightarrow M^2$ defined by

$$x = F^{-1}X = \frac{(b.b)X + (X.X)b}{b.b + 2(X.b) + X.X}$$

which is singular on the cones $K(0)$ and $K(-b)$.

We investigate the intersection of the world line $X.b = 0$ with the forward cone at $-b$, which occurs on the branch $X_0 + b_0 = -(X_1 + b_1)$; the intersection is at $X_1 = -b_0, X_0 = -b_1$ (figure 5).

Consider the point $(-b_1 + \epsilon b_1, -b_0 + \epsilon b_0) \in M_2$, where $\epsilon > 0$, and transform back to the system of 0

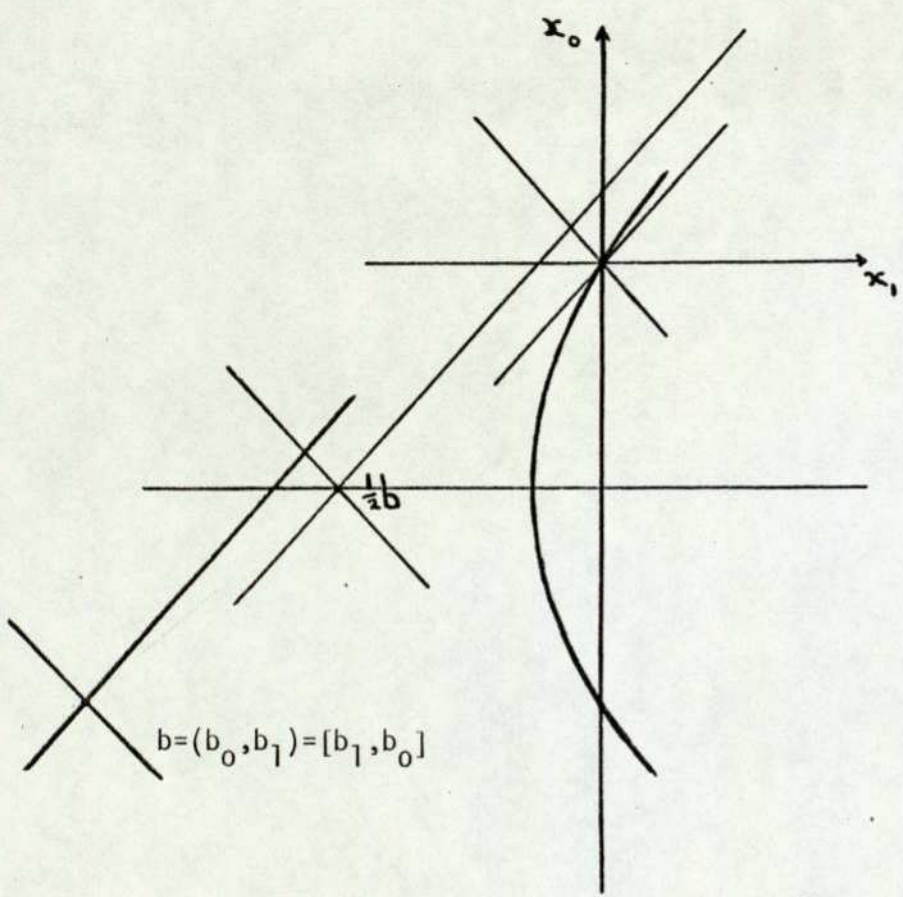


Figure 4

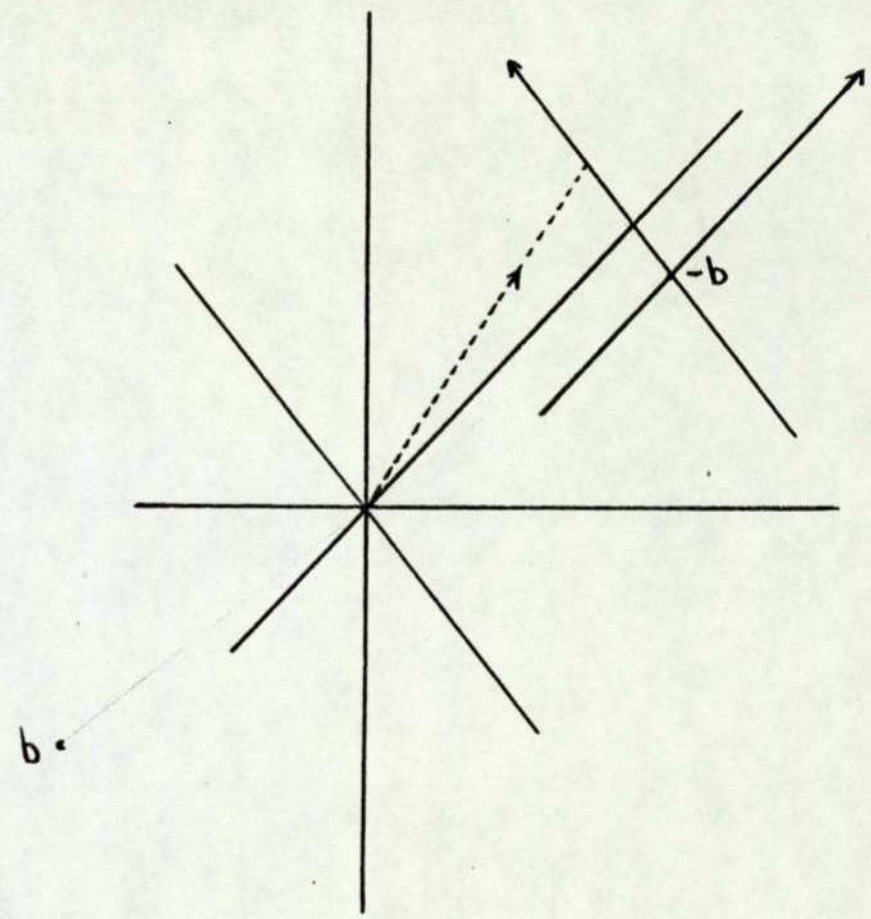


Figure 5

$$x_0 = \frac{-(1-\epsilon)b_1 - (1-\epsilon)^2 b_0}{1 - (1-\epsilon)^2}$$

$$x_1 = \frac{-(1-\epsilon)b_0 - (1-\epsilon)^2 b_1}{1 - (1-\epsilon)^2}$$

At this stage of the progress of the particle in the system of $0'$, what speed has the particle achieved in the system of 0 ?

Clearly,

$$\begin{aligned} \frac{dx_1}{dx_0} &= \frac{b_0 - 2x_0}{b_1 - 2x_1} \\ &= \frac{b_0\{1 - (1-\epsilon)^2\} + 2(1-\epsilon)b_1 + 2(1-\epsilon)^2 b_0}{b_1\{1 - (1-\epsilon)^2\} + 2(1-\epsilon)b_0 + 2(1-\epsilon)^2 b_1}, \end{aligned}$$

so that $\lim_{\epsilon \rightarrow 0} \frac{dx_1}{dx_0} = 1.$

We conclude that the map $X \rightarrow x$ implies no physical limitations as far as super light speed is concerned in the future of $0'$, either.

6.9.2 Causality violation.

We have seen that the singular cones also have the property of partitioning Minkowski space into regions in which causality is preserved, but that in crossing the singular surfaces causality is violated ; so it is with the map F .

