



City Research Online

City St George's, University of London

Citation: Sailis, F. (2026). Completing the Bootstrap Program for TT-Deformed Massive Integrable Quantum Field Theories. (Unpublished Doctoral thesis, City St George's, University of London)

This is the accepted version of the paper.

This version of the publication may differ from the final published version. To cite this item please consult the publisher's version.

Permanent repository link: <https://openaccess.city.ac.uk/id/eprint/36637/>

Copyright and Reuse: Copyright and Moral Rights remain with the author(s) and/or copyright holders. Copies of full items can be used for personal research or study, educational, or not-for-profit purposes without prior permission or charge, unless otherwise indicated, provided that the authors, title and full bibliographic details are credited, a hyperlink and/or URL is given for the original metadata page and the content is not changed in any way. For full details of reuse please refer to [City Research Online policy](#).

City St George's, University of London

School of Science and Technology
Department of Mathematics
Doctoral Thesis

Completing the Bootstrap Program for
 $\overline{\text{TT}}$ -Deformed Massive Integrable Quantum Field
Theories



Submitted by:
Fabio Sailis

Supervisor:
Olalla Castro-Alvaredo

Academic year: 2024-2025

Submission Date: 21/01/2026

“I never before wrote a book like this; and, since it is not the custom for mathematicians to introduce their works with statements of a personal nature, I could have spared myself the trouble... I wanted to understand—only to understand, nothing more...”

His Master’s Voice, Stanisław H. Lem

Declaration of Authenticity

This report contains genuine work conducted originally by the author. The work presented herein has not been submitted and/or accepted for the award of any other degree or diploma in any university. To the best of my knowledge and belief, this thesis contains no materials previously published or written by other person, except where due references has been made.

Abstract

This thesis addresses the problem of extending the Form Factor Bootstrap Program (FFBP), the main tool for computing correlation functions in integrable quantum field theories (IQFTs), to the case of irrelevant deformations, with particular focus on generalised $T\bar{T}$ -deformations. Such models pose new challenges, including the emergence of a minimal length scale and the loss of strict locality. Relying solely on the exact S -matrix, we construct the minimal form factors of $T\bar{T}$ -deformed theories and analyze their properties in detail. This resolves longstanding ambiguities in previous constructions and leads to a deeper understanding of both deformed and standard IQFTs.

Acknowledgements

First and foremost, I would like to thank my supervisor, Prof. Olalla Castro-Alvaredo, for giving me the opportunity to pursue these studies and for offering me much more than simple academic guidance. Beyond being a brilliant scientist, she has been a warm and strong presence who has taken care of me throughout these three years and has always been concerned about how things were going, especially when results were not immediately on the horizon. I would also like to express my sincere gratitude to Dr. Stefano Negro. Although, formally, he has been a collaborator, in practice he has played a far more significant role in my academic development. He has always been available for discussions, questions, and clarifications, and his constant presence has been invaluable throughout my work. If Prof. Castro-Alvaredo has been my academic guide in the fullest sense, Stefano has been an equally important reference figure from whom I have learned a great deal, both scientifically and professionally.

Secondly, I wish to thank the people at City, St. George's, University of London, who have made these years intellectually stimulating and personally enriching. In particular, I am grateful to Bethan, Michele, Manyjot, Ves, Blair, Dmitrii, Diogo, Maddalena, Elohim, Charles, Aman, and Ellie. With them, I have shared not only discussions and ideas related to research, but also friendship, humour, and a sense of everyday companionship that has made the working environment lively and welcoming. Thanks to all of you, the days have been anything but monotonous. I would also like to thank the friends and fellow researchers I met through conferences and academic events: Fred, Luca, Riccarda, Andrea, Octavio, Michelangelo, Sunny, Riccardo, David, Anna, Miklos, Konstantinos and Federico. What began as professional encounters often developed into genuine friendships, which I deeply value and hope to carry forward beyond this PhD experience.

Finally, I am deeply grateful to all those who encouraged and supported me to embark on this journey despite the distance: my parents, my sister, and my friends, who have always been a constant presence even across a continent and two seas.

List of Publications Related to this Doctoral Thesis

The following publications are those presented as part of this PhD thesis:

1. O. A. Castro-Alvaredo, S. Negro, F. Sailis,
Form factors and correlation functions of $T\bar{T}$ -deformed integrable quantum field theories, JHEP **9** (2023) 048.
2. O. A. Castro-Alvaredo, S. Negro, F. Sailis,
Entanglement entropy from form factors in $T\bar{T}$ -deformed integrable quantum field theories, JHEP **11** (2023) 129.
3. O. A. Castro-Alvaredo, S. Negro, F. Sailis,
Completing the bootstrap program for $T\bar{T}$ -deformed massive integrable quantum field theories, J. Phys. A: Math. Theor. **57** (2024) 26.
4. O. A. Castro-Alvaredo, S. Negro, F. Sailis,
Boundary Quantum Field Theories Perturbed by $T\bar{T}$: Towards a Form Factor Program, Nucl. Phys. B **1017** (2025) 116924.
5. F. Sailis, O. A. Castro-Alvaredo, S. Negro,
Complete Minimal Form Factors for Irrelevant Perturbations of IQFTs, Nucl. Phys. B, Volume 1019, (2025) 117131.

Contents

Acknowledgements	i
List of Publications Related to this Doctoral Thesis	iii
List of Figures	x
Preface	xi
0 A Very Short Introduction to Quantum Fields	1
0.1 What Is a Quantum Field?	2
0.1.1 Canonical Quantisation	2
0.2 The Renormalisation Group and Universality	4
0.2.1 Conformal Field Theory	5
0.3 IQFTs as Perturbed CFTs	9
0.3.1 Away From Criticality	9
0.3.2 Zamolodchikov’s c -theorem and Δ -sum Rules	13
1 Integrable Quantum Field Theories in 1+1 Dimensions	17
1.1 Building a Quantum Field Theory	17
1.1.1 Asymptotic States and the S-matrix	17
1.1.2 Asymptotic and Interpolating Fields	19
1.2 General Properties of Integrable Field Theories	21
1.2.1 Unitarity and Lorentz Invariance	21
1.2.2 Integrability in 1+1-dimensional QFTs	22
1.3 The Analytic Properties of the 2-particle S-matrix	26
1.3.1 Symmetries and Analyticity	26
1.3.2 The Zamolodchikov-Faddeev Algebra	31
1.4 General Solutions for Purely Elastic Scattering	31
1.4.1 Diagonal S-matrices	32
1.4.2 On the Uniqueness of Elastic S-matrices	35
1.5 Scattering Amplitudes in the Presence of Boundaries	37
1.5.1 Boundary Scattering Amplitudes	37
1.5.2 The Boundary Ising Model	40
2 Irrelevant Deformations of IQFTs	45
2.1 Generalised $T\bar{T}$ -deformations of IQFTs	45
2.1.1 Composite Fields from the Integrals of Motion	46
2.1.2 The $T\bar{T}$ -flow	47
2.1.3 The Inviscid Burgers Equation and the Hagedorn Transition	48
2.2 CDD-factors from $T\bar{T}$ -deformations	52

2.2.1	$T\bar{T}$ -deformations and Dressing of the ZF-algebra	53
3	Completing the Bootstrap Program for $T\bar{T}$-deformed IQFTs	61
3.1	The Form Factor Bootstrap Program for IQFTs	62
3.1.1	The Form Factor's Axioms	62
3.1.2	The Karowski-Weisz Construction	67
3.1.3	Form Factors and Locality	70
3.1.4	Identifying the Operator Content	77
3.2	Applying the Form Factor Bootstrap Program to $T\bar{T}$ -deformed IQFTs . .	80
3.2.1	Form Factors of the $T\bar{T}$ -deformed Ising Field Theory	83
3.2.2	The Factorisation Property	98
3.2.3	Extension to Branch Point Twist Fields	102
3.2.4	Entanglement Entropy from BPTFs	108
3.2.5	Comparison with Previous Results in the Literature	112
3.2.6	Some Thoughts	115
4	Complete Minimal Form Factors of $T\bar{T}$-deformed IQFTs	119
4.1	A Straightforward Extension to Theories with Boundaries	121
4.1.1	Reflection Amplitudes, Irrelevant Perturbations and the Boundary Form Factor Bootstrap Program	122
4.1.2	A New Representation for the Boundary Minimal Form Factors . .	124
4.1.3	More General Boundary Conditions	128
4.2	Complete Minimal Form Factors for Irrelevant Deformations of IQFTs . .	130
4.2.1	A Heuristic Derivation: MFFs and their Fourier Transforms	131
4.2.2	The $T\bar{T}$ -Deformed Ising Field Theory	137
4.2.3	Alternative Construction from the Karowski-Weiss Approach	142
4.2.4	Correlation Functions and their Asymptotic Properties	145
	Conclusion	147
A	LSZ Reduction Formulas	153
B	Expectation Value of $T\bar{T}$ in Two-Dimensional Quantum Field Theories	157
C	$T\bar{T}$-deformed Theories as QFTs on Noncommutative Minkowski Space	161
C.1	Noncommutative Minkowski space and deformed quantum fields	161
C.2	Wedge localization and geometric interpretation	163
C.3	Two-particle scattering and integrable structure in two dimensions	164
D	Correlation Functions and Cumulants for More General $T\bar{T}$-Perturbations	167
D.1	Cut-off for a Generalised $T\bar{T}$ -Perturbation: $\alpha, \alpha_s \neq 0$	167
D.2	Cut-off for a Generalised $T\bar{T}$ -Perturbation: $\alpha_s, \beta_s \neq 0$	167
E	A $T\bar{T}$ Picture of the Boundary Ising Model with Generic Boundaries	169
F	Technical Details and Supplementary Material to Section 4.2	171
F.1	The $T\bar{T}_{2n-1}$ -Deformed Ising Field Theory	171
F.2	Revisiting the Sinh-Gordon Model	172
F.2.1	Factorizing the Kinematical Pole	172
F.2.2	$T\bar{T}$ -like Solution of the sinh-Gordon Model	174
F.2.3	A Factorized Solution	175

F.2.4	Another Solution	176
F.3	Generalisation to Branch Point Twist Fields	177
Bibliography		194

List of Figures

1.1	The space-time diagram for a two-particle scattering process.	23
1.2	Space-time diagram illustrating a multiparticle factorised scattering.	25
1.3	Equivalence of the 3-particle scattering amplitudes representing the Yang-Baxter equations.	26
1.4	Graphical representation of the 2-particle elastic S-matrix element defined by (1.29).	27
1.5	The simplest 2-sheet Riemann surface of the 2-particle S-matrix.	28
1.6	S-matrix analytic properties in the s -plane. U and C stand for unitarity and crossing transformations.	30
1.7	S-matrix analytic properties in the θ -plane. U and C stand for unitarity and crossing transformations.	30
1.8	1-particle reflection off the boundary at $x = 0$	38
1.9	1-particle reflection off the boundary at $t = 0$	39
1.10	Boundary Yang-Baxter equations.	40
1.11	Boundary Unitarity.	41
1.12	Boundary Cross-Unitarity.	41
1.13	Domain wall configuration interpreted as a zero-momentum particle emitted by the boundary state.	43
3.1	Generalised form factor of the operator $\mathcal{O}$	62
3.2	Form factor of the operator $\mathcal{O}$	63
3.3	Graphical representation of the Watson equation for the braiding.	64
3.4	Graphical representation of the Watson equation for the monodromy. This equation can also be interpreted as defining the action of a map $\mathcal{K} : F_{n+2}^{\mathcal{O}} \rightarrow F_n^{\mathcal{O}}$	65
3.5	Graphical representation of the recursive equation for the kinematical pole.	67
3.6	The contour C in equation (3.34).	68
3.7	Branch-cut generated by the insertion of the $U(1)$ twist field at the origin.	72
3.8	Deformed branch-cut.	73
3.9	Triangular region on which we are performing the $U(1)$ transformation $\Phi(x) \mapsto e^{i\omega} \Phi(x)$	75
3.10	Different effect of the deformation on the scattering depending on the sign of the coupling α	82

3.11	The functions $z_2^\mu(\alpha)$ and $z_2^\varepsilon(\alpha)$. In the unperturbed Ising model we have that the two-point functions scale exactly as $r^{-\frac{1}{4}}$ and r^{-2} for μ and ε respectively so that $z_2^\mu(0) = \frac{1}{16}$ and $z_2^\varepsilon(0) = \frac{1}{2}$. From the two-particle contributions, it seems that there is still power-law scaling at short distances but the decay becomes faster the larger is $-\alpha$. Note that the definitions (3.170) and (3.171) of $z_2^\mu(\alpha)$ and $z_2^\varepsilon(\alpha)$ are not valid at $\alpha = 0$: there is a sharp change at $\alpha = 0$ which is a peculiar feature of the Ising model. . . .	97
3.12	The functions $F(\alpha)$ and $c(\alpha)$. They both tend to zero for large $-\alpha$ and to their exact values in the unperturbed theory for $\alpha = 0$: $F(0) = \frac{1}{16} = 0.0625$ and $c(\alpha) = \frac{1}{2} = 0.5$	98
3.13	The functions $c_2(\alpha)$ for the Ising model.	102
3.14	The function (3.224) for $\alpha = 0, -0.001, -0.01, -0.1$ and -1 , with the value 0 corresponding to the red dots and the value 1 corresponding to the blue dots. The exact formula (3.225) is also plotted as a solid black line, which fits the red dots perfectly. The figure on the right is a blow up of the small n region of the left figure. All functions are linear for n large and the slopes are: $b(0) = \frac{1}{48} = 0.0208333$, $b(-0.001) = 0.0195262$, $b(-0.01) = 0.0169936$, $b(-0.1) = 0.0112533$ and $b(-1) = 0.00331503$	108
3.15	The function (3.224) for $n = 2$ (red), $n = 5$ (blue), $n = 7$ (green) and $n = 10$ (black). The figure on the right shows clearly how all functions scale similarly with α . We find that all functions are extremely well fitted by quadratic polynomials of the form $f(n) + k(n)x + p(n)x^2$ with $x = \log(-\alpha)$ which are presented as dashed lines.	109
3.16	The function $\hat{c}(\alpha)$ for $\alpha < 0$ in the Ising field theory, as defined in (3.252). Although a distinct function from the function obtained from Zamolodchikov's c -theorem, they share the same main features. As expected $\hat{c}(0) = 1/2$	114
4.1	The absolute value squared of the function $\hat{u}(\theta, B) := \frac{u(\vartheta, B)}{i + \sinh \theta}$ with $u(\vartheta, B) = \exp \rho(\vartheta, B)$ evaluated numerically from (4.32). The colours correspond to different values of B : $B = 1$ (red), $B = 0.7$ (blue), $B = 0.5$ (green) and $B = 0.2$ (black).	129
4.2	Comparison of MFFs with the inclusion of cosh terms, for α positive and negative, on different lines $\text{Im} \eta = 0, -\frac{1}{4}, -\frac{1}{2}, -\frac{3}{4}, -1$	138
4.3	Comparison of MFFs with and without the inclusion of cosh terms on the line $\eta \in \mathbb{R}$ (i.e. $\text{Im} \theta = i\pi$).	139
4.4	Comparison of MFFs with and without the inclusion of cosh terms on the line $\theta \in \mathbb{R}$	140
4.5	Plots of the MFF obtained from (4.87) as a function of $\theta \in \mathbb{R}$ for various values of $\alpha > 0$	141

Preface

Nearly one hundred years ago—in 1927—the *Proceedings of the Royal Society of London* published what is widely considered the birth of Quantum Field Theory (QFT), namely P.A.M. Dirac’s pioneering work on the quantisation of the electromagnetic field [1]. And we are still here, carefully filling the gaps and exploring the limits and potential—if any, rather well hidden—of this vast research framework.

Nowadays, QFT has been recognised as one of the most successful approaches to help us understand physical reality. By embodying both the principles of special relativity and quantum mechanics, its applicability goes way beyond the description of “just” fundamental interactions and embraces all kinds of phenomena related to the *large distance* behavior of many degrees of freedom. At the same time, the mathematical foundations of these theories remain somewhat obscure and not fully rigorous in the way that, for example, classical analysis or even much of quantum mechanics are. Many constructions—such as interacting fields, path integrals, and renormalisation procedures—are defined formally or perturbatively, and a fully rigorous non-perturbative construction exists only in a limited number of cases. Integrable Quantum Field Theories (IQFT) in 1+1-dimensions constitute a small subset that offers both elegant, well-defined mathematical tools and interesting and physically relevant applications.

Already from the “prehistory” of modern scientific thought, the observation of celestial motion “*challenged men to search for the impersonal **invariants** behind events*” [2]. In our contemporary times, this approach to science becomes even more fundamental when we consider the disparity in scales of the phenomena we aim to understand: from the inconceivable vastness of the universe to the extremely small realm of the fundamental constituents of matter, these scales lie far beyond human experience, leaving ever less room for physical intuition grounded in our everyday world. Invariants guide us where we would otherwise lose our way: in the depths of outer space or in the realm of the infinitesimal. Notice, however, that while getting lost in space still resonates with human experience, even thinking of a corresponding analogy for the microscopic world gives some difficulties. Integrable field theories are models with infinitely many invariants¹, and this fact clearly makes them very attractive to theoretical physicists and mathematicians. Their appeal stems not only from the possibility of obtaining exact solutions, but also from the deep insights they provide into non-perturbative phenomena, the rich mathematical structures they reveal, and the way they serve as guiding examples for more complex, non-integrable systems.

The seminal work of A.B. Zamolodchikov [3] proposed a viewpoint of IQFTs as massive perturbations of critical points, universally described by conformal field theories (CFT).

¹While initially the term “invariants” is used in a more conceptual sense, in the context of integrable models these invariants acquire a precise mathematical meaning as conserved quantities, such as higher-spin charges, which are central to the exact solvability of the models as we will see later.

These perturbations are relevant and associated with specific fields of the underlying CFT having conformal dimension $\Delta < 1$. For over three decades, such integrable models have been extensively studied, and a variety of well-established techniques have been developed to analyse them, including the Thermodynamic Bethe Ansatz (TBA) [4, 5] and the Form Factor Bootstrap Program (FFBP) [6, 7, 8, 9, 10]. These methods allow for the computation of exact spectra, correlation functions, and other non-perturbative quantities, providing a deep understanding of the dynamics of these theories. However, relatively recently, scientists have also begun to consider irrelevant deformations—of both conformal and massive quantum field theories. These theories present a number of challenges: they are generally non-renormalizable, are believed to often exhibit non-local features, and can display various pathological behaviors. Nevertheless, they preserve integrability, which naturally raises the question: can we learn more about these theories using standard integrable techniques? The answer is likely negative, or at least, it is unlikely that the same level of success achieved with conventional IQFTs can be reproduced. A direct reason for this is that these irrelevant deformations effectively introduce a minimal length scale—controlled by the deformation coupling—posing challenges analogous to those encountered in *quantum gravity*. Standard QFT techniques rely on the ability to probe arbitrarily short distances, but the deformation imposes a finite resolution, complicating renormalisation and the definition of local observables. Consequently, constructing exact, non-perturbative quantities in such deformed theories becomes highly nontrivial, and many familiar tools of conventional QFT require—if possible—substantial adaptation. Nevertheless, integrability is preserved, providing a guiding principle to explore these ideas.

Well, certainly I am not aiming to solve quantum gravity; the goal here is precise and limited, but still technically challenging. As the title of this thesis suggests, we shall attempt to employ the FFBP, a standard framework for computing correlation functions in IQFTs. This is by no means an easy task, since, as discussed, irrelevant deformations are expected to introduce non-local features, particularly at the level of observables [11]. Can the general framework withstand these challenges? Can we still extract a meaningful characterisation of the observables of these models? In what follows, I present the results obtained in five works [12, 13, 14, 15, 16]. As the title indicates, the program is still being *completed*, and for this reason the present thesis—or book, as I prefer to think of it—remains necessarily incomplete. Nevertheless, especially during the final year of my PhD, after a long period of difficulties, we managed to obtain results that renewed our optimism and suggest that much more can yet be achieved.

This thesis is structured as follows. In writing it, I have preferred to adopt a “book-like” approach: my aim was to make it possible for a reader—ideally someone with a background in theoretical physics but without prior expertise in the specific field—to follow the discussion and, in principle, to understand the main ideas. For this reason, I begin with *Chapter 0*, where the notion of quantum fields is introduced and the minimal background needed for the research part of the thesis is established. Since our goal is to adapt IQFT techniques to a different setting, I felt that a general introduction to CFTs and IQFTs was essential.

In *Chapter 1* I provide a detailed review of scattering amplitudes in IQFTs, with particular attention to diagonal theories without bound states, which lie at the core of this work. Scattering amplitudes in the presence of boundaries are also discussed.

Chapter 2 introduces the $T\bar{T}$ deformation and briefly reviews some of its key properties—at least those that will be directly relevant to our analysis. This chapter is intentionally concise, since the essential input for our purposes is the deformation of scattering amplitudes derived from perturbed models.

In *Chapter 3* I introduce the FFBP, together with its main techniques and results, and then present our first attempt to apply it to the study of form factors in $T\bar{T}$ -deformed theories [12, 13, 14]. Although these preliminary solutions raised some technical challenges—such as the appearance of square-roots, divergences in specific cases, and a proliferation of free parameters—they also provided important insights and highlighted the key difficulties that any consistent approach must overcome.

Chapter 4 builds on these observations and presents progress towards resolving some of the aforementioned issues. In particular, the issue of the free parameters that afflicted earlier form factor solutions has been resolved through a refined construction of the minimal form factors, eliminating these ambiguities and significantly strengthening the consistency of the framework. However, as acknowledged before, not everything is settled, and some subtleties persist.

Finally, in the *Conclusions* I summarise the main results and argue that there is strong evidence for the non-local nature of observables in these models. At the same time, this does not rule out the possibility of computing correlation functions, and the thesis closes by outlining a number of open problems together with possible directions for future investigation.

Chapter 0

A Very Short Introduction to Quantum Fields

Our starting point is a statistical model defined on some lattice, where the *lattice spacing* “ a ” defines a natural **microscopic length-scale**. The model’s discrete degrees of freedom sit on the lattice sites and interact with each other. From a mechanical point of view the interactions are represented by some coupling constants that can be controlled by varying some external parameters. The interaction between the constituents defines another length scale, the *correlation length* ξ . When the system is close enough to a critical point we have that $\xi \gg a$ and therefore a continuous formulation of the model in terms of “field” $\varphi(\mathbf{x})$ becomes natural. As mentioned in the introduction, throughout this work we shall always consider a $1 + 1$ -dimensional spacetime. In this setting we denote a point as a two-vector $\mathbf{x} = (x^0, x^1)$, where the component x^0 represents the Euclidean time and x^1 the spatial direction. The second component x^1 remains the standard spatial coordinate of the two-dimensional theory. This convention will be adopted consistently in what follows.

In the functional formalism of Euclidean QFT [17, 18], if $\{s_i\}$ is a set of discrete classical variables for a given model, near a critical point we have that the partition function is

$$Z = \sum_{s_i} e^{-H(\{s_i\})} \simeq \int \mathcal{D}\varphi_i(\mathbf{x}) e^{-\mathcal{A}[\{\varphi_i\}; \{g_n\}]} \quad , \quad \xi \gg a, \quad (0.1)$$

where the model is described by an Action

$$\mathcal{A}[\{\varphi_i\}; \{g_n\}] = \int d^2x \mathcal{L}(\varphi_i(x), \partial_\mu \varphi_i(x), \partial_\mu \partial_\nu \varphi_i(x), \dots; \{g_n\}) . \quad (0.2)$$

Here $\mathcal{L}$ is the lagrangian density, a function over a set of “fundamental fields” φ_i (scalar, spinorial, etc.) and their interaction strength, specified by a set of coupling constants $\{g_n\}$. Since we are working in the Euclidean spacetime $\mathbb{R}^2$, it will often be convenient to use the shorthand notation d^2x for integration, without distinguishing it from $d\mathbf{x}$ unless explicitly required. In the same symbolic spirit, we shall also write x instead of the full bi-vector $\mathbf{x}$ whenever no confusion can arise.

The Lagrangian density $\mathcal{L}$ is typically a polynomial in the fields and their derivatives. To give a simple example, consider the Landau-Ginzburg theory described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi(x) \partial^\mu \varphi(x) + g_1 \varphi(x) + \frac{m}{2} \varphi^2(x) + g_3 \varphi^3(x) + \dots + g_n \varphi^n(x) + \dots , \quad (0.3)$$

where φ is a scalar field, m is the mass parameter and the coupling constants g_i are responsible for the possible interactions. Already at this level we encounter problems: the measure $\mathcal{D}\varphi$ is not a well-defined object a priori. The idea is that these continuous theories are *effective* and valid up to a certain scale, the microscopic length-scale a . Having a continuous relativistic theory it is customary to require translational invariance. This implies that plane waves become a natural basis to describe the field and then we can decompose the fields in its Fourier components

$$\varphi(x) = \int_{|k| < \Lambda} \frac{d^4 k}{(2\pi)^4} \tilde{\varphi}(k) e^{ikx}, \quad (0.4)$$

where the physical cut-off is related to the microscopic length-scale as

$$|k| \leq \Lambda \simeq \frac{1}{a}. \quad (0.5)$$

This is at the basis of effective field theories as initially thought by Wilson and Kogut in the seminal work [17] and this kind of set-up would be the one we are referring to when later we will speak of “Wilsonian” QFTs.

0.1 What Is a Quantum Field?

After introducing effective field theories as large-scale limits of statistical systems, we would like to return to the more foundational question: what is a quantum field, in general? While effective field theories provide powerful low-energy descriptions, the notion of a quantum field itself precedes this framework. It is therefore important to clarify what we mean by a quantum field beyond its role in effective descriptions.

0.1.1 Canonical Quantisation

As previously mentioned, the idea was to unify quantum mechanics and special relativity. Taking the simple example of a scalar field ϕ of mass m obeying the Klein-Gordon (KG) equation considered as a classical field equation

$$(\square + m^2)\phi(x) = 0, \quad (0.6)$$

where $\square$ is the 2-dimensional Laplace operator (or d'Alembertian), one imposes quantisation rules on the field in the form of canonical commutation relations. An elegant Lorentz-invariant scheme due to Peierls [19] consists in finding commutation relations between fields at arbitrary space-times points

$$[\phi(x), \phi(x')] = i\Delta(x - x'), \quad (0.7)$$

where Δ is the difference between the retarded and the advanced Green's function of the KG equation (0.6). This is a singular integral kernel, solution of the KG equation with initial conditions

$$\Delta(x)|_{x^0=0} = 0, \quad \left. \frac{\partial \Delta}{\partial x^0} \right|_{x^0=0} = \delta(x^1). \quad (0.8)$$

Then two important remarks follow

- The commutator of fields at different space-time points is a c-number, i.e. a multiple of the identity operator, and therefore it commutes with all observables;

- The commutator vanishes for space-like separations

$$[\phi(x), \phi(x')] = 0, (x - x')^2 < 0. \quad (0.9)$$

While the first property is exclusive to free fields, the second one is general and has a simple but deep physical meaning: measurements of a field in a given region of space-time cannot disturb a measurement in another region if the two regions have no causal connection. This is one of the fundamental properties of local QFT, sometimes called **microcausality** and implies **locality** of the theory under consideration. We will return to this notion later on as we set up our working framework, where locality is essential for the consistent construction of scattering amplitudes but also observables such as correlation functions. Throughout the thesis, we will also revisit this principle in more detail, especially in relation to situations where it may fail: either due to non-local interactions or in the presence of deformations that challenge standard locality.

Now, due to the field equations (0.6), the support of the Fourier transform in momentum space is concentrated on the two hyperboloids

$$p^0 = \pm \sqrt{(p^1)^2 + m^2}, \quad (0.10)$$

and as anticipated before, it is possible to decompose the field into Fourier modes:

$$\phi(x) = \frac{1}{2\pi} \int \frac{dk}{\sqrt{2p^0}} (a(k)e^{-ik \cdot x} + a^\dagger(p)e^{ik \cdot x}). \quad (0.11)$$

The Hermitian creation and annihilation operators satisfy the Canonical Commutation Relations (CCR)

$$[a(p), a(q)] = [a^\dagger(p), a^\dagger(q)] = 0, \quad [a(p), a^\dagger(q)] = 2p^0 2\pi \delta(q - p). \quad (0.12)$$

The CCR algebra just defined needs to be supplemented with the requirement of existence of a vector in the representation space, the vacuum $|0\rangle$ such that, for any k , $a(k)|0\rangle = 0$. The Fock space $\mathcal{H}_F$ is then generated by the actions of their Hermitian conjugates $a^\dagger(k)$. This is the essence of the *canonical quantisation* of a free scalar field ϕ and it leads to a theory of a quantum system of an arbitrary number of identical, spinless free particles obeying Bose statistics. After this brief review it should be clear that the quantum field ϕ at a point defined above is not a proper observable. Physically speaking it should be clear that, by simple quantum mechanical arguments, measurements at a point require infinite energy. But mathematically speaking, $\phi(x)$ is not an operator in $\mathcal{H}_F$ but rather an *operator-valued distribution*. Therefore we conclude that fields are not physical objects and that *their role is to implement the principle of locality* and the nature of different basic fields needed in a theory is related to the charge structure, not to the empirical spectrum of particles [20]. The physical information is in fact contained in the expectation values of products of fields, whose most notable example is given by the Wightman functions¹ [21]

$$w^n(x_1, \dots, x_n) = \langle 0 | \phi(x_1) \cdots \phi(x_n) | 0 \rangle. \quad (0.13)$$

¹Strictly speaking, in the Euclidean setting one refers to *Schwinger functions*, which are related to Wightman functions in Minkowski space by analytic continuation (Wick rotation). The rigorous connection is provided by the Osterwalder–Schrader reconstruction theorem [20].

To “solve” a QFT, in principle, does not only mean to determine the possible states and energies (or masses) of particles or bound states in the theory via the calculation of its S-matrix (when available) and to understand how the theory behaves in different regimes. One of the main goals (probably the main goal) is to find a good description of its observables, i.e. the calculation of all the correlation functions. Returning to the Euclidean formulation, considering a collection of observables $\mathcal{O}_i(x_i)$ we can rewrite a correlation function as

$$\langle 0 | \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) | 0 \rangle \equiv \frac{1}{Z} \int \mathcal{D}\varphi (\mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_m)) e^{-\mathcal{A}[\varphi]}, \quad (0.14)$$

where the partition function, as defined in (0.1) gives the correct normalisation. If we consider the simple example of the free bosonic field defined by (0.6)–(0.12) it is not difficult to see that $\phi(x)$ itself with the normal product of its powers $\{:\phi^n(x): \}^2$ define, together with the Identity operator $\mathbb{I} \equiv \phi^0$ a vector space, and every observable can be expanded as a series

$$\mathcal{O}(x) = \sum_{k=0}^{\infty} a_k : \phi^k(x) : . \quad (0.15)$$

In the framework of algebraic quantum field theory (AQFT), the field operator $\phi(x)$ and its normal-ordered powers $:\phi^n(x):$ are understood as generating elements of a net of C^* -algebras. These normal-ordered operators do not directly correspond to physical observables, but rather serve as a basis from which the entire space of observables is constructed within the algebraic framework [20]. The concept of a net of C^* -algebras comes into play when considering bounded regions of spacetime. Each region R corresponds to a subalgebra $\mathcal{A}(R)$, and the observables in that region are formed by elements of the algebra associated with R . The field $\phi(x)$ then generates observables for any region by acting on the subalgebra of that region. Then, *local observables* are associated, like in (0.9), to bounded space-time regions so that operators associated with space-like separated regions commute. The full set of observables in the theory is the union of the algebras $\bigcup_R \mathcal{A}(R)$ for all regions R , and the algebraic structure ensures that the theory respects locality and causality by construction. Importantly, the algebraic relations (such as commutation and anticommutation) among the field operators and their products dictate the physical interactions and the dynamics of the theory. In general it is a very hard task to construct local observable for models beyond the free particle cases but in the two-dimensional case some progress has been made in the last few decades.

0.2 The Renormalisation Group and Universality

The **Renormalisation Group** (RG) method is a beautiful construction that I describe below. Returning to the Wilsonian framework, changing the scale (energy) at which the system is observed, a flow in the parameter space $\{g\} = (g_1, \dots, g_n, \dots)$ of the theory is generated. Mathematically, the nature of the RG transformations is completely equivalent to a *dynamical system*: starting from a given point in the parameter space $\{g\}$, also called the *theory space* or the *action space*, one can indeed expect arbitrary behaviour for the trajectories, with oscillations, discontinuities, chaotic behaviors or strange curves.

²These are local scalar densities where the normal ordering is defined by the subtraction of the divergent terms from the product of operators. Starting from $:\phi^2(x): = \lim_{y \rightarrow x} \phi(x)\phi(y) - \langle 0 | \phi(x)\phi(y) | 0 \rangle$, we can systematically define them by using the Wick theorem.

However in many interesting cases, especially the ones of physical relevance, a smooth convergence towards the so-called *fixed points* is observed. Critical points g^* , points that are invariant under this flow, describe **universal** behaviors and physically correspond to phase transitions in the model under study. At the critical point the correlation length $\xi \rightarrow \infty$ and the continuous description in terms of fields becomes more natural. The fixed points for which instead $\xi = 0$ are called *trivial fixed points*. As in dynamical systems the fixed points can be further classified by their stability nature: we can have *attractive*, *repulsive* or *mixed* points. By linearizing the action of the RG transformations it is then possible to determine their nature. If in the neighbourhood of a critical point g^* the iteration of the RG transformations converges again to g^* we speak of an attractive fixed point, while if the action of the RG flow moves the point away from it we have a repulsive fixed point; a mixed fixed point contains both trajectories.

0.2.1 Conformal Field Theory

The classification of operators (through the RG flow) helps us understand the scaling behavior of various physical quantities near criticality. Specifically, the RG-flow tells us how the different types of couplings (interaction terms) appearing for instance in (0.3) behave at different scales. Starting from a critical point, described by a Conformal Field Theory (CFT) [22, 23], we can classify the typology of fields, depending on how they behave under conformal transformations³. Again, locality plays a fundamental role in the construction. Assume the existence of a numerable set of fields $\mathcal{O}_i(x)$ that constitute a basis for the model, like for the free boson in (0.15). Then, given two fluctuating fields $\mathcal{O}_1(x_1)$ and $\mathcal{O}_2(x_2)$ placed at two distinct points, clearly their separation is smaller than the diverging correlation length, and considering the limit $x_1 \rightarrow x_2$ it seems natural to expect the composite field to become local. Heuristically one is then driven to take

$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) = \sum_k C_{12}^k(x_1 - x_2)\mathcal{O}_k(x_2). \quad (0.16)$$

This is the powerful *Operator Product Expansion* (OPE). Conformal invariance, together with translation symmetry fixes the *structure constants* appearing in (0.16) to be

$$C_{12}^k(x_1 - x_2) = \frac{c_{12}^k}{|x_1 - x_2|^{d_1+d_2-d_k}}, \quad (0.17)$$

where c_{12}^k are pure numbers and the d_i are the so-called **scaling dimensions** of the fields. The power-law behavior of correlation functions is a typical feature of critical models. Indeed, considering the two-point function of a scalar operator for a unitary CFT we have

$$\langle \mathcal{O}_i(r)\mathcal{O}_i(0) \rangle \sim \frac{1}{r^{2d_i}}, \quad (0.18)$$

where $r = |x - y|$ is the relative distance between the fields and d is the scaling function associated to the field $\mathcal{O}$. Under the action of the RG-flow, operators are classified depending on their scaling dimensions. In 2D the classification proceeds as follows:

³Remember that we restrict our attention to 1+1-dimensional theories. In particular, when discussing conformal field theories (CFTs), all statements refer exclusively to the two-dimensional case, where conformal transformations are precisely given by holomorphic or antiholomorphic (analytic) functions. This special property, along with many other properties that follow from it, does not hold in higher dimensions.

- If $d_i < 2$ a field is called **relevant**. Physically this means that under the action of the RG-flow, the interaction strength becomes stronger “zooming out”, that is at large distances (low energies) while at short distances the coupling tends to zero. Returning to the classification of fixed points mentioned just above it is then clear that the action of a relevant operator move the theory away from the critical point: the fixed point action is unstable under the insertion of these operators in the action.
- If $d_i = 2$ a field is called **marginal** and the coupling remains unchanged.
- If $d_i > 2$ a field is called **irrelevant**. These are the exact opposite of the relevant ones. Their interaction strength goes to zero at large distances (low energies) while they become important at short distances (high energies).

More specifically, a scaling operator $\Phi(x)$ is labeled by a pair $(\Delta_\Phi, \bar{\Delta}_\Phi)$, called the **left** and **right conformal dimensions** (or conformal weights), corresponding to the scaling dimension Δ_Φ and the spin s_Φ as

$$d_\Phi = \Delta_\Phi + \bar{\Delta}_\Phi, \quad s_\Phi = \Delta_\Phi - \bar{\Delta}_\Phi. \quad (0.19)$$

While the *scaling dimension* controls how a local operator transforms under global dilatations: $x \rightarrow \lambda x$, $\mathcal{O}_i(x) \rightarrow \lambda^{-d_i} \mathcal{O}_i(\lambda x)$, conformal dimensions refine the notion of scaling dimension by encoding local symmetry data. If the CFT is unitary (we will always consider that this is the case) all the conformal dimensions are non-negative $\Delta_\Phi, \bar{\Delta}_\Phi \geq 0$ and only the identity field $\mathbb{I}$ has both left and right dimensions equal to zero (except for some specific cases like in the Gaussian model). The local conformal symmetry of a two-dimensional conformal field theory is generated by an infinite-dimensional algebra of conserved charges, known as the *Virasoro algebra*. This algebra is generated by the modes of the holomorphic and anti-holomorphic components of the **energy-momentum tensor**. Local fields of the CFT do not satisfy the Virasoro algebra themselves, but are organised into lowest weight representations of it⁴:

$$[L_m, L_n] = (n - m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{n+m,0}. \quad (0.20)$$

The constant c , known as the *central charge*, quantifies the quantum anomaly⁵ in the algebra of conserved charges associated with conformal transformations. Physically, c measures the effective number of local degrees of freedom and plays a key role in characterizing conformal field theories. In conformally invariant theories the energy-momentum tensor $T_{\mu\nu}$ is symmetric and traceless ($T^\mu_\mu := \Theta = 0$). In terms of $z, \bar{z} = x^0 \pm ix^1$ coordinates it decomposes into holomorphic and anti-holomorphic components as

⁴In many conformal field theories, such as Wess–Zumino–Witten and coset models, the fundamental local symmetry is described by an affine Kac–Moody algebra of conserved currents [23]. The Virasoro algebra then arises as a composite structure, constructed from the currents via the Sugawara or coset construction.

⁵The Virasoro algebra is nothing but the quantum extension of the classical Witt algebra describing global conformal transformations (isomorphic to $sl(2, \mathbf{C})/\mathbb{Z}_2$) [23]. A *quantum anomaly* occurs when a symmetry present at the classical level fails to be preserved upon quantisation. The classical algebra acquires a central extension, captured in our case by the central charge c .

$$\begin{aligned}
T_{zz} &= \frac{1}{4}(T_{00} - 2iT_{10} - T_{11}) := T(z), \\
\bar{T}_{\bar{z}\bar{z}} &= \frac{1}{4}(T_{00} + 2iT_{10} - T_{11}) := \bar{T}(\bar{z}), \\
T_{z\bar{z}} &= \frac{1}{4}(T_{00} + T_{11}) := \Theta(z, \bar{z})/4,
\end{aligned} \tag{0.21}$$

and it satisfies the continuity equations

$$\bar{\partial}T(z) = 0 = \partial\bar{T}(\bar{z}), \tag{0.22}$$

where $\partial = \partial_0 - i\partial_1$ and $\bar{\partial} = \partial_0 + i\partial_1$. The generators of the Virasoro algebra are obtained from the Laurent expansion of these components. For instance

$$T(z) = \sum_{n \in \mathbb{Z}} z^{-n-2} L_n \iff L_n = \oint_z \frac{dw}{2\pi i} (w-z)^{n+1} T(w). \tag{0.23}$$

The OPE (0.16) of the stress-energy tensor defines the central charge:

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots, \tag{0.24}$$

where the presence of the $(z-w)^{-4}$ term reflects the conformal (trace) anomaly. Because conformal symmetry in two dimensions factorises into independent holomorphic and antiholomorphic sectors, the theory becomes chiral: the energy-momentum tensor splits into two conserved components, $T(z)$ and $\bar{T}(\bar{z})$, each generating an independent Virasoro algebra. The *space of fields* of a CFT then decomposes into a sum of irreducible weight representation spaces of $\text{VIR} \times \bar{\text{VIR}}$:

$$\Sigma_{\text{CFT}} = \bigoplus_{\alpha} [\Phi_{\alpha}], \tag{0.25}$$

where $[\Phi_{\alpha}]$ is known as *conformal class* or family of a certain *primary operator* satisfying the equations

$$L_n \Phi_{\alpha} = \bar{L}_n \Phi_{\alpha} = 0, \quad \text{for } n > 0, \tag{0.26}$$

$$L_0 \Phi_{\alpha} = \Delta_{\alpha} \Phi_{\alpha}, \quad \bar{L}_0 \Phi_{\alpha} = \bar{\Delta}_{\alpha} \Phi_{\alpha}. \tag{0.27}$$

Each operator family consists of such a field and infinitely many *descendant operators* obtained through repeated action on the primary by the Virasoro generators. Therefore, a basis in the space of descendants of Φ_{α} is given by the operators

$$L_{-i_1} \dots L_{-i_I} \bar{L}_{-j_1} \dots \bar{L}_{-j_J} \Phi_{\alpha}, \tag{0.28}$$

with $0 < i_1 \leq i_2 \leq \dots \leq i_I$ and $0 < j_1 \leq j_2 \leq \dots \leq j_J$. The *levels*

$$(l, \bar{l}) = \left(\sum_{n=1}^I i_n, \sum_{n=1}^J j_n \right), \tag{0.29}$$

determine the conformal dimensions of the descendants (0.28):

$$(\Delta, \bar{\Delta}) = (\Delta_{\alpha} + l, \bar{\Delta}_{\alpha} + \bar{l}). \tag{0.30}$$

In general, the number of independent operators at a given level $(l, \bar{l})$ is $p(l)p(\bar{l})$, where $p(l)$ is the number of positive integer partitions of l . Notice that the energy-momentum tensor $T(z)$ is not a primary field, but rather a level-2 descendant of the identity operator, corresponding to the state $L_{-2}\mathbb{I}$. This reflects the fact that conformal symmetry is not imposed externally but emerges from conserved currents intrinsic to the theory, i.e. it is dynamically realised by the theory's own degrees of freedom. Indeed from (0.23),

$$T(z) = (L_{-2}\mathbb{I})(z) = \oint_z \frac{dw}{2\pi i} (w - z)^{-1} T(w), \quad (0.31)$$

and more generally

$$(L_{-n}\mathbb{I})(z) = \frac{1}{(n-2)!} \partial^{n-2} T(z). \quad (0.32)$$

As expected, all the descendant fields of the identity are composite operators built from the holomorphic⁶ component of the stress-energy tensor and its derivatives. We can therefore build the infinite-dimensional space of (left) descendant fields Λ which admits a decomposition

$$\Lambda = \bigoplus_{s=0}^{\infty} \Lambda_s, \quad (0.33)$$

in terms of eigen-spaces of the $s = 0$ Virasoro generator:

$$L_0 \Lambda_s = s \Lambda_s, \quad \bar{L}_0 \Lambda_s = 0. \quad (0.34)$$

By definition, the space Λ is an irreducible Virasoro module, a so-called *Verma module*, with the highest weight equal to zero. All the fields in a given sub-space have conformal weights $(s, 0)$ and spin s . They all satisfy a continuity equation like (0.22), $\bar{\partial}\Lambda = 0$, therefore they are chiral but they are not all linearly independent. In fact, from (0.32), we can see that all the fields for $n > 2$ are total ∂ derivatives. It is better to consider the quotient space $\hat{\Lambda} = \Lambda / L_{-1}\Lambda$ where we took out the sub-space of total derivatives $L_{-1}\Lambda \in \Lambda$. The new space $\hat{\Lambda}$ enjoys the same decomposition of (0.33) in terms of sub-spaces $\hat{\Lambda}_s$ which also satisfy (0.34). In the same way as in (0.23), every field $T_s \in \hat{\Lambda}_s$ ⁷ gives rise to an infinite set of generators

$$T_s(z) = \sum_{n \in \mathbb{Z}} z^{-n-s} \mathcal{L}_{s,n} \quad \Longleftrightarrow \quad \mathcal{L}_{s,n} = \oint_z dw (w - z)^{n+s-1} T_s(w), \quad (0.35)$$

with $n = 0, \pm 1, \pm 2, \dots$ in such a way that, given a local field $\chi(z, \bar{z})$ in a particular CFT, the operators

$$(\mathcal{L}_{s,n}\chi)(z, \bar{z}) = \oint_z dw (w - z)^{n+s-1} T_s(w) \chi(z, \bar{z}), \quad (0.36)$$

⁶We will focus only on the left chiral (holomorphic) components of the chiral algebra. It is clear from the discussion above that the same construction can be adapted for the right chiral (anti-holomorphic) sector.

⁷Here we are simplifying a bit the notation by labelling each field just by its spin. In general we should also specify the dimension of the subspaces $\hat{\Lambda}_s$ which can be deduced from the corresponding *character formula*.

constitute an infinite set of linearly independent *integrals of motion* (IM) of that given CFT.

0.3 IQFTs as Perturbed CFTs

In the fundamental work [3] (one of many as we will see), A. B. Zamolodchikov showed how starting from a given CFT and perturbing it with a relevant field it is possible to flow to an IQFT. Integrability in short means infinitely many constraints, giving rise to infinitely many IM which survive the relevant perturbation. Integrable field theories possess, at least in principle, a high degree of *solvability* and are at the centre of our discussion in the next chapters. We will begin here by providing some introductory background to this framework.

0.3.1 Away From Criticality

We quickly saw that CFTs are associated to fixed points of the Renormalisation Group action and that a great deal of information can be deduced from the very elegant and beautiful structure of the theories at the conformal point. What about the vicinity of these fixed points? A given theory is defined by a set of local observables $\mathcal{O}_i$: operators in the theory are associated with couplings g_i and their behavior under RG-flow depends on their scaling dimensions d_i . In particular, relevant operators tend to push the flow away, outlining unstable directions, while for irrelevant ones the couplings tend to shrink, bringing the flow back towards the fixed point, signaling therefore attractive stable trajectories⁸. Consider the subspace of Σ_{CFT} consisting of the relevant fields only $\bigoplus_r \Phi_r$, i.e. those fields such that their scaling dimension is $d_r < 2$. We can interpret this space as a tangent space corresponding to an *unstable manifold* taken at the fixed point and a shift along it is described by relevant perturbations

$$\mathcal{A} = \mathcal{A}_{\text{CFT}} + \sum_r \lambda_r \int d^2x \Phi_r(x). \quad (0.37)$$

However, all known examples consist of perturbations by a single relevant fields. In fact, the integrable structure is very delicate and more than one relevant perturbation tends to destroy it⁹ [24]. Therefore we will restrict our discussion to the case of a unitary CFT perturbed by a single relevant operator:

$$\mathcal{A} = \mathcal{A}_{\text{CFT}} + \lambda \int d^2x \Phi(x). \quad (0.38)$$

As we already said, in a unitary CFT all conformal weights are non-negative. For a spinless primary field we also have $\Delta_\Phi = \bar{\Delta}_\Phi := \Delta$, hence $d_\Phi = 2\Delta$ and a field is relevant if $\Delta < 1$. Looking the action (0.38), by dimensional analysis we can argue that

⁸In the following we will not consider the case of marginal flows for which further analysis is needed: at linear order, the RG flow does not “know” whether the coupling will grow, shrink or stay constant.

