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STABILITY OF SLOWLY VARYING CHANNEL
FLOWS: A NUMERICAL APPROACH

by

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Submitted for the degree of Ph.D.

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CONTENTS

	<u>Page</u>
TITLE	1
LIST OF TABLES	3
LIST OF ILLUSTRATIONS	4
ACKNOWLEDGEMENTS	6
DECLARATION	7
ABSTRACT	8
1. INTRODUCTION	10
1.1 Hydrodynamic Stability: a Brief History	10
1.2 Channels of Slowly Varying Width	14
1.3 Layout of Thesis	18
1.4 Discussion of Stability Problem	21
2. SUMMARY OF THE PAPERS EAGLES & WEISSMAN (1975) AND EAGLES & SMITH (1980)	29
2.1 Eagles & Weissman (1975)	30
2.2 Eagles & Smith (1980)	33
3. THE STRAIGHT-WALL DIVERGENT CHANNEL	35
3.1 Equations of Motion	35
3.2 The Divergent Channel	38
3.3 The Steady State Flow	40
3.4 The Stability of the Flow	49
3.5 Numerical Solution	55
3.6 Growth Rates and Wave Number	69
3.7 Numerical Results	72
3.8 The Coupled-Equation Method	95
4. THE CURVED-WALL CHANNEL	101
4.1 The Steady State Flow	101
4.1.1 Numerical Solution of the Stream Function Equation	107
4.1.2 Solution of the Stream Function Equation by an Expansion Method	120
4.2 The Stability of the Flow	127
4.3 Numerical Solution	132
4.4 Growth Rates	140
4.5 Numerical Results	142
5. AN INITIAL-BOUNDARY-VALUE PROBLEM IN A STRAIGHT-WALL DIVERGENT CHANNEL	154
5.1 Governing Equations	154
5.2 Numerical Solution	157
5.3 Numerical Results	166
6. CONCLUSIONS	175
REFERENCES	178

<u>TABLES</u>		<u>Page</u>
Table 3.3.1.	The velocity profile, $g(\eta)$.	48
Table 3.7.1.	The critical Reynolds numbers based on $\hat{E}$ and the corresponding values of α .	92
Table 3.8.1.	Values of the stream function evaluated at the channel centre, $\psi(\xi, 0)$, for $R=15$, $\alpha=0.314$, $\beta=0.1$, $K_0 = 1+i0$.	99
Table 4.1.1.1.	The basic flow stream function, $F(X, \eta)$, for $\lambda=8, 16$ at $X=-2, 0$ and $+2$.	119
Table 4.1.2.1.	The stream function of the basic flow evaluated at $X=0$, $F(0, \eta)$, for $\lambda=1$ and 4 .	126
Table 4.5.1.	Values of R_{crit} and corresponding β_{crit} and ε_{crit} .	150
Table 5.3.1.	Values of $\psi(\xi, 0)$ at time $t=0.008$ with $R=45$, $\gamma=\alpha R=4.093$, $\xi_0=-7.5$, $\xi_1=7.5$.	169
Table 5.3.2.	The effect of a fixed upstream boundary versus a moving upstream boundary.	171

<u>ILLUSTRATIONS</u>	<u>Page</u>
Figure 1.4.1. Typical neutral curve for quasi-parallel approach.	23
Figure 1.4.2. (a), (b) $\beta(\alpha)$ drawn on (α_r, α_i) axes. (c), (d) $\alpha(\beta)$ drawn on (β_r, β_i) axes.	24
Figure 3.2.1. The straight-wall divergent channel.	39
Figure 3.3.1. Velocity profiles $g(\eta)$ for $\gamma = 3.57$ and 4.093.	46
Figure 3.3.2. Velocity profiles $g(\eta)$ for $\gamma = 4.71$: $R=10$ and 30.	47
Figure 3.5.1. The mesh arrangement in the $\xi-\eta$ plane.	56
Figure 3.5.2. Computational molecules for partial derivatives.	57
Figure 3.5.3. The form of the coefficient matrix $\underline{T}$.	63
Figure 3.5.4. The right-hand side column vector $\underline{Q}$.	65
Figure 3.5.5. The matrix $\underline{P}$, used to store the band of matrix $\underline{T}$.	66
Figure 3.7.1. The disturbance stream function evaluated at $\eta=0$, $\psi(\xi, 0)$, for $\alpha=0$, $R=45$, $\beta=1.5$, $K_0=1.794 - i0.024$.	74
Figure 3.7.2. The disturbance stream function evaluated at $\eta=0$, $\psi(\xi, 0)$, for $\gamma=4.093$, $R=45$, $\beta=1.5$, $K_0 = 1.794 - i0.024$.	75
Figure 3.7.3. The disturbance stream function evaluated at $\eta=0$, $\psi(\xi, 0)$, for $\gamma=4.093$, $R=100$, $\beta=1.5$, $K_0=1.798-i0.157$.	76
Figure 3.7.4. The growth rate of ψ , $GR(\psi)$, evaluated at $\eta=0$, for $\gamma=4.093$, $R=45$, $\beta=1.5$.	78
Figure 3.7.5. The wave number of ψ , $WN(\psi)$, evaluated at $\eta=0$, for $\gamma=4.093$, $R=45$, $\beta=1.5$.	79
Figure 3.7.6. The growth rate of ψ , $GR(\psi)$, evaluated at $\eta=0$, for $\gamma=4.093$, $R=100$, $\beta=1.5$.	80
Figure 3.7.7. The wave number of ψ , $WN(\psi)$, evaluated at $\eta=0$, for $\gamma=4.093$, $R=100$, $\beta=1.5$.	81

	<u>Page</u>
Figure 3.7.8. The growth rate of ψ , $GR(\psi)$, evaluated at $\eta=0$ for three K_0 values, with $\gamma=4.093$, $R=25$, $\beta=0.1$.	83
Figure 3.7.9. The disturbance stream function, ψ , at $\xi=0$.	85
Figure 3.7.10. Typical growth rates, GR , of ψ (at $\eta=0$), E and $\hat{E}$ for $\gamma=4.093$, $R=45$, $\beta=0.1$.	87
Figure 3.7.11. Neutral curves based on ψ (at $\eta=0$), E and $\hat{E}$ for $\gamma=3.57$.	88
Figure 3.7.12. Neutral curves based on ψ (at $\eta=0$), E and $\hat{E}$ for $\gamma=4.093$.	89
Figure 3.7.13. Neutral curves based on ψ (at $\eta=0$), E and $\hat{E}$ for $\gamma=4.71$.	90
Figure 3.7.14. The overall critical Reynolds number based on the relative energy $\hat{E}$.	93
Figure 4.1. The curved-wall channel.	102
Figure 4.1.1.1. The coefficient matrix $\underline{A}$ and the column vector of unknowns $\underline{r}$.	116
Figure 4.1.1.2. Velocity profiles $HU(=F_\eta)$ versus η for the three values of $\lambda, 8, 12$ and 16 at $X=0$.	117
Figure 4.1.1.3. The downstream development of the velocity profile, $U(=\frac{1}{H} F_\eta)$, for $\lambda=16$.	118
Figure 4.3.1. Computational molecules for partial derivatives.	134
Figure 4.5.1. The growth rate of the kinetic energy density E : $\lambda=8$, $R=65$, $\beta=1$.	146
Figure 4.5.2. The growth rate of the kinetic energy density E : $\lambda=12$, $R=50$, $\beta=1.2$.	147
Figure 4.5.3. The growth rate of the kinetic energy density E : $\lambda=16$, $R=35$, $\beta=1.3$.	148
Figure 4.5.4. The overall critical Reynolds number based on the kinetic energy density E .	151
Figure 4.5.5. Overall critical Reynolds number versus ϵ based on ψ (at the channel centre), E and $\hat{E}$.	152
Figure 5.3.1. The disturbance stream function at the channel centre, $\psi(\xi, 0, t)$, for $t=5, 10, 15, 20$, with $R=45$, $\gamma=\alpha R=4.093$.	172

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DECLARATION

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ABSTRACT

The hydrodynamic stability of flows in channels of slowly varying width is presented in this thesis. Two classes of channel are considered, a straight-wall divergent channel and a channel with curved walls. A particular curved-wall channel is taken, whose walls have equations

$$y = \pm [1 + \frac{1}{2} \tanh (\epsilon x)] ,$$

where y is the cross-stream variable, x the downstream variable, and ϵ is constant. These are the same physical situations as were considered by Eagles & Weissman (1975) (straight-wall channel) and Eagles & Smith (1980) (curved-wall channel). However the results presented here are obtained by a completely different method than was used in either paper.

The straight-wall channel is taken first. Having introduced the new co-ordinates and set up the governing equations for two-dimensional flow, the steady-state solution for the velocity profile is obtained numerically. The linearized equation for the disturbance stream function is obtained, and this solved by first obtaining the finite difference analogue equations and solving these by Gauss elimination. Growth rates based on the disturbance stream function, the disturbance kinetic energy density and the relative kinetic energy density are defined, and then the results presented. Comparisons are made with results from Eagles & Weissman with favourable agreement, and a graph of the critical Reynolds number based on the relative

kinetic energy density versus the semi-divergence angle of the channel is presented.

For the curved-wall channel, the same order is followed as for the straight-wall channel. The steady state stream function equation is solved by two methods, one numerical and the other an expansion method. The results for the disturbance stream function and critical Reynolds number show reasonable agreement with those of Eagles & Smith. The critical Reynolds number drawn against ϵ is shown.

Finally, the effect of a localized initial perturbation is studied, which necessitates the solution of an initial-boundary-value problem, the linearized equation for the disturbance stream function. This was first replaced by two coupled-equations, one of which was solved by an explicit finite difference scheme, the other (Poisson's equation) by a direct method based on Fourier analysis. A graph showing the evolution of the disturbance as it travels downstream is presented.

1. INTRODUCTION

1.1 Hydrodynamic Stability: a Brief History

It is an experimentally established fact that the laminar flow solutions of the Navier-Stokes equations do not occur at very large Reynolds numbers. Indeed, the most naturally occurring flows are fully or partially turbulent. Apart from their mathematical interest, however, laminar flow solutions are important because they can be observed at low Reynolds numbers, verifying the hypotheses behind the Navier-Stokes equations. Also, a study of their stability as R , the Reynolds number, increases gives information regarding the transition from laminar to turbulent motion, or at least the early part of that transition.

The mathematical investigation of hydrodynamic stability can be formulated by taking a given steady-state solution of the equations of motion and superimposing a disturbance of some kind, which results in a set of non-linear equations for the disturbance. If the disturbance ultimately decays to zero, the flow is said to be stable, but if a disturbance results which is permanently different from zero, the flow is said to be unstable. It does not follow that instability leads to turbulent motion, because another, probably more complex, form of laminar motion may be the result. Indeed, this is often found to be the case.

What is generally sought, then, is the critical value of

R , R_{crit} , at which laminar flow begins to exhibit instability. The value of R_{crit} depends upon the undisturbed primary flow, and on the type of disturbance to which the primary flow is subjected. The problem is very often simplified by assuming that, for small disturbances, the equations governing the behaviour of the disturbance may be linearized; that is the terms quadratic or higher in the disturbances and their derivatives may be neglected. Such is the case in the present work.

It was Helmholtz (1868) who made the first major contribution to the study of hydrodynamic stability. There was an awareness of the problem before this time, but this was indicated just through general description. Indeed, as early as the fifteenth century there were drawings of vortices by Leonardo da Vinci.

Some time after the theoretical papers by Helmholtz, a great deal of most important work was undertaken in this field, particularly by Reynolds (1883) and Rayleigh (1878, 1880, 1887, 1892a, 1892b, 1892c, 1895, 1911, 1913, 1914, 1915, 1916a, 1916b, 1917). It was Reynolds who discovered the first experimental evidence of "sinuous" motion in water and is generally credited with a first description of random or "turbulent" flow. He found the important number bearing his name by dimensional analysis, pointing out that the disorder begins when this pure number exceeds a critical value, and that special stresses must be taken into consideration. However it is Rayleigh who may be

thought of as the founder of hydrodynamic stability theory with the great number of papers which he published between 1878 and 1917. The stability of flames attracted his attention, at the age of thirty six, which led to a first paper on jets. His contributions cover many topics, including the importance of inflexion points in the velocity profile, and the instability of rotating flows between cylinders.

The fact that the existence of a critical Reynolds number could not be explained easily, and that the problem involved both the effect of the second derivative of the mean flow and of the viscous forces, became apparent around 1907. The important equation arrived at independently by Orr (1907) and by Sommerfeld (1908), now known as the Orr-Sommerfeld equation, remained unsolved for twenty-two years, until Tollmien (1929) calculated the first neutral eigenvalues and obtained a critical Reynolds number. This success was made easier by the thesis of Tietjens (1925), who studied diffusion in a flow of constant shear, computing an important function, and by the analysis of Heisenberg (1924). Meanwhile Taylor (1923) produced a first confirmation of a linear theory by his experiments on vortices between concentric rotating cylinders. This was in fact a dual effort where theory and experiment were matched simultaneously.

It was later that Lin (1944, 1945) reviewed and improved the mathematical procedure used by Tollmien, and laid the foundations for a general expansion of the stability analysis.

At this time the stability of Poiseuille flows had become a particularly controversial issue and Lin put it in order by his newer and more general analysis. Any additional doubts with respect to this system were finally settled by the first use of a digital computer in hydrodynamic stability. The results of Heisenberg and Lin were in fact found to be correct.

Also at this time it became apparent that the critical Reynolds number marked only the threshold of "sinuous" motion and not that of turbulence. This was made perfectly clear by the experimental results of Schubauer & Skramstad (1947), who used a vibrating ribbon to impress a controlled perturbation in the study of boundary layer stability.

The attack on compressible flows was started by the work of Landau (1944) and Lees (1947), and continued by Dunn & Lin (1955). Worthy of mention among the great many who studied magnetic, gravitational and convective effects are Bénard (1901) and Chandrasekhar (1961).

In 1955 the comprehensive monograph by Lin appeared, which gave a systematic evaluation of the entire field of hydrodynamic stability. Also at about this time there was enough information available about instability in a continuous medium to group the studies into three main categories:

- (i) the oscillations of parallel flows, or nearly parallel flows: channel flows, boundary layers, jets and wakes;

- (ii) the class with curved streamlines, such as vortices between rotating cylinders or boundary layers along curved walls;
- (iii) those cases where the mean flow is truly zero, for example Bénard cells and convective instabilities.

After establishing the instability of Poiseuille flow, computers were used to make gradually further impacts, among which the works of Brown (1959, 1961a, 1961b, 1962, 1965), Mack (1960, 1965a, 1965b) and Kaplan (1964) are worthy of note.

The present investigation is a numerical attack on the stability of flows in two types of channel with slowly varying width, the first a straight-wall divergent channel and the second a channel with curved walls which become parallel at $\pm\infty$. These involve the same physical situations as were studied by Eagles & Weissman (1975) (the straight-wall channel) and by Eagles & Smith (1980) (the curved-wall channel), but are solved by an entirely different method. More is to be said of channels of varying width in the next section 1.2, while the two papers cited above, for convenience, are summarized separately in Chapter 2.

1.2 Channels of Slowly Varying Width

In the past, investigations of the linear stability of steady flows that are nearly parallel, such as boundary layers, jets or wakes, have been undertaken by use of what may be

termed the "quasi-parallel" approach. In this method it is argued that the stability at some point depends only upon the local properties of the flow, since the basic flow is changing slowly, and consequently the mean flow is taken to be parallel (Schlichting, 1968). Using this method are, among others, Howard (1959) for the stability of a jet, Mattingly & Criminale (1972) for the stability of an incompressible two-dimensional wake, and Eagles (1966) who, it is believed, was the first to investigate the stability of flow in a straight-wall divergent channel.

However, this method neglects the effect induced by evolution of the basic flow on the instantaneous properties of the disturbance. A number of authors have sought to overcome this deficiency for flows which are slowly varying with time or space by use of a slowly-varying approximation known as the WKB method. This was originally a method of approximating to the solutions of a second-order linear differential equation, which was first applied to Schrödinger's equation

$$\frac{d^2\phi}{dx^2} + f(x)\phi = 0,$$

where $f(x)$ is a positive slowly varying function of x , by Wentzel (1926), Kramers (1926) and Brillouin (1926). This method was first suggested for stability problems by Benney & Rosenblat (1964), but does not appear to have been used until Rosenblat & Herbert (1970) applied it to the study of thermal convection in which the temperature of one boundary

is slowly oscillating in time. The method was developed by Bouthier (1972) for a general steady, spatially dependent shear flow, who then applied it (Bouthier, 1973) to the stability of the boundary layer on a flat plate. Among other authors to use the WKB method in their investigations of stability are Gaster (1974), Nayfeh, Mook & Saric (1974), Drazin (1974), Seminara & Hall (1975) and Hall & Parker (1976).

It was this WKB method which was applied by Eagles & Weissman (1975) to the study of flows in a slowly diverging channel, and by Eagles & Smith (1980) for the stability of flows in a curved-wall channel (see Chapter 2 of the present work for further details). The major part of the present work is devoted to the same stability problems as in Eagles & Weissman and Eagles & Smith, but by using a numerical method of solution of the disturbance stream function equations. This present method of attack removes an objection to the WKB method (Smith, 1979b) about the way in which R^{-1} (where R is the Reynolds number) is treated (see the last part of section 1.4 here). Indeed, Smith (1979a) has stated that there would seem to be only two real alternatives to the slowly varying approximation approach for a basic flow which is not quite parallel: one is to solve the complete set of partial differential equations directly, and the other is to examine the behaviour of the disturbances when the Reynolds number is asymptotically large. The latter course was followed by Smith in his study of the stability of the Blasius boundary layer, while the former was taken in the present work for the straight-wall divergent channel and the curved-wall channel. Indeed this appears to be

the first investigation for slowly varying flows by a direct numerical method of solution for the disturbance stream function equation, although numerical methods have been adopted for the solution of the Navier-Stokes equations in nonlinear stability studies for non-parallel flow (for example Fasel, 1976).

For both the cases considered here, fixed frequency disturbances are considered and the growth rate downstream is calculated. There are mathematical arguments (Eagles, 1979) to support the intuitive feeling that if this spatial growth rate is positive the flow is unstable. However, such growth rates vary with the disturbance quantity considered and with the spatial variables, so the meaning is not clear cut. This is discussed in section 1.4.

Also considered in the present work is the determination of the effect of a localized initial perturbation on the steady flow in a straight-wall divergent channel by the numerical solution of an initial-boundary-value problem. The effect of such a disturbance in a Blasius boundary layer has been examined in some detail by Criminale & Kovasznay (1962) by using an asymptotic form of solution which, as Gaster & Grant (1975) pointed out, is strictly only valid far downstream. Hence Gaster (1975) proposed a linear model of a wave packet in a laminar boundary layer. His model wave packet was chosen to conform to a particular experimental situation where a packet was artificially excited by a localized pulsed disturbance at the wall boundary. The method used involved two simplifying

approximations: firstly it was assumed that a flat input spectrum of modes was created at the source, and secondly the spectral behaviour of individual modes was computed from the quasi-parallel flow eigenvalues which had then to be corrected for the influence of the boundary layer development.

As far as is known, there is at present no published work relating to slowly varying non-parallel flows which have dealt with the complete linear initial-boundary-value disturbance equations for the stream function by a direct numerical method and with an initial localized disturbance.

1.3 Layout of Thesis

In this section a brief outline of the layout of the thesis is given so that an overall impression of the work carried out may be obtained.

As already stated, the present investigation is of the same physical situation, although by a completely different method, as those studied by Eagles & Weissman (1975) (straight-wall divergent channel) and Eagles & Smith (1980) (curved-wall channel), and these two papers are summarized in Chapter 2.

The straight-wall divergent channel is dealt with in Chapter 3. After the equations of motion have been set up, and the new co-ordinates introduced, the steady-state

solutions, the Jeffrey-Hamel solutions, for flow in a wedge are obtained numerically. The full equation for the velocity profile is solved, and not the equation with error $O(\alpha^2)$ as is the case in Eagles & Weissman. The linearized equation for the disturbance stream function is then obtained, with a discussion on the upstream and downstream boundary conditions used, and this is followed by a description of the numerical method of solution, based on a direct Gaussian elimination method with interchanges. Definitions of the three growth rates used are given, based on the disturbance stream function, the disturbance kinetic energy density and the relative kinetic energy density. After that the numerical results are presented, with comparisons made between the results of the present work and those of Eagles & Weissman, the diagrams presented showing the closeness of the agreement. Finally in this chapter, a coupled-equation method of solution is briefly mentioned, a method which was rejected due to excessive computer run time, but which provided a very useful check on the method which was used.

Chapter 4 deals with the stability in a curved-wall channel. The steady-state solution is obtained first, with two methods presented for solving the steady state stream function equation. One method, an expansion method, although slightly limited in its use, provided a useful check on the other finite difference numerical method. The equation for the disturbance stream function is found next, with a

description of the numerical method of solution after that. Again three growth rates are defined, based on the same three flow quantities as before, although in Eagles & Smith only the growth rate based upon the disturbance kinetic energy density is used. Finally, the numerical results are given, a comparison with the Eagles & Smith solutions made, and the results extended to include those based on the disturbance stream function and relative kinetic energy density. Again the degree to which the two sets of results agree, those of the present work and those of Eagles & Smith, may be seen.

In the next chapter, 5, the initial-boundary-value problem in a straight-wall divergent channel is studied. The time dependent partial differential equation for the disturbance is first transformed into two coupled equations, the initial localised perturbation is discussed, and the numerical method of solution is presented. This involves one equation, the one involving the time derivative, being solved by an explicit finite difference method, and the other equation, Poisson's equation, being solved by a fast direct method based on Fourier analysis. The results obtained and the checks carried out are then described.

In the final chapter, 6, a general conclusion and discussion is made.

1.4 Discussion of Stability Problem

Most of the past work on hydrodynamic stability has dealt with real wave numbers and complex growth rates with respect to time (for example, Lin, 1955; Curle, 1957; Stuart, 1963; Mattingly & Criminale, 1972; Eagles, 1973; Drazin, 1974; Seminara & Hall, 1975; Hall & Parker, 1976), the idea being that an arbitrary disturbance can be synthesised by a superposition of modes (see Stewartson & Stuart, 1971). For non-parallel flow, however, it is easier to deal with real, fixed frequency waves, and observe their spatial growth or decay. One difficulty with this approach is that if a wave grows downstream, it decays upstream, and vice versa, (and also a wave may grow or decay downstream in different parts of the flow field). For example, the waves discussed in Eagles & Weissman all eventually decay downstream, but would grow indefinitely upstream.

There is, however, an intuitive feeling that if the wave grows in the direction of the base flow, then the base flow is "unstable". This may be made more precise in the case of the quasi-parallel approach by the analysis presented now, which is due to Eagles (1979), where first the equivalence of the critical value of the Reynolds number, R , for 'spatially growing' and 'temporally growing' waves is shown, followed by the relevance of real frequency waves to the stability problem.

Consider a disturbance stream function

$$\psi = e^{i\alpha x - i\beta t} f(y) \quad (1.4.1)$$

to a base flow $w(y)$, considered parallel. Then α and β are connected by the Orr-Sommerfeld problem

$$(D^2 - \alpha^2)^2 f = i\alpha R \left[\left\{ w(y) - \frac{\beta}{\alpha} \right\} (D^2 - \alpha^2) f - w''(y) f \right], \quad (1.4.2)$$

$$f'(0) = f'''(0) = 0; \quad f(1) = f'(1) = 0,$$

where $D \equiv d/dy$, x, y are rectangular Cartesian co-ordinates with y the cross-stream variable and x the downstream variable, and t denotes the time. That root β , as a function of α , which has greatest β_i (imaginary part) is concentrated on.

Well established numerical results show the typical situation for a neutral curve shown in figure 1.4.1. On the line ABCD, as α is increased, β_i is first negative, increases to a positive maximum, and then decreases until it becomes negative again. An extrapolation may be made to sketch lines $\beta_i = \text{constant}$ and $\beta_r = \text{constant}$ in the complex α plane.

Figure 1.4.2(a) shows the situation for $R > R_C$ (where R_C is the critical Reynolds number), and figure 1.4.2(b) when $R = R_C$. These may be interpreted for α as a function of β to give figures 1.4.2(c) ($R > R_C$) and 1.4.2(d) ($R = R_C$).

Thus it is seen that if a search is made for α as a function of β (real) then if there is a range of α for which $\alpha_i < 0$, it must follow that $R > R_C$, where R_C may be thought to be defined by figure 1.4.1. The conclusion is then made by Eagles (1979) that as far as quasi-parallel theory is concerned, using real β and searching for complex α yields the same R_C

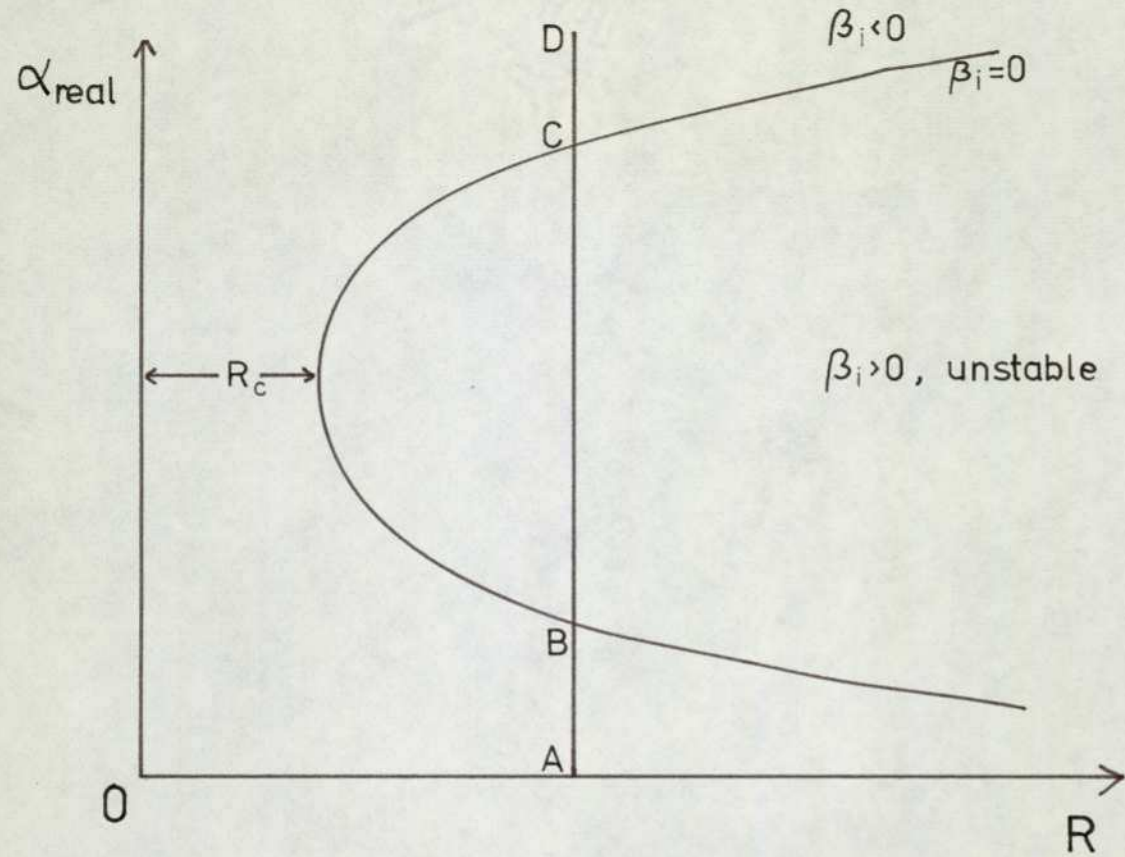
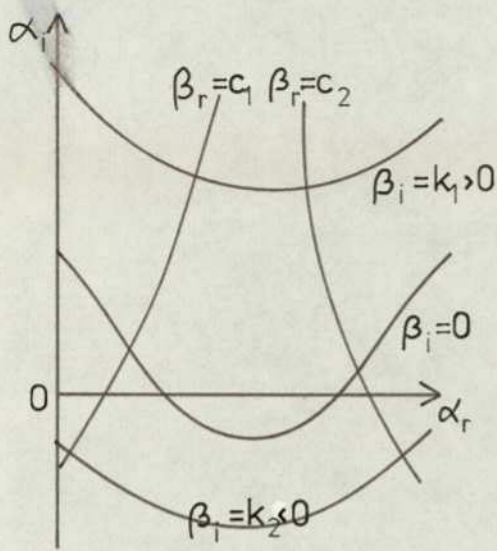
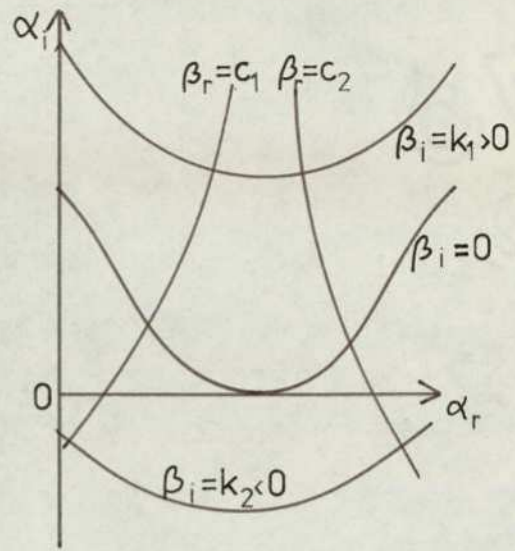


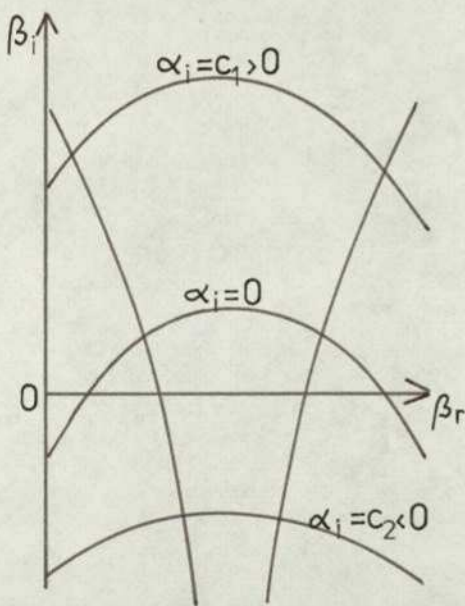
Figure 1.4.1: Typical neutral curve for "quasi-parallel" approach. R_c = critical value of the Reynolds number R .



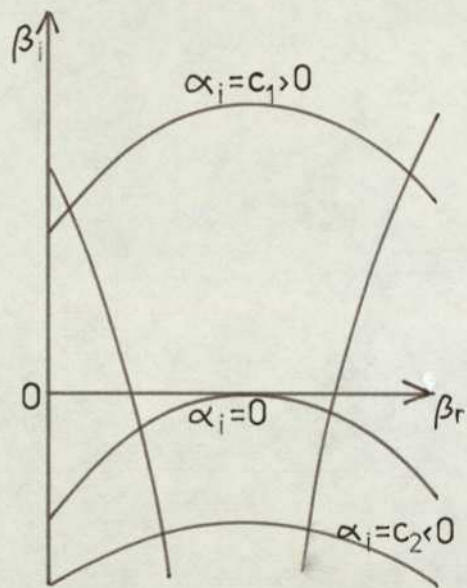
(a)



(b)



(c)



(d)

Figure 1.4.2: (a), (b) $\beta(\alpha)$ drawn on (α_r, α_i) axes for
 (a) $R > R_c$, (b) $R = R_c$;
 (c) (d) $\alpha(\beta)$ drawn on (β_r, β_i) axes for
 (c) $R > R_c$, (d) $R = R_c$.

as using real α and searching for complex β .

The relevance of real frequency waves to the stability problem is then shown by Eagles (1979) as follows.

Consider again the small disturbance of the stream function given by (1.4.1) superimposed on the velocity $w(y)$. Then the Orr-Sommerfeld problem (1.4.2) determines α as a (many valued) function of β . Then $\alpha(\beta)$ is taken to be that root with greatest value of $-\alpha_i$ which, it is assumed, uniquely defines $\alpha(\beta)$ in some region of the complex plane containing the real axis of β . Then

$$\int_{-\infty}^{\infty} e^{i\alpha(\beta)x - i\beta t} f(y) d\beta$$

is also a solution of the disturbance equations, provided that the integral converges. A simplified integral is now considered

$$I = \int_{-\infty}^{\infty} e^{i\alpha(\beta)x - i\beta t} d\beta. \quad (1.4.3)$$

At $x = 0$ this is a δ -function in t , which provides a suitable impulsive disturbance to the flow, which could be physically approximated. It is in this sense that it is said that a 'general disturbance' will contain components of a real frequency type - or can be regarded as containing components of real frequency. It could equally well be regarded as a certain integral of 'real wave number' waves.

The Reynolds number, R , is considered now to be slightly

greater than the critical value, R_C . Eagles (1979) then goes on to show, by use of the method of steepest descent for approximating (1.4.3) (see Jeffreys & Jeffreys, 1950), that a 'wave packet' type of disturbance is given, and further that

$$|I|_{\max \text{ over } x} \approx \exp(\sigma^2 x)$$

for some real σ^2 dependent upon α and β , which therefore grows with distance. (Also σ^2 is such that if $R = R_C$, $\sigma^2 = 0$ and so the 'packet height' remains constant). Thus the situation given leads to a growing wave packet. Hence, if there exist values of α with $\alpha_1 < 0$ for a range of real β , the flow is unstable.

Although a generalization to non-parallel flow is not easy, it is plausible that the main idea carries over, that a positive growth rate downstream of an Eagles & Weissman type disturbance would indicate instability to an arbitrary disturbance.

Another possible approach, however, is to imagine that at $\xi = \xi_0$ (say), where ξ is the downstream variable used in the straight-wall channel analogous to x , some sort of wave maker is in operation at a fixed frequency (for example, an acoustic signal or vibrating ribbon - see Gaster, 1965). In terms of the Eagles & Weissman theory, this would presumably invoke a sum of modes of the type

$$\int_{-\infty}^{\infty} e^{-i\beta t + S(X;\beta)} f(\eta, X) d\beta, \quad (X = \alpha\xi),$$

(where α now refers to the semi-divergence angle of the straight-wall channel and η is the new cross-stream variable) and if the situation a long time after the wave maker has been switched on is considered, it would reasonably be expected that the component with the largest growth rate would dominate, at least at some distance from the wave maker. Thus the exact conditions specified at the wave maker would not be important, since all but the "largest" mode would die away with increasing ξ . In the work presented here, therefore, the boundary conditions at $\xi = \xi_0$ are described in terms of the wave maker idea, and, by analogy with Eagles & Weissman, the flow is described as unstable if there exist regions in which a positive growth rate is found over some range of ξ .

The downstream boundary conditions also need some discussion. If the present work was restricted to the limited object of trying to check the results of Eagles & Weissman and Eagles & Smith, then $\psi \rightarrow 0$ as $\xi \rightarrow \infty$ should be set. However, this is open to the objection that the solution is somehow being forced to be similar to Eagles & Weissman and Eagles & Smith. Therefore a less restrictive condition, which allows, but does not force, $\psi \rightarrow 0$ as $\xi \rightarrow \infty$, was also used as a check, namely

$$\frac{\partial \psi}{\partial \xi} = iK\psi, \quad \frac{\partial^2 \psi}{\partial \xi^2} = -K^2\psi \text{ at } \xi = \xi_1,$$

where K is some arbitrary complex number.

It should be emphasised that it eventually turned out, after extensive trials, that the exact nature of the boundary conditions at ∞ and at ξ_0 were unimportant. The "growth rates" were found to be almost independent of the boundary conditions over a large range of ξ , indicating that this growth rate is a fundamental property of the system, and that the theory of Eagles & Weissman in choosing the "greatest" mode does appear relevant.

The other point needing discussion is the way in which R^{-1} is treated like a constant as $\alpha \rightarrow 0$ in Eagles & Weissman. In their $O(1)$ equation

$$L\Psi_1 \equiv \left[R^{-1} (D^2 - K^2)^2 + i(\beta_1 - wK) (D^2 - K^2) + iKw_{\eta\eta} \right] \Psi_1 = 0,$$

$$\Psi_1 = D\Psi_1 = 0 \text{ at } \eta = \pm 1,$$

where $D \equiv \partial/\partial\eta$, the term $R^{-1} \frac{\partial^4 \Psi}{\partial \eta^4}$ is retained since, it was argued, although $R^{-1} = O(\alpha)$ and is therefore of higher order, this term contains the highest order derivative and will be dominant somewhere in the flow field. It has been suggested (Smith, 1979a, 1979b), however, that the discrepancies which exist between theory and experiment might be attributed to this treatment of R , assuming it to be at one moment large, and at another of $O(1)$. It should be pointed out that in the present approach, no such assumptions were made, and that α and R were treated as independent parameters. Indeed, no approximations were used, except for the basic, steady flow for the curved-wall channel.

2. SUMMARY OF THE PAPERS EAGLES & WEISSMAN (1975)
AND EAGLES & SMITH (1980)

The major part of the present work is devoted to the same physical problems that were dealt with by Eagles & Weissman (1975) and by Eagles & Smith (1980), but by a completely different theoretical method of a numerical solution of the governing partial differential equations. Each of the two papers is concerned with investigations into the stability of two-dimensional flows in channels, the former with flows in a straight-wall, divergent channel, and the latter with flows in a channel with curved walls. The two are summarized in the following two sub-sections.

2.1 Eagles & Weissman (1975)

This paper deals with the stability of flow, which is slowly varying, in a straight-walled, divergent channel, of divergence angle 2α (see figure 3.1.1). The study was carried out by use of a slowly varying approximation, a modification of the method known as the 'WKB', 'ray' or multiple scaling method. This method was used in order to overcome some of the deficiencies in the "quasi-parallel" approach, in which the mean flow is taken to be parallel and analysis then made by normal modes. Two deficiencies referred to in the paper are, firstly, that the "quasi-parallel" method can only determine whether a wave is growing or decaying at a particular point; it cannot determine the solution as a function of the downstream variable. Secondly, it cannot take into account, or even estimate, the effect of the variation of the basic state on the local growth.

The slowly varying approximation method, developed primarily for conservative systems which support neutral waves (Keller, 1958; Lewis, 1964; Bretherton, 1966, 1968), but which appears to work just as well for flows that are unstable, may be applied to flows with a small rate of change in time or in the direction of wave propagation (although the overall change need not be small). In the equation for the disturbance stream function $\psi(\xi, \eta, t)$, where ξ is the streamwise variable, η the cross-stream variable, t the time (see section 3.2 of the present work for the transformation of the two-dimensional

polar coordinates (r, ϕ) to the (ξ, η) coordinate system), since the coefficients are independent of time and slowly varying with ξ , the authors sought constant frequency solutions of the form

$$\psi(\xi, \eta, t) = \Psi(\eta, X) \exp \left[i \left\{ \theta(\xi) - \beta t \right\} \right] + \text{c.c.}$$

where c.c. refers to the complex conjugate of the preceding term, $X = \alpha \xi$ is the slow variable, $\theta(\xi)$ is the complex phase function and describes the fast variation, while its derivative, the wave number K , is assumed to be slowly varying; that is

$$K = \frac{d\theta}{d\xi} = K(X).$$

Further, they considered waves of real frequency, waves that would be generated by a wave maker in an experiment, and so took β to be real.

Approximations were then made by letting

$$\Psi = \Psi_1 + \alpha \Psi_2 + \dots$$

Thus followed a sequence of problems, in which equations and solutions were taken to be up to $O(\alpha)$, which included eigenvalue problems and an "amplitude equation". Also for the basic, steady state flow, the equation was taken to $O(\alpha)$ (the $O(\alpha)$ version of equation (3.3.1) here). These flows are the Jeffrey-Hamel profiles, and are described here

in section 3.3. Note that in the present work, no such approximations are used, for the basic flow or the disturbance equation, and equation (3.4.4) is solved numerically.

The authors point out the problem of what to use as a measure of amplitude. Shen (1961) argues that, since the basic state is changing, the growth or decay of a disturbance should be measured relative to the basic flow. Following his suggestions, a growth rate based on the ratio of the disturbance kinetic energy to basic flow kinetic energy was used, together with two other growth rates, one based on the kinetic energy itself, and the other based on the stream function. The definitions for these growth rates are presented here in section 3.6.