Let $X = (-b_1 + \epsilon b_1, -b_0 + \epsilon b_0)$ be a point on the world line $X.b = 0$ prior to the singular line intersection $(-b_1, -b_0)$ and let $Y = (-b_1 - \epsilon b_1, -b_0 - \epsilon b_0)$ be immediately after the intersection ; clearly $X < Y$ (i.e. $X - Y$ is timelike, and $X_0 < Y_0$).

Then

$$x_0 = \frac{-(1-\epsilon)b_1 - (1-\epsilon)^2 b_0}{1 - (1-\epsilon)^2} ; x_1 = \frac{-(1-\epsilon)b_0 - (1-\epsilon)^2 b_1}{1 - (1-\epsilon)^2}$$

with a similar designation for y .

We obtain

$$(x_0 - y_0)^2 - (x_1 - y_1)^2 = \frac{4k^2}{\epsilon^2 (4 - \epsilon^2)^2} [-4\epsilon^2 + \epsilon^4] < 0$$

for appropriately small ϵ .

Thus $x - y$ is spacelike i.e. $x \not\prec y$, and causality is violated by F^{-1} .

In the system of O' , $-b_1$ seconds will have elapsed before the particle's world line reaches the singular cone $K(-b)$, at which event the causality violation occurs.

Now, from § 6.4, the magnitude of the uniform acceleration is measured by $\frac{2}{b_1 \sqrt{1 - u^2}}$, which is small when b_1 is large; thus the causality boundary is not soon reached for small accelerations, and for zero acceleration, causality is preserved.

6.9.3 Remark.

To check that the physical procedures are not contradictory, we confirm that as $X_0 \rightarrow -b_1$ in the system of O' , $x_0 \rightarrow \infty$ in the frame of O . Consider the point $X = (-b_1 - \delta, -b_0 - \frac{b_0 \delta}{b_1})$ on the world line $X.b = 0$, prior to $(-b_1, -b_0)$.

$$\text{Then } X.X = -b.b + 2b_1\delta + \delta^2 - \frac{2b_0^2\delta}{b_1} - \frac{b_0^2\delta^2}{b_1^2},$$

$$\text{now } x = \frac{(b.b)X + (X.X)b}{b.b + 2b_1\overset{0}{x} + X.X}$$

and we obtain

$$x_0 = b_1 \left[\frac{(b_1 + b_0) + 0(\delta)}{2\delta + 0(\delta^2)} \right]$$

$$\text{and } x_1 = b_1 \left[\frac{(b_1 + b_0) + 0(\delta)}{2\delta + 0(\delta^2)} \right]$$

so that $x_1 \sim x_0 \rightarrow +\infty$ as $\delta \rightarrow 0$; this is exactly what we would expect, to accord with the fact that $\lim_{\delta \rightarrow 0} \frac{dx_1}{dx_0} = 1$.

6.9.4 Nature of the vector b .

We have seen that, if a particle has a world line $x.x = x.b$ in the system of 0, then the ratio $\frac{b_0}{b_1}$ gives the velocity of the particle at 0 (and this is invariant for all systems special conformally related to that of 0, and for dilatations of the system of 0, but not, of course, for Lorentz related systems), and the magnitude $b.b = b_0^2 - b_1^2$ gives a measure of the reciprocal of the uniform acceleration, according to 0 (and this is not invariant for systems special conformally related to that of 0, nor for dilatations of the system of 0, but it is invariant for Lorentz related systems). As we have explained in § 6.7, the whole group C_0 preserves *uniformity* of acceleration and *local causality*.

It is interesting to notice that the vector b lies on the world line $x.x = x.b$, but since the *centre* of the hyperbola is $[\frac{1}{2} b_1, \frac{1}{2} b_0]$, it is clear that $x = b$ is on the second branch of the hyperbola, *not* the world line branch.

$$\text{Now, under } x' = \frac{x - \frac{b}{b.b}(x.x)}{1 - \frac{2x.b}{b.b} + \frac{x.x}{b.b}}$$

the map of λb is

$$(\lambda b)' = \frac{\lambda b}{1-\lambda}, \text{ provided that } \lambda \neq 1.$$

We observe that as $\lambda \rightarrow 1^-$, $(\lambda b)' \rightarrow -\infty$, and as $\lambda \rightarrow 1^+$, $(\lambda b)' \rightarrow +\infty$. We have already noted in § 6.9.4 the corresponding result that as $x' \rightarrow -b$, $x \rightarrow \pm\infty$.

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Chapter 7. Inertial frames.

7.1 Perspective.

We shall suppose that every one of a certain class of observers with a pair of coincident (zero) events is equipped or *may equip himself with* his own Minkowski space, M_4 . Suppose we then take as a fundamental law of nature *the principle of local causality* which is *abstracted* as follows :-

a pair of observers O, O' of the class compare observations by way of the locally causal automorphism group (§§ 5.7 - 5.9).

Thus the principle of local causality implies local applicability of the group C_0 , from § 5.18, Theorem C ; this, in turn, has two implications,

- (i) null lines in the systems of O, O' are locally correspondent,
- (ii) the coordinate systems for O, O' are those that, for plane world lines in each system, are kinematically appropriate to *relative uniform acceleration*, with the same local applicability ; they, too, are locally correspondent.

We shall refer to the local correspondence of relatively uniformly accelerated frames as *the principle of C^* -relativity*.

Clearly the principle of C^* -relativity invokes a wider class of equivalent coordinate systems than the principle of special relativity in one sense, inasmuch as, if the relative uniform acceleration is zero, then we have the "special relativistic" situation ; but a narrower class in another, in that coincident O, O' are necessary for the operation of C^* -relativity and that operation is necessarily local.

We also have a reverse implication that the property of null line invariance and the principle of C^* -relativity together imply the principle of local causality.

We recall at this point that it is a mathematical abstraction, Minkowski space, that we are dealing with i.e. a set of ordered quadruples of real numbers together with a scalar product : not an overtly physical situation.

It is important to emphasise that our method is a mathematically abstract but physically naive development ; the usefulness of the development is that it provides an insight into the way in which, even at the level of naive physics, the C^* -relativity is shown to be inappropriate.

We shall make a formal elementary calculation based on hyperbolic arcs in a plane and then associate the calculation with a thought experiment in the C^* -relativity physics. We shall find that we obtain a contradiction, and conclude that the principle of C^* -relativity is unsound at the most basic level of naive physics.

Thus we shall be led to make a new categorization of equivalent systems and to define *inertial observer* and *inertial frame*.

7.2 Comparison of clocks in uniformly accelerated motion.

Let A be the origin of M_2 and consider a continuous world line through A which consists of three connected hyperbolic arcs as in figure 6. We shall interpret this geometrical situation as being descriptive of an observer O' who is first uniformly

accelerated, then uniformly decelerated, then uniformly accelerated with respect to 0 (an observer with world line AR) in such a way that the world lines of 0 and 0' coincide at A and R. We shall calculate the timelike coordinate of R in the system associated with 0 and interpret it as the reading on a standard clock carried by 0, on the assumption that 0 is the reference observer. Next we use the map to a system in which 0' is always at rest and calculate the timelike coordinate for the progression AR in this system ; we interpret this reading as that on a standard clock carried by 0'.