⁹Here we are not taking into account some other “borderline cases” like for instance integrable RG flows between two CFTs (e.g. from the tricritical Ising model to the critical Ising model) or exactly marginal + relevant combinations. In all these borderline cases, the apparent “multi-deformation” nature disappears upon closer inspection — either the couplings are related by symmetry, one perturbation is exactly marginal (not a second relevant direction), or the structure forces them into a single integrable potential.

the relevant coupling λ should scale as $\lambda \sim (m^2)^{1-\Delta}$, i.e. the coupling constant must have dimensions $(1-\Delta, 1-\Delta)$. The relevant nature of the operators makes it so that the model described by (0.38) approaches the original CFT in the UV while we expect a drastic change at the large-distance scales. This fact means that we should also expect the absence of ultraviolet divergences in the perturbative expansion of (0.38), i.e. the theory should be *super-renormalizable*. The first effect that we encounter perturbing a CFT is the breaking of the conformal invariance by the generation of a mass gap: from the same dimensional analysis just performed, it is clear that the expected mass scales should go as $m \sim \lambda^{\frac{1}{2(1-\Delta)}}$. A finite mass gap means a finite correlation length $\xi \sim 1/m$, (m being the mass of the lightest particle that couples to the field Φ) that will be reflected in the asymptotic behavior of 2-point functions at large distances:

$$\langle \mathcal{O}_i(r) \mathcal{O}_i(0) \rangle_c = \begin{cases} r^{-2d_i}, & r \rightarrow 0, \\ e^{-r/\xi}, & r \rightarrow \infty, \end{cases} \quad (0.39)$$

with $\mathcal{O}_i$ being a local operator of the theory. The fact that the theory is integrable means that an infinite amount of IM are still present in the theory. In particular, the operator content and therefore the IM of these models are in a one-to-one correspondence with the one that the CFT one is starting with. Although more advanced results have been obtained during the years [25, 26, 27, 28] (mainly in the context of the Form Factor Bootstrap Program that we will discuss later on) we propose here the qualitative argument given in the original work [3], that is, in any case, the main foundation upon which all subsequent studies were built.

We want now to argue that the operator space of a given CFT Σ_{CFT} is in one-to-one correspondence to the operator space of an off-critical gapped IQFT that we will address as Σ_{IQFT} . Therefore, let us consider the analogue of the quotient space $\hat{\Lambda}$ in the massive case (0.38). For the perturbed theory we will still have fields $T_s \in \Lambda_s$ but they will not satisfy anymore the continuity eq. $\bar{\partial}\Lambda = 0$. We will rather have

$$\bar{\partial}T_s = \lambda R_{s-1}^{(1)} + \dots + \lambda^n R_{s-1}^{(n)} + \dots \quad (0.40)$$

where the $R_{s-1}^{(n)}$ are some local fields in Σ_{CFT} with conformal dimensions $(s - n(1 - \Delta), 1 - n(1 - \Delta))$ and then spin $s - 1$. Remember also that we are considering a unitary CFT and therefore all these fields must have non-negative weights

$$s - n(1 - \Delta) \geq 0, \quad 1 - n(1 - \Delta) \geq 0, \quad (0.41)$$

consequently, we can always find a value of n high enough in such a way that the right conformal dimension of the field $R_{s-1}^{(n)}$ become negative, i.e. the field vanishes and the series (0.40) is finite. The n at which this occurs is the smallest integer that satisfies

$$n > \frac{1}{1 - \Delta}. \quad (0.42)$$

In the simplest possible case only the first term in (0.40) will appear and therefore the continuity eq. will be reduced to

$$\bar{\partial}T_s = \lambda R_{s-1}^{(1)}. \quad (0.43)$$

We will focus on this case only. From (0.42) it is clear that this will always be the case if the relevant perturbing field is chosen in such a way that

$$0 < \Delta < \frac{1}{2}, \quad (0.44)$$

namely, its scaling dimension is chosen to be $0 < d_\Phi < 1$. What is left now is to identify explicitly the field. The obvious method is to check by expanding perturbatively (0.38) around the CFT by means of the conformal perturbation theory (CPT). For any arbitrary correlation function of fields T_s we have

$$\langle T_s \cdots \rangle = \langle T_s \cdots \rangle_{\text{CFT}} + \lambda \int dw \int d\bar{w} \langle T_s(z) \Phi(w, \bar{w}) \cdots \rangle_{\text{CFT}}, \quad (0.45)$$

where $\Phi(w, \bar{w})$ is the relevant perturbing field of (0.38) and $\langle \cdots \rangle_{\text{CFT}}$ are the correlation functions in the original CFT. From the definition of the T_s (0.35), we can deduce the OPE

$$T_s \Phi(w, \bar{w}) = \sum_{n \in \mathbb{Z}} (z - w)^{n-s} (\mathcal{L}_{s,-n} \Phi)(w, \bar{w}). \quad (0.46)$$

Then, substituting (0.46) in (0.45) and using the formal identity¹⁰

$$\bar{\partial}(w - z)^{-m-1} = -2\pi i \frac{1}{m!} \partial^m \delta^{(2)}(z - w), \quad (0.47)$$

we find that eq. (0.45) reduce to

$$\bar{\partial} T_s(z, \bar{z}) = \lambda \oint_z \frac{dw}{2\pi i} T_s(z) \Phi(w, \bar{z}) \iff R_{s-1}(z, \bar{z}) = \oint_z \frac{dw}{2\pi i} T_s(z) \Phi(w, \bar{z}), \quad (0.48)$$

i.e. the field arising in (0.40) is identified in terms of the IM of the unperturbed CFT. From the general form of the conservation laws in the CFT, if we want the fields T_s to still be conserved after the perturbation, we clearly need the fields R_{s-1} to be total derivatives, i.e. $R_{s-1} = \partial \Theta_{s-2}$, with Θ_{s-2} being a local field of spin $s - 2$. Therefore, we end up with the general form of the continuity equation for any integral of motion of a given IQFT obtained as deformations of a CFT:

$$\bar{\partial} T_{s+1} = \partial \Theta_{s-1}. \quad (0.49)$$

In principle, motivated by these results, we can think of checking the quantum integrability of a $1 + 1$ -dimensional QFT built from eq. (0.38) in the following fashion: one takes a chiral field of the original CFT, T_{s+1} (with spin s), and computes the OPE in (0.46) in the usual way. If the residue on the r.h.s. of (0.48) turns out to be a total derivative, then T_{s+1} is an integral of motion of the perturbed CFT. The drawback is that this is not very efficient unless we already have some idea of the spins s at which conserved quantities should appear. In this direction, there exists a “counting argument” due also to A.B. Zamolodchikov [3, 29], which is derived as follows: we know from eq. (0.43) that $T_{s+1} \in \hat{\Lambda}_{s+1}$, whereas the field $\mathcal{R}_s \in \hat{\Lambda}_s$. We can therefore interpret (0.43) and (0.49), as resulting from the action of linear operators f_s such that

$$f_{s+1} : \hat{\Lambda}_{s+1} \longrightarrow \hat{\Lambda}_s. \quad (0.50)$$

As usual, we can associate to this map a kernel, $\text{Ker } f_{s+1}$, which will contain the fields in $\hat{\Lambda}_{s+1}$ satisfying $\bar{\partial} T_{s+1} = 0$ modulo total derivatives, that is, those fields which

¹⁰A sketch of the derivation of this identity can for instance be found in [29].

remain conserved in the perturbed CFT. Consequently, the existence of spin- s conserved quantities in the perturbed CFT will be guaranteed provided

$$\dim \text{Ker} f_{s+1} \neq 0, \quad (0.51)$$

which means that equation (0.51) is a necessary and sufficient condition for the existence of spin- s conserved quantities in the massive QFT. Associated to the map f_{s+1} we will also find an image, $\text{Im} f_{s+1}$, defined as the subspace of $\hat{\Lambda}_s$ containing those fields $\mathcal{R}_s$ which arise on the r.h.s. of equation (0.43). Obviously,

$$\dim \hat{\Lambda}_{s+1} = \dim \text{Ker} f_{s+1} + \dim \text{Im} f_{s+1}. \quad (0.52)$$

Since the image contains always fields in the subspace $\hat{\Lambda}_s$, i.e., $\dim \text{Im} f_{s+1} \leq \dim \hat{\Lambda}_s$, whenever the condition

$$\dim \hat{\Lambda}_{s+1} > \dim \hat{\Lambda}_s \quad (0.53)$$

is satisfied, we can surely claim that some spins- s conserved charge of the underlying CFT exists, since the constraint (0.53) ensures that (0.51) is fulfilled. However, the opposite statement is not true in general, since equation (0.51) could hold even if (0.53) does not. Therefore, the counting argument [3] provides a necessary condition for proving the quantum integrability of a perturbed CFT without any prior knowledge of the dimensionalities of the subspaces $\hat{\Lambda}_s$. Such dimensionalities are available once the characters of the irreducible representations of the Virasoro algebra associated to the unperturbed CFT are known. As a simple example we can consider the case of the analytic component of the stress-energy tensor $T(z)$, i.e. the spin $s = 1$ descendant $T_2 \in \hat{\Lambda}_1$ in our notation. Considering the perturbed theory (0.38), following (0.45) we have

$$\langle T(z) \cdots \rangle_\lambda = \langle T(z) \cdots \rangle_{\text{CFT}} + \lambda \int dw \int d\bar{w} \langle T(z) \Phi(w, \bar{w}) \cdots \rangle_{\text{CFT}}. \quad (0.54)$$

The OPE [23, 18] we need is

$$\begin{aligned} T(z) \Phi(w, \bar{w}) &= \frac{\Delta}{(z-w)^2} \Phi(w, \bar{w}) + \frac{1}{z-w} \partial \Phi(w, \bar{w}) + \dots \\ &= \frac{\Delta}{(z-w)^2} \Phi(z, \bar{z}) + \frac{1-\Delta}{z-w} \partial \Phi(z, \bar{z}) + \dots, \end{aligned} \quad (0.55)$$

where in the second line we Taylor expanded the relevant perturbing field around z . Now, substituting the OPE inside (0.48), solving the simple contour integration we immediately obtain

$$\bar{\partial} T(z) = \lambda(\Delta - 1) \partial \Phi(z, \bar{z}). \quad (0.56)$$

Now, remembering that the stress-energy tensor, as defined in (0.21), away from the critical point satisfies the continuity equation

$$\bar{\partial} T + \frac{1}{4} \partial \Theta = 0, \quad (0.57)$$

by comparison with (0.56) we get that the trace of the stress-energy tensor is given by¹¹

¹¹Even though the calculation given here is based on CPT and therefore is exact only at first order, the equality shown in (0.58) is actually true in general as it is shown for example in the original work [3].

$$\Theta = 4\lambda(1 - \Delta)\Phi(z, \bar{z}). \quad (0.58)$$

Notice that our notation for the spin s integrals of motion (IM) in the perturbed models (0.49) is inspired precisely by the conservation law of the stress-energy tensor (0.57), even though for the generic conservation laws we preferred to invert the signs of the currents appearing on the right-hand side of (0.49). This difference in notation arises directly from the different ways the stress-energy tensor is expressed in various contexts. In particular, in Chapter 2 we will adopt a slightly different notation for which the notation in (0.49) will also be natural for the stress-energy tensor.

0.3.2 Zamolodchikov's c -theorem and Δ -sum Rules

Zamolodchikov's c -theorem [30] extends the notion of the central charge of a CFT to a more general function $c(\mathbf{g})$, called c -function, defined along the whole renormalisation group flow. In a conformal theory the central charge measures the number of degrees of freedom and is fixed at the critical point. When the theory is perturbed away from criticality, one can still define a quantity $c(\mathbf{g})$ which reduces to the central charge at the fixed points but is otherwise a function of the couplings $\mathbf{g} = (g_1, \dots, g_n)$.

The theorem states that in any unitary and Lorentz-invariant 1+1-dimensional theory this function decreases monotonically along the flow. If t denotes the logarithm of the energy scale, $t = \log \mu$, this monotonicity can be expressed as

$$\frac{dc(\mathbf{g})}{dt} \leq 0. \quad (0.59)$$

The ultraviolet fixed point therefore has a larger value of c than the infrared one,

$$c_{\text{UV}} > c_{\text{IR}}. \quad (0.60)$$

This means that the number of effective degrees of freedom is reduced as one moves from high to low energies. Moreover, this function is *stationary at critical points* $\mathbf{g} = \mathbf{g}^*$ at which the QFT becomes a CFT. Consider the so-called β -functions $\beta_i(\mathbf{g})$ [17], i.e. the vector fields associated to the coupling space

$$\beta_i(\mathbf{g}) = -\frac{dg_i}{dt}, \quad (0.61)$$

i.e. the “velocities” of change of the renormalised couplings along the flow. The stationarity condition of the c -functions can be equivalently expressed in terms of β -functions as:

$$\beta_i(\mathbf{g}^*) = 0 \quad \Longleftrightarrow \quad \left[\frac{dc(\mathbf{g})}{dg_i} \right]_{\mathbf{g}=\mathbf{g}^*} = 0. \quad (0.62)$$

We can give a quick sketch of the proof of these statements as follows. Consider, in the generic IQFT the following functions built out of the components of the stress-energy tensor defined previously in (0.23):

$$F(z, \bar{z}) := z^4 \langle T(z, \bar{z}) T(0, 0) \rangle, \quad (0.63)$$

$$G(z, \bar{z}) := z^3 \bar{z} \langle T(z, \bar{z}) \Theta(0, 0) \rangle, \quad (0.64)$$

$$H(z, \bar{z}) := z^2 \bar{z}^2 \langle \Theta(z, \bar{z}) \Theta(0, 0) \rangle. \quad (0.65)$$

The various components are related to each other by the continuity equation (0.57) (and its anti-holomorphic version), which imply the following differential equations

$$\begin{aligned}\frac{dF}{d\log\tau} + \frac{1}{4}\left(\frac{dG}{d\log\tau} - 3G\right) &= 0, \\ \frac{dG}{d\log\tau} - G + \frac{1}{4}\left(\frac{dH}{d\log\tau} - 2H\right) &= 0,\end{aligned}\tag{0.66}$$

where explicitly $\tau = (m^2 z \bar{z}) = (mr)^2$, with m being a fixed mass scale and r the radial distance. This adimensional combination defines a sort of *distance from the critical point*. Then defining the combination

$$c(\tau) \equiv 2F - G - \frac{3}{8}H,\tag{0.67}$$

we have

$$\frac{dc}{d\log\tau} = -\frac{3}{4}H.\tag{0.68}$$

Because of the unitarity assumption, the function H is a positive quantity and therefore this new function (0.67), is a non-decreasing function of the distance τ at fixed values of the coupling constants. At the critical point the stress-energy tensor is traceless, i.e. $\Theta = 0$, and therefore $G = H = 0$ while $F = c/2$ (c being the central charge). Indeed this function assumes the value of the central charge at the conformal point.¹²

In a generic QFT, the Callan-Symanzik equation [31, 32, 17] describes how correlation functions depend on the (standard) energy scale μ . For an n -point correlation function one has

$$\left(\mu \frac{\partial}{\partial \mu} + \beta_i(g) \frac{\partial}{\partial g_i} + \sum_{k=1}^n \gamma_k\right) \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = 0,\tag{0.69}$$

where the γ_k are the *anomalous dimensions* of the operators¹³. The idea is to apply the Callan-Symanzik equation to the two-point function of the stress-energy tensor components (0.63-0.65). The variation with respect to the scale μ can be expressed using the β -functions and anomalous dimensions, linking the running of the couplings g_i to the behavior of these correlators. Zamolodchikov use this insight to define the c -function as a particular combination of these two-point functions which depends on g through the correlation functions. In particular, the Callan-Symanzik equation for the function $c(\tau, \mathbf{g})$ reads

$$\left(\tau \frac{\partial}{\partial \tau} - \beta_i(g) \frac{\partial}{\partial g_i}\right) c(\tau, \mathbf{g}) = 0.\tag{0.70}$$

¹²Note also that the distance τ can also be related to the RG-time as defined in (0.61). In fact, in our problem we have the natural physical scale $\sqrt{\tau}=mr$ and therefore we can treat this square root as the inverse (up to constants) of μ .

¹³The anomalous dimension $\gamma(g)$ represents the correction to the conformal dimensions along the renormalisation group flow, so that $\Delta(g) = \Delta_{CFT} + \gamma(g)$. Thus, the total scaling dimension receives corrections from $\gamma(g)$.

that is, the scale derivative of $c(\tau, g)$ can be written in terms of β -functions as

$$\tau \frac{d}{d\tau} c(\tau, g) = \beta_i(g) \frac{\partial}{\partial g_i} c(\tau, g), \quad (0.71)$$

and Zamolodchikov showed that

$$\beta_i(g) \frac{\partial}{\partial g_i} c(g) = -\frac{3}{4} \beta_i(g) G^{ij}(g) \beta_j(g), \quad (0.72)$$

where $G_{ij}(g)$ is a positive-definite metric on coupling space constructed from correlators and we suppressed the dependence on the scale. Notice that, in contrast with (0.69), no explicit γ term appears in (0.72), because the effect of the anomalous dimensions of the perturbing operators is already encoded in the β -functions. The derivation uses only the flow of the couplings and the two-point correlators, so γ does not need to be written separately. Eq. (0.72) shows explicitly how the Callan-Symanzik equation connects the RG flow (via the β -functions) with the monotonic decrease of the c -function defined from the energy-momentum tensor correlators. Eqs. (0.68) and (0.72) constitute an explicit proof of the c -theorem (0.59). Integrating eq. (0.68) we can obtain the *difference between the ultraviolet and the infrared Virasoro central charges*

$$\Delta c = c_{UV} - c_{IR} = \frac{3}{4} \int_{-\infty}^{+\infty} d\tau H(\tau) = \frac{3}{2} \int_0^{+\infty} dr r^3 \langle \Theta(r) \Theta(0) \rangle. \quad (0.73)$$

Recall that our theory in the infrared is described by an IQFT, i.e. a model with zero central charge and therefore (0.73) gives the UV central charge of the underlying CFT (that we initially perturbed with a relevant operator) in terms of the correlation function of the trace of the stress-energy tensor only.

In a very similar fashion to the derivation of the c -theorem, one can derive an analogous result for the conformal dimension of a primary field $\mathcal{O}$ of the underlying CFT [24]:

$$\Delta_{UV}^{\mathcal{O}} - \Delta_{IR}^{\mathcal{O}} = -\frac{1}{2\langle \mathcal{O} \rangle} \int_0^{\infty} dr r \langle \Theta(r) \mathcal{O}(0) \rangle. \quad (0.74)$$

This is the so-called Δ -sum rule. In [24], starting from the correlators of a scaling operator $\mathcal{O}$ with the stress-tensor components T and Θ , the continuity equation of the stress-energy tensor leads to a differential relation between these correlators, exactly as in (0.66). Using the fact that Θ is proportional to the perturbing operator (see eq. (0.58)), its short-distance OPE with $\mathcal{O}$ fixes the UV behavior. In the simplest case (no mixing), the integration of the differential equations yields (0.74): the difference between the UV and IR conformal dimensions of $\mathcal{O}$ equals an integral of the correlator $\langle \Theta(x) \mathcal{O}(0) \rangle$. Again, in a massive theory this directly gives the UV conformal dimension, since $\Delta_{IR} = 0$.

Chapter 1

Integrable Quantum Field Theories in $1+1$ Dimensions

One of the most influential physical concepts of the past century is that of quantum fields. Fields mediate the connections between directly observable events (as a bridge between experiment and prediction), even though fields are not measurable objects by themselves. But as we saw in Chapter 0, the definition of what a quantum field is and its range of validity are no easy matter; furthermore their mathematical formulation is not always well-defined. Therefore, we want to be as concrete as possible. In Section 1.1, without going too deep into the details, we will review some basic concepts usually found in any QFT textbook that are pivotal for the introduction of the main character of the chapter: the S-matrix. In Section 1.2 integrability and its consequences we will be introduced while in Section 1.3 we will study the S-matrix's analytic structure. At this point we will be ready to classify some solutions in Section 1.4, in particular we will focus on diagonal S-matrices. We will finish the chapter with Section 1.5 with an extension of the models to the presence of boundaries.

1.1 Building a Quantum Field Theory

In this section we will develop the basic concepts necessary to build a consistent quantum field theory in terms of scattering amplitudes. Starting from the physical intuition given by a collision experiment we will introduce the main character, the S-matrix, and we will go through the main axioms and assumptions necessary to obtain a suitable theory. The basic concepts here presented can be found in any QFT textbook but we preferred to report them again because they are not only fundamental for the construction of scattering amplitudes but to the whole framework of *dispersion techniques* in quantum field theory as we will see in Chapter 3, where they will be used again to derive the Form Factors equations from first principles.

1.1.1 Asymptotic States and the S-matrix

Let us consider a convenient way of characterizing the states prepared by a physicist setting up an experiment in her laboratory and those in terms of which the results of the experiment are analyzed. The typical experiment is a scattering measurement: we have a target, a certain number of some kind of particles at rest, and we shoot a beam

of particles with a given momentum, for a certain time frame. These are the initial condition that completely fix the system, in the sense that, given this information, the same experiment can be repeated, in principle, infinitely many times. In a sense this means also that an experiment is defined by prescribing the motion of the colliding particles a long time before the collision actually takes place. Being concerned only with the specification of the system we are not inquiring at all about what happens during or after the collision. We expect them to eventually collide, probably in a very complicated way, but this does not invalidate the specification of the system in terms of *particles moving freely* or equivalently *independently of each other a long time before the collision*. This way of reasoning leads us to introduce the following fundamental axiom:

Axiom Ia:

The states which, as $t \rightarrow -\infty$, i.e. a long time in the past, had given numbers of given particles of given momenta form a complete set. We denote such states as

$$|p_1, p_2, \dots, p_n\rangle_{a_1 a_2 \dots a_n}^{\text{in}}. \quad (1.1)$$

A long time before the collision, this state contained one particle each of momentum $p_1, p_2, \dots, p_n$ with quantum numbers $a_1, a_2, \dots, a_n$; all moving freely. With the same reasoning, we can think that looking at the system after a long enough time after the scattering has taken place, it will eventually consist of particles moving freely in given numbers with given momenta. This is the typical description that an experimentalist is looking forward to, counting and analyzing the “clicks” coming out of its detectors. We can likewise believe that the complete specification of this final configuration of the system defines the system itself as well as the initial configuration. Therefore we can formulate another fundamental axiom:

Axiom Ib:

The states which, as $t \rightarrow +\infty$, i.e. a long time in the future, had given numbers of given particles of given momenta form a complete set. We denote such states as

$$|p'_1, p'_2, \dots, p'_m\rangle_{a'_1 a'_2 \dots a'_m}^{\text{out}}. \quad (1.2)$$

Therefore, we have a notion of *asymptotic state*¹. The number of particles with a given momentum must naturally be an integer. The set of in-states can be thought of as eigenvectors, in Hilbert space, of number operators $N_{p_i}^{\text{in}}$ each having non-negative integer eigenvalues. States differing by one particle are connected with each other by the operators $a_{\text{in}}(p_i)$, $a_{\text{in}}^\dagger(p_i)$ satisfying the usual CCR-algebra (0.12). The same is true for the set of operators that defines the out-states. The fact that the operator algebras of the in and out operators are identical, means that they should be connected by a unitary transformation, which we denote by S . Choosing to order the momenta $p_1 > p_2 > \dots > p_n$, we define the S-matrix as a unitary and linear operator that maps final asymptotic states into initial asymptotic states (and viceversa):

$$|\dots\rangle^{\text{in}} = S|\dots\rangle^{\text{out}}, \quad (1.3)$$

or in components

¹Note that by definition these states are time-independent, i.e. we are working in the Heisenberg picture.

$$\begin{aligned}
|p_1, p_2, \dots, p_n\rangle_{a_1 a_2 \dots a_n}^{\text{in}} &= \sum_{m=2}^{\infty} \sum_{p'_1 > p'_2 > \dots > p'_m} S_{a_1 a_2 \dots a_n}^{b_1 b_2 \dots b_m}(p_1, p_2, \dots, p_n; p'_1, p'_2, \dots, p'_m) \\
&\quad \times |p'_1, p'_2, \dots, p'_m\rangle_{b_1 b_2 \dots b_m}^{\text{out}}, \tag{1.4}
\end{aligned}$$

where on the right side we are actually integrating in the outgoing momenta $p'_1, \dots, p'_m$. This operator contains all the information available from scattering experiments. In general scattering processes are taken to include all collisions, both elastic and inelastic, but integrable field theories are characterised by the fact that the latter are absent, due to heavy constraints that make the theory in principle *solvable*. This construction is by no means rigorous because we have ignored a fundamental fact: if particles can be regarded as heading towards a collision, then they must be localised in space, therefore the uncertainty principle does not allow their momenta to be exactly prescribed. Another way to think about this setup is by assuming that the interactions between the particles occur only at *short range*. This implies that, having a collision in a very small region of space-time, then, very far away from the interaction, we expect to see free particles. Last but not least, as already pointed out in the introduction, a further axiom needs to be introduced, which logically we label as

Axiom 0:

There should exist a particular state $|0\rangle$, called the vacuum, with the following properties

- It is invariant under Lorentz transformations;
- Its eigenvalue w.r.t. all the dynamical observables (e.g. energy, angular momentum) is zero, while its eigenvalue w.r.t. discrete symmetry operations (like space reflection) is $+1$.
- It is normalizable: $\langle 0|0\rangle = 1$.
- It is the lowest energy eigenstate and it is unique.

The vacuum acts as the “no-particle” state for both the in and out operators simultaneously: $a^{\text{in}}(p)|0\rangle = a^{\text{out}}(p)|0\rangle = 0$, while the successive application of creation operators give rise to the multi-particle states defined in (1.1)-(1.2). Then the completeness postulate included in Axiom I guarantees that all vectors in the Hilbert space can be written as linear combinations of basis vectors either in the form (1.1) or by (1.2).

1.1.2 Asymptotic and Interpolating Fields

To formalise the connection between particle states and local observables, we can introduce *asymptotic fields* $\phi_{\text{in/out}}$ (also called *auxiliary fields*). These are precisely the field operators built out of the (in) and (out) CCR-algebras and therefore they are free fields: they are related to physical particles (subject to interactions) only by the physical interpretation we gave to the in/out-states in the previous section, i.e. they exist as long-time limit of *interacting* or *interpolating fields* that we will introduce shortly. We don’t want to indulge to much into the details of the free field construction but it is important to stress that these fields satisfy all the usual desirable properties already discussed in Subsection 0.1.1 (see also Appendix A). The main aim of this subsection is to delve for the first time

into non-trivial field theory by introducing the following axiom, the *asymptotic condition*:

Axiom II:

Given any pair of states $|A\rangle$ and $|B\rangle$ of interest to the experimentalist, i.e. any of the in/out states, there exists an operator field $\phi(x)$, called the interpolating field, such that

$$\lim_{t \rightarrow \pm\infty} \int dx f(x) \overset{\leftrightarrow}{\partial}_0 \langle A | \{ \phi(x) - \phi_{\text{in/out}}(x) \} | B \rangle, \quad (1.5)$$

where $f(x)$ is any normalizable solution of the free particle Klein-Gordon equation (0.6).

The role of this field is basically to *mediate the evolution of the physical system between its directly detectable initial and final configurations*. Nevertheless observe that to do it, we must first smear out the field operators by folding them into the normalizable wave $f(x)$, and second, we must apply this condition not to the operators themselves but rather to their matrix elements (weak convergence). A further necessary requirement for these fields is the *stability* of 1-particle states. To define it, it is customary to introduce a *source* term $\eta(x)$ by

$$(\square + m^2)\phi(x) = \eta(x), \quad (1.6)$$

and then taking matrix elements between the vacuum and 1-particle state, we need that

$$\langle 0 | \eta(x) | p \rangle = 0. \quad (1.7)$$

Normally, working on Lagrangian field theories, the dynamics is specified by expressing the source as a function of the interpolating fields but in our context this is not required. The only essential assumption which we need to make is that $\eta(x)$ is a **local** operator, in the sense that it depends on the interpolating fields only at the point x . For example it is forbidden to take for $\eta(x)$ a quasi-local operator like $\int dx' G(x - x') \phi^n(x')$, where $G(x - x')$ is a c-number function [33]. Lastly we need to stress once more the following and fundamental *microcausality axiom*:

Axiom III:

Irrespective of any dynamical detail of the theory, the commutator of the fields shall vanish for space-like separations:

$$[\phi(x), \phi(y)] = 0, (x - y)^2 < 0. \quad (1.8)$$

Taking into account the *locality* required for the source operator $\eta(x)$, the following condition follows:

Axiom IV:

Considering a local source $\eta(x)$, the microcausality axiom implies:

$$[\eta(x), \phi(y)] = 0, (x - y)^2 < 0. \quad (1.9)$$

From basic relativistic requirements, we must avoid propagation of signals exceeding the speed of light, and axioms **III** and **IV** ensure that experiments subject to the laws of quantum mechanics do not violate this causality condition. In Appendix A I present the first direct application of these principles: the LSZ reduction formulas [34].

1.2 General Properties of Integrable Field Theories

We are now ready to outline the general properties satisfied by the S-matrix. We will first introduce the general properties usually satisfied by any relativistic quantum field theory, and then we will see how integrability imposes drastic simplification on the scattering amplitudes: in the end, the only important object needed will be the 2-particle S-matrix that, thanks to symmetries, can be fixed “almost” uniquely.

1.2.1 Unitarity and Lorentz Invariance

Unitarity as usual means that, given a generic asymptotic state, that for brevity we write as $|\psi\rangle = \sum_n a_n |n\rangle$ ² we have *conservation of probabilities*

$$1 = \sum_n |\langle n|S|\psi\rangle|^2, \quad (1.10)$$

or equivalently $S^\dagger S = 1$. This property is often referred in literature as *physical unitarity*. Having a relativistic theory we also expect *Lorentz invariance* to play a fundamental role. If we denote by L a generic Lorentz transformation such that $L|n\rangle = |n'\rangle$, requiring invariance for the S-matrix means that

$$\langle m'|S|n'\rangle = \langle m|S|n\rangle. \quad (1.11)$$

To appreciate better this constraint we can consider a simple two-to-two scattering process: $p_1 + p_2 \rightarrow p_3 + p_4$. As usual when studying scattering theory, it is useful to introduce the Mandelstam variables:

$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad u = (p_1 - p_4)^2, \quad (1.12)$$

where $p_i = (p_i^0 = E_i, p_i^1)$. Because of the usual conservation law $p_1 + p_2 = p_3 + p_4$ and the fact that momenta are on-shell $p_i^2 = m_i^2$, then $s + t + u = \sum_{i=1}^4 m_i^2$ and the scattering amplitudes depend only on this Lorentz invariant combinations that represent the total energy squared in the centre-of-mass frame and the momentum transfer squared (between the particle 1 and 3 and between the particle 1 and 4 respectively). Usually, the parametrisation $(E_i, p_i^1) = (m_i \cosh \theta_i, m_i \sinh \theta_i)$ in terms of the rapidity variable θ is the next step. Now the Mandelstam variables read

$$s = m_1^2 + m_2^2 + 2m_1 m_2 \cosh \theta_{12}, \quad (1.13)$$

$$t = m_1^2 + m_3^2 - 2m_1 m_3 \cosh \theta_{13}, \quad (1.14)$$

$$u = m_1^2 + m_4^2 - 2m_1 m_4 \cosh \theta_{14}, \quad (1.15)$$

where $\theta_{ij} = \theta_i - \theta_j$. Indeed, Lorentz invariance implies that relativistic invariants can only depend on rapidity differences. In the following, we will also impose the usual discrete symmetries: *charge conjugation*, *parity reversal* and *time reversal* symmetries together with the fundamental property of relativistic S-matrices, the so-called *crossing symmetry*; but we postpone their characterisation to later. In fact, it turns out that, thanks to integrability, the only important object we will need to characterise arbitrary scattering of particle is the 2-particle S-matrix and we will enforce on it all these symmetries to fix

²Clearly the vectors $|n\rangle$ are orthogonal basis vectors that from the completeness postulate generate the Hilbert space of the asymptotic states (1.1) and (1.2).

it almost completely. Another very important assumption is the *macroscopic causality property of space-time* [35].

Axiom V: Macrocausality

The scattering amplitude is non-zero only if the outgoing particles are created after the incoming ones.

Considering a scattering amplitude

$$a_3, \dots, a_{m+2} \langle p_3, \dots, p_{m+2} | S | p_1, p_2 \rangle_{a_1, a_2}, \quad (1.16)$$

where the momenta are ordered as $p_1 > p_2; p_3 > p_4 > \dots > p_{m+2}$. This assumption implies that the time τ_{12} when the incoming particle 1 collides with the particle 2 has to be smaller than the time τ_{23} when the slowest incoming particle (the particle 2) interacts with the fastest outgoing particle (the particle 3), $\tau_{23} \leq \tau_{12}$. A violation of this inequality is allowed on microscopic but not macroscopic time scales.

1.2.2 Integrability in 1+1-dimensional QFTs

While in classical integrable theories the meaning of integrability is very clear and well established - we have the Liouville-Arnold theorem for finite dimensional systems [36] and the inverse scattering method for infinite-dimensional ones [37] - to translate the notion to the quantum world is not easy at all and up to this date it is still an object of debate [38, 39, 40]. The general idea is that these models should be characterised by the existence of an infinite number of independent, conserved and mutually commuting charges Q_s , i.e. Integrals of Motion (IM). As a consequence they can be diagonalised simultaneously $Q_s |p\rangle_a = q_s^{(a)}(p) |p\rangle_a$ and if they are local, i.e. they can be expressed as integrals of local densities, they satisfy an additivity condition:

$$Q_s |p_1, \dots, p_n\rangle_{a_1, \dots, a_n} = (q_s^{(a_1)}(p_1) + \dots + q_s^{(a_n)}(p_n)) |p_1, \dots, p_n\rangle_{a_1, \dots, a_n}. \quad (1.17)$$

Now, we arrive at the reason why 1+1-dimensional theories are special: by the Coleman-Mandula theorem [41], in $d > 2$ the existence of even one conserved charge, either a second or higher order tensor, the S-matrix trivialise: $S=1$! Therefore 1+1-dimensional QFTs constitute a very peculiar case where integrability implements constraints that are “strong” but “not too much”: the S-matrix is non-trivial but it is *elastic* and *factorised*. We will now go through these properties that are the trademark of integrability.

Elasticity means that there is *no particle production* and that the total momentum is conserved between incoming and outgoing particle states. If we consider the single charge Q_s :

$$\frac{dQ_s}{dt} = 0. \quad (1.18)$$

Hence, an initial eigenstate with a given eigenvalue will evolve into a superposition of states sharing the same eigenvalue

$$\sum_{i \in \text{in}} q_s^{(a_i)}(p_i) = \sum_{j \in \text{out}} q_s^{(b_j)}(p'_j). \quad (1.19)$$

However, we have infinitely many of these constraints and therefore we must have $m = n$, $p_i = p'_j$ and $q_s^{(a_i)} = q_s^{(b_j)}$ for any $i = 1, \dots, n$.

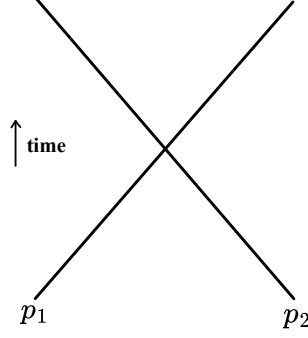


Figure 1.1: The space-time diagram for a two-particle scattering process.

Another, more rigorous argument comes from [42, 43]. Consider a localised wave packet with position and momentum centred around x_0 and p_0 respectively. Its wave function is

$$\psi(x) = \int_{-\infty}^{\infty} dp e^{-a^2(p-p_0)^2 + ip(x-x_0)}. \quad (1.20)$$

Here we have taken, for convenience, a Gaussian form of the momentum-space wave function. If the incoming and outgoing particles are localised in space by placing them in wave packets described by eq. (1.20), then the space-time diagram of the process is as shown in Fig. 1.1. Each one of the lines in Fig. 1.1 represents the centre of the wave packet of a particle either before any collision for the incoming particles or after all collisions for the outgoing particles. Acting on the state described by (1.20) with an operator $\exp\{icQ_s\}$, where c is a constant and Q_s is a conserved tensor of order s , we obtain³ a new state with wave function

$$\tilde{\psi}(x) = \int_{-\infty}^{\infty} dp e^{-a^2(p-p_0)^2 + ip(x-x_0) + icp^s}. \quad (1.21)$$

The wave packet is concentrated near the value of x where the phase is stationary at $p = p_0$. Therefore we find that if $\psi(x)$ is concentrated near $x = x_0$ (by construction) then $\tilde{\psi}(x)$ is concentrated near $x = x_0 - scp_0^{s-1}$, i.e. for $n > 1$ the centre of the wave packet is shifted by an amount that depends on its mean momentum p_0 (for $n = 1$, the charge is just the momentum and all packets are moved by the same amount). Therefore, applying the same operator to a multiparticle state, with wave packets of different momenta, all of them will be moved relatively with respect to each other. Now, consider for instance the scattering amplitude (1.16). Since the charge Q_s commutes with the S-matrix, we can use it to rearrange initial and final configurations without changing the resulting amplitude:

$$a_{3,\dots,a_{m+2}} \langle p_3, \dots, p_{m+2} | S | p_1, p_2 \rangle_{a_1, a_2} = a_{3,\dots,a_{m+2}} \langle p_3, \dots, p_{m+2} | e^{-icQ_s} S e^{icQ_s} | p_1, p_2 \rangle_{a_1, a_2}. \quad (1.22)$$

This means that with a suitable choice of c , the time t_{23} at which the fastest outgoing particle interact with the slowest incoming one, can in principle be smaller than t_{12} , the time of interaction between the two incoming particles. But then, by the *macrocausality*

³Just note that, from the definition of the conserved charge, we have $e^{icQ_s}|p\rangle = e^{icp^s}|p\rangle$, since under a Lorentz transformation, the charge Q_s transforms as s copies of the total momentum $P = Q_1$.

axiom, this amplitude vanishes. Thus, the only case in which the amplitude (1.22) is non-zero is when we only have two outgoing particles with the same momenta p_1, p_2 as the incoming ones. This is indeed an *elastic* scattering.

Factorised scattering means that *the multiparticle S-matrix elements can be expressed in terms of two-particle ones*. This fact greatly simplifies the dynamics of the theory. To clarify this point, we will use the heuristic semi-classical argument pointed out by Zamolodchikov and Zamolodchikov [44]. A more rigorous result can be found in [45]. Consider a system of N interacting particles with finite interaction range L . The configuration space of this model is $\mathbb{R}^N$, therefore, the position of the particles is described by $\{x\} = \{x_1, \dots, x_N\}$ and in addition there are $N!$ disconnected domains in this space where all the particles are far enough from each other in such a way that they effectively behave like free particles without interactions. This last consideration is completely equivalent to the assumption of existence of asymptotic states as previously formulated. Each of these domains can be identified by the succession of inequalities $x_{\sigma_1} < x_{\sigma_2} < \dots < x_{\sigma_N}$, where $|x_{\sigma_{i+1}} - x_{\sigma_i}| \gg L$ and σ_i permutations of the N integers. We denote this domain by X_σ . The free motion of particles in these domains can then be described by a wave function $\Psi(x_1, \dots, x_N)$, with $\{x\} \in X_\sigma$. Now, if the incident momenta are ordered as $p_1 > p_2 > \dots > p_N$, the wave function in each domain can only be a superposition of a finite number of N -particle plane waves:

$$\Psi_\sigma(x_1, \dots, x_N) = \sum_{\sigma'} A(\sigma, \sigma') \exp [i(p_{\sigma'_1} x_{\sigma_1} + \dots + p_{\sigma'_N} x_{\sigma_N})], \quad (1.23)$$

where the sum is carried over all permutations σ' of the fixed set of momenta $p_1, p_2, \dots, p_N$ constrained by elasticity, i.e. the set of momenta can only be reshuffled by the collision processes. The coefficients $A(\sigma, \sigma')$ are functions of the domain X_σ and of the permutation σ' . Specifically, the coefficient $A(\sigma, I)$ (I here is the identical transposition) describes the incident wave function in the domain X_σ and to obtain the scattering wave one needs to have $A(\sigma, I) = 0$ if $\sigma \neq 1$ and $A(1, 1) = 1$. Instead, outgoing waves are described by coefficients $A(\sigma, \tilde{I})$ ($\tilde{I}$ here is the inverse transposition $\tilde{I}(1, 2, \dots, N) = (N, N-1, \dots, 1)$) in X_σ and therefore they are recognised as elements of the N -particle S-matrix. Taking the 2-particle case (with particles of the same mass)

$$\Psi_{x_1 \ll x_2}(x_1, x_2) = e^{i(p_1 x_1 + p_2 x_2)} + S_R(p_1, p_2) e^{i(p_2 x_1 + p_1 x_2)}, \quad (1.24)$$

$$\Psi_{x_1 \gg x_2}(x_1, x_2) = S_T(p_1, p_2) e^{i(p_1 x_1 + p_2 x_2)}.$$

Here S_R and S_T are two-particle S-matrix elements corresponding to backward scattering (reflection) and forward scattering (transmission) respectively. This setup would have been completely exact if we had considered a 1-dimensional problem of non-relativistic particles interacting via a δ -function potential [46]: basically, the scattering process can be thought of as a system of plane waves propagating from one of these asymptotic/free regions to the others passing through “boundary” interaction regions, a system of semi-penetrable hypersurfaces positioned at $x_i = x_j$ (for any i, j). The point is that *the propagation path can always be chosen in a way such that it goes through interaction regions where only two particles are close enough to interact*. In the case of (1.24), this region is identified by

$$|x_1 - x_2| \ll L, \quad |x_1 - x_j| \gg L, \quad |x_2 - x_j| \gg L, \quad |x_i - x_j| \gg L, \quad i, j = 3, 4, \dots, N. \quad (1.25)$$

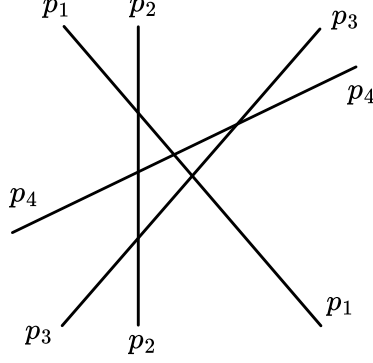


Figure 1.2: Space-time diagram illustrating a multiparticle factorised scattering.

In the end, the knowledge of the two-particle S-matrix is sufficient to determine all the coefficients $A(\sigma, \sigma')$ and therefore to obtain the full multi-particle S-matrix. Every N-particle element would be given by a product of $N(N-1)/2$ two-particle ones:

$$S_N(p_1, p_2, \dots, p_N) = \prod_{i=1}^{N-1} \prod_{j=i+1}^N S_2(p_i, p_j). \quad (1.26)$$

In Fig. (1.2) we have a typical representation of this kind of scattering picture. Clearly, to prove formally this result we would have needed to consider at least the case of a 3-particle scattering as in [45]. But to give an idea, we can make use again of the action of the symmetry operator $\exp\{icQ_s\}$. If we consider a 3-particle state, it should be possible to show that, by the action of the symmetry operator for different values of c one should be able to pass from one configuration to the others. This equivalence is formalised by the famous *Yang-Baxter equation* (YBE) [47, 48]:

$$S_2(p_2, p_3)S_2(p_1, p_3)S_2(p_1, p_2) = S_2(p_1, p_2)S_2(p_1, p_3)S_2(p_2, p_3). \quad (1.27)$$

The full power of this equation is best appreciated when considering matrix elements involved in a generic non-diagonal process, where exchanges of flavors between particles are possible. Indeed, making explicit the components, the YBE reads

$$\sum_{c_1, c_2, c_3} S_{a_1, a_2}^{c_1, c_2}(\theta_{12}) S_{c_1, a_3}^{b_1, c_3}(\theta_{13}) S_{c_2, c_3}^{b_1, b_3}(\theta_{23}) = \sum_{c_1, c_2, c_3} S_{a_2, a_3}^{c_2, c_3}(\theta_{23}) S_{a_1, c_3}^{c_1, b_3}(\theta_{13}) S_{c_1, c_2}^{b_1, b_2}(\theta_{12}), \quad (1.28)$$

where we also passed to rapidity variables. To generalise this result to amplitudes of an arbitrary number of particle is straightforward. For instance, given a 5-particle process, it is always possible to decompose it in products of 4-particle ones by displacing a particle and repeating the procedure again and again we arrive at the fundamental 2-particle components. From this construction we can also understand better why the S-matrix trivialise in dimensions $d > 2$. In fact, in higher dimensions it is possible to choose **non-crossing trajectories** for the particles, which in the plane is impossible, and in the end every scattering event can be made equivalent to one where the particles never cross each others.

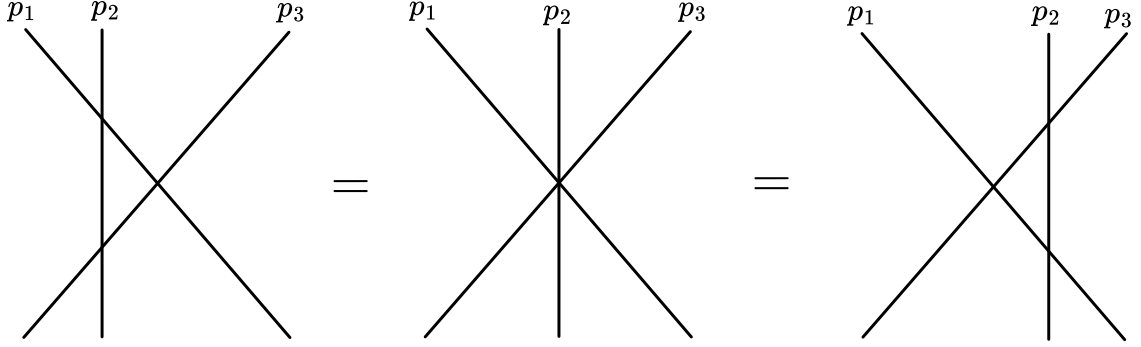


Figure 1.3: Equivalence of the 3-particle scattering amplitudes representing the Yang-Baxter equations.