A critical Reynolds number was defined, so that as long as the Reynolds number is larger than this critical value, there is a band of growing waves for any value of X (although this may not indicate ^{non-linear} instability). Also a critical Reynolds number can be defined for each flow quantity, indicating when that particular flow quantity will be able to grow. A graph was presented showing the critical Reynolds number based on the relative kinetic energy distribution plotted against the semi-divergence angle α .

The results obtained by Eagles & Weissman, the approximate solutions for waves of constant, real frequency as they travel

down a slowly diverging channel, indicate that although waves may pass through regions of growth and regions of decay, they all eventually decay as they travel far downstream from the wave maker. Also discovered was the fact that the growth rates of the various flow quantities considered (the stream function, the absolute kinetic energy, the relative kinetic energy) are each different, and that, at some point in the channel, any one or more of the quantities may be growing while the others are decaying.

It was pointed out by the authors, however, that although waves of all frequencies eventually decay downstream, in the unstable region waves may become large enough for non-linear effects to be important, although the results of Eagles (1973) indicate that the non-linearity will have a stabilizing effect, for some values of α .

2.2 Eagles & Smith (1980)

Both the basic, steady flow and its stability, in a curved-wall channel, are dealt with in this paper. The equations of the walls of the channel were taken as $y = \pm H(X)$, where $X = \epsilon x$ is a slow streamwise variable, x being the original streamwise variable, and ϵ being a small quantity. The special case considered in the paper was with

$$H(X) = 1 + \frac{1}{2} \tanh X,$$

with the flow at $X = \pm\infty$ expected to be Poiseuille flow

(see figure 4.1).

The steady state solution was obtained by solving an $O(\epsilon)$ equation (equation 4.1.6 of the present work), and this is described here in section 4.1. For the stability of the flow, the same slowly varying approximation as for the straight-wall channel was used, with equations and solutions used to $O(\epsilon)$. In the present work, the same equation as Eagles & Smith is solved for the steady state, but no $O(\epsilon)$ approximation is used for the stability calculations, the disturbance stream function equation (equation 4.2.4 here) being solved numerically.

For the particular channel indicated above, Eagles & Smith calculated the critical Reynolds number below which all fixed frequency waves have kinetic energy density $E(X)$ decaying downstream. The results indicate that with a Reynolds number above the critical value, waves will have growing $E(X)$ as the wave progresses downstream, for some limited range of X . This, they pointed out, is a different situation from the straight-walled channel, when with the Reynolds number greater than the critical value, growing disturbances would be expected to occur at all X -stations.

3. THE STRAIGHT-WALL DIVERGENT CHANNEL

3.1 Equations of Motion

The Navier-Stokes equations for incompressible, viscous flow with zero external forces, are

$$\frac{d\underline{v}'}{dt'} = - \frac{1}{\rho} \text{grad } p' + \nu \nabla_1'^2 \underline{v}' , \quad (3.1.1)$$

where ρ is the constant fluid density, p' the pressure, $\underline{v}'$ the velocity vector, ν the kinematic viscosity. The total or substantive derivative following the motion, $\frac{d}{dt'}$, sometimes called the "Stokes derivative", consists of an unsteady term and a convective term

$$\frac{d}{dt'} \equiv \frac{\partial}{\partial t'} + (\underline{v}' \cdot \text{grad}) .$$

The equation is used in two-dimensional polar coordinates (r', ϕ') , and so

$$\nabla_1'^2 \equiv \frac{\partial^2}{\partial r'^2} + \frac{1}{r'} \frac{\partial}{\partial r'} + \frac{1}{r'^2} \frac{\partial^2}{\partial \phi'^2} .$$

($\nabla_1'^2$ is used here since ∇^2 is used later for a system with transformed co-ordinates).

The equation of continuity is

$$\text{div } \underline{v}' = 0. \quad (3.1.2)$$

These equations can be expressed in non-dimensional form in terms of M , where $2M$ is the volumetric flow rate per unit thickness, and a representative length b (which is half the distance across channel at the ^{imaginary} wave-maker to be discussed in section 3.2), where

$$\underline{v} = \left(\frac{b}{M}\right) \underline{v}', \quad r = \frac{r'}{b}, \quad t = \left(\frac{M}{b^2}\right) t', \quad p = \left(\frac{b^2}{\rho M^2}\right) p'.$$

Equations (3.1.1) and (3.1.2) then become

$$\frac{d\underline{v}}{dt} = -\text{grad } p + \frac{1}{R} \nabla_1^2 \underline{v} \quad (3.1.3)$$

$$\text{div } \underline{v} = 0, \quad (3.1.4)$$

where $R = \frac{M}{\nu}$ is the Reynolds number.

A convenient form of equation (3.1.3) is

$$\begin{aligned} \frac{\partial \underline{v}}{\partial t} - \underline{v} \wedge \text{curl } \underline{v} &= -\text{grad } (p + \frac{1}{2} \underline{v} \cdot \underline{v}) \\ &\quad - \frac{1}{R} \text{curl curl } \underline{v}. \end{aligned} \quad (3.1.5)$$

Defining the vorticity vector by the relation

$$\underline{\zeta} = \text{curl } \underline{v},$$

the pressure is eliminated from the Navier-Stokes equations by taking the curl of equation (3.1.5), yielding

$$\frac{\partial \underline{\zeta}}{\partial t} - \text{curl } (\underline{v} \wedge \underline{\zeta}) = -\frac{1}{R} \text{curl curl } \underline{\zeta},$$

or

$$\frac{\partial \underline{\zeta}}{\partial t} = (\underline{\zeta} \cdot \text{grad}) \underline{v} - (\underline{v} \cdot \text{grad}) \underline{\zeta} + \frac{1}{R} \nabla_1^2 \underline{\zeta}. \quad (3.1.6)$$

This equation is the vorticity transport equation.

As stated, the cases considered are two-dimensional, and thus

$$(\underline{\zeta} \cdot \text{grad}) \underline{v} = 0,$$

so that the vorticity transport equation becomes

$$\left(\frac{\partial}{\partial t} + \underline{v} \cdot \text{grad} \right) \underline{\zeta} = \frac{1}{R} \nabla_1^2 \underline{\zeta}. \quad (3.1.7)$$

In polar coordinates, $\underline{v} = (U_r, U_\phi, 0)$, and the continuity equation (3.1.4) becomes

$$\frac{\partial}{\partial r} (r U_r) + \frac{\partial U_\phi}{\partial \phi} = 0. \quad (3.1.8)$$

In terms of the stream function Ψ , where

$$U_r = \frac{1}{r} \frac{\partial \Psi}{\partial \phi}, \quad U_\phi = -\frac{\partial \Psi}{\partial r}, \quad \underline{\zeta} = (0, 0, -\nabla_1^2 \Psi),$$

satisfying (3.1.8) identically, equation (3.1.7) takes the form

$$\left[\nabla_1^2 - R \frac{\partial}{\partial t} - R \left(\frac{1}{r} \frac{\partial \Psi}{\partial \phi} \frac{\partial}{\partial r} - \frac{1}{r} \frac{\partial \Psi}{\partial r} \frac{\partial}{\partial \phi} \right) \right] \nabla_1^2 \Psi = 0. \quad (3.1.9)$$

3.2 The Divergent Channel

The flow considered is two-dimensional in a straight-wall divergent channel of divergence angle 2α . To investigate the stability of the basic flow, which is steady and symmetric, an imaginary wave-maker is taken to be situated at a position in the channel such that the channel width is approximately $2b$. Upstream from this position, the walls of the channel curve smoothly until they are parallel (see Figure 3.2.1).

Coordinates ξ and η are introduced by the following definitions:

$$\xi = \frac{1}{\alpha} \ln(\alpha r), \quad \eta = \frac{\phi}{\alpha};$$

the walls of the channel thus becoming $\eta = \pm 1$, with the imaginary wave-maker situated at $\xi = 0$. Using these coordinates yields the following:

$$\frac{\partial}{\partial r} \equiv e^{-\alpha\xi} \frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial \phi} \equiv \frac{1}{\alpha} \frac{\partial}{\partial \eta},$$

$$\nabla_1^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2}$$

$$\begin{aligned} &\equiv e^{-2\alpha\xi} \left(\frac{\partial^2}{\partial \xi^2} - \alpha \frac{\partial}{\partial \xi} \right) + \alpha e^{-\alpha\xi} \cdot e^{-\alpha\xi} \frac{\partial}{\partial \xi} + \alpha^2 e^{-2\alpha\xi} \frac{1}{\alpha^2} \frac{\partial^2}{\partial \eta^2} \\ &\equiv e^{-2\alpha\xi} \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right). \end{aligned}$$

∇^2 is defined as $\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}$, and thus

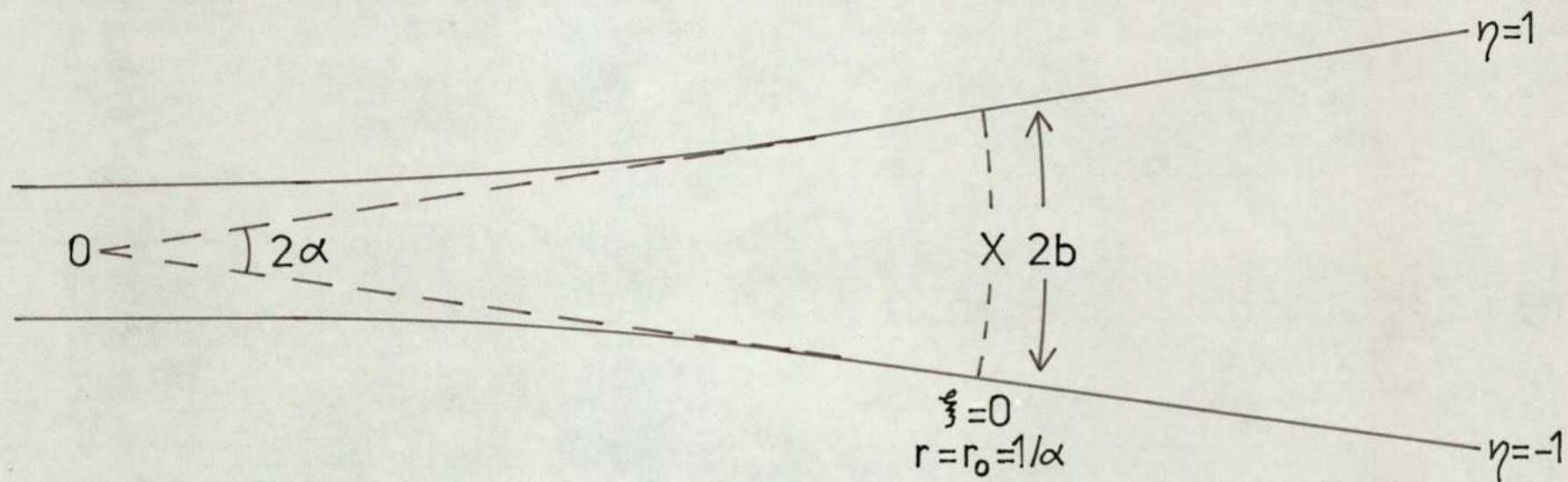


Figure 3.2.1: The straight-wall divergent channel.
X marks the position of a wave maker.

$$\nabla_1^2 \equiv e^{-2\alpha\xi} \nabla^2, \quad (3.2.1)$$

$$\frac{1}{r} \frac{\partial \Psi}{\partial \varnothing} \frac{\partial}{\partial r} \equiv e^{-2\alpha\xi} \frac{\partial \Psi}{\partial \eta} \frac{\partial}{\partial \xi}, \quad (3.2.2)$$

$$\frac{1}{r} \frac{\partial \Psi}{\partial r} \frac{\partial}{\partial \varnothing} \equiv e^{-2\alpha\xi} \frac{\partial \Psi}{\partial \xi} \frac{\partial}{\partial \eta}. \quad (3.2.3)$$

On substituting (3.2.1) - (3.2.3) into equation (3.1.9), the following equation is obtained:

$$\left[\nabla^2 - e^{2\alpha\xi} R \frac{\partial}{\partial t} - R \left(\frac{\partial \Psi}{\partial \eta} \frac{\partial}{\partial \xi} - \frac{\partial \Psi}{\partial \xi} \frac{\partial}{\partial \eta} \right) \right] (e^{-2\alpha\xi} \nabla^2 \Psi) = 0. \quad (3.2.4)$$

The boundary conditions, which specify the total flow rate and the condition of zero slip at the walls, are

$$\Psi = \pm 1, \quad \frac{\partial \Psi}{\partial \eta} = 0 \text{ when } \eta = \pm 1.$$

3.3 The Steady State Flow

The steady, ξ -independent solutions of equation (3.2.4), denoted by Ψ_0 , represent the stream function of the basic flow. These solutions are the Jeffrey-Hamel solutions for flow in a wedge (see Fraenkel, 1962, 1963). By taking

$g(\eta) = \frac{\partial \psi}{\partial \eta}$, it is found that g satisfies the equation

$$g'''' + 4\alpha^2 g' + 2\alpha R g g' = 0, \quad (3.3.1)$$

(where a prime denotes differentiation with respect to η)
with boundary conditions

$$g(\pm 1) = 0, \quad \int_{-1}^{+1} g \, d\eta = 2.$$

The solutions are to be restricted to symmetric velocity profiles here, so that the equation may be solved for half the region, from $\eta = 0$ (channel centre) to $\eta = 1$ (wall). The boundary conditions then become

$$g(1) = 0, \quad g'(0) = 0, \quad \int_0^1 g \, d\eta = 1.$$

Following Eagles & Weissman (1975), a parameter γ is introduced, where $\gamma = \alpha R$, so that the solution of equation (3.3.1) may be expressed as $g(\eta; R, \gamma)$. The parameter γ is restricted to a certain range $-\infty < \gamma < \gamma_1$ in Fraenkel's (1962) theory, where $\gamma > 0$ corresponds to outflow and $\gamma < 0$ to inflow; the value of γ_1 depends on R . For example with $R = 10$, $\gamma_1 = 5.05$; with $R = 8$, $\gamma_1 = 4.936$; with $R = \infty$, $\gamma_1 = 5.46$. Within the range $-\infty < \gamma < 5.46$, $g(\eta; R, \gamma)$ is an analytic function of γ , but at $\gamma = 5.46$, $\frac{\partial g}{\partial \gamma}$ becomes infinite, although

g itself remains finite. As Eagles (1966) points out, the flow cannot be analytically continued by any symmetric Jeffrey-Hamel function beyond $\gamma = 5.46$ owing to this singularity in g . Also, as in Eagles & Weissman (1975), only the outflow solutions are considered here, and thus equation (3.3.1) is to be solved with $0 \leq \alpha R < 5.46$.

Although it is possible to solve equation (3.3.1) by the use of elliptic functions, it would appear to be easier to solve the equation directly by a numerical method, and this is done in the present work.

In order to solve equation (3.3.1) using the Runge-Kutta method, it would be necessary to have the starting values $g(0)$, $g'(0)$, $g''(0)$ specified, and hence it would be possible to evaluate each $g(\eta)$, $g'(\eta)$, $g''(\eta)$ at equally spaced values of η up to $g(1)$, $g'(1)$, $g''(1)$. However, in this case only $g'(0)$ is known together with two other conditions, and so it is necessary to use an iterative scheme based on the 4th order Runge-Kutta method as follows.

A and C are taken as the starting values of $g(0)$ and $g''(0)$ respectively, and the following functions are defined:

$$F_1(A, C) \equiv g(1)$$

$$F_2(A, C) \equiv \int_0^1 g \, d\eta - 1.$$

Note that with exact values for $g(0)$ and $g''(0)$,

$F_1(A,C) = F_2(A,C) = 0$. Thus some initial "guess" is made for the values of A and C, and then better approximations to these values are obtained.

If δA and δC are, respectively, small increments in A and C, then, by Taylor's Theorem:

$$F_1(A+\delta A, C+\delta C) \approx F_1(A,C) + \delta A \frac{\partial F_1}{\partial A} + \delta C \frac{\partial F_1}{\partial C}, \quad (3.3.2)$$

$$F_2(A+\delta A, C+\delta C) \approx F_2(A,C) + \delta A \frac{\partial F_2}{\partial A} + \delta C \frac{\partial F_2}{\partial C}. \quad (3.3.3)$$

The iterative method employed is then as follows:

(i) Equation (3.3.1) is solved by Runge-Kutta with $g(0) = A$, $g'(0) = 0$, $g''(0) = C$.

(ii) $F_1(A,C)$, $F_2(A,C)$, $\frac{\partial F_1}{\partial A}$, $\frac{\partial F_1}{\partial C}$, $\frac{\partial F_2}{\partial A}$, $\frac{\partial F_2}{\partial C}$ are evaluated.

(iii) Assuming that $F_1(A+\delta A, C+\delta C) = F_2(A+\delta A, C+\delta C) = 0$, equations (3.3.2) and (3.3.3) are solved simultaneously for δA and δC .

(iv) Better approximations to A and C are taken as $A+\delta A$, $C+\delta C$ respectively.

(v) If $|F_1(A,C)| < \epsilon$ and $|F_2(A,C)| < \epsilon$, where ϵ is some small prescribed number, the iteration is complete; if not, return to (i) with the new values of A and C.

In order to evaluate $\frac{\partial F_i}{\partial A}, \frac{\partial F_i}{\partial C}$ ($i=1,2$)

in equations (3.3.2) and (3.3.3), equation (3.3.1) is solved with the four sets of values $(A-h, C)$, $(A+h, C)$, $(A, C-h)$, $(A, C+h)$ as the starting values $(g(0), g''(0))$, where h is a small step length in A and C ; central difference formulae are then used for the derivatives:

$$\frac{\partial F_i}{\partial A} = \frac{F_i(A+h, C) - F_i(A-h, C)}{2h} + O(h^2),$$

$$\frac{\partial F_i}{\partial C} = \frac{F_i(A, C+h) - F_i(A, C-h)}{2h} + O(h^2).$$

($i=1,2$)

The integral $\int_0^1 g \, d\eta$ is evaluated by using Simpson's rule.

Many sets of values for (α, R) could be accommodated during any one computer run, the initial values of A and C for a given (α, R) being the final values obtained for the previous set (α, R) .

The ICL 1905F computer of the City University was used for the solution, with a step length $\delta\eta = 0.025$. Test runs were made by ignoring the $O(\alpha^2)$ term in equation (3.3.1) and comparing the results with those obtained by Eagles & Weissman and Fraenkel. A further test was made by putting $\alpha = 0$ in equation (3.3.1) and so obtaining Poiseuille flow results. Agreement was obtained in both tests.

Checks were also made by using $\delta\eta = 0.0125$, giving agreement to five decimal places or better with those results using $\delta\eta = 0.025$.

The solutions for $g(\eta)$, $g'(\eta)$, $g''(\eta)$ are stored in order to be used as data for the stability problem, as is

$$\Gamma = \int_0^1 [g(\eta)]^2 d\eta \text{ which is also required (see section 3.6).}$$

Typical velocity profiles are shown in figures 3.3.1 and 3.3.2 for the values $\gamma = \alpha R = 3.57, 4.093, 4.71$, these being the cases which are to be compared with the results obtained by Eagles & Weissman.

Note that γ was fixed, R varied and the corresponding values of α found. For the smaller values of γ (3.57 and 4.093), it is found that there is little variation in the solution for g as R varied. However at $\gamma = 4.71$, there are noticeable differences in the solutions for different R values. Hence only one curve is drawn for each of $\gamma = 3.57$ and 4.093, while two typical curves are drawn for $\gamma = 4.71$ ($R=10$ and $R=30$). Note, too, that the solution for $\gamma = 4.71$, $R=10$ exhibits some reversed flow near the wall.

Also presented in table 3.3.1 are numerical values for typical velocity profiles, $g(\eta)$, covering the range of γ considered in the present work for stability considerations, namely $\gamma = 3.25, R=50$; $\gamma = 4.093, R=45$; $\gamma = 4.8, R=10$.

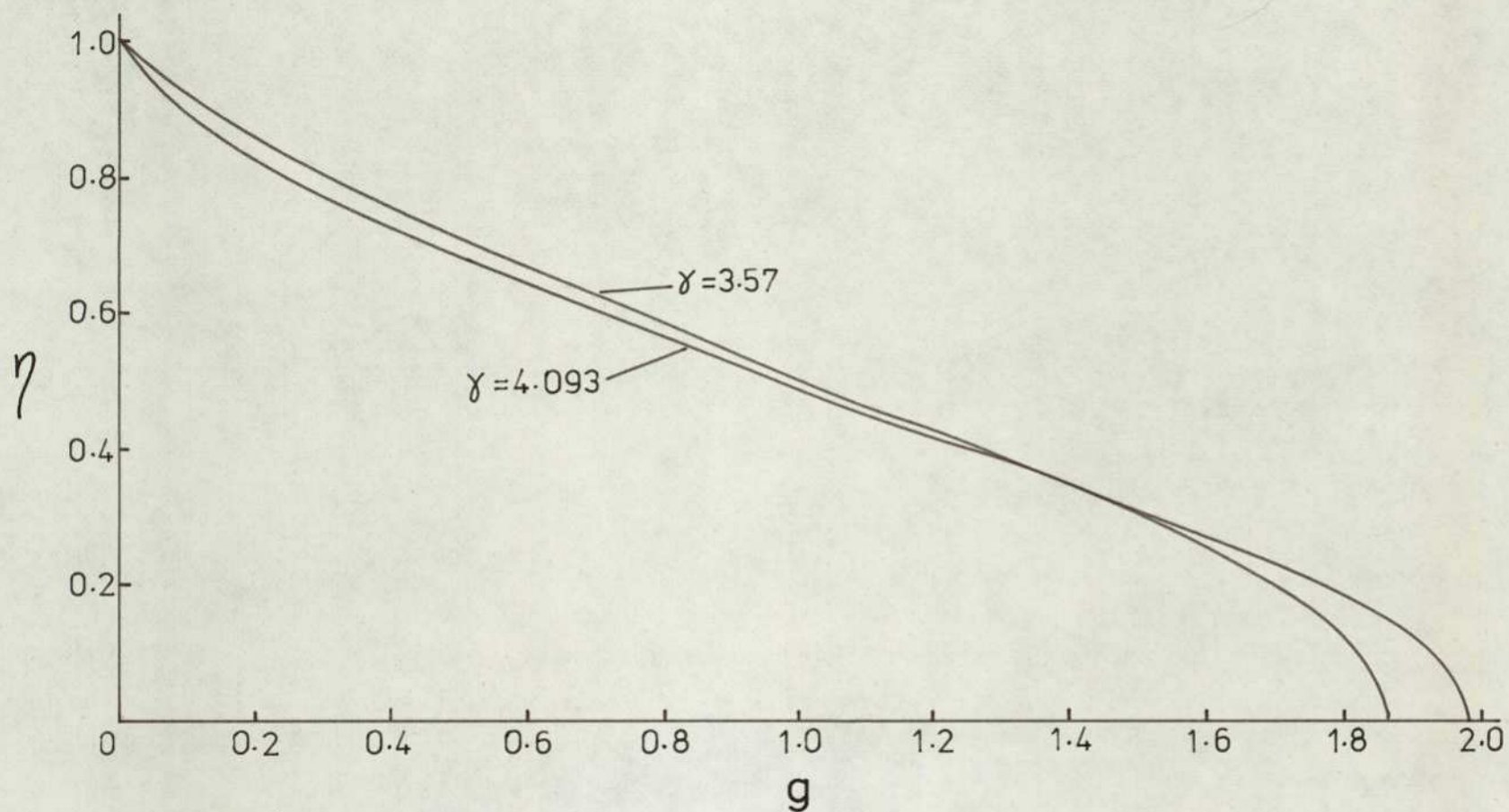


Figure 3.3.1: Velocity profiles $g(\eta)$ for $\gamma = 3.57$ and 4.093 .

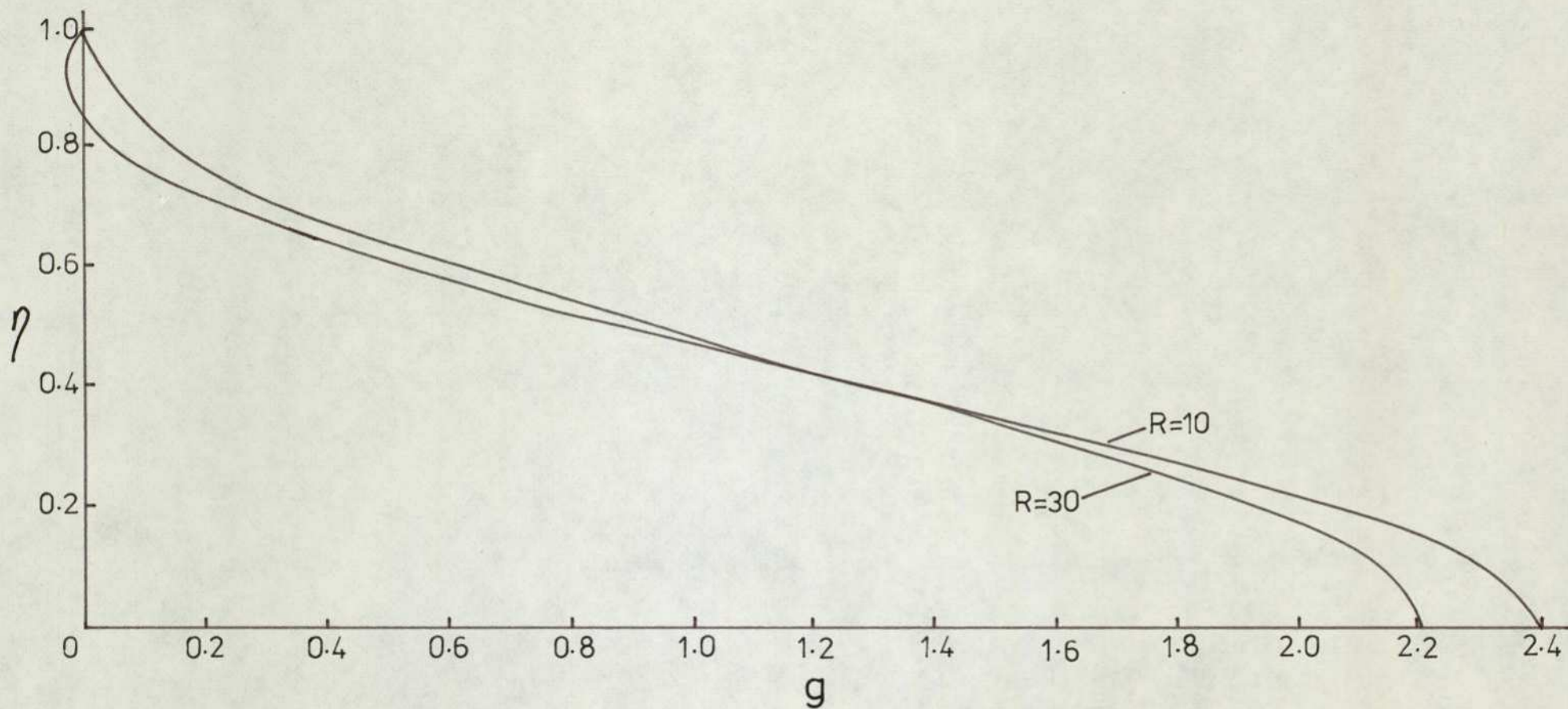


Figure 3.3.2: Velocity profiles $g(\eta)$ for $\gamma = 4.71$: $R = 10$ and 30 .

η	$g(\eta)$		
	$\gamma=3.25, R=50$	$\gamma=4.093, R=45$	$\gamma=4.8, R=10$
0.0	1.8000	1.9765	2.4792
0.1	1.7622	1.9230	2.3762
0.2	1.6531	1.7710	2.0912
0.3	1.4847	1.5428	1.6857
0.4	1.2737	1.2689	1.2350
0.5	1.0390	0.9804	0.8055
0.6	0.7980	0.7037	0.4432
0.7	0.5651	0.4582	0.1738
0.8	0.3507	0.2555	0.0084
0.9	0.1612	0.1020	-0.0498
1.0	0.0	0.0	0.0

Table 3.3.1 The velocity profile, $g(\eta)$, for the values of R & γ shown.

3.4 The Stability of the Flow

In order to investigate the stability of the basic flow, a small disturbance is superimposed on the flow by allowing

$$\Psi = \Psi_0(\eta) + \Psi_1(\xi, \eta, t)$$

in equation (3.2.4) and linearizing in Ψ_1 .

By observing that

$$\nabla^2 (e^{-2\alpha\xi} \nabla^2 \Psi) = e^{-2\alpha\xi} \left[\left(\frac{\partial}{\partial \xi} - 2\alpha \right)^2 + \frac{\partial^2}{\partial \eta^2} \right] \nabla^2 \Psi,$$

equation (3.2.4) is first re-written as:

$$\begin{aligned} & \frac{\partial^2 (\nabla^2 \Psi)}{\partial \xi^2} - 4\alpha \frac{\partial}{\partial \xi} (\nabla^2 \Psi) + 4\alpha^2 \nabla^2 \Psi + \frac{\partial^2}{\partial \eta^2} (\nabla^2 \Psi) - e^{2\alpha\xi} R \frac{\partial}{\partial t} (\nabla^2 \Psi) \\ & - R \left[\frac{\partial \Psi}{\partial \eta} \left(\frac{\partial}{\partial \xi} (\nabla^2 \Psi) - 2\alpha \nabla^2 \Psi \right) - \frac{\partial \Psi}{\partial \xi} \frac{\partial}{\partial \eta} (\nabla^2 \Psi) \right] = 0. \end{aligned} \quad (3.4.1)$$

On substituting $\Psi = \Psi_0(\eta) + \Psi_1(\xi, \eta, t)$, where as in section 3.3 $g = \frac{\partial \Psi_0}{\partial \eta}$, and noting that

$\nabla^2 \Psi = \nabla^2 \Psi_0 + \nabla^2 \Psi_1 = g' + \nabla^2 \Psi_1$, equation (3.4.1) becomes:

$$\begin{aligned} & \frac{\partial^2}{\partial \xi^2} (\nabla^2 \Psi_1) - 4\alpha \frac{\partial}{\partial \xi} (\nabla^2 \Psi_1) + 4\alpha^2 (g' + \nabla^2 \Psi_1) + g'''' + \frac{\partial^2}{\partial \eta^2} (\nabla^2 \Psi_1) \\ & - e^{2\alpha\xi} R \frac{\partial}{\partial t} (\nabla^2 \Psi_1) - R \left[\left(g' + \frac{\partial \Psi_1}{\partial \eta} \right) \left(\frac{\partial}{\partial \xi} (\nabla^2 \Psi_1) - 2\alpha g' - 2\alpha \nabla^2 \Psi_1 \right) \right. \\ & \left. - \frac{\partial \Psi_1}{\partial \xi} (g'' + \frac{\partial}{\partial \eta} \nabla^2 \Psi_1) \right] = 0. \end{aligned} \quad (3.4.2)$$

On rearranging and dividing through by R, equation (3.4.2) becomes:

$$\begin{aligned}
 & \left[R^{-1} \left\{ \left(\frac{\partial}{\partial \xi} - 2\alpha \right)^2 + \frac{\partial^2}{\partial \eta^2} \right\} - e^{2\alpha \xi} \frac{\partial}{\partial t} - g(\eta) \left(\frac{\partial}{\partial \xi} - 2\alpha \right) \right] \nabla^2 \psi_1 \\
 & + g''(\eta) \frac{\partial \psi_1}{\partial \xi} + 2\alpha g'(\eta) \frac{\partial \psi_1}{\partial \eta} = \left[\frac{\partial \psi_1}{\partial \eta} \left(\frac{\partial}{\partial \xi} - 2\alpha \right) - \frac{\partial \psi_1}{\partial \xi} \frac{\partial}{\partial \eta} \right] \nabla^2 \psi_1 \\
 & - R^{-1} \left[g'''(\eta) + 4\alpha^2 g'(\eta) + 2\alpha R g(\eta) \cdot g'(\eta) \right]. \quad (3.4.3)
 \end{aligned}$$

The first expression on the right-hand side of equation (3.4.3) consists of terms of the order of ψ_1^2 , and these are assumed to be negligible, and the second term on the right-hand side is zero by section 3.3.

The equation for the disturbance then becomes:

$$\begin{aligned}
 & \left[R^{-1} \left\{ \left(\frac{\partial}{\partial \xi} - 2\alpha \right)^2 + \frac{\partial^2}{\partial \eta^2} \right\} - e^{2\alpha \xi} \frac{\partial}{\partial t} - g(\eta) \left(\frac{\partial}{\partial \xi} - 2\alpha \right) \right] \nabla^2 \psi_1 \\
 & + g''(\eta) \frac{\partial \psi_1}{\partial \xi} + 2\alpha g'(\eta) \frac{\partial \psi_1}{\partial \eta} = 0, \quad (3.4.4)
 \end{aligned}$$

with boundary conditions

$$\psi_1 = \frac{\partial \psi_1}{\partial \eta} = 0 \quad \text{at} \quad \eta = \pm 1.$$

Some modification is first made to equation (3.4.4) before the solution is attempted. Since the coefficients are independent of time, fixed frequency solutions of the form

$$\psi(\xi, \eta) e^{-i\beta t} + \text{complex conjugate}$$

are sought.

Thus, by putting $\psi_1 = \psi(\xi, \eta) e^{-i\beta t}$, equation (3.4.4) becomes

$$R^{-1} \nabla^4 \psi - \left\{ 4\alpha R^{-1} + g(\eta) \right\} \frac{\partial}{\partial \xi} (\nabla^2 \psi) + \left\{ 4\alpha^2 R^{-1} + 2\alpha g(\eta) + i\beta e^{2\alpha \xi} \right\} \nabla^2 \psi \\ + g''(\eta) \frac{\partial \psi}{\partial \xi} + 2\alpha g'(\eta) \frac{\partial \psi}{\partial \eta} = 0, \quad (3.4.5)$$

and the boundary conditions become

$$\psi = \frac{\partial \psi}{\partial \eta} = 0 \quad \text{at} \quad \eta = \pm 1.$$

Further, since the solution is to be symmetric about the line $\eta=0$ (the channel centre), solutions are found only for $0 \leq \eta \leq 1$. The boundary conditions at the channel centre ($\eta=0$) and the wall ($\eta=1$) are then taken as:

$$\psi = \frac{\partial \psi}{\partial \eta} = 0 \quad \text{at} \quad \eta = 1, \\ \frac{\partial \psi}{\partial \eta} = \frac{\partial^3 \psi}{\partial \eta^3} = 0 \quad \text{at} \quad \eta = 0.$$

Boundary conditions are required also up-stream, at the

line $\xi=0$, and down-stream, at the line $\xi=\xi_1$ (say).

Up-stream at $\xi=0$, ψ will be a given function of η . Also since the coefficients of equation (3.4.5) are slowly varying with ξ , as in Eagles & Weissman, constant frequency solutions of the form

$$\psi(\xi, \eta) \approx \psi_1(\eta, X) \exp \left\{ i\theta(\xi) \right\}$$

are expected, where $X = \alpha\xi$, $\theta(\xi)$ is the complex phase function, and the wave number

$K(X)$ is defined by $K = \frac{d\theta}{d\xi}$.

Then

$$\frac{\partial \psi}{\partial \xi} = \left(\alpha \frac{\partial \psi_1}{\partial X} + i K \psi_1 \right) e^{i\theta}.$$

Since $\frac{\partial \psi_1}{\partial X}$ is unknown, an approximation to the derivative

$\frac{\partial \psi}{\partial \xi}$ is made by neglecting $\alpha \frac{\partial \psi_1}{\partial X}$, yielding

$$\frac{\partial \psi}{\partial \xi} \approx i K \psi.$$

Thus at $\xi=0$ the following boundary conditions are used:

$\psi = \psi(0, \eta) = \text{given function}$

$$\frac{\partial \psi}{\partial \xi} = i K_0 \psi(0, \eta).$$

As explained elsewhere, the precise boundary conditions

employed at $\xi=0$ were found not to affect the growth rate solution appreciably over most of the flow field. Thus once the test cases had been considered, K was varied in the preliminary numerical work, and was not chosen to be the K occurring in Eagles & Weissman.

For the down-stream boundary conditions, the same method may be employed, so that at a point $\xi = \xi_1$ far downstream the boundary conditions

$$\frac{\partial \psi}{\partial \xi} = i K_1 \psi(\xi_1, \eta) ,$$

$$\frac{\partial^2 \psi}{\partial \xi^2} = - K_1^2 \psi(\xi_1, \eta)$$

can be used. Similar boundary conditions were used by Fasel (1976).

Alternatively, since the disturbances are expected to decay sufficiently far downstream from the wave-maker, the following boundary conditions may be used:

$$\psi = \frac{\partial \psi}{\partial \xi} = 0 \quad \text{at} \quad \xi = \xi_1 .$$

Both sets of boundary conditions were tried in the present work, the former being used in a test run which will be described later, the latter being that finally adopted as the most convenient, so long as it is possible to take the

solution sufficiently far downstream. Again, more will be said of this later.

Also it is necessary to note here that $K(X)$ is not a known function of X , and thus the value of K_0 (and of K_1) is not a known quantity. Experimentation was carried out with different K_0 values, and, sufficiently far from the wave-maker, these did not affect the solutions downstream. Once again more will be reported on this in section 3.7 where the results are presented.

To summarize, equation (3.4.5) is to be solved over the rectangular region $0 \leq \xi \leq \xi_1$, $0 \leq \eta \leq 1$, subject to the boundary conditions

$$\left. \begin{aligned} \psi &= \text{given function} \\ \frac{\partial \psi}{\partial \xi} &= i K_0 \psi \end{aligned} \right\} \quad \text{at } \xi = 0, \quad (3.4.6a)$$

$$\frac{\partial \psi}{\partial \eta} = \frac{\partial^3 \psi}{\partial \eta^3} = 0 \quad \text{at } \eta = 0, \quad (3.4.6b)$$

$$\psi = \frac{\partial \psi}{\partial \eta} = 0 \quad \text{at } \eta = 1, \quad (3.4.6c)$$

$$\text{either } \psi = \frac{\partial \psi}{\partial \xi} = 0 \quad \text{at } \xi = \xi_1, \quad (3.4.6d)$$

$$\begin{aligned} \text{or } \quad & \left. \begin{aligned} \frac{\partial \psi}{\partial \xi} &= i K_1 \psi \\ \frac{\partial^2 \psi}{\partial \xi^2} &= -K_1^2 \psi \end{aligned} \right\} \quad \text{at } \xi = \xi_1. \quad (3.4.6e) \end{aligned}$$

3.5 Numerical Solution

A finite difference method is used to transform the partial differential equation (3.4.5), together with the boundary conditions (3.4.6), to a set of finite difference equations.

Letting h, k be the step lengths in the ξ -, η - directions respectively, dividing the rectangular region $0 \leq \xi \leq \xi_1$, $0 \leq \eta \leq 1$ by a mesh with N h -steps and M k -steps (see figure 3.5.1), and denoting $\psi(rh, sk)$ by $\psi_{r,s}$, etc., the central finite differences used are pictorially represented by the computational molecules in figure 3.5.2, the numbers in the various positions representing the multipliers that are to be applied to the values of ψ at these stations (see Ames, 1969: 15-19).