7.2.1 0's reading.

First arc:

The arc has equation

$$x \cdot x = -b_1 x_1, \text{ where } b_1 \text{ is a negative number ;}$$

and parametric equations

$$x_0 = -\frac{1}{2} b_1 \sinh \theta, \quad x_1 = -\frac{1}{2} b_1 (-1 + \cosh \theta).$$

The first arc is followed from the point A to the point P, where $\tau_0 = -\frac{1}{2} b_1 \sinh \alpha$, $\tau_1 = -\frac{1}{2} b_1 (-1 + \cosh \alpha)$; we make the convention that the coordinates of P in the diagram are written (τ_0, τ_1) , instead of the usual practice. It is easy to see that P is in the semi-infinite future region between the pairs of lines $x_0^2 - x_1^2 = 0$ and $x_0^2 - (x_1 - b_1)^2 = 0$ [these are the null "cones" at 0 and $B_1 \equiv (0, +b_1)$].

In addition, the gradient of the curve at P, u (which measures the speed of a material particle for which the

arc is world line) is $\frac{-2\tau_0}{b_1 - 2\tau_1}$;

thus $u = \tanh \alpha$. .

Second arc:

The arc has equation $X_0^2 - X_1^2 = b_1 X_1$, referred to axes at its vertex, which is the point $(-b_1 \sinh \alpha, -b_1 \{-1 + \cosh \alpha\})$; the equation referred to P as origin is

$$\begin{aligned} [Y_0 + \frac{1}{2} b_1 \sinh \alpha]^2 - [Y_1 + \frac{1}{2} b_1 (-1 + \cosh \alpha)]^2 \\ = b_1 [Y_1 + \frac{1}{2} b_1 (-1 + \cosh \alpha)] , \end{aligned}$$

for which an appropriate parametric form is

$$Y_0 + \frac{1}{2} b_1 \sinh \alpha = -\frac{1}{2} b_1 \sinh \phi$$

$$Y_1 + \frac{1}{2} b_1 (-1 + \cosh \alpha) = \frac{1}{2} b_1 (-1 + \cosh \phi).$$

Thus the parameter associated with P is $\phi = -\alpha$, and that associated with Q is $\phi = \alpha$.

It is clear that the hyperbolic arc PQ is confined to the semi-infinite future region bounded by the line-pairs $Y_0^2 - Y_1^2 = 0$ and $(Y_0 + b_1 \sinh \alpha)^2 - (Y_1 + b_1 \cosh \alpha)^2 = 0$ [these are the null "cones" at P and B_2 , whose coordinates in the Y system are respectively $(0,0)$ and $(-b_1 \sinh \alpha, -b_1 \cosh \alpha)$].

Third arc:

The third arc has equation $Z_0^2 - Z_1^2 = -b_1 Z_1$ referred to its vertex R ; it is clear that the progress along the hyperbolic arc QR is exactly similar to that along OP. We note also that QR is confined to the semi-infinite past region between the cones at R and B_3 ,

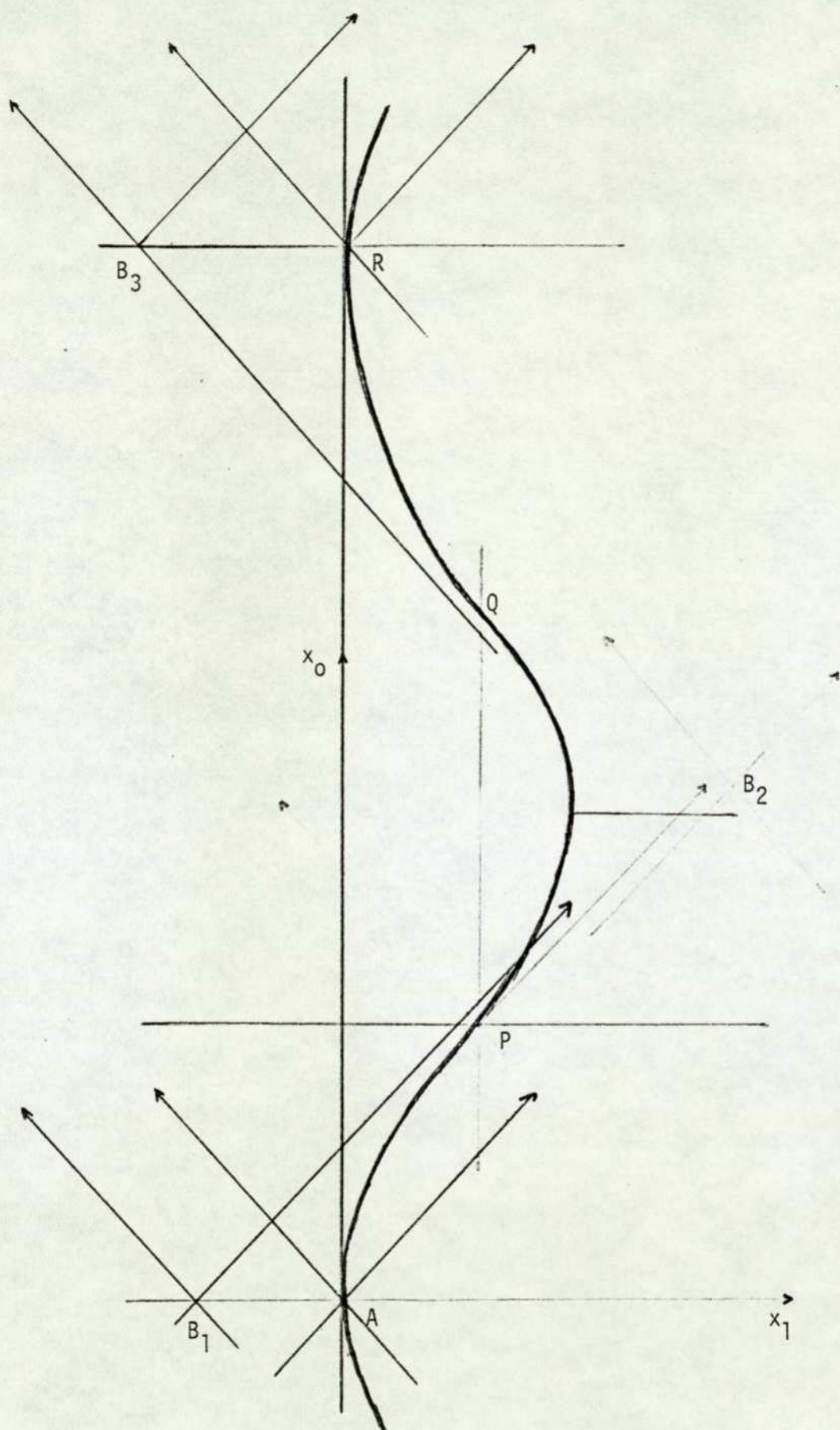


Figure 6

where $B_3 \equiv (0, b_1)$ in the Z coordinate system.

Time lapse according to 0.

If 0 considers himself to be an unaccelerated observer, the time lapse according to his clock between the events 0 and R will be $-2b_1 \sinh \alpha$.

7.2.2 0's reading.

Suppose that 0 and 0' coincide initially but 0' is the observer whose world line according to 0 is the previously described hyperbolic arc path. There is a zero acceleration system and a proper system associated with each arc of the world line as in figures 7 and 8.