1.3 The Analytic Properties of the 2-particle S-matrix

The 2-particle S-matrix is all we need to describe scattering processes of an integrable quantum field theory completely. Thus, we are left with the task of characterizing it, relying mostly on its symmetries and analytical properties. From here onwards we will primarily make use of the rapidity parametrisation already defined previously. Because of Lorentz invariance, the Mandelstam invariants depend only on difference of rapidities. Therefore, following the general definition (1.3), in a 2-particle elastic scattering of two incoming (outgoing) particles with rapidities θ_1, θ_2 (θ_3, θ_4), because of the constraints already discussed, we have, $\theta_1 = \theta_3$ and $\theta_2 = \theta_4$. The 2-particle S-matrix is then just a function $S = S(\theta_1 - \theta_2)$ and for $\theta_1 > \theta_2$ its action is defined by

$$|\theta_1, \theta_2\rangle_{i,j}^{\text{in}} = S_{ij}^{kl}(\theta_1 - \theta_j) |\theta_1, \theta_2\rangle_{k,l}^{\text{out}}. \quad (1.29)$$

In terms of the Mandelstam variables defined in (1.13)-(1.15), we have $u = 0$ and $t = s(i\pi - \theta_{12})$ ⁴. Therefore the S-matrix depends only on one invariant: $S = S(s)$.

1.3.1 Symmetries and Analyticity

Now we can make use of the symmetries to fix it as much as possible. We start requiring CPT-symmetry.

Parity symmetry (P-symmetry) indicates invariance under space-reflections. This means, looking at (1.4), that the process should be the same if we exchange the particle i with the particle j and the particle j with the particle l , or in components

$$S_{ij}^{kl}(\theta) = S_{ji}^{lk}(\theta). \quad (1.30)$$

⁴Not to be confused with the time-intervals defined previously

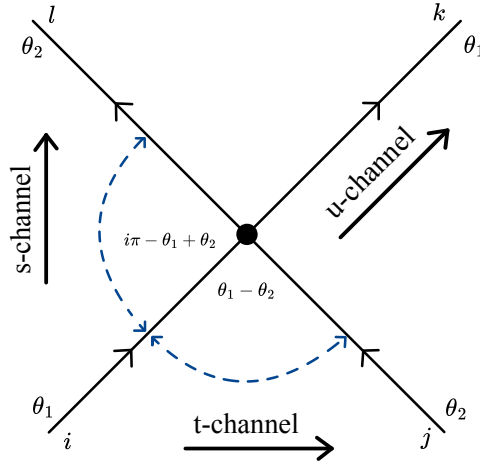


Figure 1.4: Graphical representation of the 2-particle elastic S-matrix element defined by (1.29).

Time-reversal (T-symmetry) indicates reflection but in time, i.e. the process should be the same if we exchange the particle i with the particle l and the particle j with the particle k , or in components

$$S_{ij}^{kl}(\theta) = S_{lk}^{ji}(\theta). \quad (1.31)$$

In a theory invariant under *charge conjugation* (C-symmetry) we expect that the scattering amplitudes is the same if the particles are substituted by their respective charge-conjugates. That is

$$S_{ij}^{kl}(\theta) = S_{\bar{i}\bar{j}}^{\bar{k}\bar{l}}(\theta), \quad (1.32)$$

where the barred indices indicates the charged conjugated particles or anti-particles (that can be considered also as moving backwards in time). These are fundamental symmetries that we will always take for granted in our theories⁵. Now we can proceed and discuss the *analytic properties* in terms of the Mandelstam variables. We just saw that in a 1 + 1-dimensional integrable field theory only one of these is independent and it is standard to choose s , the energy squared of the centre of mass (the square of the forward-channel momentum). Remembering the definition in terms of rapidities (1.13),

$$s = m_i^2 + m_j^2 + 2m_i m_j \cosh \theta_{ij}, \quad (1.33)$$

we see that, to have a physical process we need $\theta_{ij} \in \mathbb{R}$ and consequently s is also real and then it also satisfies the inequality: $s \geq (m_i + m_j)^2$. To study the analytic properties of the function $S(s)$ we need to analytically continue it to the complex plane.

In the complex s -plane S is a singlevalued, meromorphic function with branch cuts on the real axis starting at the kinematical thresholds $s \geq (m_i + m_j)^2$ (the “direct” s -channel)

⁵Note that in IQFTs such as the homogeneous sine-Gordon model or the Federbush model, an S -matrix can be constructed even in the absence of explicit PT-symmetry; in these models the factorisation and integrability properties suffice to determine a consistent scattering theory. For further reading, see e.g. [49, 50, 51, 52, 29].

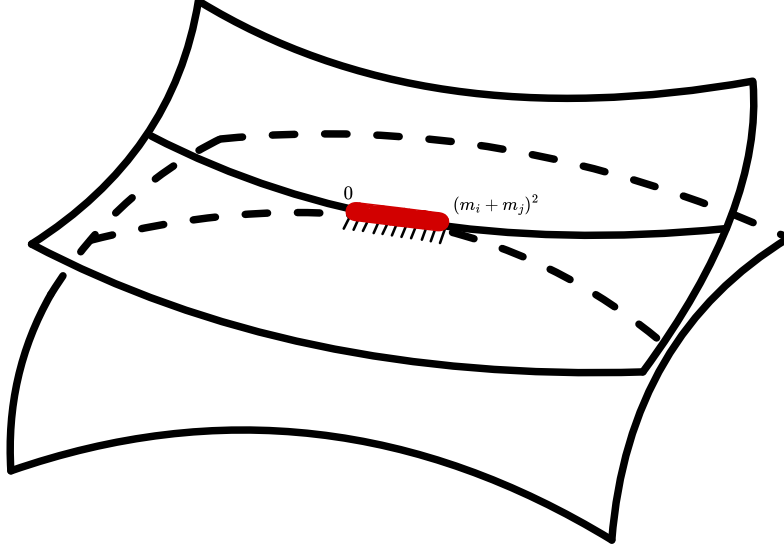


Figure 1.5: The simplest 2-sheet Riemann surface of the 2-particle S-matrix.

and $s \leq (m_i - m_j)^2$ (the “crossed” t/u -channel). The *physical region* is therefore defined by $s^+ = s + i0$, i.e. slightly above the right cut (see Fig. 1.6) and the first sheet of the Riemann surface for S is consequently called the “*physical sheet*”.

Analyticity of the S-matrices in this circumstances means that apart from the thresholds, these objects are analytical in the entire plane apart from a minimal number of singularities (poles) dictated by specific physical processes.

Unitarity in the physical region implies

$$S_{ij}^{kl}(s^+)(S_{kl}^{mn})^*(s^+) = \delta_i^m \delta_j^n. \quad (1.34)$$

Then from the analyticity assumption just stated above, the S-matrix in the physical region is the *boundary value* of an analytic function in the whole s -plane and we can upgrade this property to **hermitian analyticity**:

$$S_{ij}^{kl}(s^*) = (S_{kl}^{ij})^*(s). \quad (1.35)$$

If we add up the T-symmetry (1.31) we obtain an even stronger property called **real analyticity** [51]

$$S_{ij}^{kl}(s^*) = (S_{ij}^{kl})^*(s), \quad (1.36)$$

hence for $s \in \mathbb{R}$, $S(s)$ is also real and $(m_i - m_j)^2 \leq s \leq (m_i + m_j)^2$.

Another crucial property we require is the **crossing symmetry**:

$$S_{ij}^{kl}(s) = S_{\bar{l}\bar{i}}^{\bar{j}\bar{k}}(t). \quad (1.37)$$

In terms of rapidities, since $t(\theta) = s(i\pi - \theta)$, this property can be rewritten as

$$S_{ij}^{kl}(\theta) = S_{\bar{l}\bar{i}}^{\bar{j}\bar{k}}(i\pi - \theta) = \mathcal{C}_{jn} S_{mi}^{nk}(i\pi - \theta) \mathcal{C}^{mn}, \quad (1.38)$$

where $\mathcal{C}$ is the charge-conjugation operator. This property is deeply related in the relativistic invariance of the theory since it is based on the invariance of the physical processes under exchange of space and time (i.e. rotation of the s -channel into the t -channel). The real analyticity property (1.36) implies that

$$S_{ij}^{kl}(s^+)(S_{lk}^{mn})^*(s^-) = \delta_i^n \delta_j^m, \quad (1.39)$$

with $s^- = s - i0$. This condition ensures the existence of the branch cut starting at $s = (m_i + m_j)^2$ (the 2-particle threshold) for the S-matrix⁶ and crossing symmetry implies that the other branch cut starting from $s = (m_i - m_j)^2$ (towards $-\infty$) must also be there. If we take for granted this structure on the s -plane is also easy to see that the branch is of square-root type. Indeed, let $S_\gamma(s)$ be the analytical continuation of $S(s)$ below the cut around the right branch point. By unitarity

$$S(s^+)S_\gamma(s^+) = 1, \quad (1.40)$$

and then

$$S_\gamma(s^-) = S^{-1}(s^-) = S(s^+), \quad (1.41)$$

by real analyticity. The last equality means that to return to the original S-matrix we need to analytically continue twice around the branch point, i.e. the branch point is of square root type. See Fig.(1.5) for a qualitative idea of the structure of the Riemann surfaces. Another perspective on the analytical structure can be obtained passing to rapidity variables. Inverting (1.13) we get

$$\begin{aligned} \theta_{ij} &= \cosh^{-1} \left(\frac{s - m_i^2 - m_j^2}{2m_i m_j} \right) \\ &= \log \left(\frac{s - m_i^2 - m_j^2 + \sqrt{(s - (m_i + m_j)^2)(s - (m_i - m_j)^2)}}{2m_i m_j} \right). \end{aligned} \quad (1.42)$$

Now, the physical sheet is mapped into the strip $0 \leq \text{Im}(\theta_{ij}) \leq \pi$, the second sheet is mapped to $\pi \leq \text{Im}(\theta_{ij}) \leq 2\pi$ and so on, with periodicity $2\pi i$. *As a function of rapidity θ , $S(\theta)$ is a meromorphic function and by real analyticity, $S(\theta)$ is real on the imaginary axis of θ* (see Fig.(1.7)). In terms of rapidity variables we can also rewrite the fundamental analytic properties of the S-matrix as

- Unitarity:

$$S_{ij}^{mn}(\theta)S_{nm}^{kl}(-\theta) = \delta_i^k \delta_j^l; \quad (1.43)$$

- Crossing:

$$S_{ij}^{mn}(\theta) = S_{i\bar{l}}^{k\bar{j}}(i\pi - \theta). \quad (1.44)$$

⁶If the theory is an interacting one.

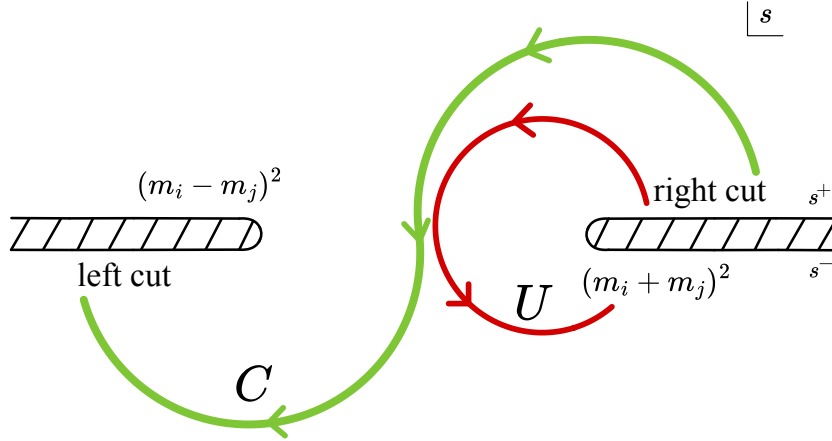


Figure 1.6: S-matrix analytic properties in the s -plane. U and C stand for unitarity and crossing transformations.

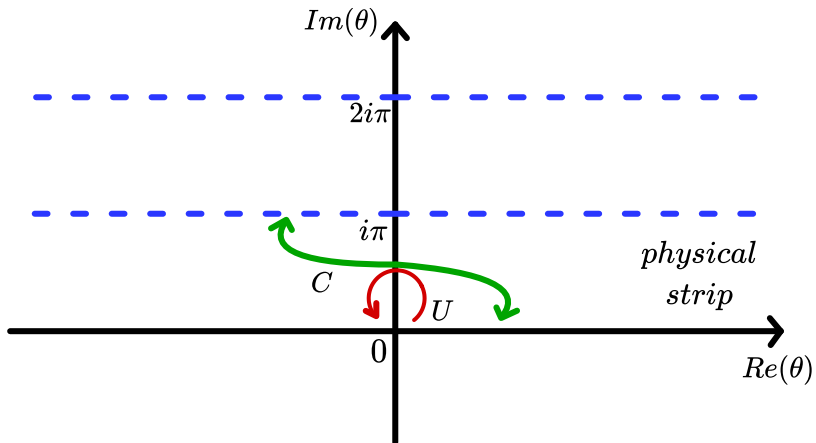


Figure 1.7: S-matrix analytic properties in the θ -plane. U and C stand for unitarity and crossing transformations.

1.3.2 The Zamolodchikov-Faddeev Algebra

Another important aspect of the formulation of scattering amplitudes in integrable field theories that we need to introduce comes from the algebraic perspective. In fact, we can describe the general structure of the factorised S-matrix by generalising the usual CCR-algebra (0.12) to the interacting case. We can think of defining the S-matrix directly as an operator that satisfies

$$A_a(\theta_i)A_b(\theta_j) = S_{ab}^{cd}(\theta_{ij})A_c(\theta_j)A_d(\theta_i), \quad (1.45)$$

where the $A_a(\theta_i)$ s define an algebra of symbols that acting onto the vacuum create asymptotic particles. The arrangement of the symbols in a product corresponds to that of particles along the spatial axis: in-states are identified with order of decreasing rapidities of particles and the order is inverted for out-states. That is

$$A_{a_1}(\theta_1)|0\rangle = 0 = \langle 0|A_{a_1}^\dagger(\theta_1), \quad (1.46)$$

while $A_{a_1}^\dagger(\theta_1)|0\rangle = |\theta_1\rangle_{a_1}$ creates a particle with rapidity θ_1 and transforms in a linear irreducible representation of the symmetry algebra. For generic asymptotic states, taking the order $\theta_1 > \theta_2 > \dots > \theta_n$, we have

$$|\theta_1, \theta_2, \dots, \theta_n\rangle_{a_1 a_2 \dots a_n}^{\text{in}} = A_{a_1}^\dagger(\theta_1)A_{a_2}^\dagger(\theta_2) \dots A_{a_n}^\dagger(\theta_n)|0\rangle, \quad (1.47)$$

$$|\theta_1, \theta_2, \dots, \theta_n\rangle_{a_1 a_2 \dots a_n}^{\text{out}} = A_{a_n}^\dagger(\theta_n) \dots A_{a_2}^\dagger(\theta_2)A_{a_1}^\dagger(\theta_1)|0\rangle, \quad (1.48)$$

$${}_{a_1 a_2 \dots a_n}^{\text{in}}\langle \theta_1, \theta_2, \dots, \theta_n| = A_{a_1}(\theta_1)A_{a_2}(\theta_2) \dots A_{a_n}(\theta_n)\langle 0|, \quad (1.49)$$

$${}_{a_1 a_2 \dots a_n}^{\text{out}}\langle \theta_1, \theta_2, \dots, \theta_n| = A_{a_n}(\theta_n) \dots A_{a_2}(\theta_2)A_{a_1}(\theta_1)\langle 0|. \quad (1.50)$$

The operators $A_{a_1}(\theta_1), A_{a_1}^\dagger(\theta_1)$ are elements of an associative non-commutative algebra, the Zamolodchikov-Faddeev (ZF) algebra [44]. To complete its definition we need to specify the full commutation relations:

$$A_a^\dagger(\theta_1)A_b^\dagger(\theta_2) = S_{ab}^{cd}(\theta_{12})A_c^\dagger(\theta_2)A_d^\dagger(\theta_1), \quad (1.51)$$

$$A_a(\theta_1)A_b^\dagger(\theta_2) = S_{ab}^{cd}(\theta_{21})A_c^\dagger(\theta_2)A_d(\theta_1) + \delta(\theta_{12})\delta_{ab}. \quad (1.52)$$

These relations clearly generalise the free bosonic/fermionic CCR, corresponding to $S = \pm 1$. In this algebraic perspective, the Yang-Baxter equations naturally arise as the associativity condition of the ZF algebra and also the unitarity condition (1.43) can be seen as a simple consequence of it.

1.4 General Solutions for Purely Elastic Scattering

Historically, the first factorised S-matrix was discovered in the context of non-relativistic one-dimensional scattering of particles through a δ -function potential [53, 47, 54] and in the context of nonlinear classical integrable models [46, 55, 56, 57, 58] by means of the inverse scattering method (ISM). The sine-Gordon equation [57] describes a relativistic, nonlinear field theory admitting soliton and anti-soliton solutions. In the 1970s, it gained prominence as a rare example of an exactly solvable quantum field theory, made even more intriguing by its equivalence to the massive Thirring model [59]. Despite its integrability, it features a rich particle spectrum including solitons, anti-solitons, and their bound states

and it is therefore no surprise that it was chosen as the ideal testing ground for new non-perturbative techniques and ultimately, for the first exact S-matrix computation. Indeed, starting from some preliminary results obtained by the quantum version of the inverse scattering method (QISM) [60], A. B. Zamolodchikov succeeded in computing the exact S-matrix elements of soliton-antisoliton scattering [61, 62]. This result was later generalised by himself and his brother to the complete S-matrix of the $O(N)$ models (the sine-Gordon model correspond to $N = 2$) in the foundational work [44]. In this thesis we will focus only on models with diagonal S-matrices with a single particle species without bound states and therefore we do not need to go deep into the machinery of the more intricate models (we will not discuss the S-matrix Bootstrap). For instance, the YBE can determine the ratios between S-matrix elements belonging to the same multiplet, but clearly they become trivial in the case we are interested in. Therefore in the following we will first introduce general solutions for diagonal S-matrices following [63, 18, 64] and then, we will return to a more general case (precisely the sine-Gordon scattering initially studied in [62]) discussed in [65] to give an idea of the procedure used and to show the uniqueness of the solutions. This last part is not there just as a mere curiosity, but in fact, we will see in the small computation presented by M. Karowski et al. a method that will be proven to be fundamental for the computations of form factors [66, 6] and to our results presented in the later chapters.

1.4.1 Diagonal S-matrices

Integrability implies the conservation of the set of momenta in a scattering process but if the scattering is *diagonal* the picture gets simplified even further. In fact, in this case the momentum and all the quantum numbers of each particle are conserved in every process individually. The scattering is purely transmissive and having now

$$S_{ij}^{kl}(\theta) = \delta_i^k \delta_j^l S_{ij}(\theta), \quad (1.53)$$

we can specify the S-matrix elements by just S_{ij} . The unitarity and crossing conditions simplify to

$$S_{ij}(\theta)S_{ij}(-\theta) = 1, \quad S_{ij}(\theta) = S_{i\bar{j}}(i\pi - \theta). \quad (1.54)$$

There are two kinds of generic meromorphic functions that satisfy (1.54), real analyticity and $2\pi i$ -periodicity [63]:

1. Functions $f(\theta)$ such that

$$f(\theta) = \prod_{x \in A} (x)_\theta, \quad \text{where } (x)_\theta = \frac{\sinh\left(\frac{\theta + ix\pi}{2}\right)}{\sinh\left(\frac{\theta - ix\pi}{2}\right)}, \quad (1.55)$$

for A a set of complex numbers invariant under conjugation. Notice that the blocks $(x)_\theta$ possess a simple pole of residue $2i \sin(x\pi)$ at $\theta = ix\pi$ and a simple zero at $\theta = -ix\pi$.

2. Functions $\Phi(\theta)$ such that

$$\Phi(\theta) = \prod_{s \in \mathcal{S}} \Phi_{\alpha_s}(\theta), \quad \text{where } \Phi_{\alpha_s}(\theta) = e^{-i\alpha_s \sinh(s\theta)}, \quad (1.56)$$

where α_s is real and s is an odd integer in some arbitrary set $\mathcal{S}$ [67].

The second kind (1.56) is usually excluded: in *local* QFTs, it follows from the Wightman axioms that scattering amplitudes are expected to be polynomially bounded in the momenta [67, 68, 69] and the second type clearly violates this bound. Nevertheless amplitudes like (1.56) will be at the centre of our discussion in the later chapters and it is at least interesting to note as they were already taken into account (and then disregarded) in early studies. But now we want to concentrate on the first kind written above. We exclude the existence of unstable particles requiring that all the poles occur on the imaginary θ -axis. In this way every x in (1.55) is real and we can restrict it to $-1 \leq x \leq 1$. The functions $(x)_\theta$ inside this range are the *basic building blocks for every purely elastic scattering theory*. They satisfy some other useful properties:

$$\begin{aligned} (x)_\theta &= (x+2)_\theta = (-x)_{-\theta}, \\ (x)_\theta (-x)_\theta &= 1, \\ (x)_{i\pi-\theta} &= (1-x)_\theta, \\ (x)_{\theta-i\pi y} (x)_{\theta+i\pi y} &= (x-y)_\theta (x+y)_\theta, \\ (0)_\theta &\equiv -(-1)_\theta \equiv 1. \end{aligned} \tag{1.57}$$

If then, at least one of the particles i or j is its own antiparticle, the additional constraint coming from crossing

$$S_{ij}(\theta) = S_{ij}(i\pi - \theta), \tag{1.58}$$

implies [70] that up to a sign, $S_{ij}(\theta)$ must be a product of building blocks in the form

$$[x]_\theta = -(x)_\theta (1-x)_\theta = \frac{\sinh \theta + i \sin x\pi}{\sinh \theta - i \sin x\pi} = \frac{\tanh \left(\frac{\theta + ix\pi}{2} \right)}{\tanh \left(\frac{\theta - ix\pi}{2} \right)}. \tag{1.59}$$

These solutions satisfy a similar set of constraints

$$\begin{aligned} [x]_\theta &= [x+2]_\theta = [1-x]_\theta = [-x]_\theta, \\ [x]_\theta [x]_{-\theta} &= 1 \end{aligned} \tag{1.60}$$

$$\begin{aligned} [x]_{\theta-i\pi y} [x]_{\theta+i\pi y} &= [x+y]_\theta [x-y]_\theta, \\ [0]_\theta &\equiv 1. \end{aligned} \tag{1.61}$$

For $0 < x < \frac{1}{2}$ the simple poles of these functions are at $\theta = i\pi x$ and $i(1-x)\pi$ with residues $2i \tan x\pi$ and $-2i \tan x\pi$ while the zeros are at $-ia\pi$ and $-i(1-x)\pi$. The analyticity property we required previously implies that the poles (and the zeros by unitarity) of a factorised S-matrix completely specify the dynamics of a given model. Therefore, its poles in the strip $-\pi < \text{Im}(\theta) \leq \pi$ of a given S-matrix element completely specify (up to a sign⁷) the building blocks $(x)_\theta$ into which the amplitude factorises.

Functions such as (1.55), (1.59) and (1.56) solve identically the unitarity and crossing equations and are known as *CDD-factors* [71]. Studying Low's scattering equation in the one-meson approximation the authors found solutions involving an infinite number of parameters. The existence of redundant solutions ignited the debate on what “physical” really meant in this context and in particular, what are (if they exist) the most general

⁷We will not discuss further this problem but an overall minus sign is seen to be equivalent to an extra factor of $(1)_\theta$.

physical principles which allow us to pick out in a unique way one “physical” admissible solution [72]. For us, as mathematical physicists, this seems the perfect time to state what we think the word “physical” means. Clearly there are two main interpretation on what a “physical” model can be for us: first, we can think of “physical models” as the ones that are realised in the real world, i.e. theories that describe natural phenomena like the fundamental interactions or the collective behaviour of matter’s constituents; secondly we can think of a “physical model” in a broader sense, for example as a QFT model that is consistent with all the criteria that we expect to hold, like relativistic invariance, unitarity, crossing and locality. In this sense the diagonal S-matrices described for instance by (1.59) define well-posed and consistent QFTs and they will be at the centre of the discussion in the next chapters. The simplest model we can build from (1.59), i.e. a model consisting of a single interacting neutral particle, is the sinh-Gordon model [73, 74]:

$$S(\theta) = \left[-\frac{b+1}{2} \right]_{\theta} = \frac{\sinh \theta - i \cos \frac{\pi b}{2}}{\sinh \theta + i \cos \frac{\pi b}{2}}, \quad (1.62)$$

where $-1 \leq b \leq 1$. Apart from free theories, this is the only “physical” interacting diagonal IQFT we will be interested in. A crucial property of the sinh-Gordon S-matrix is that it does not admit any further non-trivial CDD factors. Indeed, the S-matrix (1.62) itself can be interpreted as a pure CDD deformation of the free massive boson theory. In particular, the free theory is recovered smoothly in the limit in which the coupling constant is sent to zero, corresponding to $b \rightarrow 0$, where the S-matrix becomes trivial.

This feature is not accidental, but rather reflects a general property of affine Toda field theories: their exact relativistic S-matrices are given by the S-matrix of an underlying conformal field theory, perturbed by a relevant operator, multiplied by a minimal CDD factor encoding the interaction. From this perspective, the sinh-Gordon model can be regarded as the simplest example of an affine Toda theory, corresponding to the $SU(2)$ case. Other well-known examples include the Bullough–Dodd model, which is related to the Yang–Lee edge singularity, and more generally Toda theories associated with higher-rank Lie algebras [73, 18]. In all these cases, the interacting S-matrix is fixed uniquely (up to trivial factors) and coincides with the CDD dressing of the corresponding perturbed minimal CFT.

Because CDD factors automatically solve unitarity and crossing symmetry, they represent, in general, the only ambiguity in the solution of the bootstrap equations for a given particle content. However, in the case of the sinh-Gordon model this ambiguity is completely exhausted by (1.62) itself, which therefore provides a unique interacting solution compatible with all physical requirements. Returning to the application to the real-world, it is known in practice that not even one of the diagonal theories here described gives rise to any realistic situation⁸: they are too unconstrained (by symmetries) and as such they should be considered as useful (at least to us) toy models. Real-world particles have internal degrees of freedom which naturally implies non-diagonal scattering. Two relevant examples are given by the already mentioned sine-Gordon model, which appears often in condensed matter applications like Josephson junction arrays [78] and the magnetic deformation of the Ising model [79] which was experimentally observed 15 years ago [80].

⁸Even though purely diagonal IQFTs do not seem to correspond to any direct physical model, their non-relativistic limits can be identified with physically relevant systems. For instance, suitable non-relativistic limits of Toda field theories and the $O(N)$ non-linear sigma model reduce to Lieb–Liniger models [75, 76, 77]. These examples provide explicit evidence of the universality of the Lieb–Liniger model in describing non-relativistic integrable theories of bosons with local interactions.

To conclude this paragraph, we briefly take a detour to address a concept that will not be discussed further in this work, but nevertheless underlies the entire framework developed so far, namely the notion of **bootstrap**, after which this thesis is named. We have seen that all the constraints discussed up to this point are completely independent of the particle content of a given theory. In particular, the diagonal two-particle scattering amplitude of a local integrable quantum field theory can be constructed in terms of the building blocks (1.55). These building blocks exhibit a simple pole at $\theta = i\pi x$. When this pole lies on the physical sheet, namely for $0 < x < 1$, it signals the presence of a **bound state** in the spectrum of the theory. Let us consider this situation and place the simple pole at $\theta = iu_{AB}^C$, where the indices refer to a scattering process in which two stable particles of masses A and B form a stable particle C , namely $A + B \rightarrow C$. The particle C is an asymptotic one-particle state already present in the spectrum of the theory. Therefore, from (1.33), the mass of this particle is given by

$$m_C^2 = m_A^2 + m_B^2 + 2m_A m_B \cos u_{AB}^C, \quad (1.63)$$

which establishes a direct relation between the masses of the particles involved in the fusion process. The quantities u_{AB}^C are commonly referred to as *fusing angles*. A further important consequence follows from crossing symmetry: depending on the channel in which the process is viewed, each of the particles A , B , or C can be interpreted as a bound state of the other two. As a result, one can write two additional equations analogous to (1.63) by permuting the particle labels. Consistency then requires the constraint

$$u_{AB}^C + u_{BC}^A + u_{AC}^B = 2\pi. \quad (1.64)$$

This interplay between the presence of bound states and crossing symmetry lies at the core of the *bootstrap method* and of the associated bootstrap equations [81]. The term “bootstrap” originates from the expression “to pull oneself up by one’s bootstraps”, historically used to describe an impossible task. In the present context, it reflects the idea that particles are held together by interactions mediated by the very same particles appearing in the spectrum, leading to a set of self-consistency conditions.

At the level of the S-matrix, this self-consistency is encoded in the S-matrix bootstrap equations [82, 61, 65], which relate scattering amplitudes involving bound states to those of their constituents. Explicitly, one has

$$S_{AD}(\theta + i\bar{u}_{AC}^B) S_{BD}(\theta - i\bar{u}_{BC}^A) = S_{CD}(\theta), \quad (1.65)$$

where $\bar{u}_{AB}^C = \pi - u_{AB}^C$.

1.4.2 On the Uniqueness of Elastic S-matrices

Now, to end this section we will show, following [65], how the absence of particle production, non-vanishing backward scattering and factorisation are enough to determinate uniquely the S-matrix of a massive fermion g . Their work was an immediate follow up to Zamolodchikov’s first results on the sine-Gordon model that was obtained not only from the meromorphicity of the S-matrix elements but also requiring exactness of the semi-classical bound-state spectrum, vanishing of the reflection amplitude at integer values of the coupling constant λ and absence of resonances (poles away from the physical sheet). They instead adopt the Thirring model point of view and prove that the meromorphicity condition is enough to determine the scattering amplitude completely. Indeed, consider

a 2-particle scattering amplitude with both transmissive $t_{g\bar{g}}(\theta)$ and reflective $r_{g\bar{g}}(\theta)$ components

$${}^{\text{out}}\langle g(p'_1)\bar{g}(p'_2)|g(p_1)\bar{g}(p_2)\rangle^{\text{in}} = \delta(p'_1 - p_1^1)\delta(p'_2 - p_2^1)t_{g\bar{g}}(\theta) + \delta(p'_1 - p_2^1)\delta(p'_2 - p_1^1)r_{g\bar{g}}(\theta). \quad (1.66)$$

If $t_{gg}(\theta)$ is the particle-particle scattering amplitude then we can write down the crossing relations

$$t_{g\bar{g}}(\theta) = r_{g\bar{g}}(i\pi - \theta), \quad t_{g\bar{g}}(i\pi - \theta) = t_{gg}(\theta) = t_{\bar{g}\bar{g}}(\theta), \quad (1.67)$$

and, taking into account also the elasticity and meromorphicity constraints, the unitarity

$$t_{g\bar{g}}(\theta)t_{g\bar{g}}(-\theta) + r_{g\bar{g}}(\theta)r_{g\bar{g}}(-\theta) = 1, \quad (1.68)$$

$$r_{g\bar{g}}(\theta)t_{g\bar{g}}(-\theta) + t_{g\bar{g}}(\theta)r_{g\bar{g}}(-\theta) = 0, \quad (1.69)$$

$$t_{gg}(\theta)t_{gg}(-\theta) = 1. \quad (1.70)$$

Notice that, if $r_{g\bar{g}}(\theta) \equiv 0$ we return to the diagonal case (1.53) and these equations admit an infinite family of solutions $t_{g\bar{g}}(\theta)$, that we already classified in the previous subsection. Instead for $r_{g\bar{g}}(\theta) \neq 0$ we introduce the ratio

$$h(\theta) = \frac{t_{g\bar{g}}(\theta)}{r_{g\bar{g}}(\theta)}, \quad (1.71)$$

an odd function of θ , that satisfies

$$t_{g\bar{g}}(\theta)t_{g\bar{g}}(i\pi + \theta) = \frac{h(\theta)}{h(i\pi - \theta)}, \quad (1.72)$$

and the quadratic functional equation

$$h(i\pi - \theta)h(i\pi + \theta) + h^2(\theta) = 1. \quad (1.73)$$

Now comes into play the YBE (1.28), that we need to use to select the compatible solutions. In fact not all the solutions of this equations are compatible with the factorised form of the S-matrix. In particular, studying the 3-particle process (and its factorisation) $g(p_1)g(p_2)\bar{g}(p_3) \rightarrow g(p_1)\bar{g}(p_2)g(p_3)$, equation (1.73) yields an additional constraint

$$h(\theta_{31} + \theta_{12}) = h(i\pi + \theta_{31})h(\theta_{12}) + h(\theta_{31})h(i\pi - \theta_{12}), \quad (1.74)$$

from which, after symmetrisation we obtain

$$\frac{h(i\pi + \theta_{31}) - h(i\pi - \theta_{31})}{h(\theta_{31})} = \frac{h(i\pi + \theta_{12}) - h(i\pi - \theta_{12})}{h(\theta_{12})} = \text{const.} \quad (1.75)$$

For convenience the constant is chosen to be equal to $2\cos\mu$. A common and unique solution of the three equations (1.73)-(1.75) is given by the function

$$h(\theta) = \frac{\sinh(\mu\theta/\pi)}{\sinh(i\mu)}. \quad (1.76)$$

The uniqueness of this solution can be seen to descend from the multiplicative additivity of the combinations $L(x) = h(i\pi - x) + e^{i\mu}h(x)$, which indeed satisfy the so-called *exponential Cauchy functional equation*: $L(x + y) = L(x) \cdot L(y)$. What is left is to show

that all the amplitudes can be obtained by this solution $h(\theta)$. We need two technical assumptions here: first, there should exist a value of μ such that $t_{g\bar{g}}(\theta)$ is analytic and non-zero on the physical strip $0 < \text{Im}(\theta) < \pi$ and then, we need that the function $[1/\sinh(z - \theta)] \log t_{g\bar{g}}(\theta)$ vanishes in absolute value as $|\text{Re}(z)| \rightarrow \infty$. If this is the case, by Cauchy's formula

$$\log t_{g\bar{g}}(\theta) = \frac{1}{2\pi i} \int_C \frac{\log t_{g\bar{g}}(z)}{\sinh(z - \theta)} dz. \quad (1.77)$$

Therefore, since $(1/2i) \log t_{g\bar{g}}(\theta) = \delta_{g\bar{g}}(\theta)$, is the phase-shift, from (1.72) and (1.76) we have

$$\delta_{g\bar{g}}(\theta) = -\frac{1}{4\pi} \int_{-\infty}^{+\infty} \frac{dz}{\sinh(z - \theta)} \log \left(\frac{\sinh(\mu\theta/\pi)}{\sinh(\mu(i\pi - \theta)/\pi)} \right), \quad (1.78)$$

or through Fourier transform and partial integration

$$\delta_{g\bar{g}}(\theta) = \frac{1}{2} \int_0^{+\infty} \frac{dx}{x} \frac{\sinh[x(i\pi - \theta)/\pi] \sinh\left[\frac{x}{2} \left(\frac{\pi}{\mu} - 1\right)\right]}{\cosh\left(\frac{x}{2}\right) \sinh\left(\frac{\pi x}{2\mu}\right)}. \quad (1.79)$$

By using Malmsten's formula⁹ for the integral representation of the logarithm of the Gamma function and setting $\lambda = \mu/\pi$ it is then possible to derive Zamolodchikov's solution [61, 62] in terms of an infinite product of Gamma functions. We want to stress [67] that the unicity of this solution is obtained by adding several restrictions: in particular, the assumption that $\log t_{g\bar{g}}(\theta)$ was damped out forced all the CDD-factors of the type (1.56) to vanish.

1.5 Scattering Amplitudes in the Presence of Boundaries

In this section we briefly review the extension of integrable scattering amplitudes to models characterised by the presence of boundaries. Infinite and homogeneous systems are a great arena but sometimes they are too “idealised” with respect to physical situations where finite size effects, inhomogeneities and defects are always present. It is therefore natural to implement these occurrences by taking into account boundary effects [18, 83, 84] (see also [85] for a survey of related studies). Even though the bulk theory (i.e. without boundaries) is integrable, and therefore is characterised by a factorised scattering, it is highly non-trivial to obtain a factorised reflection at an arbitrary boundary. Most boundaries spoil integrability and only a restricted set of *integrable boundary conditions* preserve the desired structure. In general these boundary conditions are specified through a boundary action functional or as perturbed conformal boundary conditions [83] but for the sake of brevity we will directly take them for granted and consider amplitudes.

1.5.1 Boundary Scattering Amplitudes

Boundary scattering amplitudes were first discussed in [86, 87] but all the material reviewed here is taken from the works [88, 83, 89]. We will work only on semi-infinite

⁹ $\log \Gamma(z) = \int_0^{+\infty} dt \left[(z-1) - \frac{1-e^{-(z-1)t}}{1-e^{-t}} \right] \frac{e^{-t}}{t}$

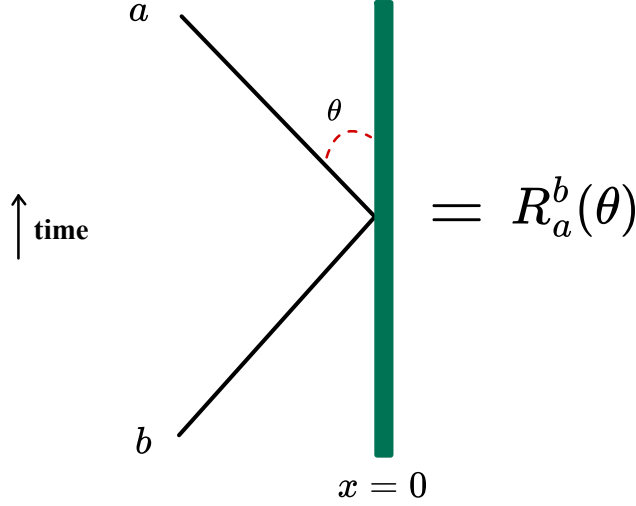


Figure 1.8: 1-particle reflection off the boundary at $x = 0$.

systems; considering a semi-infinite Minkowski space-time (x, t) , $t = -iy$, $x < 0$, we define asymptotic states as in Section 1.1. We have an impenetrable *boundary in space* and we introduce a boundary Hilbert space $\mathcal{H}_B$ associated with the half-line $y = \text{const}$, $x \in (-\infty, 0]$. The initial state

$$|\theta_1, \theta_2, \dots, \theta_n\rangle_{a_1, a_2, \dots, a_n}^{B, \text{in}} = A_{a_1}^\dagger(\theta_1), A_{a_2}^\dagger(\theta_2), \dots, A_{a_n}^\dagger(\theta_n)|0\rangle_B, \quad (1.80)$$

where the subscript B indicates that $|\dots\rangle \in \mathcal{H}_B$ and $|0\rangle_B$ is the ground state in the boundary Hilbert space. Again, the asymptotic state can be seen as created by a ZF-algebra analogous to (1.45), where in (1.80) we have $\theta_1 > \theta_2 > \dots > \theta_n$. The picture is the following: at time $t = -\infty$ particles are created and travel towards the boundary positioned at $x = 0$ – all the rapidities are positive. At $x = 0$ the particles are reflected and in the infinite future, $t \rightarrow \infty$, the state becomes a superposition of out-states

$$|-\theta'_1, -\theta'_2, \dots, -\theta'_n\rangle_{b_1, b_2, \dots, b_n}^{B, \text{out}}, \quad (1.81)$$

of particles moving away from the boundary with negative rapidities. Notice that because of integrability the scattering is purely elastic and therefore the set $\theta'_1, \theta'_2, \dots, \theta'_n$ can differ from the initial state just by permutations.

We can think of the boundary as an infinitely heavy impenetrable particle B sitting at the origin and then write the boundary vacuum as if it was “created” by a so-called “boundary creating operator” B :

$$|0\rangle_B = B|0\rangle. \quad (1.82)$$

Then this operator satisfies the following exchange relation with the ZF-algebra

$$A_a(\theta)B = R_a^b(\theta)A_b(-\theta)B, \quad (1.83)$$

where the coefficients $R_a^b(\theta)$ ¹⁰ are interpreted as the amplitudes of a 1-particle reflecting off the boundaries as shown in Figure 1.8. Interestingly this is not the only set-up we can

¹⁰It is important to note that from the structure of the boundary asymptotic states, if $m_a \neq m_b$ the amplitude $R_a^b(\theta)$ vanishes.

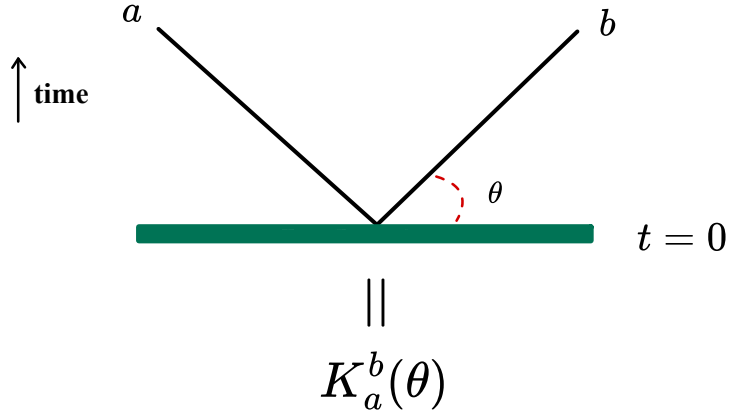


Figure 1.9: 1-particle reflection off the boundary at $t = 0$.

build from; we can also place a *boundary in time*, at $t = 0$: this is the crossed channel with respect to the previous situation, where the previous scattering process appears now as a virtual emission pair from the boundary, with amplitude $K_a^b(\theta)$ (see Figure 1.9).

Being the crossed channel, this amplitude is related to the previous one by analytic continuation to the domain $-\text{Im}(\theta) = i\pi/2, \text{Re}(\theta) < 0$ – as

$$K_a^b(\theta) = R_a^b\left(\frac{i\pi}{2} - \theta\right). \quad (1.84)$$

In this picture the boundary condition is encoded in a *boundary state* $|B\rangle$ that is contained in the Hilbert space of the bulk theory $\mathcal{H}$. Hence this state is a superposition of the asymptotic states (1.1) of the bulk theory but not all the states can enter: because of integrability and relativistic invariance only pairs of equal-mass particles with opposite rapidities or zero-rapidity particles¹¹ can enter the combinations. In the end one can write down the boundary state as an expansion whose coefficients are precisely the amplitudes $K_a^b(\theta)$:

$$|B\rangle = \left(1 + \frac{1}{2} \int_{-\infty}^{+\infty} d\theta K_a^b(\theta) A_a^\dagger(\theta) A^b(-\theta) + \dots\right) = \exp\left[\frac{1}{2} \int_{-\infty}^{+\infty} d\theta K_a^b(\theta) A_a^\dagger(\theta) A^b(-\theta)\right], \quad (1.85)$$

i.e., the boundary state is a linear superposition of infinitely-many multiparticle states but because of integrability, the multiparticle amplitudes are all factorised into an ordered infinite products of 2-particle amplitudes $K_a^b(\theta)$.

As in the infinite system, the fundamental amplitudes (now we have both bulk and boundary scattering available) $S_{a_1 a_2}^{b_1 b_2}(\theta)$ and $R_a^b(\theta)$ are constrained by a set of equations.

1. *Boundary Yang-Baxter equation (bYBE):*

$$R_{a_2}^{c_2}(\theta_2) S_{a_1 c_2}^{c_1 d_2}(\theta_1 + \theta_2) R_{c_1}^{d_1}(\theta_1) S_{d_2 d_1}^{b_2 b_1}(\theta_1 - \theta_2) = S_{a_1 a_2}^{c_1 c_2}(\theta_1 - \theta_2) R_{c_1}^{d_1}(\theta_1) S_{c_2 d_1}^{d_2 b_1}(\theta_1 + \theta_2) R_{d_2}^{b_2}(\theta_2), \quad (1.86)$$

¹¹This possibility will be discussed briefly in the next subsection in the simple case of the Ising model.

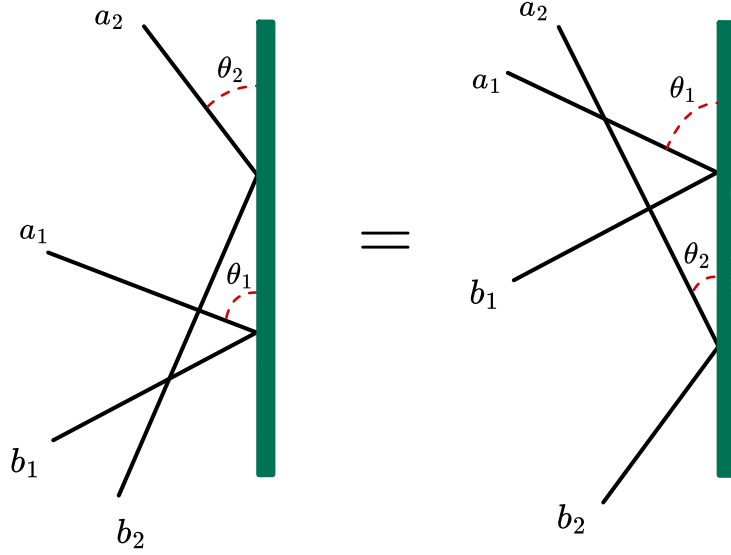


Figure 1.10: Boundary Yang-Baxter equations.

2. *Boundary Unitarity:*

$$R_a^c(\theta)R_c^b(-\theta) = \delta_a^b, \quad (1.87)$$

3. *Boundary Cross-Unitarity:*

$$K_{ab}(\theta) = S_{ab}^{cd}(2\theta)K_{cd}(-\theta), \quad (1.88)$$

represented graphically in Figure 1.10, 1.11 and 1.12 respectively.

Like in the bulk case, the equations fix completely solutions up to a CDD-ambiguity, i.e. we can always multiply a given solution by a function $\Phi_B(\theta)$ satisfying trivially (1.87) and

$$\Phi_B(\theta) = \Phi_B(i\pi - \theta). \quad (1.89)$$

1.5.2 The Boundary Ising Model

The simplest example we can consider is the Ising model at $T \neq T_c$ in zero magnetic field, a free theory of massive Majorana fermions $\Psi(x)$. The bulk S-matrix is just $S = -1$ and the equations (1.87) and (1.88) simplify to

$$R(\theta)R(-\theta) = 1, \quad R\left(\frac{i\pi}{2} - \theta\right) = -R\left(\frac{i\pi}{2} + \theta\right). \quad (1.90)$$

Having a free theory, we have linearity in the fermionic modes and there are two obvious boundary conditions which preserve integrability of the bulk theory, namely

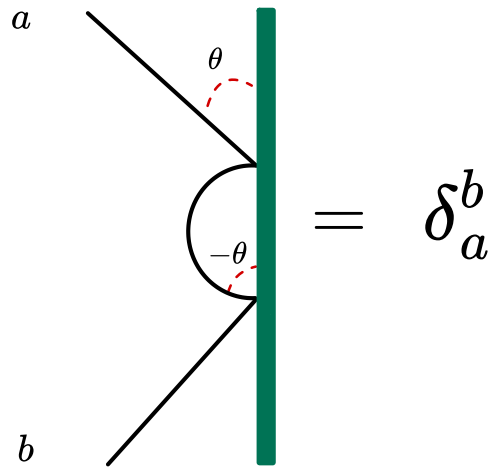


Figure 1.11: Boundary Unitarity.