Using these finite difference approximations for equation (3.4.5) yields finite difference equations of the form:

$$\begin{aligned}
 & A_s \psi_{r-2,s} + B_s \psi_{r-1,s-1} + C_{r,s} \psi_{r-1,s} + B_s \psi_{r-1,s+1} \\
 & + D \psi_{r,s-2} + E_{r,s} \psi_{r,s-1} + F_{r,s} \psi_{r,s} + G_{r,s} \psi_{r,s+1} + D \psi_{r,s+2} \\
 & + H_s \psi_{r+1,s-1} + I_{r,s} \psi_{r+1,s} + H_s \psi_{r+1,s+1} + J_s \psi_{r+2,s} = 0, \\
 & (r = 1 \text{ (1)} N-1, s = 0 \text{ (1)} M-1),
 \end{aligned}
 \tag{3.5.1}$$

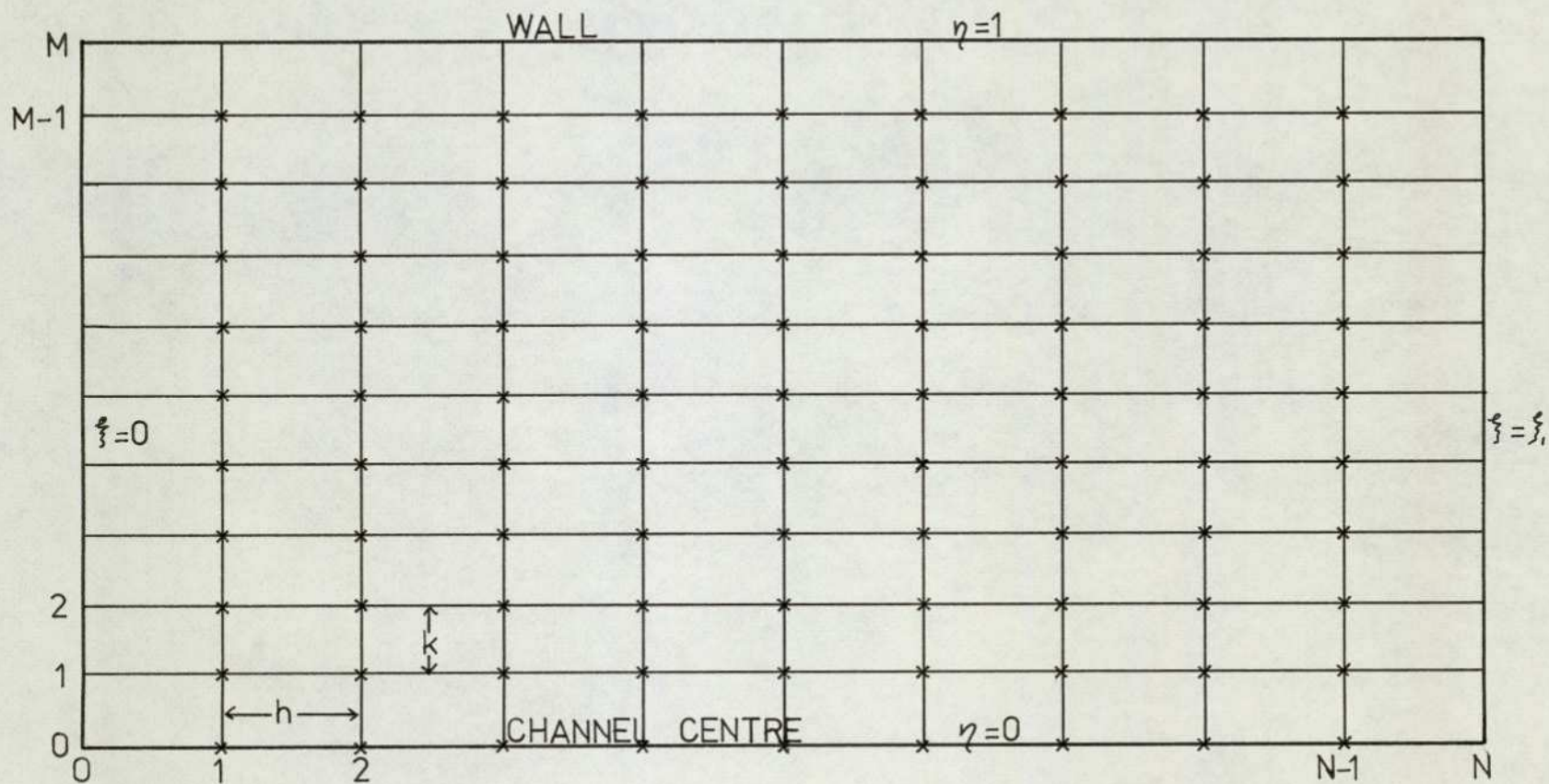


Figure 3.5.1: The mesh arrangement in the ξ - η plane.

The unknown points are marked x.

$$h = \delta\xi, \quad k = \delta\eta.$$

$$\left. \frac{\partial \psi}{\partial \xi} \right|_{r,s} \equiv \begin{array}{c} \text{---} \left(-\frac{1}{2h} \right)_{r-1,s} \text{---} \left(0 \right)_{r,s} \text{---} \left(\frac{1}{2h} \right)_{r+1,s} \end{array}$$

$$\left. \frac{\partial \psi}{\partial \eta} \right|_{r,s} \equiv \begin{array}{c} \left(\frac{1}{2k} \right)_{r,s+1} \\ \text{---} \\ \left(0 \right)_{r,s} \\ \text{---} \\ \left(-\frac{1}{2k} \right)_{r,s-1} \end{array}$$

$$\nabla^2 \psi \Big|_{r,s} \equiv \begin{array}{c} \left(\frac{1}{k^2} \right) \\ \text{---} \\ \left(\frac{1}{h^2} \right)_{r-1,s} \text{---} \left(-2\left(\frac{1}{h^2} + \frac{1}{k^2} \right) \right)_{r,s} \text{---} \left(\frac{1}{h^2} \right)_{r+1,s} \\ \text{---} \\ \left(\frac{1}{k^2} \right) \end{array}$$

Figure 3.5.2: For legend, see next page.

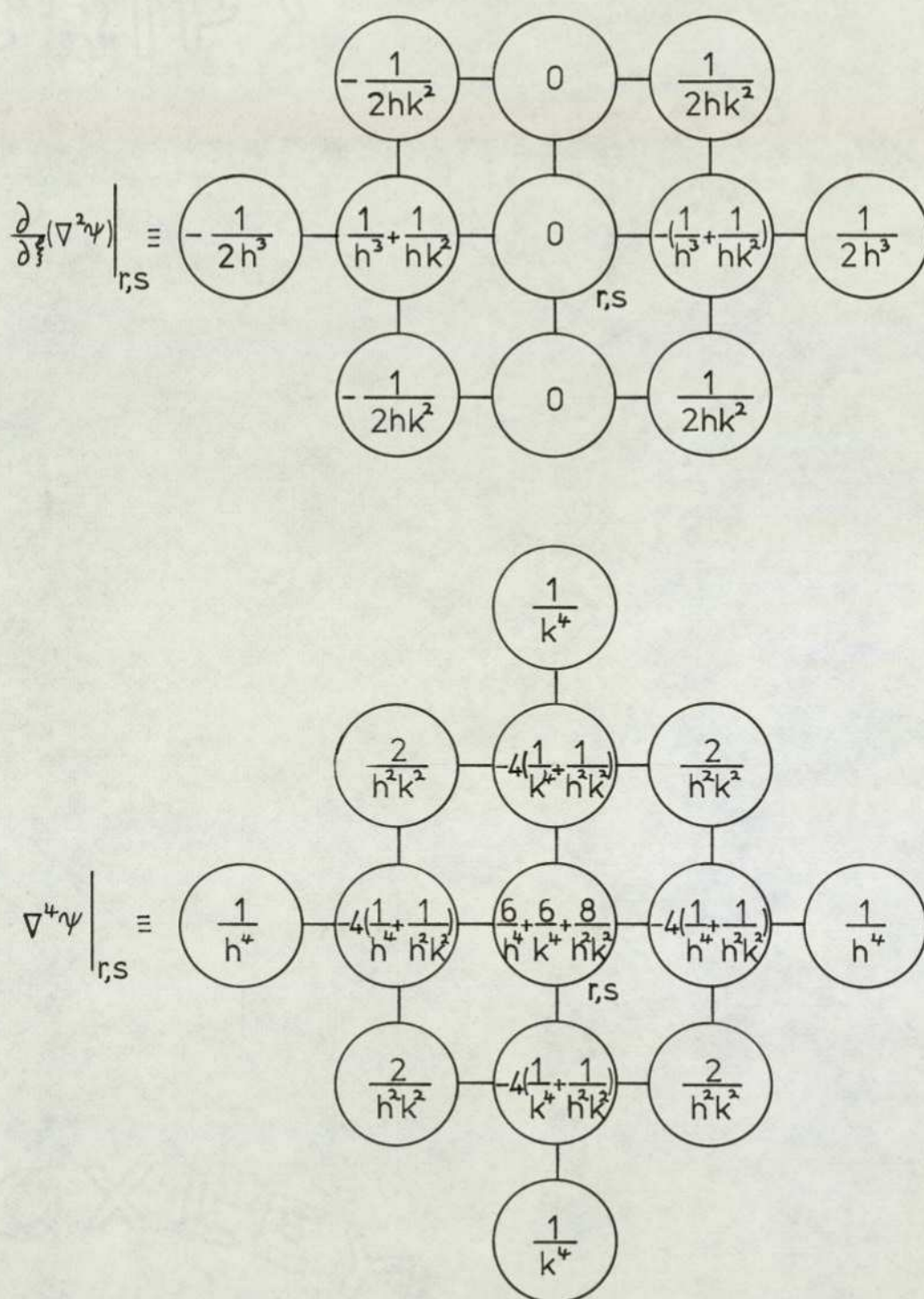


Figure 3.5.2: Computational molecules for partial derivatives.

where

$$A_s = \frac{R^{-1}}{h^3} \left(\frac{1}{h} + 2\alpha \right) + \frac{1}{2h^3} g_s, \quad (3.5.2a)$$

$$J_s = \frac{R^{-1}}{h^3} \left(\frac{1}{h} - 2\alpha \right) - \frac{1}{2h^3} g_s, \quad (3.5.2b)$$

$$B_s = \frac{2R^{-1}}{hk^2} \left(\frac{1}{h} + \alpha \right) + \frac{1}{2hk^2} g_s, \quad (3.5.2c)$$

$$H_s = \frac{2R^{-1}}{hk^2} \left(\frac{1}{h} - \alpha \right) - \frac{1}{2hk^2} g_s, \quad (3.5.2d)$$

$$C_{r,s} = \frac{4R^{-1}}{h} \left\{ \frac{\alpha^2}{h} - \left(\frac{1}{h^2} + \frac{1}{k^2} \right) \left(\frac{1}{h} + \alpha \right) \right\} + \frac{1}{h} \left\{ \frac{2\alpha}{h} - \left(\frac{1}{h^2} + \frac{1}{k^2} \right) \right\} g_s \\ - \frac{1}{2h} g''_s + i \frac{\beta}{h^2} e^{2\alpha\xi_r}, \quad (3.5.2e)$$

$$I_{r,s} = \frac{4R^{-1}}{h} \left\{ \frac{\alpha^2}{h} - \left(\frac{1}{h^2} + \frac{1}{k^2} \right) \left(\frac{1}{h} - \alpha \right) \right\} + \frac{1}{h} \left\{ \frac{2\alpha}{h} + \left(\frac{1}{h^2} + \frac{1}{k^2} \right) \right\} g_s \\ + \frac{1}{2h} g''_s + i \frac{\beta}{h^2} e^{2\alpha\xi_r}, \quad (3.5.2f)$$

$$D = \frac{R^{-1}}{h^4}, \quad (3.5.2g)$$

$$E_{r,s} = \frac{4R^{-1}}{k^2} \left\{ \alpha^2 - \left(\frac{1}{h^2} + \frac{1}{k^2} \right) \right\} + \frac{2\alpha}{k^2} g_s - \frac{\alpha}{k} g'_s + \frac{i\beta}{k^2} e^{2\alpha\xi_r}, \quad (3.5.2h)$$

$$G_{r,s} = \frac{4R^{-1}}{k^2} \left\{ \alpha^2 - \left(\frac{1}{h^2} + \frac{1}{k^2} \right) \right\} + \frac{2\alpha}{k^2} g_s + \frac{\alpha}{k} g'_s + \frac{i\beta}{k^2} e^{2\alpha\xi_r}, \quad (3.5.2i)$$

$$F_{r,s} = R^{-1} \left\{ 6 \left(\frac{1}{h^4} + \frac{1}{k^4} \right) + \frac{8}{h^2 k^2} - 8\alpha^2 \left(\frac{1}{h^2} + \frac{1}{k^2} \right) \right\} \\ - 4\alpha \left(\frac{1}{h^2} + \frac{1}{k^2} \right) g_s - i 2\beta \left(\frac{1}{h^2} + \frac{1}{k^2} \right) e^{2\alpha\xi_r}. \quad (3.5.2j)$$

The boundary conditions in finite difference form become

$$\psi_{-1,s} = \psi_{1,s} - 2hiK_o \psi_{o,s} \left. \vphantom{\psi_{-1,s}} \right\} \text{ for all } s \text{ (upstream)} \quad (3.5.3a)$$

$$\psi_{o,s} = \text{given values}$$

$$\left. \begin{aligned} \psi_{r,-2} &= \psi_{r,2} \\ \psi_{r,-1} &= \psi_{r,1} \end{aligned} \right\} \text{ for all } r \text{ (channel centre)} \quad (3.5.3b)$$

$$\left. \psi_{r,M} = 0 \right\} \text{ for all } r \text{ (wall)} \quad (3.5.3c)$$

$$\psi_{r,M+1} = \psi_{r,M-1}$$

Either:

$$\left. \begin{aligned} \psi_{N,s} &= 0 \\ \psi_{N+1,s} &= \psi_{N-1,s} \end{aligned} \right\} \text{ for all } s \text{ (downstream equivalent to (3.4.6d))} \quad (3.5.3d)$$

or: for the transform of boundary condition (3.4.6e) the following is used:

Let K_1 be the complex wave number at the downstream boundary. Then

$$\left(\frac{\partial \psi}{\partial \xi} \right) \bigg|_{\xi=\xi_1} = i K_1 \psi(\xi_1, n)$$

$$\text{and } \left(\frac{\partial^2 \psi}{\partial \xi^2} \right) \bigg|_{\xi=\xi_1} = - K_1^2 \psi(\xi_1, n).$$

In finite difference form these yield:

$$\psi_{N+1,s} - \psi_{N-1,s} = i 2h K_1 \psi_{N,s}$$

$$\psi_{N+1,s} + \psi_{N-1,s} = (2 - h^2 K_1^2) \psi_{N,s}.$$

Solving for $\psi_{N,s}$ and $\psi_{N+1,s}$ gives

$$\psi_{N,s} = \left(\frac{2}{2 - h^2 K_1^2 - i 2h K_1} \right) \psi_{N-1,s} = K \psi_{N-1,s} \quad \left. \vphantom{\frac{2}{2 - h^2 K_1^2 - i 2h K_1}} \right\} \text{ for all } s. \quad (3.5.3e)$$

$$\psi_{N+1,s} = (1 + i 2h K_1 K) \psi_{N-1,s}$$

The finite difference equations (3.5.1), with unknown mesh

points (r,s) , where $1 \leq r \leq N-1$, $0 \leq s \leq M-1$, may be written in matrix form:

$$\underline{T} \underline{U} = \underline{Q}, \quad (3.5.4)$$

where $\underline{Q}$ is a non-zero column matrix, since it incorporates the values $\psi(0,\eta)$ and $\left(\frac{\partial\psi}{\partial\xi}\right)\bigg|_{\xi=0}$. The other boundary conditions

are included within the coefficient matrix $\underline{T}$. The forms of the matrices $\underline{T}$ and $\underline{Q}$ are shown in figures 3.5.3 and 3.5.4 respectively.

The method of solution employed is to reduce $\underline{T}$ to triangular form by partial pivoting, performing the same row operations on the right-hand side vector $\underline{Q}$, and then using back substitution (i.e. the method of Gaussian elimination with interchanges).

Since $\underline{T}$ is sparse (see figure 3.5.3), its elements were stored in a matrix $\underline{P}$, only the submatrices within the diagonal band of matrix $\underline{T}$ being stored, without storing all the submatrices of zeros outside the band. If no row interchanges are to be used, $\underline{P}$ would be of dimension $\{M(N-1)\} \times (4M+1)$, but if, as is the case in the present work, row interchanges are to be employed, $\underline{P}$ becomes $\{M(N-1)\} \times (6M+1)$ (see figure 3.5.5). In either case, a considerable reduction in storage is obtained over having to store the complete $\{M(N-1)\} \times \{M(N-1)\}$ matrix $\underline{T}$. The change

$$\underline{I} = \begin{bmatrix} \underline{C}_1 & \underline{D}_1 & \underline{E} & & & & \\ \underline{B}_2 & \underline{C}_2 & \underline{D}_2 & \underline{E} & & & \\ \underline{A} & \underline{B}_3 & \underline{C}_3 & \underline{D}_3 & \underline{E} & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & \underline{A} & \underline{B}_{M-3} & \underline{C}_{M-3} & \underline{D}_{M-3} & \underline{E} \\ & & & & \underline{A} & \underline{B}_{M-2} & \underline{C}_{M-2} & \underline{D}_{M-2} \\ & & & & & \underline{A} & \underline{B}_{M-1} & \underline{C}_{M-1} \end{bmatrix}$$

with:-

$$\underline{A} = \begin{bmatrix} A_0 & & & \\ & A_1 & & \\ & & \ddots & \\ & & & A_{M-1} \end{bmatrix}$$

$$\underline{E} = \begin{bmatrix} J_0 & & & \\ & J_1 & & \\ & & \ddots & \\ & & & J_{M-1} \end{bmatrix}$$

$$\underline{B}_r = \begin{bmatrix} C_{r,0} & 2B_0 & & & \\ B_1 & C_{r,1} & B_1 & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & B_{M-2} & C_{r,M-2} & B_{M-2} \\ & & & & B_{M-1} & C_{r,M-1} \end{bmatrix}$$

$$\underline{D}_r = \begin{bmatrix} I_{r,0} & 2H_0 & & & \\ H_1 & I_{r,1} & H_1 & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & H_{M-2} & I_{r,M-2} & H_{M-2} \\ & & & & H_{M-1} & I_{r,M-1} \end{bmatrix}$$

Figure 3.5.3: For legend, see next page.

$$\begin{aligned}
\underline{C}_1 &= \begin{bmatrix} (F_{1,0}+A)(G_{1,0}+E_{1,0}) & 2D & & & \\ E_{1,1}(F_{1,1}+A+D) & G_{1,1} & D & & \\ D & E_{1,2}(F_{1,2}+A) & G_{1,2} & D & \\ & & & & \ddots \\ & & D & E_{1,M-3}(F_{1,M-3}+A) & G_{1,M-3} & D \\ & & & D & E_{1,M-2}(F_{1,M-2}+A) & G_{1,M-2} \\ & & & & D & E_{1,M-1}(F_{1,M-1}+A+D) \end{bmatrix} \\
\underline{C}_r &= \begin{bmatrix} F_{r,0}(G_{r,0}+E_{r,0}) & 2D & & & \\ E_{r,1}(F_{r,1}+D) & G_{r,1} & D & & \\ D & E_{r,2} & F_{r,2} & G_{r,2} & D \\ & & & & \ddots \\ & & D & E_{r,M-3} & F_{r,M-3} & G_{r,M-3} & D \\ & & & D & E_{r,M-2} & F_{r,M-2} & G_{r,M-2} \\ & & & & D & E_{r,M-1} & (F_{r,M-1}+D) \end{bmatrix} \\
&\text{for } r=2(1)N-2; \\
\underline{C}_{N-1} &= \begin{bmatrix} (F_{N-1,0}+J)(G_{N-1,0}+E_{N-1,0}) & 2D & & & \\ E_{N-1,1}(F_{N-1,1}+J+D) & G_{N-1,1} & D & & \\ D & E_{N-1,2}(F_{N-1,2}+J) & G_{N-1,2} & D & \\ & & & & \ddots \\ & & D & E_{N-1,M-3}(F_{N-1,M-3}+J) & G_{N-1,M-3} & D \\ & & & D & E_{N-1,M-2}(F_{N-1,M-2}+J) & G_{N-1,M-2} \\ & & & & D & E_{N-1,M-1}(F_{N-1,M-1}+J+D) \end{bmatrix}
\end{aligned}$$

Figure 3.5.3: The form of the coefficient matrix $\underline{T}$.

$$\underline{Q} = \left[\underline{Q}_1 \quad \underline{Q}_2 \quad \underline{Z}_1 \quad \cdot \quad \cdot \quad \cdot \quad \underline{Z}_1 \right]^T$$

$\xleftarrow{\hspace{10em}} (N-1) \text{ blocks} \xrightarrow{\hspace{10em}}$

where :-

$$\underline{Q}_1 = \begin{bmatrix} 2hA_0 \left(\frac{\partial \psi}{\partial \xi} \right)_{0,0} - B_0 \psi_{0,-1} - C_{1,0} \psi_{0,0} - B_0 \psi_{0,1} \\ 2hA_1 \left(\frac{\partial \psi}{\partial \xi} \right)_{0,1} - B_1 \psi_{0,0} - C_{1,1} \psi_{0,1} - B_1 \psi_{0,2} \\ \vdots \\ 2hA_{M-1} \left(\frac{\partial \psi}{\partial \xi} \right)_{0,M-1} - B_{M-1} \psi_{0,M-2} - C_{1,M-1} \psi_{0,M-1} - B_{M-1} \psi_{0,M} \end{bmatrix}$$

$$\underline{Q}_2 = \begin{bmatrix} -A_0 \psi_{0,0} \\ -A_1 \psi_{0,1} \\ \vdots \\ -A_{M-1} \psi_{0,M-1} \end{bmatrix}$$

$$\underline{Z}_1 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \begin{array}{c} \updownarrow \\ M \end{array}$$

Figure 3.5.4: The right-hand side column vector $\underline{Q}$.

$$\underline{P} = \begin{bmatrix} \underline{P}_1 & \underline{P}_2 & \underline{P}_3 & \cdot & \cdot & \cdot & \underline{P}_{N-2} & \underline{P}_{N-1} \end{bmatrix}^T$$

where :-

$$\underline{P}_1 = \begin{bmatrix} \underline{Z}_1 & \underline{Z}_{M-2} \underline{Z}_3 & \underline{Z}_{M-4} \underline{H}_1 & \underline{Z}_{M-4} \underline{I}_1 & \underline{Z}_{M-2} \underline{J} & \underline{Z}_{2M} \end{bmatrix}$$

$$\underline{P}_2 = \begin{bmatrix} \underline{Z}_1 & \underline{Z}_{M-2} \underline{G}_2 & \underline{Z}_{M-4} \underline{H}_2 & \underline{Z}_{M-4} \underline{I}_2 & \underline{Z}_{M-2} \underline{J} & \underline{Z}_{2M} \end{bmatrix}$$

⋮

$$\underline{P}_r = \begin{bmatrix} \underline{F} & \underline{Z}_{M-2} \underline{G}_r & \underline{Z}_{M-4} \underline{H}_r & \underline{Z}_{M-4} \underline{I}_r & \underline{Z}_{M-2} \underline{J} & \underline{Z}_{2M} \end{bmatrix}$$

for $r = 3(1)N-3$;

⋮

$$\underline{P}_{N-2} = \begin{bmatrix} \underline{F} & \underline{Z}_{M-2} \underline{G}_{N-2} & \underline{Z}_{M-4} \underline{H}_{N-2} & \underline{Z}_{M-4} \underline{I}_{N-2} & \underline{Z}_{M-2} \underline{Z}_1 & \underline{Z}_{2M} \end{bmatrix}$$

$$\underline{P}_{N-1} = \begin{bmatrix} \underline{F} & \underline{Z}_{M-2} \underline{G}_{N-1} & \underline{Z}_{M-4} \underline{H}_{N-1} & \underline{Z}_{M-4} \underline{Z}_3 & \underline{Z}_{M-2} \underline{Z}_1 & \underline{Z}_{2M} \end{bmatrix}$$

with :-

$$\underline{F} = \begin{bmatrix} A_0 & A_1 & \cdot & \cdot & \cdot & A_{M-1} \end{bmatrix}^T \quad \underline{J} = \begin{bmatrix} J_0 & J_1 & \cdot & \cdot & \cdot & J_{M-1} \end{bmatrix}^T$$

$$\underline{G}_r = \begin{bmatrix} 0 & C_{r,0} & 2B_0 \\ B_1 & C_{r,1} & B_1 \\ \vdots & \vdots & \vdots \\ B_{M-2} & C_{r,M-2} & B_{M-2} \\ B_{M-1} & C_{r,M-1} & 0 \end{bmatrix} \quad \underline{I}_r = \begin{bmatrix} 0 & I_{r,0} & 2H_0 \\ H_1 & I_{r,1} & H_1 \\ \vdots & \vdots & \vdots \\ H_{M-2} & I_{r,M-2} & H_{M-2} \\ H_{M-1} & I_{r,M-1} & 0 \end{bmatrix}$$

Figure 3.5.5: For legend, see next page.

$$\begin{aligned}
\underline{H}_1 &= \begin{bmatrix} 0 & 0 & (F_{1,0} + A) & (G_{1,0} + E_{1,0}) & 2D \\ 0 & E_{1,1} & (F_{1,1} + A + D) & G_{1,1} & D \\ D & E_{1,2} & (F_{1,2} + A) & G_{1,2} & D \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ D & E_{1,M-3} & (F_{1,M-3} + A) & G_{1,M-3} & D \\ D & E_{1,M-2} & (F_{1,M-2} + A) & G_{1,M-2} & 0 \\ D & E_{1,M-1} & (F_{1,M-1} + A + D) & 0 & 0 \end{bmatrix} \\
\underline{H}_r &= \begin{bmatrix} 0 & 0 & F_{r,0} & (G_{r,0} + E_{r,0}) & 2D \\ 0 & E_{r,1} & (F_{r,1} + D) & G_{r,1} & D \\ D & E_{r,2} & F_{r,2} & G_{r,2} & D \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ D & E_{r,M-3} & F_{r,M-3} & G_{r,M-3} & D \\ D & E_{r,M-2} & F_{r,M-2} & G_{r,M-2} & 0 \\ D & E_{r,M-1} & (F_{r,M-1} + D) & 0 & 0 \end{bmatrix} \\
&\text{for } r=2(1)N-2; \\
\underline{H}_{N-1} &= \begin{bmatrix} 0 & 0 & (F_{N-1,0} + J) & (G_{N-1,0} + E_{N-1,0}) & 2D \\ 0 & E_{N-1,1} & (F_{N-1,1} + J + D) & G_{N-1,1} & D \\ D & E_{N-1,2} & (F_{N-1,2} + J) & G_{N-1,2} & D \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ D & E_{N-1,M-3} & (F_{N-1,M-3} + J) & G_{N-1,M-3} & D \\ D & E_{N-1,M-2} & (F_{N-1,M-2} + J) & G_{N-1,M-2} & 0 \\ D & E_{N-1,M-1} & (F_{N-1,M-1} + J + D) & 0 & 0 \end{bmatrix} \\
\underline{Z}_n &= \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix} \begin{matrix} \uparrow \\ M \\ \downarrow \end{matrix} \\
&\quad \leftarrow n \rightarrow
\end{aligned}$$

Figure 3.5.5: The matrix $\underline{P}$, used to store the band of matrix $\underline{T}$.

in the dimension of $\underline{P}$ when row interchanges are performed may be seen if one considers figure 3.5.3. Due to the band form of $\underline{T}$, when row interchanges are made, each row being changed has the number of elements increased by at most $2M$ in number. To accommodate this, therefore, the width of $\underline{P}$ is increased from $4M+1$ to $6M+1$. All the row operations are then performed within the matrix $\underline{P}$, thus reducing $\underline{P}$ to what may be termed "quasi-triangular" form.

Remembering that the elements of $\underline{P}$ are complex numbers, each one requiring two computer storage locations, it is found that, using the CDC 7600 computer of the University of London Computer Centre, the large core memory is just large enough to accommodate $\underline{P}$ if $N=80$, $M=10$, $\underline{P}$ then being 790×61 in dimension. This presents no problem with $h(=\delta\xi) = 0.2$, so long as ψ has decayed sufficiently at $\xi = 16$. This of course depends upon the values of γ , R and β being used, and in most cases is found not to be the case.

It is thus decided to use backing storage for $\underline{P}$, and this is accomplished by storing a block of rows on each record of the backing mass-storage. During the process of reducing $\underline{P}$ to its "quasi-triangular" form, when checking the rows of the first block, a search is made through the rows of the next two blocks only. Thus $\underline{P}$ is brought into the central processor by first bringing down three blocks, row operations are performed on the first block, which is then rewritten onto the backing store, the other two blocks in

the central processor are shifted up, and the next block brought down. This process is repeated until all (N-1) blocks have been dealt with. The working storage for $\underline{P}$ is thus reduced to dimension $3M \times (6M+1)$, with double the number of locations required in the computer since $\underline{P}$ is complex. A much larger value of N can be used with this method, and the maximum value used, which is found to be sufficient for all cases considered, is 300.

3.6 Growth Rates and Wave Number

Following Eagles & Weissman, three growth rates are used, those based on the disturbance stream function ψ , the disturbance kinetic energy density E , and the relative kinetic energy density $\hat{E}$, the last being suggested by Shen (1961) as a suitable measurement of growth. These quantities, together with the wave number, are defined as follows.

The growth rate of ψ , $GR(\psi)$, is defined as

$$GR(\psi) \equiv \frac{1}{|\psi|} \frac{\partial |\psi|}{\partial \xi} , \quad (3.6.1)$$

where $|\psi|$ is the wave amplitude that would be observed in an experiment (the "total" or "physical" amplitude).

The mean kinetic energy density, averaged over time and integrated across the channel, is defined as

$$E \equiv \frac{1}{2} \int_{-\alpha}^{\alpha} (\overline{u^2} + \overline{v^2}) r \, d\phi, \quad (3.6.2)$$

where an overbar indicates an average over a period.

Now

$$u^2 = \frac{1}{r^2} \left(\frac{\partial \Psi_1}{\partial \phi} \right)^2 = e^{-2\alpha\xi} \left(\frac{\partial \Psi_1}{\partial \eta} \right)^2,$$

$$v^2 = \left(\frac{\partial \Psi_1}{\partial r} \right)^2 = e^{-2\alpha\xi} \left(\frac{\partial \Psi_1}{\partial \xi} \right)^2,$$

where $\Psi_1 = \psi e^{-i\beta t} + \text{complex conjugate}$,

and thus

$$\overline{u^2} = 2 e^{-2\alpha\xi} \left| \frac{\partial \psi}{\partial \eta} \right|^2 \quad \text{and} \quad \overline{v^2} = 2 e^{-2\alpha\xi} \left| \frac{\partial \psi}{\partial \xi} \right|^2.$$

Therefore from (3.6.2):

$$\begin{aligned} E &\equiv \frac{1}{2} \int_{-1}^1 2 \left(\left| \frac{\partial \psi}{\partial \eta} \right|^2 + \left| \frac{\partial \psi}{\partial \xi} \right|^2 \right) e^{-2\alpha\xi} e^{\alpha\xi} \, d\eta \\ &\equiv 2 e^{-\alpha\xi} \int_0^1 (|\psi_\eta|^2 + |\psi_\xi|^2) \, d\eta. \end{aligned} \quad (3.6.3)$$

The growth rate of E , $GR(E)$, is then defined as

$$GR(E) \equiv \frac{1}{2E} \frac{dE}{d\xi}. \quad (3.6.4)$$

The kinetic energy density of the basic flow, E_0 , is given by

$$\begin{aligned}
 E_0 &\equiv \frac{1}{2} \int_{-\alpha}^{\alpha} U_0^2 r d\phi \equiv \frac{1}{2} \int_{-1}^1 e^{-2\alpha\xi} [g(\eta)]^2 e^{\alpha\xi} d\eta \\
 &\equiv e^{-\alpha\xi} \int_0^1 [g(\eta)]^2 d\eta.
 \end{aligned} \tag{3.6.5}$$

The relative kinetic energy density, $\hat{E}$, is then defined as

$$\hat{E} \equiv E/E_0.$$

Putting $\Gamma \equiv \int_0^1 [g(\eta)]^2 d\eta$, then

$$E_0 \equiv e^{-\alpha\xi} \Gamma$$

$$\text{and } \hat{E} \equiv \frac{2}{\Gamma} \int_0^1 (|\psi_\eta|^2 + |\psi_\xi|^2) d\eta. \tag{3.6.6}$$

The growth rate of $\hat{E}$, $GR(\hat{E})$, is then given by

$$GR(\hat{E}) \equiv \frac{1}{2\hat{E}} \frac{d\hat{E}}{d\xi}. \tag{3.6.7}$$

The factor $\frac{1}{2}$ was introduced in the definitions of the growth rates of E and $\hat{E}$ ((3.6.4) and (3.6.7)) by Eagles & Weissman in order that all three growth rates may be compared, since they found that

$$GR(\dots) = -K_i + O(\alpha)$$

when the factor $\frac{1}{2}$ was included.

Again following Eagles & Weissman, denoting the argument of ψ by $\arg\psi$, the observed or "physical" wave number, $WN(\psi)$, is then defined as

$$WN(\psi) \equiv \frac{\partial (\arg\psi)}{\partial \xi} . \quad (3.6.8)$$

For the numerical calculations of the growth rates and wave number, central differences are used for the derivatives and Simpson's rule for the integrals.

3.7 Numerical Results

Test runs were first made to investigate the effect of different mesh sizes on the solution for the stream function, ψ . The values finally adopted were $h = 0.2$, $k = 0.1$. From the test runs it was found that calculations with these interval sizes sliced in half lead to maximum relative deviations in the computed values always less than 1%.

The numerical method was then checked by taking some test cases with which to compare the computed values of the stream function, ψ . Firstly, flow in a channel with parallel walls was considered by allowing $\alpha = 0$ in the governing equation (3.4.5). Since, for parallel flow, the growth rate should be independent of ξ throughout the channel, the downstream boundary condition (3.4.6e) was taken, with K at $\xi = \xi_1$, (K_1) , having the same value as at $\xi = 0$, (K_0) . With $\alpha = 0$ in equation (3.4.5), the coefficients are independent of ξ , and

so solutions of the form

$$\psi(\xi, \eta) = e^{iK\xi} f(\eta)$$

may be expected, the K being the same as that used for the upstream and downstream boundary conditions. The values used for this run were $R = 45$, $\beta = 1.5$. The values of $\psi(0, \eta)$ for the upstream boundary condition, and of the basic flow velocity profile, $g(\eta)$, were those from Eagles & Weissman consistent with $R = 45$, $\beta = 1.5$, $\gamma (= \alpha R) = 4.093$, as was the constant value taken for K ($= 1.794 - i 0.024$). In figure 3.7.1 a graph of the values of ψ at the channel centre ($\eta=0$) is presented, together with some plotted points of $e^{iK\xi} f(\eta)$, and, as can be seen, the two results are in very close agreement.

Checks were then carried out to compare the results obtained here with those obtained by Eagles & Weissman. The results of two such comparisons are shown in figures 3.7.2 and 3.7.3. In the former, the values used were $R = 45$, $\gamma = 4.093$, $\beta = 1.5$, $K_0 = 1.794 - i 0.024$, $\psi(0, \eta)$ being obtained from Eagles & Weissman; in the latter, the values were $R = 100$, $\gamma = 4.093$, $\beta = 1.5$, $K_0 = 1.798 - i 0.157$, with again $\psi(0, \eta)$ from Eagles & Weissman. The values of K_0 used were, in both cases, those from Eagles & Weissman results. Both sets of downstream boundary conditions, (3.4.6d) and (3.4.6e), were tried, the latter with various values of K_1 . However, ξ_1 was chosen so that the stream

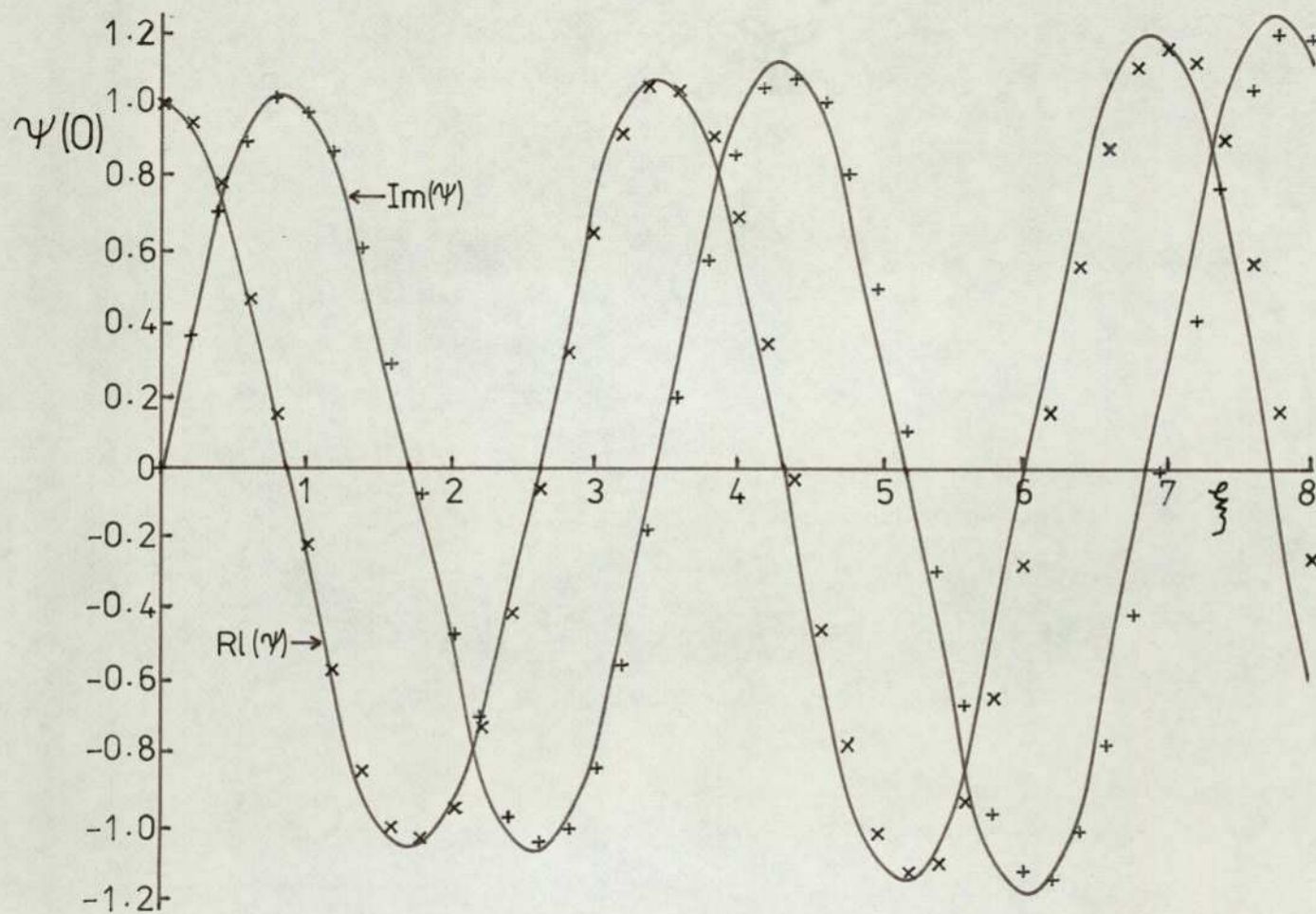


Figure 3.7.1: The disturbance stream function evaluated at $\eta = 0$, $\psi(0)$, for $\alpha = 0$, $R = 45$, $\beta = 1.5$, $K_0 = 1.794 - i 0.024$.

$$\left. \begin{array}{l} \times \operatorname{Re}(\psi) \\ + \operatorname{Im}(\psi) \end{array} \right\} \psi = e^{iK\xi} f(\eta).$$

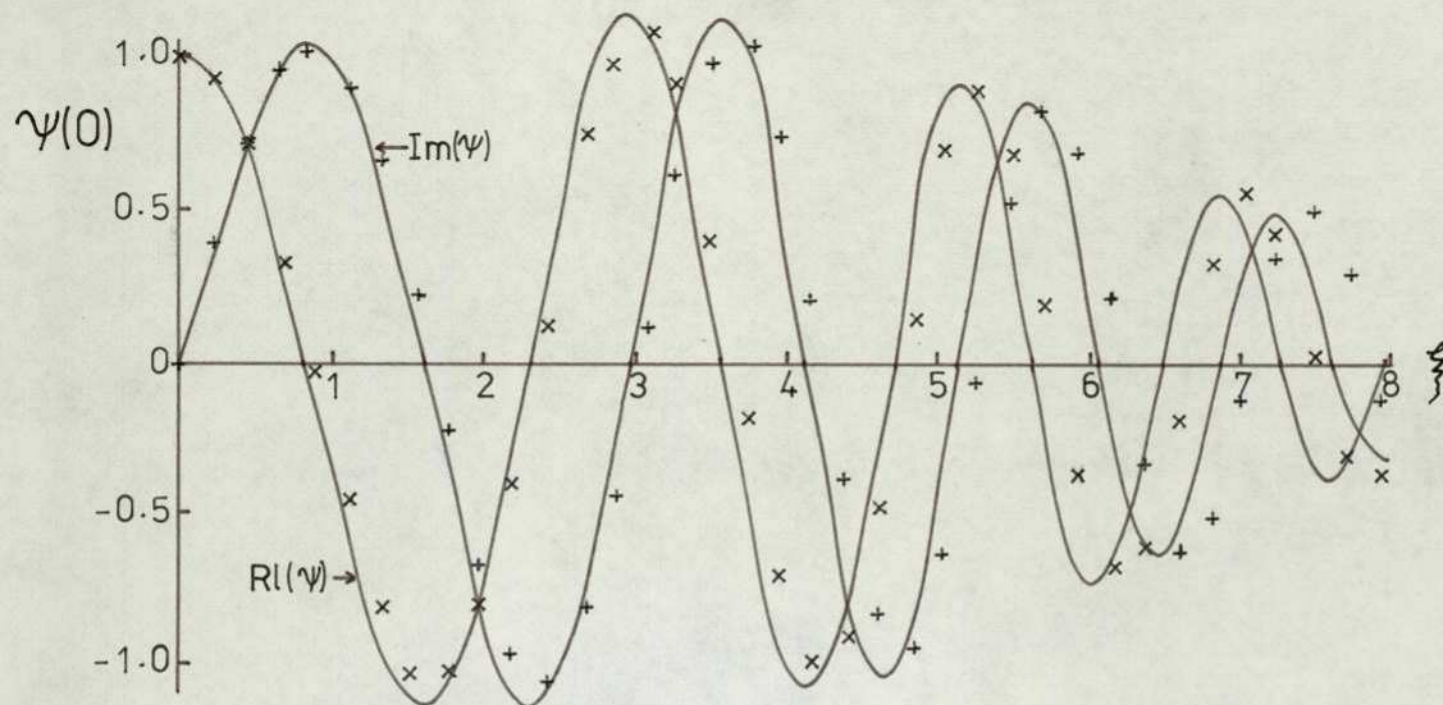


Figure 3.7.2: The disturbance stream function evaluated at $\eta = 0$, $\psi(0)$, for $\gamma = 4.093$, $R = 45$, $\beta = 1.5$, $K_O = 1.794 - i 0.024$.