First zero-acceleration system : (this is also the proper system)

Consider the special conformal map

$$x' = \frac{(b.b)x - (x.x)b}{b.b - 2x.b + x.x}, \text{ where } b.b = -b_1^2;$$

the map of the arc of $x.x = -b_1 x_1$ is the appropriate segment of $x'_1 = 0$.

$$\text{Also } x'_0 = \frac{-b_1^2 x_0}{-b_1^2 + b_1 x_1}, \text{ and if } x_0 = \tau_0, x_1 = \tau_1,$$

$$\text{we have } x'_0 = \frac{-b_1 \sinh \alpha}{1 + \cosh \alpha}$$

Second zero-acceleration system :

$$\text{Consider } Y' = \frac{(B.B)Y - (Y.Y)B}{B.B - 2Y.B + Y.Y}$$

$$\text{where } B_0 = -b_1 \sinh \alpha; B_1 = -b_1 \cosh \alpha; B.B = -b_1^2.$$

In this case, the map of the arc of $Y.Y = Y.B$ is the appropriate segment of $Y'.B = 0$.

Also

$$Y'_0 = \frac{-b_1^2 Y_0 + (Y.Y)b_1 \sinh \alpha}{-b_1^2 + Y_0 b_1 \sinh \alpha - Y_1 b_1 \cosh \alpha}$$

$$Y'_1 = \frac{-b_1^2 Y_1 + (Y.Y)b_1 \cosh \alpha}{-b_1^2 + Y_0 b_1 \sinh \alpha - Y_1 b_1 \cosh \alpha}$$

i.e. $Y'_0 = \frac{-b_1^2 Y_0 + (-Y_0 b_1 \sinh \alpha + Y_1 b_1 \cosh \alpha)b_1 \sinh \alpha}{-b_1^2 + Y_0 b_1 \sinh \alpha - Y_1 b_1 \cosh \alpha}$

and $Y'_1 = \frac{-b_1^2 Y_1 + (-Y_0 b_1 \sinh \alpha + Y_1 b_1 \cosh \alpha)b_1 \cosh \alpha}{-b_1^2 + Y_0 b_1 \sinh \alpha - Y_1 b_1 \cosh \alpha}$

Let V be the event, which in P's system, occurs as the vertex of the hyperbolic arc PQ. We find that, in the zero-acceleration system, V has coordinates

$$T'_0 = \frac{-b_1 \sinh \alpha \cosh \alpha}{1 + \cosh \alpha}, \quad T'_1 = \frac{-b_1 \sinh^2 \alpha}{1 + \cosh \alpha}$$

Second proper system :

The Lorentz transformation to the proper system has parameter $u = -\tanh \alpha$, and if the coordinates of V in the proper system are t'_0, t'_1 , then

$$t'_0 = \frac{1}{\sqrt{1 - \tanh^2 \alpha}} \cdot \frac{-b_1 \sinh \alpha \cosh \alpha}{1 + \cosh \alpha} - \frac{\tanh \alpha}{\sqrt{1 - \tanh^2 \alpha}} \cdot \frac{b_1 \sinh^2 \alpha}{1 + \cosh \alpha}$$

$$= \frac{-b_1 \sinh \alpha}{1 + \cosh \alpha}$$

We have symmetry in the readings about V and conclude that the net time lapse in the second proper system between events P and Q is

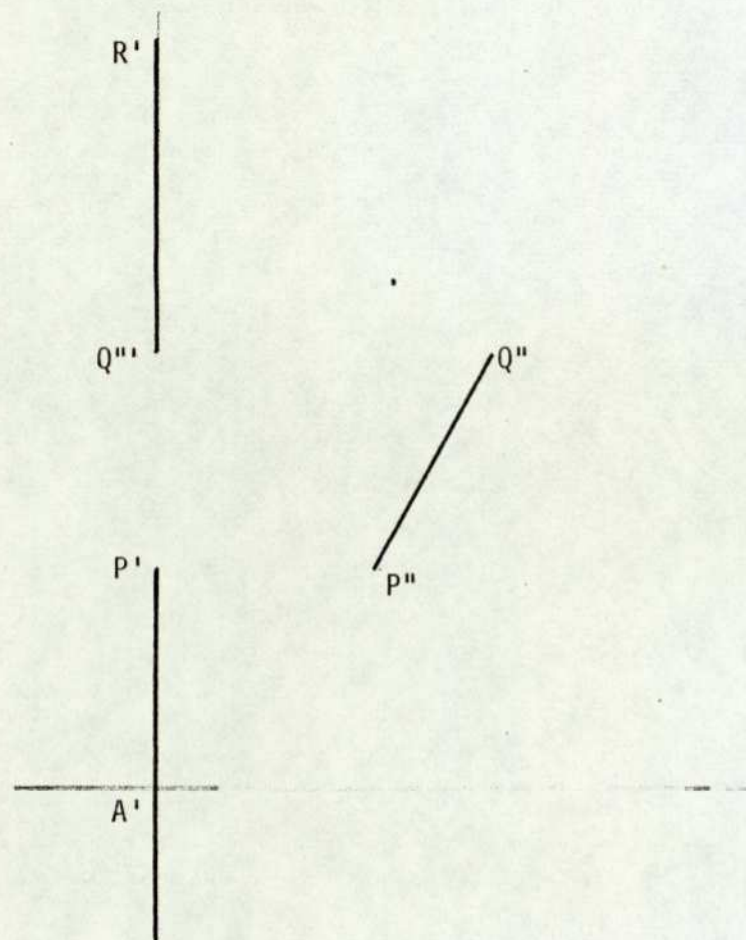


Figure 7

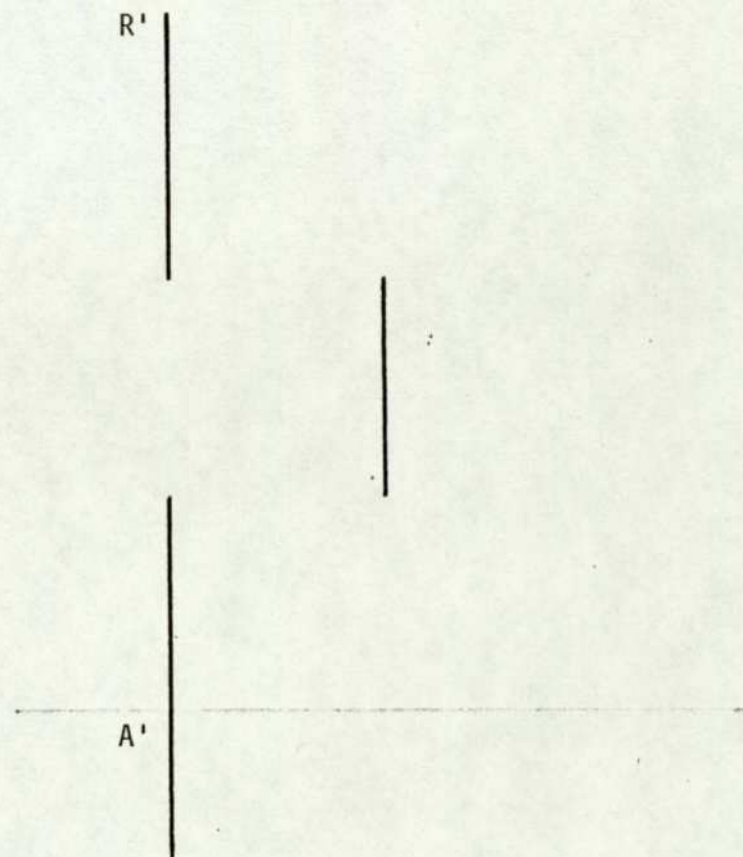


Figure 8

$$\frac{-2b_1 \sinh \alpha}{1 + \cosh \alpha} .$$

Third proper system.