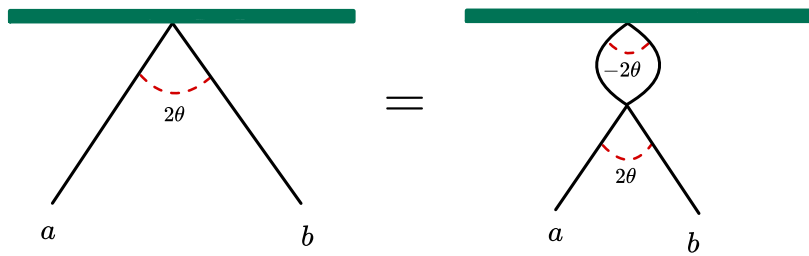


Figure 1.12: Boundary Cross-Unitarity.

$$(\Psi(x) + \bar{\Psi}(x))_{x=0} = 0, \quad \text{and} \quad (\Psi(x) - \bar{\Psi}(x))_{x=0} = 0. \quad (1.91)$$

In the following we will refer to the first one as the *fixed boundary* condition and to the second as the *free boundary* condition. Indeed, if we express the original spin operator of the Ising model in terms of the Majorana fermions we can see that the first case corresponds to the case where all the spins at the boundary are all equal to +1 (or -1) while in the other case the spins at the boundary can take any arbitrary value. We will not derive here the amplitudes but it is easy to see that to do it we just need to take the mode expansion of the Majorana fermions in terms of their creation/annihilation operators, substitute them into the boundary conditions (1.91) and then compare them with the exchange relation of the boundary creating operator (1.83).

Imposing the **fixed boundary condition** the reflection amplitude that solves (1.83) is

$$R_{\text{fixed}}(\theta) = i \tanh \left(\frac{i\pi}{4} - \frac{\theta}{2} \right). \quad (1.92)$$

The amplitude in the crossed channel then is simply $K_{\text{fixed}}(\theta) = i \tanh \frac{\theta}{2}$. Instead, imposing **free boundary condition**, the solution is given by

$$R_{\text{free}}(\theta) = -i \coth \left(\frac{i\pi}{4} - \frac{\theta}{2} \right), \quad (1.93)$$

with the corresponding $K_{\text{free}}(\theta) = -i \coth \frac{\theta}{2}$. Notice that this amplitude possesses a pole at $\theta = 0$ which points out at the existence of a zero-momentum 1-particle state in the boundary state (1.85), i.e.

$$|B_{\text{free}}\rangle \sim (1 + A^\dagger(0) + \dots) |0\rangle, \quad (1.94)$$

The physical picture [83] is that spin configurations associated to this boundary condition contain an infinitely long fluctuating domain wall which separates two domains with opposite magnetisation. The domain wall is interpreted as a zero-momentum particle emitted by the boundary as is shown in Figure 1.13.

Perhaps surprisingly, these two are not the only boundary conditions that preserve integrability in the Ising model. In fact the presence of a **boundary magnetic field** does not spoil integrability. At the level of the action, this is possible because even with the addition of this boundary field the action remains quadratic (in terms of the fermions). With a similar procedure as in the other two cases, it is then possible to arrive to a solution that actually interpolates between the other two boundary conditions:

$$R_h(\theta) = i \tanh \left(\frac{i\pi}{4} - \frac{\theta}{2} \right) \frac{\kappa - i \sinh \theta}{\kappa + i \sinh \theta}, \quad \text{with } \kappa = 1 - \frac{h^2}{2m}, \quad (1.95)$$

where h is the magnetic field. Varying h from 0 (free) to ∞ (fixed), as just anticipated, this amplitude interpolates between the two other boundary conditions realizing a *boundary RG flow*. The crossed-channel's amplitude reads

$$K_h(\theta) = i \tanh \left(\frac{\theta}{2} \right) \frac{\kappa + \cosh \theta}{\kappa + \cosh \theta}, \quad (1.96)$$

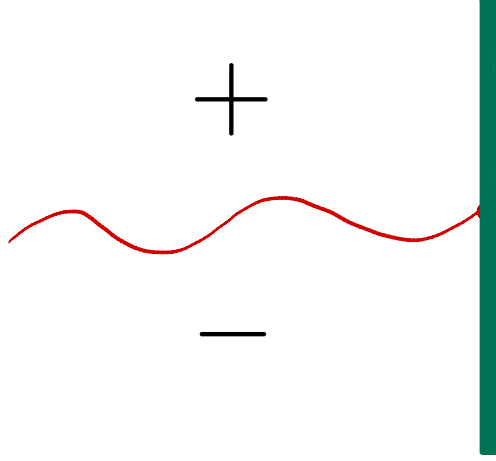


Figure 1.13: Domain wall configuration interpreted as a zero-momentum particle emitted by the boundary state.

and therefore for any non-zero value of h the amplitude no longer has a pole at $\theta = 0$ since the magnetic field splits the degeneracy of the two vacua from which we start at $h = 0$. As a final remark, we can note that for $h < h_c = \sqrt{2m}$ we can use the parametrisation $\kappa = \cos v$ and we can study the pole in the physical strip at $\theta = i(\pi/2 - v)$ that corresponds to a boundary bound state. This phenomenon is usually understood as a form of “*boundary hysteresis*”. For $h > h_c$ the pole disappears but we still have a pair of resonances localised at $\theta = -i\pi/2 \pm \theta_0$, where now κ is parametrised as $\kappa = -\cosh \theta_0$ with θ_0 positive and real.

Chapter 2

Irrelevant Deformations of IQFTs

In this chapter we will introduce the *generalised $T\bar{T}$ -deformations*: irrelevant perturbations of a given CFT or QFT (in the RG-flow sense). Recently, a particular class of such perturbations has attracted considerable interest: those generated by the field $T\bar{T}$, where T and $\bar{T}$ are the holomorphic and antiholomorphic components of the stress energy tensor. Note that we have adopted a different notation for the $T\bar{T}$ -deformation and for the operator itself $T\bar{T}$, since the class of deformations we are about to introduce are not given by the mere addition of this operator to the action but, as we shall see, arise from a more involved procedure. The earliest study of this field's properties, particularly its vacuum expectation value, was carried out in [90] for generic 2D quantum field theory and quickly followed by a systematic study of the form factors of the $T\bar{T}$ operator itself [91, 92].

Generalised $T\bar{T}$ -deformations and their properties have been studied from several viewpoints: in the context of 2D classical and quantum field theory¹ [102, 100, 103], in the framework of the ODE/IM correspondence [104] (see [105] for a review), employing the thermodynamic Bethe ansatz (TBA) approach [106, 107, 108, 109, 110, 111, 112, 113], from the viewpoint of perturbed conformal field theory [114, 115, 116, 117, 118, 119, 120, 121] and also in the context of string theory [122, 123, 124, 125], holography [126, 127, 128, 129, 130, 131, 132, 133, 134], quantum gravity [135, 101, 136, 137, 138, 139], out-of-equilibrium conformal field theory [140, 141], long-range spin chains [142, 143, 144, 145], and the generalised hydrodynamics (GHD) approach [146, 147, 148, 149].

2.1 Generalised $T\bar{T}$ -deformations of IQFTs

In this section, following closely [150] we will introduce generalised $T\bar{T}$ -deformations in the context of IQFTs.

¹While extensions of the $T\bar{T}$ -deformation to 1-dimensional, quantum-mechanical systems yields well-defined, controllable theories [93, 94, 95], the proposed higher dimensional generalisations [96, 97] present several complications, mainly consequence of the fewer amount of restrictions on the short-distance singularities arising in the OPE of energy-momentum tensor components. Some promising advances have been made in [98, 99] that considers extensions of the $T\bar{T}$ deformation in higher dimensions as field-dependent perturbations of the space-time geometry, in the same spirit as the geometric interpretation of [100] and the topological gravity picture of [101]

2.1.1 Composite Fields from the Integrals ff Motion

As we already saw, the most important characteristic of these theories is the presence of an infinite set of IM, generated by local currents, that is pairs of local fields $\{T_{s+1}(z), \Theta_{s-1}(z)\}$ connected by a continuity equation

$$\bar{\partial}T_{s+1}(z) = \partial\Theta_{s-1}(z), \quad (2.1)$$

where the label s represents the *spin* of the corresponding fields. Usually we are thinking of a subset $\mathcal{S} \equiv \{s\} \subset \mathbb{Z}$ which in general differs depending on the model under study but at least for the values $s = \pm 1$ we can easily recognise the equations (2.1) to represent the conservation of energy-momentum which should be expected by any standard QFT under general assumptions. In this case, the currents are precisely the components of the energy-momentum tensor $T_{\mu\nu}$ that for later convenience we reparametrise as

$$T = -2\pi T_{zz}, \quad \bar{T} = -2\pi T_{\bar{z}\bar{z}}, \quad \Theta = 2\pi T_{z\bar{z}}. \quad (2.2)$$

Considering a parity invariant theory (P-symmetric) the set $\mathcal{S}$ would result symmetric under P-reflection: $s \rightarrow -s$, therefore, it is convenient for the following to fix properly the notation. For $s > 0$ we write Θ_{-s-1} as $\bar{T}_{s+1}$ and T_{-s+1} as $\bar{\Theta}_{s-1}$, removing all negative entries from $\mathcal{S}$. Then for $s < 0$ the continuity equations (2.1) will read

$$\partial\bar{T}_{s+1}(z) = \bar{\partial}\bar{\Theta}_{s-1}(z). \quad (2.3)$$

From (2.1) and (2.3) follows directly that the integrals

$$P_s = \frac{1}{2\pi} \int_C T_{s+1}(z)dz + \Theta_{s-1}(z)d\bar{z}, \quad (2.4)$$

$$\bar{P}_s = \frac{1}{2\pi} \int_C \bar{T}_{s+1}(z)d\bar{z} + \bar{\Theta}_{s-1}(z)dz, \quad (2.5)$$

are invariant under trivial deformations of the integration contour C and therefore define local IM. This is precisely the construction we introduced in Chapter 0, in particular, eq. (0.49) was derived as the extension of the chiral continuity equation to the massive perturbation. The other ingredient for integrability is the commutativity of these charges P_s , i.e. we need

$$[P_s, P_{s'}] = 0, \quad \forall s, s' \in \mathcal{S}. \quad (2.6)$$

Notice also here that (2.4) and (2.5) are the natural generalisation to the massive case of (0.36). Having local IM of the form (2.4),(2.5) implies for the P_s (similar equations hold for the $\bar{P}_s$):

$$[P_{s'}, T_{s+1}(z)] = \partial A_{s',s}(z), \quad [P_{s'}, \Theta_{s-1}(z)] = \bar{\partial} A_{s',s}(z), \quad (2.7)$$

and

$$[P_{s'}, \bar{T}_{s+1}(z)] = \bar{\partial} B_{s',s}(z), \quad [P_{s'}, \bar{\Theta}_{s-1}(z)] = \partial B_{s',s}(z), \quad (2.8)$$

where $A_{s',s}$ and $B_{s',s}$ are some local fields. Note also that, in general, the commutator of a conserved charge P_s with any local field $\mathcal{O}$ can be defined to be

$$[P_{s'}, \mathcal{O}(z_0)] = \frac{1}{2\pi} \int_{C_{z_0}} [T_{s+1}(z)dz + \Theta_{s-1}(z)d\bar{z}] \mathcal{O}(z_0). \quad (2.9)$$

In the case of QFTs with “conventional UV behaviour” the components $\{T_{s+1}(z), \Theta_{s-1}(z)\}$ both have scale dimensions equal to $s + 1$. Now, having defined these conserved currents and their properties we want to use them to construct *composite scalar fields* taking products at different space time points, say z and z' and then take the limit $z \rightarrow z'$. In general this should be a procedure plagued by divergencies and ambiguities but the combination

$$\lim_{z \rightarrow z'} (T_{s+1}(z) \bar{T}_{s-1}(z') - \Theta_{s-1} \bar{\Theta}_{s-1}(z')) = X_s(z') + \text{derivatives terms}, \quad (2.10)$$

up to derivatives, is a well defined and not ambiguous object. This result was obtained originally by Al. B. Zamolodchikov [90] in the case $s = \pm 1$ and we review it in Appendix B. The composite fields X_s have mass dimensions $[\text{mass}]^{2(s+1)}$ and are therefore *irrelevant* in the RG-flow sense. Hence when used to perturb a given action, these operators will not influence the long-distance/IR-behaviour but instead they will alter drastically the UV properties. This means that, in general (not always), UV completeness will be lost. The $s = 1$ case is special: it defines the $T\bar{T}$ -deformation, as we will see in the next section. This operator first emerged in Zamolodchikov’s analysis of massless RG flows [151], particularly in the context of flows driven by a spontaneously broken supersymmetry between minimal models. For instance for the tricritical Ising to Ising flow, the dominant operator in the IR is precisely the composite field $T\bar{T}$:

$$\mathcal{L} = \frac{1}{2\pi} (\psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi} + \alpha \psi \partial \psi \bar{\psi} \bar{\partial} \bar{\psi}), \quad (2.11)$$

where $\psi \bar{\partial} \psi \bar{\psi} \partial \bar{\psi} \sim T\bar{T}$. This Lagrangian, the so-called Volkov-Akulov model [152], describes the goldstino emerging as the massless fermionic excitation resulting from the broken supersymmetry.

To end this section we want to introduce a very interesting and universal property satisfied by this family of operators. Consider an IQFT defined on an infinite cylinder, i.e. with the spatial coordinate on a circle of circumference R , $(x, y) \rightarrow (x + R, y)$ —the energy spectrum is then discrete. It can be shown [90, 150] that, taking generic diagonal matrix elements $\langle n | \dots | n \rangle$ of these operators, where $|n\rangle$ is an arbitrary non-degenerate eigenstate of the energy and momentum operators², we have

$$\langle n | X_s | n \rangle = \langle n | T_{s+1} | n \rangle \langle n | \bar{T}_{s+1} | n \rangle - \langle n | \Theta_{s-1} | n \rangle \langle n | \bar{\Theta}_{s-1} | n \rangle. \quad (2.12)$$

In the case $s = 1$ this property will be enough to completely characterise the energy-spectrum of the theory. Notice also that, by rotational invariance, the expectation values $\langle X_s \rangle$ will all vanish in the infinite system size limit, $R \rightarrow \infty$ for $s > 1$.

2.1.2 The $T\bar{T}$ -flow

We are finally ready to introduce the models we want to study in the later chapters. Consider an IQFT in the spirit of [3], i.e. as a relevant perturbation of some known CFT. Now, we can think of perturbing it again, say with the $s = 1$ operator defined in (2.10). This operator is a very special one, it is precisely equal to the determinant of the stress-energy tensor of the “seed theory”. Our intention is to *generate a flow*, that is, we want to consider trajectories $\mathcal{A}_\alpha$ in the action space Σ , where α denotes the parameter along the trajectory, such that at any point of the curve, the tangent vector is proportional to

²Assume also the normalisation $\langle n | n \rangle = 1$.

$$\frac{d}{d\alpha} \mathcal{A}_\alpha = \frac{1}{\pi^2} \int d^2 z (\det T)_\alpha . \quad (2.13)$$

This equation defines the so-called $T\bar{T}$ -deformation, where the name comes from CFTs, where the determinant is precisely given by the product of just the chiral components of $T^{\mu\nu}$ (the trace Θ is zero in the conformal theory). These deformations preserve integrability and are universal (in the sense that at least the $s = 1$ case, is almost always clearly defined, by deriving from the existence of the stress-energy tensor itself). At this point some observations need to be made. Being irrelevant these deformations affect the UV properties of the theory and therefore it is not obvious why one should try to study the massive field theory. Indeed at very high energies (compared to the physical mass of the particles) the deformed massive theory is expected to possess the same UV singularities as the deformed CFT [153]. The massive theory is associated with a relevant perturbation [3] that selects a spectrum of particles which describes the IR. In the end, to fix uniquely the fields X_s we need to consider an IQFT, where these fields can be constructed explicitly in terms of the local IM densities (2.4)-(2.5) [150]. Another important aspect is that IQFTs' solvability is related to the possibility of reconstructing all the interesting physical quantities to a single object, the S-matrix. In the next section we will prove that these irrelevant deformations modify the scattering amplitudes in a very precise way, that is, they amount to multiplicative CDD-factors of the type (1.56) and therefore, in principle, the IQFTs techniques can be carried out [106].

2.1.3 The Inviscid Burgers Equation and the Hagedorn Transition

As anticipated, we want to study the expectation value (2.12) in the $s = 1$ case. Having the theory defined on a cylinder of radius R , it was proven in [90] that

$$\langle n | X_1 | n \rangle = \langle n | T | n \rangle \langle n | \bar{T} | n \rangle - \langle n | \Theta | n \rangle \langle n | \bar{\Theta} | n \rangle , \quad (2.14)$$

where inside the bracket we have precisely the components of the stress-energy tensor (2.2) and $X_1 := \det T$. Being in a finite-size geometry we expect a discrete spectrum; we denote $E_n = E_n(R, \alpha)$ the energy of the stationary states $|n\rangle$ and $P_n = P_n(R) = 2\pi l_n/R$ ($l_n \in \mathbb{Z}$) the corresponding quantised spatial momenta. In [90] is shown that

$$\langle n | X_1 | n \rangle = -\frac{\pi^2}{R} \left(E_n \partial_R E_n + \frac{P_n^2}{R} \right) . \quad (2.15)$$

The next crucial step was then taken more than ten years later simultaneously and independently in the two fundamental works [106, 150]. In fact, according to the definition (2.13), an infinitesimal shift of α is generated precisely by X_1 and when the operator is sandwiched between eigenstates, the result will be proportional to the variation of the energy with respect to the deformation itself. Therefore we are led to a close differential equation for the deformed energy levels in terms of the undeformed ones that act as initial condition at the RG-time $\alpha = 0$ [106, 150]:

$$\partial_\alpha E(R, \alpha) + E(R, \alpha) \partial_R E(R, \alpha) + \frac{P^2}{R} = 0 . \quad (2.16)$$

The equation is the same for all n so we dropped the index. Equation (2.16) is known as the *forced inviscid Burgers equation*, known for describing an incompressible fluid in

the absence of viscosity. In our context the energy $E(R, \alpha)$ plays the role of the velocity field and α is the time variable. Notice also that the term proportional to P can be interpreted as a “driving force”, with dimensions of a momentum flux and therefore can be interpreted as a pressure. The general solution for any P can be worked out with the method of characteristics but for $P = 0$ the equation (2.16) reduces now to the standard inviscid Burgers equation

$$\partial_\alpha E(R, \alpha) + E(R, \alpha) \partial_R E(R, \alpha) = 0. \quad (2.17)$$

It is now instructive to derive the solutions of this equation explicitly to study their properties: the inviscid Burgers equation is known to possess solutions that develop shock waves, i.e. singularities. This pathology has physical meaning and it is one of the main features of these theories. The method of characteristics is the simplest way to treat a partial differential equation (PDE) as (2.17): a quasi-linear first order hyperbolic equation. It is a powerful method that relies on finding special curves, the so-called *characteristics*, in the (α, R) plane, along which the PDE reduces to a family of ordinary differential equation (ODE). Once the ODE is found, it can be solved along the characteristic curves and we can revert back the solution to the original PDE. Take again equation (2.17)

$$\frac{\partial E(R, \alpha)}{\partial \alpha} + E(R, \alpha) \frac{\partial E(R, \alpha)}{\partial R} = 0. \quad (2.18)$$

Considering an arbitrary curve $\alpha \rightarrow R(\alpha)$, then by chain rule, the total derivative of E along the curve is

$$\frac{dE(R, \alpha)}{d\alpha} = \frac{\partial E(R, \alpha)}{\partial \alpha} + \frac{dR}{d\alpha} \frac{\partial E(R, \alpha)}{\partial R}. \quad (2.19)$$

Now, the idea is to pick $dR/d\alpha$ in such a way that the PDE will imply $dE/d\alpha = 0$, i.e. $E(R, \alpha)$ would be constant along it. From the PDE (2.17)

$$\frac{dE(R, \alpha)}{d\alpha} = \left(\frac{dR}{d\alpha} - E(R, \alpha) \right) \frac{\partial E(R, \alpha)}{\partial R}. \quad (2.20)$$

For this to vanish for all E (except from trivial constant cases) we need to choose

$$\frac{dR}{d\alpha} = E(R, \alpha), \quad (2.21)$$

and then (2.20) will follow, that is $E(R, \alpha)$ will be constant along each characteristic curve $R(\alpha)$. Now we need to solve a system of ODEs along the characteristics:

$$\frac{dR}{d\alpha} = E(R, \alpha), \quad \text{and} \quad \frac{dE(R, \alpha)}{d\alpha} = 0. \quad (2.22)$$

We now need initial conditions. We impose the energy profile at $\alpha = 0$ to be $E(R, 0) = E_0(R)$ and each characteristic is uniquely labeled by its starting point:

$$R(0) = R_0, \quad E(R_0, 0) = E_0(R_0). \quad (2.23)$$

Recall now that the energy is constant along the characteristics:

$$E(R(\alpha), \alpha) = E_0(R_0). \quad (2.24)$$

Therefore, integrating the characteristic equation (2.22) we obtain

$$R(\alpha) = R_0 + \alpha E_0(R_0). \quad (2.25)$$

Since $E(R, \alpha) = E_0(R_0)$, we can now invert the relation (2.25) to get the **implicit** equation

$$E(R, \alpha) = E_0(R - \alpha E(R, \alpha)). \quad (2.26)$$

This equation gives the universal $T\bar{T}$ -deformed spectrum (on the cylinder) of any possible model as a function of the undeformed one. Notice also that to get this result we never used integrability even though the same solution can be found by IQFT techniques as the TBA and the NLIE [106]. We are finally ready to discuss some possible solutions and their breakdown, i.e. how shock singularities develop. The problem is that, the method of characteristics **works only as long as characteristics do not intersect**. Once they cross, the solution becomes multi-valued corresponding to a break-down of the classical method. In fact we found that each characteristic is defined by (2.25): since the energy is constant along each curve, we can recover the solution at a given point (R, α) by finding the characteristic that passes through it, i.e. finding the corresponding R_0 . To do this we must invert the function (2.25) and therefore **invertibility** is the crucial property that guarantees that characteristics do not cross each other. To check invertibility we need to check the derivative of (2.25) with respect to R_0 :

$$R'(\alpha) := \frac{dR}{dR_0} = 1 + \alpha \frac{dE_0(R_0)}{dR_0}. \quad (2.27)$$

When this derivative vanishes the mapping ceases to be locally invertible, i.e. two or more characteristics intersect at the same point (R, α) . Therefore the shock condition is

$$1 + \alpha \frac{dE_0(R_0)}{dR_0} = 0. \quad (2.28)$$

This means that there exists a **critical value** of the deformation parameter $\alpha = \tilde{\alpha}$ at which the first shock appears:

$$\tilde{\alpha} = - \left(\frac{dE_0(R_0)}{dR_0} \right)^{-1}. \quad (2.29)$$

We can understand the nature of the singularity by expanding the characteristic curve R around the critical point R_c defined by

$$1 + \alpha \frac{dE_0(R_0)}{dR_0} \Big|_{R_0=R_c} = 0. \quad (2.30)$$

Then we can Taylor expand $R(R_0)$ around this point:

$$R = R_0 + \alpha E_0(R_0) \simeq R_c + \alpha E_0(R_c) + (1 + \alpha E'_0(R_0))(R_0 - R_c) + \frac{\alpha}{2} E''_0(R_c)(R_0 - R_c)^2 + \dots \quad (2.31)$$

but at the shock point by definition $1 + \alpha E'_0(R_c) = 0$, hence the linear term in (2.31) vanishes and we are left with

$$R \simeq R_c + \alpha E_0(R_c) + \frac{\alpha}{2} E''_0(R_c)(R_0 - R_c)^2 + \dots \quad (2.32)$$

Inverting this expression for R_0 we get to leading order

$$R - R_c - \alpha E_0(R_c) \simeq \frac{\alpha}{2} E_0''(R_c) (R_0 - R_c)^2, \quad (2.33)$$

and solving for $R_0 - R_c$:

$$R_0 - R_c \simeq \sqrt{\frac{2(R - R_c - \alpha E_0(R_c))}{\alpha E_0''(R_c)}}. \quad (2.34)$$

Now, we can plug this solution in (2.24) to find, expanding again

$$\begin{aligned} E(R, \alpha) &= E_0(R_0) = E_0(R_c) \simeq E_0(R_c) + E_0'(R_c)(R_0 - R_c) + \dots \\ &\simeq E_0(R_c) + E_0'(R_c) \sqrt{\frac{2(R - R_c - \alpha E_0(R_c))}{\alpha E_0''(R_c)}} + \dots \end{aligned} \quad (2.35)$$

This shows that the energy develops a **square-root branch point** at the shock location. We are finally ready to see a concrete example. Consider now the undeformed theory to be a unitary 2-dimensional CFT. For the initial conditions we have [23]:

$$E_n(R, 0) = E_n(R) = \frac{2\pi}{R} \left(\Delta_n - \frac{c}{12} \right), \quad P_n(R, 0) = P_n(R) = \frac{2\pi}{R} s_n, \quad (2.36)$$

where $\Delta_n = h_n + \bar{h}_n$ is the scaling dimension, $s_n = h_n - \bar{h}_n$ is the spin and c is the central charge. Considering the ground-state in the simplest setting possible, like a free boson (the same reasoning works also for more involved theories where $\Delta_0 \neq 0$), we have

$$E_0(R) = -\frac{\pi c}{6R}. \quad (2.37)$$

Imposing the shock condition (2.28) we get

$$\frac{dE_0(R_0)}{dR_0} = \frac{\pi c}{6R^2} > 0, \quad (2.38)$$

and therefore from (2.29), as soon as α becomes negative the singularity develops.

Always with the method of characteristics, and initial conditions (2.36) we can find a general solution of (2.16) [154]. Actually, there will be two solutions $E_n^\pm(R, \alpha)$ for the equation, corresponding to the two branches of square root:

$$E_n^\pm(R, \alpha) = \frac{R}{2\alpha} \left[1 \pm \sqrt{1 - \frac{4\alpha E_n(R)}{R} + \frac{4\alpha^2 P_n^2}{R^2}} \right], \quad (2.39)$$

but we can select the “physical branch” by Taylor expanding for small deformations $\alpha \sim 0$ and seeing which one of the two reduce to the original CFT data. Indeed, taking the ground-state, we can see that the branch with the minus sign goes like

$$E_0^-(R, \alpha) \sim \frac{R}{2\alpha} \left[1 - \left(1 - \frac{2\alpha E_0(R)}{R} - \dots \right) \right] = E_0(R) + \dots \quad (2.40)$$

Hence, it correctly reproduces the undeformed theory, while the other branch, in the same limit, is seen to diverge as $\sim R/\alpha$. As already mentioned, these shock singularities are not just a mathematical feature but rather a signal of the breakdown of theory. To end this section we want to proof that the appearance of shock singularities in the finite-volume spectrum is precisely related to the breakdown of the theory at short distances

and in fact it signals an Hagedorn transition [81, 155]. To see this we need to compute the density of states: the irrelevant nature of the perturbation causes an exponential growth of the density of states leading to a *maximal temperature*, known as *Hagedorn temperature*. This is a signal of a phase transition where conventional thermodynamics break down, often interpreted as the emergence of a stringy/non-local regime³. To compute the density of states we need first to define the partition function on a torus. Actually, $T\bar{T}$ -deformed CFTs enjoy a lot of interesting and deep modular properties but we don't refer the interested reader to [115, 154] for the details.

To conclude this section, let us emphasise that an integrable quantum field theory deformed by a $T\bar{T}$ operator exhibits remarkably different physical properties depending on the sign of the deformation parameter. Consider a theory with undeformed ground-state energy given by (2.37). Taking the physical branch, the deformed ground-state energy for $P_0 = 0$ is

$$E_0^-(R, \alpha) = \frac{R}{2\alpha} \left[1 - \sqrt{1 + \frac{2\pi\alpha c}{3R^2}} \right]. \quad (2.41)$$

If $\alpha < 0$, the function eventually develops a branch point at a certain value of α ; this is precisely the shock-singularity we described above. On the other hand, for $\alpha > 0$, the ground-state energy is well-defined for arbitrarily large values of α . However, in the high-energy limit $R \rightarrow 0$ we obtain

$$\lim_{R \rightarrow 0} E_0^-(R, \alpha > 0) = -\sqrt{\frac{\pi c}{6\alpha}}, \quad (2.42)$$

i.e. a *finite* α -dependent ground-state energy $-e_0(\alpha)$ in the UV. This behavior is not the one you would expect from a standard local QFT where the vacuum energy becomes very large as you probe very short distances (the UV limit). This normally happens because there are infinitely many high-energy modes — the so-called zero-point fluctuations — contributing to the energy. As a result, the ground state energy usually diverges in the UV, as one can see for instance from eq. (2.37). This occurs because the deformation introduces a new “minimal length” scale, $\alpha \sim \ell^2$, and when the finite volume R is sent to zero, this length remains as the only relevant scale, effectively taking its place. Notice that in the limit $\alpha \rightarrow 0$, this density diverges exactly as the Casimir energy would have diverged as a function of R .

2.2 CDD-factors from $T\bar{T}$ -deformations

The main motivation behind our interest in this class of deformations of integrable quantum field theories (IQFTs) lies in the fact that they correspond, at the level of the S-matrix, to simple multiplicative CDD factors. Specifically, it is understood that, deforming a given IQFT with S-matrix $S_0(\theta)$ with a generic number of generalised $T\bar{T}$ -deformations of spin $s \in \mathcal{S}$, as defined by (2.10) and (2.13), corresponds to the following deformation of the scattering amplitude:

³The meaning of the Hagedorn temperature can vary significantly across different models. In some cases, it signals a first-order phase transition, while in others it corresponds to a second-order or even a more subtle crossover, depending on the underlying microscopic details and the nature of the deformation [155, 106, 150, 156, 157].

$$S_\alpha(\theta) := S_0(\theta)\Phi_\alpha(\theta) \quad \text{with} \quad \log \Phi_\alpha(\theta) = -i \sum_{s \in \mathcal{S}} \alpha_s m^{2s} \sinh(s\theta). \quad (2.43)$$

Notice that this CDD-factors are precisely the ones already mentioned in (1.56). Here $\mathcal{S}$ denotes the set of the spins of the local conserved quantities and it is a fundamental *datum* that depends on the specific theory under consideration. In the interpretation of IQFTs as massive perturbations of CFTs, these are the conserved charges that are not destroyed by the deformation process [3, 158]. In many cases, such as the thermal Ising and sinh-Gordon models, $\mathcal{S}$ coincides with the set of odd natural numbers, but this is not always the case⁴. Already in [161], it was observed that deforming the Ising field theory (whose S-matrix is simply $(S = -1)$ with a sinh-Gordon-like CDD factor (1.62) yields a “bosonic” type S-matrix. These theories were generally labelled “unphysical”⁵ due to the appearance of complex values in the finite-volume ground state energy. Nevertheless, it was understood that this S-matrix corresponds to a massive Ising model perturbed by a particular combination of irrelevant operators, with $T\bar{T}$ being the leading term in the perturbative expansion. However, these CDD-factors were first employed within the framework of integrability techniques in the context of string theory [162, 163]. Specifically, it was discovered that the finite-size spectrum of free massless bosons deformed by a simple scalar CDD factor can be exactly computed via the Thermodynamic Bethe Ansatz (TBA)⁶, reproducing the well-known Nambu–Goto energy levels. This approach, initially developed to describe the closed string spectrum, was later extended to open strings and to arbitrary 2D CFTs, offering a novel integrable perspective on effective string models for confining flux tubes [164, 106]. In [165] certain theories of free fields $\phi(Q, x)$ on noncommutative Minkowski spaces with different noncommutativity parameters Q were studied. These models were proven to satisfy a weaker form of localisation called *wedge localisation*. Interestingly, these models can also be viewed as non-local (but wedge-local) field theories on a flat Minkowski space with a non-trivial S-matrix given precisely by the $s = 1$ term in (2.43), i.e. the CDD factor coming from a “pure” $T\bar{T}$ -deformation. Wedge locality is a weaker notion of locality: instead of demanding that fields commute at arbitrary spacelike separation, one requires commutativity only when the fields are localised in opposite wedge-regions of Minkowski space. Note that a wedge is not a bounded region - it extends infinitely - so this condition is strictly weaker than the usual point-locality. The existence of non-trivial “local” observables and their characterisation is one of the main open questions in the study of these models. The construction here described is briefly summarised in Appendix C.

2.2.1 $T\bar{T}$ -deformations and Dressing of the ZF-algebra

We want to provide here an original (and unpublished so far) way of proving the relation between $T\bar{T}$ -deformations and the CDD-factors (2.43). The general idea comes from [166]. We start from an integrable theory with ZF-algebra (1.45):

⁴For example in the magnetic deformation of Ising model, $\mathcal{S} = \{1, 7, 11, 13, 17, 19, 23, 29\} \bmod 30$ [3]. Remarkably, these numbers are the *Coxeter exponents* of the Lie algebra E_8 . This is just one instance of the deep relationship between IQFTs and the structure of Lie algebras, particularly evident for models in the Toda family. The interested reader can find more details in, e.g., [159, 160].

⁵As clarified in subsection 1.4.1, by “physical QFT” we mean a consistent local quantum field theory.

⁶T the first TBA analysis on models with S-matrices of the type (2.43) were performed in [162, 164], before the “official” introduction of the $T\bar{T}$ -deformation in [150, 106] although the fact that this operator possessed a key role had already been noted [151, 161, 164].

$$\begin{aligned} Z(p_1)Z(p_2) &= S(p_1, p_2)Z(p_2)Z(p_1), \\ Z(p_1)Z^\dagger(p_2) &= S(p_2, p_1)Z^\dagger(p_2)Z(p_1) + \delta(p_1 - p_2), \end{aligned} \quad (2.44)$$

along with the usual S-matrix properties

$$S^{-1}(p_1, p_2) = (S(p_1, p_2))^* = S(p_2, p_1) = S(-p_1, -p_2). \quad (2.45)$$

Notice that in the present analysis we depart from our usual rapidity variables and revert to momenta. This choice is motivated by two reasons. First, the perturbative computation we follow is more naturally expressed in terms of two-momentum conservation, where working directly with momenta avoids unnecessary changes of variables. Second, the original reference [166] on which this section is based also employs momentum variables, so adopting the same convention allows for a more direct comparison with its results. This change is purely technical: as we know, momenta and rapidities are related one-to-one for massive particles, and no ambiguity arises. For consistency with the rest of this work, however, we shall eventually return to rapidity variables when presenting the final result to facilitate comparison and maintain a uniform notation. That said, we want now to study possible deformations of this algebra that still satisfy all the standard properties and preserve integrability. Infinitesimally we have

$$\begin{aligned} Z'(p) &= Z(p) + \epsilon W(p), \\ S'(p_1, p_2) &= S(p_1, p_2) + \epsilon T(p_1, p_2). \end{aligned} \quad (2.46)$$

In order to satisfy the ZF algebra they must fulfill the following relations [166]:

$$\begin{aligned} W(p_1)Z(p_2) + Z(p_1)W(p_2) &= T(p_1, p_2)Z(p_2)Z(p_1) + S(p_1, p_2)(Z(p_2)W(p_1) + W(p_2)Z(p_1)), \\ W(p_1)Z^\dagger(p_2) + Z(p_1)W^\dagger(p_2) &= T(p_2, p_1)Z^\dagger(p_2)Z(p_1) + S(p_2, p_1)(Z^\dagger(p_2)W(p_1) + W^\dagger(p_2)Z(p_1)), \end{aligned} \quad (2.47)$$

and unitarity conditions for T

$$(T(p_1, p_2))^* = T(p_2, p_1) = T(-p_1, -p_2) = -T(p_1, p_2)S^{-2}(p_1, p_2). \quad (2.48)$$

Note that infinitesimal changes of the S-matrix are especially subtle and not taken for granted if we consider a free boson because for every other integrable theory except this one the S-matrix at zero momentum is $S(0) = -1$. Therefore we can say that the transition from a free bosonic point of an integrable theory to another one is discontinuous at the S-matrix level⁷ and in this general construction we are explicitly thinking of S-matrices that are “fermionic” (at zero rapidity). Returning to the main point, we are interested in those transformations that in [166] are called “*generators of S-matrix changes*”. These are transformations that induce a non-zero function T in the S-matrix and they are unique in the sense that any infinitesimal transformation that has the same effect must involve this ones⁸. The ansatz is the following

$$W(p) = \left(\sum_q \alpha_{p,q} Z^\dagger(q) Z(q) \right) Z(p). \quad (2.49)$$

⁷In fact it is proven [167] that it is possible to obtain the S-matrix of all possible IQFT with smooth deformations of the free fermion.

⁸This is in contrast with the usual aim of [83, 166] where the desired deformation of the states needs to preserve the S-matrix.

Inserting this ansatz in the conditions above gives a very strict constraint which is completely encoded in a relation for the coefficients in the sum:

$$\alpha_{p,q} + \alpha_{p,q}^* = 0. \quad (2.50)$$

Hence, one can also set directly $\alpha_{p,q} = ia_{p,q}$ with a real functions. This infinitesimal deformation of the ZF algebra preserves conserved charges by construction and the corresponding finite transformation reads

$$Z'(p) = \mathcal{P} \left[\exp \left(i \int dt \hat{\mathcal{L}}_p(t) \right) \right] Z(p), \quad \text{where} \quad \hat{\mathcal{L}}_p(t) = \sum_q \alpha_{p,q} Z^\dagger(t) Z(q), \quad (2.51)$$

where $\mathcal{P}[\dots]$ denotes path-ordering integration and t is a continuous parameter along some path. This produces a change in the original S-matrix given by

$$S'(p, q) = \exp \left[i \int ds (a_{q,p}(s) - a_{p,q}(s)) \right] S(p, q). \quad (2.52)$$

This closely resembles the result of [103], where the scattering amplitudes are dressed via topological gauging of the Poincaré symmetry, implemented through a path integral deformation by bilinear current operators. In contrast, our approach is inspired by boundary integrable quantum field theories and quantum quenches, where the boundary or quench state can be expressed as a dressing of the initial state by coherent pairs of particle–antiparticle excitations⁹. Therefore the transformation *convert the old particle into a new particle of the same momentum accompanied (or “dressed”) by a particle-antiparticle*. In the context of the $\text{T}\bar{\text{T}}$ -deformation this should correspond to “graviton-antigravitons” pairs, i.e. the original particle gets gravitationally dressed. Hence, the aim of this section is two-fold: first we will prove perturbatively that by dressing the ZF-algebra with $\text{T}\bar{\text{T}}$ we correctly reproduce the CDD-factors (2.43): to prove this point we will consider the simplest and most tractable theory (at least perturbatively) there is, i.e. a massive free bosonic theory. Secondly, we will use the general idea to propose an exact and universal deformation of the ZF-algebra that correctly implies it in the spirit of [166].

Consider the free massive boson Hamiltonian

$$H_0 = \frac{1}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 + \frac{1}{2} m^2 \phi^2, \quad (2.53)$$

and deform it with

$$T\bar{T} = \varepsilon^{\mu\rho} \varepsilon^{\nu\sigma} T_{\mu\nu} T_{\rho\sigma} = \frac{1}{2} [(\partial_\mu \phi \partial^\mu \phi)^2 - m^4 \phi^4]. \quad (2.54)$$

We want to dress the ZF operators with the interaction following [166]. We first define auxilliary operators

$$B_+^\dagger(k) = \Omega_+ A^\dagger(k) \Omega_+^\dagger, \quad B_-^\dagger(k) = \Omega_- A^\dagger(k) \Omega_-^\dagger, \quad (2.55)$$

where $\Omega_\pm$ are the Møller operators

$$\Omega_\pm = \lim_{T \rightarrow \pm\infty} e^{-i \int_{-T}^0 dt H_{\text{int}}}. \quad (2.56)$$

⁹We already saw a similar construction for the boundary state (1.85).

