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Scattering of D0-branes and Strings

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ABSTRACT: It has been known for about thirty years that a scattering amplitude involving D0-branes and closed strings suffers from infrared divergences beyond tree level. These divergences arise because the conventional world-sheet approach cannot account for the difference between the D0-brane's momentum before and after scattering. We show that, by using string field theory, the divergence can be removed and the amplitude rendered finite and unambiguous. We illustrate this using the simplest possible example in bosonic string theory: a three-point function with one incoming and one outgoing D0-brane and an incoming or outgoing closed string tachyon.

KEYWORDS: Bosonic Strings, D-Branes, String Field Theory

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1 Introduction and summary

It has been known for many years that the scattering of D0-branes and closed strings suffers from an infrared divergence [1, 2] at the next-to-leading order. The reason for this divergence is also well understood and has been discussed in the original papers and subsequently [3–12]. Physically, we expect that during such scattering the momentum of the final D0-brane will differ from that of the initial D0-brane. However, in the standard world-sheet approach the computation is done by summing over Riemann surfaces with boundaries, with D0-brane boundary conditions that have *fixed* momenta. The leading contribution comes from disk amplitudes and the next-to-leading order contribution comes from the annulus amplitude. Hence, there is no scope of using different boundary conditions corresponding to incoming and outgoing D0-branes. It is a common phenomenon that use of incorrect external states leads to infrared divergences and this case is no exception. One finds that the annulus amplitude computed using fixed D0-brane boundary condition is infrared divergent [1].

String field theory (SFT) is well suited to address this problem.¹ In this framework, the basic degrees of freedom are closed string fields and 0+1 dimensional open string fields

¹This has been suspected by many people — see in particular questions and comments by Emil Martinec [13] during Strings 2020 and by Igor Klebanov [14] during Strings 2025.

living on the D0-brane. Among the open string fields are massless fields describing transverse displacement of the D0-brane. Momentum carrying D0-branes can be regarded as coherent excitations of these fields. Therefore, the problem of computing scattering amplitudes of D0-branes and closed strings, with the incoming and outgoing D0-branes carrying different momenta, can be reduced to the problem of computing the matrix element of the interaction term involving closed and open string fields between two different coherent states of the open string field.

In this paper we illustrate this by computing the three point function in bosonic string theory for which two of the external states are D0-branes carrying different momenta and the third state is a closed string tachyon. Generically three point functions of this kind vanish due to kinematic constraints, but we can avoid this by taking the external momenta to be complex. For example, if M is the mass of the D0-brane and if we set $\alpha' = 1$ so that the closed string tachyon has $\text{mass}^2 = -4$, then we can take the incoming D0-brane and closed string tachyon to have momenta

$$p_{\text{D0,in}} \equiv (E_{\text{D0,in}}, \vec{p}_{\text{D0,in}}) = (M, \vec{k}_1), \quad p_{\text{tach,in}} = (0, \vec{k}), \quad (1.1)$$

and the outgoing D0-brane to have momenta

$$p_{\text{D0,out}} = (M, \vec{k}_1), \quad (1.2)$$

with

$$\vec{k}_1^2 = 0, \quad \vec{k}^2 = 4, \quad (\vec{k} + \vec{k}_1)^2 = 0. \quad (1.3)$$

String world-sheet theory offers a way to calculate the leading contribution to this amplitude: it is the disk one point function of the closed string vertex operator V with boundary conditions corresponding to a D0-brane at rest. Explicitly, this is given by²

$$2\pi \delta(k^0) \times \frac{1}{2} g_s M, \quad (1.4)$$

where g_s is the string coupling, defined in the normalization convention for string amplitudes given in [16, 17] and reviewed in appendix A. The expression given in (1.4) calculates the T-matrix and the S-matrix element is obtained by multiplying (1.4) by i . Note that (1.4) ignores the fact that the D0-brane states have spatial momenta $\vec{k}_1$ and $\vec{k}_2 \equiv \vec{k}_1 + \vec{k}$. Since the D0-brane mass M is of order $1/g_s$ and a momentum $\vec{k}_i$ corresponds to a velocity $\vec{v}_i = \vec{k}_i/M \sim g_s$, this discrepancy does not affect the leading order result. However, the absence of an explicit momentum conserving delta function captures the fact that we are treating the D0-brane as a rigid classical object instead of a quantum state.

Now consider the next order correction. Naively this will be given by the annulus one point function of the closed string tachyon vertex operator V . Let us parametrize the annulus as a strip $0 \leq \text{Re } w \leq \pi$ in the complex w plane with the identification $w \equiv w + 2\pi i t$, and let the closed string vertex operator V be inserted as a point w_0 with $\text{Re } w_0 = 2\pi x$. The standard rules of string theory require us to insert appropriate ghost insertions that make

²Using eq. (8.7.26) of [15] and eq. (4.123) of [16], we get $M^2 = 2^{12} \pi^{23}/g_s^2$, but we shall not use this relation in our analysis.

the one point function into a volume form in the moduli space. If $F(x, t) dx dt$ denotes this volume form, then the amplitude takes the form

$$\int_0^\infty dt \int_0^{1/4} dx F(x, t), \quad (1.5)$$

where we have restricted the integral over x to be from 0 to 1/4 using the reflection symmetry $x \rightarrow 1/2 - x$ on the annulus. Explicit computation gives

$$F(x, t) = 2\pi\delta(k^0) \frac{g_s \eta'_c}{\sqrt{2\pi}} t^{-1/2} \eta(it)^{-24} \left[\frac{\vartheta_1(2x|it)}{\vartheta'_1(0|it)} \right]^{-2}, \quad \eta'_c \equiv \frac{1}{2\pi}, \quad (1.6)$$

where

$$\vartheta_1(z|\tau) = -2 e^{i\pi\tau/4} \sin(\pi z) \prod_{n=1}^{\infty} \{(1 - e^{2\pi i n \tau})(1 - 2 e^{2\pi i n \tau} \cos(2\pi z) + e^{4\pi i n \tau})\}, \quad (1.7)$$

and

$$\eta(\tau) = e^{\pi i \tau / 12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}), \quad (1.8)$$

are, respectively, the odd Jacobi theta function and Dedekind eta function. We now see that the integral over $F(x, t)$ has divergences from the $x \rightarrow 0$ and / or $t \rightarrow \infty$ region. There are also divergences from the $t \rightarrow 0$ region associated with the closed-string tachyon of the bosonic string theory, which we will treat using Witten's $i\epsilon$ prescription [18].

Our goal will be to use SFT to extract an unambiguous, finite answer for (1.5). The main step in this analysis is to represent the amplitude as a sum of SFT Feynman diagrams and remove the contribution due to the massless and tachyonic open string modes in the internal propagators. The contribution from the tachyon and massless ghosts can be treated using the standard tools of SFT that was used, e.g., for D-instantons. Special treatment is needed for dealing with the collective modes that describe the motion of the D0-brane in the transverse directions. We quantize them using standard tools of a non-relativistic quantum mechanics that allows us to take the incoming and outgoing D0-branes as momentum eigenstates. The matrix element of the SFT action interaction terms between these states can then be used to compute the relevant part of the effective action for external closed strings. After computing the desired scattering amplitude using this effective action, we arrive at a finite, unambiguous result for the amplitude.

We shall now summarize our results. After using SFT to remove infrared divergences, the amplitude up to the first subleading order in the string coupling is given by

$$(2\pi)^{26} \delta^{(26)}(k_{\text{in}} - k_{\text{out}}) \left[\mathcal{F}_{(0)} + \mathcal{F}_{\text{annulus}} \right]. \quad (1.9)$$

Here k_{in} and k_{out} are the total incoming and outgoing momenta and the incoming and outgoing D0-branes states are delta-function normalized as in the case of a non-relativistic point particles. $\mathcal{F}_{(0)}$ is the leading contribution from the disk amplitude:

$$\mathcal{F}_{(0)} = \frac{1}{2} g_s M, \quad (1.10)$$

M being the mass of the D0-brane. $\mathcal{F}_{\text{annulus}}$ is the annulus contribution, given by

$$\mathcal{F}_{\text{annulus}} = \lim_{\substack{\alpha, \tilde{\lambda} \rightarrow \infty \\ \alpha/\tilde{\lambda} = \text{fixed}}} \left[\mathcal{F}_{(a)} + \mathcal{F}_{(b)} + \mathcal{F}_{(c)} + \mathcal{F}_{(d)} + \mathcal{F}_{(e)} + \mathcal{F}_{(f)} + \mathcal{F}_{(g)} + \mathcal{F}_{\text{jac}} + \mathcal{F}_{\text{cor}} \right], \quad (1.11)$$

where,

$$\mathcal{F}_{(a)} = \frac{1}{4} g_s \eta'_c \tilde{\lambda} \left\{ -i + \operatorname{erfi} \left(\sqrt{2 \ln \alpha} \right) \right\}, \quad (1.12)$$

erfi being the imaginary error function,

$$\operatorname{erfi}(z) \equiv -\frac{2}{\sqrt{\pi}} i \int_0^{iz} e^{-u^2} du = \frac{2}{\sqrt{\pi}} \int_0^z e^{u^2} du, \quad (1.13)$$

$$\mathcal{F}_{(b)} = -g_s \eta'_c \frac{1}{2\sqrt{2}} (1 + \alpha^{-2}) \tilde{\lambda} \int_{t_c}^{\frac{1}{2\pi} \ln(\alpha^2 - 1/2)} dt t^{-1/2} \eta(it)^{-24}, \quad (1.14)$$

$$\mathcal{F}_{(c)} = -\frac{1}{2} g_s \eta'_c \int_{1/(2\tilde{\lambda})}^1 \frac{d\beta}{4\beta^2} (1 + \beta^2) \left\{ -i + \operatorname{erfi} \left(\sqrt{2 \ln \alpha + 2 \ln \frac{4\tilde{\lambda}^2 + 1}{4\tilde{\lambda}} + 2 \ln \frac{2\beta}{1 + \beta^2}} \right) \right\}, \quad (1.15)$$

$$\mathcal{F}_{(d)} = \frac{g_s \eta'_c}{\sqrt{2\pi}} \int_A^{1/4} dx \int_{t_c}^{B(x)} dt t^{-1/2} \eta(it)^{-24} \left[\frac{\vartheta_1(2x|it)}{\vartheta'_1(0|it)} \right]^{-2}, \quad (1.16)$$

$$A \equiv (2\pi \tilde{\lambda})^{-1} (1 - \alpha^{-2}), \quad (1.17)$$

$$e^{2\pi B(x)} \equiv \alpha^2 \tilde{\lambda}^2 \sin^2(2\pi x) \left(1 + \frac{1}{4\tilde{\lambda}^2} \right)^2 \left[1 + 2 \left\{ \cot^2(2\pi x) - \tilde{\lambda}^2 f(\tan \pi x)^2 \right\} \alpha^{-2} \tilde{\lambda}^{-2} \left(1 + \frac{1}{4\tilde{\lambda}^2} \right)^{-2} \right], \quad (1.18)$$

$$\mathcal{F}_{(e)} = \frac{1}{8} g_s \eta'_c \tilde{\lambda} \frac{1}{\sqrt{2\pi}} (\ln \alpha)^{-1/2}, \quad (1.19)$$

$$\mathcal{F}_{(f)} = \frac{\eta'_c g_s}{\sqrt{2\pi}} \tilde{\lambda}^2 \int_{1/(2\tilde{\lambda})}^1 d\beta f(\beta)^2 \frac{1}{1 + \beta^2} \frac{1}{\sqrt{\ln \alpha + \ln \frac{4\tilde{\lambda}^2 + 1}{4\tilde{\lambda}} + \ln \frac{2\beta}{1 + \beta^2}}}, \quad (1.20)$$

$$\mathcal{F}_{(g)} = -\frac{1}{4} g_s \eta'_c \tilde{\lambda} \frac{1}{\sqrt{2\pi}} (\ln \alpha)^{-1/2}, \quad (1.21)$$

$$\begin{aligned} \mathcal{F}_{\text{jac}} = & \frac{1}{\sqrt{2\pi}} \left[-2g_s \eta'_c \int_{1/(2\tilde{\lambda})}^1 d\beta \left(\ln \alpha + \ln \frac{4\tilde{\lambda}^2 + 1}{4\tilde{\lambda}} + \ln \frac{2\beta}{1 + \beta^2} \right)^{-1/2} \right. \\ & \left. \left\{ \frac{25}{8\beta} - \frac{25}{8} \beta + \vec{k}^2 \tan^{-1} \beta \right\} \left(\frac{1}{\beta} - \frac{2\beta}{1 + \beta^2} \right) \right. \\ & \left. + \pi \vec{k}^2 g_s \eta'_c \left(\ln \alpha + \ln \frac{4\tilde{\lambda}^2 + 1}{4\tilde{\lambda}} \right)^{1/2} \right], \quad \vec{k}^2 = 4, \end{aligned} \quad (1.22)$$

and

$$\begin{aligned} \mathcal{F}_{\text{cor}} = & g_s \eta'_c \left[\frac{1}{4\sqrt{2\pi}} \tilde{\lambda}^2 \alpha^{-2} \int_{1/(2\tilde{\lambda})}^1 \frac{d\beta}{1 + \beta^2} \left\{ \ln \alpha + \ln \tilde{\lambda} + \ln \frac{2\beta}{1 + \beta^2} \right\}^{-3/2} f(\beta)^4 \right. \\ & \left. - \frac{1}{16\tilde{\lambda}} \left\{ -i + \operatorname{erfi} \left(\sqrt{2 \ln \alpha} \right) \right\} + \frac{1}{6\sqrt{2}} \tilde{\lambda}^{-1} \int_{t_c}^{\frac{1}{2\pi} \ln \alpha^2} dt t^{-1/2} e^{2\pi t} \right]. \end{aligned} \quad (1.23)$$

Here $f(\beta)$ is an arbitrary function of β subject to the condition:

$$f(1/2\tilde{\lambda}) = \frac{4\tilde{\lambda}^2 - 3}{8\tilde{\lambda}^2}, \quad f(1) = 0, \quad (1.24)$$

and α and $\tilde{\lambda}$ are parameters that need to be taken to be large but are otherwise arbitrary. More precisely, if we take $\alpha, \tilde{\lambda} \sim \gamma$ for some large number γ , then the expression inside the square bracket in (1.11) gives the correct expression for $\mathcal{F}_{\text{annulus}}$ up to correction terms of order γ^{-1} , possibly multiplied by powers of $\ln \gamma$. These corrections represent contributions from massive open string modes in internal propagators but are suppressed for large $\alpha, \tilde{\lambda}$. $\alpha, \tilde{\lambda}$ and the function $f(\beta)$ arise in the formulation of SFT, but the final result is expected to be independent of these parameter since SFTs corresponding to different choices of these parameters are related by field redefinitions. We have explicitly checked that the total contribution to the term inside the square bracket in (1.11) is independent of α, γ and $f(\beta)$ up to terms of order γ^{-1} . We can in principle avoid having to take the large $\alpha, \tilde{\lambda}$ limit by adding appropriate terms of order γ^{-1} to the expression inside the square bracket in (1.11) that will make it fully independent of the choice of $\alpha, \tilde{\lambda}$ and $f(\beta)$, but we have not done this.

The appearance of the imaginary error function can be traced to the open string tachyon propagating in the loop. We treat it using the usual $i\epsilon$ prescription and use the identity

$$\int_{-i\infty}^{i\infty} \frac{d\omega}{2\pi} \kappa^{2+2\omega^2} \frac{1}{\omega^2 + 1 + i\epsilon} = \frac{i}{2} \left\{ -i + \operatorname{erfi} \left(\sqrt{2 \ln \kappa} \right) \right\}, \quad \text{for } \kappa > 1. \quad (1.25)$$

Finally, we need to explain the lower cut-off t_c on the integration over t . The integral is singular from the $t = 0$ end due to the closed string tachyon. These can also be dealt with using SFT³ but we use Witten's $i\epsilon$ prescription to deal with these singularities. This requires integrating t up to some small number $1/\Lambda$, then change variable to $s = 1/t$ and carry out the integration over s from Λ to $\Lambda + i\infty$. So we can take

$$t_c = (\Lambda + i\infty)^{-1}. \quad (1.26)$$

The final result can be shown to be independent of Λ .

Numerical evaluation suggests the following result for $\mathcal{F}_{\text{annulus}}(i\vec{k})$:

$$\mathcal{F}_{\text{annulus}}(i\vec{k}) \approx (7.28219 - 2.75650 i) g_s \eta'_c \approx (1.15900 - 0.43871 i) g_s. \quad (1.27)$$

The imaginary part comes from closed string tachyons in intermediate states.

2 General strategy

In this section we shall describe the origin of the divergence in the integral (1.5) and the general strategy that we shall follow to resolve this, leaving the details of the analysis to later sections.

The expression for the integrand $F(x, t)$ was given in (1.6) and the actual derivation of this will be given in section 5.1. However, one can determine the general structure of the

³Note that while the presence of the tachyon renders the theory inconsistent, e.g. we do not have a unitary theory, there is no difficulty in getting finite amplitudes involving internal tachyons using SFT. In the path integral, this can be regarded as carrying out the integration over the tachyon field along its steepest descent contour.

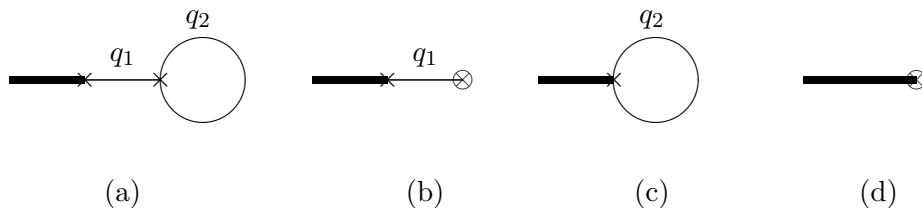


Figure 1. This figure shows the Feynman diagrams contributing to the annulus one point function of a closed string. The thick line represents external closed strings and the thin lines denote internal open strings. The interaction vertex with a $\times$ represents part of a disk amplitude, while the interaction vertex with $\otimes$ represents part of an annulus amplitude.

singularities of $F(x, t)$ even without explicit computation. This is best understood using the language of SFT where the amplitude is expressed as a sum over the Feynman diagrams of SFT of open and closed strings and the divergences appear from Schwinger parameter representation of the internal open string propagators. The relevant Feynman diagrams were constructed in [19, 20] and are shown in figure 1. In the full SFT there are also diagrams with internal closed string propagators, but when the number of non-compact space-time dimensions is larger than two, there are no divergences associated with the closed string propagators and we can integrate out these modes and include their contribution as part of the interaction vertex. The exceptions are closed string tachyons whose effect will be discussed separately in section 5.3. In the Siegel gauge, an open string propagator is proportional to,

$$L_0^{-1} = \int_0^1 dq q^{L_0-1}, \quad (2.1)$$

and the divergences appear from the $q = 0$ end of the integral due to $L_0 \simeq 0$ or $L_0 \leq 0$ states. The relation between the variables x, t and variables q_1 and q_2 associated with the two propagators shown in figure 1, were found in [19] for small x and large t , with the result:

$$v \equiv e^{-2\pi t} \simeq q_2/\alpha^2, \quad x \simeq q_1/(2\pi\tilde{\lambda}), \quad (2.2)$$

where α and $\tilde{\lambda}$ are two arbitrary large parameters used in the construction of SFT, with the final result expected to be independent of the choice of these parameters. Therefore, the singularities of the Feynman diagram corresponding to small q_1 will control the singularity of $F(x, t)$ for small x and the singularities corresponding to small q_2 correspond to the singularity of $F(x, t)$ for large t . There are similar relations when only one of the variables x and v becomes small. These will be discussed in section 5.

Note that (2.1) is valid for positive L_0 . For $L_0 \leq 0$ the right hand side of (2.1) diverges, and the infrared divergences that we encountered earlier in (1.5) from the $t = \infty$ and $x = 0$ region all stem from such contributions. The general strategy in SFT is to take the left hand sides of (2.1) as the correct expression that should replace the divergent integral on the right hand side of (2.1). This may still leave us with divergences from the $L_0 = 0$ states. We need to identify the origin of these infrared divergences in SFT and treat them correctly.