This is formally the same as the first proper system, and we may conclude that the total time elapsed on the proper clock for O' is

$$\frac{-4b_1 \sinh \alpha}{1 + \cosh \alpha} \quad \text{or} \quad \frac{-4b_1 u}{1 + \sqrt{1 - u^2}} .$$

This is a positive number which is strictly less than $-2b_1 \sinh \alpha$ (we cannot have $\alpha = 0$).

7.2.3 Validity of the transforms.

We have proved in §§6.9 - 6.9.3 the rather surprising fact that the presence of the singular cones imposes no physical constraint on the problem, inasmuch as $u < 1$ guarantees both that the speed of light is not attained in either kinematical system and that the causality violation boundaries are not breached. Thus it is of no consequence to the foregoing analysis that the speed u is arbitrarily close to 1 (it cannot be equal to 1) nor that the acceleration is arbitrarily large.

7.3 Consequence for C*-relativity.

The consequence of the result of the previous paragraphs for the theory of C*-relativity is simple, and devastating ! We may not conclude that in the C*-relativity, clocks are not proper measuring devices ; proper or not, a standard clock is a standard clock ! The only conclusion is that C*-relativity leads to a self contradictory kinematical theory, if clocks are to be any use at all.

We are forced, therefore, to select a proper subset of the coordinate systems thus far considered to be equivalent, and give them privileged status. We shall refer to these privileged systems in the forthcoming paragraphs as *inertial systems* or *frames*.

Equally, we may consider the wider class of coordinate systems associated with the principle of local causality as insufficiently restrictive for a viable kinematical theory.

7.3.1 Remark.

The privileged coordinate systems give rise to privileged geometries with which to model space-time kinematics, thus if M_2 is a privileged system and if $x < y$ then x, y lie in a privileged timelike straight line inherited from the privileged geometry.

As far as the privileged clocks are concerned, if T is the reading on a privileged clock which contains x, y in its history then the reading on every other clock of the locally causal equivalence class, τ is such that $\tau \leq T$.

Further $\tau = T$ iff $u = 0$ i.e. $\alpha = 0$.

We now make the following postulate.

7.3.2 Postulate of the inertial system.

If x, y are two events such that $x < y$ then there exists a member of the locally causal group such that x, y have the form $(x_0, 0, 0, 0)$, $(y_0, 0, 0, 0)$ in that system and $T = y_0 - x_0$ is *maximal* for the class of world lines encompassing both x and y in the manner of §§7.2 - 7.2.3. Such a system is referred to as an *inertial system* or *inertial frame*.

7.3.3 Remark.

It is clear that there is an equivalence class of inertial

systems appropriate to a pair of events x,y for which $x < y$ in M_4 but the systems differ only in that their spatial axes are rotated to new directions. It is also clear that once we have settled on the inertial frame(s) for a pair of events that coordinate system will be the inertial frame for every event on the inertial line through x and y .

7.3.4 Kinematical interpretation.

Thus far, it is recognized that only the inertial system(s) appropriate to a pair of (causally) related events x,y are to be considered equivalent for the description of space-time kinematics, those which categorize a relative (non-zero) uniform acceleration are not in the set of equivalent systems. We must now forge the link from one inertial system, that associated with x,y to another, that associated with u,v , where $u < v$ but u,v are not on the inertial line for x,y . We now state our second postulate.

7.4 The principle of relativity.

All inertial systems are equivalent for the description of space-time kinematics.

7.4.1 Deductions from the principle of relativity.

We would like to be able to say that if I,I' are any two inertial frames then the allowable transformations are affine, but this is not as easy as it looks (Hegerfeldt ; 1972) since we are assured only that all inertial lines (timelike lines) are mapped to inertial lines. However, Hegerfeldt (1972) has shown that that fact is sufficient to ensure affine transformations. In our system, if we had concerned ourselves with local phenomena,

as in §§7.2 - 7.2.3 we would have the immediate implication that the transformation group no longer contains special conformal maps, so that the group would be $L^{\uparrow*}$ locally. For remote inertial frames the transformation group is then $P^{\uparrow*}$.

7.5 The theory of special relativity.

Having modified the principle of local causality as a viable physical law we are manoeuvred into the position of adopting the following postulates, in addition to the postulate of the inertial frame

- (i) null lines in the systems of locally causal observers O and O' are locally correspondent.
- (ii) the principle of relativity for inertial frames.

However, we now know that the transformation group between inertial frames is $P^{\uparrow*}$, and we know that this is the *globally* causal covariance group (Zeeman; 1964). Thus we need not leave postulate (i) as it stands but replace it by

- (i)' null lines in the systems of inertial observers O and O' are globally correspondent.

Now we may follow Hegerfeldt (1972) and state that (i)' also implies that the scalar factor is unity, so that the transformation group is in fact $P^{\uparrow}$.

7.6 Discussion.

Our development could be looked upon as having its foundations in a Robb-type (Robb, 1936) axiom system ; can one say that it is an axiom system in which an individual's appreciation of causality is endemic rather than his appreciation of external physical

phenomena ? In a sense one can, for an individual's personal appreciation of a before/after relation for *himself* is intellectually appealing , yet there is the heaviest possible reliance on an absolute geometry with the Euclidean parallel property. This is as much a matter of external judgement and collaborative concert as is an acceptance of Newton's laws, or inertial frames, or the phenomena associated with the propagation of light. Nevertheless the author has found it instructive to work with this idea of the personal judgement of a before/after relation and the principles of local and global causality. The more traditional developments of the special theory of relativity, of course, very quickly introduce the concept of inertial frame (Pauli, 1958 : 4 ; Rindler, 1960 : 6) ; that is most natural, since the principle of *special* relativity, the one that we are concerned with, only applies to inertial frames. Now Infeld and Peblanski (1960 : 69) speak of

"... the strange concepts of action at a distance, or an *inertial coordinate system* ..." (my italics), a statement that is echoed in Pauli (1958 : 4)

"... It is unsatisfactory that one cannot regard *all* systems as completely equivalent or at least give a logical reason for selecting a particular set of them."

However, they are looking toward the theory of general relativity for which a pseudo-Riemannian space is the model; as far as Minkowski space is concerned, as we have seen, any attempt to widen the class of equivalent frames beyond the collection of inertial frames is fraught with difficulties at the level of kinematics. Indeed, consistency is only restored in that case, if we move into the realm of dynamics, and consider the concept

of "external force" and make new postulates as to how such forces affect measuring rods and clocks. None of this is surprising, for physicists have never surmised that no experiment can be devised to detect (non-zero) relative uniform acceleration.

7.7 Detection of non-inertial frames.

Rindler (1960 : 6) states that
"We shall now accept the existence of at least one inertial frame as an *experimental* fact : the frame of the "fixed" stars is known to be inertial *to a high degree of accuracy*". (My italics).