Acting on the vacuum state of the interacting theory, the operators $B_+^\dagger(k)$, $B_-^\dagger(k)$ create “in” and “out” states respectively. We consider, as usual, the interaction Hamiltonian as normal ordered, so that we have at first order, using the Hausdorsf-Baker formula

$$B_\pm^\dagger(k) = A^\dagger(k) - i\frac{\alpha}{2} \int_{\mp\infty}^0 dt \int dx [(\partial_\mu \phi \partial^\mu \phi)^2 : -m^4 : \phi^4 :], A^\dagger(k)] . \quad (2.57)$$

For simplicity let's divide and calculate separately the two contributions. We have

$$X_\pm^\dagger(k) = i\frac{\alpha}{2} \int_{\mp\infty}^0 dt \int dx [(\partial_\mu \phi \partial^\mu \phi)^2 :, A^\dagger(k)] ; \quad (2.58)$$

$$Y_\pm^\dagger(k) = im^4 \frac{\alpha}{2} \int_{\mp\infty}^0 dt \int dx [:\phi^4 :, A^\dagger(k)] . \quad (2.59)$$

Actually (2.59) was already calculated in [166] and therefore we just need to calculate X and then take the difference. We define

$$\phi(x, t) = \frac{1}{2\pi} \int \frac{dk}{\sqrt{2E_k}} (e^{ik \cdot x} A^\dagger(k) + e^{-ik \cdot x} A(k)) , \quad (2.60)$$

$$\partial_\mu \phi(x, t) = \frac{i}{2\pi} \int \frac{dk}{\sqrt{2E_k}} k_\mu (e^{ik \cdot x} A^\dagger(k) - e^{-ik \cdot x} A(k)) , \quad (2.61)$$

with $E_k^2 = k^2 + m^2$ and $[A^\dagger(k), A(p)] = 2\pi\delta(k - p)$. We get

$$\begin{aligned} [(\partial_\mu \phi \partial^\mu \phi)^2 :, A^\dagger(k)] &= 4 \int \frac{dk_1 dk_2 dk_3}{(2\pi)^3 \sqrt{2^4 E_{k_1} E_{k_2} E_{k_3} E_k}} \times \\ &\times \left[-(k_1 \cdot k_2)(k_3 \cdot k) A^\dagger(k_1) A^\dagger(k_2) A^\dagger(k_3) e^{i(k_1+k_2+k_3-k) \cdot x} \right. \\ &+ 3(k_1 \cdot k_2)(k_3 \cdot k) A^\dagger(k_1) A^\dagger(k_2) A(k_3) e^{i(k_1+k_2-k_3-k) \cdot x} \\ &- [3(k_1 \cdot k_2)(k_3 \cdot k) + (k_1 \cdot k)(k_2 \cdot k_3)] A^\dagger(k_1) A(k_2) A(k_3) e^{i(k_1-k_2-k_3-k) \cdot x} \\ &\left. + 2(k_1 \cdot k_2)(k_3 \cdot k) A(k_1) A(k_2) A(k_3) e^{i(-k_1-k_2-k_3-k) \cdot x} \right] . \end{aligned}$$

Where $(k_i \cdot k_j) = (k_i)_\mu (k_j)^\mu$ is the scalar product of the momenta. Now the idea is to invert some of the momenta inside the integral so that, integrating in space we can factorise a common Dirac delta. We obtain

$$\begin{aligned} X_\pm^\dagger(k) &= i\frac{\alpha}{2} 4 \int_{\mp\infty}^0 dt \frac{e^{-iE_k t}}{\sqrt{2E_k}} \int \frac{dk_1 dk_2 dk_3 (2\pi) \delta(k + k_1 + k_2 + k_3)}{(2\pi)^3 \sqrt{2^3 E_{k_1} E_{k_2} E_{k_3}}} \times \\ &\times \left[-((-k_1) \cdot (-k_2))((-k_3) \cdot k) A^\dagger(-k_1) A^\dagger(-k_2) A^\dagger(-k_3) e^{-i(-E_{k_1}-E_{k_2}-E_{k_3})t} \right. \\ &+ 3((-k_1) \cdot (-k_2))(k_3 \cdot k) A^\dagger(-k_1) A^\dagger(-k_2) A(k_3) e^{-i(-E_{k_1}-E_{k_2}+E_{k_3})t} \\ &- [3((-k_1) \cdot k_2)(k_3 \cdot k) + ((-k_1) \cdot k)(k_2 \cdot k_3)] A^\dagger(-k_1) A(k_2) A(k_3) e^{-i(-E_{k_1}+E_{k_2}+E_{k_3})t} \\ &\left. + 2(k_1 \cdot k_2)(k_3 \cdot k) A(k_1) A(k_2) A(k_3) e^{-i(+E_{k_1}+E_{k_2}+E_{k_3})t} \right] . \end{aligned}$$

Now we can make use of the “adiabatic switching” prescription which consist in the introduction a factor $e^{-\epsilon|t|}$ in the t-integrand. According to the identity

$$\int_{\mp\infty}^0 dt e^{-\epsilon|t|-iEt} = \frac{i}{E \pm i\epsilon}, \quad (2.62)$$

we find

$$\begin{aligned} X_{\pm}^{\dagger}(k) = & 2\alpha \frac{1}{\sqrt{2E_k}} \int \frac{dk_1 dk_2 dk_3 (2\pi) \delta(k + k_1 + k_2 + k_3)}{(2\pi)^3 \sqrt{2^3 E_{k_1} E_{k_2} E_{k_3}}} \times \\ & \times \left[+ \frac{((-k_1) \cdot (-k_2))((-k_3) \cdot k) A^{\dagger}(-k_1) A^{\dagger}(-k_2) A^{\dagger}(-k_3)}{E_k - E_{k_1} - E_{k_2} - E_{k_3}} \right. \\ & - 3 \frac{((-k_1) \cdot (-k_2))(k_3 \cdot k) A^{\dagger}(-k_1) A^{\dagger}(-k_2) A(k_3)}{E_k - E_{k_1} - E_{k_2} + E_{k_3} \pm i\epsilon} \\ & + \frac{[3((-k_1) \cdot k_2)(k_3 \cdot k) + ((-k_1) \cdot k)(k_2 \cdot k_3)] A^{\dagger}(-k_1) A(k_2) A(k_3)}{E_k - E_{k_1} + E_{k_2} + E_{k_3}} \\ & \left. - 2 \frac{(k_1 \cdot k_2)(k_3 \cdot k) A(k_1) A(k_2) A(k_3)}{E_k + E_{k_1} + E_{k_2} + E_{k_3}} \right]. \end{aligned}$$

The only term where the shift $\pm i\epsilon$ is required is the second one, because it is the only one with a singularity (at $k_1 = -k$, $k_2 = -k_3$ or $k_2 = -k$, $k_1 = -k_3$). Using then the formal identity

$$\lim_{\epsilon \rightarrow 0^+} \frac{1}{E \pm i\epsilon} = \mathcal{P} \left(\frac{1}{E} \right) \mp i\pi \delta(E), \quad (2.63)$$

we obtain

$$\begin{aligned} X_{\pm}^{\dagger}(k) = & X^{\dagger}(k) \pm 2\alpha \frac{3i\pi}{\sqrt{2E_k}} \int \frac{dk_1 dk_2 dk_3}{(2\pi)^2 \sqrt{2^3 E_{k_1} E_{k_2} E_{k_3}}} \\ & \times ((-k_1) \cdot (-k_2))(k_3 \cdot k) A^{\dagger}(-k_1) A^{\dagger}(-k_2) A(k_3) \delta(k + k_1 + k_2 + k_3) \delta(E_k - E_{k_1} - E_{k_2} + E_{k_3}) \\ = & X^{\dagger}(k) \pm \frac{3i\alpha}{8} \int \frac{dk_3}{(2\pi) E_k E_{k_3}} \left| \frac{k}{E_k} - \frac{k_3}{E_{k_3}} \right| A^{\dagger}(k) A^{\dagger}(k_3) A(k_3) \\ = & X^{\dagger}(k) \pm \frac{3i\alpha}{8} \int \frac{dq}{(2\pi)} \frac{(q \cdot k)^2}{|k E_q - q E_k|} A^{\dagger}(k) A^{\dagger}(q) A(q), \end{aligned}$$

with

$$\begin{aligned} X^{\dagger}(k) = & \frac{2\alpha}{\sqrt{2E_k}} \int \frac{dk_1 dk_2 dk_3 (2\pi) \delta(k + k_1 + k_2 + k_3)}{(2\pi)^3 \sqrt{2^3 E_{k_1} E_{k_2} E_{k_3}}} \times \\ & \times \left[+ \frac{((-k_1) \cdot (-k_2))((-k_3) \cdot k) A^{\dagger}(-k_1) A^{\dagger}(-k_2) A^{\dagger}(-k_3)}{E_k - E_{k_1} - E_{k_2} - E_{k_3}} \right. \\ & - 3 \mathcal{P.V.} \frac{((-k_1) \cdot (-k_2))(k_3 \cdot k) A^{\dagger}(-k_1) A^{\dagger}(-k_2) A(k_3)}{E_k - E_{k_1} - E_{k_2} + E_{k_3}} \\ & + \frac{[3((-k_1) \cdot k_2)(k_3 \cdot k) + ((-k_1) \cdot k)(k_2 \cdot k_3)] A^{\dagger}(-k_1) A(k_2) A(k_3)}{E_k - E_{k_1} + E_{k_2} + E_{k_3}} \\ & \left. - 2 \frac{(k_1 \cdot k_2)(k_3 \cdot k) A(k_1) A(k_2) A(k_3)}{E_k + E_{k_1} + E_{k_2} + E_{k_3}} \right]. \end{aligned}$$

In the same way, we can write

$$Y_{\pm}^{\dagger}(k) = Y^{\dagger}(k) \pm \frac{3i\alpha}{8} \int \frac{dq}{(2\pi)} \frac{m^4}{|kE_q - qE_k|} A^{\dagger}(k) A^{\dagger}(q) A(q), \quad (2.64)$$

with

$$\begin{aligned} Y^{\dagger}(k) = & \frac{2\alpha m^4}{\sqrt{2E_k}} \int \frac{dk_1 dk_2 dk_3 (2\pi) \delta(k + k_1 + k_2 + k_3)}{(2\pi)^3 \sqrt{2^3 E_{k_1} E_{k_2} E_{k_3}}} \times \\ & \times \left[+ \frac{A^{\dagger}(-k_1) A^{\dagger}(-k_2) A^{\dagger}(-k_3)}{E_k - E_{k_1} - E_{k_2} - E_{k_3}} - 3\mathcal{P}.\mathcal{V} \cdot \frac{A^{\dagger}(-k_1) A^{\dagger}(-k_2) A(k_3)}{E_k - E_{k_1} - E_{k_2} + E_{k_3}} \right. \\ & \left. + \frac{A^{\dagger}(-k_1) A(k_2) A(k_3)}{E_k - E_{k_1} + E_{k_2} + E_{k_3}} - 2 \frac{A(k_1) A(k_2) A(k_3)}{E_k + E_{k_1} + E_{k_2} + E_{k_3}} \right]. \end{aligned}$$

The final result is

$$B_{\pm}^{\dagger}(k) = A^{\dagger}(k) - [X_{\pm}^{\dagger}(k) - Y_{\pm}^{\dagger}(k)] = B^{\dagger}(k) \mp \frac{3i\alpha}{8} \int \frac{dq}{(2\pi)} \frac{(q \cdot k)^2 - m^4}{|kE_q - qE_k|} A^{\dagger}(k) A^{\dagger}(q) A(q), \quad (2.65)$$

with $B^{\dagger}(k) = X^{\dagger}(k) - Y^{\dagger}(k)$. We can also use rapidity variables to rewrite it as

$$B_{\pm}^{\dagger}(\theta) = B^{\dagger}(\theta) \mp \frac{3i\alpha m^4}{8} \int \frac{d\theta'}{(2\pi)} |\sinh(\theta - \theta')| A^{\dagger}(\theta) A^{\dagger}(\theta') A(\theta'). \quad (2.66)$$

Now, we can define the ZF-operator

$$\begin{aligned} Z^{\dagger}(\theta) &= B^{\dagger}(\theta) - \frac{3i\alpha}{8} \int \frac{d\theta'}{(2\pi)} \sinh(\theta - \theta') A^{\dagger}(\theta) A^{\dagger}(\theta') A(\theta') \\ &= B_{\pm}^{\dagger}(\theta) - \frac{3i\alpha}{8} \int \frac{d\theta'}{(2\pi)} (\sinh(\theta - \theta') \mp |\sinh(\theta - \theta')|) A^{\dagger}(\theta) A^{\dagger}(\theta') A(\theta') \end{aligned} \quad (2.67)$$

If we remain at the first order in α we see that this operator satisfy the ZF-algebra

$$Z^{\dagger}(\theta_1) Z^{\dagger}(\theta_2) = (S(\theta_{12}))^* Z^{\dagger}(\theta_2) Z^{\dagger}(\theta_1), \quad \text{with} \quad S(\theta_{12}) = 1 - \frac{3i\alpha}{4} \sinh \theta_{12} + O(\alpha^2). \quad (2.68)$$

Therefore we have now another way to see the dressing by $T\bar{T}$ of the CCR algebra. This gives also the right S-matrix (2.43) for the free boson at first order (The coefficient can be reabsorbed as we wish in the α). Perturbatively we found that effectively $T\bar{T}$ generate the S-matrix that we wanted but gave us also a way to deduce the exact form of the dressing in the form (2.51). Actually this kind of dressing is not new and can be traced back to methods of ISM [168]. The same method applies in general for 1+1-dimensional QFT [169] but also in the context of $T\bar{T}$ -deformations was actually already introduced (abstractly) in [165]. Basically, if we consider a free bosonic field with CCR-algebra (0.12) it is always possible (formally) to define a bilinear form

$$W_{\alpha}(\theta) = \exp \left\{ \int d\theta' G_{\alpha}(\theta - \theta') A^{\dagger}(\theta') A(\theta') \right\}, \quad (2.69)$$

and then

$$W_{\alpha}(\theta) A^{\dagger}(\theta') = e^{G_{\alpha}(\theta - \theta')} A^{\dagger}(\theta') W_{\alpha}(\theta), \quad (2.70)$$

and

$$W_\alpha(\theta) = : \exp \left\{ \int d\theta' \left(e^{G_\alpha(\theta-\theta')} - 1 \right) A^\dagger(\theta') A(\theta') \right\} : . \quad (2.71)$$

In this way we can generate the ZF-algebra $Z^\dagger(\theta) = W_\alpha(\theta) A_\alpha^\dagger(\theta)$ where $G_\alpha(\theta - \theta') \rightarrow \ln F_\alpha(\theta - \theta')$ is exactly the exponent of the CDD-factor. The new ZF-algebra will satisfy for example

$$Z^\dagger(\theta) Z^\dagger(\theta') = e^{2 \ln \Phi_\alpha} Z^\dagger(\theta') Z^\dagger(\theta) . \quad (2.72)$$

With the choice $F_\alpha(\theta - \theta') = \sqrt{\Phi_\alpha(\theta)}$ from (2.43) we get precisely the result we wanted. It is worth noting that our construction not only shows how a deformation of the action is explicitly related to the previously defined CDD factors, but also that **such factors preserve integrability**. This, however, is not surprising, as these CDD factors, as already saw in Chapter 1 (see 1.56) satisfy trivially the S-matrix equations.

Chapter 3

Completing the Bootstrap Program for $T\bar{T}$ -deformed IQFTs

In a IQFT the only important data is the S-matrix of the model: the logarithm of the S-matrix is in fact the only object needed for the TBA analysis already discussed previously, but there is more to add. The knowledge of the scattering data is also a fundamental step towards the determination of correlation functions. An essential ingredient for the full solution of a 1+1-dimensional IQFT is the determination of its *generalised form factors* [6, 10, 170, 171], i.e. the matrix elements of local or semi-local operators $\mathcal{O}$, evaluated between *out* and *in* states:

$$F_{m+n}^{\mathcal{O}}(\theta'_1, \dots, \theta'_m | \theta_{m+1}, \dots, \theta_n) = {}^{out}\langle \theta'_1, \dots, \theta'_m | \mathcal{O}(0) | \theta_{m+1}, \dots, \theta_n \rangle^{in}. \quad (3.1)$$

See Figure 3.1 for a graphical representation. Indeed, it can be easily seen that correlation functions can be expanded as spectral series of form factors. Inserting a complete set of intermediate states

$$1 = \sum_{n=0}^{\infty} \int \int_{-\infty}^{\infty} \frac{d\theta_1 \dots d\theta_n}{n!(2\pi)^n} |\theta_1, \dots, \theta_n\rangle^{in} {}^{in}\langle \theta_1, \dots, \theta_n|, \quad (3.2)$$

into a 2-point function of two operators $\mathcal{O}_1, \mathcal{O}_2$ we obtain

$$\langle 0 | \mathcal{O}_1(r) \mathcal{O}_2(0) | 0 \rangle = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{d\theta_1 \dots d\theta_n}{n!(2\pi)^n} F_n^{\mathcal{O}_1}(\theta_1, \dots, \theta_n) \left(F_n^{\mathcal{O}_2^\dagger}(\theta_1, \dots, \theta_n) \right)^* e^{-rm \sum_{j=1}^n \cosh \theta_j} \quad (3.3)$$

where $r = \sqrt{x_0^2 + x_1^2}$ and

$$F_n^{\mathcal{O}}(\theta_1, \dots, \theta_n) = \langle 0 | \mathcal{O}(0) | \theta_1, \dots, \theta_n \rangle^{in} \quad (3.4)$$

is the actual *form factor* (see Fig. 3.2), i.e. the matrix element of a local or semi-local operator placed at the origin, between an n -particle state and the vacuum with $\theta_1 > \theta_2 > \dots > \theta_n$. By relativistic invariance we just need to consider the relative distance r between the operators and the exponential damping in (3.3) is obtained considering the action of a Poincaré translation U_{T_x} on the operator $U_{T_x} \mathcal{O}(0) U_{T_x}^{-1} = \mathcal{O}(x)$ and on the ZF-algebra

$$U_{T_x} A^\dagger(\theta_i) U_{T_x}^{-1} = e^{-ip^\mu(\theta_i)x_\mu} A^\dagger(\theta_i), \quad (3.5)$$

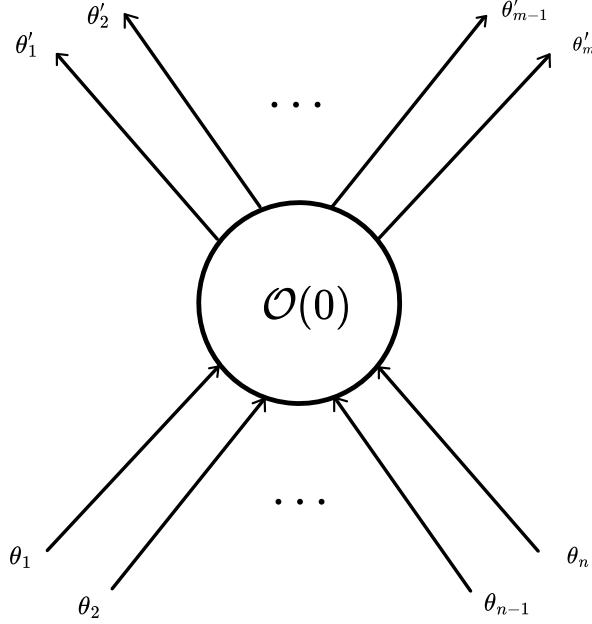


Figure 3.1: Generalised form factor of the operator $\mathcal{O}$.

that results into

$$\langle 0 | \mathcal{O}(x) | \theta_1, \dots, \theta_n \rangle^{in} = e^{-ip^\mu(\theta_i)x_\mu} F_n^{\mathcal{O}}(\theta_1, \dots, \theta_n). \quad (3.6)$$

The integrals that enter the spectral series are all convergent in principle, but a complete proof of convergence of these spectral series was proposed just recently [172, 173]. All the rest of this thesis will be devoted to the study of these objects. As we did in the first half of the thesis we will first introduce the main object in the context of IQFTs and then we will try to understand how we can adapt what it is known to a $\text{T}\bar{\text{T}}$ -deformed theory.

3.1 The Form Factor Bootstrap Program for IQFTs

As already stressed previously, the very ambitious goal we have is to solve a IQFT completely by specifying its observables and we want to do it expanding the correlation functions in terms of form factors (3.4).

3.1.1 The Form Factor's Axioms

Like in the S-matrix case, we expect these objects to fulfill a series of constraints imposed by symmetry and general physical requirement. Indeed, some of the properties we derive for these objects are based on very fundamental requirements, namely the validity of the LSZ-reduction formalism and the validity of the *maximal analyticity assumption* already

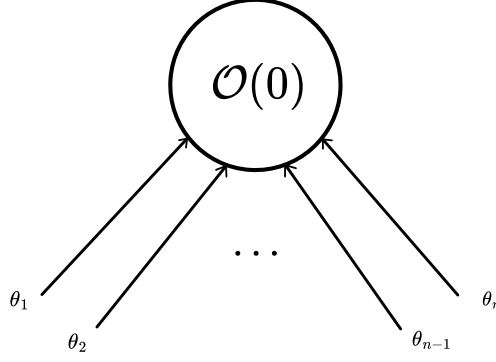


Figure 3.2: Form factor of the operator $\mathcal{O}$.

employed in subsection 1.3.1 which now we extend:

Maximal analyticity: *the S -matrix and the form factors are analytic functions everywhere except at the points where they possess singularities due to physical intermediate states.*

Considering a scalar operator placed at the origin $\mathcal{O}(0)$ we define a 2-particle *generalised form factor* as the boundary value of an analytic function

$$\langle 0 | \mathcal{O}(0) | \theta_1, \theta_2 \rangle^{\text{in}} = F_2^{\mathcal{O}}(s_{12} + i\epsilon), \quad (3.7)$$

where $s_{12} = (p_1 + p_2)^2 = 2m^2(1 + \cosh \theta_{12})$ is the usual Mandelstam variable. Under the assumption of analyticity, an interchange between “in” and “out” states (equivalently an inversion of the order of rapidities) brings us on the other half of the plane, i.e. through analytic continuation to the lower edge of the unitary cut we obtain

$$\langle 0 | \mathcal{O}(0) | \theta_1, \theta_2 \rangle^{\text{out}} = F_2^{\mathcal{O}}(s_{12} - i\epsilon). \quad (3.8)$$

This property can be easily deduced from the LSZ-reduction formalism (see Appendix A) by looking at the connected components of these elements. More generically, the **crossing property** for the connected components yields

$${}^{\text{out}}\langle \theta'_1, \dots, \theta'_m | \mathcal{O}(0) | \theta_{m+1}, \dots, \theta_n \rangle_{\text{conn}}^{\text{in}} = F^{\mathcal{O}}(s_{ij} + i\epsilon, t_{rs} - i\epsilon, s_{kl} + i\epsilon), \quad (3.9)$$

where $1 \leq i < j \leq m$, $1 \leq r \leq m < s \leq n$, $m \leq k < l \leq n$ and t_{rs} is the transmitted momentum. Usually we can suppress completely the “in/out” notation for the states assuming that form factors are analytic functions of the rapidities which assume physical values when the ordering prescription is (1.50). In practice we can restrict our attention to matrix elements of the form (3.4) because the change in sign (and then direction) of the particles can be implemented in rapidity variables by shifts of $i\pi$ in the crossed channel. Therefore (3.9) can be rewritten as

$$F_{m+n}^{\mathcal{O}}(\theta'_1, \dots, \theta'_m | \theta_{m+1}, \dots, \theta_n) = F^{\mathcal{O}}(\theta'_1 + i\pi, \dots, \theta'_m + i\pi, \theta_{m+1}, \dots, \theta_n). \quad (3.10)$$

Let's now list some other fundamental properties.

Relativistic covariance implies that, for a generic operator of spin s , a boost of rapidity Λ gives:

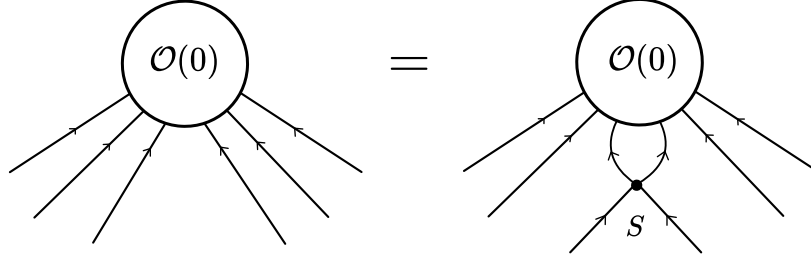


Figure 3.3: Graphical representation of the Watson equation for the braiding.

$$F_n^{\mathcal{O}}(\theta_1 + \Lambda, \dots, \theta_n + \Lambda) = e^{s\Lambda} F_n^{\mathcal{O}}(\theta_1, \dots, \theta_n). \quad (3.11)$$

As expected the form factors of a scalar operator (i.e. $s = 0$) only depend on rapidity differences. The **braiding property**

$$F_n^{\mathcal{O}}(\theta_1, \dots, \theta_i, \theta_{i+1}, \dots, \theta_n) = S(\theta_i - \theta_{i+1}) F_n^{\mathcal{O}}(\theta_1, \dots, \theta_{i+1}, \theta_i, \dots, \theta_n), \quad (3.12)$$

is a simple consequence of the integrable structure of the theory (see Fig. 3.3).

In fact, considering the simple 2-particle case, from the definition of the ZF-algebra (1.45) is clear that

$$\begin{aligned} \langle 0 | \mathcal{O}(0) | \theta_1, \theta_2 \rangle &= \langle 0 | \mathcal{O}(0) | A^\dagger(\theta_1) A^\dagger(\theta_2) | 0 \rangle \\ &= S(\theta_{12}) \langle 0 | \mathcal{O}(0) | A^\dagger(\theta_2) A^\dagger(\theta_1) | 0 \rangle = S(\theta_{12}) \langle 0 | \mathcal{O}(0) | \theta_2, \theta_1 \rangle. \end{aligned} \quad (3.13)$$

We can make this discussion a bit more precise. Returning to the initial definition (3.7), if we insert a complete set of out-states between the operator insertion and the 2-particle state we can determine, for all ¹ $s \geq 4m^2$, the discontinuity of $F_2^{\mathcal{O}}(s)$ across the cut

$$F_2^{\mathcal{O}}(s + i\epsilon) = \langle 0 | \mathcal{O}(0) | \theta_1 \theta_2 \rangle^{\text{out out}} \langle \theta_1, \theta_2 | \theta_1, \theta_2 \rangle^{\text{in}} = F_2^{\mathcal{O}}(s - i\epsilon) S(s + i\epsilon). \quad (3.14)$$

This is the only branch-cut of the 2-particle form factor in the s -plane. Always considering just the 2-particle form factor, if we instead consider the crossing symmetry property (3.9)-(3.10) together with eqs. (3.7)-(3.8) we get

$${}^{\text{out}} \langle p_2^\mu | \mathcal{O}(0) | p_1^\mu \rangle^{\text{in}} = \langle 0 | \mathcal{O}(0) | p_1^\mu, -p_2^\mu \rangle^{\text{in}} = F_2^{\mathcal{O}}(4m^2 - s - i\epsilon), \quad (3.15)$$

$${}^{\text{in}} \langle p_2^\mu | \mathcal{O}(0) | p_1^\mu \rangle^{\text{out}} = \langle 0 | \mathcal{O}(0) | p_1^\mu, -p_2^\mu \rangle^{\text{out}} = F_2^{\mathcal{O}}(4m^2 - s + i\epsilon). \quad (3.16)$$

¹In general we should have: “lowest inelastic threshold” $> s \geq 4m^2$, but in integrable theories there are no inelastic transitions.

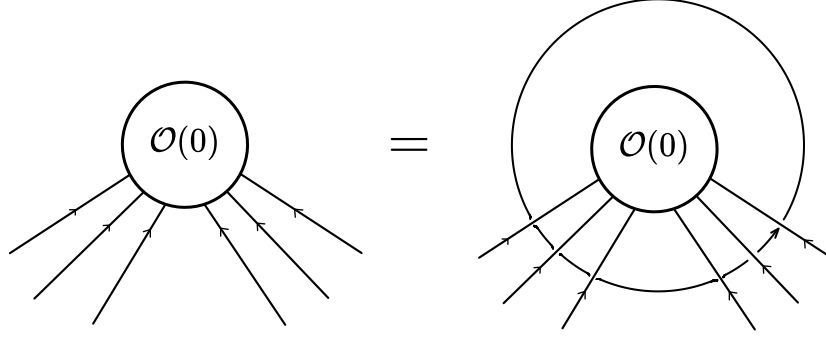


Figure 3.4: Graphical representation of the Watson equation for the monodromy. This equation can also be interpreted as defining the action of a map $\mathcal{K} : F_{n+2}^{\mathcal{O}} \rightarrow F_n^{\mathcal{O}}$.

But these two matrix elements should coincide (it can be shown by using the completeness of the in and out basis states and the fact that ${}^{\text{out}}\langle p_2^\mu | p_1^\mu \rangle^{\text{in}} = \delta(p_1^\mu - p_2^\mu)$) and therefore we get the **monodromy property**

$$F_2^{\mathcal{O}}(4m^2 - s - i\epsilon) = F_2^{\mathcal{O}}(4m^2 - s + i\epsilon), \quad (3.17)$$

or in term of rapidities

$$F_2^{\mathcal{O}}(\theta_2, \theta_1 - i\pi) = F_2^{\mathcal{O}}(\theta_1 + i\pi, \theta_2). \quad (3.18)$$

Notice that this equation was obtained just for $\theta_1 - \theta_2 > 0$ but this result holds true on the whole strip $0 \leq \text{Im}(\theta_{12}) \leq 2\pi$ by analytic continuation. If in (3.18) we shift θ_1 by $i\pi$ we can recast it in the more “standard” form

$$F_2^{\mathcal{O}}(\theta_1 + 2i\pi, \theta_2) = F_2^{\mathcal{O}}(\theta_2, \theta_1). \quad (3.19)$$

More generally, as in the case of eq. (3.12), also the monodromy property holds for each 2-particle sub-channel, i.e.

$$F_n^{\mathcal{O}}(\theta_1 + 2\pi i, \theta_2, \dots, \theta_n) = F_n^{\mathcal{O}}(\theta_2, \dots, \theta_n, \theta_1). \quad (3.20)$$

Equations (3.12) and (3.20) are usually referred as *generalised Watson’s equations* [174, 6] (see also Fig. 3.3 and Fig. 3.4). Finally, if we consider diagonal theories without bound states, form factors are meromorphic functions of rapidity differences θ_{ij} in the strip $0 \leq \text{Im}(\theta_{ij}) \leq 2\pi$ with simple poles arising only at $\theta_{ij} = i\pi$; the corresponding particle–antiparticle pair is not an independent asymptotic state anymore: together they effectively annihilate. From the point of view of crossing, this situation is equivalent to a three-particle configuration (cluster) in which two of the particles form a cluster that is kinematically indistinguishable from a single particle (see Fig. 3.5). In other words, the invariant mass of the cluster approaches that of a single on-shell excitation, schematically

$$(p_1 + p_2 + p_3)^2 \longrightarrow m_4^2. \quad (3.21)$$

This explains why the residue of the pole is governed by the reduction from an $(n+2)$ -particle to an n -particle form factor: the “missing pair” has effectively been replaced by a single particle state. To understand this better let’s consider eq. (3.10) with $m = n = 0$ and taking $\mathcal{O} = \mathbb{I}$, i.e. the identity operator. We have

$$F_2^{\mathbb{I}}(\theta'|\theta) \equiv {}^{\text{out}}\langle\theta'|\theta\rangle^{\text{in}} = 2\pi\delta(\theta' - \theta). \quad (3.22)$$

This extremely simple example shows that when a particle with rapidity θ appears both in the initial and in the final state, a singular contribution is generally “produced” in the right hand side of (3.10). Basically, the particle $A(\theta)$ in the final state, crossing to the initial one becomes $A(\theta + i\pi)$ and annihilates the particle of the same type $A(\theta)$ that is already there. More formally [10, 175], from the LSZ-reduction formalism follows that, taking a set of ordered rapidities $\theta' \geq \theta > \theta_1 > \dots > \theta_n$, and considering the full matrix elements (with also the disconnected contributions) we have

$$F_{n+2}^{\mathcal{O}}(\theta'|\theta, \theta_1, \dots, \theta_n) = F_{n+2}^{\mathcal{O}}(\theta' + i\pi, \theta, \theta_1, \dots, \theta_n) + 2\pi\delta(\theta - \theta')F_n^{\mathcal{O}}(\theta_1, \dots, \theta_n), \quad (3.23)$$

$$F_{n+2}^{\mathcal{O}}(\theta'|\theta_n, \dots, \theta_1, \theta) = F_{n+2}^{\mathcal{O}}(\theta_n, \dots, \theta_1, \theta, \theta' - i\pi) + 2\pi\delta(\theta - \theta')F_n^{\mathcal{O}}(\theta_n, \dots, \theta_1). \quad (3.24)$$

We can then use (3.12) and the unitarity of the S-matrix to rewrite (3.24) as

$$\begin{aligned} F_{n+2}^{\mathcal{O}}(\theta|\theta_n, \dots, \theta_1, \theta) &= F_n^{\mathcal{O}}(\theta, \theta_1, \dots, \theta_n, \theta' - i\pi) \\ &\quad + 2\pi\delta(\theta - \theta') \prod_{j=1}^n S(\theta_j - \theta) F_n^{\mathcal{O}}(\theta_n, \dots, \theta_1). \end{aligned} \quad (3.25)$$

In the end, if we subtract (3.23) and (3.25) we get

$$\begin{aligned} F_{n+2}^{\mathcal{O}}(\theta' + i\pi, \theta, \theta_1, \dots, \theta_n) &= F_{n+2}^{\mathcal{O}}(\theta, \theta_1, \dots, \theta_n, \theta' - i\pi) \\ &\quad + 2\pi\delta(\theta - \theta') \left[\prod_{j=1}^n S(\theta_j - \theta) - 1 \right] F_n^{\mathcal{O}}(\theta_1, \dots, \theta_n). \end{aligned} \quad (3.26)$$

For $\theta' \neq \theta$ this equation is clearly the same as (3.20) but it also implies that the form factor $F_{n+2}^{\mathcal{O}}(\theta' + i\pi, \theta, \theta_1, \dots, \theta_n)$ possesses an *annihilation pole* at $\theta' = \theta + i\pi$ with residue given by

$$-i \operatorname{Res}_{\theta'=\theta+i\pi} F_{n+2}^{\mathcal{O}}(\theta' + i\pi, \theta, \theta_1, \dots, \theta_n) = \left(1 - \prod_{j=1}^n S(\theta_j - \theta) \right) F_n^{\mathcal{O}}(\theta_1, \dots, \theta_n). \quad (3.27)$$

This equation plays a crucial role, as it provides a recursive relation that allows one to construct higher-particle form factors starting from the lower-particle ones. The three equations (3.12), (3.20), and (3.27) are commonly referred to as the *form factor axioms* or *form factor consistency equations*, though—as we have just seen—they are more aptly described by the latter term. Indeed, we derived them from fundamental properties of the IQFT, rather than postulating them as axioms.

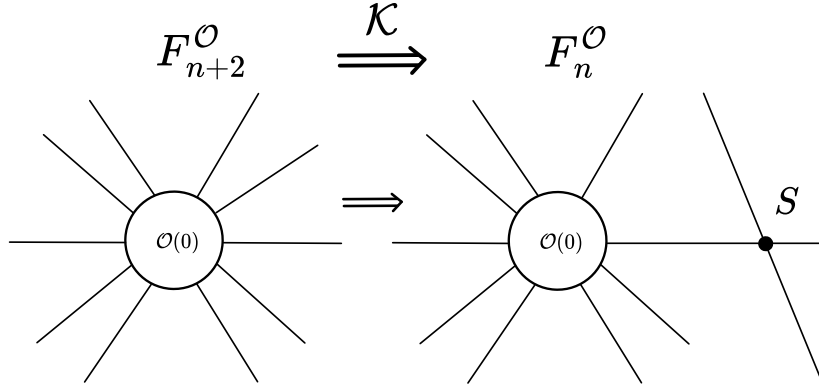


Figure 3.5: Graphical representation of the recursive equation for the kinematical pole.

In practice, however, the standard approach often treats these equations as given: one typically solves Watson's equation in the two-particle case and then employs the kinematical residue equation to generate higher-particle form factors. Yet, this procedure does not always yield consistent results. In particular, as we will see, the nature of the observables enters explicitly into the equations, which turn out to be not entirely operator-independent. In fact, the locality property of the observable plays a crucial role. This is further confirmed by the fact that our derivation was explicitly carried out for a *local* operator within the LSZ formalism.

3.1.2 The Karowski-Weisz Construction

The first step in the construction of form factors consists in solving the Watson's equation in the lowest-particle number case, i.e. the 2-particle one. The method was developed in the foundational paper [6] and we are now going to review it here. We are dealing with a diagonal S-matrix without bound-states and therefore the Watson's equations (3.12)-(3.20) in the case $n = 2$ simplify to

$$F_2^{\mathcal{O}}(\theta) = S(\theta)F_2^{\mathcal{O}}(-\theta), \quad F_2^{\mathcal{O}}(i\pi - \theta) = F_2^{\mathcal{O}}(i\pi + \theta). \quad (3.28)$$

Now, from [6] we state the following theorem

Theorem (Karowski and Weisz) :

If $F_2^{\mathcal{O}}$ is meromorphic in the physical strip $0 \leq \text{Im}(\theta) \leq \pi$ with possible poles (or zeros) only in the imaginary axis and $F_2^{\mathcal{O}} = O(\exp \exp |\theta|)$ for $\text{Re}(\theta) \rightarrow \infty$, the solutions of (3.28) are uniquely determined by the poles at $\theta = i\alpha_k$ (and zeros) up to a normalisation constant. They can be written as

$$F_2^{\mathcal{O}}(\theta) = K_2^{\mathcal{O}}(\theta)F_{\min}(\theta), \quad (3.29)$$

*where the **minimal solution** of (3.28) (with $F_{\min}(i\pi) = 1$) has no poles (and zeros) in the physical strip and $K_2^{\mathcal{O}}(\theta)$ is a solution of*

$$K_2^{\mathcal{O}}(\theta) = K_2^{\mathcal{O}}(-\theta) = K_2^{\mathcal{O}}(2\pi i + \theta). \quad (3.30)$$

Notice that for interacting theories generally $S(0) = -1$ and therefore the braiding equation in (3.29) forces the **minimal form factor** (MFF) $F_{\min}(\theta)$ to have a zero at the 2-particle threshold:

$$F_{\min}(\theta) \simeq 0, \quad \theta \rightarrow 0. \quad (3.31)$$

Having no poles or zero in the physical strip this object constitutes the first fundamental piece from which a complete solution is built. Indeed the first step is always to find it, by solving the equations

$$F_{\min}(\theta) = S(\theta)F_{\min}(-\theta) = F_{\min}(2\pi i - \theta). \quad (3.32)$$

Note that the first equality in (3.32) define a Riemann-Hilbert problem for the corresponding function, with $S(\theta)$ playing the role of “jump function” [176]. This can be used to explicitly write the solution as an integral representation involving the logarithm of the jump function and it is precisely how we are going to prove (3.29): we will solve the Riemann-Hilbert problem fixing uniquely the MFF from the knowledge of the S-matrix. Instead the remaining factors $K_2^{\mathcal{O}}(\theta)$ satisfy Watson’s equations but with $S = 1$ and therefore they are completely symmetric functions in rapidity variables with period $2\pi i$. As the minimal part does not possess any pole they must have all the physical poles expected for the form factors.

Let $i\alpha_1, \dots, \alpha_L$ be the location of the poles of $F_2^{\mathcal{O}}(\theta)$ in the physical strip and

$$K_2^{\mathcal{O}}(\theta) = \text{const} \prod_{k=1}^L \left[\sinh \frac{1}{2}(\theta - i\alpha_k) \sinh \frac{1}{2}(\theta - i\alpha_k) \right]^{-1}, \quad (3.33)$$

which solves (3.30). Then $F_{\min}(\theta) = F_2^{\mathcal{O}}(\theta)/K_2^{\mathcal{O}}(\theta)$ is analytic in the physical strip $0 \leq \text{Im}(\theta) \leq \pi$ and satisfies (3.32). Hence, by Cauchy’s theorem,

$$\frac{\partial}{\partial \theta} \log F_{\min}(\theta) = \frac{1}{8\pi i} \int_C \frac{\log F_{\min}(z)}{\sinh^2 \frac{1}{2}(z - \theta)} dz. \quad (3.34)$$

Here C is a contour enclosing the strip $0 \leq \text{Im}(\theta) \leq 2\pi$ as shown in Fig. 3.6.

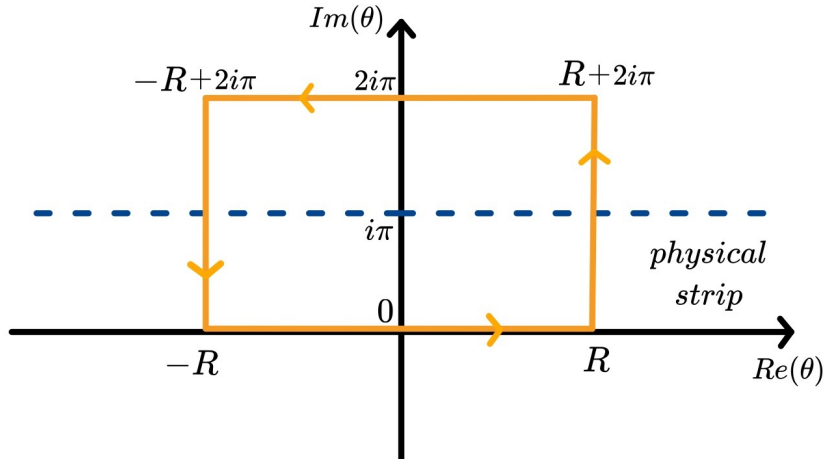


Figure 3.6: The contour C in equation (3.34).

The denominator $\sinh^2 \frac{1}{2}(z - \theta)$ has double poles at $z = \theta + 2\pi i n$ ($n \in \mathbb{N}$). For $\theta > 0$ only the pole at $z = \theta$ falls within the contour. Then, by Cauchy's residue theorem, we obtain the derivative on the l.h.s. Let us examine this construction better. Because of the equations (3.32), we have certain asymptotic properties the form factor must satisfy. If the S -matrix tends to a constant in the high energy limit $\theta \rightarrow \infty$, then we must have $F_{\min}(\theta) = O(\exp \exp |\theta|)$ for $\text{Re}(\theta) \rightarrow \infty$. This technical assumption is already contained in Karowski and Weisz's theorem and notice that a similar assumption was also used in Chapter 1 to fix completely the S -matrix of subsection 1.4.2 (not coincidentally this strategy was taken from another paper by the same authors). This is very important when computing the integral along the contour above. We can divide the contour C in four pieces: $C = C_1 + C_2 + C_3 + C_4$. Given $z = x + iy$ we have:

- two horizontal segments: $C_1 = \{x : -R \leq x \leq R\}$ and $C_3 = \{x + 2i\pi : -R \leq x \leq R\}$;
- two vertical segments: $C_2 = \{R + iy : 0 \leq y \leq 2\pi\}$ and $C_4 = \{-R + iy : 0 \leq y \leq 2\pi\}$;

with C oriented counterclockwise. So, our integral can be written also as

$$\int_C dz w(z) = \int_{-R}^R dx w(x) + \int_R^{-R} dx w(x + 2\pi i) + i \int_0^{2\pi} dy w(R + iy) + i \int_{2\pi}^0 dy w(-R + iy), \quad (3.35)$$

with

$$w(z) = \frac{\log F_{\min}(z)}{8\pi i \sinh^2 \frac{1}{2}(z - \theta)}. \quad (3.36)$$

The idea is that, if $w(z)$ decays fast enough for large R , we can discard the vertical contributions. That is by the estimation lemma we can write

$$\left| \int_{C_2} dz w(z) \right| \leq \int_0^{2\pi} dy |w(R + iy)| \xrightarrow{R \rightarrow \infty} 0, \quad (3.37)$$

and similarly for the integral along C_4 . For our function $w(z)$ we have that

$$\left| \frac{\log F_{\min}(R + iy)}{\sinh^2 \frac{1}{2}(R - \theta + iy)} \right| = \frac{o(e^{|R|})}{e^{|R|}} \xrightarrow{R \rightarrow \infty} 0, \quad (3.38)$$

because the MFF has the asymptotic behavior described above. Therefore, the integral is given by the sum of the horizontal contributions. These contributions are related to each other by the form factor equations (3.32) giving

$$\begin{aligned} \frac{1}{8\pi i} \int_C \frac{\log F_{\min}(z)}{\sinh^2 \frac{1}{2}(z - \theta)} dz &= \frac{1}{8\pi i} \int_{-\infty}^{\infty} \frac{dx}{\sinh^2 \frac{1}{2}(x - \theta)} \log \frac{F_{\min}(x)}{F_{\min}(x + 2\pi i)} \\ &= \frac{1}{8\pi i} \int_{-\infty}^{\infty} \frac{\log S(x)}{\sinh^2 \frac{1}{2}(x - \theta)} dx, \end{aligned} \quad (3.39)$$

which therefore provides a representation of the derivative of the MFF in terms of the scattering matrix. This is exactly what we wanted, i.e. we uniquely fixed the minimal solution $F_{\min}(\theta)$ from the S -matrix. In addition, if the S -matrix admits an integral representation

$$i\delta(\theta) := \log S(\theta) = \int_0^{\infty} \frac{dt}{t} g(t) \sinh \frac{t\theta}{i\pi}, \quad (3.40)$$

that is, we can write it as a Fourier sine-transform, then we can obtain straight away a similar representation for the MFF:

$$\rho(\theta) := \log F_{\min}(\theta) = \int_0^\infty \frac{dt}{t} \frac{g(t)}{\sinh t} \sin^2 \frac{t\theta}{2\pi}. \quad (3.41)$$

Notice that this derivation is very similar to the one (not coincidentally by the same authors) used to completely fix the diagonal elements of the S -matrix of the sine-Gordon model (1.79).

3.1.3 Form Factors and Locality

Having fixed the MFF, we are led to the task of computing “the rest”. Already at the level of the 2-particle form factors, we saw from the Karowski–Weisz construction (3.29) that the pole structure is entirely conveyed into the symmetric function $K_2^\mathcal{O}$ satisfying (3.30). Since the MFF does not contain any poles and its determination depends only on the S -matrix (3.39), these functions must carry all the information about the observable we wish to compute, and therefore their determination is the natural next step. Recall that, by construction², there must be a simple pole at $\theta_{ij} = i\pi$ arising from three-particle clusters that reach a kinematical configuration equivalent to a single particle (3.27). In other words, these processes correspond to the crossing channels of the S -matrix (see Fig. 3.5). Therefore, *all the poles of the form factors are determined by the scattering theory itself and do not depend on the operator*. But then, how can we identify the operator for which we are computing the form factors? A first and very strong indication comes, as always, from locality. Indeed, not all operators in a given theory are “strictly local”.

We already defined locality as an axiom (1.8) in Chapter 1. Our definition expressed the usual notion that operators localised at different positions should not interfere in the dynamics, i.e. we want our theory to fulfill *causality*. However, when considering more than one field, it is natural to ask whether they preserve this property *with respect to each other*. This leads to the notion of **mutual locality**: two fields O_1 and O_2 , are said to be mutually local if

$$[O_1(x), O_2(x')] = 0 \quad (\text{bosons}), \quad \{O_1(x), O_2(x')\} = 0 \quad (\text{fermions}), \quad (3.42)$$

for space-like separation. Thus, mutual locality is a relative notion: it expresses the compatibility of two fields with each other under the principle of causality (depending on statistics). But in two-dimensional quantum field theories the situation is special: besides the usual Bose and Fermi statistics, one can consistently realise generalised statistics, such as parastatistics and braid (anyon-like) statistics, due to the non-trivial topology of particle exchanges in $1+1$ -dimensions: we can introduce the concept of **semi-locality**. Considering an index $\omega_{1,2}$, we call two fields O_1 and O_2 semi-local with respect to each other if

$$O_1(x)O_2(x') = (-1)^{f_1 f_2} e^{-2\pi i \omega_{1,2} \Theta(x-x')} O_2(x')O_1(x), \quad x \neq x'; \quad (3.43)$$

or

$$O_1(x)O_2(x') = (-1)^{f_1 f_2} e^{2\pi i \omega_{1,2} \Theta(x'-x)} O_2(x')O_1(x), \quad x \neq x'. \quad (3.44)$$

²As already stressed previously, we are considering only diagonal theories without bound states.

Here $f_i \in \{0, 1\}$ is the fermion number of the operator O_i , and Θ is the Heaviside step function. This phase factor encodes the semi-local commutation rules: circling one operator around the other produces a nontrivial monodromy proportional to $\omega_{1,2}$.

We can get some further intuition of this concept from Feynman's path integral formulation of QFT [177]. Returning to the standard Minkowski signature, time-ordered vacuum expectation values can be expressed via the path integral:

$$\langle \text{vac} | O_1(x_1, t_1) \cdots O_n(x_n, t_n) | \text{vac} \rangle = \int [d\Phi] e^{i\mathcal{A}[\Phi]} \prod_{k=1}^n O_k(x_k, t_k), \quad (3.45)$$

so the semi-locality condition implies, for any operator insertions at times different from zero:

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \int [d\Phi] e^{i\mathcal{A}[\Phi]} \left(\cdots O_1(x, \varepsilon) O_2(x', 0) \cdots \right) \\ = e^{\pm 2\pi i \omega_{1,2} \Theta(\pm(x' - x))} \lim_{\varepsilon \rightarrow 0^+} \int [d\Phi] e^{i\mathcal{A}[\Phi]} \left(\cdots O_1(x, -\varepsilon) O_2(x', 0) \cdots \right). \end{aligned} \quad (3.46)$$

The choice of $\pm$ signs is consistent with the orientation of the spatial axis, and the result holds for bosonic as well as fermionic fields. Using the standard prescription for time ordering, we can analytically continue to imaginary time, replacing $\varepsilon \mapsto -i\varepsilon$:

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \int [d\Phi] e^{i\mathcal{A}[\Phi]} \left(\cdots O_1(x, -i\varepsilon) O_2(x', 0) \cdots \right) \\ = e^{\pm 2\pi i \omega_{1,2} \Theta(\pm(x' - x))} \lim_{\varepsilon \rightarrow 0^+} \int [d\Phi] e^{i\mathcal{A}[\Phi]} \left(\cdots O_1(x, -i\varepsilon) O_2(x', 0) \cdots \right), \end{aligned} \quad (3.47)$$

for $x \neq x'$. It is now convenient to define the functional integral

$$F(x, t) := \int [d\Phi] e^{i\mathcal{A}[\Phi]} (\cdots O_1(x, t) O_2(0, 0) \cdots), \quad (3.48)$$

and analytically continue the time variable t from a space-like region $|x| > |t|$ to purely imaginary time $t \rightarrow -i\tau$. Notice that in imaginary time the distance between two operators is always space-like. While the action $\mathcal{A}[\Phi]$ defines local displacements of fields via the equations of motion, semi-locality shows that moving a field around a point along a closed path can generate a nontrivial phase. In fact, introducing complex coordinates

$$z = x + i\tau, \quad \bar{z} = x - i\tau, \quad (3.49)$$

the operator F then exhibits a monodromy when rotated around the origin:

$$F(e^{i\alpha} z, e^{-i\alpha} \bar{z}) \Big|_{\alpha: 0 \rightarrow 2\pi} = e^{2\pi i \omega_{1,2}} F(z, \bar{z}), \quad (3.50)$$

indicating that the function $F(z, \bar{z})$ naturally lives on a branched cover of the complex plane, with a branch cut along $\mathbb{R}^+$ or $\mathbb{R}^-$ according to (3.47). By “local displacements”, we mean its continuation through that cut, defined as the finite limit of its normal derivative (given by the equations of motion) as we approach the cut; in other words, we consider its value on a covering of the Euclidean plane with a branch point at $(0, 0)$.