$\times \text{Re}(\psi)$
 $+ \text{Im}(\psi)$

} Values from Eagles & Weissman.

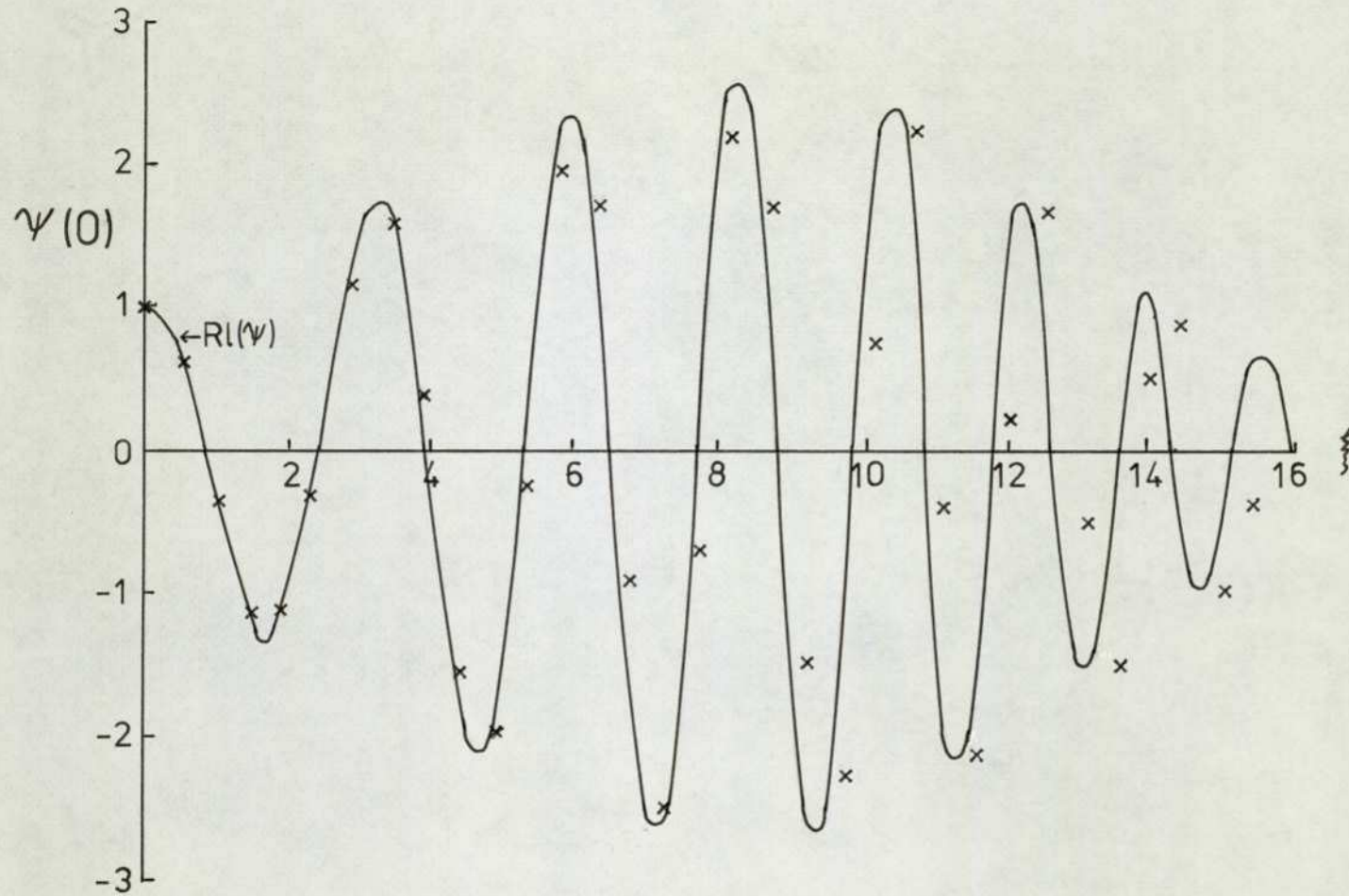


Figure 3.7.3: The disturbance stream function evaluated at $\eta = 0$, $\psi(0)$, for $\gamma = 4.093$, $R = 100$, $\beta = 1.5$, $K_0 = 1.798 - i 0.157$. $\times \text{Rl}(\psi)$ values from Eagles & Weissman.

function had decayed well upstream from this point, and so this did not affect the solutions upstream (more is said on this below). Again, good agreement can be seen between the two sets of results for both these cases.

As a further comparison, the growth rate based on the stream function ψ , $GR(\psi)$, together with the wave number, $WN(\psi)$, (as defined in section 3.6), evaluated at the channel centre, were calculated for the two cases cited above, and compared with the corresponding values from Eagles & Weissman. These results may be seen in figures 3.7.4, 3.7.5, 3.7.6 and 3.7.7. It should be restated here that in the Eagles & Weissman calculations an equation with error $O(\alpha^2)$ was solved for the basic flow, that an expansion method which ignored terms of $O(\alpha^2)$ was used for the disturbance stream function, and further that the growth rate calculations ignored terms of $O(\alpha^2)$. In the present approach, however, no such approximations are made.

In addition, the stream function results were checked against those obtained by a completely independent method of solution, the coupled-equation approach, the details of which may be found in section 3.8. As can be seen from the values in table 3.8.1, there is excellent agreement between the two methods.

For the general solutions to be obtained, in order to find some neutral curves and the overall critical Reynolds

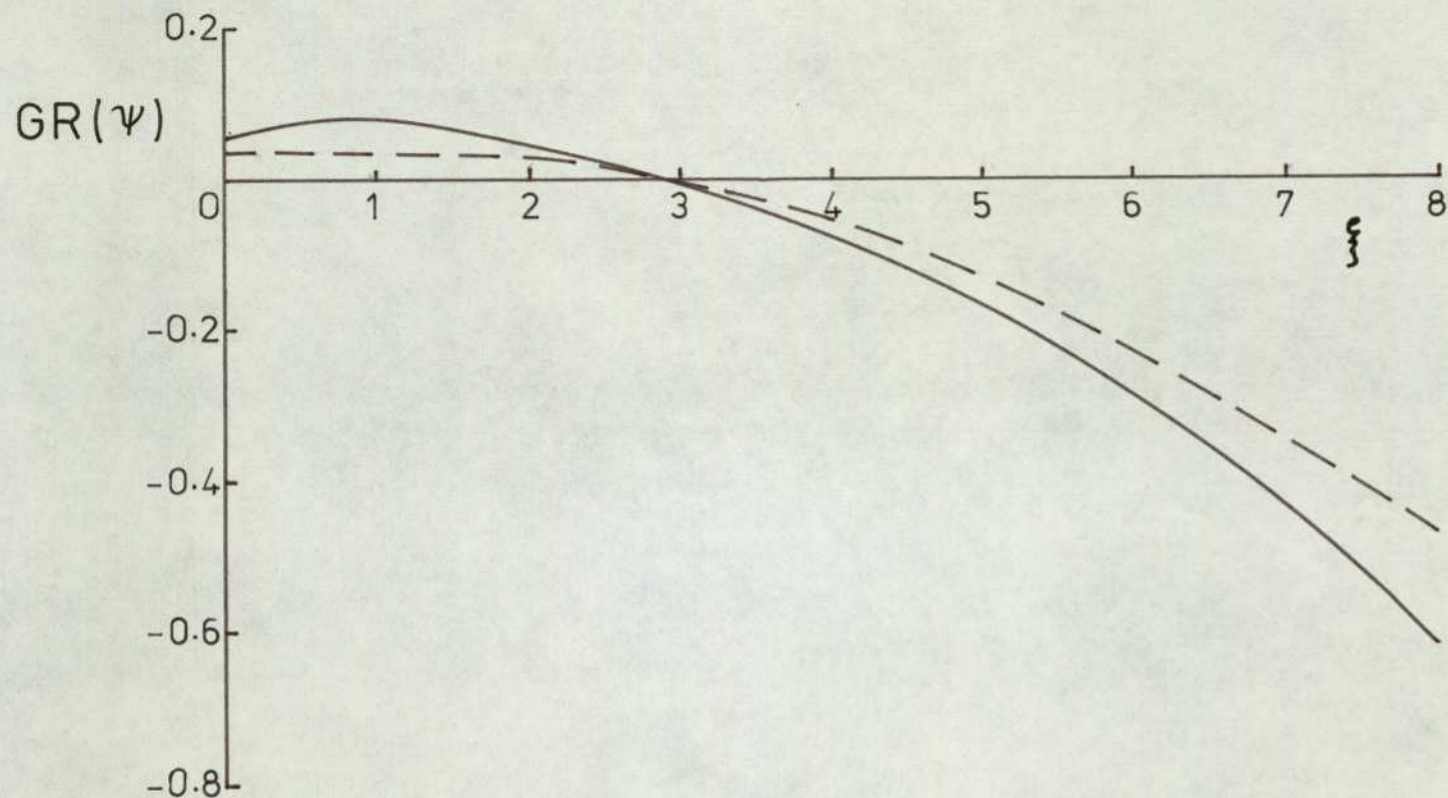


Figure 3.7.4: The growth rate of ψ , $GR(\psi)$, evaluated at $\eta = 0$,
for $\gamma = 4.093$, $R = 45$, $\beta = 1.5$.
— present work; --- Eagles & Weissman.

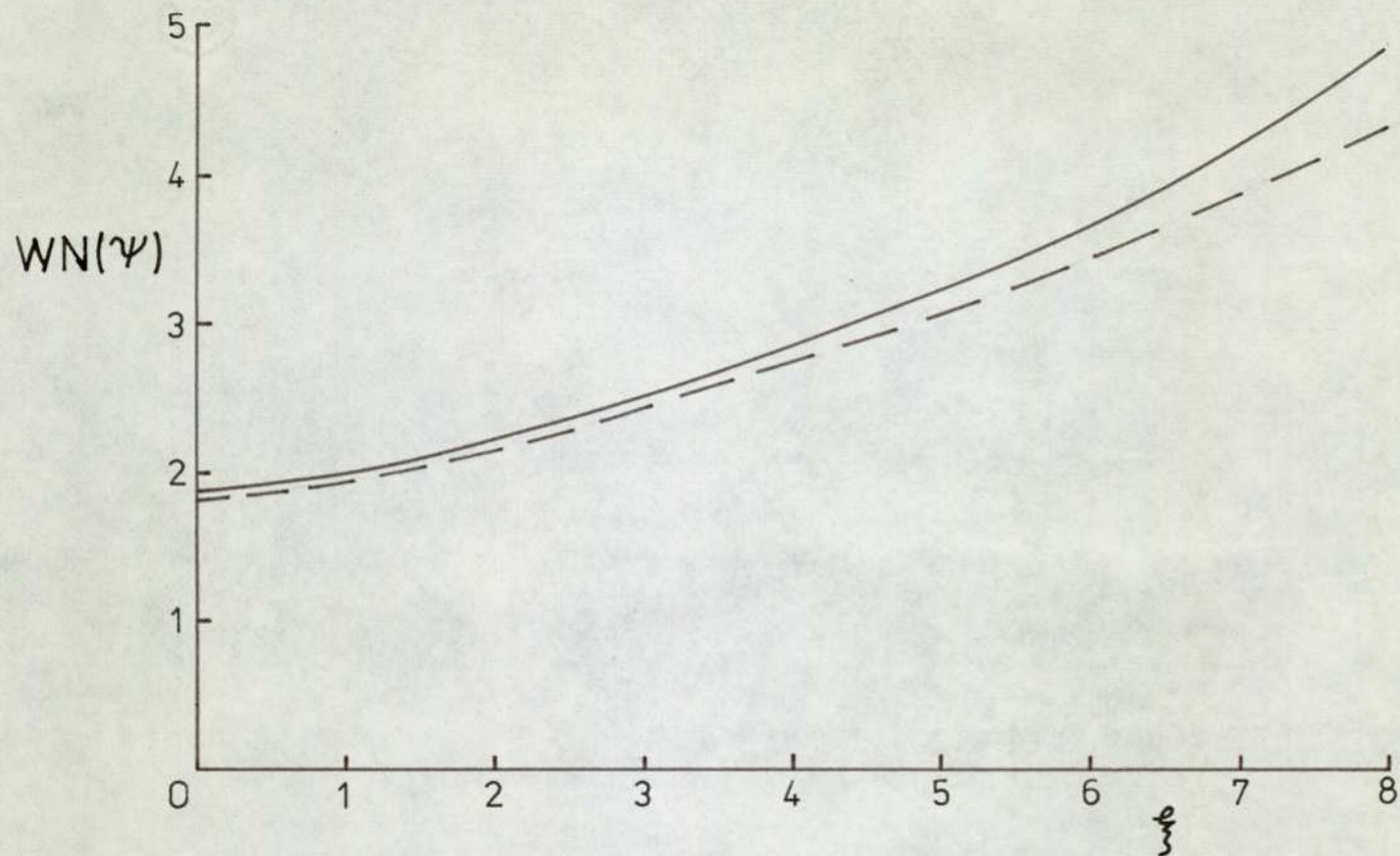


Figure 3.7.5: The wave number of ψ , $WN(\psi)$, evaluated at $\eta = 0$ for $\gamma = 4.093$, $R = 45$, $\beta = 1.5$. _____ present work;
 - - - - - Eagles & Weissman.

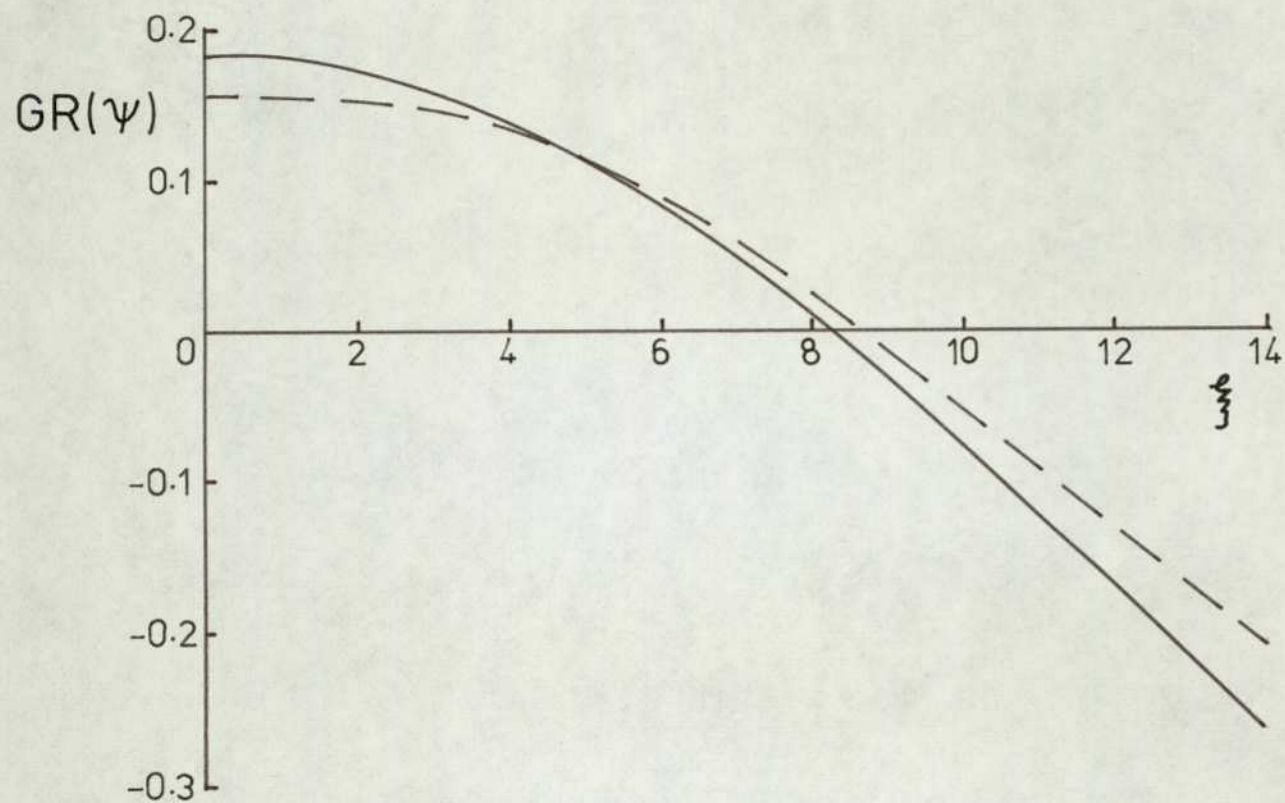


Figure 3.7.6: The growth rate of ψ , $GR(\psi)$, evaluated at $\eta=0$,
for $\gamma = 4.093$, $R = 100$, $\beta = 1.5$.
_____ present work; ---- Eagles & Weissman.

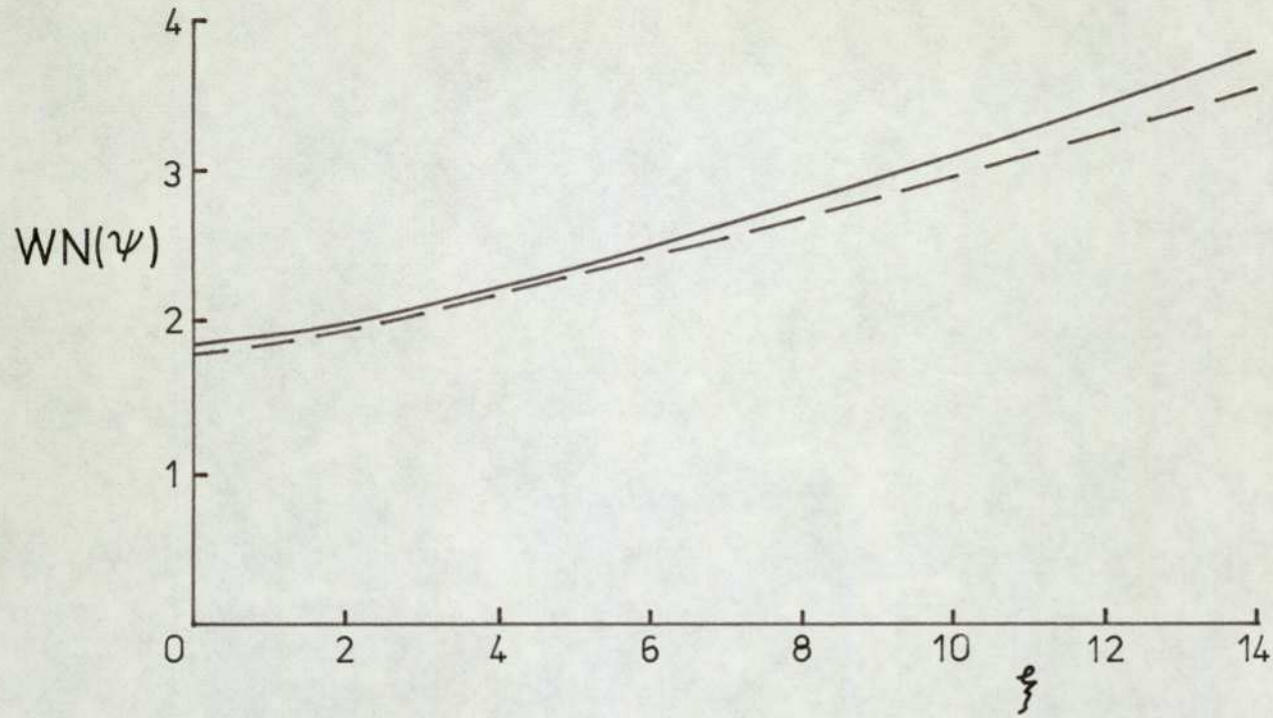


Figure 3.7.7: The wave number of ψ , $WN(\psi)$, evaluated at $\eta = 0$ for $\gamma = 4.093$, $R = 100$, $\beta = 1.5$. — present work; - - - Eagles & Weissman.

number which again will be compared with the Eagles & Weissman results, it is felt that it would be unreasonable to use the values from Eagles & Weissman for $\psi(0, \eta)$, K_0 , K_1 (where this is used), as it would appear that the two solutions are being "forced" to be the same. Thus extensive checks were carried out on the influence of $\psi(0, \eta)$ and K_0 downstream from $\xi = 0$, and of using either of the downstream boundary conditions, (3.4.6d) and (3.4.6e) (with various K_1 values) upstream from $\xi = \xi_1$. As far as $\psi(0, \eta)$ is concerned, it was found that by taking different functions, the growth rate results were not significantly affected downstream from $\xi = 0$. Similarly, different values for K_0 were tried, and again the results for the growth rates were not affected at sufficient distance downstream from $\xi = 0$. The result of taking three different values for K_0 for $\gamma = 4.093$, $R = 25$, $\beta = 0.1$ is shown in figure 3.7.8. The growth rates start off differently, but eventually all merge into the same curve. The distance downstream at which this happens depends upon the values of the other parameters, γ , R and β , and so for each value of γ taken (for which the neutral curves are to be drawn and for which the critical Reynolds number is to be calculated), a range of values of R and β are used so that the various values taken for K_0 do not affect the values found for which each growth rate is zero (these being the points plotted to obtain the neutral curves - this is discussed

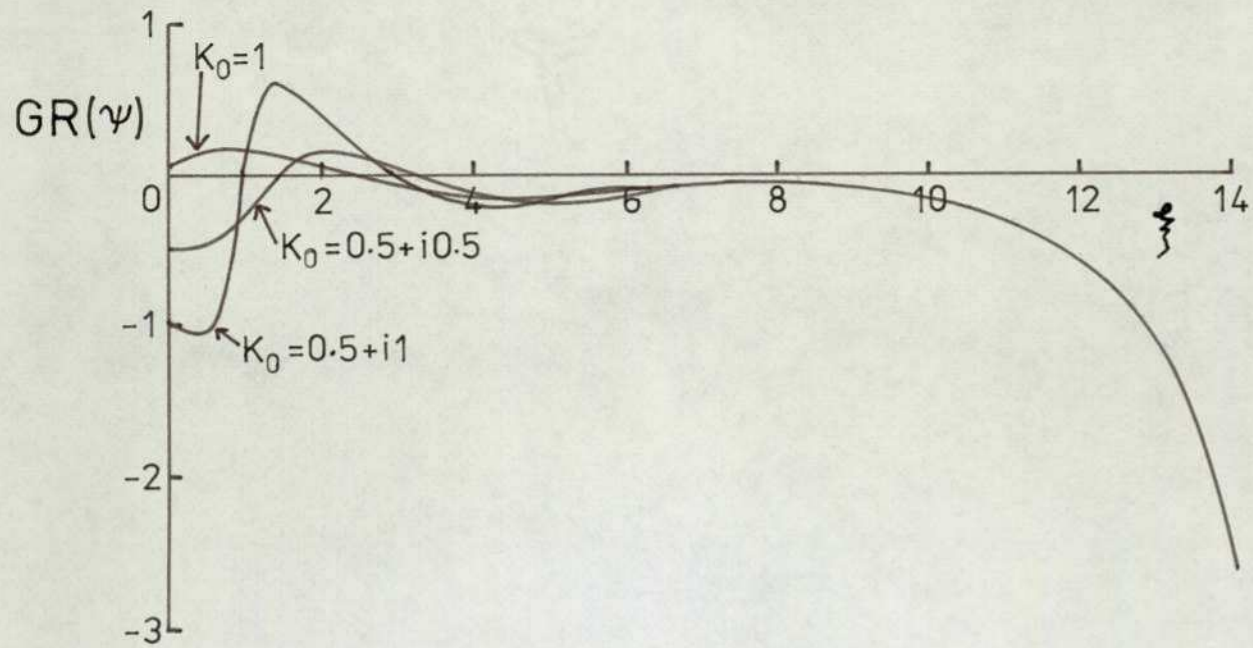


Figure 3.7.8: The growth rate of ψ , $GR(\psi)$, evaluated at $\eta = 0$, for the three values of K_0 shown, with $\gamma = 4.093$, $R = 25$, $\beta = 0.1$.

later in this section). For other cases within that range of values of R , with a suitable value for β , it was deemed reasonable to take $K_0 = 1 + i0$. In Eagles & Weissman the solution for ψ is in the form $A(X) f(\eta; \beta_1(X), K(X)) + O(\alpha)$, where $X = \alpha\xi$, $\beta_1 = \beta e^{2X}$, and the function which has the same basic shape as their eigenfunctions f is used extensively for $\psi(0, \eta)$ namely

$$\psi(0, \eta) = 0.5(1 + \cos \pi \eta) - i0.05(1 - \cos 2\pi \eta),$$

and this is illustrated in figure 3.7.9.

Downstream at $\xi = \xi_1$, both sets of boundary conditions, (3.4.6d) and (3.4.6e), were tried, and it was found that, so long as the value of ξ_1 was consistent with the fact that the solution had decayed upstream from this, the solutions upstream for stream function and growth rates were unaffected by the choice of boundary conditions. Indeed, tests were made with various values of K_1 for boundary condition (3.4.6e), and again the solutions upstream were not affected by the choice of K_1 .

What is required from the calculations are the positions at which each growth rate is zero, so that, for given γ , the neutral curves can be drawn, dividing the plane into regions of growth and regions of decay. Thus

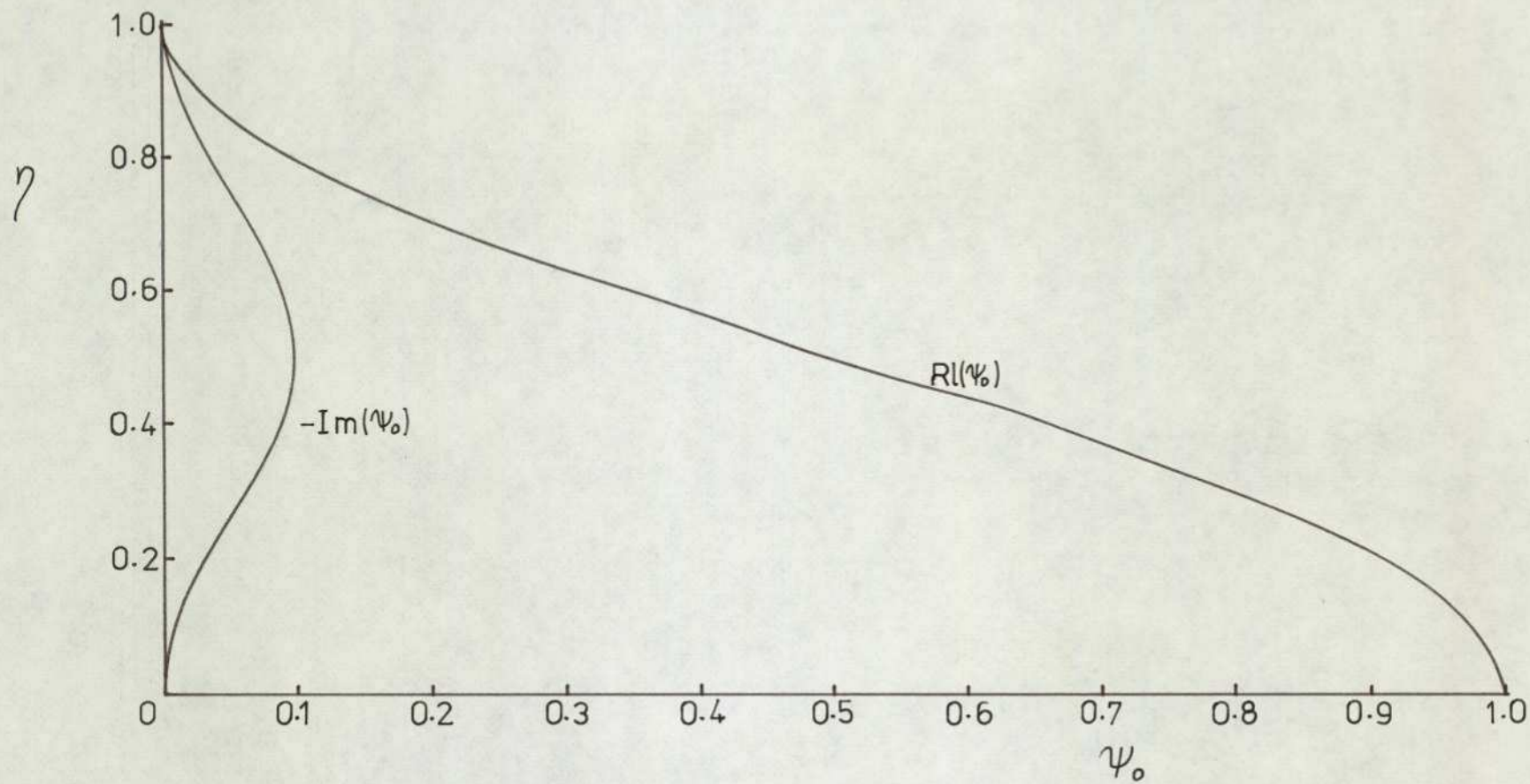


Figure 3.7.9: The disturbance stream function, ψ , at $\xi = 0$:
 $\psi_0 \equiv \psi(0, \eta) \equiv 0.5 (1 + \cos \pi \eta) + i 0.05 (1 - \cos 2 \pi \eta)$.

for each γ experimentation was made with a range of values of R , selecting suitable β and ξ_1 so that the positions of the zeros were not affected by the boundary conditions upstream at $\xi = 0$ and downstream at $\xi = \xi_1$. Shown in figure 3.7.10 are typical growth rates of E , $\hat{E}$ and ψ (evaluated at $\eta = 0$). The neutral curves are then plotted as follows. In equation (3.4.5) set $\beta_1 = \beta e^{2\alpha\xi}$, which is used instead of the variable ξ , so that it can be seen that each growth rate (with $GR(\psi)$ evaluated at the channel centre, $\eta = 0$) is a function of β_1, R and α . Eagles & Weissman introduced $\gamma = \alpha R$, since for fixed γ they have one velocity profile, $g(\eta)$ (or their $w(\eta)$ to $O(\alpha)$). Thus it is a convenient parameter to use, rather than α , and to fix γ so that with each value of R , α is determined by $\alpha = \gamma R^{-1}$. Each growth rate is therefore expressed as

$$GR(\dots) \equiv F(\beta_1, R, \gamma),$$

and the neutral curves are drawn for each γ by determining the value of $\beta_1 = \beta e^{2\alpha\xi}$ for which each growth rate is zero for various R values. A graph is then drawn on the (β_1, R) axes. Three different sets of curves, for the values $\gamma = 3.57, 4.093, 4.71$, are shown in figures 3.7.11, 3.7.12, 3.7.13. These are the three values used by Eagles & Weissman, and shown in each figure are the results of the present work (continuous curves) and the results of Eagles & Weissman (dashed curves). The curves indicate

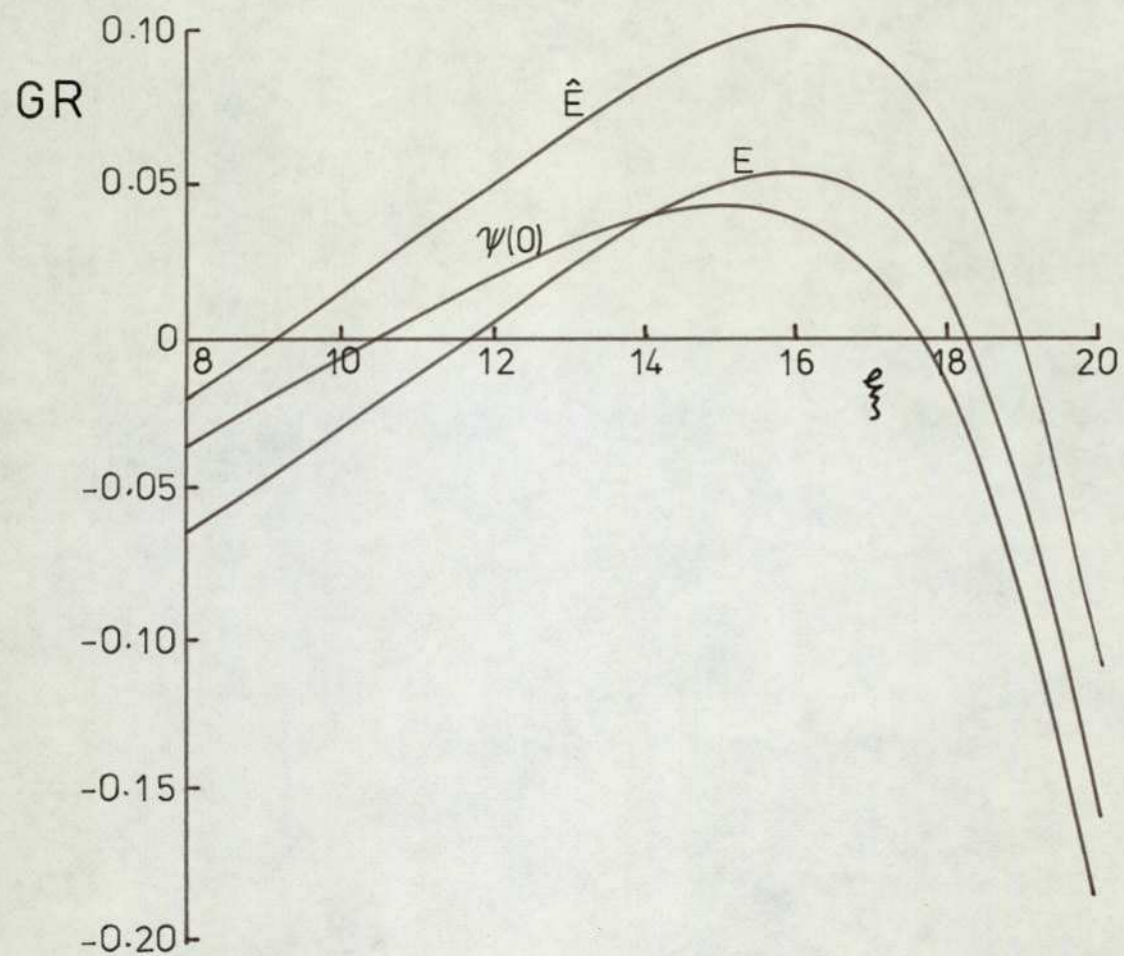


Figure 3.7.10: Typical growth rates, GR, of ψ (at $\eta=0$), E and $\hat{E}$ for $\gamma = 4.093$, $R = 45$, $\beta = 0.1$.

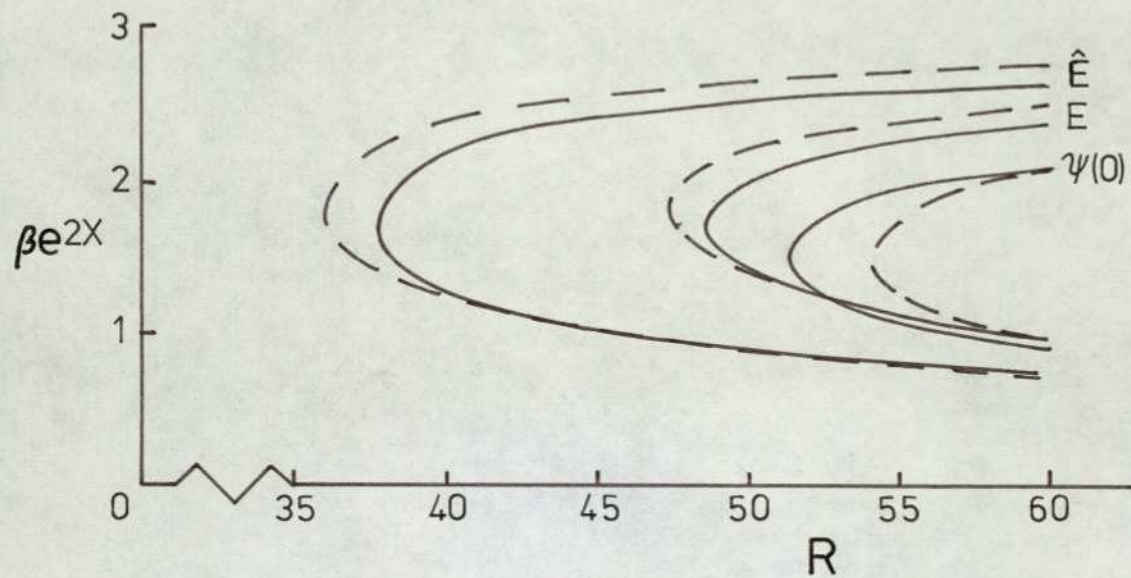


Figure 3.7.11: Neutral curves based on ψ (at $\eta=0$), E and $\hat{E}$ for $\gamma = 3.57$.
 Inside the curves there is growth, outside there is decay.
 ——— present work; - - - Eagles & Weissman.

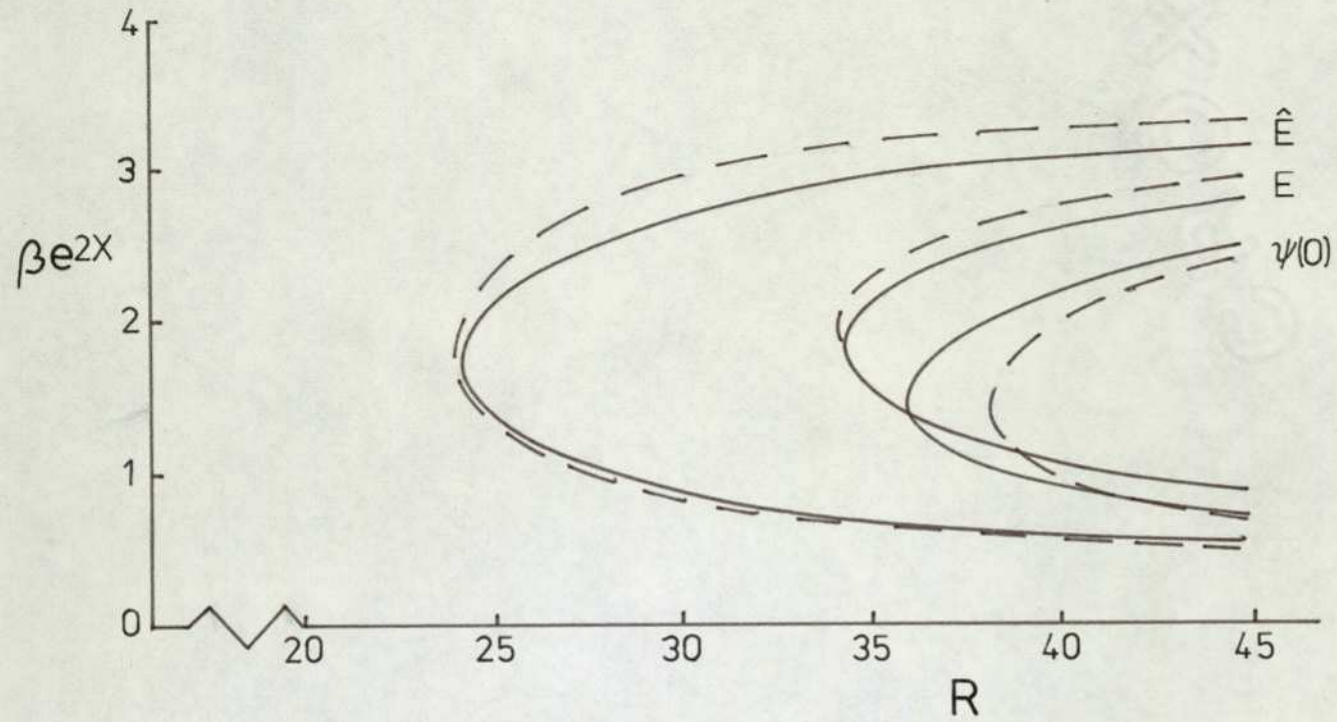


Figure 3.7.12: Neutral curves based on ψ (at $\eta=0$), E and $\hat{E}$ for $\gamma = 4.093$.
 — present work; --- Eagles & Weissman.