In our development there is theoretically an operational method for detecting an inertial frame by checking that the clock reading is maximal between pairs of causally related events. We have referred to this method as "naive physics" and it may be true that the more naive the physics the less the likelihood of importing into the theory folkloric and intuitive conceptions (which may be misconceptions).

To answer the question : "what other means are open to us to distinguish a system that is *non-inertial* ?" we can look back to Galileo's test (Kilmister, 1970 : 3). Galileo considers the example of an observer on board a ship in calm, non-tidal waters; if the observer were confined to his cabin below decks in the company of a goldfish bowl, then a uniform motion of the ship would surely leave the surface of the water in the bowl level, but a uniform acceleration (for example) would cause the water to "pile up" at the rear end (with respect to the direction of the acceleration) of the bowl. Closer still to our conception, if a spaceship

were in the vicinity of the neutral point for gravitational attraction from the Sun, the Earth and the Moon, then a small free object, say a pen, would betray a forward acceleration from the rocket thrust by retreating to the rearward boundary of the capsule.

There is another, more sophisticated result that would enable us to distinguish between a particle (frame) that has a non-zero uniform acceleration with respect to an inertial frame and a particle that has a *zero* uniform acceleration in the inertial frame, unfortunately the result has been clouded in some controversy. Pauli (1958 : 93) states categorically, quoting a result of Born (1909) , that a charge in uniformly accelerated relativistic motion does not radiate energy ; Synge (1964 : 421) is equally sure that it does, although Anderson (1967 : 247) indicates exactly how the confusion of interpretation has arisen. It is likely that both Synge (1964) and Anderson (1967) were influenced by a paper of Fulton and Rohrlich (1960) , in which the problem of radiation from a uniformly accelerated charge is thoroughly examined. An account of the problem and its explanation is beyond the scope of this thesis.

7.8 The transformation between an inertial frame and a uniformly accelerated frame.

We conclude our development by using the theory of §§6.3 - 6.5 to establish explicit transformations between an inertial frame I and a frame A which is in uniformly accelerated one-dimensional motion relative to I.

7.8.1 Maps from one inertial frame to another : apparent length contraction.

Consider the Lorentz transformation which maps one inertial frame I to another I' , with $O \equiv O'$; $x' = \Lambda_u x$, where

$$x'_0 = \frac{x_0}{\sqrt{1-u^2}} - \frac{u}{\sqrt{1-u^2}} x_1 ,$$

$$x'_1 = \frac{x_1}{\sqrt{1-u^2}} - \frac{u}{\sqrt{1-u^2}} x_0 .$$

The map of the line $x_1 = 0$ is $x'_1 = -ux'_0$,

the map of the line $x_1 = \xi$ is $x'_1 = -ux'_0 + \sqrt{1-u^2} \xi$.

Thus, given a rod of length ξ at rest in the I frame ; at any instant in the frame of I' , the rod will appear to have length $\sqrt{1-u^2} \xi$.

Our analysis of the map from a uniformly accelerated frame to an inertial frame might be considered as a preliminary to that between two uniformly accelerated frames. In any case, we shall incorporate in the method of transformation a quality that will provide the aforementioned relativistic length contraction when we are dealing with the special case of *two* inertial frames.

7.8.2 World lines of constituent particles.

We shall say that a frame A is uniformly accelerated with respect to an inertial frame I if every point of A has the same uniform acceleration with respect to I .

Let K be the uniform acceleration of every point of the frame

of A relative to I, then (Pauli, 1958 : 75)

$$\frac{dv}{dx_0} = K(1 - v^2)^{3/2}$$

where v is the speed of C relative to I at x_0 .

$$\text{Thus } \int \frac{dv}{(1 - v^2)^{3/2}} = \int K dx_0$$

and we obtain (via $v = \cos\theta$)

$$\frac{v}{\sqrt{1 - v^2}} = Kx_0 + \alpha_0 ; \alpha_0 \text{ arbitrary,}$$

$$\text{whence } v^2 = (Kx_0 + \alpha_0)^2 (1 - v^2)$$

$$\text{and } \frac{dx_1}{dx_0} = \frac{Kx_0 + \alpha_0}{\sqrt{1 + (Kx_0 + \alpha_0)^2}},$$

$$\text{so that } x_1 + \alpha_1 = \frac{1}{K} \sqrt{1 + (Kx_0 + \alpha_0)^2}, \text{ with } \alpha_1 \text{ arbitrary.}$$

Thus the world line of each particle of the frame of A has the form

$$K^2(x_1 + \alpha_1)^2 - (Kx_0 + \alpha_0)^2 = 1,$$

$$\text{i.e. } \left(x_0 + \frac{\alpha_0}{K}\right)^2 - (x_1 + \alpha_1)^2 = -\frac{1}{K^2},$$

and the individual world line appropriate to a given particle in the frame will be determined by the initial (inertial frame) conditions appropriate to that particle viz α_0, α_1 .

7.8.3 The geometric analysis.

For convenience, we write $\frac{1}{K} = k$ and consider the world line of a particle of A's frame as observed in the frame of I ; assume that the particle passes through 0 with zero speed.

Here $\alpha_0 = 0$ and $\alpha_1 = k$, so that the world line is the hyperbola

$$x_0^2 - (x_1 + k)^2 = -k^2 \quad \dots (i).$$

Consider now a hyperbola through $(0, \xi)$ in the frame of I, viz

$$x_0^2 - (\bar{x}_1 + k)^2 = -(\xi + k)^2 \quad \dots (ii),$$

so that (x_0, x_1) are the coordinates of a point on the hyperbola through 0 and $(x_0, \bar{x}_1)$ are those for a point on the hyperbola through $(0, \xi)$. Thus $\bar{x}_1$ is a function of x_1 and ξ . Clearly both hyperbolas have centre $(0, -k)$ in the x_0/x_1 system. We obtain

$$(\bar{x}_1 + k)^2 - (x_1 + k)^2 = \xi^2 + 2\xi k$$

$$\text{i.e.} \quad (\bar{x}_1 - x)(\bar{x}_1 + x + 2k) = \xi^2 + 2\xi k \quad \dots (iii).$$

Now write

$$\frac{dx_1}{dx_0} = \frac{x_0}{x_1 + k} = u$$

$$\text{and} \quad \frac{d\bar{x}_1}{dx_0} = \frac{x_0}{\bar{x}_1 + k} = \bar{u}.$$

We notice that if $\bar{x}_1 > x_1$, then for a given value of x_0 , $\bar{u} < u$; the hyperbola through $(0, \xi)$ is steeper in the x_0/x_1 diagram than the hyperbola through $(0, 0)$, (Figure 9).