In Euclidean signature, the path integral representation of correlation functions becomes

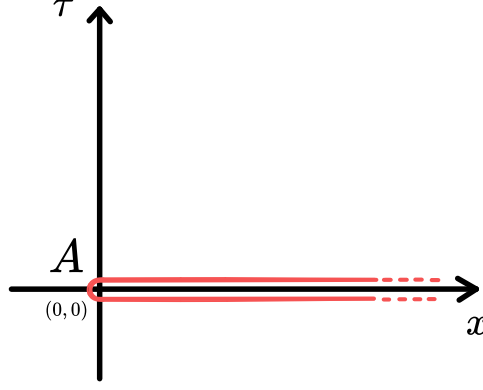


Figure 3.7: Branch-cut generated by the insertion of the $U(1)$ twist field at the origin.

$$\int [d\Phi] e^{-\mathcal{A}_E[\Phi]} \prod_{k=1}^n O_k^E(x_k, \tau_k). \quad (3.51)$$

Taking Φ to be a complex scalar field, the monodromy (3.50) can be realised concretely by introducing a **twist field** $\mathcal{T}^E$, associated to a global $U(1)$ symmetry, $\Phi \mapsto e^{i\alpha}\Phi$ (with the electric charge set to $q = 1$). Basically, this twist field corresponds to the insertion of a source of magnetic field at the origin³ $(0, 0)$:

$$\int [d\Phi] e^{-\mathcal{A}_E[\Phi]} \mathcal{T}^E(0, 0) \prod_{k=1}^n O_k^E(x_k, \tau_k) = \int_{C_A(w)} [d\Phi] e^{-\mathcal{A}_E[\Phi]} \prod_{k=1}^n O_k^E(x_k, \tau_k). \quad (3.52)$$

The name twist field comes precisely from this property: $C_A(w)$ denotes the quasi-periodicity (literally a *twisted* boundary condition) constraint on the fields Φ in the functional integral, namely:

$$C_A(w) := \begin{cases} \Phi(x, 0^-) = e^{2\pi i \omega} \Phi(x, 0^+), & x > 0, \\ \Phi(x, 0^-) = \Phi(x, 0^+), & x < 0. \end{cases} \quad (3.53)$$

See Fig. 3.7 for a graphical representation. Operationally, the functional integral is defined by removing the ray $x > 0$, evaluating the path integral with fixed boundary conditions on the two sides of the cut so that (3.53) is satisfied, and then integrating over all configurations on each side while keeping (3.53) enforced. It is then immediate that the field $\mathcal{T}^E$ defined in this way has semi-locality index ω with respect to the fundamental field Φ , and semi-locality index 0 with respect to any $U(1)$ -invariant field. Moreover, since the Hamiltonian density $h(x)$ has to be $U(1)$ -invariant, $\mathcal{T}^E$ is local with respect to it. As a consequence, $\mathcal{T}^E$ can be inserted anywhere in the path integral without spoiling the causal structure. Its correlation functions are independent of the shape of any cuts or branch points in the Euclidean plane, up to global $U(1)$ transformations of the fields that the moving cut may cross.

Indeed, consider cutting not only along $A : x > 0$, but also along a path B starting and ending on the line $x > 0$, thereby isolating a region bounded by B and by a segment

³The $U(1)$ twist field can be viewed as inserting a magnetic charge (or twist defect) at the origin and then, for consistency, one imagines a compensating charge placed at infinity; since it is at infinity, its only effect is an overall normalisation of correlation functions, without altering their local properties.

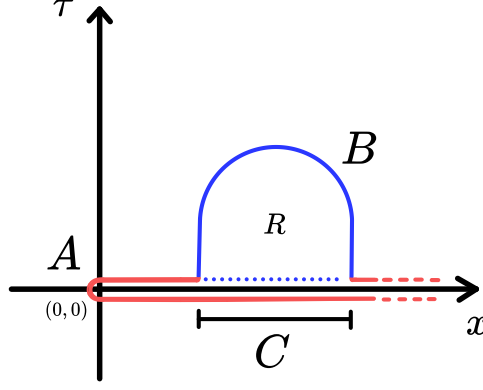


Figure 3.8: Deformed branch-cut.

$C \subset A$. Fix configurations along all these cuts with the quasi-periodicity condition (3.53) across A and with continuity across B , i.e.

$$\begin{cases} \Phi|_{C^-} = e^{2\pi i \omega} \Phi|_{C^+}, \\ \Phi|_{B^-} = \Phi|_{B^+}. \end{cases} \quad (3.54)$$

Integrating over all configurations on these cuts reproduces the functional integral (3.52). In fact, placing a cut at B with a continuity condition and summing over its configurations is equivalent to inserting a complete set of states, hence does not modify the path integral. Renaming (3.52) as

$$I_A \equiv \int_{C_A(\omega)} [d\Phi] e^{-\mathcal{A}_E[\Phi]} \prod_{k=1}^n O_k^E(x_k, \tau_k), \quad (3.55)$$

consider now the path B starting and ending on $x > 0$, and let $C \subset A$ be the segment of A enclosed by B (see Figure 3.8). Denoting by R the region bounded by B and C and by R^c its complement, we can rewrite the functional integral as

$$I_A = \int ([d\Phi]_R [d\Phi]_{R^c}) e^{-\mathcal{A}_E[\Phi]} \prod_k O_k^E, \quad (3.56)$$

with boundary conditions (3.54). Here the integration is meant as integration over fields in R and R^c with fixed boundary values, followed by integration over such boundary values (which corresponds to the insertion of a complete set of states):

$$I_A = \int \mathcal{D}\varphi_B \mathcal{D}\varphi_A \left(\int_{\substack{\Phi|_B = \varphi_B \\ \Phi|_{A \setminus C} = \varphi_A}} [d\Phi]_R e^{-\mathcal{A}_E[\Phi]_R} \right) \left(\int_{\substack{\Phi|_B = \varphi_B \\ \Phi|_{A \setminus C} = \varphi_A}} [d\Phi]_{R^c} e^{-\mathcal{A}_E[\Phi]_{R^c}} \prod_k O_k^E \right), \quad (3.57)$$

where φ_B, φ_A denote boundary values along B and $A \setminus C$, respectively.

Since R is an isolated region, we can then make a $U(1)$ transformation in it

$$\Phi \mapsto \tilde{\Phi} \equiv e^{i\omega\chi_R} \Phi, \quad \chi_R(x, \tau) = \begin{cases} 1 & (x, \tau) \in R, \\ 0 & \text{elsewhere.} \end{cases} \quad (3.58)$$

such that $\mathcal{A}_E[\tilde{\Phi}] = \mathcal{A}_E[\Phi]$ and $[d\tilde{\Phi}] = [d\Phi]$, i.e. the action and the measure remain invariant. After applying $\Phi \mapsto \tilde{\Phi}$ in R , the boundary conditions (3.54) become

$$\begin{cases} \tilde{\Phi}|_{B^-} = e^{2\pi i \omega} \tilde{\Phi}|_{B^+}, \\ \tilde{\Phi}|_{C^-} = \tilde{\Phi}|_{C^+}. \end{cases} \quad (3.59)$$

Thus the discontinuity (twist) is “*moved*” from $A \setminus C$ to B , while on C we now have continuity. Now the functional integral reads

$$I_A = \int_{C_{B \cup (A \setminus C)}(\omega)} [d\Phi] e^{-\mathcal{A}_E[\Phi]} \prod_k O_k^E \equiv I_{B \cup (A \setminus C)}, \quad (3.60)$$

where $C_{B \cup (A \setminus C)}(\omega)$ denotes quasi-periodicity along $B \cup (A \setminus C)$ and continuity along C . In the end we have that

$$I_A = I_{B \cup (A \setminus C)}. \quad (3.61)$$

That is, the functional integral with the original cut A is equal to the one with the deformed cut, up to a $U(1)$ transformation inside the isolated region, which leaves the action and the measure unchanged.

Actually it is even possible to remove the cut but in exchange we get the insertion of the Noether current associated to the local $U(1)$ transformation needed for “the removal”. Consider the path integral (3.52) in the presence of the cut, and perform the $U(1)$ transformation $\Phi(x) \mapsto e^{i\omega} \Phi(x)$ only inside an infinite triangular region bounded by:

1. the line $A : x > 0$ (which coincides with the cut),
2. the line $B : z = e^{i\alpha l}, l > 0$, and
3. infinity.

This triangular region (see Figure 3.9) can be taken, for instance, as lying above the cut A . Because the transformation is a symmetry, the bulk action is invariant. However, three modifications appear:

- a contribution on the boundary B coming from the kinetic (derivative) term in the action,
- a modification of the boundary condition just above A , and
- a $U(1)$ transformation of any fields that may be located inside the triangle.

Now, if we choose the parameter ω of the $U(1)$ transformation so that the boundary condition just above A becomes equal to that just below A , the discontinuity of the fields across the cut disappears. As a consequence, the path integral reduces to an ordinary one, without cut, but with an insertion of a boundary contribution along B :

$$\mathcal{O}_E(0,0) = \exp \left(2\pi i \omega \int_B dx^\mu j^\nu \epsilon_{\mu\nu} \right), \quad (3.62)$$

where j^ν is the Noether current associated with the global $U(1)$ invariance, and $\epsilon_{\mu\nu}$ selects the component of the current normal to B . An appropriate regularisation of the exponential is of course required. It is now clear that the insertion that replaces the cut is nothing but a Wilson line operator for the $U(1)$ current: the exponential of the line

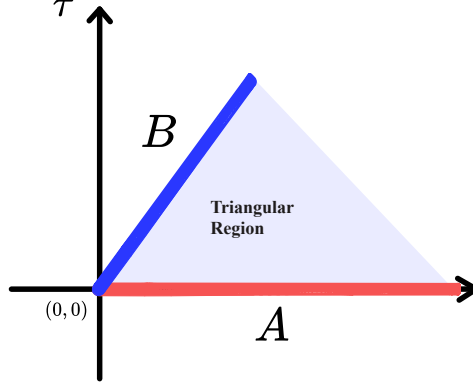


Figure 3.9: Triangular region on which we are performing the $U(1)$ transformation $\Phi(x) \mapsto e^{i\omega}\Phi(x)$.

integral of j^ν along B . Moving B across other fields implements a $U(1)$ transformation on them, while integrating the current along a closed loop L yields

$$\exp \left(i\omega \oint_L dx^\mu j^\nu \epsilon_{\mu\nu} \right), \quad (3.63)$$

which acts as a $U(1)$ transformation on all fields enclosed by L . This establishes that the operator $\mathcal{O}_E$ can be equivalently described as a disorder operator implementing a magnetic-type charge, or, more geometrically, as the insertion of a $U(1)$ Wilson line in the path integral.

As a concrete example, consider the free Majorana (Ising) theory with $\mathbb{Z}_2$ symmetry $\psi \mapsto -\psi$. One can define a twist field σ such that the Euclidean correlator

$$\langle \text{vac} | \sigma(0, 0) \psi(x, \tau) \psi(x', \tau') | \text{vac} \rangle \quad (3.64)$$

solves the Euclidean equations of motion on a double cover of the plane branched at the origin. Circling the branch point once changes the sign of the fields, reflecting the semi-local nature of σ [178, 179, 180].

As we just saw, the distinction between local and semi-local fields originates from the principle of locality, which requires commutativity (or graded commutativity) of observables at space-like separations, ensuring compatibility with relativistic causality. Semi-local fields relax this requirement: they remain consistent with causality but develop non-trivial exchange phases or branch-cut monodromies when taken around other operators. Such behavior reflects the presence of distinct charge sectors that cannot be connected by strictly local observables. In the examples considered here, these semi-local properties are realised through a global $U(1)$ symmetry, or in the free Majorana (Ising) theory through the $\mathbb{Z}_2$ symmetry, where twist fields create excitations in *superselection sectors* associated with the respective symmetry [20, 181]. In this way, semi-locality provides a natural bridge to the operator-algebraic notion of *superselection rules*, which classify the different sectors of a quantum field theory according to their locality and symmetry properties.

It is therefore clear that, depending on the locality property of a given observable, its form factors will follow different consistency equations. In particular, the equations for the form factors, which involve analytic continuations in the rapidity variables, are sensitive to the monodromy imposed by the corresponding twist field. This is because continuing a rapidity θ_i by $2\pi i$ in the complex plane corresponds to moving the operator associated with that particle around the insertion point of $\mathcal{O}$ in Euclidean space. Hence, if $\mathcal{O}$ is semi-local, this encircling picks up a non-trivial phase determined by the semi-locality index $\omega_{\mathcal{O}}$. Hence, the n -particle form factor satisfies a generalised cyclicity condition:

$$F_n^{\mathcal{O}}(\theta_1 + 2\pi i, \theta_2, \dots, \theta_n) = \gamma_{\mathcal{O}} F_n^{\mathcal{O}}(\theta_2, \dots, \theta_n, \theta_1), \quad (3.65)$$

where $\gamma_{\mathcal{O}} = e^{2\pi i \omega_{\mathcal{O}}}$ is the so-called *factor of local commutativity*⁴ first introduced in [182]. Similarly, the kinematical pole equation, must also reflect the monodromy of the operator:

$$\text{Res}_{\theta'=\theta+i\pi} F_{n+2}^{\mathcal{O}}(\theta' + i\pi, \theta, \theta_1, \dots, \theta_n) = i \left(1 - \gamma_{\mathcal{O}} \prod_{j=1}^n S(\theta_j - \theta) \right) F_n^{\mathcal{O}}(\theta_1, \dots, \theta_n). \quad (3.66)$$

Thus, the semi-locality or locality of the operator directly controls the analytic structure and permutation properties of its form factors, ensuring that all monodromies and residues are consistent with the operator's symmetry and superselection sector.

It is important to stress again that the locality or semi-locality of an operator is not an absolute property, but is defined relative to the fields considered fundamental in the theory. In particular, the factor of local commutativity appearing in (3.65) and (3.66) is always measured with respect to the fundamental field $\Phi(x)$ associated with the particle creation operators $Z^\dagger(\theta)$. Concretely, consider the operator $\mathcal{O}(y)$ and the field $\phi(x)$ associated with the ZF particle creation operators $Z^\dagger(\theta)$. The factor of local commutativity $\gamma^{\mathcal{O}}$ is defined through the equal-time exchange relation

$$\Phi(x) \mathcal{O}(y) = \gamma^{\mathcal{O}} \mathcal{O}(y) \Phi(x) \quad \text{for} \quad x^1 > y^1 \quad \text{and} \quad x^0 = y^0, \quad (3.67)$$

where, formally, the fundamental field admits the mode expansion

$$\Phi(x) \sim \int d\theta \left[Z(\theta) e^{-ip \cdot x} + Z^\dagger(\theta) e^{ip \cdot x} \right]. \quad (3.68)$$

This relation makes it explicit that the notion of semi-locality is always measured relative to the fundamental excitations of the theory. As an example, for the order/disorder operators in the Ising field theory, σ and μ , we have $\gamma_{\sigma/\mu} = \pm 1$. However, $\gamma_{\mathcal{O}}$ can be a more general phase as it happens, for instance, in the Federbush model [49, 183, 184]. One can also have exchange relations more complicated than (3.67), where fields do not simply pick up a phase but become different fields. For instance, this is what happens in the case of *branch point twist fields* [185] which we will introduce later in the chapter to try to investigate the entanglement properties of T $\bar{\text{T}}$ -deformed IQFTs.

A direct consequence is that equations (3.65) and (3.66) cannot be satisfied if this factor is not included [183, 184]. In other words, the locality properties of the observables enter explicitly into the form factor equations and their understanding is pivotal for the computation of closed exact expressions for n -particle form factors.

⁴At times in the literature the factor of local commutativity is defined to be directly $\omega_{\mathcal{O}}$.

3.1.4 Identifying the Operator Content

We have just seen that the form factor equations are not entirely operator-independent, as one might have thought at first sight: locality properties play a crucial role. But is this sufficient? In practice, it is often “almost sufficient”, in the sense that, if we assume that *the superselection sectors of the underlying CFT remain separated after introducing the mass scale*, then we already have a clear idea of which families of observables characterise a given model. What is left, in such cases, is essentially to extract the numerical data that confirm these expectations. The two standard consistency checks were already discussed in Chapter 0, namely the *c-theorem* (0.73) and the *Δ -sum rules* (0.74). Before we can identify which solutions correspond to which operators, however, we must first examine more carefully equation (3.66) and the general structure of its solutions.

The most general solution of Watson equations (3.12) and (3.65) is given by

$$F_n^{\mathcal{O}}(\theta_1, \dots, \theta_n) = H_n^{\mathcal{O}} K_n^{\mathcal{O}}(\theta_1, \dots, \theta_n) \prod_{i < j} F_{\min}(\theta_{ij}), \quad (3.69)$$

where the $H_n^{\mathcal{O}}$ are rapidity-independent constants while the remaining factors $K_n^{\mathcal{O}}$, as we already saw in (3.30) for the 2-particle case, satisfy Watson equation but with $S = 1$ and therefore they are *completely symmetric and periodic functions* in the variables θ_{ij} with period $2\pi i$. With this property in mind we can consider them directly as functions of $\cosh \theta_{ij}$. Moreover, because the only poles present are the kinematical ones encapsuled by the recursive eq. (3.66), we can directly parametrise them as

$$K_n^{\mathcal{O}}(\theta_1, \dots, \theta_n) = \frac{Q_n^{\mathcal{O}}(\theta_1, \dots, \theta_n)}{D_n(\theta_1, \dots, \theta_n)}, \quad (3.70)$$

where the denominator D_n is a polynomial in $\cosh \theta_{ij}$ fixed only by the pole structure of the S-matrix and $Q_n^{\mathcal{O}}$ is another polynomial, the actual object that we need to compute using (3.66), that completely specify the properties of the operator $\mathcal{O}$. Being the $K_n^{\mathcal{O}}$ symmetric under permutations of rapidities it is customary to introduce the variables $x_i \equiv e^{\theta_i}$ so that both numerator and denominator in eq. (3.70) can be written in terms of a basis of *elementary symmetric polynomials* in the new variables. Denoting the order k elementary symmetric polynomial of n symbols as $\sigma_k^{(n)}(x_1, \dots, x_n)$, we can define them as stemming from the generating function

$$\prod_{i=1}^n (x + x_i) = \sum_{k=0}^n x^{n-k} \sigma_k^{(n)}(x_1, \dots, x_n). \quad (3.71)$$

By convention, usually all $\sigma_k^{(n)}$ with $k > n$ or $n < 0$ are zero. Explicitly we have

$$\sigma_0^{(n)}(x_1, \dots, x_n) = 1, \dots, \sigma_k^{(n)}(x_1, \dots, x_n) = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k}. \quad (3.72)$$

Therefore the $\sigma_k^{(n)}$ are homogeneous linear polynomials in x_i of total degree k . The polynomials $Q_n^{\mathcal{O}}$ of (3.70) are further constrained by the asymptotics of form factors. For instance, if we consider the relativistic invariance property shown in (3.11), in the simple case of a scalar operator we have

$$F_n^{\mathcal{O}}(\theta_1 + \Lambda, \dots, \theta_n + \Lambda) = F_n^{\mathcal{O}}(\theta_1, \dots, \theta_n), \quad (3.73)$$

which implies the equality of the total degrees of numerator and denominator in (3.70).

Let us now turn to eq. (3.27). In the class of models under consideration, the kinematical residue equation encodes the presence of the *only* singularities of the form factors, namely the kinematical poles. We may introduce a map $\mathcal{K}$ implementing the recursive relation

$$F_{n+2}^{\mathcal{O}} = \mathcal{K} F_n^{\mathcal{O}}, \quad (3.74)$$

which allows one to generate the entire hierarchy of solutions starting from an initial datum at $n = 1$. All the form factor equations - (3.12), (3.20) and (3.74) - are *linear* in nature. Consequently, for any fixed particle number n , their solutions form a linear space⁵. Recall that at criticality the operator space of the CFT can be organised in terms of the irreducible representations of the Virasoro algebra. Provided the relevant perturbation does not mix different superselection sectors, one may then associate to each family of primaries (at the conformal point) the vectors spanning these linear spaces.

However, some care is required: in fact, multiple solutions may correspond to the same initial condition. Within these linear spaces one may find elements that belong to the *kernel* of the operator $\mathcal{K}$ [26], namely functions $\hat{F}_n$ such that

$$\mathcal{K} \hat{F}_n = 0. \quad (3.75)$$

Such functions do not carry any pole and hence satisfy the recursive equation in a trivial manner; they are simply symmetric polynomials in the variables x_i . The subspace generated by these kernel solutions can be further classified by fixing the total and partial degrees of the polynomials. Whenever the kernel is nontrivial, the most general solution of the recursive equation at level n takes the form

$$F_n = \tilde{F}_n + \sum_i \alpha_i \hat{F}_n, \quad (3.76)$$

where $\tilde{F}_n$ is a reference solution of the recursion. In this situation, operator identification is achieved only once all the coefficients α_i are fixed at every level n . Establishing the isomorphism between the operator content of the CFT and that of its relevant perturbation thus requires an explicit counting of these solutions [28].

Up to this point our discussion has mostly concerned operators corresponding to primaries of the underlying CFT. A natural question then arises: what about the descendants? How are their form factors characterised within the same framework? Although we will only touch upon this topic briefly, it is worth saying a few words for completeness, since descendant operators also play a role in the full operator content of the theory. The interested reader may consult the existing literature for a more detailed account [18, 26, 25, 91, 92, 170, 186, 27]. The descendant are obtained by *boosting* the fundamental solutions, i.e. given the “space of form factors” $\mathcal{P}$, we can decompose it as

$$\mathcal{P} = \bigoplus_s \mathcal{P}_s, \quad (3.77)$$

where the FF $F_n^{\mathcal{O}} \in \mathcal{P}_s$ if $\mathcal{O}$ has spin s [170]. A boost on the rapidity variables is realised by $\theta_i \rightarrow \theta_i + \varepsilon$ and in the new basis we have $x_i \rightarrow e^\varepsilon x_i$. As already said the elementary symmetric polynomials (3.71) are homogeneous in their variables and then under a Lorentz boost they transform as

⁵An operator $\mathcal{O}$ is uniquely determined once all its matrix elements $F_n^{\mathcal{O}}$ are specified.

$$\sigma_n(x_1, \dots, x_n) \rightarrow e^{n\epsilon} \sigma_n(x_1, \dots, x_n). \quad (3.78)$$

This means that given a form factor $F_n^\mathcal{O}$ of an operator $\mathcal{O}$ of spin s that satisfies the consistency equations, it is possible to define a new function in the “boosted space” $\mathcal{P}_{s+s'}$ which still satisfies Watson’s equations and it is given by

$$F_n^{\mathcal{O}'}(x_1, \dots, x_n) = I_n^{s'}(x_1, \dots, x_n) F_n^\mathcal{O}(x_1, \dots, x_n) \quad (3.79)$$

provided that I_n^s is composed out of elementary symmetric polynomials too. An additional constraint comes from the requirement of “invariance under the kinematical residue equation”

$$I_{n+2}^s(-x, x, x_1, \dots, x_n) = I_n^s(x_1, \dots, x_n). \quad (3.80)$$

In this way it is possible to make an ansatz (always in terms of symmetric polynomials) that covers all the possible solutions for descendant fields. Eq. (3.79) endows the space of matrix elements of local operators in an integrable massive theory with a natural grading [187, 170]. Concretely, once an invariant polynomial I_n^s is specified, eq. (3.79) generates the form factors of a new operator $\mathcal{O}'_s$. In analogy with the CFT terminology, it is therefore appropriate to interpret $\mathcal{O}'_s$ as the descendant of the spinless field $\mathcal{O}$. A particularly interesting case arises when $\mathcal{O}$ is taken to be the trace of the stress-energy tensor: in this case, the form factors obtained from eq. (3.79) can be identified with the matrix elements of higher conserved currents. Remember that even the stress-energy tensor itself is not a primary field, and its form factors can therefore be expressed in terms of those of a scalar descendant operator. In particular, all components of $T_{\mu\nu}$ can be reconstructed from the form factors of its trace Θ , which is directly related to the relevant primary field defining the theory (0.58). The recursive equations then reduce the problem of fixing the one- and two-particle form factors of Θ ; the latter is normalised by requiring consistency with the definition of the total energy. By contrast, the one-particle form factor remains undetermined, reflecting the intrinsic ambiguity in defining the stress-energy tensor.

Finally, while general methods for Vacuum Expectation Values (VEVs) are lacking, exact results are available in specific models such as the Ising, sine–Gordon, Bullough–Dodd, and certain RSOS theories [188, 189, 190]. In general, in IQFTs the vacuum expectation value (VEV) of a local operator $\mathcal{O}(x)$ is determined by dimensional analysis. If $\mathcal{O}$ has scaling dimension Δ , then

$$\langle \mathcal{O} \rangle \propto m^{2\Delta}, \quad (3.81)$$

where m is the mass scale of the theory. In particular, since the trace of the stress–energy tensor Θ has $\Delta = 1$, its VEV scales as $\langle \Theta \rangle \sim m^2$.

To conclude this discussion of the operator content, we shall outline a set of standard consistency checks that can be used to identify the physical observables corresponding to a given solution of the form factor equations. Already in Chapter 0 we encountered two fundamental consistency checks for integrable QFTs, namely the c -theorem (0.73) and the Δ -sum rule (0.74). Through the spectral decomposition of 2-point functions (3.3), these results can be expressed directly in terms of form factors. In addition, there are two further selection rules that follow from the asymptotic behavior of form factors. Given an observable $\mathcal{O}$, one expects that [191]

$$\lim_{|\theta_i| \rightarrow \infty} F_n^{\mathcal{O}}(\theta_1, \dots, \theta_n) \leq \text{const. } e^{\Delta_{\mathcal{O}}|\theta_i|}, \quad (3.82)$$

i.e. the UV growth of the matrix elements of a given scaling operator $\mathcal{O}$ of the underlying CFT should be exponentially bounded in terms of its scaling dimension $2\Delta_{\mathcal{O}}$ ⁶. This is the so-called *asymptotic bound*. When the operator space splits into distinct sectors associated with the internal symmetries of the model (i.e. the superselection sectors previously discussed), this bound is often sufficient to single out the correct form factor solutions [192]. In the absence of internal symmetries, for a relevant operator Φ with $\Delta_{\Phi} < 1$, one can also exploit the *asymptotic factorisation property* [192, 193, 26, 194]:

$$\lim_{\lambda \rightarrow \infty} F_{r+l}^{\Phi}(\theta_1 + \lambda, \dots, \theta_r + \lambda, \theta_{r+1}, \dots, \theta_{r+l}) = \frac{1}{\langle \Phi \rangle} F_r^{\Phi}(\theta_1, \dots, \theta_r) F_l^{\Phi}(\theta_{r+1}, \dots, \theta_{r+l}), \quad (3.83)$$

where $\langle \Phi \rangle$ denotes the vacuum expectation value.

3.2 Applying the Form Factor Bootstrap Program to $T\bar{T}$ -deformed IQFTs

Up to this point, the discussion has been mainly concerned with the general framework and the existing literature. We now turn to the original results obtained over the last few years. The first stage of this program began in 2022 with a series of three papers [12, 14, 13], in which we attempted to apply the Form Factor Bootstrap Program (FFBP), in its standard formulation, to our particular “new” class of models. As already emphasised in Chapter 1, the CDD-factors (1.56) were known since the earliest studies of elastic S -matrices in 1+1-dimensions [67], but were systematically disregarded because they were regarded as non-physical.⁷ In the last few decades, however, the origin and relevance of such CDD-factors has become much clearer, particularly in the context of high-energy physics. In particular, the works [150, 106] established that the properties of a 1+1-dimensional QFT perturbed by the irrelevant deformations presented in Chapter 2 are under exceptionally good control. Remarkably, the $T\bar{T}$ -deformation can be regarded as *solvable*, in the sense that physical observables of interest — such as the finite-volume spectrum (see Eq. (2.39)), the S -matrix [150, 106, 135], and partition functions [101, 115, 195] — can all be determined exactly in terms of their undeformed counterparts. The property of solvability extends to the generalised $T\bar{T}$ -deformations, obtained by perturbing an integrable QFT with composite operators constructed from higher-spin conserved currents [107, 108]. Performing a (generalised) $T\bar{T}$ -deformation in an integrable QFT is equivalent to modifying the two-body scattering matrix precisely by such CDD factors, as shown in (2.43) [150, 106, 109, 110]. It is worth noting that we were not the first to attempt the computation of correlation functions in $T\bar{T}$ -deformed theories using the form factor approach. Indeed, already in [162]⁸, the FFBP was applied in its massless variant [196].

In a massless theory, since the mass gap is zero, the unitarity and crossing cuts of the S -matrix in the s -plane join at the origin, so that the Riemann surface splits into two

⁶Here we restrict to scalar operators.

⁷Here, the notion of *physical* is again related to the expectation of *locality*, which is generally assumed to govern physical laws.

⁸This is, to the best of my knowledge, the first paper in which these CDD factors were rediscovered after a long period of neglect since their last appearance in the literature.

distinct parts and a new parametrisation is required [151, 197]. We have free massless right- and left-movers that can only interact through a “boundary”. This parametrisation also takes into account the fact that one needs to send $m \rightarrow 0$ while keeping energy and momentum finite. One then finds that by replacing θ with $\beta \pm \beta_0/2$ and taking the limits $m \rightarrow 0$, $\beta_0 \rightarrow \infty$ in such a way that $M \equiv me^{\beta_0/2}$ remains finite, the excitations can be classified as right-movers, with $p^0 = p^1 = \frac{M}{2}e^\beta$, and left-movers, with $p^0 = -p^1 = \frac{M}{2}e^{-\beta}$.

Notice that now the Mandelstam variable is $s_{ij} = M^2 e^{\beta_i - \beta_j}$. In this context, namely a $\text{T}\bar{\text{T}}$ -deformed massless boson (equivalently a Nambu–Goto string in light-cone gauge), the CDD-factor turns out to be [162]

$$\Phi(\beta) = e^{i\ell_s^2 M^2 e^\beta}, \quad (3.84)$$

where ℓ_s is the string scale⁹. They therefore attempted to compute form factors, starting from the MFF, which should still satisfy an equation like (3.32). The problem is that the procedure discussed previously (see Subsection 3.1.2) does not seem to be reproducible in this case, due to the poor convergence properties of this CDD-factor. Again, this is a feature of a “non-local” theory in the sense of Hepp [68, 69]. These factors are precisely the ones that Karowski and Weisz refused to include when formulating their generalised version of Watson’s theorem (3.29). A possible solution to this problem will be presented in Chapter 4. Therefore, the only available strategy is to attempt to guess what a minimal solution of (3.32) could look like. In [162] they found the minimal solution

$$F_{\min}(\beta) = e^{-\ell_s^2 M^2 \frac{\beta}{2\pi} e^\beta}. \quad (3.85)$$

At large momenta this function decays extremely fast: it is a double exponential in rapidity. However, it grows at the same rate when $\text{Im}(\beta) = \pi$. In the end, their conclusion was that there was no reason to assume polynomial growth for the Q -part of possible form factor solutions, and they stated that, “at least in its standard form”, the technique could not be used to determine form factors and hence off-shell observables. This is more of a conjecture than an exact formal statement, but nevertheless a very convincing one, especially in light of our previous discussion about locality and the way form factors are built.

In our first three works we started from exactly the same “initial conditions”: we only knew the S -matrix, without any certainty regarding the applicability of the FFBP. As stated in the Preface, this remains an ongoing line of research and no definitive conclusions can yet be drawn. Nevertheless, as we shall see, this initial effort provided valuable insights and led to interesting results. We began by addressing the problem in the simplest possible setting, namely the deformation of the Ising field theory, i.e. a theory of free massive Majorana fermions [12]. We then showed how the results obtained in this case could be extended to general diagonal theories without bound states [13], and finally we considered the inclusion of branch point twist fields [14].

For simplicity, we are going to focus IQFTs with single particle spectrum. After a single $\text{T}\bar{\text{T}}$ perturbation the scattering matrix $S_0(\theta)$ is modified to

$$S_\alpha(\theta) := S_0(\theta) e^{-iam^2 \sinh \theta}. \quad (3.86)$$

i.e. the simple $\alpha = \alpha_1$ of the more general case (2.43). Written in terms of energy-momentum vectors of the two scattering particles $p_\mu^i = m(\cosh \theta_i, \sinh \theta_i)$, with $i = 1, 2$,

⁹In the literature the phase of the S -matrix is often taken positive while throughout our works we always adopted the opposite notation as in (2.43).

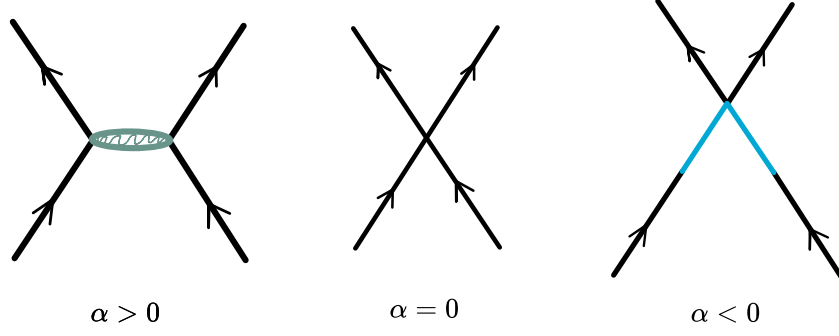


Figure 3.10: Different effect of the deformation on the scattering depending on the sign of the coupling α .

the $\text{T}\bar{\text{T}}$ -deformation (3.86) reads $\exp[-i\alpha\varepsilon^{\mu\nu}p_\mu^1p_\nu^2]$, where $\theta = \theta_1 - \theta_2$. Although this exponential form is formally reminiscent of the pure scattering phases that can appear in models with current-current interactions [50, 198, 184], where the effect of the interaction is exhausted by a topological or charge-dependent phase, in the present case the phase carries genuine kinematical information and cannot be absorbed into a constant redefinition of asymptotic states. For simplicity, from now on we will take the mass scale $m = 1$. This deformation introduce a very uncommon double-exponential dependence on the rapidity that radically changes the S -matrix asymptotic behaviour and has a stark effect on the theory's RG flow. Generally speaking, whereas the IR regime is left unaltered by the perturbation, in the short-distance limit the theory displays unusual features, which are incompatible with the existence of a UV CFT¹⁰ [162, 163, 135]. This is in agreement with the fact that the CDD factors (2.43) correspond to perturbations by irrelevant operators, whose presence is not felt at large distances but is expected to severely alter the properties of the UV. In a conventional, UV-complete IQFT the small R behaviour of $E(R)$ is dominated by a simple pole $E(R) \sim -\frac{\pi c}{6R}$ that signals the presence of a CFT with central charge c in the UV. As it has been shown in various works [106, 154, 202] (see also Subsection 2.1.3), in $\text{T}\bar{\text{T}}$ -deformed theories there are two possible scenarios for small R properties of $E(R, \alpha)$, depending to the sign of α in (3.86):

- for $\alpha > 0$, from (2.42), we see that in the UV limit, the ground-state energy is finite $\lim_{R \rightarrow 0} E(R, \alpha) = -e_0(\alpha)$, implying that the theory possesses a finite amount of degrees of freedom;
- for $\alpha < 0$ a square-root branch point appears at $R = R_* \sim |\alpha|^{-1/2} > 0$, signalling the presence of a Hagedorn growth [81] of the density of states at high energy (see Eq. (2.35)).

The GHD viewpoint [146] complements this picture, suggesting that the abnormal UV limit of $\text{T}\bar{\text{T}}$ -deformed IQFTs can be imputed to point-like particles acquiring a positive or negative size (i.e. free space is added), for $\alpha > 0$ and $\alpha < 0$ respectively (see Fig. 3.10). For what concerns the generalised $\text{T}\bar{\text{T}}$ -deformations with S -matrices given by (2.43), it was shown [109] that they can only display the second of the above behaviours. In fact, for S -matrices (2.43) with a finite set of couplings α , the standard TBA equations only

¹⁰The nature of the UV regime has also been explored in a careful analysis of the spectrum in $\text{T}\bar{\text{T}}$ -deformed 2D Yang-Mills [199, 200, 201].

make sense if the coupling with larger index s is negative $\alpha_s < 0$. One possible way to study the regime $\alpha_s > 0$ is to employ the *generalised TBA* [203] approach. Note that neither of the behaviours listed above is compatible with Wilson’s paradigm of local QFTs [17]. For this reason – and thanks to their property of being solvable and controllable at any energy scale – the generalised $\text{T}\bar{\text{T}}$ -deformed theories can be considered as a sensible extension of the Wilsonian notion of a local QFT. Our work constitutes the first systematic attempt (in the sense that it encompasses a large set of theories, fields and perturbations) at computing the form factors of local and semi-local fields in generalised $\text{T}\bar{\text{T}}$ -perturbed IQFTs. Although the following work focuses particularly on the Ising model some of the key results are clearly more general and will be discussed immediately after. The form factors and correlation functions of a theory with similar unusual features (the sinh-Gordon model, with deformation $\Phi_\alpha(\theta) = -1$) were computed for the first time in [161], and found to give rise to a number of problems, including non-convergent correlators and form factors involving free parameters that could not be fully fixed. The present work will allow us to understand similar issues in a much wider context.

3.2.1 Form Factors of the $\text{T}\bar{\text{T}}$ -deformed Ising Field Theory

Consider an IQFT with a single-particle spectrum and deformed S -matrix given by (2.43), with $m = 1$. We can think of the n -particle form factor of a field $\mathcal{O}$ as defined as:

$$F_n^\mathcal{O}(\theta_1, \dots, \theta_n; \alpha) := \langle 0 | \mathcal{O}(0) | \theta_1, \dots, \theta_n | 0 \rangle, \quad (3.87)$$

where we have now included a dependence on the deformation parameters. Assuming that the form factor equations still hold for the fields of the perturbed theory, we should be able to compute the quantities (3.87) by following the usual bootstrap program just introduced in the first part of this chapter, taking as starting point the deformed S -matrix (2.43). Therefore we can then use them as building blocks to construct correlation functions, allowing in particular to study the short and long-distance limits of these correlation functions and to understand how those limits are affected by the perturbation.

In the absence of bound states, the form factor equations for local and semi-local fields in this theory should be, in principle, be written as

$$F_n^\mathcal{O}(\theta_1, \dots, \theta_i, \theta_{i+1}, \dots, \theta_n; \alpha) = S_\alpha(\theta_i - \theta_{i+1}) F_n^\mathcal{O}(\theta_1, \dots, \theta_{i+1}, \theta_i, \dots, \theta_n; \alpha), \quad (3.88)$$

$$F_n^\mathcal{O}(\theta_1 + 2\pi i, \theta_2, \dots, \theta_n; \alpha) = \gamma_\mathcal{O} F_n^\mathcal{O}(\theta_2, \dots, \theta_n, \theta_1; \alpha), \quad (3.89)$$

and

$$\lim_{\bar{\theta} \rightarrow \theta} (\bar{\theta} - \theta) F_{n+2}^\mathcal{O}(\bar{\theta} + i\pi, \theta, \theta_1, \dots, \theta_n; \alpha) = i \left(1 - \gamma_\mathcal{O} \prod_{j=1}^n S_\alpha(\theta - \theta_j) \right) F_n^\mathcal{O}(\theta_1, \dots, \theta_n; \alpha). \quad (3.90)$$

In the standard form factor bootstrap reviewed earlier in this chapter, the two-particle form factor $F_2^\mathcal{O}(\theta)$ is first factorised into an operator-dependent prefactor $K_2^\mathcal{O}(\theta)$ (as defined in (3.30)) and a universal minimal solution of Watson’s equations. The first step is therefore to search for a MFF, i.e. a solution of (3.88) and (3.89) without poles in the physical strip. Unfortunately the usual Karowski and Weisz procedure does not seem here possible to pursue seeing the double exponential dependence in rapidity of the CDD factor itself that is strictly forbidden by the theorem (3.29). But as in [162], we can at least guess what a minimal solution that solves the equation

$$F_{\min}(\theta; \alpha) = S_{\alpha}(\theta) F_{\min}(-\theta; \alpha) = F_{\min}(2\pi i - \theta; \alpha), \quad (3.91)$$

should look like. Since the S -matrix is factorised as in (2.43), it is natural to make the ansatz

$$F_{\min}(\theta; \alpha) := F_{\min}(\theta; \mathbf{0}) \varphi(\theta; \alpha), \quad (3.92)$$

where the function $\varphi(\theta; \alpha)$ solves the equations

$$\varphi(\theta; \alpha) = \Phi_{\alpha}(\theta) \varphi(-\theta; \alpha) = \varphi(2\pi i - \theta; \alpha). \quad (3.93)$$

It is easy to show that the simplest solution to (3.93) is

$$\varphi(\theta; \alpha) = \exp \left[-\frac{(i\pi - \theta)}{2\pi} \sum_{s \in \mathcal{S}} \alpha_s \sinh(s\theta) \right]. \quad (3.94)$$

Notice that this solution is precisely the correct one that reproduce (3.85) in the massless limit. However, more generally, given a solution (3.94) it is always possible to multiply it by a function of the following form

$$C(\theta; \beta) := \exp \left[\sum_{s \in \mathbb{Z}} \beta_s \cosh(s\theta) \right]. \quad (3.95)$$

In [12] we named these functions “CDD-factor” of the minimal form factor itself because it satisfies trivially the MFF equations. It is clear that we have some free parameters that we are unable to fix and for now we will not consider the larger family of solutions generated by (3.95) focusing on the “minimal” solution which is necessary to solve the equations. Nevertheless, it is possible to give a physical interpretation for the factor $C(\theta; \beta)$. Remember in fact that, every CDD-factor coming from a generalised $\text{T}\bar{\text{T}}$ -deformation of spin s is in one-to-one correspondence with a given conserved current defined by (2.1) and (2.3). The terms $\alpha_s \sinh(s\theta)$ in the sum over s in (2.43) are the one-particle eigenvalues of local conserved quantities of spin s . However, those one-particle eigenvalues are more generically linear combinations of the form $\alpha_s \sinh(s\theta) + \beta_s \cosh(s\theta)$. The fact that we can always add a sum of terms $\beta_s \cosh(s\theta)$ is a reflection of this “ambiguity” in our choice of the conserved quantities. It also means that given a solution to the form factor equations for a certain field for the choice $\beta = \mathbf{0}$, new solutions can be constructed by “switching on” some of the parameters β_i . These new solutions will correspond to the same field but to a different perturbation. Although these additional factors do not affect the property of minimal analyticity, since they introduce no extra singularities in the physical strip, they do modify the asymptotic behaviour of the solution. This, however, should not be regarded as unexpected: the starting point already involves an S -matrix that does not satisfy the standard constraints underlying the Karowski-Weisz construction, and accordingly the remaining part of the minimal form factor is not required to display the usual asymptotic behaviour of a conventional MFF.

For the free boson (+) and free fermion (−) theories, with scattering matrices $S_{\mathbf{0}}(\theta) = \pm 1$, the minimal form factors are

$$F_{\min}^+(\theta; \alpha) = \varphi(\theta; \alpha), \quad F_{\min}^-(\theta; \alpha) = -i \sinh \frac{\theta}{2} \varphi(\theta; \alpha). \quad (3.96)$$

The key observation is then that the function $\varphi(\theta; \alpha)$ is common to all theories and describes the universal way in which the minimal solution to the form factor equations is modified by the presence of irrelevant perturbations. Before moving on, we remark that, if $\mathcal{S}$ contains only odd spins, as we are going to assume henceforth, the function φ satisfies the following identity that will be useful in subsequent computations:

$$\varphi(\theta; \alpha)\varphi(\theta + i\pi; \alpha) = \sqrt{\Phi_\alpha(\theta)}. \quad (3.97)$$

Furthermore, with our normalisations

$$\varphi(i\pi; \alpha) = \varphi(\theta; \mathbf{0}) = 1. \quad (3.98)$$

Solving the Kinematical Residue Equation

We now turn to the problem of computing higher particle form factors. Being this a universal deformation, we expect this result to still be true at least for any interacting diagonal theory without bound state, as we will see later.

Consider the form factor equations (3.88), (3.89) and (3.90) for higher particle form factors of local or semi-local operators. *Assuming* there is a kinematic pole whenever any two rapidities differ by $i\pi$ ¹¹, we can make the standard type of ansatz

$$F_n^\mathcal{O}(\theta_1, \dots, \theta_n; \alpha) = H_n^\mathcal{O} Q_n^\mathcal{O}(x_1, \dots, x_n; \alpha) \prod_{i < j} \frac{F_{\min}(\theta_{ij}; \alpha)}{x_i + x_j}, \quad (3.99)$$

where the functions $Q_n^\mathcal{O}(x_1, \dots, x_n; \alpha)$ are symmetric and $2\pi i$ -periodic functions of all the variables $x_j := e^{\theta_j}$. As already discussed, they can be written in terms of a basis of elementary symmetric polynomials in these variables. The symbols $H_n^\mathcal{O}$ in (3.99) are constants, independent of the rapidity variables. Plugging this ansatz into the kinematic residue equation (3.90) we can reshape the left hand side into the following form

$$\begin{aligned} F_{n+2}^\mathcal{O}(\bar{\theta} + i\pi, \theta, \theta_1, \dots, \theta_n; \alpha) &= H_{n+2}^\mathcal{O} Q_{n+2}^\mathcal{O}(-\bar{x}, x, x_1, \dots, x_n; \alpha) \prod_{1 \leq i < j \leq n} \frac{F_{\min}(\theta_{ij}; \alpha)}{x_i + x_j} \\ &\times \left[\prod_{j=1}^n \frac{F_{\min}(\theta - \theta_j; \alpha) F_{\min}(\bar{\theta} + i\pi - \theta_j; \alpha)}{(x + x_j)(-\bar{x} + x_j)} \right] \frac{F_{\min}(i\pi; \alpha)}{-\bar{x} + x}, \end{aligned} \quad (3.100)$$

where $x = e^\theta$ and $\theta_{ij} := \theta_i - \theta_j$. Then the equation becomes

$$\begin{aligned} \lim_{\bar{\theta} \rightarrow \theta} \frac{\bar{\theta} - \theta}{x - \bar{x}} H_{n+2}^\mathcal{O} Q_{n+2}^\mathcal{O}(-\bar{x}, x, x_1, \dots, x_n; \alpha) &\prod_{j=1}^n \frac{F_{\min}(\theta - \theta_j; \alpha) F_{\min}(\bar{\theta} + i\pi - \theta_j; \alpha)}{(x + x_j)(-\bar{x} + x_j)} \\ &= i \left(1 - \gamma_\mathcal{O} \prod_{j=1}^n S_\alpha(\theta - \theta_j) \right) H_n^\mathcal{O} Q_n^\mathcal{O}(x_1, \dots, x_n; \alpha). \end{aligned} \quad (3.101)$$

¹¹Note that this is typically not the case for the two-particle form factor of the trace of the stress-energy tensor Θ or more generally of any local fields (i.e. fields where $\gamma_\mathcal{O} = 1$), since in this case the r.h.s. of the kinematic residue equation (3.90) with $n = 2$ is vanishing. The kinematic pole in the two-particle form factor is generally only present for symmetry fields, which sit at the origin of branch cuts in the complex plane. However, for theories with a non-trivial S -matrix, kinematic residue poles will be present for the higher particle form factors, even for local fields.

where we used (3.98). Finally, noting that $\lim_{\bar{\theta} \rightarrow \theta} \frac{\bar{\theta} - \theta}{x - \bar{x}} = -\frac{1}{x}$ and using the properties (3.92), (3.97) and (3.98), we arrive at

$$\begin{aligned} & -x^{-1} H_{n+2}^{\mathcal{O}} Q_{n+2}^{\mathcal{O}}(-x, x, x_1, \dots, x_n; \boldsymbol{\alpha}) \prod_{j=1}^n \frac{F_{\min}(\theta - \theta_j; \mathbf{0}) F_{\min}(\bar{\theta} + i\pi - \theta_j; \mathbf{0})}{x_j^2 - x^2} \\ & = i \left[\prod_{j=1}^n \Phi_{\boldsymbol{\alpha}}(\theta - \theta_j)^{-\frac{1}{2}} - \gamma_{\mathcal{O}} \prod_{j=1}^n \Phi_{\boldsymbol{\alpha}}(\theta - \theta_j)^{\frac{1}{2}} S_0(\theta - \theta_j) \right] H_n^{\mathcal{O}} Q_n^{\mathcal{O}}(x_1, \dots, x_n; \boldsymbol{\alpha}). \end{aligned} \quad (3.102)$$

We will now show how to find solutions to this equation for the $T\bar{T}$ -deformed Ising field theory.