Before going into the remedy, let us use (2.2) to discuss what kind of divergence we expect the integral in (1.5) to possess. For this analysis we shall not keep track of the contribution

from interaction vertices; hence we shall be able to extract the leading divergent pieces only up to overall multiplicative constants. The open string fields living on the D0-brane are (0+1) dimensional fields. In momentum space they are functions of the energy variable ω which is manifestly conserved in amplitudes. From figure 1 we see that the open string propagator 1 is forced to carry zero energy since the external closed string carries zero energy (1.1). Furthermore, open strings on D0-branes do not carry momenta along non-compact directions, so massless open strings along this propagator have strictly vanishing L_0 and the integrand for small x takes the form

$$dq_1 q_1^{-1} \simeq dx x^{-1}, \quad (2.3)$$

where we used (2.2). This gives $F(x, t) \sim x^{-1}$ for $x \rightarrow 0$ and the integral diverges in this region. In bosonic string theory we also have open string tachyon modes along propagator 1 carrying $L_0 = -1$. From (2.1) we see that these will produce divergent integrands of the form

$$dq_1 q_1^{-2} \simeq \frac{1}{2\pi\tilde{\lambda}} dx x^{-2}. \quad (2.4)$$

The presence of the arbitrary constant $\tilde{\lambda}$ may sound surprising, but this is cancelled by the factors coming from interaction vertices. We shall see this explicitly in section 5.

Let us now turn to propagator 2. It carries a loop energy ω that needs to be integrated. In the $\alpha' = 1$ unit, massless open strings along this propagator have $L_0 = -\omega^2$ and hence (2.1) leads to a divergent integrand of the form⁴

$$dq_2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} q_2^{-\omega^2-1} = -2\pi dt \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \alpha^{-2\omega^2} e^{2\pi t \omega^2} \simeq -\frac{1}{\sqrt{2}} i dt t^{-1/2} \quad \text{for large } t, \quad (2.5)$$

where we used $q_2 = \alpha^2 e^{-2\pi t}$ for large t from (2.2). The i arises due to the fact that the integral needs to be defined by the euclidean rotation $\omega \rightarrow i\omega_E$ of the integration contour. This shows $F(x, t) \sim t^{-1/2}$ for large t . Hence, the integral diverges in this region, as was originally noted in [1, 2]. On the other hand a tachyon appearing in propagator 2 will have $L_0 = -\omega^2 - 1$, and integration over ω will lead to a contribution of the form

$$dq_2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} q_2^{-\omega^2-2} = -2\pi dt \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \alpha^{-2-2\omega^2} e^{2\pi t(\omega^2+1)} \simeq -\frac{1}{\sqrt{2}} i \alpha^{-2} dt t^{-1/2} e^{2\pi t} \quad (2.6)$$

for large t .

Thus the t integral diverges from the large t region. The singularities appearing in (2.3)–(2.6) match the singularities of $F(x, t)$ given in (1.6) in the small x and large t limit.

To remedy these problems, we need to first identify the massless and tachyonic open string fields that are responsible for this divergence. In bosonic open string theory the massless string fields can be represented as

$$|\Psi\rangle = |\Psi_s\rangle + |\Psi_{ns}\rangle, \quad (2.7)$$

⁴The large t behaviour arises from the small ω region; so even though there may be additional ω dependence from the vertex factors in the graphs, they do not affect the leading contribution for large t .

where the Siegel gauge field $|\Psi_s\rangle$ takes the form⁵

$$|\Psi_s\rangle = \int \frac{d\omega}{2\pi} \left[y^i(\omega) \alpha_{-1}^i c_1 |\omega\rangle + a_0(\omega) \alpha_{-1}^0 c_1 |\omega\rangle + i q(\omega) |\omega\rangle + i p(\omega) c_{-1} c_1 |\omega\rangle \right],$$

$$|\omega\rangle \equiv e^{-i\omega X^0}(0)|0\rangle, \quad \alpha_{-1}^\mu |\omega\rangle \equiv i\sqrt{2} \partial X^\mu e^{-i\omega X^0}(0)|0\rangle, \quad (2.8)$$

and the out of Siegel gauge field $|\Psi_{ns}\rangle$ takes the form:

$$|\Psi_{ns}\rangle = \int \frac{d\omega}{2\pi} \left[\tilde{y}^i(\omega) \alpha_{-1}^i c_0 c_1 |\omega\rangle + \tilde{a}_0(\omega) \alpha_{-1}^0 c_0 c_1 |\omega\rangle + i \tilde{q}(\omega) c_{-1} c_0 c_1 |\omega\rangle + i \tilde{p}(\omega) c_0 |\omega\rangle \right]. \quad (2.9)$$

Here $|\omega\rangle$ represents the Fock vacuum carrying energy ω , α_n^μ are oscillators associated with world-sheet scalar fields X^μ , and b, c are world-sheet diffeomorphism ghosts. y^i represent the zero modes associated with the transverse position of the D0-brane along the flat directions, a_0 is the U(1) gauge field living on the D0-brane world-volume and p, q are the ghost fields associated with gauge fixing of the U(1) gauge symmetry on the D0-brane world-volume. While propagating along the propagators 1 or 2, these fields will be responsible for the divergences in the $x \rightarrow 0$ and $t \rightarrow \infty$ limit. The fields $\tilde{y}^i, \tilde{p}, \tilde{a}_0$ and $\tilde{q}$ are the anti-fields of y^i, p, a_0 and q respectively. In Siegel gauge $|\Psi_{ns}\rangle$ is set to zero. Thus we see that the gauge fixing follows the usual Batalin-Vilkovisky (BV) formalism where we integrate over half of the fields, setting their anti-fields to zero.

We shall use the normalization conventions given in [16], in which

$$\langle \omega | c_{-1} c_0 c_1 | \omega' \rangle = -2\pi K \delta(\omega + \omega'), \quad (2.10)$$

where $\langle \dots \rangle$ is the disk correlation function, and K is a constant, related to the D0-brane mass M and the closed string coupling g_s by the relation [16]:

$$K = -g_s \frac{M}{2\sqrt{\eta_c}}, \quad \eta_c \equiv \frac{i}{2\pi}. \quad (2.11)$$

For all the world-sheet fields we use the normalization given in [16]. In computing correlation functions on the upper half plane, we shall often use the doubling trick to express the correlation function in the full complex plane via the replacement,

$$\begin{aligned} \bar{b}(x+iy) &\rightarrow b(x-iy), & \bar{c}(x+iy) &\rightarrow c(x-iy), & \bar{\partial} X^\mu(x+iy) &\rightarrow \pm \partial X^\mu(x-iy), \\ e^{ik \cdot X}(x+iy) &= e^{ik \cdot (X_L + X_R)}(x+iy) &\rightarrow e^{ik \cdot X_R}(x+iy) e^{\pm ik \cdot X_R}(x-iy), \end{aligned} \quad (2.12)$$

where we choose $+$ sign for coordinates with Neumann boundary condition and $-$ sign for coordinates with Dirichlet boundary condition. Here X_R and X_L denote the holomorphic and anti-holomorphic components of X . In this spirit, the open string vertex operator $e^{i\omega X^0}(x)$, inserted on the real line in the upper half plane, may be regarded as $e^{2i\omega X_R^0}(x)$ in the full complex plane.

⁵Since we shall work in a flat background metric, all our spatial indices will be raised and lowered by the flat metric δ_{ij} and we shall not distinguish between upper and lower indices.

The action S will be normalized such that e^S is the weight factor in the Euclidean path integral. In this convention, the kinetic term of the string field in the Siegel gauge gives⁶

$$\frac{1}{2} \langle \Psi_s | Q_B | \Psi_s \rangle = K \int \frac{d\omega}{2\pi} \left[\frac{1}{2} \omega^2 y^i(-\omega) y^i(\omega) - \frac{1}{2} \omega^2 a_0(-\omega) a_0(\omega) + \omega^2 p(\omega) q(-\omega) \right]. \quad (2.13)$$

The ω^2 factors in the quadratic terms are responsible for the ω^{-2} terms in the propagator, which in turn produce the divergences in the $x \rightarrow 0$ and / or $t \rightarrow \infty$ limit via (2.3) and (2.5). Indeed, not only the divergent part but the complete world-sheet expression for the amplitudes can be shown to emerge from the Feynman rules in Siegel gauge.

We now describe how to treat the divergences coming from the y^i , a_0 , p and q fields. We begin with the p, q fields. In the Faddeev-Popov formalism, these arise from fixing the gauge transformation associated with the parameter

$$|\Lambda\rangle = i \int \frac{d\omega}{2\pi} \theta(\omega) |\omega\rangle + \dots, \quad (2.14)$$

by setting to zero the out of Siegel gauge field $\tilde{p}$ appearing in (2.9). Indeed, using the gauge transformation law $\delta|\Psi\rangle = Q_B|\Lambda\rangle$, we get

$$\delta\tilde{p} = -\omega^2 \theta. \quad (2.15)$$

Therefore, fixing the $\tilde{p} = 0$ gauge produces a jacobian factor ω^2 which is captured by the integration over the ghost fields p, q , and in turn leads to the infrared divergence problems discussed earlier. The remedy we shall follow is to fix a different gauge, setting

$$a_0 = 0. \quad (2.16)$$

This has two effects. First of all it removes integration over p and a_0 . The second effect is that we now have to integrate over the out of Siegel gauge field $\tilde{p}$ and a new ghost field which we can identify as the anti-field $\tilde{a}_0$ of the field a_0 , appearing in (2.9). The string field at level 0 in this new gauge now takes the form

$$|\Psi_{new}\rangle = \int \frac{d\omega}{2\pi} \left[y^i(\omega) \alpha_{-1}^i c_1 |\omega\rangle + i q(\omega) |\omega\rangle + \tilde{a}_0(\omega) \alpha_{-1}^0 c_0 c_1 |\omega\rangle + i \tilde{p}(\omega) c_0 |\omega\rangle \right]. \quad (2.17)$$

It will also be important to determine the integration measure over these fields. For this we note that from the perspective of the BV formalism, the part of the level zero string field that has been fixed to zero in this gauge is

$$|\Psi_{gf}\rangle = \int \frac{d\omega}{2\pi} \left[\tilde{y}^i(\omega) \alpha_{-1}^i c_0 c_1 |\omega\rangle + i \tilde{q}(\omega) c_{-1} c_0 c_1 |\omega\rangle + a_0(\omega) \alpha_{-1}^0 c_1 |\omega\rangle + i p(\omega) c_{-1} c_1 |\omega\rangle \right]. \quad (2.18)$$

In this expansion, $\tilde{y}^i$, p , a_0 and $\tilde{q}$ are the anti-fields of y^i , $\tilde{p}$, $\tilde{a}_0$ and q respectively up to constant normalizations and signs. Following the standard rules of BV quantization, the gauge invariant measure can now be written as

$$\int Dy^i D\tilde{y}^i D\tilde{p} Dp D\tilde{a}_0 Da^0 Dq D\tilde{q} \prod_i \delta(\tilde{y}^i) \delta(p) \delta(a^0) \delta(\tilde{q}) = \int Dy^i D\tilde{p} D\tilde{a}_0 Dq, \quad (2.19)$$

without any additional Jacobian factor.

⁶In this computation we need to account for the fact that when the Grassmann odd fields p, q pass through the BRST charge Q_B , we get an extra minus sign.

In this gauge the action contains the terms

$$\frac{1}{2} \langle \Psi_{new} | Q_B | \Psi_{new} \rangle = K \int \frac{d\omega}{2\pi} \left[\frac{1}{2} \omega^2 y^i(-\omega) y^i(\omega) - \tilde{p}(-\omega) \tilde{p}(\omega) + i \sqrt{2} \omega \tilde{a}_0(-\omega) q(\omega) \right], \quad (2.20)$$

showing that $\tilde{p}$ plays the role of an auxiliary field and the kinetic term of the new Faddeev-Popov ghosts q and $\tilde{a}_0$ is proportional to ω . The latter is related to the fact that for the gauge transformation parameter $|\Lambda\rangle$ given in (2.14), $\delta|\Psi\rangle = Q_B|\Lambda\rangle$ gives $\delta a_0 \propto i\omega\theta$. The path integral over the $\tilde{p}$ field now gives a finite result. On the other hand the $\tilde{a}_0$ - q ghost propagator is proportional to $\omega/(\omega^2 + i\epsilon)$ and hence the potentially divergent integrand involving the $\tilde{a}_0$ - q propagator has the form $d\omega f(\omega) \omega/(\omega^2 + i\epsilon)$ for some smooth function $f(\omega)$. The integral over ω now has no divergence from the $\omega \simeq 0$ region.

This leaves us to deal with the modes labelled by y^i . Physically the origin of the divergence is clear. In the approach discussed so far, the D0-brane is taken to be a static object that does not backreact during the scattering process. In particular, the D0-brane boundary condition breaks translation invariance along the non-compact spatial directions and hence violates conservation of spatial momenta. This is reflected in the fact that in the standard world-sheet approach both the initial and the final D0-brane states are taken to be zero momentum objects. In actual practice, the D0-brane will suffer a recoil, leading to momentum conservation. The choice of ‘wrong external states’ typically leads to infrared divergences in the amplitudes and this is precisely what is responsible for the divergence from the y^i propagators.

To remedy this problem, we note that the y^i ’s are the collective modes of the D0-brane associated with the motion in the transverse direction. As is well known, the collective modes of solitons cannot be treated using perturbation theory, instead they have to be quantized separately. So we need to first remove the contribution of the y^i ’s from the open string propagator. This removes the remaining terms responsible for the divergences and allows us to integrate over the other open string modes to construct an effective theory of the collective modes. We then quantize the collective modes. This will generate the momentum eigenstates of the D0-brane that appear in the initial and final states of a scattering process and will allow us to have initial and final D0-branes carrying different momenta, leading to overall momentum conservation. This will be discussed in section 3.

A further complication arises due to the fact that the y^i ’s that enter the expansion of the string field are not directly the collective coordinates, but are related to them by field redefinition. Since our starting point is the SFT action, we need to find this field redefinition and then express the SFT action in terms of the collective modes, including the effect of the Jacobian factor due to the change of variables in the path integral. In section 4 we shall find this field redefinition by comparing the coupling of y^i to a set of closed string states to the expected coupling of the collective modes to the closed string states.

In bosonic string theory we also have integration over the open string tachyon field $T(\omega)$ that appears in the expansion of the string field as

$$|\Psi\rangle = \int \frac{d\omega}{2\pi} T(\omega) c_1 |\omega\rangle. \quad (2.21)$$

The action involving the tachyon field takes the form:

$$\frac{K}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} (1 + \omega^2) T(-\omega) T(\omega), \quad (2.22)$$

leading to a tachyon propagator $-(\omega^2 + 1)^{-1}$. This can be traced to the fact that $c_1|\omega\rangle$ has L_0 eigenvalue $-(\omega^2 + 1)$ and is responsible for the leading divergences appearing in (2.4) and (2.6) via (2.1). Let us begin with (2.4) that comes from a tachyon propagator with $\omega = 0$. The remedy of this is to replace the right hand side of (2.1) by the left hand side of (2.1) with $L_0 = -1$, i.e. after representing the contribution from the $x = 0$ region in terms of the variable q_1 coming from SFT, we simply replace $\int_0^1 dq_1 q_1^{-2}$ by -1 .

For (2.6), i.e. when the tachyon appears in propagator 2, the situation is a bit more complicated. As before, we replace $\int_0^1 dq_2 q_2^{-\omega^2-2}$ by $(\omega^2 + 1)^{-1}$. Although the integral over ω looks divergence free, in SFT the interaction vertices will typically contain terms proportional to $\exp(C\omega^2)$ for some positive constant C . Hence, the end points of the ω integration contour must approach $\pm i\infty$ in order that the integral converges [21]. If we take the integration contour to be along the imaginary ω axis then the integrand develops a pole on the integration contour at $\omega = \pm i$. Hence we need to choose an integration contour avoiding these poles. We shall use the standard $i\epsilon$ prescription where we replace $p^2 + m^2$ in the denominator by $p^2 + m^2 - i\epsilon$. This corresponds to replacing $(\omega^2 + 1)^{-1}$ by $(\omega^2 + 1 + i\epsilon)^{-1}$. This ensures that we do not encounter any poles during the Wick rotation from the real ω axis to the imaginary ω axis. For definiteness, we shall work with this choice of contour. However, given that the tachyonic mode represents an instability of the D0-brane and the system is not physical, other choice of contour may also be possible.

3 Bosonic collective modes

Our goal in this section will be to deal with the divergences associated with the y^i propagators. For this we first integrate out all the open string fields other than the y^i 's to construct an effective action of y^i and the closed string tachyon field Σ whose amplitude we are trying to compute. The analysis of the previous section shows that this process does not encounter any infrared divergence. If we try to compute amplitudes from this effective action using perturbation theory, we shall encounter infrared divergences due to the ω^{-2} singularity in the y^i propagators. Therefore we cannot treat the y^i 's using perturbation theory. We shall now describe how this difficulty is resolved.

If y^i 's had been exactly the collective modes, then we could remove the contribution of the y^i propagators from the internal lines of an open string propagator and quantize the y^i 's separately to construct momentum eigenstates of the D0-brane, which can then be used for computing a scattering amplitude. The complication arises from the fact that the y^i 's are not exactly the collective modes. For example, the action should be invariant under a rigid translation of a collective mode associated with broken translation invariance, but the full SFT action is not invariant under such translations of y^i . Let $\chi^i(t)$ be the actual collective mode, related to $y^i(t)$ and other string field components by a field redefinition. As will be discussed in section 4, we can find this field redefinition using perturbation theory, and the Jacobian due to the change of variable from y^i to χ^i will give an additional term in the

effective action. This resulting action should be invariant under the transformation⁷

$$\Sigma(t, \vec{x}) \rightarrow \Sigma(t, \vec{x} + \vec{a}), \quad \chi^i(t) \rightarrow \chi^i(t) - a^i, \quad (3.1)$$

for any constant vector $\vec{a}$. Also, the action is invariant under a time reversal symmetry

$$\Sigma(t, \vec{x}) \rightarrow \Sigma(-t, \vec{x}), \quad \chi^i(t) \rightarrow \chi^i(-t), \quad (3.2)$$

and parity symmetry

$$\Sigma(t, \vec{x}) \rightarrow \Sigma(t, -\vec{x}), \quad \chi^i(t) \rightarrow -\chi^i(t). \quad (3.3)$$

An action of this type is

$$\begin{aligned} & -\frac{1}{2} \int dt d^D x \left[\eta^{\mu\nu} \partial_\mu \Sigma(t, \vec{x}) \partial_\nu \Sigma(t, \vec{x}) + m_\Sigma^2 \Sigma(t, \vec{x})^2 \right] + \frac{M}{2} \int dt \partial_t \vec{\chi}(t) \cdot \partial_t \vec{\chi}(t) \\ & + \int dt \mathcal{F}(\vec{\nabla}) \Sigma(t, \vec{\chi}(t)) + \int dt \partial_t \chi^i(t) \partial_t \chi^j(t) \mathcal{F}(\vec{\nabla})_{ij} \Sigma(t, \vec{\chi}(t)) + \dots, \end{aligned} \quad (3.4)$$

where we have kept terms up to linear order in Σ and quadratic order in χ^i (other than those appearing in the argument of Σ) since these are the terms relevant for the Feynman diagrams of figure 1. Here m_Σ is the mass of Σ , $M \propto g_s^{-1}$ is the mass of the D0-brane and $\mathcal{F}(\vec{\nabla})$, $\mathcal{F}(\vec{\nabla})_{ij}$ are polynomials of spatial derivative operators acting on Σ . D is the number of non-compact spatial dimensions which will eventually be set to 25. $\mathcal{F}(\vec{\nabla}) \Sigma(t, \vec{\chi}(t))$ means that we first compute $\mathcal{F}(\vec{\nabla}) \Sigma(t, \vec{x})$ and then replace $\vec{x}$ by $\vec{\chi}$. These spatial derivatives will translate into factors of spatial momenta in the final amplitude. On the other hand, since the external closed string carries zero energy, the time derivative of Σ can be ignored. Also, we have used the time reversal symmetry to exclude terms linear in $\partial_t \chi^i$ and ignored terms proportional to $\partial_t^n \chi^i$ for $n \geq 2$ since they can be removed by field redefinition. The full theory contains many more terms consistent with the symmetry (3.1) involving higher powers of $\partial_t \chi^i$. However, these terms will not contribute to the Feynman diagrams appearing in figure 1 and hence will not be relevant for our discussion. They will of course become important at higher order in g_s expansion.