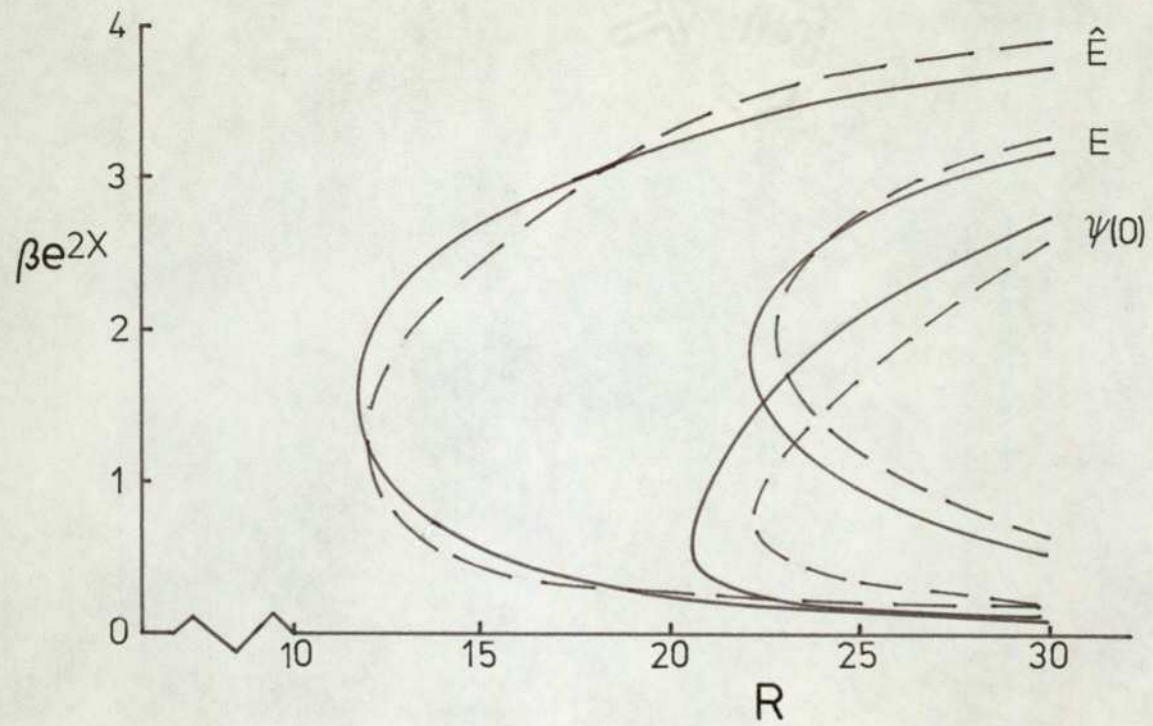


Figure 3.7.13: Neutral curves based on ψ (at $\eta=0$), E and $\hat{E}$ for $\gamma = 4.71$.
 ——— present work; --- Eagles & Weissman.

the regions of growth for the various flow quantities - inside the curves there is growth, outside there is decay. Also, for fixed β , the evolution of a wave as it travels downstream is given along a line of constant R , with waves which pass through a region of growth ultimately decaying. As can be seen from the three figures, there is qualitative agreement between the two methods, particularly when it is remembered that Eagles & Weissman use methods with an error of $O(\alpha^2)$, and the present work uses a direct numerical method with no such approximations.

The critical Reynolds number for each flow quantity is defined to be that value of R such that, with the Reynolds number greater than this value, there is a band of waves which grow over some range of values of ξ . However, this may not indicate instability. Thus for each value of γ , a critical Reynolds number is defined for each flow quantity, indicating when that particular flow quantity will be able to grow. The calculations as for the three values of γ given above were repeated for other γ values, the critical Reynolds numbers based on $\hat{E}$ were found, together with the corresponding values of α . These results are given in table 3.7.1. Plotted in figure 3.7.14 is the critical Reynolds number based on $\hat{E}$, versus α . Also shown on this graph are the three values from Eagles & Weissman. Here there is good agreement for this overall critical Reynolds number curve, although it

γ	R_{crit}	α_{crit}
3.25	49.5	0.066
3.57	37.8	0.094
3.75	32.6	0.115
4.093	24.3	0.168
4.25	21.0	0.202
4.5	15.8	0.285
4.71	11.7	0.403
4.8	10.3	0.466

Table 3.7.1:

The critical Reynolds numbers
 (R_{crit}) based on $\hat{E}$ and the corresponding
values of α (α_{crit}) for the values of
 $\gamma (= \alpha R)$ given.

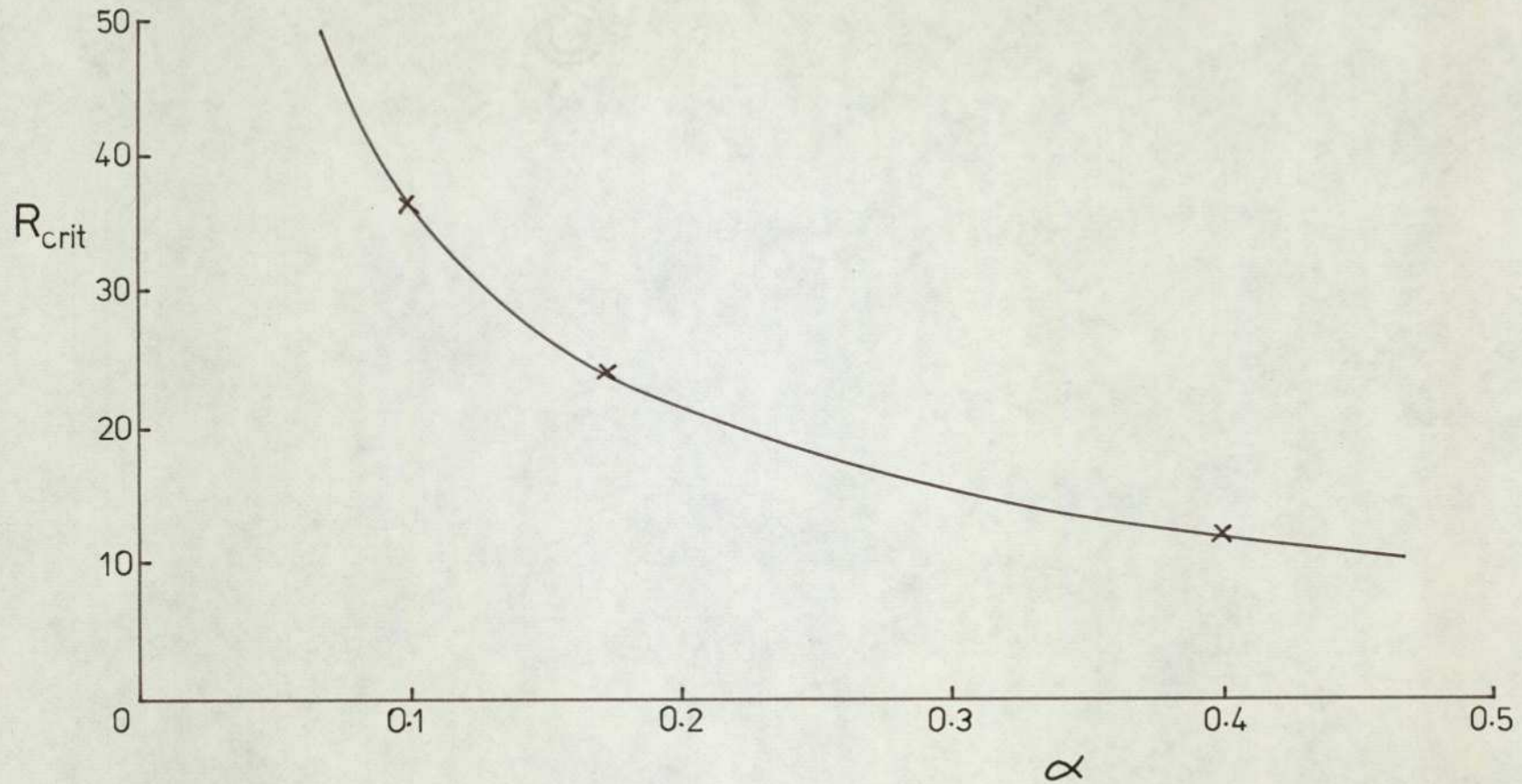


Figure 3.7.14: The overall critical Reynolds number, R_{crit} , based on the relative energy E . x the results of Eagles & Weissman.

must be pointed out that the agreement to such a degree most probably is in part a coincidence, since the calculations of Eagles & Weissman neglect terms of $O(\alpha^2)$ in the growth rate, and in part due to the scale used for the diagram.

A further point is of interest. With $\gamma = 4.8$, it was not possible to find points on the lower branch of the neutral curve for $\hat{E}$ above $R = 10.8$, although both points were obtained for the values of R from R_{crit} to 10.8. Eagles (1966), in his study of the stability of flows in a straight-wall divergent channel by the "quasi-parallel" approach, points out that when $\gamma > 4.71$ (and the velocity profile of the basic flow has reversed flow near the walls), amplified waves with a negative wave velocity occur when R is greater than a value larger than the critical Reynolds number. In the present case, this would result in a neutral curve going below the R -axis, with a negative $\beta e^{2\alpha\xi}$. This may be an explanation as to why these lower branch points cannot be found, although it should be pointed out that the failure to find them may be due to the influence of the boundary conditions at $\xi = 0$ giving incorrect growth rates and not having settled down by the time the value of ξ on the lower branch is reached, even when a value of β as small as 10^{-6} is used.

3.8 The Coupled-Equation Method

The method described in section 3.5 which was finally adopted for the solution of equation (3.4.5) is not that which was first tried. One such method, which does give results but was ultimately rejected due to the length of run time on the computer, will be described briefly in this section.

In equation (3.4.5), set

$$\nabla^2 \psi = \omega, \quad (3.8.1)$$

giving

$$\begin{aligned} R^{-1} \nabla^2 \omega - \left\{ 4\alpha R^{-1} + g(\eta) \right\} \frac{\partial \omega}{\partial \xi} + \left\{ 4\alpha^2 R^{-1} + 2\alpha g(\eta) + i\beta e^{2\alpha\xi} \right\} \omega \\ = -g''(\eta) \frac{\partial \psi}{\partial \xi} - 2\alpha g'(\eta) \frac{\partial \psi}{\partial \eta}, \end{aligned} \quad (3.8.2)$$

so that equations (3.8.1) and (3.8.2) form two coupled equations to be solved for ψ and ω . The boundary conditions for ψ are

$$\psi = 0 \quad \text{at } \eta = 1,$$

$$\frac{\partial \psi}{\partial \eta} = 0 \quad \text{at } \eta = 0,$$

$$\psi(0, \eta) = \text{given function},$$

$$\psi(\xi_1, \eta) = 0,$$

while the boundary conditions for ω are obtained using $\omega = \nabla^2 \psi$ following Greenspan (1968), and making use of the other boundary conditions for ψ as given in section 3.4.

The general method of solution employed is an iterative scheme. First an initial "guess" is made for ω and ψ (in fact both are put equal to zero everywhere), and then each is updated in turn, using the latest values of the other, by solving equation (3.8.2) and then equation (3.8.1) until values obtained for ψ and ω are to the required degree of accuracy. Thus the scheme may be set out as follows, having first set up ψ and $\frac{\partial \psi}{\partial \xi}$ at $\xi = 0$ as in the single equation method of section 3.4.

- (i) Calculate new boundary values for ω using the latest values of ψ .
- (ii) Calculate a better approximation to ω using equation (3.8.2) and the boundary conditions found in (i) above.
- (iii) Calculate a better approximation to ψ using equation (3.8.1) and the fixed boundary conditions for ψ .
- (iv) Return to (i) if ψ and ω are not to the prescribed required accuracy.

There is further discussion in section 5.2 on how the boundary

conditions for ψ are used at step (iii), and how they are employed to calculate the boundary conditions for ω at step (i).

In order to solve equation (3.8.1) (Poisson's equation) and equation (3.8.2), central differences are used for the derivatives to reduce each to linear simultaneous equations. At first, each set of equations was solved by a successive over-relaxation iterative method (SOR method), although for equation (3.8.2) the optimum relaxation factor could not be found a priori, and indeed would change for different values of R , α and β . Each was tested separately for convergence. It is then found that when the two are combined to form the overall iterative process given above in the steps (i)-(iv), the values obtained for ψ and ω diverge. It is therefore necessary to employ an under-relaxation technique, and so at steps (ii) and (iii), the values of ω and ψ adopted are the weighted averages of the "new" and "old" values. This then introduces two further relaxation or averaging factors (one for ω and one for ψ), and again there is no known analytic method of calculating the optimum values of these factors.

It is thus decided to solve equations (3.8.1) and (3.8.2) (or rather the finite difference equations obtained from them) by the direct method of Gaussian elimination.

Each is again taken separately, with results checked against those found by the SOR method, and agreement obtained. This then eliminated two of the relaxation factors, and left two remaining. The optimum values of these two averaging factors would change for different values of R, α, β and step lengths used, and, as pointed out above, they could not be found analytically. However, much numerical experimentation was carried out, and convergence of the overall scheme was achieved. This provided a very useful, independent check on the method of solution finally adopted, and in table 3.8.1 is presented a comparison of the solutions obtained by the two different methods. The results given are with $R = 15$, $\alpha = 0.314$, $\beta = 0.1$, $K_0 = 1+i0$, and as can be seen there is excellent agreement between the two methods. As an example of the numerical experimentation carried out, with the relaxation factor for ψ kept at 0.1, convergence was obtained in 237 iterations with the relaxation factor for ω ($RF\omega$) equal to 0.6, 320 iterations with $RF\omega = 0.7$, while with $RF\omega = 0.4$ the values diverged.

Although the coupled-equation method does work, it was felt that if numerical experimentation had to be undertaken for each run in order to obtain overall convergence, and since each separate equation for ω and ψ was being solved by a direct method, it was logical to solve the single equation (3.4.5) by a direct method. The coupled-equation method was

ξ	$\psi(\xi, 0)$			
	$\text{Re}(\psi)$		$\text{Im}(\psi)$	
	(a)	(b)	(a)	(b)
0.0	1.0	1.0	0.0	0.0
1.0	0.640694	0.640694	0.998280	0.998281
2.0	-0.455809	-0.455810	1.398530	1.398534
3.0	-1.486120	-1.486127	0.449692	0.449698
4.0	-0.725872	-0.725883	-1.218603	-1.218607
5.0	1.215649	1.215657	-0.269553	-0.269564
6.0	-0.728331	-0.728333	0.589218	0.589231
7.0	0.380516	0.380523	0.139096	0.139093
8.0	0.020362	0.020363	0.004523	0.004522

Table 3.8.1 Values of the stream function evaluated at the channel centre, $\psi(\xi, 0)$, for $R = 15$, $\alpha = 0.314$, $\beta = 0.1$, $K_0 = 1+i0$.

Columns (a): Coupled equation method;

columns (b): single equation method.

thus rejected in favour of the method finally adopted. However, the solution of Poisson's equation (3.8.1) was to prove useful in the solution of the initial-value problem described in Chapter 5, although both methods of solution indicated here (direct and SOR) were finally rejected in favour of a Fourier transform method, details of which may be found in section 5.2.

4. THE CURVED-WALL CHANNEL

This section deals with the basic, steady flow and its stability in a channel with curved walls. Taking the usual two-dimensional Cartesian Coordinate System (x,y) , a slow variable $X = \epsilon x$ is introduced, where ϵ is a small parameter, and the equations of the walls are then taken to be $y = \pm H(X)$. Following Eagles & Smith (1980), a particular class of channel is considered with equation

$$H(X) = 1 + \frac{1}{2} \tanh X,$$

so that the flow at $X = \pm\infty$ is expected to be Poiseuille flow. The type of channel is illustrated in figure 4.1.

4.1 The Steady State Flow

The non-dimensionalized Navier-Stokes equations for two-dimensional, steady, incompressible viscous flow in the absence of external forces are

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{\partial p}{\partial x} + \frac{1}{R} \nabla^2 u, \quad (4.1.1)$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = - \frac{\partial p}{\partial y} + \frac{1}{R} \nabla^2 v, \quad (4.1.2)$$

and the continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \quad (4.1.3)$$

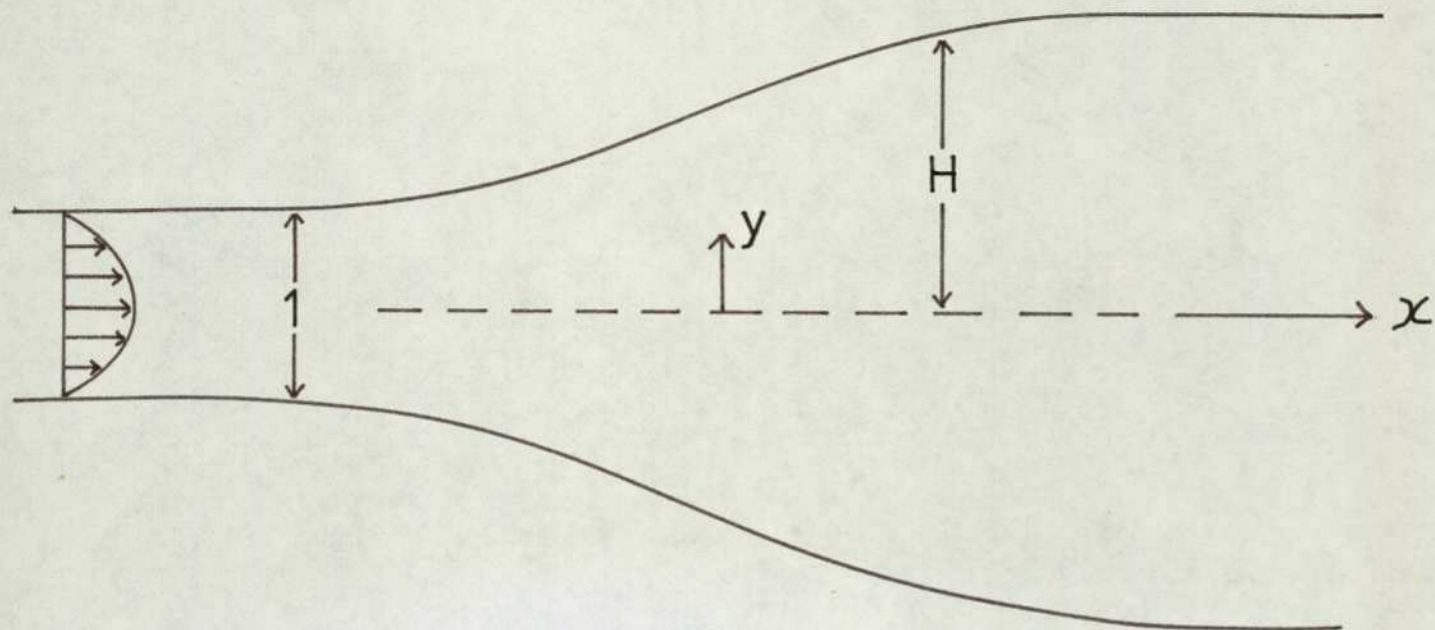


Figure 4.1. The curved-wall channel.

The velocity vector is (u,v) , p denotes the fluid pressure, and x,y are rectangular Cartesian co-ordinates in the streamwise and transverse directions respectively. The non-dimensionalization has been performed by

$$u = \left(\frac{a}{M}\right) u', \quad v = \left(\frac{a}{M}\right) v', \quad x = \frac{x'}{a}, \quad y = \frac{y'}{a}, \quad p = \left(\frac{a^2}{\rho M^2}\right) p',$$

where the primes denote dimensional quantities, $2M$ is the volumetric flow rate per unit thickness of the oncoming Poiseuille flow, a is the undisturbed channel width, ρ is the fluid density. The Reynolds number, R , is defined by $R = M/\nu$, ν being the kinematic viscosity. The non-dimensional stream function, ψ_0 (non-dimensionalized with respect to M), is also introduced at this point, where in the usual way

$$u = \frac{\partial \psi_0}{\partial y}, \quad v = -\frac{\partial \psi_0}{\partial x}.$$

The flow considered here is the steady planar laminar flow through a fixed channel of slowly varying width. Far upstream the channel walls are taken to be parallel and to contain incompressible fluid in plane Poiseuille motion. This oncoming flow is then described by

$$\psi_0 \rightarrow 3y - 4y^3, \quad u \rightarrow 3 - 12y^2, \quad v \rightarrow 0, \quad \frac{\partial p}{\partial x} \rightarrow -\frac{24}{R} \quad (4.1.4)$$

as $x \rightarrow -\infty$.

As in Eagles & Smith, the type of channel considered is one whose typical axial length scale is $O(R)$, i.e. the slopes of the channel walls are $O(R^{-1})$ typically. The two main reasons given for this are, first, that longer typical axial length scales generally produce a basic flow which is essentially just a minor perturbation of the original Poiseuille flow (e.g. Wilson, 1969; Tutty, 1980) while, second, shorter length scales tend to lead to multi-structured flow fields (Smith, 1976a, 1976b, 1979c), often involving separation and detached shear layers. What is required in this channel is the solution to equations (4.1.1), (4.1.2) and (4.1.3), with boundary conditions as the oncoming Poiseuille flow (4.1.4), no slip conditions at the wall, and a downstream condition of boundedness as $x \rightarrow \infty$.

The slow variable $X = \epsilon x$ is now introduced, and it is supposed that the channel width $2H$ is a function of X . Also R and ϵ are taken to be related in the form $\lambda = \epsilon R$, where λ is an $O(1)$ constant. In fact different flow fields will be obtained by altering λ . Asymptotic expansions in powers of ϵ are taken for u, v, p and ψ_0 , these substituted into equations (4.1.1), (4.1.2) and (4.1.3), with the result that the expansions are in the form

$$\begin{aligned} u &= U(X, y) + O(\epsilon^2), & v &= \epsilon V(X, y) + O(\epsilon^3), \\ p &= P(X) + O(\epsilon^2), & \psi_0 &= \Psi_0(X, y) + O(\epsilon^2). \end{aligned}$$

The lowest order streamwise momentum equation then reduces to the boundary layer equation

$$U U_X - \psi_{0X} U_Y = -\frac{dP}{dX} + \frac{1}{\lambda} U_{YY}, \quad \text{with } U = \psi_{0Y}. \quad (4.1.5)$$

The boundary conditions are

$$\psi_0(-\infty, y) = 3y - 4y^3, \quad U(-\infty, y) = 3-12y^2,$$

$$\frac{dP}{dX}(-\infty) = -24\lambda^{-1},$$

$$\psi_0 = -1, \quad U = 0 \quad \text{at } y = -H(X),$$

$$\psi_0 = 1, \quad U = 0 \quad \text{at } y = +H(X).$$

The solution of equation (4.1.5) with its boundary conditions can be obtained in principle by marching forward from the incoming solution since the governing (boundary layer) equation is locally parabolic in the X-direction, provided there is no flow reversal. The y-co-ordinate is now transformed to make the walls become the co-ordinate lines, by use of

$$\eta = y/H(X).$$

Also ψ_0 is denoted by $F(X, \eta)$, which yields the governing equation as

$$F_{\eta\eta\eta} + \lambda \left[H(F_{\eta\eta} F_X - F_{\eta} F_{\eta X}) + H^2 F_{\eta}^2 \right] - \lambda H^3 \frac{dP}{dX} = 0 \quad (4.1.6)$$

where the prime denotes differentiation with respect to X .

The boundary conditions are

$$F(X, \pm 1) = \pm 1, \quad F_\eta(X, \pm 1) = 0 \quad (4.1.7a)$$

$$F(-\infty, \eta) = \frac{3}{2}\eta - \frac{1}{2}\eta^3 \quad (4.1.7b)$$

$$\frac{dP}{dX}(-\infty) = 24\lambda^{-1},$$

where (4.1.7a) ensures constant flux and no slip at the walls, while (4.1.7b) matches with the oncoming Poiseuille flow.

Alternative boundary conditions to (4.1.7a) may be used, and such is the case in the present study, by using a symmetry condition at $\eta = 0$, yielding

$$F(X, 1) = 1, \quad F_\eta(X, 1) = 0, \quad (\text{wall}) \quad (4.1.8a)$$

$$F(X, 0) = F_{\eta\eta}(X, 0) = 0, \quad (\text{channel centre}). \quad (4.1.8b)$$

Note that equation (4.1.6) is the integral with respect to η of the vorticity equation:

$$F_{\eta\eta\eta\eta} + 2\lambda H' F_\eta F_{\eta\eta} + \lambda H(F_{\eta\eta\eta} F_X - F_\eta F_{\eta\eta X}) = 0. \quad (4.1.9)$$

In the present work, equation (4.1.6) is solved numerically following a method by Smith (1974), which provides the values required for the stability problem, while equation (4.1.9) is solved by an expansion method, which, with the terms found, is valid for a limited range of values of λ , but which provides a very useful check on the numerical results for the solution of equation (4.1.6). The method of solution of each equation

now follows.

4.1.1 Numerical Solution of the Stream Function Equation

The equation to be solved, giving the stream function, $F(X,\eta)$, of the basic, steady flow, is:

$$F_{\eta\eta\eta} + \lambda \left[H(F_{\eta\eta} F_X - F_{\eta} F_{\eta X}) + H' F_{\eta}^2 \right] - \lambda H^3 \frac{dP}{dX} = 0, \quad (4.1.1.1)$$

with boundary conditions

$$F(X,1) = 1, \quad F_{\eta}(X,1) = 0 \quad (4.1.1.2a)$$

$$F(X,0) = F_{\eta\eta}(X,0) = 0 \quad (4.1.1.2b)$$

$$F(-\infty, \eta) = \frac{3}{2}\eta - \frac{1}{2}\eta^3 \quad (4.1.1.2c)$$

$$\frac{dP}{dX}(-\infty) = -24\lambda^{-1}. \quad (4.1.1.2d)$$

Following Smith (1974), the equation (4.1.1.1) is replaced by three first order central difference equations in the variables F, G, S and P , where $G = F_{\eta}$ and $S = G_{\eta} = F_{\eta\eta}$. Thus the equations to be solved are:

$$G = F_{\eta},$$

$$S = G_{\eta},$$

$$\frac{\partial S}{\partial \eta} + \lambda H' G^2 = \lambda H^3 \frac{dP}{dX} + \lambda H (G \frac{\partial G}{\partial X} - S \frac{\partial F}{\partial X}).$$

The finite difference approximations to these three equations are:

$$G_{r,s-\frac{1}{2}} = \frac{F_{r,s} - F_{r,s-1}}{k}, \quad (4.1.1.3)$$

$$S_{r,s-\frac{1}{2}} = \frac{G_{r,s} - G_{r,s-1}}{k}, \quad (4.1.1.4)$$

$$\begin{aligned} \frac{S_{r-\frac{1}{2},s} - S_{r-\frac{1}{2},s-1}}{k} + \lambda H'_{r-\frac{1}{2}} (G^2)_{r-\frac{1}{2},s-\frac{1}{2}} &= \lambda H^3_{r-\frac{1}{2}} \left(\frac{P_r - P_{r-1}}{h} \right) \\ &+ \lambda H_{r-\frac{1}{2}} \left[G_{r-\frac{1}{2},s-\frac{1}{2}} \left(\frac{G_{r,s-\frac{1}{2}} - G_{r-1,s-\frac{1}{2}}}{h} \right) \right. \\ &\quad \left. - S_{r-\frac{1}{2},s-\frac{1}{2}} \left(\frac{F_{r,s-\frac{1}{2}} - F_{r-1,s-\frac{1}{2}}}{h} \right) \right], \quad (4.1.1.5) \end{aligned}$$

where h and k are the step lengths in the X - and η -coordinates respectively, so that $X_r = X_{r-1} + h$, and $X_{r-\frac{1}{2}} = X_r - \frac{1}{2}h$ etc., and $s = 0, 1, \dots, M$ where the interval $0 \leq \eta \leq 1$ has been divided into M equal intervals of length $k = 1/M$ so that $\eta_s = sk$. To facilitate the computation the values of the variables (e.g. $S_{r-\frac{1}{2},s}$) at points midway between mesh points are averaged in terms of their values at the surrounding mesh points (e.g. $S_{r-1,s}$ and $S_{r,s}$).

Thus equations (4.1.1.3), (4.1.1.4) and (4.1.1.5) are to be solved for the unknowns

$$\left\{ F_{r,s}, G_{r,s}, S_{r,s} \right\}_{s=0}^M, P_r$$

for given r , knowing the solutions at $(r-1)$.

The equations then become:

$$\frac{k}{2} (G_{r,s} + G_{r,s-1}) = F_{r,s} - F_{r,s-1},$$

or

$$F_{r,s-1} - F_{r,s} + \frac{k}{2} G_{r,s-1} + \frac{k}{2} G_{r,s} = 0, \quad (4.1.1.6)$$

$$G_{r,s-1} - G_{r,s} + \frac{k}{2} S_{r,s-1} + \frac{k}{2} S_{r,s} = 0, \quad (4.1.1.7)$$

while (4.1.1.5) becomes

$$\begin{aligned} & \frac{1}{2k} (S_{r,s} + S_{r-1,s} - S_{r,s-1} - S_{r-1,s-1}) \\ & + \lambda H'_{r-\frac{1}{2}} \left[(G^2)_{r,s-\frac{1}{2}} + (G^2)_{r-1,s-\frac{1}{2}} \right] \\ & = \frac{\lambda H^3_{r-\frac{1}{2}}}{h} (P_r - P_{r-1}) \\ & + \frac{\lambda}{2} H_{r-\frac{1}{2}} \left[(G_{r,s-\frac{1}{2}} + G_{r-1,s-\frac{1}{2}}) \left(\frac{G_{r,s-\frac{1}{2}} - G_{r-1,s-\frac{1}{2}}}{h} \right) \right. \\ & \left. - (S_{r,s-\frac{1}{2}} + S_{r-1,s-\frac{1}{2}}) \left(\frac{F_{r,s-\frac{1}{2}} - F_{r-1,s-\frac{1}{2}}}{h} \right) \right]. \end{aligned}$$

On multiplying through by $2k$ and collecting the unknowns on the left-hand side, this equation becomes:

$$\begin{aligned} & S_{r,s} - S_{r,s-1} + \lambda k H'_{r-\frac{1}{2}} \left[(G^2)_{r,s-\frac{1}{2}} \right] - \frac{\lambda k}{h} H_{r-\frac{1}{2}} \left[(G_{r,s-\frac{1}{2}})^2 - S_{r,s-\frac{1}{2}} (F_{r,s-\frac{1}{2}} \right. \\ & \left. - F_{r-1,s-\frac{1}{2}}) \right] - S_{r-1,s-\frac{1}{2}} F_{r,s-\frac{1}{2}} - \frac{2\lambda k}{h} H^3_{r-\frac{1}{2}} P_r = W_{r-1,s-\frac{1}{2}}, \end{aligned} \quad (4.1.1.8)$$

where:

$$W_{r-1,s-\frac{1}{2}} = - (S_{r-1,s} - S_{r-1,s-1})^{-\lambda k H^1}_{r-\frac{1}{2}} \left[(G^2)_{r-1,s-\frac{1}{2}} \right] \\ - \frac{\lambda k}{h} H_{r-\frac{1}{2}} \left[(G_{r-1,s-\frac{1}{2}})^2 - S_{r-1,s-\frac{1}{2}} F_{r-1,s-\frac{1}{2}} \right] - \frac{2\lambda k}{h} H^3_{r-\frac{1}{2}} P_{r-1}.$$

An iteration process is used to solve these equations.

The nth approximations to the true solutions of equations

(4.1.1.6), (4.1.1.7) and (4.1.1.8) are denoted by

$F_{r,s}^{(n)}, G_{r,s}^{(n)}, S_{r,s}^{(n)}, P_r^{(n)}$. The (n+1)th approximation is

then written as

$$F_{r,s}^{(n+1)} = F_{r,s}^{(n)} + \delta F_{r,s}^{(n)} \quad \text{etc.,}$$

these substituted into the equations and linearized, equation

(4.1.1.8) giving:

$$S_{r,s}^{(n)} + \delta S_{r,s}^{(n)} - S_{r,s-1}^{(n)} - \delta S_{r,s-1}^{(n)} + \lambda k H^1_{r-\frac{1}{2}} \left[(G^2)^{(n)} + \delta G^2^{(n)} \right]_{r,s-\frac{1}{2}} \\ - \frac{\lambda k}{h} H_{r-\frac{1}{2}} \left[(G_{r,s-\frac{1}{2}}^{(n)} + \delta G_{r,s-\frac{1}{2}}^{(n)})^2 - (S_{r,s-\frac{1}{2}}^{(n)} + \delta S_{r,s-\frac{1}{2}}^{(n)}) (F_{r,s-\frac{1}{2}}^{(n)} + \delta F_{r,s-\frac{1}{2}}^{(n)} - F_{r-1,s-\frac{1}{2}}) \right. \\ \left. - S_{r-1,s-\frac{1}{2}} (F_{r,s-\frac{1}{2}}^{(n)} + \delta F_{r,s-\frac{1}{2}}^{(n)}) \right] - \frac{2\lambda k}{h} H^3_{r-\frac{1}{2}} (P_r^{(n)} + \delta P_r^{(n)}) = W_{r-1,s-\frac{1}{2}}.$$

Now:-

$$\left[(G^2)^{(n)} + \delta (G^2)^{(n)} \right]_{r,s-\frac{1}{2}} = \frac{1}{2} \left[\left\{ (G^2)^{(n)} + \delta (G^2)^{(n)} \right\}_{r,s} + \left\{ (G^2)^{(n)} + \delta (G^2)^{(n)} \right\}_{r,s-1} \right] \\ = \frac{1}{2} \left[(G^2)_{r,s}^{(n)} + (G^2)_{r,s-1}^{(n)} + 2(G^{(n)} \delta G^{(n)})_{r,s} + 2(G^{(n)} \delta G^{(n)})_{r,s-1} \right]$$

$$= (G^2)_{r,s-\frac{1}{2}}^{(n)} + G_{r,s}^{(n)} \delta G_{r,s}^{(n)} + G_{r,s-1}^{(n)} \delta G_{r,s-1}^{(n)}$$

and

$$\begin{aligned} (G_{r,s-\frac{1}{2}}^{(n)} + \delta G_{r,s-\frac{1}{2}}^{(n)})^2 &= (G_{r,s-\frac{1}{2}}^{(n)})^2 + 2G_{r,s-\frac{1}{2}}^{(n)} \delta G_{r,s-\frac{1}{2}}^{(n)} \\ &= (G_{r,s-\frac{1}{2}}^{(n)})^2 + G_{r,s-\frac{1}{2}}^{(n)} (\delta G_{r,s}^{(n)} + \delta G_{r,s-1}^{(n)}), \end{aligned}$$

where the expressions have been linearized by ignoring second order terms $(\delta G)^2$ etc.