The form factors of the Ising field theory have been extensively studied in the literature [178, 179]. In the so-called disordered phase, form factors and correlation functions of the order (σ) and disorder (μ) fields were computed in [182, 180]. Around the same time, form factors of descendant fields in the CFT sense were computed in [25] and shown to be in one-to-one correspondence with the Virasoro irreducible representations characterising the critical theory. The Ising field theory is a free fermion theory with scattering matrix $S_0(\theta) = -1$, and $F_{\min}(\theta; \mathbf{0}) = -i \sinh \frac{\theta}{2}$. For this model, the set $\mathcal{S}$ of the spins of local conserved quantities coincides with the odd positive integers

$$\mathcal{S} = \{s = 2n - 1 | n \in \mathbb{Z}^+\}. \quad (3.103)$$

With these specifications, equation (3.102) simplifies to

$$\begin{aligned} & -x^{-1} H_{n+2}^{\mathcal{O}} Q_{n+2}^{\mathcal{O}}(-x, x, x_1, \dots, x_n; \boldsymbol{\alpha}) \prod_{j=1}^n \frac{i}{4xx_j} \\ & = i \left[\prod_{j=1}^n \Phi_{\boldsymbol{\alpha}}(\theta - \theta_j)^{-\frac{1}{2}} - \gamma_{\mathcal{O}} (-1)^n \prod_{j=1}^n \Phi_{\boldsymbol{\alpha}}(\theta - \theta_j)^{\frac{1}{2}} \right] H_n^{\mathcal{O}} Q_n^{\mathcal{O}}(x_1, \dots, x_n; \boldsymbol{\alpha}), \end{aligned} \quad (3.104)$$

that we can further split into two equations:

$$\begin{aligned} Q_{n+2}^{\mathcal{O}}(-x, x, x_1, \dots, x_n; \boldsymbol{\alpha}) & = x^{n+1} \prod_{j=1}^n x_j \\ & \times \left[\prod_{j=1}^n \Phi_{\boldsymbol{\alpha}}(\theta - \theta_j)^{-\frac{1}{2}} - \gamma_{\mathcal{O}} (-1)^n \prod_{j=1}^n \Phi_{\boldsymbol{\alpha}}(\theta - \theta_j)^{\frac{1}{2}} \right] Q_n^{\mathcal{O}}(x_1, \dots, x_n; \boldsymbol{\alpha}) \end{aligned} \quad (3.105)$$

and

$$H_{n+2}^{\mathcal{O}} = 4^n i^{-1-n} H_n^{\mathcal{O}}. \quad (3.106)$$

The latter is easily solved

$$H_{2n}^{\mathcal{O}} = 4^{n(n-1)} i^{-n^2} H_0^{\mathcal{O}} \quad \text{and} \quad H_{2n+1}^{\mathcal{O}} = 4^{n^2} i^{-n(n+1)} H_1^{\mathcal{O}}, \quad (3.107)$$

whereas the equation for the polynomials $Q_n^{\mathcal{O}}(x_1, \dots, x_n; \boldsymbol{\alpha})$ requires some additional input about the field $\mathcal{O}$. Let us recall the field content of the Ising field theory (besides the fermion itself) consists of the energy ε , the spin σ and the disorder μ fields. Their form factors are well known

$$F_{2n}^{\mu}(\theta_1, \dots, \theta_{2n}; \mathbf{0}) = i^n \prod_{i < j} \tanh \frac{\theta_{ij}}{2}, \quad F_{2n+1}^{\mu}(\theta_1, \dots, \theta_{2n+1}; \mathbf{0}) = 0, \quad (3.108)$$

$$F_{2n}^{\sigma}(\theta_1, \dots, \theta_{2n}; \mathbf{0}) = 0, \quad F_{2n+1}^{\sigma}(\theta_1, \dots, \theta_{2n+1}; \mathbf{0}) = i^n \prod_{i < j} \tanh \frac{\theta_{ij}}{2}, \quad (3.109)$$

and

$$F_2^{\varepsilon}(\theta; \mathbf{0}) = -i \sinh \frac{\theta}{2}, \quad F_{n>2}^{\varepsilon}(\theta_1, \dots, \theta_n; \mathbf{0}) = 0. \quad (3.110)$$

In the Ising model, there is a $\mathbb{Z}_2$ symmetry and the fields organise themselves into two symmetry sectors. The energy ε and the spin σ fields both have $\gamma_{\varepsilon} = \gamma_{\sigma} = 1$ whereas the field μ has $\gamma_{\mu} = -1$. Due to the symmetry the fields ε and μ have only non-vanishing even-particle form factors (more precisely, since ε is a bilinear in the fermion field it only has non-vanishing two-particle form factor) whereas σ has only odd-particle ones. The generalised TT-deformations do not break the $\mathbb{Z}_2$ symmetry so these properties are all preserved. Instead of focusing on the energy field itself, it is common to study the properties of the trace of the stress-energy tensor Θ which is a field proportional to ε . Its form factors are very simple

$$F_2^{\Theta}(\theta; \mathbf{0}) = -2\pi i \sinh \frac{\theta}{2}, \quad F_{n>2}^{\Theta}(\theta_1, \dots, \theta_n; \mathbf{0}) = 0. \quad (3.111)$$

For dimensional reasons, the form factor above must be proportional to m^2 (recall that we have set $m = 1$ throughout the paper). Consequently, in the massless limit $m \rightarrow 0$ all the form factors of Θ vanish identically, which is consistent with vanishing of the stress-energy tensor trace in the CFT limit.

For Θ the equation (3.105) becomes

$$\begin{aligned} Q_{2n+2}^{\Theta}(-x, x, x_1, \dots, x_{2n}; \boldsymbol{\alpha}) &= \\ &= 2ix^{2n+1} \sin \left(\sum_{s \in S} \frac{\alpha_s}{2} \left(\sum_{j=1}^{2n} \sinh s(\theta - \theta_j) \right) \right) \prod_{j=1}^{2n} (x_j) Q_{2n}^{\Theta}(x_1, \dots, x_{2n}; \boldsymbol{\alpha}) \end{aligned} \quad (3.112)$$

and

$$H_{2n}^{\Theta} = 4^{n(n-1)} i^{-n^2} H_0^{\Theta} \quad \text{with} \quad H_0^{\Theta} = F_0^{\Theta}. \quad (3.113)$$

The equation (3.112) only holds for $n > 0$. Indeed, the operator Θ is special in that its two-particle form factor has no kinematic pole while higher particle ones do. Hence, we expect to have the standard normalisation $F_0^{\Theta} = 2\pi = F_2^{\Theta}(i\pi; \boldsymbol{\alpha})$. The two-particle form factor should be generalisation of (3.111) with the standard form

$$F_2^{\Theta}(\theta; \boldsymbol{\alpha}) = 2\pi f(\theta; \boldsymbol{\alpha}) \frac{F_{\min}(\theta; \boldsymbol{\alpha})}{F_{\min}(i\pi; \boldsymbol{\alpha})}. \quad (3.114)$$

where the function $f(\theta; \boldsymbol{\alpha})$ is constrained by $f(i\pi; \boldsymbol{\alpha}) = f(\theta; \mathbf{0}) = 1$ and should be an even, $2\pi i$ -periodic function of θ . We will see later that consistency with higher particle

solutions requires this function to be non-trivial. The equation (3.112) determines the higher-particle form factors which, as the sine factor suggests, will all vanish for $\alpha_i = 0$, in agreement with (3.111). For the field μ we have $\gamma_\mu = -1$ and only even-particle form factors. Then the equations (3.105, 3.107) become:

$$\begin{aligned} Q_{2n+2}^\mu(-x, x, x_1, \dots, x_{2n}; \alpha) &= \\ &= 2x^{2n+1} \cos \left(\sum_{s \in \mathcal{S}} \frac{\alpha_s}{2} \left(\sum_{j=1}^{2n} \sinh s(\theta - \theta_j) \right) \right) \prod_{j=1}^{2n} (x_j) Q_{2n}^\mu(x_1, \dots, x_{2n}; \alpha) \end{aligned} \quad (3.115)$$

and

$$H_{2n}^\mu = 4^{n(n-1)} i^{-n^2} H_0^\mu \quad \text{with} \quad H_0^\mu = \langle \mu \rangle_\alpha. \quad (3.116)$$

where $\langle \mu \rangle_\alpha = F_0^\mu(\alpha)$ is the vacuum expectation value of μ , which may depend on α in the deformed theory. In the case of the field σ , we only have odd particle numbers and a trivial factor of local commutativity $\gamma_\sigma = 1$, so we also get a cosine

$$\begin{aligned} Q_{2n+1}^\sigma(-x, x, x_1, \dots, x_{2n+1}; \alpha) &= \\ &= 2x^{2n} \cos \left(\sum_{s \in \mathcal{S}} \frac{\alpha_s}{2} \left(\sum_{j=1}^{2n-1} \sinh s(\theta - \theta_j) \right) \right) \prod_{j=1}^{2n-1} (x_j) Q_{2n-1}^\sigma(x_1, \dots, x_{2n-1}; \alpha) \end{aligned}$$

and

$$H_{2n+1}^\sigma = 4^{(n-1)^2} i^{-n(n-1)} H_1^\sigma \quad \text{with} \quad H_1^\sigma = F_1^\sigma(\alpha). \quad (3.117)$$

Given the stark contrast (in behavior as functions of rapidity) between the MFF of the IFT and (3.94), we are led to consider solutions to the form factor equations that factorise into the unperturbed solutions times a function that depends on the deformation. This factorisation is particularly natural for the Ising field theory, as we will see below. We can write

$$Q_{2n}^\mu(x_1, \dots, x_{2n}; \alpha) := D_{2n}^\mu(x_1, \dots, x_{2n}; \alpha) P_{2n}^\mu(x_1, \dots, x_{2n}), \quad (3.118)$$

where

$$D_{2n+2}^\mu(-x, x, x_1, \dots, x_{2n}; \alpha) = 2 \cos \left(\sum_{s \in \mathcal{S}} \frac{\alpha_s}{2} \left(\sum_{j=1}^{2n} \sinh s(\theta - \theta_j) \right) \right) D_{2n}^\mu(x_1, \dots, x_{2n}; \alpha), \quad (3.119)$$

and

$$P_{2n+2}^\mu(-x, x, x_1, \dots, x_{2n}) = x^{2n+1} \sigma_{2n}^{(2n)} P_{2n}^\mu(x_1, \dots, x_{2n}). \quad (3.120)$$

The equations for the operator σ are identical to (3.118 – 3.120) with $2n \mapsto 2n-1$ and so are the solutions. The equation (3.120) can be solved easily, starting with the cases

$$P_2^\mu(-x, x) = x P_0^\mu \quad \text{and} \quad P_3^\sigma(-x, x, x_1) = x^2 x_1 P_1^\sigma, \quad (3.121)$$

and proceeding recursively. For instance

$$P_2^\mu(x_1, x_2) = -i(\sigma_2^{(2)})^{\frac{1}{2}} P_0^\mu, \quad (3.122)$$

and so

$$P_4^\mu(-x, x, x_1, x_2) = -i(x^2 x_1 x_2)^{\frac{3}{2}} P_0^\mu \quad \Rightarrow \quad P_4^\mu(x_1, x_2, x_3, x_4) = (\sigma_4^{(4)})^{\frac{3}{2}} P_0^\mu. \quad (3.123)$$

It is then easy to show that the general expressions are

$$\begin{aligned} P_{2n}^\mu(x_1, \dots, x_{2n}) &= i^{-n^2} (\sigma_{2n}^{(2n)})^{\frac{2n-1}{2}} P_0^\mu, \\ P_{2n+1}^\sigma(x_1, \dots, x_{2n+1}) &= (-1)^n (\sigma_{2n+1}^{(2n+1)})^n P_1^\mu. \end{aligned} \quad (3.124)$$

Then, by requiring consistency with unperturbed solutions we can fix the constants to $P_0^\mu = 1$ and $P_1^\mu = i^{-n(n-1)}$. The α -dependent part is solved by

$$D_{2n}^\mu(x_1, \dots, x_{2n}; \alpha) = 2^n \sqrt{\prod_{i=1}^{2n} \cos \left(\sum_{s \in \mathcal{S}} \frac{\alpha_s}{2} \sum_{j=1}^{2n} \sinh(s\theta_{ij}) \right)}, \quad (3.125)$$

and

$$D_{2n+1}^\sigma(x_1, \dots, x_{2n+1}; \alpha) = 2^n \sqrt{\prod_{i=1}^{2n+1} \cos \left(\sum_{s \in \mathcal{S}} \frac{\alpha_s}{2} \sum_{j=1}^{2n+1} \sinh(s\theta_{ij}) \right)}. \quad (3.126)$$

Putting everything together, the form factors of the disorder and order fields are

$$\begin{aligned} F_{2n}^\mu(\theta_1, \dots, \theta_{2n}; \alpha) &= i^n \langle \mu \rangle_\alpha \sqrt{\prod_{i=1}^{2n} \cos \left(\sum_{s \in \mathcal{S}} \frac{\alpha_s}{2} \sum_{j=1}^{2n} \sinh(s\theta_{ij}) \right)} \\ &\quad \times \prod_{i < j} \tanh \frac{\theta_{ij}}{2} \varphi(\theta_{ij}; \alpha). \end{aligned} \quad (3.127)$$

and

$$\begin{aligned} F_{2n+1}^\sigma(\theta_1, \dots, \theta_{2n+1}; \alpha) &= i^n F_1^\sigma(\alpha) \sqrt{\prod_{i=1}^{2n+1} \cos \left(\sum_{s \in \mathcal{S}} \frac{\alpha_s}{2} \sum_{j=1}^{2n+1} \sinh(s\theta_{ij}) \right)} \\ &\quad \times \prod_{i < j} \tanh \frac{\theta_{ij}}{2} \varphi(\theta_{ij}; \alpha), \end{aligned} \quad (3.128)$$

where we used the identities

$$\prod_{1 \leq i < j \leq 2n} (x_i + x_j) = \prod_{1 \leq i < j \leq 2n} \sqrt{x_i x_j} 2 \cosh \frac{\theta_{ij}}{2} = 2^{n(2n-1)} (\sigma_{2n}^{(2n)})^{\frac{2n-1}{2}} \prod_{1 \leq i < j \leq 2n} \cosh \frac{\theta_{ij}}{2}. \quad (3.129)$$

As we can see from the above expressions, the unperturbed solutions given in (3.108) and (3.109) are correctly recovered in the $\alpha \rightarrow 0$ limit. Note that this factorisation

property is model-independent. The sole exception is the field Θ in the Ising field theory. In the case of σ and μ , the deforming factor reads

$$G_n(\theta_1, \dots, \theta_n; \alpha) := \sqrt{\prod_{i=1}^n \cos \left(\sum_{s \in \mathcal{S}} \frac{\alpha_s}{2} \sum_{j=1}^n \sinh(s\theta_{ij}) \right)} \prod_{i < j} \varphi(\theta_{ij}; \alpha). \quad (3.130)$$

The equation for the form factors of the trace of the stress-energy tensor suggests a similar solution procedure as for μ and σ . However, a special feature of this field is that it is local, hence its two-particle form factor has no kinematic pole, even if higher-particle ones do. Since their kinematic pole residue is non-vanishing, according to (3.112), higher particle form factors are non-vanishing in the $\text{T}\bar{\text{T}}$ -perturbed theory. Hence, we need to solve (3.112) for $n > 1$. Comparing (3.112) with (3.115) and reviewing the solution to (3.115) given by (3.127) it is clear that the simplest, consistent solution to the higher-particle form factor equations will be of the form

$$F_{2n}^\Theta(\theta_1, \dots, \theta_{2n}; \alpha) = A_{2n}(\alpha) \sqrt{\prod_{i=1}^{2n} \sin \left(\sum_{s \in \mathcal{S}} \frac{\alpha_s}{2} \sum_{j=1}^{2n} \sinh(s\theta_{ij}) \right)} \prod_{i < j} \tanh \frac{\theta_{ij}}{2} \varphi(\theta_{ij}; \alpha), \quad n > 1 \quad (3.131)$$

where $A_{2n}(\alpha)$ is a normalisation constant to be fixed later. Taking this solution and writing the kinematic residue equation that relates the four-particle to the two-particle form factor we get that the latter is given by

$$F_2^\Theta(\theta; \alpha) = -A_4(\alpha) \left| \sin \left(\sum_{s \in \mathcal{S}} \frac{\alpha_s}{2} \sinh(s\theta) \right) \right| \tanh \frac{\theta}{2} \varphi(\theta; \alpha). \quad (3.132)$$

This function has no pole at $i\pi$, as required but, unless $A_4(\alpha)$ is chosen carefully, it will give zero when the parameters $\alpha_s \rightarrow 0$. One naive way to deal with this is to choose

$$-A_4(\alpha) = \frac{2C(\alpha)}{\alpha_{s'}}, \quad (3.133)$$

and then take the limit α_s to zero for each term on the sum over s , with $s = s'$ the spin associated with the last limit taken. This limit then gives

$$\lim_{\alpha_{s'} \rightarrow 0} \lim_{\alpha_{s \neq s'} \rightarrow 0} F_2^\Theta(\theta; \alpha) = F_2^\Theta(\theta; \mathbf{0}) = C(0) \left| \sinh \frac{s'\theta}{2} \right| \tanh \frac{\theta}{2}. \quad (3.134)$$

This result is problematic: not only it does not reproduce the correct form factor at $\alpha = 0$ but is dependent on the order in which the parameters α_s are taken to zero, a property that seems quite unnatural. We must then conclude that the ansatz (3.131) is not the correct solution to the form factor equations for $n > 1$. The question is then, how should (3.131) be modified so that it is consistent with a two-particle form factor which has all the correct properties? Given that the oscillatory part of the solution (3.131) is crucial for the solution of the higher particle form factors, it seems that we must accept the presence of an oscillatory part also in the two-particle form factor. We assume then the following form

$$F_2^\Theta(\theta; \alpha) = 2\pi f(\theta; \alpha) \frac{F_{\min}(\theta; \alpha)}{F_{\min}(i\pi; \alpha)}, \quad (3.135)$$

where $f(\theta; \alpha)$ is a function that satisfies

$$f(i\pi; \alpha) = f(\theta; \mathbf{0}) = 1, \quad (3.136)$$

as well as being symmetric and $2\pi i$ -periodic in θ . According to the discussion above, it must also include the oscillatory factor in (3.131), specialised to $n = 1$. It is not too difficult to realise that the function we need is

$$f(\theta; \alpha) = \left| \frac{\sin\left(\sum_{s \in \mathcal{S}} \frac{\alpha_s}{2} \sinh(s\theta)\right)}{\sum_{s \in \mathcal{S}} \frac{\alpha_s}{2} \sinh(s\theta)} \right| \cosh \frac{\theta}{2}, \quad (3.137)$$

which gives the two-particle form factor

$$F_2^\Theta(\theta; \alpha) = -2\pi i \left| \frac{\sin\left(\sum_{s \in \mathcal{S}} \frac{\alpha_s}{2} \sinh(s\theta)\right)}{\sum_{s \in \mathcal{S}} \frac{\alpha_s}{2} \sinh(s\theta)} \right| \sinh \frac{\theta}{2} \varphi(\theta; \alpha). \quad (3.138)$$

Comparing (3.138) to (3.132) we see that we need to modify the ansatz (3.131) so as to produce the function (3.138). In order to do this, while still solving the higher-particle form factor equations we need to find a function $f_{2n}(x_1, \dots, x_{2n}; \alpha)$ such that we can multiply (3.131) by it, while still solving all equations. In other words, we need this function to satisfy

$$f_{2n+2}(-x, x, x_1, \dots, x_{2n}; \alpha) = f_{2n}(x_1, \dots, x_{2n}; \alpha), \quad (3.139)$$

and

$$f_2(x_1, x_2; \alpha) = \frac{\cosh \frac{\theta_{12}}{2}}{\left| \sum_{s \in \mathcal{S}} \frac{\alpha_s}{2} \sinh(s\theta_{12}) \right|}. \quad (3.140)$$

A crucial observation to proceed is that the ratio of elementary symmetric polynomials

$$\frac{\sigma_1^{(2n)} \sigma_{2n-1}^{(2n)}}{\sigma_{2n}^{(2n)}}, \quad (3.141)$$

solves (3.139). This is because under the reduction

$$(x_1, x_2, \dots, x_{2n+2}) \mapsto (-x, x, x_1, \dots, x_{2n})$$

the elementary symmetric polynomials behave as follows:

$$\sigma_{2n+1}^{(2n+2)} \mapsto -x^2 \sigma_{2n-1}^{(2n)}, \quad \sigma_{2n+2}^{(2n+2)} \mapsto -x^2 \sigma_{2n}^{(2n)}, \text{ and } \sigma_1^{(2n+2)} \mapsto \sigma_1^{(2n)}. \quad (3.142)$$

In fact, any function of the ratio (3.140) will solve (3.139). Additionally, this facts will still hold true if the elementary symmetric polynomials are taken to be functions of the powers x_i^s rather than just $x_i = e^{\theta_i}$. Let us then be slightly more general and denote as $\sigma_{i,s}^{(2n)}$ the i th elementary symmetric polynomial of the $2n$ variables $\{e^{s\theta_1}, \dots, e^{s\theta_{2n}}\}$. It is a matter of simple algebra to verify that for any odd integer values of s , the function

$$g_{2n}(x_1, \dots, x_{2n}; s) = \frac{\sigma_{1,s}^{(2n)} \sigma_{2n-1,s}^{(2n)}}{\sigma_{2n,s}^{(2n)}}, \quad (3.143)$$

solves (3.139). We now proceed to write (3.140) in terms of these factors. For $n = 1$ we have that

$$g_2(x_1, x_2; s) = \frac{(\sigma_{1,s}^{(2)})^2}{\sigma_{2,s}^{(2)}} = \frac{(e^{s\theta_1} + e^{s\theta_2})^2}{e^{s\theta_1 + s\theta_2}} = 4 \cosh^2 \frac{s\theta_{12}}{2}. \quad (3.144)$$

so that

$$4 \sinh^2 \frac{s\theta_{12}}{2} = g_2(x_1, x_2; s) - 4, \quad (3.145)$$

and

$$\sqrt{g_2(x_1, x_2; s)(g_2(x_1, x_2; s) - 4)} = 2|\sinh(s\theta_{12})|. \quad (3.146)$$

This is precisely the sort of factor we need in our two-particle form factor. In fact, in order to have a consistent set of form factors, we need to multiply our solution (3.131) by the function

$$f_{2n}(x_1, \dots, x_{2n}; \alpha) = \frac{\sqrt{\frac{\sigma_1^{(2n)} \sigma_{2n-1}^{(2n)}}{\sigma_{2n}^{(2n)}}}}{\left| \sum_{s \in S} \frac{\alpha_s}{4} \sqrt{\left(\frac{\sigma_{1,s}^{(2n)} \sigma_{2n-1,s}^{(2n)}}{\sigma_{2n,s}^{(2n)}} - 4 \right) \frac{\sigma_{1,s}^{(2n)} \sigma_{2n-1,s}^{(2n)}}{\sigma_{2n,s}^{(2n)}}} \right|}, \quad (3.147)$$

where $\sigma_i^{(2n)} \equiv \sigma_{i,1}^{(2n)}$ and we set $f_0(\alpha) = 1$. For $n = 1$ this function correctly reproduces (3.140) and the two-particle form factor reduces to (3.138) with an appropriate choice of normalisation. Note however that, strictly-speaking, there is a subtlety here. The solution (3.147) only reproduces (3.140) if all the parameters α_s have the same sign when non-vanishing. This includes of course the case when only one parameter is non-vanishing, such as the $T\bar{T}$ -perturbation, which is the case most commonly studied. The reason for this restriction is that for $n = 1$ the square root in the denominator of (3.147) gives (3.146) and so the denominator of (3.147) reduces to the absolute value of a sum of absolute values of hyperbolic sine functions. This equals the denominator of (3.140), that is the absolute value of the sum only when all α_s are either non-positive or non-negative.

Let us then summarise what we found. All the form factors of the field Θ , with the standard normalisation $F_0 = F_2^\Theta(i\pi) = 2\pi$, are given by

$$\begin{aligned} F_{2n}^\Theta(\theta_1, \dots, \theta_{2n}; \alpha) &= 2\pi i i^n \frac{\sqrt{\prod_{i=1}^{2n} \sin \left(\sum_{s \in S} \frac{\alpha_s}{2} \sum_{j=1}^{2n} \sinh(s\theta_{ij}) \right)}}{\left| \sum_{s \in S} \frac{\alpha_s}{4} \sqrt{\left(\frac{\sigma_{1,s}^{(2n)} \sigma_{2n-1,s}^{(2n)}}{\sigma_{2n,s}^{(2n)}} - 4 \right) \frac{\sigma_{2n}^{(2n)} \sigma_{1,s}^{(2n)} \sigma_{2n-1,s}^{(2n)}}{\sigma_{2n,s}^{(2n)} \sigma_1^{(2n)} \sigma_{2n-1}^{(2n)}}} \right|} \\ &\times \prod_{i < j} \tanh \frac{\theta_{ij}}{2} \varphi(\theta_{ij}; \alpha). \end{aligned} \quad (3.148)$$

Note that, although the final formula contains α -dependence both in the numerator and denominator, it is easy to see $F_{2n}^\Theta(\theta_1, \dots, \theta_{2n}; \mathbf{0}) = 0$ for $n > 1$.

A First Look at the Asymptotics

Having found some solutions we can now try to study some of their features. Consider a generic field $\mathcal{O}$. We are going to employ the form factors found in the previous section to write the cumulant expansion of the two-point function. For fields with non-vanishing VEV we can write

$$\log \frac{\langle \mathcal{O}(0)\mathcal{O}(r) \rangle_{\alpha}}{\langle \mathcal{O} \rangle_{\alpha}^2} = \sum_{\ell=1}^{\infty} c_{\ell}^{\mathcal{O}}(r; \alpha), \quad (3.149)$$

where $c_{\ell}^{\mathcal{O}}(r; \alpha)$ are the cumulants,

$$\begin{aligned} c_{\ell}^{\mathcal{O}}(r; \alpha) &:= \frac{1}{\ell!} \int_{-\infty}^{\infty} \frac{d\theta_1}{2\pi} \cdots \int_{-\infty}^{\infty} \frac{d\theta_{\ell}}{2\pi} e^{-r \sum_{i=1}^{\ell} \cosh \theta_i} h_{\ell}^{\mathcal{O}}(\theta_1, \dots, \theta_{\ell}; \alpha) \\ &= \frac{1}{\pi \ell!} \int_{-\infty}^{\infty} \frac{d\theta_1}{2\pi} \cdots \int_{-\infty}^{\infty} \frac{d\theta_{\ell-1}}{2\pi} K_0(r d_{\ell}) h_{\ell}^{\mathcal{O}}(\theta_1, \dots, \theta_{\ell-1}, 0; \alpha). \end{aligned} \quad (3.150)$$

Here we introduced the notation

$$d_{\ell} = \sqrt{\left(\sum_{j=1}^{\ell-1} \cosh \theta_j + 1 \right)^2 - \left(\sum_{j=1}^{\ell-1} \sinh \theta_j \right)^2}, \quad (3.151)$$

and $K_0(x)$ is the modified Bessel function. The second line of (3.150) follows from integration of one rapidity variable, since the functions $h_{\ell}(\{\theta_i\}; \alpha)$ depend on rapidity differences only. This expression for the cumulants can be found in many places, for instance [204]. In the Ising model, due to $\mathbb{Z}_2$ symmetry, we usually have either even- or odd-particle cumulants only. For instance for the field μ

$$\langle \mu \rangle_{\alpha}^2 h_2^{\mu}(\theta_1, \theta_2; \alpha) := |F_2^{\mu}(\theta_1, \theta_2; \alpha)|^2, \quad (3.152)$$

$$\begin{aligned} \langle \mu \rangle_{\alpha}^4 h_4^{\mu}(\theta_1, \theta_2, \theta_3, \theta_4; \alpha) &:= \langle \mu \rangle_{\alpha}^2 |F_4^{\mu}(\theta_1, \theta_2, \theta_3, \theta_4; \alpha)|^2 - |F_2^{\mu}(\theta_1, \theta_2; \alpha)|^2 |F_2^{\mu}(\theta_3, \theta_4; \alpha)|^2 \\ &\quad - |F_2^{\mu}(\theta_1, \theta_3; \alpha)|^2 |F_2^{\mu}(\theta_2, \theta_4; \alpha)|^2 \\ &\quad - |F_2^{\mu}(\theta_1, \theta_4; \alpha)|^2 |F_2^{\mu}(\theta_2, \theta_3; \alpha)|^2, \end{aligned} \quad (3.153)$$

whereas for Θ we just need to replace $\mu \mapsto \Theta$ and $\langle \mu \rangle_{\alpha} \mapsto 2\pi$. For the field σ on the other hand, only the odd cumulants are not vanishing, the first two of which are

$$h_1^{\sigma}(\theta_1; \alpha) := 1, \quad (3.154)$$

$$|F_1^{\sigma}(\alpha)|^6 h_3^{\sigma}(\theta_1, \theta_2, \theta_3; \alpha) := |F_1^{\sigma}(\alpha)|^4 |F_3^{\sigma}(\theta_1, \theta_2, \theta_3; \alpha)|^2 - |F_1^{\sigma}(\alpha)|^6. \quad (3.155)$$

Note that, since the 1-point function of σ vanishes, the normalisation in the cumulant expansion (3.149) should be replaced by the norm squared of the one-particle form factor

$$\log \frac{\langle \sigma(0)\sigma(r) \rangle_{\alpha}}{|F_1^{\sigma}(\alpha)|^2} = \sum_{\ell=1}^{\infty} c_{\ell}^{\sigma}(r; \alpha). \quad (3.156)$$

Under a generalised $\text{T}\bar{\text{T}}$ -perturbation, a free theory will cease to be such, in the sense that many of the nice properties free theories enjoy – such as a trivial S -matrix – are lost to the deformation. In particular, the cumulant expansion does not simplify in any

obvious way, a fact which is well-known to happen for the Ising fields in the usual free model [182] and on its replica version, studied in the context of entanglement measures (see i.e. [205, 206]). These simplifications are generally due to the Pfaffian structure of the form factors that follows from Wick's theorem in free theories. Such properties no longer hold in generalised $T\bar{T}$ -perturbations. Instead, as far as we can see, with our form factor solutions, there is no simple closed general expression for the cumulants. It is however interesting to write down at least the lower ones. As we shall see, they exhibit interesting properties, some of which will extend to the whole cumulant expansion. Let us look at the lower-particle contributions to the expansion for the fields Θ and μ . For the field μ we have

$$c_2^\mu(r; \alpha) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\theta_1}{2\pi} \int_{-\infty}^{\infty} \frac{d\theta_2}{2\pi} e^{-2r \cosh \frac{\theta_{12}}{2} \cosh \frac{\hat{\theta}_{12}}{2}} \cos^2 \left(\sum_{s \in \mathcal{S}} \frac{\alpha_s}{2} \sinh(s\theta_{12}) \right) \times \prod_{s \in \mathcal{S}} e^{\frac{\alpha_s \theta_{12}}{\pi} \sinh(s\theta_{12})} \tanh^2 \frac{\theta_{12}}{2}. \quad (3.157)$$

As usual, one integral can be carried out by introducing new variables $x = \theta_{12}$ and $y = \hat{\theta}_{12} := \theta_1 + \theta_2$, so the cumulant becomes

$$c_2^\mu(r; \alpha) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dx K_0(2r \cosh \frac{x}{2}) \cos^2 \left(\sum_{s \in \mathcal{S}} \frac{\alpha_s}{2} \sinh(sx) \right) \prod_{s \in \mathcal{S}} e^{\frac{\alpha_s x}{\pi} \sinh(sx)} \tanh^2 \frac{x}{2}. \quad (3.158)$$

A very similar behaviour is found in the first cumulant contribution to the two-point function of Θ . Following the same steps, we arrive at

$$c_2^\Theta(r; \alpha) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dx K_0(2r \cosh \frac{x}{2}) \left[\frac{\sin \left(\sum_{s \in \mathcal{S}} \frac{\alpha_s}{2} \sinh(sx) \right)}{\sum_{s \in \mathcal{S}} \frac{\alpha_s}{2} \sinh(sx)} \right]^2 \prod_{s \in \mathcal{S}} e^{\frac{\alpha_s x}{\pi} \sinh(sx)} \sinh^2 \frac{x}{2}. \quad (3.159)$$

Regarding the convergence of the integrals (3.158) and (3.159) it is easy to see that this is determined by the exponential factor and, in particular, by the sign of the parameter α_{s^*} where s^* is the largest spin which is present in the product of exponentials. If $\alpha_{s^*} > 0$ the integral is divergent, whereas the opposite is true for $\alpha_{s^*} < 0$. Notice that this fact echoes what was mentioned in the introduction: the TBA for generalised $T\bar{T}$ deformations with more than 1 coupling are only well defined if $\alpha_{s^*} < 0$. We must mention however, that the asymptotics of these expressions and higher terms in the form factor expansion is much less trivial when the set of spins s involved in the products above is infinite, as it will now depend on the analytic properties of the resummation of the product of exponentials. For the purposes of this paper and in order to avoid such subtleties, we will always assume that we are considering a finite set of non-vanishing couplings, even if the total set $\mathcal{S}$ is infinite.

Let us now focus on the standard $T\bar{T}$ perturbation corresponding to having a single non-vanishing parameter $\alpha_1 := \alpha$. According to the picture put forward in [146] the $\alpha > 0$ regime corresponds to a theory whose UV physics is characterised by objects of finite length. It is then not surprising that short distances cannot be probed and that in this limit the two-point function should diverge. At the TBA level [106] this finite length leads to a finite ground state energy.

In fact, for $\alpha > 0$, even the IR limit can be problematic, if we also probe high momenta. In this case we will again encounter this finite length ($l = \sqrt{\alpha}$), leading to a divergent form factor expansion. In other words, there is no local QFT describing the deep UV limit of the theory and if we try to probe this theory by approaching either small values of r for any momenta, or large values of r and high momenta we always find a divergence. For large momenta, we can even estimate a characteristic scale or cut-off – a function of r/α – at which this divergence dominates as we now show.

If we take the two-particle cumulant of μ and expand the Bessel function at leading order for large r we find

$$c_2^\mu(r \gg 1; \alpha > 0) \approx \frac{1}{4\pi^2} \sqrt{\frac{\pi}{2r}} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{\cosh \frac{x}{2}}} \cos^2 \left(\frac{\alpha}{2} \sinh x \right) e^{-2r \cosh \frac{x}{2} + \frac{\alpha x}{\pi} \sinh x} \tanh^2 \frac{x}{2}. \quad (3.160)$$

Convergence of the integral is determined by the balance between the two terms in the exponent. While for $\alpha < 0$ the exponent is negative and the exponential rapidly decreasing, for $\alpha > 0$ the integral is convergent for as long as

$$2r \cosh \frac{x}{2} > \frac{\alpha x}{\pi} \sinh x. \quad (3.161)$$

In other words, there exist a maximum allowed rapidity x , solution of the equation

$$\frac{\pi r}{\alpha} = x \sinh \frac{x}{2}. \quad (3.162)$$

For $x \gg 0$ we can approximate the sinh by an exponential, arriving at

$$\frac{\pi r}{\alpha} = \frac{x}{2} e^{\frac{x}{2}}. \quad (3.163)$$

The solutions to the equation above are known in the literature as Lambert W -function $W_k(z)$ for $k \in \mathbb{Z}$. This is a multi-valued function, hence the index k . For real values of x and $\frac{\pi r}{\alpha} > 0$ the correct branch of the solution is the principal one $x = 2W_0(\pi r/\alpha)$. This scales as

$$W_0(a) \approx a \quad \text{for } a \ll 1, \quad \text{and} \quad W_0(a) \approx \log a \quad \text{for } a \gg 1. \quad (3.164)$$

Quite naturally, the cut-off depends on the ratio of the two independent scales in the system r/α . If r is large compared to α this cut-off will be very large as well since the finite length of the fundamental excitation is not seen, even for large momenta. However, if α is large compared to r the particles' finite length is probed for momenta that are not necessarily large and the cumulant expansion can only be made convergent by the introduction of the rapidity cut-off

$$\Lambda = 2W_0(\pi r/\alpha). \quad (3.165)$$

The fact that Lambert's W -function plays a role is rather suggestive given that it features in a variety of problems, many of which are intuitively connected to the physics of $\text{T}\overline{\text{T}}$ perturbations (see [207] for a review). For instance, a key application is to delay problems which are similar in spirit to the “delay” induced by the phase-shift $\Phi_\alpha(\theta)$ and by the finite length of the rods, that best describe the effect of the $\text{T}\overline{\text{T}}$ perturbation in the GHD context. The Lambert function also features directly in previous discussions of $\text{T}\overline{\text{T}}$ perturbations, such as in [146] where it enters the formula for the free energy in a hard-rod model, in the solutions of JT gravity [135, 208] and in [98] as the explicit inversion

of the classical flow equation for the deformed Lagrangian. Although the characteristic momentum cut-off (3.165) is obtained from the two-particle contribution, the qualitative picture presented here should extend to higher-particle terms. It is indeed possible to write a simple argument showing this. Consider the definition (3.150) in the large r limit. Expanding the Bessel function, for large r , all cumulants will include an r -dependent exponential of the form e^{-rd_ℓ} with d_ℓ defined in (3.151). Suppose that one of the rapidities is very large, say $\theta_1 \gg 1$ and $\theta_1 \gg \theta_{i \neq 1}$. Then, $\cosh \theta_1, \sinh \theta_1$ are also much larger than $\cosh \theta_{i \neq 1}$ and $\sinh \theta_{i \neq 1}$. By neglecting these terms in the definition of d_ℓ , we have the approximation

$$e^{-rd_\ell} \approx e^{-2r \cosh \frac{\theta_1}{2}} \quad \text{for } \theta_1 \gg 1, \theta_1 \gg \theta_{i \neq 1}. \quad (3.166)$$

The cumulant will also contain the factor

$$\prod_{1 \leq i \leq j \leq \ell-1} e^{\frac{\alpha \theta_{ij}}{\pi} \sinh \theta_{ij}} \prod_{i=1}^{\ell-1} e^{\frac{\alpha \theta_i}{\pi} \sinh \theta_i} \approx e^{-\frac{\ell \alpha \theta_1}{\pi} \sinh \theta_1}, \quad (3.167)$$

where the approximation is made under the same assumptions as above. Then the two exponentials can be compared just as for the two-particle cumulant and a similar cut-off Λ – with α replaced by $\ell \alpha$ – can be found.

Following the analysis of [146], for $\alpha < 0$ we are now dealing with a theory of particles having a “negative” length. From the viewpoint of correlation functions, both the UV and IR regimes can be probed without any issues at all. In fact, the form factor series is more rapidly convergent for $\alpha < 0$ than for $\alpha = 0$ since the minimal form factor typically decays much faster for large rapidities than any other function in the integrand. We have that, for all cumulants in (3.150)

$$\lim_{\theta_i \rightarrow \infty} h_\ell^\mathcal{O}(\theta_1, \dots, \theta_\ell; \alpha) = 0 \quad \forall i \text{ \& } \alpha < 0. \quad (3.168)$$

For many theories, the statement above also holds at $\alpha = 0$ but the Ising model is an exception to this rule since the cumulants of μ, σ and Θ all either diverge exponentially (Θ) or tend to a constant (μ and σ) for large rapidities.

Now that (3.168) holds, we can safely expand the Bessel functions $K_0(2rd_\ell)$ in (3.150) for small r in order to investigate the UV scaling of correlators,

$$K_0(rd_\ell) = -\log r - \gamma + \log 2 - \log d_\ell. \quad (3.169)$$

We find that the leading contributions to the cumulant expansion at short distances are all proportional to a factor $\log r$. This implies a power-law scaling of the correlators, just as one would expect if the UV limit was a CFT. This is an interesting though unexpected result, given that we know a $\text{T}\bar{\text{T}}$ -deformed theory is not supposed to have a UV fixed point in the standard sense. These results suggest that for large energies, the $\text{T}\bar{\text{T}}$ deformation does have properties that are reminiscent of critical systems. In other words, two-point functions will still scale as power-laws $r^{-4z^\mathcal{O}(\alpha)}$ at short distances, with exponent $z^\mathcal{O}(\alpha) < \Delta^\mathcal{O}$ – for $\alpha < 0$ – where $\Delta^\mathcal{O}$ is the UV dimension of the field in the unperturbed theory. We also see that $z^\mathcal{O}(\alpha) \rightarrow 0$ for $-\alpha \rightarrow \infty$. The IR limit is straightforwardly analysed as well, with two-point functions decaying exponentially at large distances, faster than in the undeformed theory. In the deep IR the theory will still be trivial, as dictated by the presence of a mass scale.

For the two-particle cumulants, the leading UV ($r \ll 1$) contributions are

$$c_2^\mu(r \ll 1; \alpha < 0) \approx -\frac{\log r}{4\pi^2} \int_{-\infty}^{\infty} dx \cos^2 \left(\frac{\alpha}{2} \sinh x \right) e^{\frac{\alpha x}{\pi} \sinh x} \tanh^2 \frac{x}{2} := -4z_2^\mu(\alpha) \log r, \quad (3.170)$$

and similarly

$$c_2^\varepsilon(r \ll 1; \alpha < 0) \approx -\frac{\log r}{4\pi^2} \int_{-\infty}^{\infty} dx \left[\frac{\sin(\frac{\alpha}{2} \sinh x)}{\alpha \cosh \frac{x}{2}} \right]^2 e^{\frac{\alpha x}{\pi} \sinh x} := -4z_2^\varepsilon(\alpha) \log r. \quad (3.171)$$

The cumulants of the energy field ε which is related to the Θ by $\Theta = 2\pi m\varepsilon$ are identical to those of Θ when $m = 1$, however, unlike Θ , ε is a primary field in CFT and should scale as $1/r^2$ in the short-distance limit for $\alpha = 0$ (whereas Θ vanishes in the CFT limit). It is thus more interesting to consider the short-distance scaling of ε than that of Θ .

Due to the exponential terms, the integrals in (3.170) and (3.171) cannot be computed analytically but they can be evaluated numerically and the results are shown in Fig. 3.11. These indicate that the short-distance scaling of the two-point function will now be dependent on α . Note that it is not possible to extrapolate these contributions to $\alpha = 0$ because the integral is not convergent in this case (see [182]). This sharp change in the behaviour of $z_2^{\mu,\varepsilon}(\alpha)$ is a special feature of the Ising model.

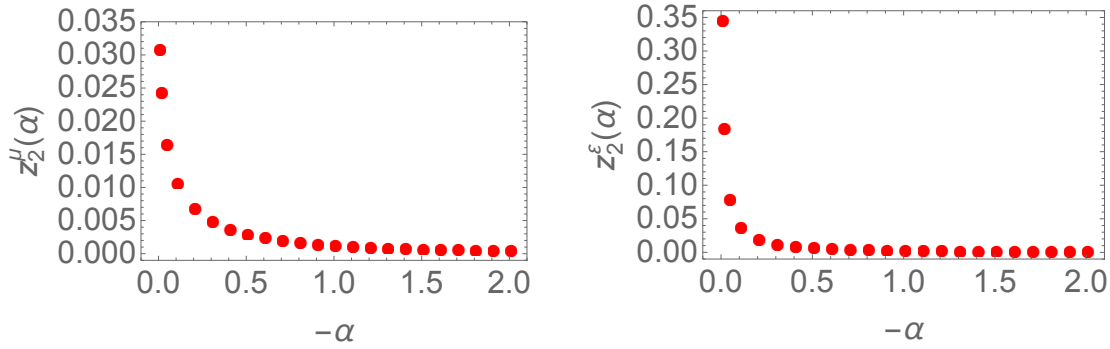


Figure 3.11: The functions $z_2^\mu(\alpha)$ and $z_2^\varepsilon(\alpha)$. In the unperturbed Ising model we have that the two-point functions scale exactly as $r^{-\frac{1}{4}}$ and r^{-2} for μ and ε respectively so that $z_2^\mu(0) = \frac{1}{16}$ and $z_2^\varepsilon(0) = \frac{1}{2}$. From the two-particle contributions, it seems that there is still power-law scaling at short distances but the decay becomes faster the larger is $-\alpha$. Note that the definitions (3.170) and (3.171) of $z_2^\mu(\alpha)$ and $z_2^\varepsilon(\alpha)$ are not valid at $\alpha = 0$: there is a sharp change at $\alpha = 0$ which is a peculiar feature of the Ising model.

Standard Consistency Checks

We can also try to apply the two consistency checks : the Δ -sum rule (0.74) and Zamolodchikov's c -theorem (0.73). Once more we limit our consideration to the simplest case of the $\text{T}\bar{\text{T}}$ deformation. Consider the field μ . After integrating in r and in one of the rapidities, from (0.74) and (3.3) we obtain the exact formula

$$F_\mu(\alpha) := \Delta_\mu^{UV} - \Delta_\mu^{IR} = \frac{1}{16\pi} \int_{-\infty}^{+\infty} dx \left| \frac{\sin(\alpha \sinh x)}{\alpha \sinh x} \right| \frac{\tanh^2 \frac{x}{2}}{\cosh \frac{x}{2}} e^{\frac{\alpha x}{\pi} \sinh x}. \quad (3.172)$$

For $\alpha = 0$ the integral can be computed exactly and we obtain the expected CFT value $\Delta_\mu = \frac{1}{16}$ while for $\alpha < 0$ we can only integrate numerically. We find that $F(\alpha)$ tends to zero for $-\alpha \rightarrow \infty$. The decay is extremely fast and can be best seen when plotting against $\log(-\alpha)$, as in Fig. 3.12 (left). The behaviours at small and large $-\alpha$ are exactly what we

would expect from the general picture developed in the analysis of the cumulants, thus the results of the Δ -sum rule are compatible with the general picture of power-law scaling at short distances.