We shall now show that the term proportional to $\mathcal{F}(\vec{\nabla})_{ij}$ also does not affect our analysis. For this we make a field redefinition

$$\chi^j \rightarrow \chi^j - M^{-1} \chi^i(t) \mathcal{F}(\vec{\nabla})_{ij} \Sigma(t, \vec{\chi}(t)). \quad (3.5)$$

This removes the terms in (3.4) proportional to $\mathcal{F}(\vec{\nabla})_{ij}$ and produces new terms involving $\partial_t \Sigma$ or cubic terms in χ^i . Since the external closed string has zero energy and since the Feynman diagrams in figure 1 involve interaction terms that are at most quadratic in the y^i 's, none of these new terms contribute to the Feynman diagrams of figure 1. The field redefinition (3.5) will give rise to a Jacobian whose effect will be to give an additional term

⁷The action should also have Lorentz invariance. However, since the D0-brane has mass of order g_s^{-1} , the Lorentz transformation will mix different orders in g_s expansion. For this reason we do not make use of Lorentz symmetry.

in the effective action proportional to $\int dt \mathcal{F}(\vec{\nabla})_{ii} \Sigma(t, \vec{\chi}(t))$, but this just renormalizes $\mathcal{F}(\vec{\nabla})$. Thus we work with the action

$$\begin{aligned}
 & -\frac{1}{2} \int dt d^D x \left[\eta^{\mu\nu} \partial_\mu \Sigma(t, \vec{x}) \partial_\nu \Sigma(t, \vec{x}) + m_\Sigma^2 \Sigma(t, \vec{x})^2 \right] + \frac{M}{2} \int dt \partial_t \vec{\chi}(t) \cdot \partial_t \vec{\chi}(t) \\
 & + \int dt \mathcal{F}(\vec{\nabla}) \Sigma(t, \vec{\chi}(t)).
 \end{aligned} \tag{3.6}$$

It is instructive to see what happens if we expand the action in powers of χ^i and evaluate the amplitude by computing Feynman diagrams. The action to quadratic order in χ^i takes the form:

$$\begin{aligned}
 & -\frac{1}{2} \int dt d^D x \left[\eta^{\mu\nu} \partial_\mu \Sigma(t, \vec{x}) \partial_\nu \Sigma(t, \vec{x}) + m_\Sigma^2 \Sigma(t, \vec{x})^2 \right] + \frac{M}{2} \int dt \partial_t \vec{\chi}(t) \cdot \partial_t \vec{\chi}(t) \\
 & + \int dt \left(\mathcal{F}(\vec{\nabla}) \Sigma(t, \vec{0}) + \chi^i(t) \mathcal{F}(\vec{\nabla}) \partial_i \Sigma(t, \vec{0}) + \frac{1}{2} \chi^i(t) \chi^j(t) \mathcal{F}(\vec{\nabla}) \partial_i \partial_j \Sigma(t, \vec{0}) \right) + \dots,
 \end{aligned} \tag{3.7}$$

where $\partial_{i_1} \dots \partial_{i_n} \Sigma(t, \vec{0}) \equiv \partial_{i_1} \dots \partial_{i_n} \Sigma(t, \vec{x})|_{\vec{x}=\vec{0}}$. In Fourier transformed space, the χ^i propagator will be of order ω^{-2} . Hence loops of the χ^i field will lead to infrared divergences, e.g. in diagrams in figure 1(c) when the propagator 2 is a χ^i field and the interaction vertex is produced by the last term in (3.7).

We now describe the remedy of the problem, which is to quantize the mode χ^i exactly instead of treating it using Feynman diagrams. While we can use the path integral formulation for both the collective coordinate χ^i and the field Σ , it will be a bit more illuminating to treat the modes χ^i using the Hamiltonian formulation. More precisely we use the Routhian formalism where we use the Hamiltonian formalism for the fields $\chi^i(t)$ and the Lagrangian formalism for the field Σ . We define the conjugate momenta

$$p_j = \frac{\partial L}{\partial(\partial_t \chi^j)} = M \partial_t \chi^j, \tag{3.8}$$

and the Routhian:

$$\begin{aligned}
 R = \sum_j p_j \partial_t \chi^j - L = \frac{p_j p_j}{2M} + \frac{1}{2} \int dt d^D x \left[\eta^{\mu\nu} \partial_\mu \Sigma(t, \vec{x}) \partial_\nu \Sigma(t, \vec{x}) + \frac{1}{2} m_\Sigma^2 \Sigma(t, \vec{x})^2 \right] \\
 - \int dt \mathcal{F}(\vec{\nabla}) \Sigma(t, \vec{\chi}(t)).
 \end{aligned} \tag{3.9}$$

We shall treat $p_i p_i / (2M)$ as the unperturbed Hamiltonian of the collective mode and take the incoming and the outgoing D0-brane to be eigenstates of energy and momentum $(\omega_{\text{in}}, \vec{k}_{\text{in}})$ and $(\omega_{\text{out}}, \vec{k}_{\text{out}})$ respectively, satisfying

$$\omega_{\text{in}} = M + \frac{\vec{k}_{\text{in}}^2}{2M}, \quad \omega_{\text{out}} = M + \frac{\vec{k}_{\text{out}}^2}{2M}. \tag{3.10}$$

Our strategy will be to first compute the matrix element of $\mathcal{F}(\vec{\nabla}) \Sigma(t, \vec{\chi}(t))$ between the incoming and outgoing D0-brane states $|\vec{k}_{\text{in}}\rangle, |\vec{k}_{\text{out}}\rangle$, and then treat $\langle \vec{k}_{\text{out}} | \int dt \mathcal{F}(\vec{\nabla}) \Sigma(t, \vec{\chi}(t)) | \vec{k}_{\text{in}} \rangle$ as a term in the effective action of Σ from which we can compute the one point function of Σ . This will give the desired amplitude.

We normalize the D0-brane states as those of a non-relativistic particle so as not to mix different orders in perturbation theory

$$\langle \vec{k}_{\text{out}} | \vec{k}_{\text{in}} \rangle = (2\pi)^D \delta^{(D)}(\vec{k}_{\text{in}} - \vec{k}_{\text{out}}), \quad (3.11)$$

and introduce the Fourier transform $\tilde{\Sigma}$ of Σ , defined through

$$\Sigma(t, \vec{x}) = \int \frac{d\omega}{2\pi} \int \frac{d^D k}{(2\pi)^D} e^{-i\omega t + i\vec{k} \cdot \vec{x}} \tilde{\Sigma}(\omega, \vec{k}). \quad (3.12)$$

This gives the linear term in the effective action of Σ to be

$$\begin{aligned} & \int dt \langle \vec{k}_{\text{out}} | \mathcal{F}(\vec{\nabla}) \Sigma(t, \vec{\chi}(t)) | \vec{k}_{\text{in}} \rangle = \int dt e^{i(\omega_{\text{out}} - \omega_{\text{in}})t} \langle \vec{k}_{\text{out}} | \mathcal{F}(\vec{\nabla}) \Sigma(t, \vec{\chi}(0)) | \vec{k}_{\text{in}} \rangle \\ &= \int dt \int \frac{d\omega}{2\pi} \int \frac{d^D k}{(2\pi)^D} e^{i(\omega_{\text{out}} - \omega_{\text{in}} - \omega)t} \mathcal{F}(i\vec{k}) \tilde{\Sigma}(\omega, \vec{k}) \langle \vec{k}_{\text{out}} | e^{i\vec{k} \cdot \vec{\chi}(t=0)} | \vec{k}_{\text{in}} \rangle \\ &= \int \frac{d\omega}{2\pi} \int \frac{d^D k}{(2\pi)^D} \mathcal{F}(i\vec{k}) \tilde{\Sigma}(\omega, \vec{k}) 2\pi \delta(\omega_{\text{out}} - \omega_{\text{in}} - \omega) (2\pi)^D \delta^{(D)}(\vec{k} + \vec{k}_{\text{in}} - \vec{k}_{\text{out}}). \end{aligned} \quad (3.13)$$

Above, the first delta function comes from the t integral and the second delta function comes from the matrix element. The relevant amplitude for an external closed string state of energy ω and momentum $\vec{k}$ is then given by,

$$\mathcal{F}(i\vec{k}) 2\pi \delta(\omega - \omega_{\text{out}} + \omega_{\text{in}}) (2\pi)^D \delta^{(D)}(\vec{k} + \vec{k}_{\text{in}} - \vec{k}_{\text{out}}). \quad (3.14)$$

This recovers the full energy-momentum conserving delta function. Thus the main task is to compute $\mathcal{F}(i\vec{k})$ using SFT.

4 Field redefinition from the string fields to collective coordinates

In this section we shall describe the procedure for finding the field redefinition that takes us from string field variables y^i to the collective coordinates χ^i .

To simplify notation, for the open string field y^i and the collective coordinate χ^i we shall use the same symbol for the field and its Fourier transform in the time variable — the argument of the field will convey information about which one we are using. We shall be looking for a relation between y^i and χ^i of the form

$$\begin{aligned} y^i(\omega) &= \sqrt{\frac{M}{K}} \chi^i(\omega) + \int \frac{d^D k}{(2\pi)^D} g^i(\omega, \vec{k}) \tilde{\Sigma}(\omega, \vec{k}) \\ &+ \int \frac{d\omega'}{2\pi} \int \frac{d^D k}{(2\pi)^D} f^i_j(\omega, \omega', \vec{k}) \chi^j(\omega') \tilde{\Sigma}(\omega - \omega', \vec{k}) + \dots, \end{aligned} \quad (4.1)$$

where $\dots$ denotes other terms containing higher powers of χ^i and / or $\tilde{\Sigma}$ that will not be needed in our analysis and f^i_j and g^i are functions that we need to determine. The normalization factor $\sqrt{M/K}$ in the first term has been fixed by comparing the kinetic term of y^i in the SFT action (2.13) and that of χ^i in the action of collective field theory given in (3.4). Rotational

invariance prevents the appearance of terms of the form $\int d\omega' C_{jk}^i(\omega, \omega') \chi^j(\omega') \chi^k(\omega - \omega')$ on the right hand side of (4.1). Treating $\tilde{\Sigma}$ as a background field, we can write, up to an overall constant normalization,

$$\begin{aligned} \prod_{i,\omega} dy^i(\omega) &\simeq \prod_{i,\omega} d\chi^i(\omega) \left[1 + \sqrt{\frac{K}{M}} \int_{-i\infty}^{i\infty} (-i) \frac{d\omega'}{2\pi} \int \frac{d^D k}{(2\pi)^D} f_{jk}^j(\omega', \omega', \vec{k}) \tilde{\Sigma}(0, \vec{k}) \right] \\ &\simeq \prod_{i,\omega} d\chi^i(\omega) \exp \left[-i \sqrt{\frac{K}{M}} \int_{-i\infty}^{i\infty} \frac{d\omega'}{2\pi} \int \frac{d^D k}{(2\pi)^D} f_{jk}^j(\omega', \omega', \vec{k}) \tilde{\Sigma}(0, \vec{k}) \right]. \end{aligned} \quad (4.2)$$

The $-i$ and the range of the integration over ω' has the following origin. While computing quantum corrections we shall use a Euclidean path integral with weight factor e^S . In this case the string fields will be labelled by Euclidean energy ω_E , related to the Lorentzian energy ω via $\omega = i\omega_E$. The trace involved in computing the Jacobian will then involve $\int_{-\infty}^{\infty} d\omega_E/(2\pi)$, which we have expressed as $-i \int_{-i\infty}^{i\infty} d\omega/(2\pi)$. The term in the exponent in (4.2) can now be interpreted as a new term in the action given by

$$-i \sqrt{\frac{K}{M}} \int_{-i\infty}^{i\infty} \frac{d\omega'}{2\pi} \int \frac{d^D k}{(2\pi)^D} f_{jk}^j(\omega', \omega', \vec{k}) \tilde{\Sigma}(0, \vec{k}). \quad (4.3)$$

Comparing this with (3.6) we see that this may be interpreted as a new contribution to $\mathcal{F}(i\vec{k})$, given by

$$\mathcal{F}_{\text{jac}} = -i \sqrt{\frac{K}{M}} \int_{-i\infty}^{i\infty} \frac{d\omega'}{2\pi} f_{jk}^j(\omega', \omega', \vec{k}), \quad (4.4)$$

that needs to be added to the annulus one point function. Therefore our goal will be to compute the functions f_{jk}^i .

The strategy for computing f_{jk}^i will be as follows. We shall first write down the general effective action for y^i and Σ consistent with the parity and time reversal symmetries (3.2), (3.3) with χ^i replaced by y^i and then look for a field redefinition (4.1) that relates the effective action of y^i 's to the effective action of the χ^i 's and the Σ given in (3.6). The general y^i and Σ dependent terms in the SFT action takes the form:

$$\begin{aligned} &\frac{K}{2} \int \frac{d\omega}{2\pi} \omega^2 y^i(\omega) y^i(-\omega) + \int \frac{d^D k}{(2\pi)^D} \mathcal{F}(i\vec{k}) \tilde{\Sigma}(0, \vec{k}) \\ &+ \int \frac{d\omega}{2\pi} \int \frac{d^D k}{(2\pi)^D} B_i^{(1)}(\omega, \vec{k}) y^i(\omega) \tilde{\Sigma}(-\omega, \vec{k}) \\ &+ \frac{1}{2} \int \frac{d\omega}{2\pi} \int \frac{d\omega'}{2\pi} \int \frac{d^D k}{(2\pi)^D} B_{ij}^{(2)}(\omega, \omega', \vec{k}) y^i(\omega) y^j(\omega') \tilde{\Sigma}(-\omega - \omega', \vec{k}) + \dots, \end{aligned} \quad (4.5)$$

where $\mathcal{F}(i\vec{k})$, $B_i^{(1)}(\omega, \vec{k})$ and $B_{ij}^{(2)}(\omega, \omega', \vec{k})$ can be determined from the SFT action. This will be done in section 5.5. Using the relation (4.1) between the y^i 's and the χ^i 's, we can

express (4.5) as

$$\begin{aligned}
 & \frac{M}{2} \int \frac{d\omega}{2\pi} \omega^2 \chi^i(\omega) \chi^i(-\omega) + \int \frac{d^D k}{(2\pi)^D} \mathcal{F}(i\vec{k}) \tilde{\Sigma}(0, \vec{k}) \\
 & + \sqrt{KM} \int \frac{d\omega}{2\pi} \int \frac{d^D k}{(2\pi)^D} \omega^2 \chi^i(-\omega) g^i(\omega, \vec{k}) \tilde{\Sigma}(\omega, \vec{k}) \\
 & + \sqrt{KM} \int \frac{d\omega}{2\pi} \int \frac{d\omega'}{2\pi} \int \frac{d^D k}{(2\pi)^D} \omega^2 \chi^i(-\omega) f^i_j(\omega, \omega', \vec{k}) \chi^j(\omega') \tilde{\Sigma}(\omega - \omega', \vec{k}) \\
 & + \sqrt{\frac{M}{K}} \int \frac{d\omega}{2\pi} \int \frac{d^D k}{(2\pi)^D} B_i^{(1)}(\omega, \vec{k}) \chi^i(\omega) \tilde{\Sigma}(-\omega, \vec{k}) \\
 & + \frac{M}{2K} \int \frac{d\omega}{2\pi} \int \frac{d\omega'}{2\pi} \int \frac{d^D k}{(2\pi)^D} B_{ij}^{(2)}(\omega, \omega', \vec{k}) \chi^i(\omega) \chi^j(\omega') \tilde{\Sigma}(-\omega - \omega', \vec{k}) + \dots \quad (4.6)
 \end{aligned}$$

We have to compare this with (3.6) after expanding this in powers of χ^i :

$$\begin{aligned}
 & \frac{M}{2} \int \frac{d\omega}{2\pi} \omega^2 \chi^i(\omega) \chi^i(-\omega) \\
 & + \int \frac{d^D k}{(2\pi)^D} \mathcal{F}(i\vec{k}) \left[\tilde{\Sigma}(0, \vec{k}) + i k_i \int \frac{d\omega}{2\pi} \chi^i(\omega) \tilde{\Sigma}(-\omega, \vec{k}) \right. \\
 & \left. - \frac{1}{2} k_i k_j \int \frac{d\omega}{2\pi} \int \frac{d\omega'}{2\pi} \chi^i(\omega) \chi^j(\omega') \tilde{\Sigma}(-\omega - \omega', \vec{k}) + \dots \right]. \quad (4.7)
 \end{aligned}$$

Comparison of (4.6) and (4.7) now gives

$$\begin{aligned}
 i k_i \mathcal{F}(i\vec{k}) &= \sqrt{KM} \omega^2 g^i(-\omega, \vec{k}) + \sqrt{\frac{M}{K}} B_i^{(1)}(\omega, \vec{k}) \\
 -\frac{1}{2} k_i k_j \mathcal{F}(i\vec{k}) &= \frac{\sqrt{KM}}{2} \{ \omega^2 f^i_j(-\omega, \omega', \vec{k}) + (\omega')^2 f^j_i(-\omega', \omega, \vec{k}) \} + \frac{M}{2K} B_{ij}^{(2)}(\omega, \omega', \vec{k}). \quad (4.8)
 \end{aligned}$$

In particular, in the second equation, after setting $\omega = -\omega'$, and tracing over i, j , we get

$$\frac{1}{2} (\omega')^2 \left(f^i_i(\omega', \omega', \vec{k}) + f^i_i(-\omega', -\omega', \vec{k}) \right) = -\frac{1}{2\sqrt{KM}} \left\{ \vec{k}^2 \mathcal{F}(i\vec{k}) + \frac{M}{K} B_{ii}^{(2)}(-\omega', \omega', \vec{k}) \right\}, \quad (4.9)$$

from which the contribution (4.4) becomes

$$\mathcal{F}_{\text{jac}} = \frac{i}{2M} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} (\omega')^{-2} \left\{ \vec{k}^2 \mathcal{F}(i\vec{k}) + \frac{M}{K} B_{ii}^{(2)}(-\omega', \omega', \vec{k}) \right\}. \quad (4.10)$$

Due to the $1/M$ factor in the normalization, we already have a factor of g_s and so we only need to compute the tree level contribution to $\mathcal{F}(i\vec{k})$ and $B_{ii}^{(2)}$ on the right hand side. Since we use the notation $\mathcal{F}(i\vec{k})$ for a general term in the effective action linear in Σ , we shall denote the tree level contribution to $\mathcal{F}$ by $\mathcal{F}_0$ and rewrite the equation as

$$\mathcal{F}_{\text{jac}} = \frac{i}{2M} \int \frac{d\omega'}{2\pi} (\omega')^{-2} \left\{ \vec{k}^2 \mathcal{F}_0(i\vec{k}) + \frac{M}{K} B_{ii}^{(2)}(-\omega', \omega', \vec{k}) \right\}. \quad (4.11)$$

We evaluate this explicitly in section 5.5.

5 Analysis of the D0-D0-tachyon amplitude

In this section we shall explicitly compute the one point function of massless states of the 26 dimensional bosonic string theory on an annulus with its boundary on a D0-brane and extract a finite result for the amplitude (1.5) following the strategy described in the earlier sections. This will determine the annulus contribution to $\mathcal{F}(i\vec{k})$, which in turn will determine the D0-D0-tachyon amplitude via (3.14).

5.1 World-sheet expression for the amplitude

We take the external incoming closed string to carry momentum $(k^0, \vec{k})$ with $k^2 = 4$. Therefore we can take its vertex operator to be of the form $c\bar{c}V$ with:

$$V = e^{ik \cdot X} = e^{-ik^0 X^0 + i\vec{k} \cdot \vec{X}}, \quad k^2 = 4. \quad (5.1)$$

We shall label the annulus by a complex coordinate w subject to the restriction:

$$0 \leq \text{Re}(w) \leq \pi, \quad w \equiv w + 2\pi i t. \quad (5.2)$$

Following the general procedure described in [16] and reviewed in appendix A, we can now express the integrand $F(x, t)$, appearing in the expression for the annulus one point function (1.5) of $c\bar{c}V$, as,⁸

$$F(x, t) = \frac{g_s \eta_c}{2\pi i} \times (-2\pi i) \times 2\pi \left\langle \left(\int_0^\pi dw b(w) + \int_0^\pi \bar{b}(\bar{w}) d\bar{w} \right) \left(\oint_x dw' b(w') + \oint_x \bar{b}(\bar{w}') d\bar{w}' \right) c\bar{c}V(2\pi x) \right\rangle_A, \quad (5.3)$$

where

$$\eta_c \equiv \frac{i}{2\pi}. \quad (5.4)$$

$\oint_x$ denotes an anti-clockwise contour around x and $\langle \cdots \rangle_A$ denotes unnormalized one point function on the annulus. The first factor of $1/2\pi i$ is the factor that accompanies the integral of $b(w)$ and $\bar{b}(\bar{w})$ inside the first parentheses, the second factor of $-2\pi i$ comes from the identification $w \equiv w - 2\pi i t$ so that a derivative of the transition function with respect to t produces a factor of $-2\pi i$ and the third factor of 2π comes from the argument of V being $2\pi x$ so that the integration measure over x is $2\pi dx$. The $\oint$ carry their own factors of $\pm 1/2\pi i$. Now the upper half plane coordinate z is related to the strip coordinate w via the relation:

$$z = e^{iw}, \quad z \equiv e^{2\pi t} z. \quad (5.5)$$

⁸We have not attempted to fix the sign of this term from first principles although this could be done following the results of [16]. Instead we have chosen the sign so that it agrees with the ones computed from Feynman diagrams. This will be seen in section 5.2.