From (i) we obtain

$$1 - u^2 = \frac{k^2}{(x_1 + k)^2}, \quad 1 - \bar{u}^2 = \frac{(\xi + k)^2}{(\bar{x}_1 + k)^2},$$

$$\text{so that, if we introduce } \gamma = \frac{1}{\sqrt{1 - u^2}} \quad \text{and} \quad \bar{\gamma} = \frac{1}{\sqrt{1 - \bar{u}^2}},$$

$$\text{then } x_1 + k = \gamma k, \quad \bar{x}_1 + k = \bar{\gamma}(\xi + k)$$

$$\text{and } x_1 + \bar{x}_1 + 2k = k\gamma + (\xi + k)\bar{\gamma}.$$

Thus, from (iii)

$$\bar{x}_1 = x_1 + \frac{\xi^2 + 2\xi k}{k\gamma + (\xi+k)\bar{\gamma}} \quad \dots \quad (iv)$$

We note two properties of this transformation of coordinates,

1.) as $\xi \rightarrow 0$, $\bar{x}_1 \rightarrow x_1$;

2.) as $k \rightarrow \infty$ (which is equivalent to $K \rightarrow 0$), we have

$$\bar{x}_1 \rightarrow x_1 + \frac{2\xi k}{k(\gamma + \bar{\gamma})}$$

$$= x_1 + \frac{2\xi}{\gamma + \bar{\gamma}} ,$$

in addition, $\bar{\gamma} \rightarrow \gamma$,

so that, in the limit

$$\bar{x}_1 = x_1 + \frac{\xi}{\gamma} .$$

Thus, for *zero* uniform acceleration, A is itself an inertial frame, and we obtain the correct relativistic length contraction at any instant

$$\bar{x}_1 - x_1 = \frac{\xi}{\gamma} , \text{ where } \xi \text{ is the length.}$$

Thus , for *zero* uniform acceleration i.e. if the acceleration is suddenly "turned off" A coincides with its comoving inertial frame, and we obtain the correct relativistic length contraction at that instant

$$\bar{x}_1 - x_1 = \frac{\xi}{\gamma}$$

where ξ is the length of a rod in the frame of A (since A and I were at initial relative rest), and $\bar{x}_1 - x_1$ is its length according to I at the subsequent "turn off" instant.

3.) For $\xi = -k$, we obtain a pair of lines

$$x_0^2 - (\bar{x}_1 - k)^2 = 0,$$

and we shall restrict the usage of the formulae to $k > 0$ and $\xi \geq -k$.

7.8.4 Remark.

We have now obtained a geometric formulation in which the individual particles of the frame of A have hyperbolic world lines in the coordinate system associated with the frame of O. The hyperbolae are concentric and they converge in a way which is entirely consistent with our expectation for vanishing acceleration.

7.8.5 The map from I to A.

The world line

$$x_0^2 - (x_1 + k)^2 = -k^2$$

has the form $x.x = x.\beta$,

with $\beta_0 = 0$ and $\beta_1 = -2k$, so that $\beta.\beta = -4k^2$.

From §6.5

$$X = \frac{(\beta.\beta)x - (x.x)\beta}{\beta.\beta - 2x.\beta + x.x}$$

is appropriate to map $x.x = x.\beta$, with $x_0 \geq 0$, onto $X_1 = 0$.

Here, the explicit form for the transformation is the pair of equations

$$\begin{aligned} X_0 &= \frac{-4k^2 x_0}{-4k^2 - 4x_1 k + x_0^2 - x_1^2}, \\ X_1 &= \frac{-4k^2 x_1 + 2(x_0^2 - x_1^2)k}{-4k^2 - 4x_1 k + x_0^2 - x_1^2}. \end{aligned}$$

It is interesting to look at the map of $x_1 = 0$, in particular; here

$$X_0 = \frac{-4k^2 x_0^2}{-4k^2 + x_0^2} \quad \text{and} \quad X_1 = \frac{2kx_0^2}{-4k^2 + x_0^2},$$

so that $X_1 = -\frac{1}{2k} x_0 X_0$.

Also $X_1 \{-4k^2 + x_0^2\} = 2kx_0^2$

i.e. $x_0^2 \{X_1 - 2k\} = 4k^2 X_1$

hence $\frac{X_1^2}{X_0^2} \{X_1 - 2k\} = X_1$,

from which we obtain $X_0^2 - (X_1 - k)^2 = -k^2$.

We may adopt the following procedure :-

1.) to map the world line

$$x_0^2 - (\bar{x}_1 + k)^2 = -k^2$$

as observed by the inertial observer (appropriate to a single particle of the system of A) to the corresponding world line as observed in the accelerated frame :

(i) apply the appropriate special conformal map,

(ii) reflect in the line $x_1 = -k$;

(see Figure 10).

2.) to map the world line

$$x_0^2 - (\bar{x}_1 + k)^2 = -(\xi + k)^2, \quad \xi \neq 0, \quad \xi > -k,$$

to the accelerated frame,

(i) translate to 0,

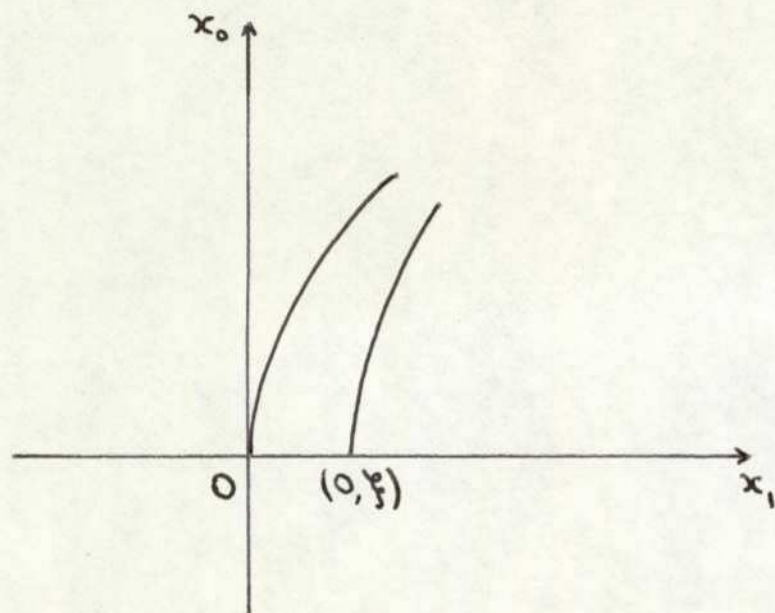
(ii) apply the appropriate special conformal map,

(iii) reflect in the line $x_1 = -k$,

(iv) translate from 0' ;

3.) to map the world line pair

$$x_0^2 - (\bar{x}_1 + k)^2 = 0,$$



Frame of I

Figure 9

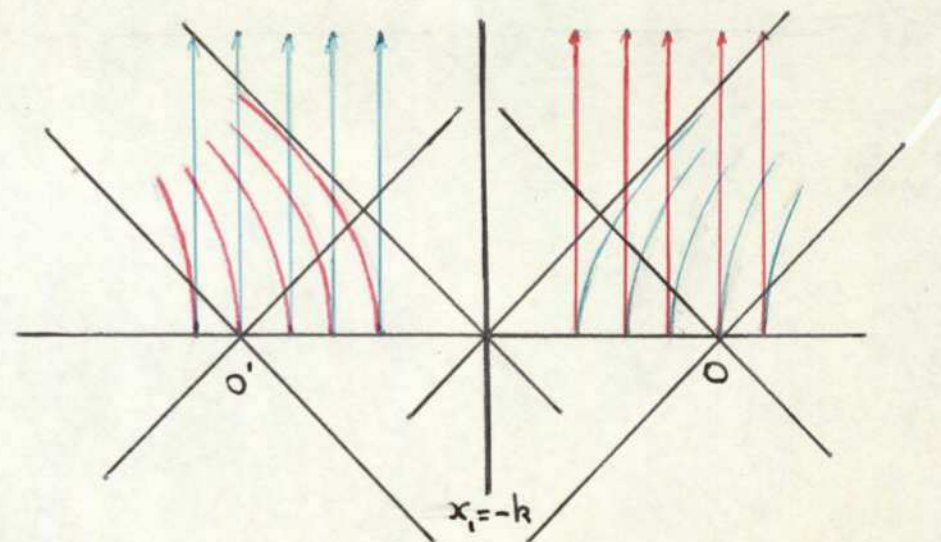
The map from the inertial (red) frame
to the accelerated (green) frame

Figure 10

- (i) translate to 0,
 - (ii) apply the appropriate special conformal map, *which is formally the identity map*,
 - (iii) reflect in the line $x_1 = -k$,
 - (iv) translate from 0'
- (this procedure is equivalent to a formal assignment of the identity map for this line pair).