Thus the equation becomes, on rearranging as an equation for $\delta F^{(n)}$, $\delta G^{(n)}$, $\delta S^{(n)}$, $\delta P^{(n)}$:-

$$\begin{aligned} &\left[\frac{\lambda k}{2h} H_{r-\frac{1}{2}} (S_{r,s-\frac{1}{2}}^{(n)} + S_{r-1,s-\frac{1}{2}}^{(n)}) \right] \delta F_{r,s-1}^{(n)} \\ &+ \left[\lambda k H'_{r-\frac{1}{2}} G_{r,s-1}^{(n)} - \frac{\lambda k}{h} H_{r-\frac{1}{2}} G_{r,s-\frac{1}{2}}^{(n)} \right] \delta G_{r,s-1}^{(n)} \\ &+ \left[-1 + \frac{\lambda k}{2h} H_{r-\frac{1}{2}} (F_{r,s-\frac{1}{2}}^{(n)} - F_{r-1,s-\frac{1}{2}}^{(n)}) \right] \delta S_{r,s-1}^{(n)} \\ &+ \left[\frac{\lambda k}{2h} H_{r-\frac{1}{2}} (S_{r,s-\frac{1}{2}}^{(n)} + S_{r-1,s-\frac{1}{2}}^{(n)}) \right] \delta F_{r,s}^{(n)} \\ &+ \left[\lambda k H'_{r-\frac{1}{2}} G_{r,s}^{(n)} - \frac{\lambda k}{h} H_{r-\frac{1}{2}} G_{r,s-\frac{1}{2}}^{(n)} \right] \delta G_{r,s}^{(n)} \\ &+ \left[1 + \frac{\lambda k}{2h} H_{r-\frac{1}{2}} (F_{r,s-\frac{1}{2}}^{(n)} - F_{r-1,s-\frac{1}{2}}^{(n)}) \right] \delta S_{r,s}^{(n)} \\ &+ \left[-\frac{2\lambda k}{h} H_{r-\frac{1}{2}}^3 \right] \delta P_r^{(n)} = W_{r-1,s-\frac{1}{2}}^{(n)} + S_{r,s-1}^{(n)} - S_{r,s}^{(n)} \\ &\quad - \lambda k H'_{r-\frac{1}{2}} (G_{r,s-\frac{1}{2}}^{(n)})^2 \end{aligned}$$

$$\begin{aligned}
& + \frac{\lambda k}{h} H_{r-\frac{1}{2}} \left[(G_{r,s-\frac{1}{2}}^{(n)})^2 - S_{r,s-\frac{1}{2}}^{(n)} (F_{r,s-\frac{1}{2}}^{(n)} - F_{r-1,s-\frac{1}{2}}) - S_{r-1,s-\frac{1}{2}} F_{r,s-\frac{1}{2}}^{(n)} \right] \\
& + \frac{2\lambda k}{h} H_{r-\frac{1}{2}}^3 P_r^{(n)} .
\end{aligned} \tag{4.1.1.9}$$

Thus equation (4.1.1.9) is of the form:

$$\begin{aligned}
& a_{r,s}^{(n)} \delta F_{r,s-1}^{(n)} + b_{r,s}^{(n)} \delta G_{r,s-1}^{(n)} + c_{r,s}^{(n)} \delta S_{r,s-1}^{(n)} + a_{r,s}^{(n)} \delta F_{r,s}^{(n)} \\
& + d_{r,s}^{(n)} \delta G_{r,s}^{(n)} + e_{r,s}^{(n)} \delta S_{r,s}^{(n)} + f_r \delta P_r^{(n)} \\
& = W_{r-1,s-\frac{1}{2}} + g_{r,s-\frac{1}{2}}^{(n)} ,
\end{aligned} \tag{4.1.1.10}$$

where:

$$a_{r,s}^{(n)} = \frac{\lambda k}{2h} H_{r-\frac{1}{2}} (S_{r,s-\frac{1}{2}}^{(n)} + S_{r-1,s-\frac{1}{2}}) ,$$

$$b_{r,s}^{(n)} = \lambda k H'_{r-\frac{1}{2}} G_{r,s-1}^{(n)} - \frac{\lambda k}{h} G_{r,s-\frac{1}{2}}^{(n)} ,$$

$$c_{r,s}^{(n)} = -1 + \frac{\lambda k}{2h} H_{r-\frac{1}{2}} (F_{r,s-\frac{1}{2}}^{(n)} - F_{r-1,s-\frac{1}{2}}) ,$$

$$d_{r,s}^{(n)} = b_{r,s}^{(n)} + \lambda k H'_{r-\frac{1}{2}} (G_{r,s}^{(n)} - G_{r,s-1}^{(n)}) ,$$

$$e_{r,s}^{(n)} = c_{r,s}^{(n)} + 2 ,$$

$$f_r^{(n)} = - \frac{2\lambda k}{h} H_{r-\frac{1}{2}}^3 ,$$

$$g_{r,s-\frac{1}{2}}^{(n)} = s_{r,s-1}^{(n)} - s_{r,s}^{(n)} - \lambda k H_{r-\frac{1}{2}}' (G_{r,s-\frac{1}{2}}^{(n)})^2 + \frac{\lambda k}{h} H_{r-\frac{1}{2}} \left[(G_{r,s-\frac{1}{2}}^{(n)})^2 - s_{r,s-\frac{1}{2}}^{(n)} (F_{r,s-\frac{1}{2}}^{(n)} - F_{r-1,s-\frac{1}{2}}^{(n)}) - s_{r-1,s-\frac{1}{2}}^{(n)} F_{r,s-\frac{1}{2}}^{(n)} \right] + \frac{2\lambda k}{h} H_{r-\frac{1}{2}}^3 P_r^{(n)}.$$

Using the same process, equations (4.1.1.6) and (4.1.1.7) become:

$$\begin{aligned} \delta F_{r,s-1}^{(n)} + \frac{k}{2} \delta G_{r,s-1}^{(n)} - \delta F_{r,s}^{(n)} + \frac{k}{2} \delta G_{r,s}^{(n)} \\ = F_{r,s}^{(n)} - F_{r,s-1}^{(n)} - \frac{k}{2} (G_{r,s}^{(n)} + G_{r,s-1}^{(n)}), \end{aligned} \quad (4.1.1.11)$$

$$\begin{aligned} \delta G_{r,s-1}^{(n)} + \frac{k}{2} \delta S_{r,s-1}^{(n)} - \delta G_{r,s}^{(n)} + \frac{k}{2} \delta S_{r,s}^{(n)} \\ = G_{r,s}^{(n)} - G_{r,s-1}^{(n)} - \frac{k}{2} (S_{r,s}^{(n)} + S_{r,s-1}^{(n)}). \end{aligned} \quad (4.1.1.12)$$

The boundary conditions are

$$F(X,1) = 1, \quad F_{\eta}(X,1) = 0, \quad F(X,0) = F_{\eta\eta}(X,0) = 0,$$

and in finite difference form these yield:

$$\begin{aligned} F_{r,0} = 0 \Rightarrow F_{r,0}^{(n)} + \delta F_{r,0}^{(n)} = 0 \\ \therefore \delta F_{r,0}^{(n)} = -F_{r,0}^{(n)}, \end{aligned} \quad (4.1.1.13)$$

$$S_{r,o} = 0 \Rightarrow \delta S_{r,o}^{(n)} = - S_{r,o}^{(n)} , \quad (4.1.1.14)$$

$$F_{r,M} = 1 \Rightarrow \delta F_{r,M}^{(n)} = 1 - F_{r,M}^{(n)} , \quad (4.1.1.15)$$

$$G_{r,M} = 0 \Rightarrow \delta G_{r,M}^{(n)} = - G_{r,M}^{(n)} . \quad (4.1.1.16)$$

The initial conditions are

$$F = \frac{3}{2}\eta - \frac{1}{2}\eta^3 \Rightarrow F_{o,s} = \frac{3}{2}\eta_s - \frac{1}{2}\eta_s^3 ,$$

$$G = \frac{3}{2} - \frac{3}{2}\eta^2 \Rightarrow G_{o,s} = \frac{3}{2} - \frac{3}{2}\eta_s^2 ,$$

$$S = -3\eta \Rightarrow S_{o,s} = -3\eta_s .$$

Thus, in order, the equations

$$(4.1.1.13), (4.1.1.14), \left\{ (4.1.1.11), (4.1.1.12), (4.1.1.10) \right\}_{s=1}^M ,$$

(4.1.1.15), (4.1.1.16) give the $(3M+4)$ equations for the $(3M+4)$ unknowns

$$\left\{ \delta F_{r,s}^{(n)} , \delta G_{r,s}^{(n)} , \delta S_{r,s}^{(n)} \right\}_{s=0}^M , P_r^{(n)} .$$

The initial values for $\delta F_{r,s}^{(0)}, \delta G_{r,s}^{(0)}, \dots$, may be taken as the final values for $\delta F_{r-1,s}, \delta G_{r-1,s}, \dots$, and the procedure is repeated until successive iterates are sufficiently close in value.

The $(3M+4)$ equations may be expressed in matrix form as

$$\underline{A} \underline{r} = \underline{q} ,$$

where the form of matrix $\underline{A}$ and column vector $\underline{r}$ is shown in figure 4.1.1.1, and $\underline{q}$ consists of the right-hand sides of equations (4.1.1.10) to (4.1.1.16), taken in the order above.

The method of solution now differs from that of Smith (1974), where in the present work a direct method of solution is used based on that used for the straight-wall channel given in section 3.5. Here, as for that case, the coefficient matrix $\underline{A}$ is sparse and of band form, with the exception of the right-hand column, which is now non-zero. The method is modified to take account of the different band width here, and also to cope with the right-hand column. The results obtained were checked graphically with those of Eagles & Smith, with good agreement. A check was also made with the method of solution for finding $F(X, \eta)$ from equation (4.1.9) which is detailed in the next section, 4.1.2, and details of this will be given there. Following Eagles & Smith, the values $\delta X = 0.01$, $\delta \eta = 0.01$, $X_{\infty} = -4.5$ were used, with an iterative tolerance of 10^{-7} . The program was run on the CDC 7600 computer of the University of London Computer Centre. Figure 4.1.1.2 illustrates the velocity profiles at $X = 0$ for three λ values ($\lambda=8, 12$ and 16), with the incoming Poiseuille flow shown as the dashed curve. The way in which the velocity profile, $U (= \frac{1}{H} F_{\eta})$, for $\lambda = 16$ develops downstream in the curved wall channel is presented in figure 4.1.1.3.

$$\underline{A} = \begin{bmatrix} 1 & 0 & 0 & & & & 0 \\ 0 & 0 & 1 & & & & 0 \\ & \underline{A}_{r,1} & & \underline{B}_{r,1} & & & \underline{F}_r \\ & & & \underline{A}_{r,2} & \underline{B}_{r,2} & & \underline{F}_r \\ & & & & \ddots & & \vdots \\ & & & & & \underline{A}_{r,M} & \underline{B}_{r,M} & \underline{F}_r \\ & & & & & & 1 & 0 & 0 & 0 \\ & & & & & & 0 & 1 & 0 & 0 \end{bmatrix}$$

where:-

$$\underline{A}_{r,s} = \begin{bmatrix} 1 & k/2 & 0 \\ 0 & 1 & k/2 \\ d_{r,s}^{(n)} & b_{r,s}^{(n)} & c_{r,s}^{(n)} \end{bmatrix}, \quad \underline{B}_{r,s} = \begin{bmatrix} -1 & k/2 & 0 \\ 0 & -1 & k/2 \\ d_{r,s}^{(n)} & b_{r,s}^{(n)} & c_{r,s}^{(n)} \end{bmatrix}, \quad \underline{F}_r = \begin{bmatrix} 0 \\ 0 \\ f_r \end{bmatrix};$$

$$\underline{r} = \left[\underline{D}_{r,0} \quad \underline{D}_{r,1} \quad \cdot \quad \cdot \quad \cdot \quad \underline{D}_{r,M} \quad \delta P_r^{(n)} \right]^T$$

where:-

$$\underline{D}_{r,s} = \left[\delta F_{r,s}^{(n)} \quad \delta G_{r,s}^{(n)} \quad \delta S_{r,s}^{(n)} \right]^T$$

Figure 4.1.1.1: The coefficient matrix $\underline{A}$ and the column vector of unknowns $\underline{r}$.

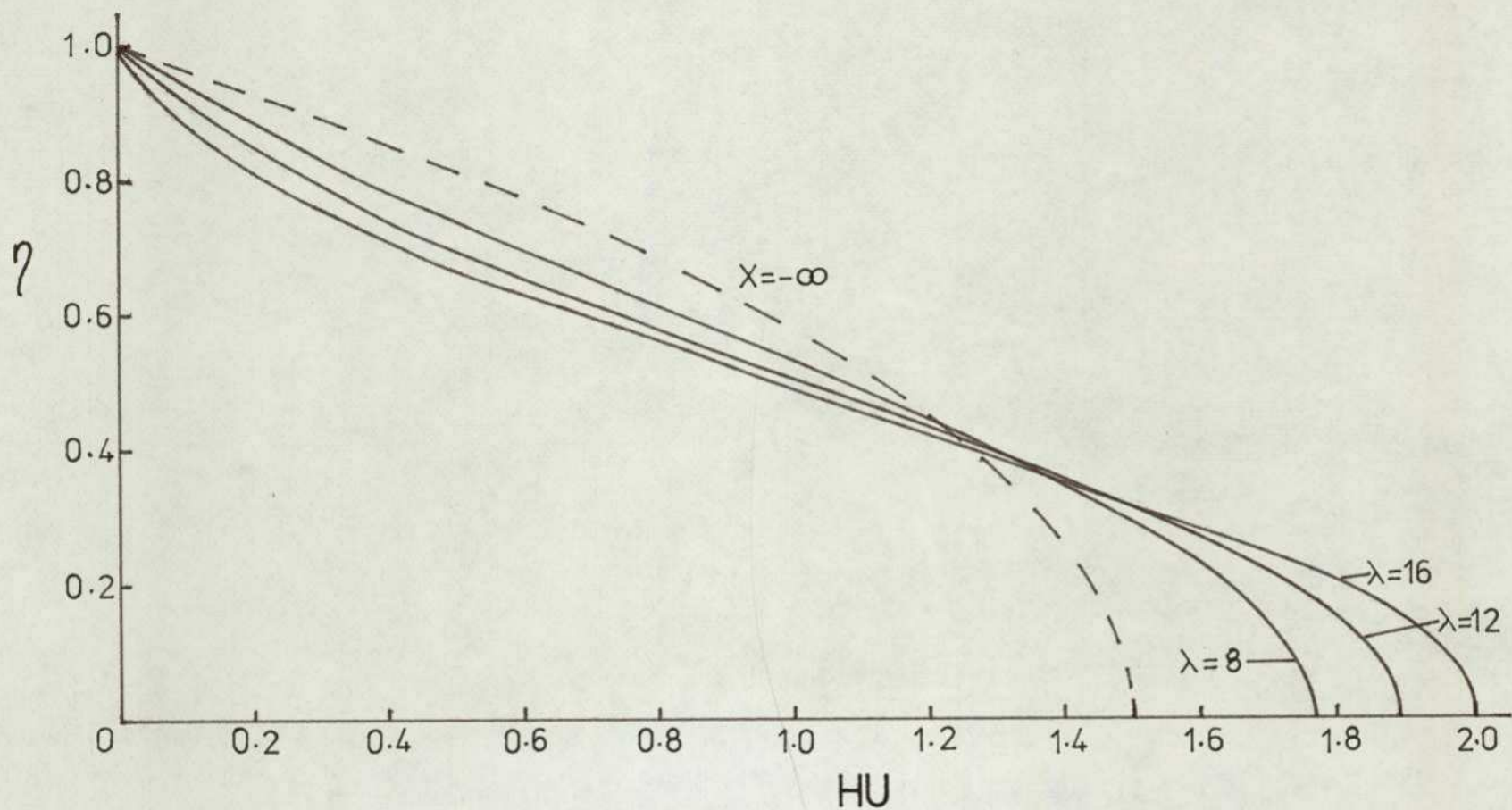


Figure 4.1.1.2: Velocity profiles HU ($=F_\eta$) versus η for the three values of λ , 8, 12 & 16 (continuous curves) at $X = 0$. The dashed curve is the incoming Poiseuille flow.

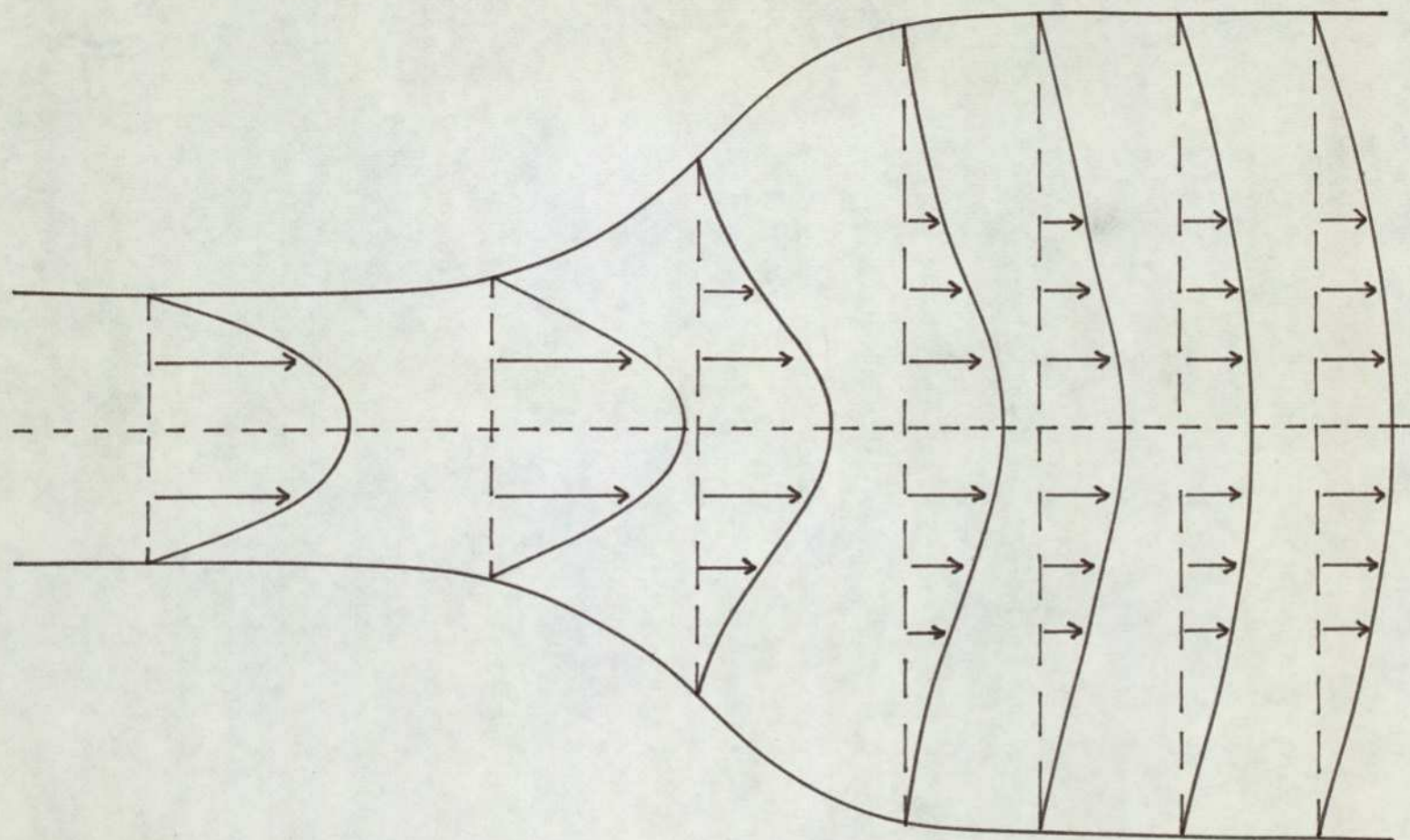


Figure 4.1.1.3: The downstream development of the velocity profile, $U (= \frac{1}{H} F_\eta)$ for $\lambda = 16$.

η	$F(X, \eta)$					
	$\lambda = 8$			$\lambda = 16$		
	$X=-2$	$X=0$	$X=+2$	$X=-2$	$X=0$	$X=+2$
0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.1506	0.1756	0.1584	0.1512	0.1984	0.1922
0.2	0.2980	0.3445	0.3124	0.2991	0.3866	0.3745
0.3	0.4392	0.5006	0.4579	0.4408	0.5555	0.5384
0.4	0.5711	0.6388	0.5911	0.5729	0.6984	0.6781
0.5	0.6905	0.7557	0.7092	0.6924	0.8117	0.7907
0.6	0.7946	0.8491	0.8096	0.7963	0.8949	0.8760
0.7	0.8804	0.9188	0.8906	0.8817	0.9502	0.9361
0.8	0.9451	0.9658	0.9503	0.9458	0.9822	0.9742
0.9	0.9858	0.9919	0.9873	0.9861	0.9967	0.9942
1.0	1.0	1.0	1.0	1.0	1.0	1.0

Table 4.1.1.1 The basic flow stream function, $F(X, \eta)$,
for $\lambda = 8, 16$ at $X = -2, 0$ and $+2$.

In table 4.1.1.1, numerical values for the basic flow stream function, $F(X, \eta)$, are given for $\lambda=8$ and $\lambda=16$, evaluated in each case at $X = -2, 0$ and $+2$.

The values which are required for the stability calculations are

$$F_X, F_{XXX}, G, G_X, S, S_X, T, \text{ where } T = S_{XX},$$

and for each value of λ these are calculated and stored on permanent file.

Also required for the growth rate of the relative energy density $\hat{E}$ are

$$\int_0^1 G^2 d\eta \quad \text{and} \quad \int_0^1 F_X^2 d\eta$$

at each X station, since these are involved in the kinetic energy of the basic flow (see section 4.4). These too are calculated and stored on permanent file. Second order central differences are used for the derivatives and Simpson's rule for the integrals.

4.1.2 Solution of the Stream Function Equation by an Expansion Method

Equation (4.1.9), the derivative with respect to η of equation (4.1.6), is rewritten here:

$$F_{\eta\eta\eta\eta} + 2\lambda H' F_{\eta} F_{\eta\eta} + \lambda H (F_{\eta\eta\eta} F_X - F_{\eta} F_{\eta\eta X}) = 0, \quad (4.1.2.1)$$

where the prime denotes differentiation with respect to X .

The boundary conditions are:

$$F(X, \pm 1) = \pm 1, \quad F_{\eta}(X, \pm 1) = 0$$

$$F(-\infty, \eta) = \frac{3}{2}\eta - \frac{1}{2}\eta^3.$$

Note that for different values of the parameter λ , different solutions to equation (4.1.2.1) are obtained.

Thus F may be expressed as

$$F \equiv F(X, \eta; \lambda).$$

F is then expressed as a power series in λ :

$$F(X, \eta; \lambda) \equiv F^{(0)}(X, \eta) + \lambda F^{(1)}(X, \eta) + \lambda^2 F^{(2)}(X, \eta) + \lambda^3 F^{(3)}(X, \eta) + \dots,$$

this substituted into equation (4.1.2.1), and the coefficients of the powers of λ then equated. This yields the following:

(a) Independent of λ ,

$$F_{\eta\eta\eta\eta}^{(0)} = 0; \quad (4.1.2.2)$$

(b) Coefficient of λ ,

$$F_{\eta\eta\eta\eta}^{(1)} + 2H' F_{\eta}^{(0)} F_{\eta\eta}^{(0)} + H (F_{\eta\eta\eta}^{(0)} F_X^{(0)} - F_{\eta}^{(0)} F_{\eta\eta X}^{(0)}) = 0; \quad (4.1.2.3)$$

(c) coefficient of λ^2 ,

$$F_{\eta\eta\eta\eta}^{(2)} + 2H' (F_{\eta}^{(0)} F_{\eta\eta}^{(1)} + F_{\eta\eta}^{(0)} F_{\eta}^{(1)}) + H (F_{\eta\eta\eta}^{(0)} F_X^{(1)} + F_X^{(0)} F_{\eta\eta\eta}^{(1)} - F_{\eta}^{(0)} F_{\eta\eta X}^{(1)} - F_{\eta\eta X}^{(0)} F_{\eta}^{(1)}) = 0; \quad (4.1.2.4)$$

(d) coefficient of λ^3 ,

$$F_{\eta\eta\eta\eta}^{(3)} + 2H' (F_{\eta}^{(0)} F_{\eta\eta}^{(2)} + F_{\eta}^{(1)} F_{\eta\eta}^{(1)} + F_{\eta}^{(2)} F_{\eta\eta}^{(0)}) + H (F_{\eta\eta\eta}^{(0)} F_X^{(2)} + F_{\eta\eta\eta}^{(1)} F_X^{(1)} + F_{\eta\eta\eta}^{(2)} F_X^{(0)} - F_{\eta}^{(0)} F_{\eta\eta X}^{(2)} - F_{\eta}^{(1)} F_{\eta\eta X}^{(1)} - F_{\eta}^{(2)} F_{\eta\eta X}^{(0)}) = 0; \quad (4.1.2.5)$$

etc.

The boundary conditions give:

$$F^{(0)}(X, \pm 1) = \pm 1, \quad F_{\eta}^{(0)}(X, \pm 1) = 0,$$

$$F^{(n)}(X, \pm 1) = 0, \quad F_{\eta}^{(n)}(X, \pm 1) = 0, \quad n = 1, 2, 3, \dots$$

Thus equations (4.1.2.2), (4.1.2.3), (4.1.2.4), (4.1.2.5) etc., give a system of equations to be solved in order. Equation (4.1.2.2) gives $F^{(0)}$, which can then be substituted into equation (4.1.2.3), and this then solved for $F^{(1)}$. $F^{(0)}$ and $F^{(1)}$ are substituted into equation (4.1.2.4), which is solved for $F^{(2)}$, and so on. This is done as far as the term $F^{(3)}$, the coefficient of λ^3 , as follows.

(i) Independent of λ .

The equation is

$$F_{\eta\eta\eta\eta}^{(0)} = 0.$$

On integrating four times and using the boundary conditions $F^{(0)}(X, \pm 1) = \pm 1$, $F_{\eta}^{(0)}(X, \pm 1) = 0$, this yields

$$F^{(0)} = \frac{1}{2}\eta(3-\eta^2). \quad (4.1.2.6)$$

(ii) Coefficient of λ .

On substituting for $F^{(0)}$ into equation (4.1.2.3), the equation for $F^{(1)}$ becomes

$$F_{\eta\eta\eta\eta}^{(1)} = 9\eta(1-\eta^2)H',$$

which was solved, using the boundary conditions

$$F_{\eta}^{(1)}(X, \pm 1) = F_{\eta}^{(1)}(X, \pm 1) = 0, \text{ to give}$$

$$F^{(1)} = \frac{\eta}{280} (15-33\eta^2+21\eta^4-3\eta^6)H'. \quad (4.1.2.7)$$

(iii) Coefficient of λ^2 .

The solutions for $F^{(0)}$ and $F^{(1)}$ are then substituted into equation (4.1.2.4) to give

$$F_{\eta\eta\eta\eta}^{(2)} = \frac{3\eta}{70} (57-204\eta^2+189\eta^4-42\eta^6) (H')^2$$

$$- \frac{3\eta}{70} (21-69\eta^2+63\eta^4-15\eta^6)H H'',$$

and this solved, with boundary conditions $F^{(2)}(X, \pm 1) =$

$$F_{\eta}^{(2)}(X, \pm 1) = 0, \text{ yielding}$$

$$F^{(2)} = \frac{n}{431200} (2875-8222n^2+8778n^4-4488n^6+1155n^8-98n^{10}) (H')^2$$

$$- \frac{n}{431200} (1213-3279n^2+3234n^4-1518n^6+385n^8-35n^{10}) HH'.$$

(4.1.2.8)

(iv) Coefficient of λ^3

With the functions for $F^{(0)}$, $F^{(1)}$ and $F^{(2)}$, equation (4.1.2.5) becomes

$$F_{nnnn}^{(3)} = \frac{3n}{215600} (31324-169770n^2+316008n^4-257796n^6+90860n^8-11319n^{10}) (H')^3$$

$$- \frac{3n}{215600} (28529-137218n^2+231462n^4-177408n^6+61985n^8-7350n^{10}) HH'H''$$

$$+ \frac{3n}{215600} (4312-19449n^2+30492n^4-22110n^6+7700n^8-945n^{10}) H^2 H''',$$

and on solving, using $F^{(3)}(X, \pm 1) = F_{\eta}^{(3)}(X, \pm 1) = 0$, gives

$$F^{(3)} = f_1^{(3)}(n) (H')^3 - f_2^{(3)}(n) HH'H'' + f_3^{(3)}(n) H^2 H''', \quad (4.1.2.9)$$

where the $f_i^{(3)}(n)$ ($i=1,2,3$) are given below.

Thus, to summarize, the following is obtained

$$F(X, n; \lambda) = F^{(0)}(n) + \lambda F^{(1)}(X, n) + \lambda^2 F^{(2)}(X, n) + \lambda^3 F^{(3)}(X, n) + \dots$$

where:

$$F^{(0)}(\eta) = \frac{1}{2}\eta(3-\eta^2),$$

$$F^{(1)}(X, \eta) = f^{(1)}(\eta) H'(X),$$

$$F^{(2)}(X, \eta) = f_1^{(2)}(\eta) [H'(X)]^2 - f_2^{(2)}(\eta) H(X)H''(X),$$

$$F^{(3)}(X, \eta) = f_1^{(3)}(\eta) [H'(X)]^3 - f_2^{(3)}(\eta) H(X)H'(X)H''(X)$$

$$+ f_3^{(3)}(\eta) [H(X)]^2 H'''(X),$$

with

$$f^{(1)}(\eta) = \frac{\eta}{280} (15 - 33\eta^2 + 21\eta^4 - 3\eta^6),$$

$$f_1^{(2)}(\eta) = \frac{\eta}{431200} (2875 - 8222\eta^2 + 8778\eta^4 - 4488\eta^6 + 1155\eta^8 - 98\eta^{10}),$$

$$f_2^{(2)}(\eta) = \frac{\eta}{431200} (1213 - 3279\eta^2 + 3234\eta^4 - 1518\eta^6 + 385\eta^8 - 35\eta^{10}),$$

$$f_1^{(3)}(\eta) = \frac{\eta}{784 \cdot 784000} (704562 - 2 \cdot 187777\eta^2 + 2 \cdot 850484\eta^4 - 2 \cdot 207010\eta^6$$

$$+ 1141140\eta^8 - 355446\eta^{10} + 57820\eta^{12} - 3773\eta^{14}),$$

$$f_2^{(3)}(\eta) = \frac{\eta}{784 \cdot 784000} (740069 - 2180596\eta^2 + 2596139\eta^4 - 1783834\eta^6$$

$$+ 835835\eta^8 - 244608\eta^{10} + 39445\eta^{12} - 2450\eta^{14}),$$

$$f_3^{(3)}(\eta) = \frac{\eta}{784 \cdot 784000} (117718 - 341483\eta^2 + 392392\eta^4 - 252837\eta^6 + 110110\eta^8$$

$$- 30485\eta^{10} + 4900\eta^{12} - 315\eta^{14}).$$

η	$F(0, \eta)$			
	$\lambda = 1$		$\lambda = 4$	
	(a)	(b)	(a)	(b)
0.0	0.0	0.0	0.0	0.0
0.1	0.152278	0.152276	0.162335	0.161969
0.2	0.301188	0.301184	0.319900	0.319223
0.3	0.443409	0.443404	0.468186	0.467298
0.4	0.575718	0.575713	0.603183	0.602211
0.5	0.695025	0.695021	0.721551	0.720627
0.6	0.798401	0.798398	0.820723	0.819960
0.7	0.883092	0.883091	0.898922	0.898392
0.8	0.946519	0.946519	0.955099	0.954818
0.9	0.986256	0.986257	0.988802	0.988720
1.0	1.0	1.0	1.0	1.0

Table 4.1.2.1

The stream function of the basic flow evaluated at $X=0$, $F(0, \eta)$, for $\lambda=1$ and 4. Columns (a) are the results from the expansion method, columns (b) from the numerical method.

The results for this method are checked against those obtained by the method given in section 4.1.1 for some values of λ . With $\lambda = 0.1$, the two results differ by at most 1 in the fifth decimal place, all values for F lying between 0 and 1. Note that for the last term included, $\max|F^{(3)}| \approx 0.5 \times 10^{-4}$. Although it may be thought that this method would be valid for small values of λ only, it is found to be useful for fairly large values as well. Given in table 4.1.2.1 are numerical values of F , evaluated at $X=0$ for the two values $\lambda=1$ and $\lambda=4$, obtained by using the two methods given here in this section (the values in columns (a)) and in the previous section, 4.1.1 (columns (b)). By comparing the two sets of results obtained by each method, it can be seen that the expansion method is surprisingly accurate for λ as large as 4. Indeed, the results of the two methods differed by at most 2 in the second decimal place for $\lambda=8$. Clearly for any larger values of λ it would be necessary to find further terms in the expansion, although in the present work only the terms up to $F^{(3)}$ are found. However, it is a very useful method in its own right, even though for a limited range of values of λ , as well as providing a most useful check on the numerical method of section 4.1.1.

4.2 The Stability of the Flow

By an analogous method to that of Section 3.1 it is possible to obtain the non-dimensionalized vorticity equation for the stream function ψ for incompressible, viscous flow in the absence

of external forces. In two-dimensional Cartesian co-ordinates this may be expressed in the form

$$\left(\frac{\partial}{\partial t} + \frac{\partial \Psi}{\partial y} \frac{\partial}{\partial x} - \frac{\partial \Psi}{\partial x} \frac{\partial}{\partial y} \right) \nabla^2 \Psi = \frac{1}{R} \nabla^4 \Psi, \quad (4.2.1)$$

with boundary conditions $\Psi = \pm 1$, $\frac{\partial \Psi}{\partial y} = 0$ at the walls, where the variables have been non-dimensionalized as in Section 4.1; R is again the Reynolds number and $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

The stability of the flow in a curved-wall channel is investigated by superimposing a small disturbance on the basic steady flow, by substituting

$$\Psi(x, y, t) = F(X, \eta) + \Psi_1(x, y, t)$$

into equation (4.2.1) and linearizing in Ψ_1 where from section 4.1, $F(X, \eta)$ is the stream function of the basic flow (with error $O(\epsilon^2)$), with $X = \epsilon x$ being the slow variable, ϵ a small parameter, and $\eta = y/H(X)$, $y = \pm H(X)$ being the equations of the walls.

This yields the equation

$$\begin{aligned} \left(\frac{1}{R} \nabla^2 - \frac{\partial F}{\partial y} \frac{\partial}{\partial x} + \frac{\partial F}{\partial x} \frac{\partial}{\partial y} - \frac{\partial}{\partial t} \right) \nabla^2 \Psi_1 + \frac{\partial}{\partial y} (\nabla^2 F) \frac{\partial \Psi_1}{\partial x} \\ - \frac{\partial}{\partial x} (\nabla^2 F) \frac{\partial \Psi_1}{\partial y} = 0, \quad (4.2.2) \end{aligned}$$

with boundary conditions

$$\psi_1 = \frac{\partial \psi_1}{\partial y} = 0 \text{ at the walls.}$$

Since the coefficients of equation (4.2.2) are independent of time, fixed frequency solutions of the form $\psi_1 e^{-i\beta t} + \text{complex conjugate}$ are sought, where, as for the straight-wall channel, β is taken to be real.

Thus, equation (4.2.2), on substituting

$$\psi_1(x, y, t) \equiv \psi_1(x, y) e^{-i\beta t},$$

and rearranging, becomes

$$\begin{aligned} \frac{1}{R} \nabla^4 \psi_1 - \frac{\partial F}{\partial y} \frac{\partial}{\partial x} (\nabla^2 \psi_1) + \frac{\partial F}{\partial x} \frac{\partial}{\partial y} (\nabla^2 \psi_1) + i\beta \nabla^2 \psi_1 + \frac{\partial}{\partial y} (\nabla^2 F) \frac{\partial \psi_1}{\partial x} \\ - \frac{\partial}{\partial x} (\nabla^2 F) \frac{\partial \psi_1}{\partial y} = 0. \end{aligned} \quad (4.2.3)$$

The equations of the walls are taken to be $y = \pm H(X)$, and therefore, to make the walls the coordinate lines, as in section 4.1, the y -coordinate is transformed by use of

$$\eta = y/H(X).$$

Also as in section 4.1, the x -coordinate is transformed by putting $X = \epsilon x$. With these new co-ordinates, and by setting $\psi_1(x, y) = \psi(X, \eta)$, equation (4.2.3) takes the form:

$$\begin{aligned}
& \frac{1}{R} \left[\epsilon^4 \frac{\partial^4 \psi}{\partial X^4} + \frac{1}{H^4} \frac{\partial^4 \psi}{\partial \eta^4} + \frac{\epsilon^2}{H^2} \left(\frac{\partial^4 \psi}{\partial X^2 \partial \eta^2} + \frac{\partial^4 \psi}{\partial \eta^2 \partial X^2} \right) - \frac{4\epsilon^2 H'}{H^3} \frac{\partial^3 \psi}{\partial X \partial \eta^2} \right. \\
& + \left. 6 \frac{\epsilon^2 (H')^2}{H^4} \frac{\partial^2 \psi}{\partial \eta^2} - 2 \frac{\epsilon^2 H''}{H^3} \frac{\partial^2 \psi}{\partial \eta^2} \right] - \frac{1}{H} \frac{\partial F}{\partial \eta} \left(\epsilon^3 \frac{\partial^3 \psi}{\partial X^3} + \frac{\epsilon}{H^2} \frac{\partial^3 \psi}{\partial X \partial \eta^2} \right. \\
& - \left. \frac{2\epsilon H'}{H^3} \frac{\partial^2 \psi}{\partial \eta^2} \right) + \epsilon \frac{\partial F}{\partial X} \left(\frac{\epsilon}{H} \frac{\partial^3 \psi}{\partial \eta \partial X^2} + \frac{1}{H^3} \frac{\partial^3 \psi}{\partial \eta^3} \right) + i\beta \left(\epsilon^2 \frac{\partial^2 \psi}{\partial X^2} + \frac{1}{H^2} \frac{\partial^2 \psi}{\partial \eta^2} \right) \\
& + \left(\frac{\epsilon}{H} \frac{\partial^3 F}{\partial \eta \partial X^2} + \frac{1}{H^3} \frac{\partial^3 F}{\partial \eta^3} \right) \epsilon \frac{\partial \psi}{\partial X} - \left(\epsilon^3 \frac{\partial^3 F}{\partial X^3} + \frac{\epsilon}{H^2} \frac{\partial^3 F}{\partial X \partial \eta^2} \right. \\
& - \left. 2 \frac{\epsilon H'}{H^3} \frac{\partial^2 F}{\partial \eta^2} \right) \frac{1}{H} \frac{\partial \psi}{\partial \eta} = 0.
\end{aligned}$$

On putting $G = \frac{\partial F}{\partial \eta}$, $S = \frac{\partial G}{\partial \eta} = \frac{\partial^2 F}{\partial \eta^2}$ and $T = \frac{\partial S}{\partial \eta} = \frac{\partial^3 F}{\partial \eta^3}$

and rearranging, this equation yields:-

$$\begin{aligned}
& \frac{\epsilon}{R} \frac{\partial^4 \psi}{\partial X^4} + \frac{\epsilon}{RH^2} \left(\frac{\partial^4 \psi}{\partial X^2 \partial \eta^2} + \frac{\partial^4 \psi}{\partial \eta^2 \partial X^2} \right) + \frac{1}{RH^4} \frac{\partial^4 \psi}{\partial \eta^4} - \frac{G\epsilon^3}{H} \frac{\partial^3 \psi}{\partial X^3} \\
& - \frac{\epsilon}{H^2} \left(4 \frac{\epsilon H'}{R} + G \right) \frac{\partial^3 \psi}{\partial X \partial \eta^2} + \frac{\epsilon^3}{H} \frac{\partial F}{\partial X} \frac{\partial^3 \psi}{\partial \eta \partial X^2} + \frac{\epsilon}{H^3} \frac{\partial F}{\partial X} \frac{\partial^3 \psi}{\partial \eta^3} \\
& + i\beta \epsilon^2 \frac{\partial^2 \psi}{\partial X^2} + \frac{1}{H^4} \left\{ 2G\epsilon H' + \frac{6\epsilon^2 (H')^2}{R} - \frac{2\epsilon^2 HH''}{R} + i\beta H^2 \right\} \frac{\partial^2 \psi}{\partial \eta^2}
\end{aligned}$$

$$+ \frac{\epsilon}{H^3} (\epsilon^2 H^2 \frac{\partial^3 F}{\partial \eta \partial X^2} + T) \frac{\partial \psi}{\partial X} - \frac{\epsilon}{H^4} (\epsilon^2 H^3 \frac{\partial^3 F}{\partial X^3} + H \frac{\partial S}{\partial X} - 2H'S) \frac{\partial \psi}{\partial \eta} = 0,$$

(4.2.4)

with boundary conditions

$$\psi = \frac{\partial \psi}{\partial \eta} = 0 \text{ at } \eta = \pm 1.$$

The same symmetry condition as for the straight-wall channel is used, so that the boundary conditions at the walls are replaced by one at the channel centre and one at a wall:-

$$\psi = \frac{\partial \psi}{\partial \eta} = 0 \text{ at } \eta = 1 \quad (\text{wall}),$$

$$\frac{\partial \psi}{\partial \eta} = \frac{\partial^3 \psi}{\partial \eta^3} = 0 \text{ at } \eta = 0 \quad (\text{channel centre}).$$

Again, as for the straight-wall channel, boundary conditions are required upstream, at $X = -\infty$, and downstream, at $X = +\infty$. The same boundary conditions are taken, with the same trials being made, that the upstream and downstream boundary conditions do not affect the solutions to any appreciable extent sufficiently far from the boundaries $X=X_0$ and $X=X_1$. Since a numerical solution is employed, it is clearly not possible to use $X = -\infty$ and $X = +\infty$, and consequently these are replaced by the finite values $X=X_0$ and $X=X_1$ respectively. More will be mentioned of the boundary conditions when the results are presented.

Thus, to summarize, the governing equation for the disturbance stream function, $\psi(X, \eta)$, is equation (4.2.4), together with the boundary conditions

$$\left. \begin{array}{l} \psi = \text{given function} \\ \frac{\partial \psi}{\partial X} = iK_0 \psi \end{array} \right\} \quad \text{at } X = X_0; \quad (4.2.5a)$$

$$\frac{\partial \psi}{\partial \eta} = \frac{\partial^3 \psi}{\partial \eta^3} = 0 \quad \text{at } \eta = 0; \quad (4.2.5b)$$

$$\psi = \frac{\partial \psi}{\partial \eta} = 0 \quad \text{at } \eta = 1; \quad (4.2.5c)$$

$$\text{either: } \psi = \frac{\partial \psi}{\partial X} = 0 \quad \text{at } X = X_1; \quad (4.2.5d)$$

$$\left. \begin{array}{l} \text{or: } \frac{\partial \psi}{\partial X} = i K_1 \psi \\ \frac{\partial^2 \psi}{\partial X^2} = - K_1^2 \psi \end{array} \right\} \quad \text{at } X = X_1, \quad (4.2.5e)$$

where K_0 and K_1 are arbitrarily chosen complex numbers.