Therefore, the expansions of b and c in the strip coordinate takes the form

$$\begin{aligned} b(w) &= (dz/dw)^2 b(z) = -e^{2iw} \sum_n b_n z^{-n-2} = -\sum_n b_n e^{-inw}, \\ \bar{b}(\bar{w}) &= (d\bar{z}/d\bar{w})^2 \bar{b}(\bar{z}) = -e^{-2i\bar{w}} \sum_n b_n \bar{z}^{-n-2} = -\sum_n b_n e^{in\bar{w}}, \\ c(w) &= (dz/dw)^{-1} c(z) = -i e^{-iw} \sum_n c_n z^{-n+1} = -i \sum_n c_n e^{-inw}, \\ \bar{c}(\bar{w}) &= (d\bar{z}/d\bar{w})^{-1} \bar{c}(\bar{z}) = i e^{i\bar{w}} \sum_n c_n \bar{z}^{-n+1} = i \sum_n c_n e^{in\bar{w}}. \end{aligned} \quad (5.6)$$

After carrying out the contour integrals we can express F as:

$$F(x, t) = -2\pi g_s \eta_c \langle (2\pi b_0) (c(2\pi x) - \bar{c}(2\pi x)) V(2\pi x) \rangle_A = 8\pi^2 i g_s \eta_c \langle (b_0 c_0) V(2\pi x) \rangle_A, \quad (5.7)$$

where we used the fact that in order to get a non-zero result for the annulus amplitude we must insert b and c zero modes. If we denote by $\langle V(x) \rangle_N$ the normalized one point function of $V(x)$ on the annulus and by $Z(t)$ the partition function of the matter ghost CFT on the annulus with a $b_0 c_0$ insertion to soak up the ghost zero modes, then we can express $F(x, t)$ as

$$F(x, t) = 8\pi^2 i g_s \eta_c Z(t) \langle V(2\pi x) \rangle_N. \quad (5.8)$$

For V of the form given in (5.1), the integration over the zero mode of X^0 produces the energy conserving delta function $2\pi\delta(k^0)$. The on-shell condition now gives

$$\vec{k}^2 = 4. \quad (5.9)$$

The one point function of $e^{i\vec{k} \cdot \vec{X}}(2\pi x)$ can be evaluated by using the doubling trick that relates it to the two point function

$$-\left\langle e^{i\vec{k} \cdot \vec{X}_R(2\pi x)} e^{-i\vec{k} \cdot \vec{X}_R(-2\pi x)} \right\rangle_T \quad (5.10)$$

on a torus T :

$$w \equiv w + 2\pi \equiv w + 2\pi i t. \quad (5.11)$$

Here we have used the fact that the image of $e^{i\vec{k} \cdot \vec{X}_L(z)}$, reflected about the imaginary axis, is $-e^{-i\vec{k} \cdot \vec{X}_R(-\bar{z})}$ for a dimension one operator. This gives

$$\langle V(2\pi x) \rangle_N = -2\pi\delta(k^0) \left[\frac{2\pi\vartheta_1(2x|it)}{\vartheta_1'(0)} \right]^{-\vec{k}^2/2} = -2\pi\delta(k^0) \frac{1}{4\pi^2} \left[\frac{\vartheta_1(2x|it)}{\vartheta_1'(0)} \right]^{-2}, \quad (5.12)$$

where we used $\vec{k}^2 = 4$. We also have the standard expression for the D0-brane annulus partition function

$$Z(t) = \eta(it)^{-24} \int_{-i\infty}^{i\infty} (-i) \frac{d\omega}{2\pi} e^{2\pi t \omega^2} = \frac{1}{2\sqrt{2\pi}} t^{-1/2} \eta(it)^{-24}, \quad (5.13)$$

where the $-i$ and the range of integration over ω has the same explanation as the one given below (4.2), namely that it expresses the integral over Euclidean energy in the Lorentzian notation. Combining (5.12) and (5.13), (5.8) becomes

$$F(x, t) = 2\pi\delta(k^0) \frac{g_s \eta_c'}{\sqrt{2\pi}} t^{-1/2} \eta(it)^{-24} \left[\frac{\vartheta_1(2x|it)}{\vartheta_1'(0|it)} \right]^{-2}, \quad \eta_c' = -i\eta_c = \frac{1}{2\pi}. \quad (5.14)$$

Using (1.7) and (1.8), we see that the integral over $F(x, t)$ has divergences from the $x \rightarrow 0$ and / or $t \rightarrow \infty$ limit. Below we shall describe how to treat these divergences using Feynman diagrams with open string propagators.

5.2 Feynman diagram analysis with tachyon propagator

We begin by comparing the leading divergence in (5.14) in the $x \rightarrow 0, t \rightarrow \infty$ limit with the divergences coming from figure 1(a). For this we need to compute the leading divergent part of figure 1(a) which comes from the open string tachyon propagating along both propagators. This has four constituents:

1. The open-closed interaction vertex which, in the notation of [16], is given by

$$\{c\bar{c}e^{ik.X}; c\}_{g=0, b=1}, \quad (5.15)$$

where we have set the energy carried by the open string along propagator 1 to 0 using energy conservation. Here $g = 0, b = 1$ implies that this involves an amplitude on a Riemann surface of genus 0 with one boundary. Using (A.4) and taking into account the extra minus sign mentioned below (A.4), this can be expressed as an upper half plane correlation function

$$-g_s^{1/2}(\eta_c)^{1/4} \langle c\bar{c}e^{ik.X}(i) f_0 \circ c(0) \rangle_{UHP}, \quad (5.16)$$

where f_0 is the conformal transformation that appears in the definition of the open-closed interaction vertex and $f_0 \circ c$ is the conformal transform of c under f_0 . We use (A.5) to get the relation between the upper half plane coordinate z and the local coordinate w_o at the open string puncture:

$$z = f_0(w_o), \quad f_0(w_o) = w_o/\lambda. \quad (5.17)$$

This gives

$$f_0 \circ c(0) = \lambda c(0). \quad (5.18)$$

The correlation function appearing in (5.16) has to be calculated using the normalization given in (2.10). This gives

$$\langle c(z_1)c(z_2)c(z_3)e^{ik.X}(z) \rangle_{UHP} = -K(z_1 - z_2)(z_2 - z_3)(z_1 - z_3) 2\pi\delta(k^0)(z - \bar{z})^{-\vec{k}^2/2}. \quad (5.19)$$

Using the doubling trick we now replace $\bar{c}(i)$ by $c(-i)$ in (5.16) and use $\vec{k}^2 = 4$ to get

$$-g_s^{1/2}(\eta_c)^{1/4} \langle c\bar{c}e^{ik.X}(i) f_0 \circ c(0) \rangle_{UHP} = -K 2\pi\delta(k^0) \frac{i}{2} \lambda g_s^{1/2}(\eta_c)^{1/4}. \quad (5.20)$$

2. The interaction vertex of three open string tachyons is given by

$$\begin{aligned} \{; ce^{i\omega_1 X^0}, ce^{i\omega_2 X^0}, ce^{i\omega_3 X^0}\}_{g=0, b=1} = g_s^{1/2} \eta_c^{3/4} \Big[& \langle f_1 \circ ce^{i\omega_1 X^0(0)} f_2 \circ ce^{i\omega_2 X^0(0)} f_3 \circ ce^{i\omega_3 X^0(0)} \rangle \\ & + \langle f_1 \circ ce^{i\omega_2 X^0(0)} f_2 \circ ce^{i\omega_1 X^0(0)} f_3 \circ ce^{i\omega_3 X^0(0)} \rangle \Big], \end{aligned} \quad (5.21)$$

where f_1, f_2, f_3 are given in (A.6):

$$w_1 \equiv f_1^{-1}(z) = \alpha \frac{2z}{2-z}, \quad w_2 \equiv f_2^{-1}(z) = -2\alpha \frac{1-z}{1+z}, \quad w_3 \equiv f_3^{-1}(z) = \alpha \frac{2}{1-2z}. \quad (5.22)$$

This gives

$$\begin{aligned} f_1(w_1) &= \alpha^{-1}w_1 - \frac{1}{2}\alpha^{-2}w_1^2 + \mathcal{O}(w_1^3), & f_2(w_2) &= 1 + \alpha^{-1}w_2 + \frac{1}{2}\alpha^{-2}w_2^2 + \mathcal{O}(w_2^3), \\ f_3(w_3) &= -\alpha w_3^{-1} + \frac{1}{2}, \end{aligned} \quad (5.23)$$

and

$$\{; ce^{i\omega_1 X^0}, ce^{i\omega_2 X^0}, ce^{i\omega_3 X^0}\}_{g=0, b=1} = 2\alpha^{3+\omega_1^2+\omega_2^2+\omega_3^2} g_s^{1/2} \eta_c^{3/4} K 2\pi\delta(\omega_1 + \omega_2 + \omega_3). \quad (5.24)$$

3. The tachyon propagator is obtained from the kinetic term of the tachyon

$$\frac{K}{2} \int \frac{d\omega}{2\pi} (\omega^2 + 1) T(-\omega) T(\omega). \quad (5.25)$$

This gives the propagator

$$-K^{-1}(\omega^2 + 1)^{-1} = K^{-1} \int_0^1 \frac{dq}{q} q^{-\omega^2-1}. \quad (5.26)$$

Putting all these results together and taking into account a symmetry factor of 1/2 in the loop in the Feynman diagram figure 1(a) we get the following expression for the contribution

$$-\int_0^1 dq_1 \int_0^1 dq_2 \int_{-i\infty}^{i\infty} (-i) \frac{d\omega'}{2\pi} q_1^{-2} q_2^{-2-\omega'^2} \frac{1}{2} \times 2\lambda \alpha^{3+2\omega'^2} g_s \eta_c \frac{i}{2} 2\pi\delta(k^0) \quad (5.27)$$

We now use the leading order relation between q_1, q_2 and x, t given in (A.12) in the limit of large α and λ :

$$e^{-2\pi t} \simeq \frac{q_2}{\alpha^2}, \quad 2\pi x = \frac{q_1}{\tilde{\lambda}}, \quad \tilde{\lambda} \equiv \alpha\lambda, \quad (5.28)$$

to express (5.27) as

$$\begin{aligned} & -2\pi\delta(k^0) \times \frac{i}{2} g_s \eta'_c \int dt \int dx \int_{-i\infty}^{i\infty} \frac{d\omega'}{2\pi} e^{2\pi t} x^{-2} e^{2\pi t \omega'^2} \\ &= \frac{1}{4\sqrt{2\pi}} 2\pi\delta(k^0) \times g_s \eta'_c \int dt \int dx e^{2\pi t} x^{-2} t^{-1/2}, \end{aligned} \quad (5.29)$$

where we performed the gaussian integral like in (5.13). This agrees with the divergent part of (5.14) in the limit of small x and large t . As shown in [19], the range $0 \leq q_1, q_2 \leq 1$ covers the region (a) shown in figure 2 in the (x, t) plane.

The integrals (5.27) and (5.29) are of course both divergent, but the correct prescription to deal with these divergences is to replace the integrals over q_1 and q_2 by the left hand side of (5.26). This gives the net contribution from the Feynman diagrams of figure 1(a) to be:

$$I_{(a)} = -2\pi\delta(k^0) \times \frac{i}{2} g_s \eta_c \times \int_{-i\infty}^{i\infty} (-i) \frac{d\omega'}{2\pi} \tilde{\lambda} \alpha^{2+2\omega'^2} \times (\omega'^2 + 1 + i\epsilon)^{-1}. \quad (5.30)$$

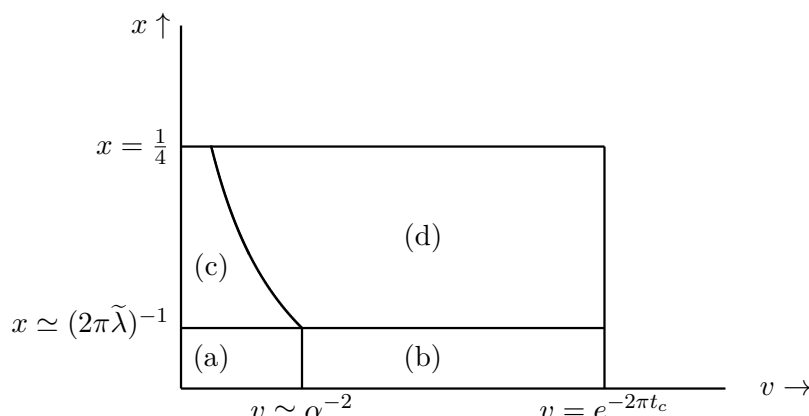


Figure 2. The moduli space regions associated with the Feynman diagrams shown in figure 1(a), (b), (c) and (d), as described in eqs. (A.12)–(A.18) in appendix A. The boundaries between different regions shown here are approximate. A more precise description of these boundaries that we use in our analysis can be found in appendix A. This figure is a reproduction of figure 8 of [19] with minor changes.

Note that we have included the $i\epsilon$ term in the open string propagator. In the $\epsilon \rightarrow 0$ limit there is a pole on the imaginary ω' axis and we need the $i\epsilon$ prescription for integrating around the singularity. As was already mentioned earlier, we have chosen the same $i\epsilon$ prescription that is used for positive mass² states. This also agrees with Witten’s $i\epsilon$ prescription [18] that effectively adds $-i\epsilon$ to L_0 . This ambiguity will be absent in a tachyon free theory like type IIA or IIB string theory. Note also that under Wick rotation $\omega' \rightarrow i\omega'_E$, $\alpha^{3+2\omega'^2}$ becomes $\alpha^{3-2\omega'^2_E}$ and produces exponential suppression in the integral for large ω'_E .

Regarding the one point function (5.30) as coming from the third term in (3.6), we see that this corresponds to a contribution to $\mathcal{F}(i\vec{k})$ of the form

$$\mathcal{F}_{(a)} = -\frac{i}{2} g_s \eta'_c \times \int_{-i\infty}^{i\infty} \frac{d\omega'}{2\pi} \tilde{\lambda} \alpha^{2+2\omega'^2} \times (\omega'^2 + 1 + i\epsilon)^{-1}. \quad (5.31)$$

After performing the integration over ω' , $\mathcal{F}_{(a)}$ can be expressed as

$$\mathcal{F}_{(a)} = \frac{1}{4} g_s \eta'_c \tilde{\lambda} \left\{ -i + \operatorname{erfi} \left(\sqrt{2 \ln \alpha} \right) \right\}, \quad (5.32)$$

where $\operatorname{erfi}(z)$ is the imaginary error function, defined as

$$\operatorname{erfi}(z) = -\frac{2}{\sqrt{\pi}} i \int_0^{iz} e^{-u^2} du. \quad (5.33)$$

In the analysis described above, we have only included the tachyonic contribution but have not considered possible contribution from massless and massive states. As already discussed in section 2, the contribution from the massless states need to be treated separately and they will not be treated using Feynman diagrams. However, there are also contribution from massive states. A massive state propagating along propagator 1 will be accompanied by a factor of $\tilde{\lambda}^{-1}$ and a massive state propagating along propagator 2 will be accompanied by a factor of α^{-2} coming from the interaction vertices. For this reason, if we drop terms carrying

inverse powers of $\tilde{\lambda}$ or α , then the massive state contribution can be ignored. In [19], where analytic expressions for the corresponding expressions were known, this was used to avoid computing the contribution from the massive modes, since one could systematically drop all terms containing inverse powers of α or $\tilde{\lambda}$. But when analytic expressions are not known and the result is computed numerically, this procedure can miss contributions with positive power of α and negative power of $\tilde{\lambda}$ or vice versa, which could give significant contribution even in the limit of large α and $\tilde{\lambda}$. This suggests that we must also include the massive state contribution to the various Feynman diagrams. This however will not solve the problem since even for determining the regions of the moduli space covered by various Feynman diagrams, [19] used an approximation in which terms containing inverse powers of α and / or $\tilde{\lambda}$ were dropped. We use the same approximation in our analysis. To compensate for this, in section 6, we shall use an indirect method for determining all such missing terms at one go. For this reason we shall not separately discuss the computation of the contribution due to massive open string states here.

Next we shall consider the contribution from the Feynman diagram shown in figure 1(c). This has two components.

1. The closed-open-open three point vertex, with the external open strings both tachyonic, will be given in the notation of [16] by,

$$\{c\bar{c}e^{ik.X}; ce^{i\omega'X^0}, ce^{-i\omega'X^0}\}_{g=0,b=1}. \quad (5.34)$$

Following (A.3), (A.17) and the sign conventions mentioned below (A.4), this is given by:

$$-2g_s\eta_c \int_{1/(2\tilde{\lambda})}^1 d\beta \left\langle c\bar{c}e^{ik.X}(i) \left(\oint_{-\beta} - \oint_{\beta} \right) dz b(z) F_1 \circ ce^{i\omega'X^0}(0) F_2 \circ ce^{-i\omega'X^0}(0) \right\rangle, \quad (5.35)$$

where the factor of 2 accounts for an equal contribution coming from the range $-1 \leq \beta \leq -1/2\tilde{\lambda}$ and, from (A.9),

$$F_1(w_1) = -\beta + \frac{4\tilde{\lambda}^2}{4\tilde{\lambda}^2 + 1} \frac{1 + \beta^2}{\alpha\tilde{\lambda}} w_1 + \mathcal{O}(w_1^2), \quad F_2(w_2) = \beta + \frac{4\tilde{\lambda}^2}{4\tilde{\lambda}^2 + 1} \frac{1 + \beta^2}{\alpha\tilde{\lambda}} w_2 + \mathcal{O}(w_2^2). \quad (5.36)$$

An explanation of the overall sign in (5.35) can be given as follows. The $(\oint_{-\beta} - \oint_{\beta})$ factor, with the contour integrals in the anti-clockwise direction and F_a 's given as in (5.36), can be identified as $\mathcal{B}_\beta$ defined in (A.2). This is inserted to the left of the open string vertex operator at $F_1(0)$. According to the rules described below (A.4), we are supposed to insert $-\mathcal{B}_\beta$ and integrate β along the direction such that the vertex operator moves along the boundary with the world-sheet kept to the *left*. In this case, as we increase β , the vertex operator at $F_1(0) = -\beta$ moves along the real axis to the left, keeping the world-sheet to the *right*. This gives an extra minus sign that converts $-\mathcal{B}_\beta$ to $\mathcal{B}_\beta$. Once we strip off the factor of $\mathcal{B}_\beta$ and the open string vertex operator at $-\beta$, we are left with an open-closed two point function on the disk, which has an extra

minus sign according to the rules given below (A.4). This explains the overall minus sign in (5.35).

After carrying out the contour integration picking up residues at $\pm\beta$, (5.35) can be expressed as

$$-2g_s\eta_c \int_{1/(2\tilde{\lambda})}^1 d\beta \left\{ \frac{\alpha^2\tilde{\lambda}^2}{(1+\beta^2)^2} \left(1 + \frac{1}{4\tilde{\lambda}^2}\right)^2 \right\}^{1+\omega'^2} \left\langle c\bar{c}e^{ik\cdot X}(i) \left(c e^{i\omega'X^0}(-\beta)e^{-i\omega'X^0}(\beta) + e^{i\omega'X^0}(-\beta) c e^{-i\omega'X^0}(\beta) \right) \right\rangle_{UHP}. \quad (5.37)$$

After evaluating the correlation function and using $\vec{k}^2 = 4$, we get

$$-2\pi\delta(k^0) 2i g_s \eta_c K \left\{ \alpha^2\tilde{\lambda}^2 \left(1 + \frac{1}{4\tilde{\lambda}^2}\right)^2 \right\}^{1+\omega'^2} \int_{1/(2\tilde{\lambda})}^1 \frac{d\beta}{(1+\beta^2)^{1+2\omega'^2}} (2\beta)^{2\omega'^2}. \quad (5.38)$$

2. The contribution from the tachyon propagator is given by (5.26).

Combining these results and taking into account an extra factor of $1/2$ that arises from the symmetry factor in the open string loop and the $-i$ that accompanies integration measure over ω' , we get the net contribution from the Feynman diagram of figure 1(c):

$$-2\pi\delta(k^0)g_s\eta'_c i \int \frac{d\omega'}{2\pi} \left\{ \alpha^2\tilde{\lambda}^2 \left(1 + \frac{1}{4\tilde{\lambda}^2}\right)^2 \right\}^{1+\omega'^2} \int_{1/(2\tilde{\lambda})}^1 \frac{d\beta}{(1+\beta^2)^{1+2\omega'^2}} (2\beta)^{2\omega'^2} \int_0^1 \frac{dq_2}{q_2} q_2^{-\omega'^2-1}. \quad (5.39)$$

Using the result in (A.13) in the limit of small q_2 ,

$$\beta = \tan(\pi x), \quad e^{-2\pi t} = q_2 \frac{(1+\beta^2)^2}{4\beta^2\alpha^2\tilde{\lambda}^2} \left(1 + \frac{1}{4\tilde{\lambda}^2}\right)^{-2}, \quad (5.40)$$

we can express the leading term in (5.39) in the small q_2 limit as,

$$\begin{aligned} & -2\pi\delta(k^0) g_s \eta'_c i \pi \times 2\pi \int dx \int dt \int \frac{d\omega'}{2\pi} e^{2\pi t(1+\omega'^2)} \frac{1}{\sin^2(2\pi x)} \\ & = 2\pi\delta(k^0) g_s \eta'_c \frac{\pi}{\sqrt{2}} \int dx \int dt \frac{1}{\sin^2(2\pi x)} t^{-1/2} e^{2\pi t}. \end{aligned} \quad (5.41)$$

The integrand agrees with the leading term in (5.14) in the large t limit. The integration range over x and t covers the region (c) in figure 2[19].