7.8.6 An algebraic formulation.

If $\phi_{-\beta/\beta.\beta}$ is the special conformal map mentioned in the previous paragraph, t_ξ is a translation along the x_1 axis of ξ , and ρ_{-k} is a reflection in $x_1 = -k$, then

$$f_\xi = t_{-\xi} \circ \rho_{-k} \circ \phi_{-\beta/\beta.\beta} \circ t_{-\xi}$$

is that map which applies to all points on the world line

$$x_0^2 - (\bar{x}_1 + k)^2 = -(\xi + k)^2.$$

Notice, however, that $\xi \geq -k$, so that the collection $F = \{f_\xi\}$ has the property that $F : Y \rightarrow Z$, where Y is that part of I for which, initially, $-k \leq x_1 < \infty$, and Z is that part of A for which, initially, $-\infty < x_1 \leq k$.

Now, formally

$$\begin{aligned} f_\xi^{-1} &= (t_{-\xi} \circ \rho_{-k} \circ \phi_{-\beta/\beta.\beta} \circ t_{-\xi})^{-1} \\ &= t_\xi \circ \phi_{\beta/\beta.\beta} \circ \rho_k \circ t_\xi, \end{aligned}$$

so that $F^{-1} = \{f_\xi^{-1}\}$ is such that $F^{-1} : Z \rightarrow Y$.

7.8.7 Remark.

We notice that the procedure described in §7.8.5 ; 1.), 2.) and 3.) maps each world line of a particle of I 's frame into the *second branch* of a hyperbola of the original system whilst the

first branch is mapped into the world line of a particle of A's frame. Indeed, figure 10 demonstrates the completeduality between the map from the inertial to the accelerated frame and vice versa. Thus both inertial and accelerated systems may be illustrated in the *same* Minkowski space diagram, and the formal results for F^{-1} given above are entirely appropriate to describe the map from the accelerated frame to the inertial one.

7.8.8 Summary.

With reference to figure 10 we may now write down the method of transforming from the inertial frame I to the uniformly accelerated frame A as the following theorem.

Theorem D.

Each constituent of the collection of maps $F : Y \rightarrow Z$,

where $Y = \{(x_0, x_1) \in I : x_0 \geq 0, x_1 + k \geq x_0\}$

and $Z = \{(X_0, X_1) \in A : X_0 \geq 0, X_1 - k \leq -X_0\}$,

given by $F = \{f_\xi\}$, where

$$f_\xi = t_{-\xi} \circ \rho_{-k} \circ \phi_{-\beta/\beta \cdot \beta} \circ t_{-\xi}$$

is everywhere bijective. The collections $F = \{f_\xi\}$ and $F^{-1} = \{f_\xi^{-1}\}$ are those appropriate to map from the inertial frame I to the uniformly accelerated frame and vice versa.

Proof :

We need only confirm that the singular cones associated with

$\phi_{-\beta/\beta \cdot \beta}$ are not activated in the procedure.

(i) $K(0) = \{x : x \cdot x = 0\}$ is only breached at $x = 0$, where

$\phi_{-\beta/\beta \cdot \beta}(0)$ is formally assigned the value 0.

$$(ii) K(\beta) = \{x : (x - \beta) \cdot (x - \beta) = 0, \text{ where } \beta_0 = 0, \beta_1 = -2k \}$$

is not breached for any point in the future of 0. It is clear that the composite maps which comprise the collections F and F^{-1} are not singular in the regions of M_2 to which they apply.

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Concluding remarks.

Postscript.

The investigations reported in this thesis were triggered by a need for a fuller understanding of three key concepts :

- (i) the mathematical structure of Minkowski space,
- (ii) the rôles of causality and the principle of causality,
- (iii) the relation between non-inertial and inertial frames.

It is hoped that valuable insights into those ideas have been transmitted by our method of development. Certainly a remark of Fulton and Rohrlich (1960 : 517),

"One concludes that the [special conformal] transformation ... cannot be regarded as a physically meaningful transformation of a world line ... into uniformly accelerated motion," seems not to have validity ; indeed the statement is modulated in Fulton, Rohrlich and Witten (1962 : 659),

"The fact that the transformation is singular is in no way disturbing. One must merely obey the injunction to stay away from the singularities in discussing any physical process.

Using this transformation, we discuss physical processes only for regions in space-time for which the transformation is non-singular". We have obeyed the injunction to the letter.

Work is proceeding on the problem of mapping from one uniformly accelerated frame to another.

Geometry and topology.

Zeeman (1967) considered the following question in relation to Minkowski space and the globally causal geometry :

"If a set of elements and a group of transformations i.e. a *geometry*

is given, can we design a *topology* that is compatible with it ?".

Here, we mean by "compatible" that the homeomorphism group for the topology is identical to the transformation group for the geometry. Zeeman proved that there was such a topology ; not surprisingly, it turned out to be non-metrizable. However, if we question its usefulness on this ground, for example, by saying that it cannot generate measure and volume for subsets of M_4 , then we must examine to what extent such "four-volumes" are useful in M_4 . It is more likely that only three-volumes, induced on space-like hypersurfaces, are required; that is certainly achieved in Zeeman's topology.

Zeeman's paper (Zeeman, 1967) evidently seeks to absorb causality and the principle of causality at a deeper level in the mathematical model for Special Relativity than we have attempted in this thesis. It would clearly be a matter of some interest to work on a comprehensive review of the mathematical structure of Minkowski space, basing it on topological rather than geometrical foundations. Of course, the physical principle of global causality is then modelled in the mathematical structure at the level of topology. It would certainly be worthwhile for physicists to have a clearer understanding of what this new degree of abstraction implies for the dynamics and physics of Special Relativity.

Recent developments.

The following authors have developed or varied the original design of Zeeman (1967), whilst following the same precepts and limiting their investigation to Minkowski space :-

Nanda (1971, 1972), Whiston (1972), Williams (1974a, 1974b),
Nanda (1979, 1980).

Also of interest is Cel'nik (1968).

The following authors have designed topologies for pseudo-Riemannian spaces associated with General Relativity whose homeomorphisms are also local preservators of causality, differentiability and conformality :-

Kronheimer and Penrose (1967), Kronheimer (1971), Carter (1971),
Woodhouse (1973), Göbel (1976a, 1976b), Hawking, King and
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