4.3 Numerical Solution

As for the straight-wall channel, a finite difference scheme is employed, transforming the partial differential equation (4.2.4), together with the boundary conditions (4.2.5), to a set of finite difference equations. For the

numerical solution it is necessary to use a finite value of X to replace the upstream $X = -\infty$, and it is found that a value of -4.5 is sufficient. The value of X used at the downstream boundary ranges from 4.5 to 10.5 , and will be denoted by X_1 . The step-lengths in the X - and η -directions are denoted, as before, by h and k respectively, and the rectangular region of interest, $-4.5 \leq X \leq X_1$, $0 \leq \eta \leq 1$, is divided into a mesh with N

steps in X and M steps in η (as in figure 3.5.1, with $X=-4.5$ replacing $\xi=0$ and $X=X_1$ replacing $\xi=\xi_1$). The value of ψ at a general point, $\psi(-4.5+rh, sk)$, is denoted, in the usual way, by $\psi_{r,s}$. The same second order, central finite differences as for the straight-wall case are again used and the computational molecules are illustrated in figure 4.3.1. On applying these to equation (4.2.4), the following equations are obtained:

$$\begin{aligned} & C_{r,s}^{(1)} \psi_{r-2,s} + C_{r,s}^{(2)} \psi_{r-1,s-1} + C_{r,s}^{(3)} \psi_{r-1,s} + C_{r,s}^{(4)} \psi_{r-1,s+1} \\ & + C_{r,s}^{(5)} \psi_{r,s-2} + C_{r,s}^{(6)} \psi_{r,s-1} + C_{r,s}^{(7)} \psi_{r,s} + C_{r,s}^{(8)} \psi_{r,s+1} + C_{r,s}^{(9)} \psi_{r,s+2} \\ & + C_{r,s}^{(10)} \psi_{r+1,s-1} + C_{r,s}^{(11)} \psi_{r+1,s} + C_{r,s}^{(12)} \psi_{r+1,s+1} + C_{r,s}^{(13)} \psi_{r+2,s} = 0, \quad (4.3.1) \end{aligned}$$

where

$$C_{r,s}^{(1)} = \frac{\epsilon^3}{2Rh_r h^4} (2\epsilon H_r + Rh G_{r,s}), \quad (4.3.2a)$$

$$C_{r,s}^{(13)} = \frac{\epsilon^3}{2Rh_r h^4} (2\epsilon H_r - Rh G_{r,s}), \quad (4.3.2b)$$

$$\left. \frac{\partial \psi}{\partial X} \right|_{r,s} \equiv \frac{1}{2h} \left\{ \begin{array}{c} \textcircled{-1}_{r-1,s} \text{---} \textcircled{0}_{r,s} \text{---} \textcircled{1}_{r+1,s} \end{array} \right\} ;$$

$$\left. \frac{\partial \psi}{\partial \eta} \right|_{r,s} \equiv \frac{1}{2k} \left\{ \begin{array}{c} \textcircled{1}_{r,s+1} \\ \textcircled{0}_{r,s} \\ \textcircled{-1}_{r,s-1} \end{array} \right\} ;$$

$$\left. \frac{\partial^2 \psi}{\partial X^2} \right|_{r,s} \equiv \frac{1}{h^2} \left\{ \begin{array}{c} \textcircled{1} \text{---} \textcircled{-2}_{r,s} \text{---} \textcircled{1} \end{array} \right\} ;$$

$$\left. \frac{\partial^2 \psi}{\partial \eta^2} \right|_{r,s} \equiv \frac{1}{k^2} \left\{ \begin{array}{c} \textcircled{1} \\ \textcircled{-2}_{r,s} \\ \textcircled{1} \end{array} \right\} ;$$

$$\left. \frac{\partial^3 \psi}{\partial X^3} \right|_{r,s} \equiv \frac{1}{2h^3} \left\{ \begin{array}{c} \textcircled{-1} \text{---} \textcircled{2} \text{---} \textcircled{0}_{r,s} \text{---} \textcircled{-2} \text{---} \textcircled{1} \end{array} \right\} ;$$

$$\left. \frac{\partial^3 \psi}{\partial \eta^3} \right|_{r,s} \equiv \frac{1}{2k^3} \left\{ \begin{array}{c} \textcircled{1} \\ \textcircled{-2} \\ \textcircled{0}_{r,s} \\ \textcircled{2} \\ \textcircled{-1} \end{array} \right\} ;$$

$$\left. \frac{\partial^4 \psi}{\partial X^4} \right|_{r,s} \equiv \frac{1}{h^4} \left\{ \begin{array}{c} \textcircled{1} \text{---} \textcircled{-4} \text{---} \textcircled{6}_{r,s} \text{---} \textcircled{-4} \text{---} \textcircled{1} \end{array} \right\} ;$$

Figure 4.3.1: For legend, see next page.

$$\begin{aligned}
 \left. \frac{\partial^4 \psi}{\partial \eta^4} \right|_{r,s} &\equiv \frac{1}{k^4} \left\{ \begin{array}{c} 1 \\ -4 \\ 6_{r,s} \\ -4 \\ 1 \end{array} \right\} ; \\
 \left. \frac{\partial^3 \psi}{\partial X \partial \eta^2} \right|_{r,s} &\equiv \frac{1}{2hk^2} \left\{ \begin{array}{ccc} -1 & 0 & 1 \\ 2 & 0_{r,s} & -2 \\ -1 & 0 & 1 \end{array} \right\} ; \\
 \left. \frac{\partial^3 \psi}{\partial \eta \partial X^2} \right|_{r,s} &\equiv \frac{1}{2h^2k} \left\{ \begin{array}{ccc} 1 & -2 & 1 \\ 0 & 0_{r,s} & 0 \\ -1 & 2 & -1 \end{array} \right\} \\
 \left. \frac{\partial^4 \psi}{\partial X^2 \partial \eta^2} \right|_{r,s} &\equiv \frac{\partial^4 \psi}{\partial \eta^2 \partial X^2} \equiv \frac{1}{h^2k^2} \left\{ \begin{array}{ccc} 1 & -2 & 1 \\ -2 & 4_{r,s} & -2 \\ 1 & -2 & 1 \end{array} \right\} .
 \end{aligned}$$

Figure 4.3.1: Computational molecules for partial derivatives.

$$C_{r,s}^{(2)} = \frac{\epsilon}{2RH_r^3 h^2 k^2} \left\{ 2\epsilon (H_r + hH'_r) + R(hG_{r,s} - \epsilon^2 k H_r^2 F'_{r,s}) \right\}, \quad (4.3.2c)$$

$$C_{r,s}^{(4)} = \frac{\epsilon}{2RH_r^3 h^2 k^2} \left\{ 2\epsilon (H_r + hH'_r) + R(hG_{r,s} + \epsilon^2 k H_r^2 F'_{r,s}) \right\}, \quad (4.3.2d)$$

$$C_{r,s}^{(10)} = \frac{\epsilon}{2RH_r^3 h^2 k^2} \left\{ 2\epsilon (H_r - hH'_r) - R(hG_{r,s} + \epsilon^2 k H_r^2 F'_{r,s}) \right\}, \quad (4.3.2e)$$

$$C_{r,s}^{(12)} = \frac{\epsilon}{2RH_r^3 h^2 k^2} \left\{ 2\epsilon (H_r - hH'_r) - R(hG_{r,s} - \epsilon^2 k H_r^2 F'_{r,s}) \right\}, \quad (4.3.2f)$$

$$\begin{aligned} C_{r,s}^{(3)} = & - \frac{\epsilon}{2RH_r^3 h^4 k^2} \left[8\epsilon \left\{ \epsilon^2 k^2 H_r^3 + h^2 (H_r + hH'_r) \right\} \right. \\ & + h^2 R \left\{ 2G_{r,s} (\epsilon^2 k^2 H_r^2 + h^2) + h^2 k^2 (\epsilon^2 H_r^2 G_{r,s}'' + T_{r,s}) \right\} \\ & \left. + i \frac{\epsilon \beta}{h^2} \right], \end{aligned} \quad (4.3.2g)$$

$$\begin{aligned} C_{r,s}^{(11)} = & - \frac{\epsilon}{2RH_r^3 h^4 k^2} \left[8\epsilon \left\{ \epsilon^2 k^2 H_r^2 - h^2 (H_r - hH'_r) \right\} \right. \\ & - h^2 R \left\{ 2G_{r,s} (\epsilon^2 k^2 H_r^2 + h^2) + h^2 k^2 (\epsilon^2 H_r^2 G_{r,s}'' + T_{r,s}) \right\} \\ & \left. + i \frac{\epsilon \beta}{h^2} \right], \end{aligned} \quad (4.3.2h)$$

$$C_{r,s}^{(5)} = \frac{1}{2RH_r^4 k^4} (2 - R \epsilon k H_r F'_{r,s}), \quad (4.3.2i)$$

$$C_{r,s}^{(9)} = \frac{1}{2RH_r^4 k^4} (2 + R\epsilon k H_r F'_{r,s}), \quad (4.3.2j)$$

$$C_{r,s}^{(6)} = \frac{1}{2RH_r^4 h^2 k^4} \left[4 \left\{ \epsilon^2 h^2 k^2 (3H_r'^2 + H_r) - 2(\epsilon k^2 H_r^2 + h^2) \right\} \right. \\ \left. + R\epsilon k^2 \left\{ 2h^2 k H_r' (2G_{r,s} - k S_{r,s}) + 2H_r F'_{r,s} (\epsilon^2 k^2 H_r^2 + h^2) \right. \right. \\ \left. \left. + h^2 k^2 H_r (\epsilon^2 H_r^2 F_{r,s}'''' + S_{r,s}') \right\} \right] + \frac{i\beta}{k^2 H_r^2}, \quad (4.3.2k)$$

$$C_{r,s}^{(8)} = \frac{1}{2RH_r^4 h^2 k^4} \left[4 \left\{ \epsilon^2 h^2 k^2 (3H_r'^2 + H_r) - 2(\epsilon k^2 H_r^2 + h^2) \right\} \right. \\ \left. + R\epsilon k^2 \left\{ 2h^2 k H_r' (2G_{r,s} + k S_{r,s}) - 2H_r F'_{r,s} (\epsilon^2 k^2 H_r^2 + h^2) \right. \right. \\ \left. \left. - h^2 k^2 H_r (\epsilon^2 H_r^2 F_{r,s}'''' + S_{r,s}') \right\} \right] + i \frac{\beta}{k^2 H_r^2}, \quad (4.3.2l)$$

$$C_{r,s}^{(7)} = \frac{2}{RH_r^4 h^4 k^4} \left[\epsilon^2 h^2 H_r^2 (3\epsilon^2 k^2 H_r^2 + 4h^2) + h^4 \left\{ 3 - 2\epsilon^2 k^2 (3H_r'^2 + H_r H_r'') \right\} \right. \\ \left. - Rh^2 k^2 \left\{ 2\epsilon H_r' G_{r,s} + i\beta H_r^2 (\epsilon^2 k^2 H_r^2 + h^2) \right\} \right], \quad (4.3.2m)$$

where $H' \equiv \frac{dH}{dX}$ etc. ; $F' \equiv \frac{\partial F}{\partial X}$ etc.

In finite difference form, the boundary conditions (4.2.5) become:

$$\left. \psi_{-1,s} = \psi_{1,s} - 2hiK_o \psi_{o,s} \right\} \text{ for all } s \text{ (upstream),} \quad (4.3.3a)$$

$$\psi_{o,s} = \text{given values}$$

$$\left. \psi_{r,-2} = \psi_{r,2} \right\} \text{ for all } r \text{ (channel centre),} \quad (4.3.3b)$$

$$\psi_{r,-1} = \psi_{r,1}$$

$$\left. \psi_{r,M} = 0 \right\} \text{ for all } r \text{ (wall),} \quad (4.3.3c)$$

$$\psi_{r,M+1} = \psi_{r,M-1}$$

either

$$\left. \psi_{N,s} = 0 \right\} \text{ for all } s \text{ (downstream),} \quad (4.3.3d)$$

$$\psi_{N+1,s} = \psi_{N-1,s}$$

or

$$\left. \begin{aligned} \psi_{N,s} &= \frac{2}{(2-h^2K_1^2-i2hK_1)} \quad \psi_{N-1,s} = K \psi_{N-1,s} \\ &\quad \left. \right\} \text{ for all } s \text{ (downstream).} \end{aligned} \right\} \quad (4.3.3e)$$

$$\psi_{N+1,s} = (1+i2hK_1K) \psi_{N-1,s}$$

As for the straight-wall channel, the equations (4.3.1) may be expressed in matrix form as

$$\underline{T} \underline{U} = \underline{Q}, \quad (4.3.4)$$

where the coefficient matrix $\underline{T}$ is of the same band form as before (see figure 3.5.3), except that the block matrices $\underline{A}$ and $\underline{E}$ now include coefficients with a downstream as well as cross-stream dependence. Thus $\underline{A}$ and $\underline{E}$ may be replaced by $\underline{A}_r$ and $\underline{E}_r$ respectively, with $r = 3(1)N-1$ for $\underline{A}_r$ and $r = 1(1)N-3$ for $\underline{E}_r$. In the former, each A_s is replaced by $C_{r,s}^{(1)}$, and in the latter, each J_s is replaced by $C_{r,s}^{(13)}$ with $s = 0(1)M-1$. The form of the other blocks may be found by comparing the corresponding coefficients in the two sets of equations, (3.5.1) and (4.3.1). Thus the first B_s is replaced by $C_{r,s}^{(2)}$, the second by $C_{r,s}^{(4)}$; $C_{r,s}$ is replaced by $C_{r,s}^{(3)}$; the first D is replaced by $C_{r,s}^{(5)}$, and so on. The right-hand side column vector $\underline{Q}$ also has the same form as before (see figure 3.5.4) with again the same change of coefficients.

In the same manner as for the straight-wall case, for computational purposes the band of matrix $\underline{T}$ is stored in a matrix $\underline{P}$, which has the same form as that given in figure 3.5.5 with the same transformation of coefficients as indicated above. The main difference between the two cases is that in equation (3.5.1), there are constant coefficients, coefficients dependent upon the cross-stream variable only, and coefficients dependent upon both variables, whereas in equation (4.3.1), all coefficients depend upon both the cross-stream and downstream variables.

Once the matrices have been set up, equation (4.3.4) is solved in exactly the same way as equation (3.5.4) for the straight-wall divergent channel, details of which are given in section 3.5. Once again the maximum value of N which is used is 300, and this is found to be sufficient for all the cases considered.

4.4 Growth Rates

Eagles & Smith used the growth rate based on the disturbance kinetic energy density, E . It is decided in the present work, however, to extend this and to use three growth rates, as used in Eagles & Weissman for the straight-wall channel, namely those based on E , on the disturbance stream function ψ , evaluated at the channel centre, and on the relative kinetic energy density, $\hat{E}$. These are defined as follows.

The growth rate of ψ , $GR(\psi)$, is defined as

$$GR(\psi) \equiv \frac{1}{|\psi|} \frac{\partial |\psi|}{\partial x} \equiv \frac{\varepsilon}{|\psi|} \frac{\partial |\psi|}{\partial X} . \quad (4.4.1)$$

The mean kinetic energy density, averaged over time and integrated across the channel, is defined as

$$E(X) \equiv \frac{1}{2} \int_0^{H(X)} (\overline{u^2} + \overline{v^2}) dy, \quad (4.4.2)$$

where u and v are the velocity components of the disturbance,

and an overbar indicates an average over a period. Now

$$u^2 = \left(\frac{\partial \psi_1}{\partial Y} \right)^2 \quad \text{and} \quad v^2 = \left(\frac{\partial \psi_1}{\partial X} \right)^2,$$

where $\psi_1(x, y, t) = \psi_1(x, y) e^{-i\beta t} + \text{complex conjugate}$.

Thus $\overline{u^2} = 2 \left| \frac{\partial \psi_1}{\partial Y} \right|^2$ and $\overline{v^2} = 2 \left| \frac{\partial \psi_1}{\partial X} \right|^2$, and on setting

$\psi_1(x, y) = \psi(X, \eta)$ gives

$$\overline{u^2} = \frac{2}{H^2} \left| \frac{\partial \psi}{\partial \eta} \right|^2 \quad \text{and} \quad \overline{v^2} = 2\epsilon^2 \left| \frac{\partial \psi}{\partial X} \right|^2.$$

Substituting these into (4.4.2) yields:

$$\begin{aligned} E(X) &\equiv H \int_0^1 \left(\frac{1}{H^2} \left| \frac{\partial \psi}{\partial \eta} \right|^2 + \epsilon^2 \left| \frac{\partial \psi}{\partial X} \right|^2 \right) d\eta \\ &\equiv H(X) \Gamma(X). \end{aligned} \quad (4.4.3)$$

The growth rate of E , $GR(E)$, is then defined as

$$GR(E) \equiv \frac{1}{E} \frac{dE}{dX} \equiv \frac{\epsilon}{E} \frac{dE}{dX}. \quad (4.4.4)$$

Denoting the velocity components of the basic flow by u_0 and v_0 , the kinetic energy density of the basic flow, E_0 , is given by

$$E_O(X) \equiv \frac{1}{2} \int_0^{H(X)} (u_O^2 + v_O^2) dy \equiv \frac{H}{2} \int_0^1 (u_O^2 + v_O^2) d\eta.$$

$$\text{Since } u_O^2 = \left(\frac{\partial F}{\partial Y}\right)^2 = \frac{1}{H^2} \left(\frac{\partial F}{\partial \eta}\right)^2 = \frac{1}{H^2} G^2,$$

$$\text{and } v_O^2 = \left(\frac{\partial F}{\partial X}\right)^2 = \epsilon^2 \left(\frac{\partial F}{\partial X}\right)^2 = \epsilon^2 F_X^2,$$

$$\text{then } E_O(X) \equiv \frac{H}{2} \int_0^1 \left(\frac{1}{H^2} G^2 + \epsilon^2 F_X^2 \right) d\eta$$

$$\equiv \frac{H}{2} \Gamma_O(X). \quad (4.4.5)$$

The relative kinetic energy density, $\hat{E}$, is then defined as

$$\hat{E} \equiv E/E_O,$$

$$\text{i.e. } \hat{E} \equiv \frac{H(X) \Gamma(X)}{\frac{1}{2} H(X) \Gamma_O(X)} \equiv \frac{2 \Gamma(X)}{\Gamma_O(X)}.$$

The growth rate of $\hat{E}$, $GR(\hat{E})$, is then defined as

$$GR(\hat{E}) \equiv \frac{1}{\hat{E}} \frac{d\hat{E}}{dx} \equiv \frac{\epsilon}{\hat{E}} \frac{d\hat{E}}{dX}. \quad (4.4.6)$$

For the numerical calculations, Simpson's rule is used for the integrals and central differences for the derivatives.

4.5 Numerical Results

As for the straight-wall channel, the effect of different

mesh sizes on the solution for ψ was first investigated with some test runs, and $h = 0.05$, $k = 0.1$ were finally adopted. As before, with these interval sizes halved the maximum relative deviations in the computed values were always less than 1%.

Since the method of solution for the curved-wall channel is identical to that for the straight-wall channel, once the equations and matrices have been set up, it is necessary only to check that the $\underline{P}$ matrix used in the computation has been stored correctly. One way in which this was done was to print out the blocks of $\underline{P}$ after they had been stored, and to check them by hand. A further check which was undertaken was to consider the same parallel flow case for both the straight-wall and the curved-wall solutions. This was achieved by taking α (the semi-divergence angle) equal to zero in the former case, and $H(X) = 1$ in the latter. In the curved-wall solution, ϵ was chosen so that the same streamwise step lengths were used in each case (where $X = \epsilon x$), that is that $\delta\xi = \delta x$. Also for each the appropriate Poiseuille flow was found for the basic, steady state. The two solutions were then compared taking various R values, and in each case the results found to be identical.

As for the straight-wall channel, extensive checks were made on the influence of the boundary conditions upstream and downstream upon the results obtained (see section 3.7). This

was done by first fixing $\lambda (= \epsilon R)$, and then taking a range of values of R and β . In each case the conclusion was the same as for the straight-wall solution: firstly upstream, that the values taken for $\psi(X_0, \eta)$ and K_0 (where $X_0 = -4.5$ for the numerical solutions) do not affect the results at sufficient distance downstream; secondly, that the choice of downstream boundary conditions (4.2.5d) or (4.2.5e) do not affect the solutions upstream, so long as the position chosen for these, at $X = X_1$, is consistent with the fact that the solution has decayed upstream from this.

Once the value for X_1 has been fixed by making some test runs as above, for the other values of R and β for each λ , the same values are used for the boundary conditions as for the straight-wall case; namely $K_0 = 1 + i0$, $\psi(X_0, \eta)$ is taken as the function illustrated in figure 3.7.9, and the downstream boundary conditions (4.2.5d) are used.

From the solutions obtained, it is required to evaluate the critical value of the Reynolds number, R_{crit} , for each of the three growth rates chosen in the present work (see section 4.4). This is the value of R below which the particular flow quantity always has a negative growth rate, and above which that flow quantity has a positive growth rate for some range of values of X and for some values of β . Also it is necessary to compare the critical Reynolds number based on the kinetic energy density, E , found in the present work with that found by Eagles & Smith, since that was the one measure of

growth rate used in their paper. Shown in figures 4.5.1, 4.5.2 and 4.5.3 are typical growth rates based on E for three of the values used for λ (8, 12 and 16), for the values of R and β given on the figures. Note that in each case there is a positive growth rate for a range of X , and that therefore the value of R used is above the critical value for that value of λ .

The method of finding the critical Reynolds number for each flow quantity is more involved than is the case for the straight-wall channel, since each growth rate is a function of the four parameters X, β, R, λ (where $\lambda = \epsilon R$ is used in place of ϵ for convenience, and following Eagles & Smith). The method used by Eagles & Smith is as follows. They first fixed λ , calculated the growth rate for three values of β for a given R ; then by interpolation determined the value of β which gave the minimum growth rate for that R ; they repeated with two other values of R and interpolated (or extrapolated) to a point (R, β) where the minimum growth rate was zero. New starting values were chosen by them and the process repeated until the critical point $(R_{\text{crit}}, \beta_{\text{crit}})$ in (R, β) - space for given X and λ was found with sufficient accuracy. This process was done for a range of λ and X , and then they interpolated over X for each λ to obtain the minimum R_{crit} for each λ . Thus they found R_{crit} and the corresponding β_{crit} for various λ -values.

The method adopted in the present work varies slightly from that above, and is as follows.

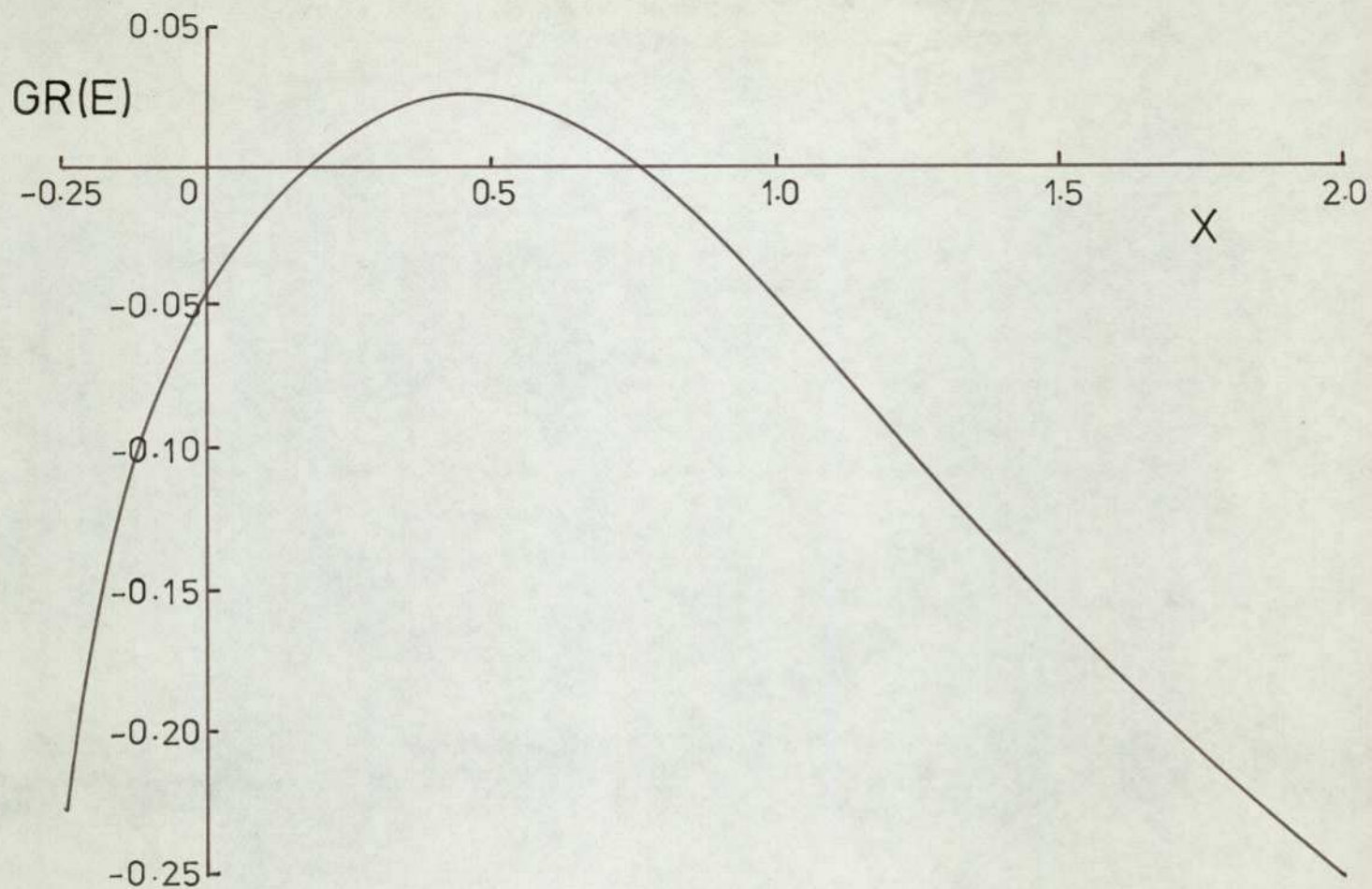


Figure 4.5.1: The growth rate of the kinetic energy density E : $\lambda = 8$, $R = 65$, $\beta = 1$.

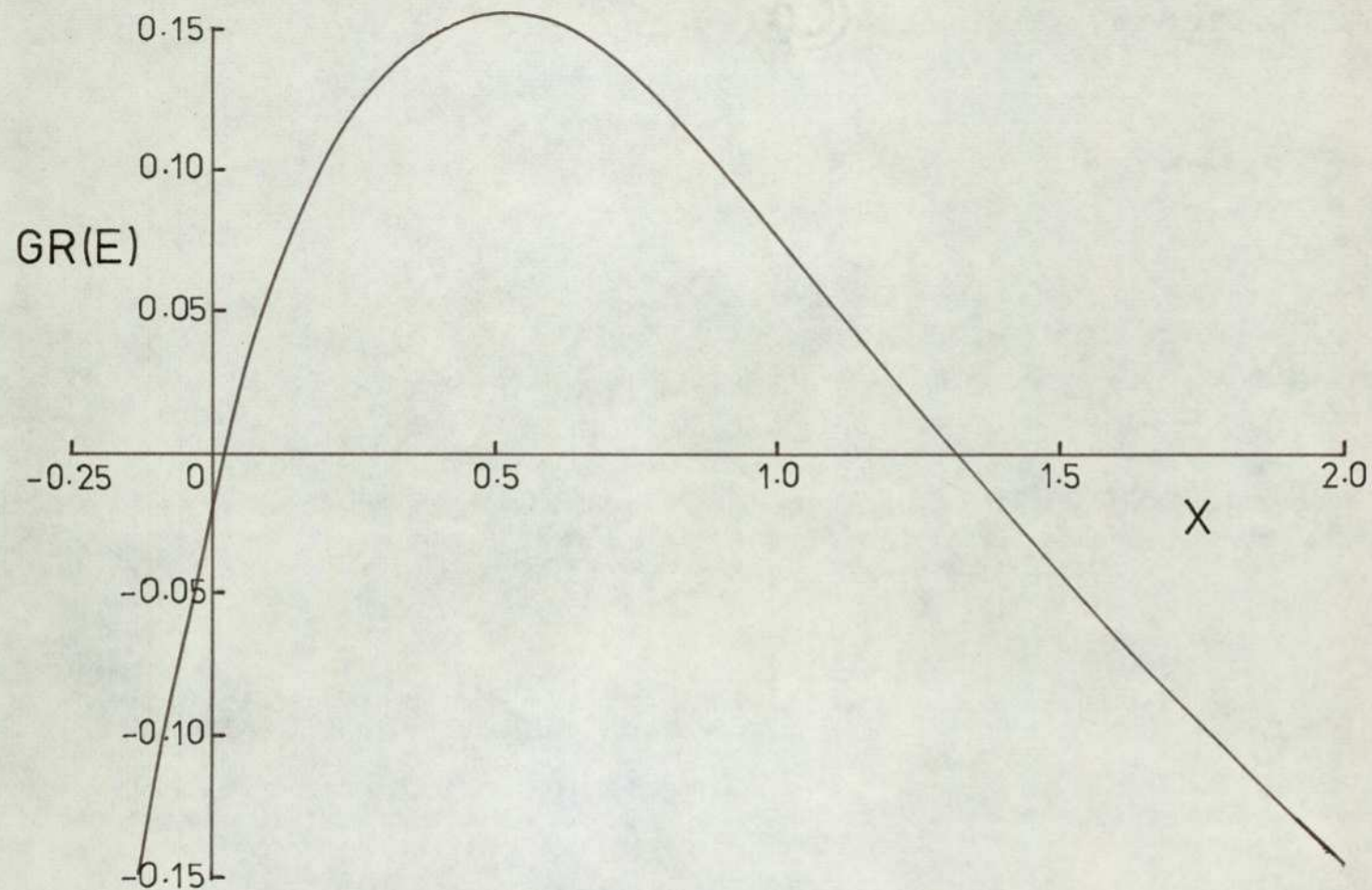


Figure 4.5.2: The growth rate of the kinetic energy density E : $\lambda = 12$, $R = 50$, $\beta = 1.2$.

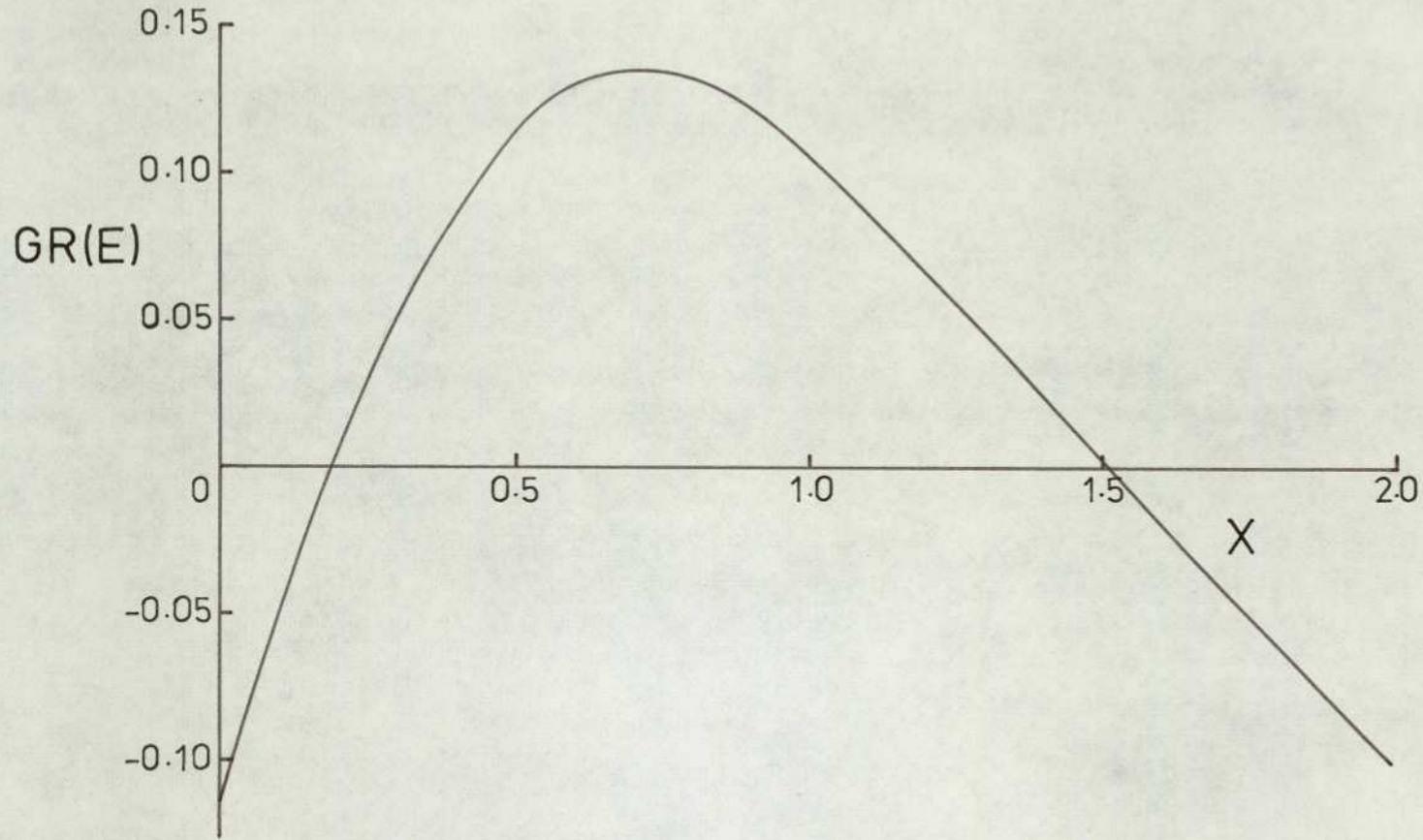


Figure 4.5.3: The growth rate of the kinetic energy density E :
 $\lambda = 16$, $R = 35$, $\beta = 1.3$.

- (i) Fix λ .
- (ii) Take a value of R and a value of β and find the maximum growth rate over X , G_X .
- (iii) Repeat (ii) with the same value of R and two other values of β , and use quadratic interpolation to find the maximum of the maximum growth rates, $\max(G_X)$, and the corresponding value of β .
- (iv) Repeat (ii) and (iii) for a second value of R .
- (v) Use linear interpolation to find the values of R and β for which $\max(G_X) = 0$.
- (vi) Repeat from (ii)-(v) with two new values of R until R_{crit} and the corresponding β_{crit} are found to sufficient accuracy.

The whole of this process is performed for various λ -values for each of the growth rates based on the disturbance stream function, ψ (evaluated at the channel centre), the kinetic energy density, E , and the relative kinetic energy density, $\hat{E}$. Some results found are given in table 4.5.1, which gives values of R_{crit} and the corresponding values of β_{crit} and $\varepsilon_{\text{crit}}$ for various values of λ .

Figure 4.5.4 illustrates the comparison between the values obtained in the present work (the continuous curve) with those obtained by Eagles & Smith (the dashed curve), where the graph of λ versus R_{crit} (based on the kinetic energy density, E) is plotted. As can be seen, there is reasonable agreement between

GROWTH RATE BASED ON

λ	ψ			E			$\hat{E}$		
	R_{crit}	β_{crit}	ϵ_{crit}	R_{crit}	β_{crit}	ϵ_{crit}	R_{crit}	β_{crit}	ϵ_{crit}
7	-	-	-	75.0	1.01	0.093	63.6	0.97	0.110
8	57.0	0.84	0.140	57.2	0.98	0.140	46.1	1.07	0.174
9	48.0	0.91	0.188	47.4	1.05	0.190	38.1	1.12	0.236
10	42.7	0.93	0.234	41.6	1.09	0.240	32.7	1.15	0.306
11	39.0	0.93	0.282	37.5	1.11	0.293	28.7	1.18	0.383
12	36.2	0.92	0.331	34.4	1.13	0.349	25.6	1.20	0.469
13	34.0	0.92	0.382	31.9	1.16	0.408	22.7	1.21	0.573
14	32.3	0.91	0.433	30.0	1.18	0.467	-	-	-
15	31.0	0.91	0.484	28.4	1.21	0.528	-	-	-
16	29.9	0.91	0.535	27.0	1.23	0.593	-	-	-

Table 4.5.1 Values of R_{crit} and corresponding β_{crit} and ϵ_{crit} for the values of λ shown, based on the growth rates of ψ (at the channel centre), E and $\hat{E}$.

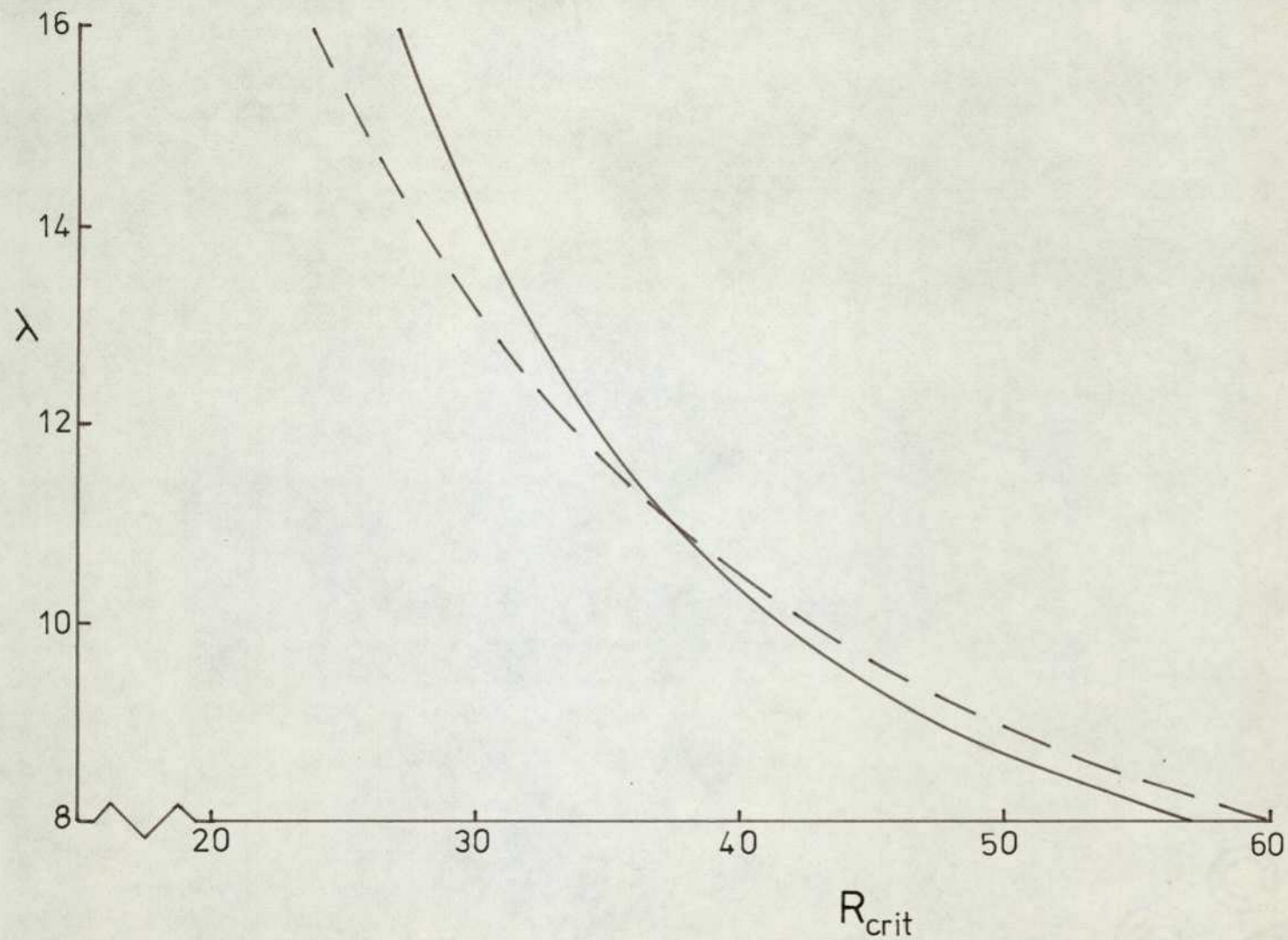


Figure 4.5.4: The overall critical Reynolds number based on the kinetic energy density E .
— present work; --- Eagles & Smith.