As before, the correct procedure to deal with the divergence in the large t limit is to replace the divergent integral over q_2 in (5.39) by the left hand side of (5.26). This gives:

$$I_{(c)} = 2\pi\delta(k^0)g_s\eta'_c i \int \frac{d\omega'}{2\pi} \left\{ \alpha^2\tilde{\lambda}^2 \left(1 + \frac{1}{4\tilde{\lambda}^2}\right)^2 \right\}^{1+\omega'^2} \int_{1/(2\tilde{\lambda})}^1 \frac{d\beta}{(1+\beta^2)^{1+2\omega'^2}} (2\beta)^{2\omega'^2} \frac{1}{\omega'^2 + 1 + i\epsilon}. \quad (5.42)$$

This corresponds to a contribution to $\mathcal{F}(i\vec{k})$ of the form:

$$\mathcal{F}_{(c)} = i g_s \eta'_c \int \frac{d\omega'}{2\pi} \left\{ \alpha^2\tilde{\lambda}^2 \left(1 + \frac{1}{4\tilde{\lambda}^2}\right)^2 \right\}^{1+\omega'^2} \int_{1/(2\tilde{\lambda})}^1 \frac{d\beta}{(1+\beta^2)^{1+2\omega'^2}} (2\beta)^{2\omega'^2} \frac{1}{\omega'^2 + 1 + i\epsilon}. \quad (5.43)$$

After carrying out the ω' integration, we get

$$\mathcal{F}_{(c)} = -\frac{1}{2} g_s \eta'_c \int_{1/(2\tilde{\lambda})}^1 \frac{d\beta}{4\beta^2} (1 + \beta^2) \left\{ -i + \operatorname{erfi} \left(\sqrt{2 \ln \alpha + 2 \ln \frac{4\tilde{\lambda}^2 + 1}{4\tilde{\lambda}} + 2 \ln \frac{2\beta}{1 + \beta^2}} \right) \right\}. \quad (5.44)$$

Next, we shall consider the contribution from the Feynman diagram of figure 1(b). It has three components.

1. The open-closed vertex is given by the same expressions as (5.20):

$$-\lambda g_s^{1/2} (\eta_c)^{1/4} \frac{i}{2} K 2\pi \delta(k^0). \quad (5.45)$$

2. The open string propagator is given by the same expression as (5.26) with $\omega = 0$:

$$-K^{-1} = K^{-1} \int_0^1 \frac{dq}{q} q^{-1}. \quad (5.46)$$

3. The open string one point vertex on the annulus is given by an expression similar to (5.3) with the closed string vertex operator replaced by the open string tachyon vertex operator c inserted at $x = 0$, the b and $\bar{b}$ integrals around x removed and a different normalization constant that can be read out from (A.3):⁹

$$\frac{g_s^{1/2} \eta_c^{3/4}}{2\pi i} \times (-2\pi i) \times \left\langle \left(\int_0^\pi dw b(w) + \int_0^\pi \bar{b}(\bar{w}) d\bar{w} \right) F_0 \circ c(0) \right\rangle_A. \quad (5.47)$$

Note that the last factor of 2π in (5.3) is absent since we do not integrate over x . F_0 can be read from (A.11):

$$w = F_0(w_o) = \frac{3}{2} i \alpha^{-2} - i \alpha^{-1} (1 - \alpha^{-2}) w_o + \dots, \quad (5.48)$$

where we have ignored terms of order α^{-4} . This gives

$$F_0 \circ c(0) = (F'_0(0))^{-1} c(w = 3i\alpha^{-2}/2) = i\alpha \left(1 + \alpha^{-2} \right) c(w = 3i\alpha^{-2}/2). \quad (5.49)$$

Using (5.6) we can reduce (5.47) to

$$g_s^{1/2} \eta_c^{3/4} 2\pi \alpha \left(1 + \alpha^{-2} \right) \langle b_0 c_0 \rangle_A = g_s^{1/2} \eta_c^{3/4} 2\pi \alpha \left(1 + \alpha^{-2} \right) Z(t). \quad (5.50)$$

Multiplying all the factors we get:

$$-2\pi \delta(k^0) i g_s \eta_c \pi \left(1 + \alpha^{-2} \right) \int_{t_c}^{\frac{1}{2\pi} \ln(\alpha^2 - 1/2)} dt Z(t) \tilde{\lambda} \int dq_1 q_1^{-2}, \quad (5.51)$$

where the upper limit on t follows from (A.16) and the lower limit t_c , designed to separate out the closed string tachyon contribution, will be discussed in section 5.3. We now use (A.14),

$$q_1 = 2\pi x \tilde{\lambda} (1 + \alpha^{-2}), \quad (5.52)$$

⁹As in the case of (5.3), the sign of (5.47) has been fixed by requiring that it matches the Feynman diagram contribution of figure 1(a) in the large t limit. In principle this could be fixed from first principles using the result of [16].

to express (5.51) as

$$\begin{aligned} & -2\pi\delta(k^0) i g_s \eta_c \frac{1}{2} \int_{t_c}^{\frac{1}{2\pi} \ln(\alpha^2-1/2)} dt Z(t) \int dx x^{-2} \\ & = 2\pi\delta(k^0) \frac{1}{4\sqrt{2}\pi} g_s \eta'_c \int_{t_c}^{\frac{1}{2\pi} \ln(\alpha^2-1/2)} dt \int dx x^{-2} t^{-1/2} \eta(it)^{-24}, \end{aligned} \quad (5.53)$$

where we used (5.13). This agrees with the small x behaviour of (5.14). The integration region over x and t covers the region (b) in figure 2(b) [19]

We shall remove the apparent divergence in this integral as $x \rightarrow 0$ by going back to eq. (5.51) and replacing $\int_0^1 dq_1/q_1^2$ by -1 according to (5.26). This gives the contribution from figure 1(b) to be

$$I_{(b)} = 2\pi\delta(k^0) i g_s \eta_c \pi \left(1 + \alpha^{-2}\right) \tilde{\lambda} \int_{t_c}^{\frac{1}{2\pi} \ln(\alpha^2-1/2)} dt Z(t). \quad (5.54)$$

This corresponds to a contribution of $\mathcal{F}(i\vec{k})$ of the form:

$$\mathcal{F}_{(b)} = -g_s \eta'_c \frac{1}{2\sqrt{2}} \left(1 + \alpha^{-2}\right) \tilde{\lambda} \int_{t_c}^{\frac{1}{2\pi} \ln(\alpha^2-1/2)} dt t^{-1/2} \eta(it)^{-24}, \quad (5.55)$$

where we used (5.13).

Finally, the contribution from figure 1(d) can now be expressed as:

$$I_{(d)} = \int_{R'_{(d)}} F(x, t) dx dt, \quad (5.56)$$

where $R'_{(d)}$ is the region described in (A.18) together with the replacement of the lower limit on t by t_c .

$$\begin{aligned} R'_{(d)} : \frac{\pi}{2} \geq 2\pi x \geq \tilde{\lambda}^{-1}(1 - \alpha^{-2}), \\ \frac{1}{\alpha^2 \tilde{\lambda}^2 \sin^2(2\pi x)} \left(1 + \frac{1}{4\tilde{\lambda}^2}\right)^{-2} \left[1 + 2 \left\{ \cot^2(2\pi x) - \tilde{\lambda}^2 f^2 \right\} \alpha^{-2} \tilde{\lambda}^{-2} \left(1 + \frac{1}{4\tilde{\lambda}^2}\right)^{-2}\right]^{-1} \leq v \leq e^{-2\pi t_c}, \\ f \equiv f(\tan(\pi x)), \quad v \equiv e^{-2\pi t}. \end{aligned} \quad (5.57)$$

We have slightly changed the form of the lower bound on v by dropping some terms containing inverse powers of α in the expression for v^{-1} , since we have been dropping these terms anyway. Since in $R_{(d)}$, x has a lower cut-off and t has an upper cut-off and a lower cut-off, there are no divergences from the small x and / or large t region. Using (5.14) we see that the corresponding contribution to $\mathcal{F}(i\vec{k})$ is:

$$\mathcal{F}_{(d)} = \frac{g_s \eta'_c}{\sqrt{2}\pi} \int_{R'_{(d)}} dx dt t^{-1/2} \eta(it)^{-24} \left[\frac{\vartheta_1(2x|it)}{\vartheta'_1(0|it)} \right]^{-2}. \quad (5.58)$$

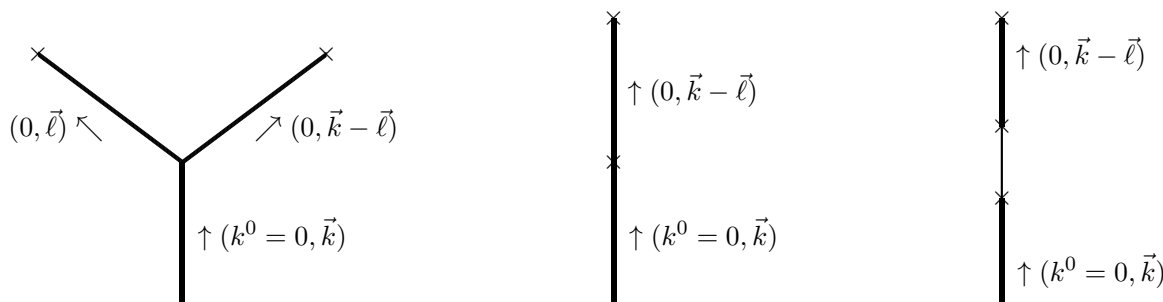


Figure 3. This figure shows the Feynman diagram involving internal closed string propagators for the contribution to the annulus one point function from the small t region. The $\times$'s denote interaction vertices associated with disk amplitudes, the thick lines represent closed strings and thin lines represent open strings. The three point interaction vertex in the leftmost diagram is the sphere three point function of three closed strings.

5.3 Divergences from the closed string channel

We now turn to the divergences arising from the $t = 0$ end of the integral. Using the modular transformation property

$$\eta(it) = t^{-1/2} \eta(i/t), \quad (5.59)$$

$$\vartheta_1(2x|it) = i t^{-1/2} \exp[-4\pi x^2/t] \vartheta_1(2x/(it)|i/t), \quad (5.60)$$

we can see that the integral indeed diverges in the $t \rightarrow 0$ limit. This has been shown explicitly in appendix B. These divergences arise from Feynman diagrams in open-closed SFT containing closed string tachyons. These have been shown in figure 3. One could carefully evaluate the contribution from the Feynman diagrams in figure 3 after defining the interaction vertices for off-shell closed and open strings as in the case of Feynman diagrams of figure 1. However, since the closed string tachyons carry momenta $\vec{\ell}$ that need to be integrated over, we can use a shortcut based on Witten's $i\epsilon$ prescription [18] that is known to be equivalent to SFT evaluation of the Feynman diagrams [22]. For the current problem this amounts to changing variables from t to $s = 1/t$ so that the divergences arise from the $s \rightarrow \infty$ limit and then changing the upper limit of integration on s to $\Lambda + i\infty$ for some large real number Λ instead of ∞ . This translates to taking $t_c = (\Lambda + i\Lambda')^{-1}$ and then taking $\Lambda' \rightarrow \infty$ limit. We shall verify in appendix B that this renders the integrals finite.

5.4 Contribution from $\tilde{p}, \tilde{a}_0, q$

In evaluating the Feynman diagrams in section 5.2 we left out the contribution from propagators of massless states y^i , a_0 and p, q following the analysis of section 3. However, the same analysis tells us that we have to include the contribution of the fields $\tilde{p}, \tilde{a}_0, q$ in the Feynman diagrams and treat the integration over y^i by relating it to the collective coordinates. In this section we shall evaluate the contribution from the Feynman diagrams shown in figure 1 when one of the internal propagators is either $\tilde{p}$ or the $\tilde{a}_0, q$ pair.

First we shall show that the propagator 1 cannot be either $\tilde{p}$ or the $\tilde{a}_0, q$ pair. For the $\tilde{a}_0, q$ pair it follows just from ghost number conservation since only ghost number 1 states can

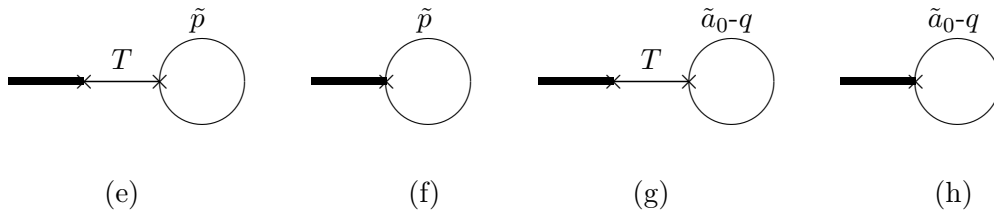


Figure 4. Additional Feynman diagrams contributing to the annulus one point amplitude. Here T denotes open string tachyon propagator.

propagate along this propagator and $\tilde{a}_0$ and q have ghost numbers 2 and 0 respectively.¹⁰ For the state $\tilde{p}$ the argument is a bit more subtle and was given in [19]. The essence of the argument is that for the open closed vertex the local coordinate w_o at the open string puncture is related to global coordinate z in the upper half plane by the simple relation $w_o = \lambda z$ (see (A.5)) and hence the vertex operator ∂c for $\tilde{p}$ takes the same form in w_o and z coordinates. Thus the evaluation of the open-closed interaction vertex involves the ghost correlator $\langle \partial c(0) c \bar{c}(i) \rangle$ on the upper half plane. It is easy to see that this vanishes.¹¹ Therefore we only have to analyze the Feynman diagrams shown in figure 4.

We begin by evaluating figure 4(e). This follows the same procedure as for (1)(a) in eqs. (5.15)–(5.30) with two differences. First the three point vertex in (5.21) is replaced by

$$\begin{aligned} & \{; i\partial c e^{i\omega_1 X^0}, i\partial c e^{i\omega_2 X^0}, c e^{i\omega_3 X^0} \}_{g=0, b=1} \\ &= -g_s^{1/2} \eta_c^{3/4} \left[\langle f_1 \circ \partial c e^{i\omega_1 X^0(0)} f_2 \circ \partial c e^{i\omega_2 X^0(0)} f_3 \circ c e^{i\omega_3 X^0(0)} \rangle \right. \\ & \quad \left. + \langle f_1 \circ \partial c e^{i\omega_2 X^0(0)} f_2 \circ \partial c e^{i\omega_1 X^0(0)} f_3 \circ c e^{i\omega_3 X^0(0)} \rangle \right]. \end{aligned} \quad (5.61)$$

Using

$$f \circ \partial c(w) = \partial c(f(w)) - \frac{f''(w)}{(f'(w))^2} c(f(w)), \quad (5.62)$$

and (5.23), we get

$$\{; i\partial c e^{i\omega_1 X^0}, i\partial c e^{i\omega_2 X^0}, c e^{i\omega_3 X^0} \}_{g=0, b=1} = -2 \alpha^{1+\omega_1^2+\omega_2^2+\omega_3^2} g_s^{1/2} \eta_c^{3/4} K 2\pi \delta(\omega_1 + \omega_2 + \omega_3). \quad (5.63)$$

The second difference arises from the tachyon propagator $-K^{-1}(\omega^2 + 1)^{-1}$ in (5.26) being replaced by $\tilde{p}$ propagator $1/(2K)$ following from (2.20). As a result, the contribution from figure 4(e) is obtained by multiplying the integrand in (5.30) by a factor of $\alpha^{-2}(\omega'^2 + 1)/2$. This gives the contribution from this diagram to be

$$I_{(e)} = -2\pi \delta(k^0) \times \frac{i}{4} g_s \eta_c \times \int_{-\infty}^{\infty} (-i) \frac{d\omega'}{2\pi} \tilde{\lambda} \alpha^{2\omega'^2}. \quad (5.64)$$

¹⁰A ghost number 1 state is needed, since the open-closed disc vertex requires a total ghost number equal to 3, the closed string vertex has ghost number 2 and there are no B insertions.

¹¹As demonstrated in [19], if we had chosen a different local coordinate at the open string puncture of the closed-open interaction vertex then $\partial c(w_o)$, expressed in the z coordinate, will have a term proportional to $c(0)$ and the contribution will not vanish.

The integration over ω' is supposed to be done after the Wick rotation $\omega' \rightarrow i\omega'_E$, and gives a finite result. The corresponding contribution to $\mathcal{F}(i\vec{k})$ is:

$$\mathcal{F}_{(e)} = -\frac{i}{4} g_s \eta'_c \int_{-i\infty}^{i\infty} \frac{d\omega'}{2\pi} \tilde{\lambda} \alpha^{2\omega'^2} = \frac{1}{8} g_s \eta'_c \tilde{\lambda} \frac{1}{\sqrt{2\pi}} (\ln \alpha)^{-1/2}. \quad (5.65)$$

We now turn to the evaluation of figure 4(f). This will follow the analysis given in (5.34)–(5.42) for the evaluation of figure 1(c) with two main differences. First, the closed-open-open three point vertex, with the external open strings both tachyonic, will be replaced by,

$$\{c\bar{c}e^{ik.X}; i\partial c e^{i\omega' X^0}, i\partial c e^{-i\omega' X^0}\}_{g=0, b=1}. \quad (5.66)$$

Following (A.3), (A.17) and the sign conventions below (A.4), this is given by:

$$-2 \eta_c g_s \int_{1/(2\tilde{\lambda})}^1 d\beta \left\langle c\bar{c}e^{ik.X}(i) \sum_{a=1}^2 \oint_{F_a(0)} \frac{\partial F_a(w_a; \beta)}{\partial \beta} dz b(z) F_1 \circ \partial c e^{i\omega' X^0}(0) F_2 \circ \partial c e^{-i\omega' X^0}(0) \right\rangle. \quad (5.67)$$

Using (5.62) and (A.9), we get

$$F_1 \circ \partial c(0) = \partial c(-\beta) - \frac{h_1}{g_1^2} c(-\beta), \quad F_2 \circ \partial c(0) = \partial c(\beta) - \frac{h_2}{g_2^2} c(\beta), \quad (5.68)$$

and,

$$\frac{\partial F_1}{\partial \beta} = -1 + \frac{1}{g_1} \frac{\partial g_1}{\partial \beta} (z + \beta) + \mathcal{O}((z + \beta)^2), \quad \frac{\partial F_2}{\partial \beta} = 1 + \frac{1}{g_2} \frac{\partial g_2}{\partial \beta} (z - \beta) + \mathcal{O}((z - \beta)^2). \quad (5.69)$$

Hence

$$\begin{aligned} & \sum_{a=1}^2 \oint_{F_a(0)} \frac{\partial F_a(w_a; \beta)}{\partial \beta} dz b(z) F_1 \circ \partial c e^{i\omega' X^0}(0) F_2 \circ \partial c e^{-i\omega' X^0}(0) \\ &= \left[\left\{ \frac{h_1}{g_1^2} + \frac{1}{g_1} \frac{\partial g_1}{\partial \beta} \right\} \left\{ \partial c(\beta) - \frac{h_2}{g_2^2} c(\beta) \right\} - \left\{ -\frac{h_2}{g_2^2} + \frac{1}{g_2} \frac{\partial g_2}{\partial \beta} \right\} \left\{ \partial c(-\beta) - \frac{h_1}{g_1^2} c(-\beta) \right\} \right] \\ & (g_1 g_2)^{-\omega'^2} e^{i\omega' X^0}(-\beta) e^{-i\omega' X^0}(\beta). \end{aligned} \quad (5.70)$$

Using this and the expression for the disk correlation function, (5.67) can be evaluated to,

$$-2\pi \delta(k^0) 8 i K \eta_c g_s \tilde{\lambda}^2 \int_{1/(2\tilde{\lambda})}^1 d\beta f(\beta)^2 \frac{1}{1 + \beta^2} (2\beta)^{2\omega'^2} \left\{ \frac{\alpha^2 \tilde{\lambda}^2}{(1 + \beta^2)^2} \left(1 + \frac{1}{4\tilde{\lambda}^2} \right)^2 \right\}^{\omega'^2}. \quad (5.71)$$

The second difference arises from the tachyon propagator $-K^{-1}(\omega^2 + 1)^{-1}$ in (5.26) being replaced by $\tilde{p}$ propagator $1/(2K)$ following from (2.20). Putting all these results together and taking into account a symmetry factor of 1/2 in the loop in the Feynman diagram figure 4(f) we get the following expression for the contribution

$$I_{(f)} = -2\pi \delta(k^0) 2 i \eta'_c g_s \tilde{\lambda}^2 \int \frac{d\omega'}{2\pi} \int_{1/(2\tilde{\lambda})}^1 d\beta f(\beta)^2 \frac{1}{1 + \beta^2} (2\beta)^{2\omega'^2} \left\{ \frac{\alpha^2 \tilde{\lambda}^2}{(1 + \beta^2)^2} \left(1 + \frac{1}{4\tilde{\lambda}^2} \right)^2 \right\}^{\omega'^2}. \quad (5.72)$$

This leads to the following contribution to $\mathcal{F}(i\vec{k})$:

$$\mathcal{F}_{(f)} = -2i\eta'_c g_s \tilde{\lambda}^2 \int \frac{d\omega'}{2\pi} \int_{1/(2\tilde{\lambda})}^1 d\beta f(\beta)^2 \frac{1}{1+\beta^2} (2\beta)^{2\omega'} \left\{ \frac{\alpha^2 \tilde{\lambda}^2}{(1+\beta^2)^2} \left(1 + \frac{1}{4\tilde{\lambda}^2}\right)^2 \right\}^{\omega'}. \quad (5.73)$$

We can perform the integration over ω' after Wick rotation and arrive at the result:

$$\mathcal{F}_{(f)} = \frac{\eta'_c g_s}{\sqrt{2\pi}} \tilde{\lambda}^2 \int_{1/(2\tilde{\lambda})}^1 d\beta f(\beta)^2 \frac{1}{1+\beta^2} \frac{1}{\sqrt{\ln \alpha + \ln \frac{4\tilde{\lambda}^2+1}{4\tilde{\lambda}} + \ln \frac{2\beta}{1+\beta^2}}}. \quad (5.74)$$

Next we turn to figure 4(g). For this we note that in the expansion of the string field given in (2.17), the vertex operators for $\tilde{a}_0$ and q appear in the combination:

$$\int \frac{d\omega}{2\pi} \left[\tilde{a}_0(-\omega) i\sqrt{2} \partial c c \partial X^0 + i q(-\omega) \right] e^{i\omega X^0}. \quad (5.75)$$

The analysis follows the same procedure as for figure 1(a) in eqs. (5.15)–(5.30) with two differences. First the three point vertex in (5.21) is replaced by

$$\begin{aligned} & \{; i\sqrt{2} \partial c c \partial X^0 e^{i\omega_1 X^0}, i e^{i\omega_2 X^0}, c e^{i\omega_3 X^0} \}_{g=0, b=1} \\ &= -\sqrt{2} g_s^{1/2} \eta_c^{3/4} \left[\langle f_1 \circ \partial c c \partial X^0 e^{i\omega_1 X^0}(0) f_2 \circ e^{i\omega_2 X^0}(0) f_3 \circ c e^{i\omega_3 X^0}(0) \rangle \right. \\ & \quad \left. - \langle f_1 \circ e^{i\omega_2 X^0}(0) f_2 \circ \partial c c \partial X^0 e^{i\omega_1 X^0}(0) f_3 \circ c e^{i\omega_3 X^0}(0) \rangle \right], \end{aligned} \quad (5.76)$$

where we have defined the vertex as the coefficient of the $\tilde{a}_0(-\omega_1)q(-\omega_2)T(-\omega_3)$ term in the action. The relative minus sign between the two terms reflect that $\tilde{a}_0$ and q are grassmann odd variables. Using

$$\begin{aligned} f \circ \partial c c(0) &= f'(0)^{-1} \partial c c(f(0)), \\ f \circ \partial X^0 e^{i\omega X^0}(0) &= (f'(0))^{1-\omega^2} \left[\partial X^0 e^{i\omega X^0}(f(0)) + \frac{1}{2} i \omega \frac{f''(0)}{(f'(0))^2} e^{i\omega X^0}(f(0)) \right], \end{aligned} \quad (5.77)$$

and (5.23), we get

$$\begin{aligned} & \{; i\sqrt{2} \partial c c \partial X^0 e^{i\omega_1 X^0}, i e^{i\omega_2 X^0}, c e^{i\omega_3 X^0} \}_{g=0, b=1} \\ &= \sqrt{2} \alpha^{1+\omega_1^2+\omega_2^2+\omega_3^2} g_s^{1/2} \eta_c^{3/4} K(i\omega_1) 2\pi\delta(\omega_1 + \omega_2 + \omega_3). \end{aligned} \quad (5.78)$$

The second difference arises from the second tachyon propagator $-K^{-1}(\omega'^2 + 1)^{-1}$ in (5.26) being replaced by the $\tilde{a}_0$ - q propagator $-i\omega_1/(\sqrt{2}K(\omega_1^2 + i\epsilon))$ following from (2.20). Finally, we do not have the factor of 1/2 from the loop since $\tilde{a}_0$ and q are different fields. As a result, the contribution from figure 4(e) is obtained by multiplying the integrand in (5.30) by a factor of $-\alpha^{-2}(\omega'^2 + 1)$. This gives the contribution from this diagram to be

$$I_{(g)} = 2\pi\delta(k^0) \times \frac{i}{2} g_s \eta_c \times \int_{-i\infty}^{i\infty} (-i) \frac{d\omega'}{2\pi} \tilde{\lambda} \alpha^{2\omega'^2}. \quad (5.79)$$

The corresponding contribution to $\mathcal{F}(i\vec{k})$ is:

$$\mathcal{F}_{(g)} = \frac{i}{2} g_s \eta'_c \times \int \frac{d\omega'}{2\pi} \tilde{\lambda} \alpha^{2\omega'^2} = -\frac{1}{4} g_s \eta'_c \tilde{\lambda} \frac{1}{\sqrt{2\pi}} (\ln \alpha)^{-1/2}. \quad (5.80)$$

We now turn to the evaluation of figure 4(h). This will follow the analysis given in (5.34)–(5.42) for the evaluation of figure 1(c) with two main differences. First, the tachyon propagator $-K^{-1}(\omega'^2 + 1)^{-1}$ is replaced by the $\tilde{a}_0(-\omega')\text{-}q(\omega')$ propagator $-i\omega'/(2\sqrt{2}K(\omega'^2 + i\epsilon))$ as in the case of figure 4(g). Second, the closed-open-open three point vertex, with the external open strings both tachyonic, will be replaced by,

$$\{c\bar{c}e^{ik.X}; i\sqrt{2}\partial c c\partial X^0 e^{i\omega'X^0}, i e^{-i\omega'X^0}\}_{g=0,b=1}. \quad (5.81)$$

Following (A.3), (A.17) and the sign conventions below (A.4), this is given by:

$$\begin{aligned} & \sqrt{2}\eta_c \int_{1/(2\tilde{\lambda})}^1 d\beta \left\langle c\bar{c}e^{ik.X}(i) \sum_{a=1}^2 \oint_{F_a(0)} \frac{\partial F_a(w_a; \beta)}{\partial \beta} dz b(z) \right. \\ & \left. \left\{ F_1 \circ \partial c c \partial X^0 e^{i\omega'X^0}(0) F_2 \circ e^{-i\omega'X^0}(0) - F_1 \circ e^{-i\omega'X^0}(0) F_2 \circ \partial c c \partial X^0 e^{i\omega'X^0}(0) \right\} \right\rangle. \end{aligned} \quad (5.82)$$

We shall first evaluate the ghost part of the correlation function. Using (5.77) and (A.9), we get

$$\begin{aligned} F_1 \circ \partial c c(0) &= \left(\frac{4\tilde{\lambda}^2}{4\tilde{\lambda}^2 + 1} \frac{1 + \beta^2}{\alpha\tilde{\lambda}} \right)^{-1} \partial c c(-\beta) \\ F_2 \circ \partial c c(0) &= \left(\frac{4\tilde{\lambda}^2}{4\tilde{\lambda}^2 + 1} \frac{1 + \beta^2}{\alpha\tilde{\lambda}} \right)^{-1} \partial c c(\beta). \end{aligned} \quad (5.83)$$

We also have

$$\frac{\partial F_1}{\partial \beta} = -1 + \frac{2\beta}{1 + \beta^2}(z + \beta), \quad \frac{\partial F_2}{\partial \beta} = 1 + \frac{2\beta}{1 + \beta^2}(z - \beta). \quad (5.84)$$

Hence

$$\begin{aligned} \sum_{a=1}^2 \oint_{F_a(0)} \frac{\partial F_a(w_a; \beta)}{\partial \beta} dz b(z) F_1 \circ \partial c c(0) &= \left(\frac{4\tilde{\lambda}^2}{4\tilde{\lambda}^2 + 1} \frac{1 + \beta^2}{\alpha\tilde{\lambda}} \right)^{-1} \left[\frac{2\beta}{1 + \beta^2} c(-\beta) + \partial c(-\beta) \right] \\ \sum_{a=1}^2 \oint_{F_a(0)} \frac{\partial F_a(w_a; \beta)}{\partial \beta} dz b(z) F_2 \circ \partial c c(0) &= \left(\frac{4\tilde{\lambda}^2}{4\tilde{\lambda}^2 + 1} \frac{1 + \beta^2}{\alpha\tilde{\lambda}} \right)^{-1} \left[\frac{2\beta}{1 + \beta^2} c(\beta) - \partial c(\beta) \right] \end{aligned} \quad (5.85)$$

We now note that the relevant ghost correlators take the form:

$$\begin{aligned} \left\langle \left[\frac{2\beta}{1 + \beta^2} c(-\beta) + \partial c(-\beta) \right] c(i)c(-i) \right\rangle &\propto \left[\frac{2\beta}{1 + \beta^2} (2i)(1 + \beta^2) - 2\beta \times (2i) \right] = 0, \\ \left\langle \left[\frac{2\beta}{1 + \beta^2} c(\beta) - \partial c(\beta) \right] c(i)c(-i) \right\rangle &\propto \left[\frac{2\beta}{1 + \beta^2} (2i)(1 + \beta^2) - 2\beta \times (2i) \right] = 0. \end{aligned} \quad (5.86)$$

Therefore this contribution vanishes:

$$I_{(h)} = 0. \quad (5.87)$$

The corresponding contribution to $\mathcal{F}(i\vec{k})$ is

$$\mathcal{F}_{(h)} = 0. \quad (5.88)$$

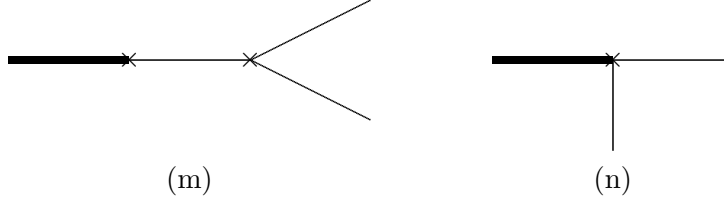


Figure 5. This figure shows two Feynman diagrams contributing to the disk amplitude with one external closed string and two external open strings.

5.5 Contribution from the Jacobian

Finally, we shall compute the contribution to the one point function of the external tachyon due to the Jacobian from change of variables, as described in (4.11).

$$\mathcal{F}_{\text{jac}} = \frac{i}{2M} \int_{-i\infty}^{i\infty} \frac{d\omega'}{2\pi} (\omega')^{-2} \left\{ \vec{k}^2 \mathcal{F}_0(i\vec{k}) + \frac{M}{K} B_{ii}^{(2)}(-\omega', \omega', \vec{k}) \right\}. \quad (5.89)$$

We begin by evaluating $\mathcal{F}_0(i\vec{k})$. This is given by the disk one point function of the closed string tachyon. Using (A.4) we get

$$2\pi \delta(k^0) \mathcal{F}_0(i\vec{k}) = \eta_c^{1/2} \langle c_0^- c \bar{c} e^{ik \cdot X} \rangle = -2\pi \delta(k^0) K \eta_c^{1/2} = \frac{1}{2} g_s M 2\pi \delta(k^0), \quad (5.90)$$

where we used $k^2 = 4$ for computing the matter sector correlator and used (2.11) in the last step.

Next, we note that $2\pi \delta(k^0) B_{ij}^{(2)}(-\omega', \omega', \vec{k})$ is the disk amplitude of the closed string tachyon with vertex operator $c\bar{c}e^{ik \cdot X}$ and a pair of open string states y^i, y^j carrying vertex operator $i\sqrt{2}\partial X^i e^{i\omega' X}$ and $i\sqrt{2}\partial X^j e^{-i\omega' X}$. This is given by the sum of the two Feynman diagrams shown in figure 5.

We begin with figure 5(m). This has three components:

1. The open closed two point vertex with an internal tachyon is given by the same expression as (5.20):

$$-K 2\pi \delta(k^0) \frac{i}{2} \lambda g_s^{1/2} (\eta_c)^{1/4}. \quad (5.91)$$

2. The open string tachyon propagator is given by $-K^{-1}$.
3. The three open string interaction vertex of a tachyon, y^i and y^j is given by

$$\begin{aligned} & \{; i\sqrt{2}\partial X^i c e^{i\omega_1 X^0}, i\sqrt{2}\partial X^j c e^{i\omega_2 X^0}, c e^{i\omega_3 X^0} \}_{g=0, b=1} \\ &= -2g_s^{1/2} \eta_c^{3/4} \left[\langle f_1 \circ c\partial X^i e^{i\omega_1 X^0(0)} f_2 \circ c\partial X^j e^{i\omega_2 X^0(0)} f_3 \circ c e^{i\omega_3 X^0(0)} \rangle \right. \\ & \quad \left. + \langle f_1 \circ c\partial X^j e^{i\omega_2 X^0(0)} f_2 \circ c\partial X^i e^{i\omega_1 X^0(0)} f_3 \circ c e^{i\omega_3 X^0(0)} \rangle \right]. \end{aligned} \quad (5.92)$$

Setting $\omega_1 = \omega', \omega_2 = -\omega'$ and $\omega_3 = 0$, this evaluates to

$$2g_s^{1/2} \eta_c^{3/4} K \alpha^{1+2\omega'^2} \delta_{ij}. \quad (5.93)$$

Combining these contributions we get the net contribution to $2\pi\delta(k^0)B_{ij}^{(2)}(-\omega', \omega', \vec{k})$ from this diagram:

$$I_{ij}^{(m)} = i K 2\pi\delta(k^0) \tilde{\lambda} \alpha^{2\omega'^2} g_s \eta_c \delta_{ij}. \quad (5.94)$$

Next we turn to figure 5(n). This is given by an expression similar to (5.35) with the pair of external closed string tachyons replaced by the y^i and y^j :

$$\begin{aligned} I_{(n)ij} &= -2 g_s \eta_c \int_{1/(2\tilde{\lambda})}^1 d\beta \left\langle c\bar{c}e^{ik.X}(i) \left(\oint_{-\beta} - \oint_{\beta} \right) dz b(z) F_1 \circ c(i\sqrt{2}\partial X^i) e^{i\omega'X^0}(0) \right. \\ &\quad \left. F_2 \circ c(i\sqrt{2}\partial X^j) e^{-i\omega'X^0}(0) \right\rangle \\ &= 4 g_s \eta_c \int_{1/(2\tilde{\lambda})}^1 d\beta \left[\frac{4\tilde{\lambda}^2}{4\tilde{\lambda}^2 + 1} \frac{1 + \beta^2}{\alpha\tilde{\lambda}} \right]^{-2\omega'^2} \\ &\quad \left\langle c\bar{c}(i)(c(\beta) + c(-\beta))e^{ik.X}(i)\partial X^i(-\beta)\partial X^j(\beta)e^{i\omega'X^0}(-\beta)e^{-i\omega'X^0}(\beta) \right\rangle. \end{aligned} \quad (5.95)$$

After evaluating the correlation function and dropping terms suppressed by inverse powers of $\tilde{\lambda}$, we get the contribution to $2\pi\delta(k^0)B_{ij}^{(2)}(-\omega', \omega', \vec{k})$ from this diagram:

$$\begin{aligned} I_{(n)ij} &= 2\pi\delta(k^0) 4 i g_s \eta_c K \int_{1/(2\tilde{\lambda})}^1 d\beta \left[\frac{4\tilde{\lambda}^2}{4\tilde{\lambda}^2 + 1} \frac{1 + \beta^2}{2\beta\alpha\tilde{\lambda}} \right]^{-2\omega'^2} \left[-\frac{1}{8\beta^2} \delta_{ij} - \frac{1}{8} \delta_{ij} + \frac{1}{1 + \beta^2} k_i k_j \right] \\ &\quad + 2\pi\delta(k^0) 4 i g_s \eta_c K \alpha^{2\omega'^2} \left[-\frac{1}{16\tilde{\lambda}} \delta_{ij} + k_i k_j \tan^{-1} \frac{1}{2\tilde{\lambda}} \right], \end{aligned} \quad (5.96)$$

where the terms in the last line have been added for later convenience. We are allowed to add these terms since they carry negative power of $\tilde{\lambda}$ and all our formulae so far allows dropping / adding such terms. For example, an exchange of massive states would add to (5.94) terms suppressed by powers of $\tilde{\lambda}$ that we have not been careful to keep.

Substituting these results into (5.89), we get

$$\begin{aligned} \mathcal{F}_{\text{jac}} &= \frac{i}{2M} \int \frac{d\omega'}{2\pi} (\omega')^{-2} \left[\frac{1}{2} g_s M \vec{k}^2 + 25 i M \tilde{\lambda} \alpha^{2\omega'^2} g_s \eta_c \right. \\ &\quad + 4 i g_s \eta_c M \int_{1/(2\tilde{\lambda})}^1 d\beta \left\{ \frac{4\tilde{\lambda}^2}{4\tilde{\lambda}^2 + 1} \frac{1 + \beta^2}{2\beta\alpha\tilde{\lambda}} \right\}^{-2\omega'^2} \left\{ -\frac{25}{8\beta^2} - \frac{25}{8} + \frac{1}{1 + \beta^2} \vec{k}^2 \right\} \\ &\quad \left. + 4 i g_s \eta_c M \alpha^{2\omega'^2} \left\{ -\frac{25}{16\tilde{\lambda}} + \vec{k}^2 \tan^{-1} \frac{1}{2\tilde{\lambda}} \right\} \right]. \end{aligned} \quad (5.97)$$

One can easily verify that the term inside the square bracket vanishes as $\omega' \rightarrow 0$ and hence the integral does not suffer from any infrared divergence. The $\tilde{\lambda}^{-1}$ suppressed terms in the last line are important for this cancellation.