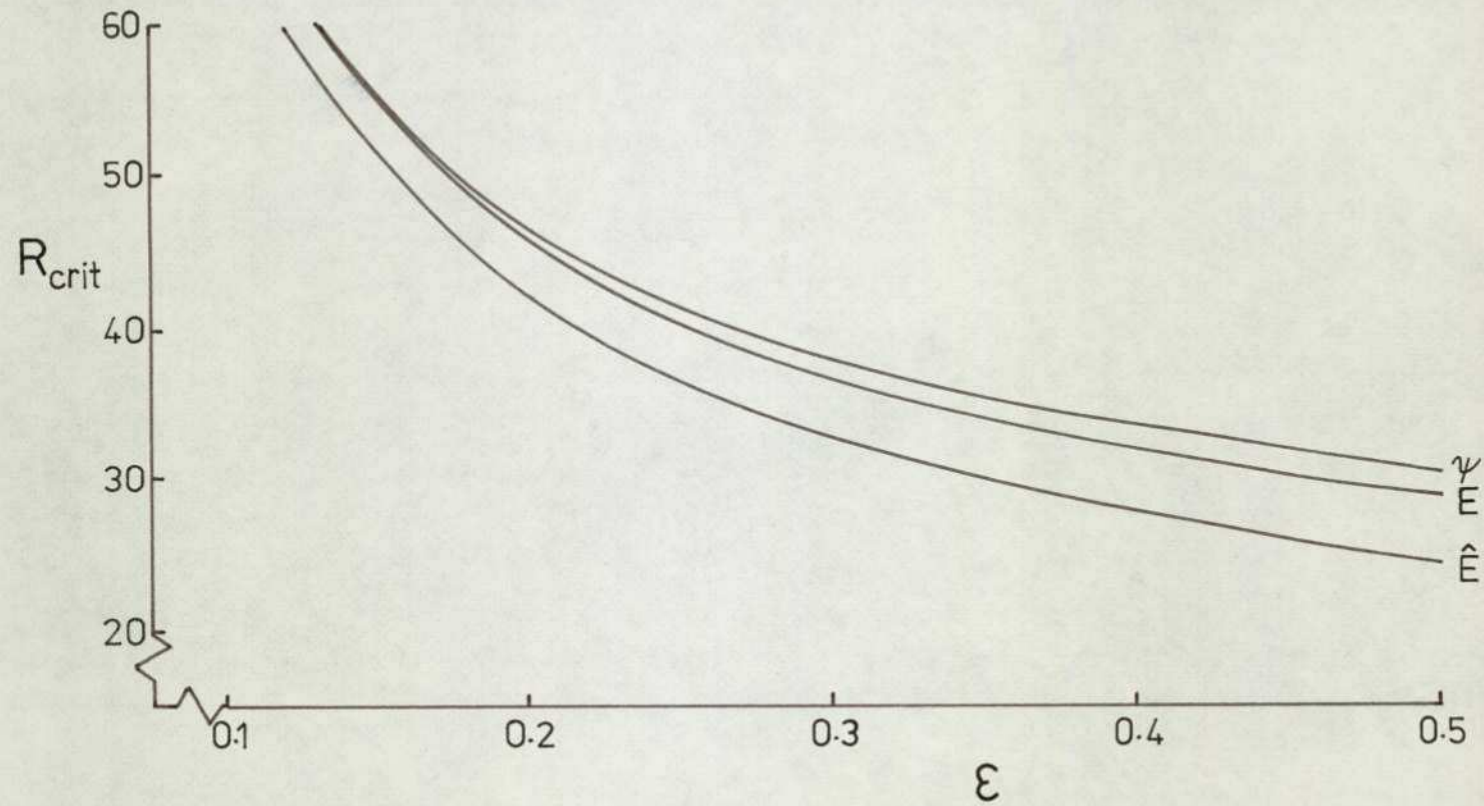


Figure 4.5.5: Overall critical Reynolds number versus ϵ based on ψ (at the channel centre), E and $\hat{E}$.

the two sets of results, although it is again worth pointing out that each is obtained by entirely different methods for the disturbance stream function equation. Eagles & Smith used an expansion method with error $O(\epsilon^2)$, and in the present work the equation is solved directly by a numerical method.

Finally, shown in figure 4.5.5 is the overall critical Reynolds number, R_{crit} , based on each of ψ (evaluated at the channel centre), E and $\hat{E}$, drawn against ϵ .

5. AN INITIAL-BOUNDARY-VALUE PROBLEM IN A STRAIGHT-WALL DIVERGENT CHANNEL.

5.1 Governing Equations.

In this section, the motion of an impulsively started disturbance as it travels downstream in a straight-wall divergent channel is considered. The same channel (see figure 3.2.1) and coordinates are used (see section 3.2). The linearized equation for the disturbance stream function, $\Psi_1(\xi, \eta, t)$, in such a channel was given in section 3.4 (equation (3.4.4)) and is restated here, where for convenience $\Psi_1(\xi, \eta, t)$ is replaced by $\psi(\xi, \eta, t)$:

$$\left[R^{-1} \left\{ \left(\frac{\partial}{\partial \xi} - 2\alpha \right)^2 + \frac{\partial^2}{\partial \eta^2} \right\} - e^{2\alpha\xi} \frac{\partial}{\partial t} - g(\eta) \left(\frac{\partial}{\partial \xi} - 2\alpha \right) \right] \nabla^2 \psi + g''(\eta) \frac{\partial \psi}{\partial \xi} + 2\alpha g'(\eta) \frac{\partial \psi}{\partial \eta} = 0, \quad (5.1.1)$$

where $g(\eta) \equiv \frac{\partial \psi_0}{\partial \eta}$, ψ_0 being the stream function of the basic, steady flow (see section 3.3). The boundary conditions, using the same symmetry condition as before, are

$$\psi = \frac{\partial \psi}{\partial \eta} = 0 \text{ at } \eta=1 \text{ (wall),}$$

$$\frac{\partial \psi}{\partial \eta} = \frac{\partial^3 \psi}{\partial \eta^3} = 0 \text{ at } \eta=0 \text{ (channel centre).}$$

In order to solve equation (5.1.1), it is first necessary to change the single equation to two coupled equations by setting

$$\nabla^2 \psi = \omega, \quad (5.1.2)$$

which yields

$$\begin{aligned} e^{2\alpha\zeta} \frac{\partial \omega}{\partial t} = R^{-1} \nabla^2 \omega - \left\{ 4\alpha R^{-1} + g(\eta) \right\} \frac{\partial \omega}{\partial \xi} \\ + \left\{ 4\alpha^2 R^{-1} + 2\alpha g(\eta) \right\} \omega + g''(\eta) \frac{\partial \psi}{\partial \xi} + 2\alpha g'(\eta) \frac{\partial \psi}{\partial \eta}. \end{aligned} \quad (5.1.3)$$

The channel centre ($\eta=0$) and wall ($\eta=1$) boundary conditions for ω are found by using the symmetry condition, so that

$$\frac{\partial \omega}{\partial \eta} = 0 \text{ at } \eta=0,$$

and the fact that $\omega = \nabla^2 \psi$ is used in an iterative manner to obtain the value of ω at $\eta=1$.

What is considered is a stationary disturbance at time $t=0$, and, with a well-defined local perturbation then applied, as time increases the evolution of the developing flow is followed by solving equations (5.1.2) and (5.1.3) numerically. Thus for time $t < 0$, the trivial solution $\psi = \omega = 0$ is used, so that the flow is completely steady before the onset of the disturbance. A very short time before $t=0$, it is imagined that unspecified external forces are applied so that at time $t=0$ the two functions $\psi(\xi, \eta, 0)$ and $\omega(\xi, \eta, 0)$ have known expressions. An ideal localized perturbation would be

$$\psi(0, 0, 0) = 1,$$

$$\psi(\xi, \eta, 0) = 0, \quad \xi \text{ and } \eta \text{ not both zero.}$$

However, since $\omega = \nabla^2 \psi$, this would make ω singular at the origin. Thus, in order that the equations may be solved numerically, a large positive constant σ is introduced, and the initial values taken are

$$\psi(\xi, \eta, 0) = \exp \left[-\sigma (\xi^2 + \eta^2) \right], \quad (5.1.4)$$

$$\omega(\xi, \eta, 0) = \nabla^2 \psi = 4\sigma \left\{ \sigma (\xi^2 + \eta^2) - 1 \right\} \exp \left[-\sigma (\xi^2 + \eta^2) \right]. \quad (5.1.5)$$

This would then give the ideal localized perturbation above in the limit $\sigma = +\infty$. In fact, for the numerical solution the value taken for σ was 20.

Equations (5.1.2) and (5.1.3), then, define an initial-boundary-value problem, and so boundary conditions are required at $\xi = \pm \infty$. Since for $t < 0$, $\psi = \omega = 0$, and an initial localized disturbance is taken, the additional boundary conditions are taken as

$$\psi(\pm\infty, \eta, t) = \omega(\pm\infty, \eta, t) = 0.$$

The general method of solution may then be set out as follows, having first set the values for R, α, σ and the initial values for ω and ψ .

- (i) Calculate new boundary conditions for ω using the latest values of ψ .
- (ii) Calculate values of ω at the next time level from equation (5.1.3) and boundary conditions for ω found in (i) above.

(iii) Calculate values of ψ at this time level from equation (5.1.2).

(iv) Return to (i) if the chosen maximum time level has not been reached.

As mentioned, each of the equations (5.1.2) and (5.1.3) is solved by numerical methods, details of which are presented in the next section, 5.2.

5.2 Numerical Solution.

For the numerical solution, the upstream and downstream boundaries at $\xi = -\infty$ and $\xi = +\infty$ are replaced by finite boundaries $\xi = \xi_0$ and $\xi = \xi_1$ respectively. Denoting the step lengths in the ξ -direction and η -direction by $\delta\xi$ and $\delta\eta$ respectively, the rectangular region $\xi_0 \leq \xi \leq \xi_1$, $0 \leq \eta \leq 1$ is divided by a mesh with N $\delta\xi$ -steps and M $\delta\eta$ -steps, as in figure 3.5.1 with $\xi = \xi_0$ now replacing $\xi=0$.

A finite difference method is used for equation (5.1.3), a first order forward scheme for the time derivative and a second order central scheme for the space derivatives. Thus

$$\frac{\partial \omega}{\partial t} \approx \frac{\omega_{i,j}^{(n+1)} - \omega_{i,j}^{(n)}}{\delta t}$$

and the space derivatives are the same as those used in Chapter 3 (see figure 3.5.2). This forms an explicit method

of solution, with the values of ω at any time level being given explicitly in terms of the values of ω and ψ at the previous time level.

The finite difference version of equation (5.1.3) thus becomes

$$\begin{aligned}
 e^{2\alpha\xi_i} \left(\frac{\omega_{i,j}^{(n+1)} - \omega_{i,j}^{(n)}}{\delta t} \right) = R^{-1} & \left(\frac{\omega_{i+1,j}^{(n)} - 2\omega_{i,j}^{(n)} + \omega_{i-1,j}^{(n)}}{\delta\xi^2} + \right. \\
 & \left. \frac{\omega_{i,j+1}^{(n)} - 2\omega_{i,j}^{(n)} + \omega_{i,j-1}^{(n)}}{\delta\eta^2} \right) \\
 - (4\alpha R^{-1} + g_j) & \left(\frac{\omega_{i+1,j}^{(n)} - \omega_{i-1,j}^{(n)}}{2\delta\xi} \right) + (4\alpha^2 R^{-1} + 2\alpha g_j) \omega_{i,j}^{(n)} \\
 + g_j & \left(\frac{\psi_{i+1,j}^{(n)} - \psi_{i-1,j}^{(n)}}{2\delta\xi} \right) + 2\alpha g_j \left(\frac{\psi_{i,j+1}^{(n)} - \psi_{i,j-1}^{(n)}}{2\delta\eta} \right), \quad (5.2.1)
 \end{aligned}$$

where with the usual notation

$$\omega_{i,j}^{(n)} = \omega(\xi_0 + i\delta\xi, j\delta\eta, n\delta t).$$

On rearranging this gives

$$\begin{aligned}
 e^{2\alpha\xi_i} \omega_{i,j}^{(n+1)} = A_j \omega_{i-1,j}^{(n)} + B \omega_{i,j-1}^{(n)} + C_{i,j} \omega_{i,j}^{(n)} + B \omega_{i,j+1}^{(n)} \\
 + D_j \omega_{i+1,j}^{(n)} + E_j (\psi_{i+1,j}^{(n)} - \psi_{i-1,j}^{(n)}) \\
 + F_j (\psi_{i,j+1}^{(n)} - \psi_{i,j-1}^{(n)}), \quad (5.2.2)
 \end{aligned}$$

$$(i = 1(1)N-1, j = 0(1)M-1),$$

where

$$A_j = \frac{\delta t R^{-1}}{\delta \xi^2} (1 + 2\alpha \delta \xi) + \frac{\delta t}{2\delta \xi} g_j,$$

$$B = \frac{\delta t R^{-1}}{\delta \eta^2},$$

$$C_{i,j} = e^{2\alpha \xi_i} + 2\delta t R^{-1} \left\{ 2\alpha^2 - \left(\frac{1}{\delta \xi^2} + \frac{1}{\delta \eta^2} \right) \right\} + 2\alpha \delta t g_j,$$

$$D_j = \frac{\delta t R^{-1}}{\delta \xi^2} (1 - 2\alpha \delta \xi) - \frac{\delta t}{2\delta \xi} g_j,$$

$$E_j = \frac{\delta t}{2\delta \xi} g''_j,$$

$$F_j = \frac{\alpha \delta t}{\delta \eta} g'_j.$$

The boundary conditions for ω are

$$\omega(\xi_0, \eta, t) = \omega(\xi_1, \eta, t) = 0, \text{ (upstream, downstream),}$$

$\frac{\partial \omega}{\partial \eta} = 0$ at $\eta = 0$, (channel centre), and $\omega = \nabla^2 \psi$ at $\eta = 1$, (wall). The boundary condition at the wall in finite difference form becomes

$$\omega_{i,M} = \frac{\psi_{i+1,M} - 2\psi_{i,M} + \psi_{i-1,M}}{\delta \xi^2} + \frac{\psi_{i,M+1} - 2\psi_{i,M} + \psi_{i,M-1}}{\delta \eta^2}.$$

Now also at $\eta = 1$

$$\psi = \frac{\partial \psi}{\partial \eta} = 0,$$

$$\text{i.e. } \psi_{i,M} = 0 \text{ and } \frac{\psi_{i,M+1} - \psi_{i,M-1}}{2\delta \eta} = 0$$

$$\Rightarrow \psi_{i,M} = 0 \quad \text{and} \quad \psi_{i,M+1} = \psi_{i,M-1}.$$

Thus at $\eta=1$ the boundary condition for ω becomes

$$\omega_{i,M} = \frac{2}{\delta\eta} \psi_{i,M-1}.$$

The other boundary conditions for ω in finite difference form are

$$\omega_{0,j} = \omega_{N,j} = 0,$$

$$\omega_{i,-1} = \omega_{i,1}.$$

If the same finite difference scheme is applied to equation (5.1.2), this would result in

$$\begin{aligned} \frac{1}{\delta\xi^2} \psi_{i-1,j} + \frac{1}{\delta\eta^2} \psi_{i,j-1} - 2 \left(\frac{1}{\delta\xi^2} + \frac{1}{\delta\eta^2} \right) \psi_{i,j} + \frac{1}{\delta\eta^2} \psi_{i,j+1} \\ + \frac{1}{\delta\xi^2} \psi_{i+1,j} = \omega_{i,j}. \end{aligned} \quad (5.2.3)$$

As mentioned in section 3.8 on the coupled equation method, this had been solved by both the successive over-relaxation method (SOR method) and by the direct method of Gaussian elimination. However, for numerical stability of equation (5.2.2), the step-length in time, δt , would have to be very small (more will be discussed on this in the next section), and so equation (5.1.2) would have to be solved a great number of times. It is felt, therefore, that it is beneficial to solve this equation, Poisson's equation, by a fast direct

method based on Fourier analysis due to Hockney (1965, 1970). This method will now be described.

The method is applicable to rectangular regions with simple boundary conditions, as is the case for the present work. To restate the problem, at each time level it is necessary to solve

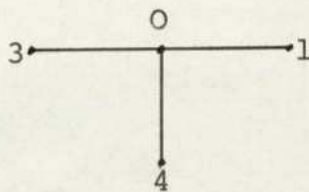
$$\frac{\partial^2 \psi}{\partial \xi^2} (\xi, \eta) + \frac{\partial^2 \psi}{\partial \eta^2} (\xi, \eta) = \omega (\xi, \eta), \quad (5.2.4)$$

for $\xi_0 \leq \xi \leq \xi_1$, $0 \leq \eta \leq 1$, where $\psi(\xi, \eta) = 0$ for

$\xi = \xi_0, \xi_1$ and $\eta = 1$, and $\frac{\partial \psi}{\partial \eta} (\xi, \eta) = 0$ for $\eta = 0$. It is

worth mentioning here that although it might appear that the boundary condition $\frac{\partial \psi}{\partial \eta} = 0$ at $\eta = 1$ has been dropped, it has in fact been incorporated above in the evaluation of the boundary condition for ω at $\eta = 1$ (giving $\omega_{i,M} = \frac{2}{\delta \eta} \psi_{i,M-1}$). This is identical to the method adopted by Greenspan (1968) for the solution of similar coupled equations. The method is described briefly here.

Along the wall $\eta = 1$,
consider the four points (ξ, η) ,
 $(\xi + \delta \xi, \eta)$, $(\xi - \delta \xi, \eta)$, $(\xi, \eta - \delta \eta)$,



numbered 0,1,3,4 respectively, in the figure. It is required to determine parameters a_0, a_1, a_3, a_4, a_5 such that

$$\left(\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} \right) \Big|_0 = a_0 \psi_0 + a_1 \psi_1 + a_3 \psi_3 + a_4 \psi_4 + a_5 \left(\frac{\partial \psi}{\partial \eta} \right) \Big|_0 .$$

Expansion of ψ_1 , ψ_3 and ψ_4 into Taylor series about the point numbered 0 and reorganisation of terms implies

$$\begin{aligned} \left(\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} \right) \Big|_0 &= \psi_0 (a_0 + a_1 + a_3 + a_4) \\ &+ \left(\frac{\partial \psi}{\partial \xi} \right) \Big|_0 \cdot (\delta \xi \cdot a_1 - \delta \xi \cdot a_3) \\ &+ \left(\frac{\partial \psi}{\partial \eta} \right) \Big|_0 \cdot (-\delta \eta \cdot a_4 + a_5) \\ &+ \left(\frac{\partial^2 \psi}{\partial \xi^2} \right) \Big|_0 \cdot \left(\frac{\delta \xi^2}{2} \cdot a_1 + \frac{\delta \xi^2}{2} \cdot a_3 \right) \\ &+ \left(\frac{\partial^2 \psi}{\partial \eta^2} \right) \Big|_0 \cdot \left(\frac{\delta \eta^2}{2} \cdot a_4 \right) \\ &+ \dots . \end{aligned}$$

In this latter equality, setting corresponding terms equal yields

$$a_0 + a_1 + a_3 + a_4 = 0,$$

$$\delta \xi \cdot a_1 - \delta \xi \cdot a_3 = 0,$$

$$-\delta \eta \cdot a_4 + a_5 = 0,$$

$$\frac{\delta \xi^2}{2} \cdot a_1 + \frac{\delta \xi^2}{2} \cdot a_3 = 1,$$

$$\frac{\delta \eta^2}{2} \cdot a_4 = 1,$$

the solution of which is

$$a_0 = -2\left(\frac{1}{\delta\xi^2} + \frac{1}{\delta\eta^2}\right), \quad a_1 = \frac{1}{\delta\xi^2}, \quad a_3 = \frac{1}{\delta\xi^2}, \quad a_4 = \frac{2}{\delta\eta^2}, \quad a_5 = \frac{2}{\delta\eta^2}.$$

Thus the following approximation is used:

$$\begin{aligned} \left(\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2}\right) \Big|_0 &= -2\left(\frac{1}{\delta\xi^2} + \frac{1}{\delta\eta^2}\right)\psi_0 + \frac{1}{\delta\xi^2}\psi_1 + \frac{1}{\delta\xi^2}\psi_3 + \frac{2}{\delta\eta^2}\psi_4 \\ &\quad + \frac{2}{\delta\eta^2} \left(\frac{\partial \psi}{\partial \eta}\right) \Big|_0. \end{aligned}$$

On using the boundary conditions for ψ along the wall,

$$\psi = \frac{\partial \psi}{\partial \eta} = 0 \text{ at } \eta = 1, \text{ gives}$$

$$\psi_0 = \psi_1 = \psi_3 = \left(\frac{\partial \psi}{\partial \eta}\right) \Big|_0 = 0,$$

and so

$$\omega_0 = \left(\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2}\right) \Big|_0 = \frac{2}{\delta\eta^2} \psi_4.$$

The boundary conditions allow $\psi(\xi, \eta)$ to be expanded in a Fourier series in either the ξ -direction, η -direction or as a double Fourier series in both directions. Since the determination of Fourier coefficients is a time consuming job on a computer, it was suggested by Hockney that the fastest computer program is obtained if the expansion is made in only one direction and if this is chosen to be the shortest. In the present case, this is the η -direction. The appropriate Fourier expansion is chosen to satisfy the boundary conditions at $\eta = 0$ and $\eta = 1$, and so the expansion is

$$\psi(\xi, \eta) = \sum_{k=0}^M \bar{\psi}^{(k)}(\xi) \cos(2k+1) \frac{\pi}{2} \eta, \quad (5.2.5)$$

and similarly for $\omega(\xi, \eta)$, where $\bar{\psi}^{(k)}(\xi)$ is the Fourier amplitude of the k th harmonic.

The substitution of (5.2.5) into the partial differential equation (5.2.4) results in a set of ordinary differential equations relating the Fourier amplitudes of $\psi(\xi, \eta)$ and $\omega(\xi, \eta)$, and which for each harmonic are independent of each other:

$$\frac{d^2 \bar{\psi}^{(k)}(\xi)}{d\xi^2} - \left\{ \frac{(2k+1)\pi}{2} \right\}^2 \bar{\psi}^{(k)}(\xi) = \bar{\omega}^{(k)}(\xi), \quad (5.2.6)$$

for $k = 0(1)M$. Using second order finite differences for the derivatives, and putting $\lambda_k = \frac{(2k+1)\pi}{2}$ and $h = \delta\xi$, yields

$$\frac{1}{h^2} (\bar{\psi}_{i+1}^{(k)} - 2\bar{\psi}_i^{(k)} + \bar{\psi}_{i-1}^{(k)}) - \lambda_k^2 \bar{\psi}_i^{(k)} = \bar{\omega}_i^{(k)},$$

$$(i = 1(1)N-1),$$

which on rearranging becomes

$$\frac{1}{h^2} \bar{\psi}_{i-1}^{(k)} - \left(\frac{2}{h^2} + \lambda_k^2 \right) \bar{\psi}_i^{(k)} + \frac{1}{h^2} \bar{\psi}_{i+1}^{(k)} = \bar{\omega}_i^{(k)}.$$

Using the boundary conditions $\bar{\psi}^{(k)}(\xi) = 0$ on the upstream and downstream boundaries, this may be expressed in matrix form as

$$\begin{bmatrix}
 -(\frac{2}{h^2} + \lambda_k^2) & & & & \\
 & \frac{1}{h^2} & & & \\
 & & -(\frac{2}{h^2} + \lambda_k^2) & & \\
 & & & \ddots & \\
 & & & & \frac{1}{h^2} & -(\frac{2}{h^2} + \lambda_k^2) & \frac{1}{h^2} \\
 & & & & & & \frac{1}{h^2} - (\frac{2}{h^2} + \lambda_k^2)
 \end{bmatrix}
 \begin{bmatrix}
 \bar{\psi}_1(k) \\
 \bar{\psi}_2(k) \\
 \vdots \\
 \bar{\psi}_{N-1}(k)
 \end{bmatrix}
 =
 \begin{bmatrix}
 \bar{\omega}_1(k) \\
 \bar{\omega}_2(k) \\
 \vdots \\
 \bar{\omega}_{N-1}(k)
 \end{bmatrix}
 \quad (5.2.7)$$

The right-hand sides, $\bar{\omega}_i(k)$, are obtained by the inverse Fourier transform of (5.2.5), giving

$$\bar{\omega}_i(k) = \frac{2}{M} \sum_{j=0}^{M_n} \omega_{i,j} \cos(2k+1) \frac{j\pi}{2M}, \quad i = 1(1)N-1, \quad (5.2.8)$$

where, with the usual notation,

$$\sum_{j=0}^n f_j = \frac{1}{2}f_0 + f_1 + f_2 + \dots + f_{n-1} + \frac{1}{2}f_n.$$

The tridiagonal matrix equation (5.2.7) is solved (for each k) by the method of Gauss elimination in the neat form as given by Varga (1962). Then, having obtained $\bar{\psi}_i(k)$ as the solutions of equation (5.2.7), $\psi_{i,j}$ is obtained by application of (5.2.5).

Summarizing, the solution of Poisson's equation (5.2.4) is obtained in three stages:

(i) Fourier analysis of right-hand side

$$\omega(\xi, \eta) \rightarrow \bar{\omega}^{(k)}(\xi).$$

(ii) Solution of k independent sets of ordinary differential equations or the corresponding tridiagonal matrix equations

$$\bar{\omega}^{(k)}(\xi) \rightarrow \bar{\psi}^{(k)}(\xi).$$

(iii) Fourier synthesis

$$\bar{\psi}^{(k)}(\xi) \rightarrow \psi(\xi, \eta).$$

Finally, to summarize the whole numerical procedure, the initial values of ψ and ω are first set up using (5.1.4) and (5.1.5). The new boundary conditions for ω at the wall ($\eta=1$) are calculated, and equation (5.2.2) is used to give the values of ω at the next time level. Equation (5.2.4) is then solved by the Fourier method to give the values of ψ at that time level. This process is repeated until the maximum allocated time level is reached.

5.3 Numerical Results

The explicit method for equation (5.1.3) detailed in the last section is clearly the simplest finite difference analogue, in that it means that each of the values of ω are updated at each time level from values obtained at the previous time level by use of equation (5.2.2). However, this method brings in problems of numerical stability, and means that a very small value of the time step-length, δt ,

must be used. The stability condition on δt in fact depends upon the coefficients of the highest order derivatives of the right-hand side of the partial differential equation, the time derivative term having been isolated on the left-hand side. Thus equation (5.1.3) may be written

$$\frac{\partial \omega}{\partial t} = e^{-2\alpha\xi} R^{-1} V^2 \omega + \dots$$

The stability condition is then (see Richtmyer, 1957)

$$\delta t < e^{2\alpha\xi} R / \left[2 \left(\frac{1}{\delta\xi^2} + \frac{1}{\delta\eta^2} \right) \right] . \quad (5.3.1)$$

Since the motion is started by a localized perturbation, the most rapid spatial rates of change would occur during the early time levels, and so $\delta\xi$ and $\delta\eta$ would have to be small, with the hope that they, and therefore δt , could be increased as t increases. This would therefore mean very small values of δt , and hence the need for the fast solution of Poisson's equation given in the last section. As examples of the restrictions on δt , with $R = 45$, $\gamma = \alpha R = 4.093$, if $\xi_0 = -5$, then with $\delta\xi = \delta\eta = 0.0125$, $\delta t < 0.0007$, and with $\delta\xi = \delta\eta = 0.025$, $\delta t < 0.0028$; if $\xi_0 = -7.5$, $\delta\xi = \delta\eta = 0.025$, then $\delta t < 0.0018$.

Extensive checks were made on the effect of changing the time step, δt , the space steps $\delta\xi$, $\delta\eta$, and on changing the positions of the upstream and downstream boundaries, ξ_0 and ξ_1 .

In all solutions presented, $R = 45$, $\gamma = 4.093$, $\sigma = 20$, and values of ψ at the channel centre are given. For example in table 5.3.1, three different results are shown at time $t=0.008$, with $\xi_0 = -7.5$, $\xi_1 = 7.5$. In column (a), the values are obtained with $\delta\xi = \delta\eta = 0.025$, $\delta t = 0.0005$; in column (b) with $\delta\xi = \delta\eta = 0.025$, $\delta t = 0.001$. The closeness of the two sets of values obtained may clearly be seen. The solutions in column (c) were found with $\delta\xi = \delta\eta = 0.025$ for five time levels, then putting $\delta\xi = \delta\eta = 0.05$, with $\delta t = 0.001$ throughout, and again there is good agreement.

As an example of another check, the results of two separate runs were checked at time $t = 0.025$. In both runs, $\delta\xi = \delta\eta = 0.025$ for five time levels, then both were doubled to 0.05. In the first run δt was kept at 0.001, while in the second run δt was changed from 0.001 to 0.005 after five time intervals. In comparing the values obtained for $|\psi|_{\max}$, it was found to differ by less than 0.5% from one run to the other.

It is the development of the disturbance as it travels downstream which is of interest, and so it is intended to follow this flow downstream. Hence, as the values close to the downstream boundary $\xi = \xi_1$ increased, it is decided to shift that boundary further downstream. A check was built into the computer program to enable this to be done. If, however, the upstream boundary $\xi = \xi_0$ were kept fixed, this would increase the number of mesh points for the solution,

ξ	$\psi(\xi, 0)$		
	(a)	(b)	(c)
1.00	0.000389	0.000389	0.000412
0.75	0.000786	0.000788	0.000854
0.50	0.007036	0.007032	0.006966
0.25	0.247440	0.247289	0.244963
0.00	0.983789	0.984026	0.988840
0.25	0.327560	0.327489	0.325070
0.50	0.008429	0.008409	0.008357
0.75	-0.000152	-0.000151	-0.000074
1.00	-0.000047	-0.000047	-0.000021

Table 5.3.1: Values of $\psi(\xi, 0)$ at time $t = 0.008$ with
 $R = 45$, $\gamma = \alpha R = 4.093$, $\sigma = 20$, $\xi_0 = -7.5$,
 $\xi_1 = 7.5$.

(a) $\delta\xi = \delta\eta = 0.025$, $\delta t = 0.0005$,

(b) $\delta\xi = \delta\eta = 0.025$, $\delta t = 0.001$,

(c) $\delta\xi = \delta\eta = 0.025$ for 5 time levels, then
 $\delta\xi = \delta\eta = 0.05$, with $\delta t = 0.001$ throughout.

and so would increase the computer run time. Tests were therefore made on the effect of moving the upstream boundary at the same rate as the downstream boundary (i.e. keeping the interval (ξ_0, ξ_1) fixed) as compared with keeping it fixed while the downstream boundary was shifted. The result of one such test given in table 5.3.2 shows that there is little difference in the values of ψ over the range of ξ of interest (where the travelling wave is) with the values upstream and downstream decaying to zero. The values are taken at time $t = 5$ with $\delta\xi = \delta\eta = 0.025$, $\delta t = 0.001$ for five time levels, then $\delta\xi = \delta\eta = 0.05$, $\delta t = 0.005$; for the moving boundary solution, $\xi_0 = -5$ at time $t = 0$ and has the value $\xi_0 = -1.5$ at time $t = 5$, and for the fixed boundary solution, $\xi_0 = -5$ throughout; for both, ξ_1 starts at $+5$. It was thus decided to move both boundaries at the same rate, keeping the distance between them constant, and hence keeping the same number of mesh points, after the initial five time levels.

Clearly with δt at such a small value, the computer run time is going to be very large, and in fact the maximum value of t reached for any single computer run was 20, all runs being made on the CDC 7600 computer of the University of London Computer Centre. This result is shown in figure 5.3.1 which clearly demonstrates the downstream development of the initial disturbance. For this run, $R=45$, $\gamma=\alpha R = 4.093$, $\sigma=20$, $\delta\xi = \delta\eta = 0.025$, $\delta t = 0.001$ for five time levels, then $\delta\xi = \delta\eta = 0.05$, $\delta t = 0.005$. The solutions are shown at four values of t , $t = 5, 10, 15$ and 20 . This diagram is similar

ξ	$\psi(\xi, 0)$	
	Moving boundary	Fixed boundary
1.0	0.019666	0.019452
1.5	0.024828	0.025400
2.0	0.032070	0.032498
2.5	0.040141	0.040209
3.0	0.050464	0.050416
3.5	0.072501	0.072475
4.0	0.099008	0.099005
4.5	0.066399	0.066399
5.0	-0.046250	-0.046250
5.5	-0.064059	-0.064059
6.0	0.019013	0.019013

Table 5.3.2. The effect of a fixed upstream boundary versus a moving upstream boundary at time $t = 5$ with $R = 45$, $\gamma = \alpha R = 4.093$, $\sigma = 20$, $\delta\xi = \delta\eta = 0.025$, $\delta t = 0.001$ for 5 time levels, then $\delta\xi = \delta\eta = 0.05$, $\delta t = 0.005$. For the fixed boundary, $\xi_0 = -5$; for the moving boundary, $\xi_0 = -5$ at $t = 0$ and $\xi_0 = -1.5$ at $t = 5$.

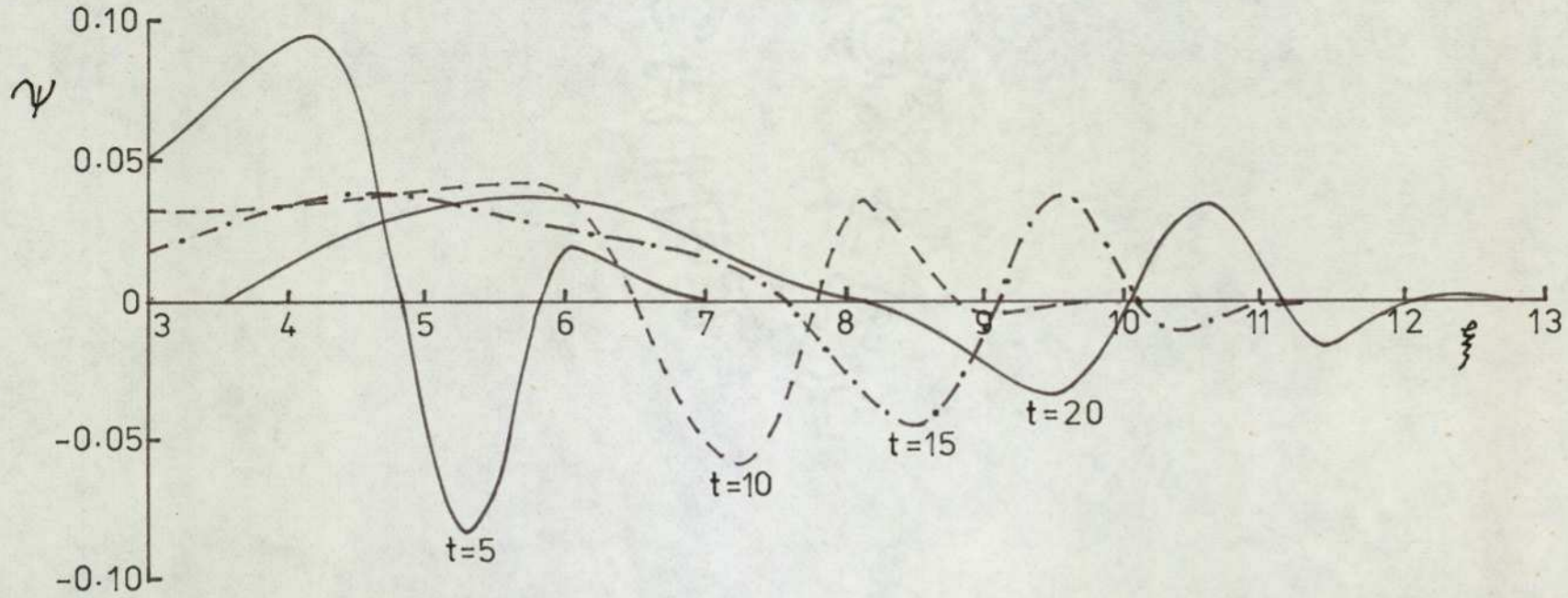


Figure 5.3.1: The disturbance stream function at the channel centre, $\psi(\xi, 0, t)$, for $t = 5, 10, 15, 20$, with $R = 45$, $\gamma = \alpha R = 4.093$.

to one presented by Betchov & Criminale (1967) for the development of an initial localized perturbation in a Blasius boundary layer.

With the values taken, $R=45$ and $\gamma = 4.093$, it is expected from the results of Chapter 3 for the straight-wall channel that $|\psi|_{\max}$ should grow as the disturbance travels downstream (see figure 3.7.12). However, as can be seen in figure 5.3.1, this has not been the case. It is not fully understood why there should be this difference between the two methods, although the remarks of Betchov & Criminale (1967) may be relevant. They noted that the maximum velocity pulse decays for some considerable time before the mode of largest amplification becomes dominant and the maximum pulse begins to grow again. It may be, therefore, that at time $t = 20$ this "largest mode" has not had time to become sufficiently dominant for $|\psi|_{\max}$ to start to grow. It was decided to see if the same were true of the kinetic energy density, E , of the disturbance. If u, v are the velocity components of the disturbance, then $E(t)$ is defined as

$$E(t) \equiv \frac{1}{2} \int_{-\alpha}^{+\alpha} \int_{-\infty}^{+\infty} (u^2 + v^2) r \, d\phi \, dr.$$

$$\text{With } u^2 = \frac{1}{r^2} \left(\frac{\partial \psi}{\partial \phi} \right)^2 = e^{-2\alpha\xi} \left(\frac{\partial \psi}{\partial \eta} \right)^2$$

$$\text{and } v^2 = \left(\frac{\partial \psi}{\partial r} \right)^2 = e^{-2\alpha\xi} \left(\frac{\partial \psi}{\partial \xi} \right)^2$$

the expression for E becomes

$$E(t) \equiv \int_{-\infty}^{+\infty} \int_0^1 (\psi_n^2 + \psi_\xi^2) d\eta d\xi.$$

Again, as for the $|\psi|_{\max}$ values, E decays over the range of t considered ($0 \leq t \leq 20$).

More investigation is clearly needed so that runs for larger values of t may be undertaken, to see if the dominant terms or components eventually show a positive growth rate. The way in which this might proceed will now be mentioned.

Further numerical experimentation could be carried out to look at the effect of further changes in the time step, δt , so that larger values of t may be reached in a given computer run time. It may be that as the upstream boundary is moved downstream, and ξ_0 consequently increased, numerical stability could be achieved with a larger value of δt , as the exponential factor in the stability condition (5.3.1) would thus be increased in value. Another possibility is in the use of an implicit scheme for equation (5.1.3), choosing one which is always stable for any δt , for example the Crank-Nicholson scheme

$$\frac{\omega_{i,j}^{(n+1)} - \omega_{i,j}^{(n)}}{\delta t} = \frac{1}{2} [(\text{RHS})^{(n+1)} + (\text{RHS})^{(n)}],$$

so that larger values of δt may be used.

6. CONCLUSIONS

Presented in this thesis are investigations into the hydrodynamic stability of flows in channels of slowly varying width, and in particular a channel with straight, divergent walls and a channel with curved walls. The particular channel chosen for the curved wall is one whose walls have equations

$$y = \pm(1 + \frac{1}{2} \tanh \epsilon x),$$

where y is the cross-stream variable, x is the downstream variable, and ϵ is constant. In each case the steady state flows are found first. For the straight-wall channel some typical velocity profiles are illustrated in figures 3.3.1 and 3.3.2. The solutions for the curved-wall channel differ from those of the straight-wall channel in that the stream function and velocity have a downstream as well as cross-stream dependence. The downstream development of the velocity profile for $\lambda (= \epsilon R) = 16$ is shown in figure 4.1.1.3, which clearly shows the influence of the change in channel width on the basic flow. As can be seen, these profiles are stretched out as the flow progresses downstream.

The stability of each of these flows is then studied by use of the linearized equations in the disturbance stream function. These equations are solved in both cases by numerical methods which do not involve approximations. The results obtained would thus be expected to be more accurate

than any which involves an approximation of some kind, such as the previously mentioned quasi-parallel approach (used for the straight-wall channel by Eagles, 1966) or the slowly varying, WKB, method (used for the straight-wall channel by Eagles & Weissman, 1975, and for the curved-wall channel by Eagles & Smith, 1980). The present work therefore makes a further contribution to the study of hydrodynamic stability by improving upon the methods previously used for channels of varying width, although it should be pointed out that the results using the WKB approach are seen to be in reasonable agreement with those presented here. The critical Reynolds number based on each of three flow quantities (the disturbance stream function, the disturbance kinetic energy density, the relative kinetic energy density) are calculated for each of the two classes of channel, the main results being shown in figures 3.7.14 and 4.5.5.

Also presented here is a study of an initial-boundary-value problem in a straight-wall divergent channel, the evolution of an initial localized perturbation as it travels downstream being traced by the direct numerical solution of the linear time dependent equation in the disturbance stream function. Although this shows a decaying travelling wave where a growing one is expected, it is pointed out that this could be due to the fact that the dominant mode has not yet been "picked out" by the solution up to the maximum value of t used ($t = 20$).

Related experimental work on the stability of channel flows

appears to be non-existent. In the case of the straight-wall channel there is the problem of setting up the original Jeffrey-Hamel steady flows. Hence the particular curved channel studied first by Eagles & Smith (1980) is believed to present a much more realizable flow situation physically. Up to the present time it is believed that no experiment has been undertaken, but it is to be hoped that this will be forthcoming so that comparisons may be made with the results presented here.

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