One can make the infrared finiteness of $\mathcal{F}_{\text{jac}}$ manifest as follows. We first perform the β integrals by parts to write

$$\begin{aligned} \mathcal{F}_{\text{jac}} = & \frac{i}{2M} \int \frac{d\omega'}{2\pi} (\omega')^{-2} \left[\frac{1}{2} g_s M \vec{k}^2 \right. \\ & - 4 i g_s \eta_c M 2 \omega'^2 \int_{1/(2\tilde{\lambda})}^1 d\beta \left\{ \frac{25}{8\beta} - \frac{25}{8} \beta + \vec{k}^2 \tan^{-1} \beta \right\} \\ & \times \left(\frac{1}{\beta} - \frac{2\beta}{1+\beta^2} \right) \left\{ \frac{4\tilde{\lambda}^2}{4\tilde{\lambda}^2+1} \frac{1+\beta^2}{2\beta\alpha\tilde{\lambda}} \right\}^{-2\omega'^2} \\ & \left. + i \pi \vec{k}^2 g_s \eta_c M \left(\frac{4\tilde{\lambda}^2}{4\tilde{\lambda}^2+1} \frac{1}{\alpha\tilde{\lambda}} \right)^{-2\omega'^2} \right]. \end{aligned} \quad (5.98)$$

After using $\eta_c = i/(2\pi)$, we can write this as

$$\begin{aligned} \mathcal{F}_{\text{jac}} = & \frac{i}{2M} \int \frac{d\omega'}{2\pi} \left[- 8 i g_s \eta_c M \int_{1/(2\tilde{\lambda})}^1 d\beta \left\{ \frac{25}{8\beta} - \frac{25}{8} \beta + \vec{k}^2 \tan^{-1} \beta \right\} \right. \\ & \times \left(\frac{1}{\beta} - \frac{2\beta}{1+\beta^2} \right) \left\{ \frac{4\tilde{\lambda}^2}{4\tilde{\lambda}^2+1} \frac{1+\beta^2}{2\beta\alpha\tilde{\lambda}} \right\}^{-2\omega'^2} \\ & \left. + i \pi \vec{k}^2 g_s \eta_c M \frac{1}{\omega'^2} \left\{ \left(\frac{4\tilde{\lambda}^2}{4\tilde{\lambda}^2+1} \frac{1}{\alpha\tilde{\lambda}} \right)^{-2\omega'^2} - 1 \right\} \right]. \end{aligned} \quad (5.99)$$

The last term can be manipulated by writing $1/\omega'^2$ as $-d\omega'^{-1}/d\omega'$ and integrating by parts. This is an allowed operation since the integral had no divergence from the $\omega' = 0$ region to start with. This gives

$$\begin{aligned} \mathcal{F}_{\text{jac}} = & \frac{i}{2M} \int \frac{d\omega'}{2\pi} \left[- 8 i g_s \eta_c M \int_{1/(2\tilde{\lambda})}^1 d\beta \left\{ \frac{25}{8\beta} - \frac{25}{8} \beta + \vec{k}^2 \tan^{-1} \beta \right\} \right. \\ & \times \left(\frac{1}{\beta} - \frac{2\beta}{1+\beta^2} \right) \left\{ \frac{4\tilde{\lambda}^2}{4\tilde{\lambda}^2+1} \frac{1+\beta^2}{2\beta\alpha\tilde{\lambda}} \right\}^{-2\omega'^2} \\ & \left. + 4 i \pi \vec{k}^2 g_s \eta_c M \left(\ln \alpha + \ln \frac{4\tilde{\lambda}^2+1}{4\tilde{\lambda}} \right) \left(\frac{4\tilde{\lambda}^2}{4\tilde{\lambda}^2+1} \frac{1}{\alpha\tilde{\lambda}} \right)^{-2\omega'^2} \right]. \end{aligned} \quad (5.100)$$

We can now carry out the ω' integration explicitly after Euclidean continuation, leading to,

$$\begin{aligned} \mathcal{F}_{\text{jac}} = & -\frac{i}{\sqrt{2\pi}} \left[- 2 i g_s \eta'_c \int_{1/(2\tilde{\lambda})}^1 d\beta \left(\ln \alpha + \ln \frac{4\tilde{\lambda}^2+1}{4\tilde{\lambda}} + \ln \frac{2\beta}{1+\beta^2} \right)^{-1/2} \right. \\ & \times \left\{ \frac{25}{8\beta} - \frac{25}{8} \beta + \vec{k}^2 \tan^{-1} \beta \right\} \left(\frac{1}{\beta} - \frac{2\beta}{1+\beta^2} \right) \\ & \left. + i \pi \vec{k}^2 g_s \eta'_c \left(\ln \alpha + \ln \frac{4\tilde{\lambda}^2+1}{4\tilde{\lambda}} \right)^{1/2} \right]. \end{aligned} \quad (5.101)$$

6 Complete annulus contribution

Based on our results so far we conclude that the complete annulus contribution to $\mathcal{F}(ik)$, after ignoring terms with inverse powers of $\tilde{\lambda}$ and / or α , is given by:

$$\mathcal{F}' \equiv \mathcal{F}_{(a)} + \mathcal{F}_{(b)} + \mathcal{F}_{(c)} + \mathcal{F}_{(d)} + \mathcal{F}_{(e)} + \mathcal{F}_{(f)} + \mathcal{F}_{(g)} + \mathcal{F}_{(h)} + \mathcal{F}_{\text{jac}}. \quad (6.1)$$

However, in numerical computations below we need to work with some particular choice of α and $\tilde{\lambda}$. Depending on this choice, either terms with a positive power of α and a negative power of $\tilde{\lambda}$ or vice versa that we have ignored could become important. This was not a problem in the analysis of [19] since these terms are expected to cancel among themselves and hence in the final expression after evaluation of the analog of (6.1), all terms containing inverse powers of α and / or $\tilde{\lambda}$ were dropped. However, unlike in the case of [19], here we shall not have an analytic expression for the various terms from where we can explicitly drop all terms containing inverse powers of α and / or $\tilde{\lambda}$. In the numerical evaluation of various terms we have to choose some large values of α and $\tilde{\lambda}$ and in that case some missing terms in the various expressions that contain positive power of α and negative power of $\tilde{\lambda}$ or vice versa can give significant contributions. To overcome this difficulty, we shall choose a particular scaling of α and $\tilde{\lambda}$, e.g. $\alpha \sim \gamma^a$, $\tilde{\lambda} \sim \gamma^b$ for some large γ and positive constants a and b , and keep all terms in $\mathcal{F}$ that scale as non-negative power of γ . For definiteness, let us choose $a = b = 1$, i.e. take

$$\alpha \sim \tilde{\lambda} \sim \gamma, \quad (6.2)$$

with γ being a large number. We can then evaluate $\mathcal{F}$ for large γ . By construction, (6.1) does not contain all the terms in $\mathcal{F}$ that survive in this limit, e.g. terms proportional to $\alpha^2/\tilde{\lambda}$ and $\tilde{\lambda}^2/\alpha$ would be absent in (6.1) even though they scale as γ . We shall now describe a general procedure to determine these missing terms.

Let us suppose that we take the expression (6.1) using the expressions for $\mathcal{F}_{(a)}$ - $\mathcal{F}_{\text{jac}}$ as computed in the last section and compute its variation $\delta\mathcal{F}'$ under arbitrary variation of α , $\tilde{\lambda}$ and f without making any further approximation. If we take $\delta\alpha$ and $\delta\tilde{\lambda}$ to scale as α and $\tilde{\lambda}$ respectively, then all terms in $\delta\mathcal{F}'$ with non-negative powers of α and $\tilde{\lambda}$ must cancel since we have been careful to keep all such terms in our analysis. However $\delta\mathcal{F}'$ may contain terms with negative powers of α and / or $\tilde{\lambda}$ since we have dropped such terms in our analysis. So once we have computed $\delta\mathcal{F}'$, we look for a term $\mathcal{F}_{\text{cor}}$ in the form of a sum of terms, each of which carries negative power of α and / or $\tilde{\lambda}$, whose variation explicitly cancels $\delta\mathcal{F}'$. We can then add $\mathcal{F}_{\text{cor}}$ to the expression for $\mathcal{F}'$ given in (6.1) to recover $\mathcal{F}$, since the latter is expected to be independent of α , $\tilde{\lambda}$ and $f(\beta)$. This procedure is completely unambiguous since the only freedom in the choice of $\mathcal{F}_{\text{cor}}$ are additive terms independent of α , $\tilde{\lambda}$ and $f(\beta)$, but we are not allowed to add such terms since they do not contain negative power of α or $\tilde{\lambda}$.

While this gives a way to recover a complete expression for $\mathcal{F}$, below we shall describe the steps to determine terms in $\mathcal{F}_{\text{cor}}$ that scale as non-negative power of γ , since the other terms will be suppressed for large γ that we shall use in our numerical analysis. Our first step will be to compute the change $\delta\mathcal{F}'$ of the expression for $\mathcal{F}'$ under arbitrary variation of α , $\tilde{\lambda}$ and f , keeping all terms that scale as non-negative power of γ . Next we verify that $\delta\mathcal{F}'$ does not contain any term carrying non-negative power of α and $\tilde{\lambda}$. Therefore, the terms in $\delta\mathcal{F}'$ will carry negative powers of α or $\tilde{\lambda}$.

We shall then explicitly add to $\mathcal{F}'$ terms that will cancel this contribution. For example, a contribution to $\delta\mathcal{F}'$ of the form

$$\frac{\delta\tilde{\lambda}}{\tilde{\lambda}} \frac{\tilde{\lambda}^2}{\alpha^2} - \frac{\delta\alpha}{\alpha} \frac{\tilde{\lambda}^2}{\alpha^2} \quad (6.3)$$

comes from a term in $\mathcal{F}'$ of the form $\frac{1}{2}\tilde{\lambda}^2\alpha^{-2}$. Hence, we should add to $\mathcal{F}'$ a term $-\frac{1}{2}\tilde{\lambda}^2\alpha^{-2}$ to cancel this, since, if we had not made any approximation in evaluating $\mathcal{F}'$, these terms would not have been present. This procedure reduces to the procedure of explicitly removing all terms containing inverse powers of α and / or $\tilde{\lambda}$ that was used in [19]. In the end, the $\mathcal{F}$ obtained this way will be the correct form of $\mathcal{F}$ up to corrections of order γ^{-1} and we can use this for numerical computation of $\mathcal{F}$ by taking γ to be large. This corresponds to taking α and $\tilde{\lambda}$ large and of the same order. We could in principle find the exact expression for $\mathcal{F}$ by adding to (6.1) a term that makes $\delta\mathcal{F}$ vanish exactly, but we have not done so.

Computation of $\delta\mathcal{F}'$ is tedious but straightforward after using the results for $\mathcal{F}_{(a)}\text{-}\mathcal{F}_{\text{jac}}$ given in section 5 and the result of ω integration given in (5.13). With the help of some integration by parts and change of variable from x to $\beta \equiv \tan(\pi x)$ in the expression for $\mathcal{F}_{(d)}$ in (5.58), we get,

$$\begin{aligned} \delta\mathcal{F}' = & -\frac{1}{\sqrt{2\pi}} g_s \eta'_c \frac{\tilde{\lambda}^2}{\alpha^2} \int_{1/(2\tilde{\lambda})}^1 \frac{d\beta}{1+\beta^2} f(\beta)^3 \delta f(\beta) \left(\ln \alpha + \ln \tilde{\lambda} + \ln \frac{2\beta}{1+\beta^2} \right)^{-3/2} \\ & - g_s \eta'_c \frac{1}{4\sqrt{2\pi}} \tilde{\lambda} (\ln \alpha)^{-3/2} \alpha^{-1} \delta\alpha \\ & + g_s \eta'_c \sqrt{\frac{1}{2\pi}} \frac{\delta\alpha}{\alpha} \int_{1/(2\tilde{\lambda})}^1 \frac{d\beta}{1+\beta^2} \left[\ln \alpha + \ln \frac{4\tilde{\lambda}^2+1}{4\tilde{\lambda}} + \ln \frac{2\beta}{1+\beta^2} \right]^{-1/2} \\ & \times \left[\frac{1}{2} \alpha^{-2} \tilde{\lambda}^2 f(\beta)^4 \left(\ln \alpha + \ln \frac{4\tilde{\lambda}^2+1}{4\tilde{\lambda}} + \ln \frac{2\beta}{1+\beta^2} \right)^{-1} \right. \\ & \left. + \frac{3}{8} \alpha^{-2} \tilde{\lambda}^2 f(\beta)^4 \left(\ln \alpha + \ln \frac{4\tilde{\lambda}^2+1}{4\tilde{\lambda}} + \ln \frac{2\beta}{1+\beta^2} \right)^{-2} \right] \\ & - g_s \eta'_c \frac{1}{24\sqrt{2\pi}} \alpha \frac{\delta\alpha}{\tilde{\lambda}} (\ln \alpha)^{-1/2} \\ & + g_s \eta'_c \sqrt{\frac{1}{2\pi}} \frac{\delta\tilde{\lambda}}{\tilde{\lambda}} \tilde{\lambda}^2 \alpha^{-2} \int_{1/(2\tilde{\lambda})}^1 \frac{d\beta}{1+\beta^2} \left[\ln \alpha + \ln \frac{4\tilde{\lambda}^2+1}{4\tilde{\lambda}} + \ln \frac{2\beta}{1+\beta^2} \right]^{-3/2} f(\beta)^4 \\ & \times \left[-\frac{1}{2} + \frac{3}{8} \left\{ \ln \alpha + \ln \frac{4\tilde{\lambda}^2+1}{4\tilde{\lambda}} + \ln \frac{2\beta}{1+\beta^2} \right\}^{-1} \right] \\ & - g_s \eta'_c \frac{1}{2\sqrt{2}} \delta\tilde{\lambda} \left(-\frac{1}{3} \tilde{\lambda}^{-2} \right) \int_{t_c}^{\frac{1}{2\pi} \ln(\alpha^2)} dt t^{-1/2} e^{2\pi t} \\ & + i g_s \eta'_c \frac{\delta\tilde{\lambda}}{8\tilde{\lambda}^2} \int \frac{d\omega'}{2\pi} \alpha^{2+2\omega'} \frac{1}{\omega'^2 + 1 + i\epsilon}. \end{aligned} \quad (6.4)$$

This can be written as,

$$\delta\mathcal{F}' = g_s\eta'_c\delta\left[-\frac{1}{4\sqrt{2\pi}}\tilde{\lambda}^2\alpha^{-2}\int_{1/(2\tilde{\lambda})}^1\frac{d\beta}{1+\beta^2}\left\{\ln\alpha+\ln\tilde{\lambda}+\ln\frac{2\beta}{1+\beta^2}\right\}^{-3/2}f(\beta)^4\right. \\ \left.-\frac{i}{8\tilde{\lambda}}\int\frac{d\omega'}{2\pi}\alpha^{2+2\omega'}\frac{1}{\omega'^2+1+i\epsilon}-\frac{1}{6\sqrt{2}}\tilde{\lambda}^{-1}\int_{t_c}^{\frac{1}{2\pi}\ln\alpha^2}dt\,t^{-1/2}e^{2\pi t}\right]. \quad (6.5)$$

This suggests that we add to $\mathcal{F}'$ a correction term

$$\mathcal{F}_{\text{cor}} = g_s\eta'_c\left[\frac{1}{4\sqrt{2\pi}}\tilde{\lambda}^2\alpha^{-2}\int_{1/(2\tilde{\lambda})}^1\frac{d\beta}{1+\beta^2}\left\{\ln\alpha+\ln\tilde{\lambda}+\ln\frac{2\beta}{1+\beta^2}\right\}^{-3/2}f(\beta)^4\right. \\ \left.+\frac{i}{8\tilde{\lambda}}\int\frac{d\omega'}{2\pi}\alpha^{2+2\omega'}\frac{1}{\omega'^2+1+i\epsilon}+\frac{1}{6\sqrt{2}}\tilde{\lambda}^{-1}\int_{t_c}^{\frac{1}{2\pi}\ln\alpha^2}dt\,t^{-1/2}e^{2\pi t}\right]. \quad (6.6)$$

Note that in $\mathcal{F}_{\text{cor}}$ we are only allowed to include terms that have either a power of α or a power of $\tilde{\lambda}$ in the denominator since these are the types of terms that we have dropped in our analysis. In particular, we cannot add a constant term to $\mathcal{F}_{\text{cor}}$. After doing the ω' integration, we get

$$\mathcal{F}_{\text{cor}} = g_s\eta'_c\left[\frac{1}{4\sqrt{2\pi}}\tilde{\lambda}^2\alpha^{-2}\int_{1/(2\tilde{\lambda})}^1\frac{d\beta}{1+\beta^2}\left\{\ln\alpha+\ln\tilde{\lambda}+\ln\frac{2\beta}{1+\beta^2}\right\}^{-3/2}f(\beta)^4\right. \\ \left.-\frac{1}{16\tilde{\lambda}}\left\{-i+\operatorname{erfi}\left(\sqrt{2\ln\alpha}\right)\right\}+\frac{1}{6\sqrt{2}}\tilde{\lambda}^{-1}\int_{t_c}^{\frac{1}{2\pi}\ln\alpha^2}dt\,t^{-1/2}e^{2\pi t}\right]. \quad (6.7)$$

Hence, the correct form of the annulus contribution to $\mathcal{F}$ is

$$\mathcal{F}_{\text{annulus}} = \mathcal{F}_{(a)} + \mathcal{F}_{(b)} + \mathcal{F}_{(c)} + \mathcal{F}_{(d)} + \mathcal{F}_{(e)} + \mathcal{F}_{(f)} + \mathcal{F}_{(g)} + \mathcal{F}_{(h)} + \mathcal{F}_{\text{jac}} + \mathcal{F}_{\text{cor}} + \mathcal{O}(\gamma^{-1}). \quad (6.8)$$

We can now use this for numerical evaluation of $\mathcal{F}$ by choosing some large values of α and $\tilde{\lambda}$ and some function $f(\beta)$ satisfying the boundary condition (A.8).

7 Numerical evaluation of $\mathcal{F}_{\text{annulus}}$

To obtain the numerical value of $\mathcal{F}_{\text{annulus}}$ summarized in (6.8), we performed the integrations numerically using *Mathematica*, and have included the corresponding code in the electronic supplementary material. We found it necessary to retain high precision (15–20 significant digits) in computing the various contributions to $\mathcal{F}_{\text{annulus}}$, as these often differ by several orders of magnitude. For example, for large α and $\tilde{\lambda}$, the largest contributions, from $\mathcal{F}_{(c)}$ and $\mathcal{F}_{(d)}$, grow as $\alpha^2\tilde{\lambda}^2$ and are many orders of magnitude larger than the sum of all the terms which is independent of α and $\tilde{\lambda}$. We note also that the imaginary part of $\mathcal{F}_{\text{annulus}}$ comes solely from the contributions to $\mathcal{F}_{(b)}$ and $\mathcal{F}_{(d)}$ from the t integration contour from $(\Lambda+i\infty)^{-1}$ to Λ^{-1} , as discussed around (1.26). These can be traced to internal closed string tachyon propagators.

In performing the explicit numerical calculations, we have used a family of functions $f_n(\beta)$

$$f_n(\beta) = \frac{4\tilde{\lambda}^2 - 3}{8\tilde{\lambda}^2(1 - (2\tilde{\lambda})^{-n})}(1 - \beta^n) \quad (7.1)$$

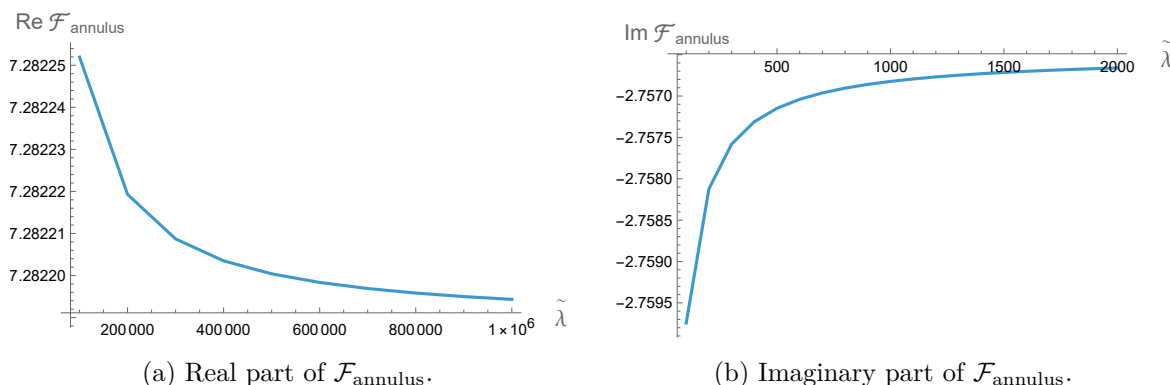


Figure 6. Real and imaginary parts of $\mathcal{F}_{\text{annulus}}$ for $t_c = 2$, $\alpha = \lambda$, and $f(x) = f_{n=1/4}(x)$.

which satisfy (A.8).

In figures (6(a)) and (6(b)), we plot the real and imaginary parts of $\mathcal{F}_{\text{annulus}}$ for $t_c = 2$, $\alpha = \lambda$, and $f(\beta) = f_{n=1/4}(\beta)$. We have verified numerically that $\mathcal{F}_{\text{annulus}}$ is independent of these choices, consistent with the general arguments presented above. The imaginary part converges significantly more rapidly than the real part, and at $\lambda = 10^6$ we find

$$\mathcal{F}_{\text{annulus}} \approx (7.28219 - 2.75650i)g_s \eta'_c \approx (1.15900 - 0.43871i)g_s. \quad (7.2)$$

Some more details of the analysis can be found in appendix C.

Acknowledgments

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A Collection of useful results

In this appendix we shall review some of the results from SFT that we use in our analysis.

We begin by writing down the expression for a general interaction term in open closed SFT. As discussed in [16], a general Riemann surface can be described as a collection of some elementary components, e.g. a sphere with three holes, disks around closed string punctures, semi-disks around open string punctures etc., glued along closed curves C_s or open curves L_m beginning and ending on a world-sheet boundary. The information on the world-sheet moduli $\{u^i\}$ is contained in the transition functions that relate the coordinate σ_s (σ_m) to the left of C_s (L_m) and τ_s (τ_m) to the right of C_s (L_m):

$$\sigma_s = F_s(\tau_s, \vec{u}), \quad \sigma_m = G_m(\tau_m, \vec{u}). \quad (\text{A.1})$$

We define:

$$\begin{aligned} \mathcal{B}_i \equiv & \sum_s \left[\oint_{C_s} \frac{\partial F_s}{\partial u^i} d\sigma_s b(\sigma_s) + \oint_{C_s} \frac{\partial \bar{F}_s}{\partial u^i} d\bar{\sigma}_s \bar{b}(\bar{\sigma}_s) \right] \\ & + \sum_m \left[\int_{L_m} \frac{\partial G_m}{\partial u^i} d\sigma_m b(\sigma_m) + \int_{L_m} \frac{\partial \bar{G}_m}{\partial u^i} d\bar{\sigma}_m \bar{b}(\bar{\sigma}_m) \right], \end{aligned} \quad (\text{A.2})$$

where $\oint_{C_s} d\sigma_s$ and $\int_{L_m} d\sigma_m$ contain intrinsic $1/2\pi i$ factors and $\oint_{C_s} d\bar{\sigma}_s$ and $\int_{L_m} d\bar{\sigma}_m$ contain intrinsic $-1/2\pi i$ factors. For n_c external closed string states $A_1^c, \dots, A_{n_c}^c$ and n_o external open string states $A_1^o, \dots, A_{n_o}^o$, we now define the p -form on the moduli space M_{g,b,n_c,n_o} of the Riemann surface of genus g , b boundaries, n_c closed string puncture and n_o open string puncture [16] (eqs. (3.69), (3.70)):

$$\begin{aligned} & \Omega_p^{(g,b,n_c,n_o)}(A_1^c, \dots, A_{n_c}^c; A_1^o, \dots, A_{n_o}^o) \\ \equiv & \eta_c^{3g-3+n_c+\frac{3}{2}b+\frac{3}{4}n_o} \frac{1}{p!} du^{i_1} \wedge \dots \wedge du^{i_p} \langle \mathcal{B}_{i_1} \dots \mathcal{B}_{i_p} A_1^c \dots A_{n_c}^c; A_1^o \dots A_{n_o}^o \rangle_{\Sigma_{g,b,n_c,n_o}}, \end{aligned} \quad (\text{A.3})$$

where $\eta_c = i/(2\pi)$ and $\langle \dots \rangle$ denotes the correlation function on the punctured Riemann surface Σ_{g,b,n_c,n_o} . The vertex operators A_i^c and A_i^o are inserted using the local coordinate system on the disks or the semi-disks on which the corresponding puncture lies. The exception to this is the disk one point function of closed strings for which we use

$$\Omega_0^{(0,1,1,0)}(A^c) = \eta_c^{1/2} \langle c_0^- A^c \rangle_{0,1,1,0}. \quad (\text{A.4})$$

Up to overall signs, the string amplitudes are obtained by integrating $g_s^{2g+b-2+n_c+\frac{1}{2}n_o} \Omega_p^{(g,b,n_c,n_o)}$ over the moduli space $\mathcal{M}_{g,b,n_c,n_o}$, with p given by the dimension of the moduli space. The interaction terms of SFT are obtained by integrating the same forms over a subspace of the moduli space $\mathcal{M}_{g,b,n_c,n_o}$, setting all the A_i^o 's to the open string field Ψ_o , all the A_i^c 's to the closed string field Ψ_c and dividing the result by the combinatoric factor $n_c!n_o!$.

For the overall signs of the amplitudes we need additional data since we have to specify what constitutes positive integration measure in the moduli space. For amplitudes involving purely closed strings and without boundaries, the moduli space has a complex structure and for a complex modulus $u = u_1 + iu_2$ we take $du_1 \wedge du_2$ to have positive measure and there is no additional sign in the amplitude. In the presence of open strings or boundaries there is no such natural choice of the sign. In [16] a detailed description of the signs of all the amplitudes were given. Here we summarize the results that we shall need. For the disk amplitude with one open and one closed strings, there is an additional minus sign besides the normalization constants given above. Every additional open string vertex operator on the boundary, whose location is parametrized by a modulus u , is accompanied by a factor of $-\mathcal{B}_u$ inserted to the immediate left of the vertex operators and the integration over u is taken to have positive measure if increasing u moves the vertex operator in a direction that keeps the world-sheet to the left [16].

Next we shall describe the local coordinate at the open string puncture(s) in different interaction vertices that appear in figure 1, following the conventions of [19]. We begin with the open-closed interaction vertex that appears in figure 1(a) and (b). Representing the

open-closed interaction vertex as a correlation function on the upper half plane labelled by z with the closed string inserted at $z = i$ and the open string inserted at $z = 0$, the local coordinate at the open string puncture is taken to be [19] (eq. (4.3))

$$w_o = \lambda z, \quad (\text{A.5})$$

where λ is an arbitrary real parameter.

For the three open string interaction vertex, described as a correlation function on the upper half plane with the open string vertex operators inserted at $z = 0$, $z = 1$ and $z = \infty$, we have the following relations between the local coordinates $w_o^{(i)}$ at the i -th open string punctures and z [19] (eq. (4.6))

$$w_o^{(1)} = \alpha \frac{2z}{2-z}, \quad w_o^{(2)} = -2\alpha \frac{1-z}{1+z}, \quad w_o^{(3)} = \alpha \frac{2}{1-2z}, \quad (\text{A.6})$$

where α is a real parameter that is taken to be large.

Next we consider the closed-open-open interaction vertex, represented by a correlation function in the upper half plane with the closed string inserted at i and the open strings inserted at $\pm\beta$ on the real line. It follows from eq. (4.10) of [19] that the local coordinates at the punctures at $-\beta$ and β are given respectively by $w_1 = F_1^{-1}(z)$ and $w_2 = F_2^{-1}(z)$, where,

$$\begin{aligned} F_1^{-1}(z) &= \alpha \tilde{\lambda} \frac{4\tilde{\lambda}^2}{4\tilde{\lambda}^2 + 1} \frac{z + \beta}{1 - \beta z - \tilde{\lambda} f(\beta)(z + \beta)}, \\ F_2^{-1}(z) &= \alpha \tilde{\lambda} \frac{4\tilde{\lambda}^2}{4\tilde{\lambda}^2 + 1} \frac{z - \beta}{1 + \beta z + \tilde{\lambda} f(\beta)(z - \beta)}, \quad \tilde{\lambda} \equiv \alpha \lambda, \end{aligned} \quad (\text{A.7})$$

where $f(\beta)$ is an arbitrary function satisfying

$$f(1/2\tilde{\lambda}) = \frac{4\tilde{\lambda}^2 - 3}{8\tilde{\lambda}^2}, \quad f(1) = 0. \quad (\text{A.8})$$

Using eq. (B.3), (B.10) of [19] we can invert these equations as,

$$\begin{aligned} F_a(w_a, \beta) &= e_a(\beta) + g_a(\beta) w_a + \frac{1}{2} h_a(\beta) w_a^2 + \mathcal{O}(w_a^3), \quad a = 1, 2 \\ e_1(\beta) &= -\beta, \quad g_1(\beta) = \frac{4\tilde{\lambda}^2}{4\tilde{\lambda}^2 + 1} \frac{1 + \beta^2}{\alpha \tilde{\lambda}}, \quad h_1(\beta) = -2 \left(\frac{4\tilde{\lambda}^2}{4\tilde{\lambda}^2 + 1} \right)^2 \frac{\beta + \tilde{\lambda} f}{(\alpha \tilde{\lambda})^2} (1 + \beta^2), \\ e_2(\beta) &= \beta, \quad g_2(\beta) = \frac{4\tilde{\lambda}^2}{4\tilde{\lambda}^2 + 1} \frac{1 + \beta^2}{\alpha \tilde{\lambda}}, \quad h_2(\beta) = 2 \left(\frac{4\tilde{\lambda}^2}{4\tilde{\lambda}^2 + 1} \right)^2 \frac{\beta + \tilde{\lambda} f}{(\alpha \tilde{\lambda})^2} (1 + \beta^2). \end{aligned} \quad (\text{A.9})$$

Finally, we consider the open string one point vertex on the annulus that appears in figure 1(b). We label the points on the annulus by a complex coordinate w with the restriction:

$$0 \leq \text{Re}(w) \leq \pi, \quad w \equiv w + 2\pi i t. \quad (\text{A.10})$$

Then the local coordinate w_o at the open string puncture at $w = 0$ is related to w via [19] (eqs. (4.60), (4.68)):

$$w_o = 2\alpha \frac{(4 + 3\alpha^{-2})\hat{z} - 4 + 3\alpha^{-2}}{(4 - \alpha^{-2})\hat{z} + 4 - 7\alpha^{-2}}, \quad \hat{z} = e^{iw}. \quad (\text{A.11})$$

In writing this we have taken the limit of large α and $\tilde{\lambda}$.

Next we shall review the relation between the parameters q_1 and q_2 associated with the propagators in figure 1 and the parameters x and t labeling the moduli space of one point function of a closed string on an annulus. We begin with figure 1(a). For this diagram we have [19] (eqs. (4.73), (4.81) together with the relation $v = e^{-2\pi t}$):

$$e^{-2\pi t} \simeq \frac{q_2}{\alpha^2} \left(1 - \frac{q_2}{2\alpha^2}\right)^{-1}, \quad 2\pi x = \frac{q_1}{\tilde{\lambda}} \left(1 - \frac{q_2}{\alpha^2}\right). \quad (\text{A.12})$$

Next we consider the relation of the parameters q_2 and the parameter β of the closed-open-open interaction vertex in figure 1(c) with the parameters x and t of the annulus one point function of the closed string. We have [19] (eqs. (4.92), (4.98)):

$$\begin{aligned} 2\pi x &= 2 \tan^{-1} \beta + 2 u f \tilde{\lambda}^{-1} - \frac{1}{\beta} (1 - \beta^2) \tilde{\lambda}^{-2} u, \quad u \equiv q_2 \alpha^{-2} \left\{1 + \frac{1}{4\tilde{\lambda}^2}\right\}^{-2}, \\ e^{-2\pi t} &= u \frac{(1 + \beta^2)^2}{4\beta^2 \tilde{\lambda}^2} \left\{1 + u \tilde{\lambda}^{-2} \frac{1}{2\beta^2} (1 - \beta^2 - 2\beta \tilde{\lambda} f)^2\right\}. \end{aligned} \quad (\text{A.13})$$

Finally, we consider the relation between the parameters t of the open string one point vertex of the annulus and the parameter q_1 of the open string propagator in figure 1(b) and the parameters x and t of the annulus one point function of the closed string. The parameter t is common; so we only need to give the relation between x and q_1 . This is given by [19] (eqs. (4.85), (4.86)):

$$2\pi x = \frac{q_1}{\tilde{\lambda}} (1 - \alpha^{-2}). \quad (\text{A.14})$$

Next we shall review the range of the integration parameters associated with the Feynman diagrams in figure 1. First of all the q_i 's in each diagram are always integrated from 0 to 1:

$$0 \leq q_1 \leq 1, \quad 0 \leq q_2 \leq 1. \quad (\text{A.15})$$

In figure 1(b), the parameter t associated with the annulus one point function of the open string is integrated over the range (eq. (4.67) of [19]):

$$R_{(b)} : \left(\alpha^2 - \frac{1}{2}\right)^{-1} \leq e^{-2\pi t} \leq 1. \quad (\text{A.16})$$

In figure 1(c), the parameter β associated with the closed-open-open interaction vertex is integrated over the range (eq. (4.99) of [19])

$$R_{(c)} : \frac{1}{2\tilde{\lambda}} \leq \beta \leq 1. \quad (\text{A.17})$$

In figure 1(d), the range of x and β are (eq. (4.102) of [19]):

$$\begin{aligned} R_{(d)} : \frac{\pi}{2} &\geq 2\pi x \geq \tilde{\lambda}^{-1} (1 - \alpha^{-2}), \\ \frac{1}{\alpha^2 \tilde{\lambda}^2 \sin^2(2\pi x)} &\left(1 + \frac{1}{4\tilde{\lambda}^2}\right)^{-2} \left[1 - 2 \left\{\cot^2(2\pi x) - \tilde{\lambda}^2 f^2\right\} \alpha^{-2} \tilde{\lambda}^{-2} \left(1 + \frac{1}{4\tilde{\lambda}^2}\right)^{-2}\right] \leq v < 1, \\ f &\equiv f(\tan(\pi x)), \quad v \equiv e^{-2\pi t}. \end{aligned} \quad (\text{A.18})$$

B Closed string tachyons

As already discussed in section 5, the integrands of $I_{(b)}$ and $I_{(d)}$ diverge as $t \rightarrow 0$, rendering the integrals divergent if we take the lower limit of t to be zero. In this appendix we shall verify that Witten's $i\epsilon$ prescription, discussed at the end of section 5, makes these integrals finite.

We begin with $I_{(d)}$ for which the integrand is

$$F(x, t) = 2\pi\delta(k^0) \frac{g_s\eta'_c}{\sqrt{2\pi}} t^{-1/2} \eta(it)^{-24} \left[\frac{\vartheta_1(2x|it)}{\vartheta'_1(0|it)} \right]^{-2}. \quad (\text{B.1})$$

To study its behaviour at small t , we make a change of variable from t to s :

$$s = \frac{1}{t}, \quad (\text{B.2})$$

so that $t \rightarrow 0$ corresponds to $s \rightarrow \infty$. The annulus can now be regarded as having circumference $2\pi/s$ and width π , but by scaling the coordinates by s we can also describe this as having circumference 2π and width πs . Thus physically πs represents the distance over which the closed string propagates. We now use the modular transformation properties (5.59) and (5.60) to write

$$\eta(it) = t^{-1/2} \eta(i/t) = s^{1/2} \eta(is), \quad (\text{B.3})$$

$$\vartheta_1(2x|it) = i t^{-1/2} \exp[-4\pi x^2/t] \vartheta_1(2x/(it)|i/t) = i s^{1/2} \exp[-4\pi x^2 s] \vartheta_1(-2ixs|is). \quad (\text{B.4})$$

Using the product representations (1.7), (1.8) we now get,

$$(\eta(it))^{-24} = s^{-12} e^{2\pi s} \prod_{n=1}^{\infty} (1 - e^{-2\pi ns})^{-24}, \quad (\text{B.5})$$

$$\vartheta_1(-2ixs|is) = 2i e^{-\pi s/4} \sinh(2\pi xs) \prod_{n=1}^{\infty} \{(1 - e^{-2\pi ns})(1 - 2e^{-2\pi ns} \cosh(4\pi xs) + e^{-4\pi ns})\}. \quad (\text{B.6})$$

Using this in (B.1), we see that $F(x, t)$ has divergence from the $s \rightarrow \infty$ limit due to the $e^{2\pi s}$ factor in (B.5). As anticipated, the origin of this can be traced to the closed string tachyon. The rest of the factor can be expanded in a power series in $e^{-2\pi s}$, $e^{-2\pi sx}$ and $e^{-2\pi s(1-x)}$ and gives a convergent expansion.

Now suppose that we express the integral over x and t as an integral over x and s and take the s integration contour to run along the positive real axis to some large number Λ and then turn the contour parallel to the imaginary axis towards $\Lambda + i\infty$. Then the offensive $e^{2\pi s}$ factor becomes oscillatory, and the integral converges due to the s^{-12} factor in (B.5). This is Witten's $i\epsilon$ prescription [18], but this is equivalent to treating the divergences using the Feynman diagrams of open closed SFT shown in figure 3.

Similar analysis can be done for the integrand of $I_{(b)}$ given in (5.54). In this case the t dependent part of the integrand is $Z(t) \propto \eta(it)^{-24}$ and hence is given by (B.5). Arguments identical to the ones given above show that this integral diverges if the upper limit of s integration is taken to be ∞ , but we can get a convergent result by taking the upper limit to be $\Lambda + i\Lambda'$ for large positive Λ, Λ' and then taking Λ' to ∞ .

$\tilde{\lambda} = 5\alpha$	$\text{Re } \mathcal{F}_{\text{annulus}}$	$\text{Im } \mathcal{F}_{\text{annulus}}$
100000	7.284015676539	-2.7565011017855
200000	7.28307221834	-2.75649947586
300000	7.2827668799	-2.75649893385
400000	7.2826168249	-2.7564986628
500000	7.282527888	-2.7564985002
600000	7.282469155	-2.756498392
700000	7.282427526	-2.756498314
800000	7.282396506	-2.756498256
900000	7.28237251	-2.756498211
1000000	7.28235341	-2.756498175

Table 1. Numerical values for $\tilde{\lambda}$, $\text{Re } \mathcal{F}_{\text{annulus}}$, and $\text{Im } \mathcal{F}_{\text{annulus}}$, with $\alpha = \tilde{\lambda}/5$.

$\tilde{\lambda} = \alpha$	$\text{Re } \mathcal{F}_{\text{annulus}}$	$\text{Im } \mathcal{F}_{\text{annulus}}$
100000	7.28225194597	-2.75650110179
200000	7.2822192953	-2.7564994759
300000	7.282208705	-2.756498934
400000	7.28220349	-2.756498663
500000	7.28220040	-2.75649850
600000	7.28219836	-2.75649839
700000	7.2821969	-2.75649831
800000	7.2821958	-2.75649826
900000	7.2821950	-2.7564982
1000000	7.2821943	-2.7564982

Table 2. Numerical values for $\tilde{\lambda}$, $\text{Re } \mathcal{F}_{\text{annulus}}$, and $\text{Im } \mathcal{F}_{\text{annulus}}$, with $\alpha = \tilde{\lambda}$.

C Numerical results

In this appendix we summarise the numerical results for estimating $\mathcal{F}_{\text{annulus}}$. A Mathematica notebook is attached to this article. Here we present the numerical estimates for $1 \times 10^5 \leq \tilde{\lambda} \leq 10^6$, in the three cases of $\alpha = \tilde{\lambda}/5$, $\tilde{\lambda}$, $5\tilde{\lambda}$. These are presented in tables 1, 2 and 3, respectively. The number of decimal places presented in the tables varies because each value is printed with the precision returned by Mathematica; we avoid zero-padding to prevent implying spurious accuracy.

We also fitted the data given in each of these tables to a trial function

$$c_0 + c_1 \tilde{\lambda}^{-1} (\ln \tilde{\lambda})^{3/2} + c_2 \tilde{\lambda}^{-1} (\ln \tilde{\lambda})^{1/2} + c_3 \tilde{\lambda}^{-1} (\ln \tilde{\lambda})^{-1/2}, \quad (\text{C.1})$$

with the c_i 's allowed to be different for different tables. However we find that for each of the three tables, we get

$$c_0 = 7.28219 - 2.75650 i. \quad (\text{C.2})$$

Hence we can take this to be the result for $\tilde{\lambda} \rightarrow \infty$.

$\tilde{\lambda} = \alpha/5$	$\text{Re } \mathcal{F}_{\text{annulus}}$	$\text{Im } \mathcal{F}_{\text{annulus}}$
100000	7.282186213	-2.7565011018
200000	7.28218737	-2.756499476
300000	7.2821878	-2.75649893
400000	7.2821880	-2.7564987
500000	7.2821881	-2.7564985
600000	7.282188	-2.7564984
700000	7.282188	-2.756498
800000	7.282188	-2.756498
900000	7.282188	-2.756498
1000000	7.28219	-2.756498

Table 3. Numerical values for $\tilde{\lambda}$, $\text{Re } \mathcal{F}_{\text{annulus}}$, and $\text{Im } \mathcal{F}_{\text{annulus}}$, with $\alpha = 5\tilde{\lambda}$.

Data Availability Statement. This article has no associated data or the data will not be deposited.

Code Availability Statement. This article has code included as electronic supplementary material.

Supplementary material and data. The following files are provided as supplementary material/data and are accessible through the article’s webpage.

- Recoil_Numerics.nb (Mathematica notebook)

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