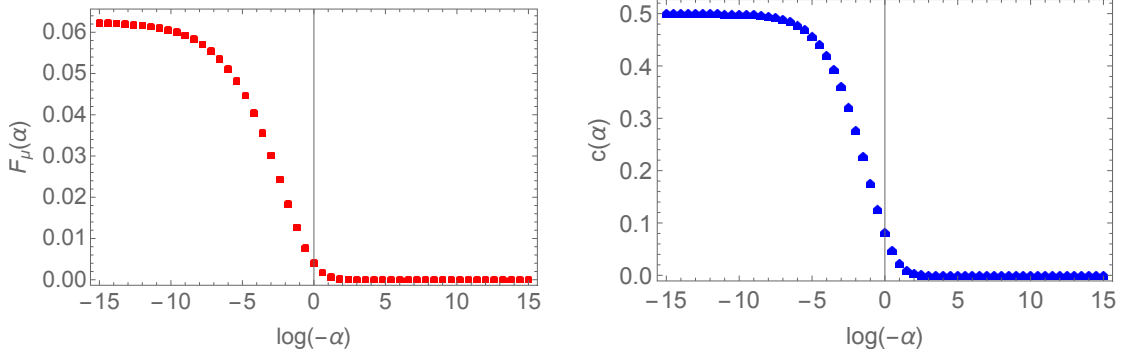


Figure 3.12: The functions $F(\alpha)$ and $c(\alpha)$. They both tend to zero for large $-\alpha$ and to their exact values in the unperturbed theory for $\alpha = 0$: $F(0) = \frac{1}{16} = 0.0625$ and $c(\alpha) = \frac{1}{2} = 0.5$

In the same way we can study Zamolodchikov's c -function. From (0.73) and (3.3), integrating once again in r and in one rapidity, we obtain the formula

$$c(\alpha) := c^{UV} - c^{IR} \approx \frac{3}{8} \int_{-\infty}^{+\infty} dx \left[\frac{\sin(\frac{\alpha}{2} \sinh x)}{\frac{\alpha}{2} \sinh x} \right]^2 \frac{\tanh^2 \frac{x}{2}}{\cosh^2 \frac{x}{2}} e^{\frac{\alpha}{\pi} x \sinh x}, \quad (3.173)$$

in the two-particle approximation. For $\alpha = 0$ we obtain the usual value $c = 1/2$ while $c(\alpha) \rightarrow 0$ for $\alpha \rightarrow -\infty$. The decay to zero as $-\alpha$ is increased is very fast and best seen in logarithmic scale, as shown in Fig. 3.12 (right).

3.2.2 The Factorisation Property

The fact that we can consider factorised solutions, like we did in (3.118) in the case of the $\text{T}\bar{\text{T}}$ -deformed Ising model, it is actually more general [13]. For any IQFT with diagonal scattering matrix and a single-particle spectrum, we can always find solutions with the general structure

$$F_n^\mathcal{O}(\theta_1, \dots, \theta_n; \alpha) = F_n^\mathcal{O}(\theta_1, \dots, \theta_n; \mathbf{0}) \Upsilon_n^\mathcal{O}(\theta_1, \dots, \theta_n; \alpha), \quad (3.174)$$

with $\Upsilon_n^\mathcal{O}(\theta_1, \dots, \theta_n; \alpha)$ a non-trivial function of α and $\Upsilon_n^\mathcal{O}(\theta_1, \dots, \theta_n; \mathbf{0}) = 1$. Here, as we did for the Ising field theory, we will restrict ourselves to the case $\beta = 0$.

The function $\Upsilon_n^\mathcal{O}(\theta_1, \dots, \theta_n; \alpha)$ can be computed by plugging (3.174) into the standard ansatz (3.99). This gives rise to the following equations

$$H_n^\alpha = H_n^0 \mathcal{H}_n^\alpha, \quad F_{\min}(\theta; \alpha) = F_{\min}(\theta; \mathbf{0}) \varphi(\theta; \alpha), \quad (3.175)$$

$$Q_n^\mathcal{O}(\theta_1, \dots, \theta_n; \alpha) = Q_n^\mathcal{O}(\theta_1, \dots, \theta_n; \mathbf{0}) \Theta_n^\mathcal{O}(\theta_1, \dots, \theta_n; \alpha), \quad (3.176)$$

or equivalently

$$\Upsilon_n^\mathcal{O}(\theta_1, \dots, \theta_n; \alpha) = \mathcal{H}_n^\alpha \Theta_n^\mathcal{O}(\theta_1, \dots, \theta_n; \alpha) \prod_{i < j}^n \varphi(\theta_i - \theta_j; \alpha) . \quad (3.177)$$

The quantities $H_n^\mathbf{0}$, $Q_n^\mathcal{O}(\theta_1, \dots, \theta_n; \mathbf{0})$ and $F_{\min}(\theta; \mathbf{0})$ are supposed to satisfy the axioms for the undeformed theory, which means

$$\mathcal{H}_n^\mathbf{0} = \Theta_n^\mathcal{O}(\theta_1, \dots, \theta_n; \mathbf{0}) = \varphi(\theta; \mathbf{0}) = 1 . \quad (3.178)$$

Imposing the form factor equations for the deformed theory we recover the equations

$$\varphi(\theta; \alpha) = \Phi_\alpha(\theta) \varphi(-\theta; \alpha) = \varphi(2\pi i - \theta; \alpha) , \quad (3.179)$$

which have already been solved to (3.94), and¹²

$$\lim_{\bar{\theta} \rightarrow \theta} \Upsilon_{n+2}^\mathcal{O}(\bar{\theta} + \pi i, \theta, \theta_1, \dots, \theta_n; \alpha) = \frac{1 - \gamma_\mathcal{O} \prod_{j=1}^n S_\alpha(\theta - \theta_j)}{1 - \gamma_\mathcal{O} \prod_{j=1}^n S_\mathbf{0}(\theta - \theta_j)} \Upsilon_n^\mathcal{O}(\theta_1, \dots, \theta_n; \alpha) . \quad (3.180)$$

This equation in turn splits into

$$\mathcal{H}_{n+2}^\alpha = \mathcal{H}_n^\alpha , \quad (3.181)$$

$$\Theta_{n+2}^\mathcal{O}(\theta + i\pi, \theta, \theta_1, \dots, \theta_n; \alpha) = \mathbb{S}_n(\theta - \theta_1, \dots, \theta - \theta_n; \gamma_\mathcal{O}) \Theta_n^\mathcal{O}(\theta_1, \dots, \theta_n; \alpha) , \quad (3.182)$$

where we introduced the quantity

$$\mathbb{S}_n(\theta_1, \dots, \theta_n; \gamma_\mathcal{O}) = \frac{\prod_{j=1}^n S_\alpha(\theta_j)^{-1/2} - \gamma_\mathcal{O} \prod_{j=1}^n S_\alpha(\theta_j)^{1/2}}{\prod_{j=1}^n S_\mathbf{0}(\theta_j)^{-1/2} - \gamma_\mathcal{O} \prod_{j=1}^n S_\mathbf{0}(\theta_j)^{1/2}} . \quad (3.183)$$

Notice that as a consequence of the unitarity $S_\alpha(-\theta) = S_\alpha(\theta)^{-1}$ and crossing symmetry $S_\alpha(i\pi - \theta) = S_\alpha(\theta)$ of the S -matrix, the function (3.183) satisfies the following properties

$$\mathbb{S}_n(\theta_1 \pm i\pi, \dots, \theta_n \pm i\pi; \gamma_\mathcal{O}) = \mathbb{S}_n(\theta_1, \dots, \theta_n; \gamma_\mathcal{O}^{-1}) , \quad (3.184)$$

$$\mathbb{S}_{n+2}(\beta, \beta \pm i\pi, \theta_1, \dots, \theta_n; \gamma_\mathcal{O}) = \mathbb{S}_n(\theta_1, \dots, \theta_n; \gamma_\mathcal{O}) , \quad (3.185)$$

which will be useful momentarily. The recursion (3.181) for the constant is trivially solved

$$\mathcal{H}_n^\alpha = 1 , \quad (3.186)$$

while the equation (3.182) for Θ requires some more effort, especially for a general undeformed S -matrix. Let us give here the solution

$$\Theta_n^\mathcal{O}(\theta_1, \dots, \theta_n; \alpha) = \sqrt{\prod_{i=1}^n \mathbb{S}_n\left(\theta_i - \theta_1, \dots, \theta_i - \theta_n; \gamma_\mathcal{O}^{(-1)^i}\right)} , \quad (3.187)$$

¹²Note that if $\gamma_\mathcal{O} = 1$ and $S(\theta) = 1$, the equation for Υ would involve division by zero. Extra care is needed in those cases.

and check that it indeed solves equation (3.182). The left-hand-side is given by the product of the following three expressions

$$\sqrt{\mathbb{S}_{n+2}(0, i\pi, \theta + i\pi - \theta_1, \dots, \theta + i\pi - \theta_n; \gamma_{\mathcal{O}}^{-1})} = \sqrt{\mathbb{S}_n(\theta - \theta_1, \dots, \theta - \theta_n; \gamma_{\mathcal{O}})}, \quad (3.188)$$

$$\sqrt{\mathbb{S}_{n+2}(-i\pi, 0, \theta - \theta_1, \dots, \theta - \theta_n; \gamma_{\mathcal{O}})} = \sqrt{\mathbb{S}_n(\theta - \theta_1, \dots, \theta - \theta_n; \gamma_{\mathcal{O}})}, \quad (3.189)$$

and

$$\sqrt{\prod_{i=1}^n \mathbb{S}_{n+2}(\theta_i - \theta - i\pi, \theta_i - \theta, \theta_i - \theta_1, \dots, \theta_i - \theta_n; \gamma_{\mathcal{O}}^{(-1)^i})} = \Theta_n^{\mathcal{O}}(\theta_1, \dots, \theta_n; \alpha), \quad (3.190)$$

where we used the properties (3.184) and (3.185). Taking the product of these three terms indeed yields the right-hand-side of (3.182) as long as $\gamma_{\mathcal{O}} = \gamma_{\mathcal{O}}^{-1}$ or $\gamma_{\mathcal{O}} = \pm 1$. More explicitly, we can write the final expression

$$\begin{aligned} \Theta_n^{\mathcal{O}}(\{\theta_i\}_n; \alpha) &= \prod_{i=1}^n \sqrt{\frac{\prod_{j=1}^n S_{\alpha}(\theta_{ij})^{1/2} - \gamma_{\mathcal{O}} \prod_{j=1}^n S_{\alpha}(\theta_{ij})^{-1/2}}{\prod_{j=1}^n S_0(\theta_{ij})^{1/2} - \gamma_{\mathcal{O}} \prod_{j=1}^n S_0(\theta_{ij})^{-1/2}}} \\ &= \prod_{i=1}^n \sqrt{\frac{\sin\left(\frac{1}{2} \sum_{j=1}^n [\delta(\theta_{ij}) - i \log \Phi_{\alpha}(\theta_{ij})] - \frac{\omega_{\mathcal{O}}}{2}\right)}{\sin\left(\frac{1}{2} \sum_{j=1}^n \delta(\theta_{ij}) - \frac{\omega_{\mathcal{O}}}{2}\right)}}, \end{aligned} \quad (3.191)$$

where $S_0(\theta) = S_0(-\theta)^{-1} = e^{i\delta(\theta)}$ and $\gamma_{\mathcal{O}} = e^{i\omega_{\mathcal{O}}}$. We also assumed that the S -matrix is fermionic in the sense that $S_0(0) = S_0(i\pi) = -1$, so $\delta(0) = \delta(i\pi) = \pi$. As noted above, the solution (3.191) also requires $\gamma_{\mathcal{O}} = \pm 1$. Thus, strictly speaking, it is valid for local and semi-local fields. A hitherto unresolved subtlety of the form factors of deformed theories is that clustering in momentum space (3.83) does not seem applicable. Due to the form of the function $\Upsilon_n^{\mathcal{O}}(\{\theta_i\}_n; \alpha)$ the form factors of the deformed theory do not generally factorise into form factors of lower particle numbers when some rapidities are sent to infinity. One possible reason is that clustering in momentum space has been related to a massless or UV limit of the theory [192], and in deformed theories the UV limit is no longer a standard CFT. This means that, for theories without bound states, there is no clear way to relate the normalisations of odd- and even-particle form factors. In the same way, also the bound (3.82) seems to lose any significance because now the main asymptotic properties, as we saw already in the case of the deformed IFT, are completely dictated by the MFF (3.94) and specifically by the sign of the deformation parameter itself.

Even in this general case we can consider the leading contributions to the long and short-distance expansions of the logarithm of a typical (normalised) two-point function $\langle \mathcal{O}(0) \mathcal{O}^{\dagger}(r) \rangle / \langle \mathcal{O} \rangle^2$. Also for simplicity, we will restrict ourselves to fields that only have even-particle form factors and we will restrict to the case when $\alpha_1 := \alpha \neq 0$ and $\alpha_i = 0$

for $i > 1$. In this case, the two-particle contribution to the logarithm of the two-point function is

$$q_2^{\mathcal{O}}(r; \alpha) = \int_{-\infty}^{\infty} \frac{dx}{(2\pi)^2} |\Theta_2^{\mathcal{O}}(x; \alpha)|^2 |\hat{F}_2^{\mathcal{O}}(x; 0)|^2 e^{\frac{\alpha x}{\pi} \sinh x} K_0(2r \cosh \frac{x}{2}), \quad (3.192)$$

where¹³

$$\Theta_2^{\mathcal{O}}(\theta; \alpha) = \begin{cases} \left| \frac{\cos \frac{1}{2}(\delta(\theta) - \alpha \sinh \theta)}{\cos \frac{\delta(\theta)}{2}} \right| & \text{for } \gamma_{\mathcal{O}} = 1, \\ \left| \frac{\sin \frac{1}{2}(\delta(\theta) - \alpha \sinh \theta)}{\sin \frac{\delta(\theta)}{2}} \right| & \text{for } \gamma_{\mathcal{O}} = -1, \end{cases} \quad (3.193)$$

and $\hat{F}_2^{\mathcal{O}}(x; 0)$ is the two-particle form factor normalised by the vacuum expectation value $\langle \mathcal{O} \rangle$.

As in the deformed Ising field theory case, the asymptotic behaviour depends crucially on the sign of α . For $\alpha > 0$, the exponential factor diverges and cannot be compensated by the undeformed form factor, which typically decays exponentially. At short distances this signals an unusual UV behaviour, while at large distances the convergence of the series depends on the interplay between the length scale, the momentum and α : for small momenta the expansion remains convergent, but for large momenta the finite size of particles is again probed and convergence is lost.

For $\alpha < 0$, on the other hand, the situation is reversed: the exponential factor ensures rapid convergence both in the UV and IR. In the IR the theory flows to a trivial fixed point, as in the undeformed case. In the UV, the form factor expansion must be resummed, leading to a logarithmic behaviour of the form

$$q_n^{\mathcal{O}}(r \ll 1; \alpha < 0) \sim -z_n^{\mathcal{O}}(\alpha) \log r, \quad (3.194)$$

with coefficients $z_n^{\mathcal{O}}(\alpha)$ given by integrals over rapidities, e.g.

$$z_2^{\mathcal{O}}(\alpha) = \int_{-\infty}^{\infty} \frac{d\theta}{(2\pi)^2} |\Theta_2^{\mathcal{O}}(\theta; \alpha)|^2 |\hat{F}_2^{\mathcal{O}}(\theta; 0)|^2 e^{\frac{\alpha \theta}{\pi} \sinh \theta}. \quad (3.195)$$

Since the exponential decays for $\alpha < 0$, one always finds

$$z_n^{\mathcal{O}}(\alpha) \leq z_n^{\mathcal{O}}(0),$$

whenever the undeformed coefficients are well-defined. Thus the UV behaviour is power-law, as in a CFT, but with an α -dependent power that tends to zero as $\alpha \rightarrow -\infty$.

Also the analysis via Zamolodchikov's c -theorem follows the same pattern as in the specific Ising example. One considers

$$c(\alpha) = \frac{3}{2} \int_0^{\infty} dr r^3 \langle \Theta(0) \Theta(r) \rangle_c, \quad (3.196)$$

which interpolates between the UV and IR central charges. For $\alpha = 0$ this reproduces c^{UV} , since $c^{IR} = 0$, while for $\alpha < 0$ it provides an α -dependent deformation of c^{UV} .

¹³Note that for the Ising model $\cos \frac{\delta(\theta)}{2} = 0$ in (3.193) but this can be compensated for by appropriate normalisation of the form factors.

The two-particle form factor of Θ carries the same oscillatory structure encountered previously,

$$F_2^\Theta(\theta; \alpha) = 2\pi \left| \frac{\sin(\frac{\alpha}{2} \sinh \theta)}{\frac{\alpha}{2} \sinh \theta} \right| \frac{F_{\min}(\theta; \alpha)}{F_{\min}(i\pi; \alpha)}, \quad (3.197)$$

and, in the two-particle approximation, yields

$$c_2(\alpha) = \frac{3}{8} \int_{-\infty}^{+\infty} dx \left[\frac{\sin(\frac{\alpha}{2} \sinh x)}{\alpha \cosh^3(x/2)} \right]^2 e^{\frac{\alpha}{\pi} x \sinh x}. \quad (3.198)$$

The outcome is qualitatively unchanged (w.r.t. the deformed IFT): numerical evaluation (Fig. 3.13) shows a smooth flow from the undeformed value $1/2$ to zero as $-\alpha$ increases, with oscillations suppressed by the exponential factor.

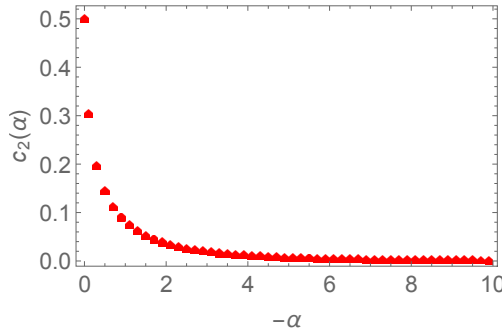


Figure 3.13: The functions $c_2(\alpha)$ for the Ising model.

3.2.3 Extension to Branch Point Twist Fields

A possible direction that we decided to pursue was the study of entanglement properties¹⁴ of the deformed theories. A perturbative approach for free theories was used in [217] and led to some interesting conclusions regarding the leading contribution to the entanglement entropies in free boson and free fermion theories. Actually, the work [218] has addressed exactly the same problems we consider here. In fact our paper [14] on which this discussion is based on was very nearly completed when [218] appeared. Despite the large overlap between our work and [218], we found important differences in some of the main results.

The paper [14] basically extends the work in [12, 13] to deal with a particular type of symmetry field, the branch point twist field (BPTF). The name BPTF, their definition as symmetry fields associated with cyclic permutation symmetry in replica theories, and their application to entanglement measures in massive IQFTs all hail back to the work [185]. The relationship between entanglement measures and correlators of fields associated with conical singularities was first discussed in [219], although the fields introduced there were distinct from the BPTF. The extension considered here is non-trivial, because the results of [13] were only applicable to local and semi-local fields, where semi-local meant specifically fields with factor of local commutativity $\gamma = \pm 1$ [182]. In order to study the form factors of BPTFs we need to consider the form factor program anew and

¹⁴Of particular relevance to the following discussion are works which have focused on entanglement measures [209, 210, 211, 212, 213, 214, 215, 216].

find solutions to the form factor equations which might a priori have a different structure from those found in [12, 13] because of the different locality properties that these fields possess. In [14] we focused on the two particle form factor (the zero and one-particle form factors are constant for spinless fields like the BPTF), leaving higher particle solutions for future study¹⁵. As we will see, the two-particle contribution can already give us valuable insights into some properties of entanglement. The results just obtained for the IFT and more complicated interacting diagonal theories without bound states certainly support the view that, at the very least, the form factors obtained starting with the S -matrix (2.43) encapsulate physical information about the models. Indeed, for the simpler $T\bar{T}$ -perturbation they capture in a very neat and clear way the distinct nature of the positive and negative coupling regimes, already highlighted earlier. Furthermore, the correlation functions obtained from these form factors have asymptotic properties which are entirely compatible with the view that the UV theory is non-local and that the elementary excitations are now extended “finite length” degrees of freedom. In addition, the form factor program, although relying heavily on locality, does not require strict locality of the operators/particles but only finite support thereof. In fact, the idea that one can apply the form factor program to extended objects has already appeared in the literature, for instance in [220], where the notion of bi-local fields is discussed (that is fields associated with the “product” of two operators located at different points in space-time). The same idea is behind the concept of twist operator employed in [221]. Even though these are different models/problems¹⁶, we were still motivated to give it a try. Again, at the time we wrote [14] we were completely clueless on possible ways to fix the free parameters in (3.95) and therefore also in here we consider $\beta_s = 0$.

Branch Point Twist Fields and their Form Factors

Fields associated to internal symmetries in quantum field theory can be characterised by their equal-time exchange relations with respect to other local fields and that these in turn lead to twist fields sitting at the origin of branch cuts in space-time as we saw in Subsection 3.1.3. The fields we are interested in here, the BPTFs, arise naturally in replica theories which consist of n identical, non-interacting copies of some chosen model. In such cases, there is a large amount of symmetry in the resulting model (in fact, under all elements of the permutation group), including cyclic permutation symmetry, which is the symmetry that turns out to be relevant in the context of entanglement. As a result of the replica process, all the fields that were present in the original (non-replicated) model acquire copy labels. In other words, if the chosen model had a local field $\phi(\mathbf{x})$, the replica theory will possess a family $\phi_j(\mathbf{x})$ with $j = 1, \dots, n$ and the identification $j \equiv j + n$. In this context the BPTF $\mathcal{T}$ and its hermitian conjugate $\mathcal{T}^\dagger$ satisfy the equal-time exchange relations

$$\phi_j(x)\mathcal{T}(y) = \begin{cases} \mathcal{T}(y)\phi_{j+1}(x) & y > x \\ \mathcal{T}(y)\phi_j(x) & y < x \end{cases} \quad \text{and} \quad \phi_j(x)\mathcal{T}^\dagger(y) = \begin{cases} \mathcal{T}^\dagger(y)\phi_{j-1}(x) & y > x \\ \mathcal{T}^\dagger(y)\phi_j(x) & y < x \end{cases} . \quad (3.199)$$

¹⁵Unfortunately we were, and we still are unable to solve the kinematical residue equation for these fields as we did in the previous subsections for more standard fields.

¹⁶In our case what is extended is rather the particles or the “scattering process itself” rather than the operators; nevertheless, the two notions are conceptually related, since bilocality of operators can be viewed as a form of spatial extension.

From these exchange relations, equations for matrix elements of $\mathcal{T}$ and $\mathcal{T}^\dagger$ can be derived. Let us define

$$F_k^{\mu_1 \dots \mu_k}(\theta_1, \dots, \theta_k; n) := \langle 0 | \mathcal{T}(0) | \theta_1, \dots, \theta_k \rangle_{\mu_1 \dots \mu_k}, \quad (3.200)$$

to be the k -particle form factor of $\mathcal{T}$, with $|\theta_1, \dots, \theta_k\rangle_{\mu_1, \dots, \mu_k}$ an in-state characterised by rapidities θ_i and quantum numbers μ_j . Typically, each quantum number carries two sets of information: the particle quantum number and its copy number; in short $\mu_j := (a, j)$. However if the theory has a single particle type, as we consider here, we can think of $\mu_j := j$ as being simply the copy number. From the relations (3.199), the form factor equations for diagonal scattering matrices can be derived and were first written in [185]:

$$F_k^{\dots \mu_i \mu_{i+1} \dots}(\dots, \theta_i, \theta_{i+1}, \dots; n) = S_{\mu_i \mu_{i+1}}(\theta_{i+1}) F_k^{\dots \mu_{i+1} \mu_i \dots}(\dots, \theta_{i+1}, \theta_i, \dots; n), \quad (3.201)$$

$$F_k^{\mu_1 \mu_2 \dots \mu_k}(\theta_1 + 2\pi i, \dots, \theta_k; n) = F_k^{\mu_2 \dots \mu_k \hat{\mu}_1}(\theta_2, \dots, \theta_k, \theta_1; n), \quad (3.202)$$

$$\text{Res}_{\bar{\theta}_0 = \theta_0} F_{k+2}^{\bar{\mu} \mu_1 \dots \mu_k}(\bar{\theta}_0 + i\pi, \theta_0, \theta_1, \dots, \theta_k; n) = i F_k^{\mu_1 \dots \mu_k}(\theta_1, \dots, \theta_k; n), \quad (3.203)$$

$$\text{Res}_{\bar{\theta}_0 = \theta_0} F_{k+2}^{\bar{\mu} \hat{\mu}_1 \dots \mu_k}(\bar{\theta}_0 + i\pi, \theta_0, \theta_1, \dots, \theta_k; n) = -i \prod_{i=1}^k S_{\mu \mu_i}(\theta_{0i}) F_k^{\mu_1 \dots \mu_k}(\theta_1, \dots, \theta_k; n), \quad (3.204)$$

where $\hat{\mu}_i := (a, i+1)$ indicates a particle label with the same quantum numbers as μ but copy number shifted by one and $\bar{\mu}_i := (\bar{a}, i)$ indicates a particle that is conjugate to the particle of index μ in the same copy. Although the kinematic residue equation for branch point twist fields now splits into two separate equations (3.203)-(3.204), the second can be derived from the first by employing Watson's equations (3.201)-(3.202). This means that typically only the equations (3.201)-(3.203) are employed in the solution procedure. Furthermore, for simplicity, these can be specialised to particles in the same copy since once those form factors are known, all others can be obtained by employing (3.201)-(3.202). The form factors of $\mathcal{T}^\dagger$ are also related to those of $\mathcal{T}$ in a simple way, as discussed in [185] and [222]. As usual, the solution to these equations starts with solving equations for two-particle form factors. We now discuss how this is done and how the known solutions generalise in the presence of $\text{T}\bar{\text{T}}$ perturbations. In this case, all equations above still hold, but, in a theory with a single particle type, the scattering matrices $S_{\mu_i \mu_j}(\theta)$ are replaced by $(S_\alpha(\theta))^{\delta_{ij}}$ and therefore the form factors must now be functions of α .

Since we want to deal with deformed theories, let us slightly modify our notation (3.200) and introduce explicitly the dependence on the couplings α . In addition, since BPTFs are spinless we know their two-particle form factor will only depend on rapidity differences, hence a single variable θ . We will then call

$$F_2^{jp}(\theta; n, \alpha), \quad (3.205)$$

the two-particle form factor of $\mathcal{T}$ in the deformed theory for two particles of the same type, labelled by distinct copy indices j, p . The minimal part of this form factor will be denoted by

$$F_{\min}^{jp}(\theta; n, \alpha), \quad (3.206)$$

and will be a solution of the two-particle form factor axioms without poles in the physical sheet. It satisfies

$$F_{\min}^{jp}(\theta; n, \alpha) = F_{\min}^{jp}(-\theta; n, \alpha) (S_\alpha(\theta))^{\delta_{jp}} = F_{\min}^{jp+1}(2\pi i - \theta; n, \alpha), \quad \forall j, p, \quad (3.207)$$

where the delta function implements the correct exchange relation (3.201) in which particles in distinct copies do not interact whereas particles in the same copy interact in a model specific manner. Using these relations as well as the fact that all copies are identical, we can show

$$F_{\min}^{\mathcal{T}|ii+p}(\theta; n, \alpha) = F_{\min}^{\mathcal{T}|jj+p}(\theta; n, \alpha), \quad \forall i, j, p, \quad (3.208)$$

$$F_{\min}^{\mathcal{T}|1j}(\theta; n, \alpha) = F_{\min}^{\mathcal{T}|11}(2\pi(j-1)i - \theta; n, \alpha), \quad \forall j \neq 1. \quad (3.209)$$

The last equation allows us to write a similar set of equations to (3.207) for form factors involving particles in the same copy only. This gives the relations

$$f_n^\alpha(\theta) = f_n^\alpha(-\theta)S_\alpha(\theta) = f_n^\alpha(-\theta + 2\pi in), \quad (3.210)$$

which imply that the minimal form factor

$$F_{\min}^{11}(\theta; n, \alpha) := f_n^\alpha(\theta), \quad (3.211)$$

must not have any pole in the extended strip $\text{Im}(\theta) \in [0, 2\pi n]$. For $\alpha = \mathbf{0}$ the relations (3.210) can be typically solved in a very similar way as for local fields and the method has been explained in many places including [185, 222]. The resulting function $f_n^{\mathbf{0}}(\theta)$ can be usually expressed in terms of an integral representation which is determined by the scattering matrix. For free theories however, this function is very simple. In particular, for the Ising field theory it is just

$$f_n^{\mathbf{0}}(\theta) = -i \sinh \frac{\theta}{2n}. \quad (3.212)$$

Since the equations (3.210) are linear, this means that their solution will typically factorise into $f_n^{\mathbf{0}}(\theta)$ (the undeformed solution) and another function $\varphi_n^\alpha(\theta)$ which satisfies

$$\varphi_n^\alpha(\theta) = \varphi_n^\alpha(-\theta)\Phi_\alpha(\theta) = \varphi_n^\alpha(-\theta + 2\pi in). \quad (3.213)$$

A simple family of solutions to this equation can be found similarly as for local fields. It is a simple generalisation to replica theories of the solutions (3.94), and it takes the form

$$\varphi_n^\alpha(\theta) = \exp \left(\frac{\theta - i\pi n}{2\pi n} \sum_{s \in \mathcal{S}} \alpha_s m^{2s} \sinh(s\theta) \right). \quad (3.214)$$

As for local and semi-local fields, the minimal form factor itself contains possible ‘‘CDD factors’’, that is factors that are both even and $2\pi in$ periodic in the variable θ . The most general combination of such factors takes the form

$$C_\beta^n(\theta) = \exp \left(\sum_{s \in \mathcal{S}'} \beta_s m^{\frac{2s}{n}} \cosh \frac{s\theta}{n} \right). \quad (3.215)$$

where $\mathcal{S}'$ may be a different set of integers than $\mathcal{S}$. Our solutions (3.214) and (3.215) are identical to those found in [218]. It is however useful to discuss a little bit why we choose (3.214) to involve only hyperbolic functions of $s\theta$ whereas (3.215) involves hyperbolic functions of $\frac{s\theta}{n}$. Indeed, if s are odd integers, which is usually the case, we could replace $\sinh(s\theta)$ by $\sinh \frac{s\theta}{n}$ and still have a solution to all required equations. The reason for choosing (3.214) as done here is two-fold: it is natural to expect that the minimal form factor satisfies

$$f_n^\alpha(\theta)(f_1^\alpha(\theta))^* \in \mathbb{R} \quad \text{for } \theta \in \mathbb{R} \quad \text{and } n \neq 1, \quad (3.216)$$

as otherwise any correlators of $\mathcal{T}$ with a local or semi-local field in the theory would give complex results. The property (3.216) holds for $\alpha = \mathbf{0}$. This property is for instance essential in the following section where we study the Δ sum rule. Without (3.216), the output of the Δ -sum rule presented below would not be a real number, even in the two-particle approximation. The choice of $\varphi_n^\alpha(\theta)$ is also supported by fundamental properties of the S -matrices and minimal form factors of IQFTs which we plan to discuss in more detail in a further publication. Furthermore, the solution (3.215) is the most general CDD-factor that we can write and of course it includes all terms of the form $\cosh(k\theta)$ when $s = kn$. Terms of the form $\cosh(k\theta)$ seem indeed more natural as they are the one-particle eigenvalues of higher local conserved quantities. In summary, we have then that any diagonal theory with a single-particle spectrum has minimal form factor

$$f_n^\alpha(\theta) = f_n^{\mathbf{0}}(\theta)\varphi_n^\alpha(\theta)C_\beta^n(\theta). \quad (3.217)$$

In the Ising model this reads explicitly

$$f_n^\alpha(\theta) = -i \sinh \frac{\theta}{2n} \exp \left(\frac{\theta - i\pi n}{2\pi n} \sum_{s \in 2\mathbb{Z}+1} \alpha_s m^{2s} \sinh(s\theta) + \sum_{s \in S'} \beta_s m^{\frac{2s}{n}} \cosh \frac{s\theta}{n} \right). \quad (3.218)$$

If we now include the relevant kinematic poles and normalisation so as to ensure that the two-particle versions of equations (3.203) and (3.204) are also satisfied, we obtain the solution

$$F_2^{11}(\theta; n, \alpha) = \frac{\langle \mathcal{T} \rangle_n^\alpha \sin \frac{\pi}{n}}{2n \sinh \left[\frac{i\pi + \theta}{2n} \right] \sinh \left[\frac{i\pi - \theta}{2n} \right]} \frac{f_n^\alpha(\theta)}{f_n^\alpha(0)}. \quad (3.219)$$

where $\langle \mathcal{T} \rangle_n^\alpha := F_0(n, \alpha)$. Note that the factor $\sin \frac{\pi}{n}$ ensures that $F_2^{11}(\theta; 1, \alpha) = 0$ which is a consistency condition for the BPTF (i.e. it becomes the identity field for $n = 1$). Apart from the minimal form factor, this form factor has exactly the usual form [185].

It is worth highlighting however, that this solution still needs to be tested against solutions to the higher particle form factor equations, which are harder to obtain. We mean by this that the two-particle form factor needs to be consistently recovered from the kinematic residue of the four-particle form factor. Until this four-particle form factor is computed some uncertainty as to the exact form of $F_2^{11}(\theta; n, \alpha)$ remains. This is similar to the behaviour we found in (3.131) for the form factors of the trace of the stress-energy tensor Θ . We saw that consistency with higher-particle form factor equations requires the standard two-particle form factor solution to be multiplied by an additional factor $f(\theta; \alpha)$ with properties

$$f(\theta; \mathbf{0}) = f(i\pi; \alpha) = 1. \quad (3.220)$$

Such a factor could be present in our case too. If so, it would need to be also an even, $2\pi i n$ -periodic function of θ . A more in-depth study of the form factor equations is required to settle this point. We will only work with the solution (3.219), the same found in [218].

We can immediately try to check our result with the Δ -sum rule. The result is a function that heavily depends on α with strong divergencies or properties that are compatible with the power-law scaling of two-point functions that we observed already. Even if the form factors of the BPTF are different from those of local and semi-local fields previously found, power-law scaling will follow through due to the exponentially decaying factors

$|\varphi_n^\alpha(\theta)|^2$ that enter the form factor expansion of the two-point function $\langle \mathcal{T}(0)\mathcal{T}(r) \rangle_n^\alpha$. For the same reason, the Δ -sum rule in the two-particle approximation is also given by a convergent integral. We have that

$$\Delta_n^\alpha = -\frac{1}{2\langle \mathcal{T} \rangle_n^\alpha} \int_0^\infty dr r \langle \Theta(r) \mathcal{T}^\dagger(0) \rangle_n^\alpha. \quad (3.221)$$

The right-hand side of (3.221) can be expressed as a sum over multi-particle contribution [223]. Again, here we will not need the most general expression and we consider the Ising model and limit our analysis to the two-particle contribution. In fact, for $\alpha_s < 0$ we expect this to constitute the leading contribution to (3.221). After some simplifications we find

$$\Delta_n^\alpha \approx -\frac{n}{32\pi^2 m^2 \langle \mathcal{T} \rangle_n^\alpha} \int_{-\infty}^\infty d\theta \frac{F_2^{\Theta|11}(\theta; \boldsymbol{\alpha}) (F_2^{\mathcal{T}|11}(\theta; n, \boldsymbol{\alpha}))^*}{\cosh^2(\theta/2)}. \quad (3.222)$$

If we now consider the $\text{T}\bar{\text{T}}$ perturbation, corresponding to having only $\alpha_1 := \alpha \neq 0$ and all $\beta = \mathbf{0}$ we then have the form factor formulae

$$F_2^{\mathcal{T}|11}(\theta; n, \alpha) = -\frac{i\langle \mathcal{T} \rangle_n^\alpha \sinh(\frac{\theta}{2n}) \cos(\frac{\pi}{2n})}{n \sinh[\frac{i\pi-\theta}{2n}] \sinh[\frac{i\pi+\theta}{2n}]} e^{-\frac{\alpha m^2}{2\pi}(i\pi-\frac{\theta}{n}) \sinh \theta}, \quad (3.223)$$

using (3.138) and (3.223) we find¹⁷

$$\Delta_n^\alpha = -\frac{1}{16\pi} \int_{-\infty}^\infty d\theta \frac{\cos(\frac{\pi}{2n}) \sinh(\frac{\theta}{2n}) \tanh \frac{\theta}{2}}{\sinh[\frac{i\pi-\theta}{2n}] \sinh[\frac{i\pi+\theta}{2n}] \cosh \frac{\theta}{2}} \left| \frac{\sin(\frac{\alpha}{2} \sinh \theta)}{\frac{\alpha}{2} \sinh \theta} \right| e^{\frac{\alpha m^2}{2\pi}(\frac{n+1}{n})\theta \sinh \theta}. \quad (3.224)$$

For $\alpha = 0$ the integral can be computed exactly as shown in [185], and it gives the exact CFT formula [224, 225]

$$\Delta_n^0 = \frac{1}{48} \left(n - \frac{1}{n} \right). \quad (3.225)$$

In [185] it was shown that the integral (3.224) can be computed exactly for $\boldsymbol{\alpha} = 0$. This is because in that case the integrand only picks up a sign under the shift $\theta \mapsto \theta + 2\pi i n$. Hence by Cauchy's theorem, the sum of the residues of the poles that cross the line under the shift give exactly twice the dimension Δ_n^0 . In the present case, unfortunately the presence of the exponential spoils this property. In the absence of an analytic formula, the integral can be evaluated numerically for different values of α and n as shown in Figs. 3.11 and 3.12. Fig. 3.11 shows the function Δ_n^α as a function of n for different negative choices of α . As observed for other fields, the values of Δ_n^α decrease as the modulus of α increases. From the figure it is clear that for large n the leading behaviour of Δ_n^α is linear in n , just as found in [218]. The curvature of the function is only visible for very small values of n , around 1. It is possible to expand (3.224) for large n to work out that the leading contributions are:

$$\Delta_n^\alpha = a(\alpha) + nb(\alpha) + \frac{1}{n}c(\alpha) + \mathcal{O}(n^{-2}), \quad (3.226)$$

with

¹⁷In [218] the authors chose to ignore the factor $\left| \frac{\sin(\frac{\alpha}{2} \sinh \theta)}{\frac{\alpha}{2} \sinh \theta} \right|$ arguing that it does not influence the final result too much. This is indeed true, but it seems a peculiar reasoning nonetheless.

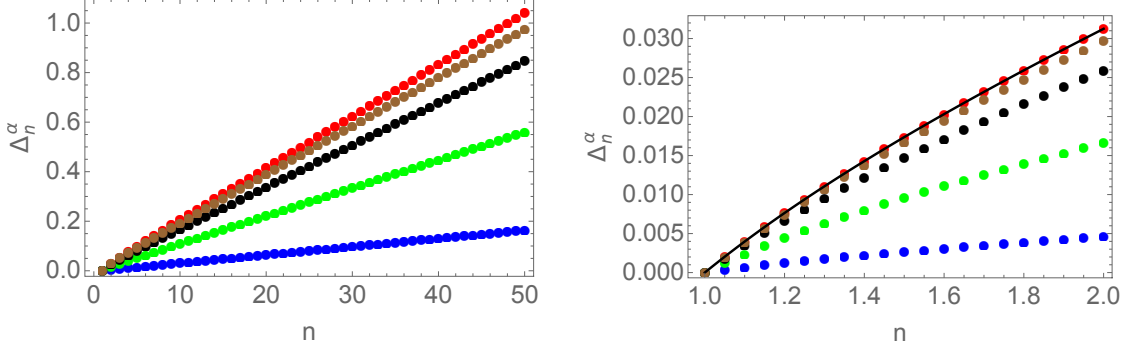


Figure 3.14: The function (3.224) for $\alpha = 0, -0.001, -0.01, -0.1$ and -1 , with the value 0 corresponding to the red dots and the value 1 corresponding to the blue dots. The exact formula (3.225) is also plotted as a solid black line, which fits the red dots perfectly. The figure on the right is a blow up of the small n region of the left figure. All functions are linear for n large and the slopes are: $b(0) = \frac{1}{48} = 0.0208333$, $b(-0.001) = 0.0195262$, $b(-0.01) = 0.0169936$, $b(-0.1) = 0.0112533$ and $b(-1) = 0.00331503$.

$$a(\alpha) := \frac{\alpha}{8\pi^2} \int_{-\infty}^{\infty} d\theta \frac{\theta^2}{\pi^2 + \theta^2} \frac{\sinh^2 \frac{\theta}{2}}{\cosh \frac{\theta}{2}} \left| \frac{\sin(\frac{\alpha}{2} \sinh \theta)}{\frac{\alpha}{2} \sinh \theta} \right| e^{\frac{\alpha\theta}{2\pi} \sinh \theta}, \quad (3.227)$$

$$b(\alpha) := \frac{1}{8\pi} \int_{-\infty}^{\infty} d\theta \frac{\theta}{\pi^2 + \theta^2} \frac{\tanh \frac{\theta}{2}}{\cosh \frac{\theta}{2}} \left| \frac{\sin(\frac{\alpha}{2} \sinh \theta)}{\frac{\alpha}{2} \sinh \theta} \right| e^{\frac{\alpha\theta}{2\pi} \sinh \theta}, \quad (3.228)$$

and

$$c(\alpha) := \frac{1}{192\pi^3 n} \int_{-\infty}^{\infty} d\theta \frac{\theta(\pi^4 + \pi^2 \theta^2 - 3\alpha^2 \theta^2 \sinh^2 \theta)}{\pi^2 + \theta^2} \frac{\sinh \frac{\theta}{2}}{\cosh^2 \frac{\theta}{2}} \left| \frac{\sin(\frac{\alpha}{2} \sinh \theta)}{\frac{\alpha}{2} \sinh \theta} \right| e^{\frac{\alpha\theta}{2\pi} \sinh \theta}. \quad (3.229)$$

The coefficients of higher order corrections in $1/n$ can be easily evaluated by further expanding the integrand of (3.224). As expected, $a(0) = 0$ and $b(0) = -c(0) = \frac{1}{48}$.

If we assume that the function (3.224) still characterises the short-distance scaling of the two-point function $\langle \mathcal{T}(0) \mathcal{T}^\dagger(r) \rangle_n^\alpha$ as well as the scaling of the vacuum expectation value $\langle \mathcal{T} \rangle_n^\alpha \sim m^{2\Delta_n^\alpha}$, then the results above already have important implications regarding the behaviour of the entropy. In order to understand this we need to briefly review the relationship between BPTF correlators and entanglement.

3.2.4 Entanglement Entropy from BPTFs

The best-known measure of the entanglement between two parts of a quantum system, that is a bipartite measure, is the entanglement entropy or von Neumann entropy. Its good properties as an entanglement measure were established in [226]. The simplest case in which it can be computed is that of a quantum model in a pure state $|\Psi\rangle$.

Let the Hilbert space of the theory be divided into two complementary subspaces $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$ and let the reduced density matrix of subsystem A be defined as

$$\rho_A = \text{Tr}_{\bar{A}}(|\Psi\rangle\langle\Psi|). \quad (3.230)$$

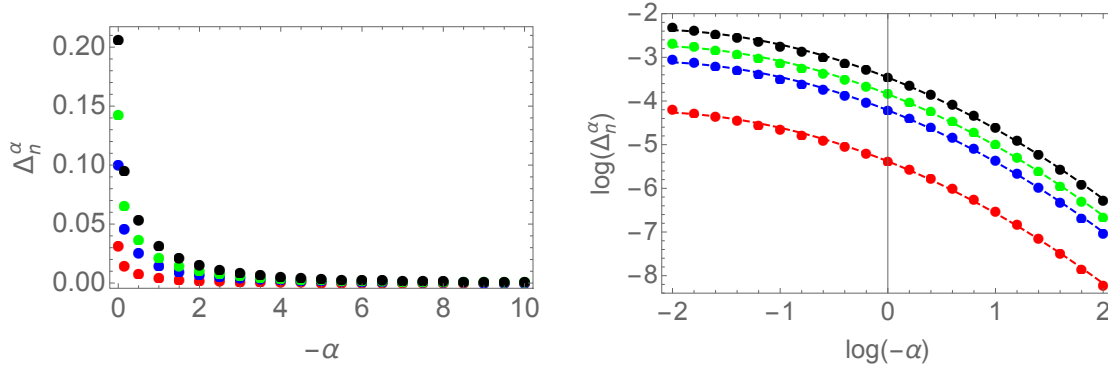


Figure 3.15: The function (3.224) for $n = 2$ (red), $n = 5$ (blue), $n = 7$ (green) and $n = 10$ (black). The figure on the right shows clearly how all functions scale similarly with α . We find that all functions are extremely well fitted by quadratic polynomials of the form $f(n) + k(n)x + p(n)x^2$ with $x = \log(-\alpha)$ which are presented as dashed lines.

Then, the von Neumann entropy is defined as

$$S(r) = -\text{Tr}_A(\rho_A \log \rho_A) , \quad (3.231)$$

where r is taken to be the length of subsystem A in a 1D theory. Related to this is the family of Rényi entropies, defined as

$$S_n(r) = \frac{\log(\text{Tr}_A(\rho_A^n))}{1 - n} . \quad (3.232)$$

The normalisation of ρ_A is chosen so that $\text{Tr}_A(\rho_A) = 1$, which implies that the von Neumann entropy can be obtained as a limit of the Rényi entropies

$$S(r) = \lim_{n \rightarrow 1} S_n(r) . \quad (3.233)$$

In the context of critical systems it was realised a long time ago [227, 228, 219] that for n integer $\text{Tr}_A(\rho_A^n)$ has a natural physical interpretation as a suitably normalised partition function in an n -sheeted Riemann surface with cyclically connected sheets. The structure of this Riemann surface is dictated by the product of n matrices ρ_A and by the trace, which induces the cyclicity property. The partition function picture applied to a configuration where the regions A and $\bar{A}$ are both complementary and fully connected, is very advantageous in the context of conformal field theory. In this case a conformal map can be employed to map the Riemann surface back to the complex plane with conical singularities at the boundary point(s) between regions A and $\bar{A}$. Henceforth both the Rényi and von Neumann entropies can be easily obtained leading to the well-known logarithmic growth of both $S(r)$ and $S_n(r)$ in critical systems [227, 228, 229, 230, 219, 231, 232].

Conformal maps can however not be used in massively perturbed and/or $\text{T}\bar{\text{T}}$ -perturbed CFTs and so an alternative approach to compute the partition function is needed. BPTFs provide such an approach. It can be shown that for a bipartition of the type described above, the function $\text{Tr}_A(\rho_A^n)$ is proportional to a correlation function of BPTFs [185, 219]. This correlation function contains as many fields as boundary points between A and $\bar{A}$. In the simplest case, when both regions are semi-infinite, and $\alpha = 0$, we require only one field and we can write

$$\text{Tr}_A(\rho_A^n) = \varepsilon^{2\Delta_n^0} \langle \mathcal{T} \rangle_n^0 , \quad (3.234)$$

where ε is a non-universal short-distance cut-off and $\langle \mathcal{T} \rangle_n^0$ is the vacuum expectation value (VEV) of the BPTF. Plugging this expression into the formula for the Rényi entropies it is easy to derive the well-known property of saturation of the entanglement entropy in gapped systems [219, 233], namely

$$S(\infty) = -\frac{c}{6} \log(m\varepsilon) + \frac{U}{2}, \quad (3.235)$$

where U is related to the proportionality constant $v(n)$ in $\langle \mathcal{T} \rangle_n^0 = v(n)m^{2\Delta_n^0}$ as

$$U := - \left. \frac{dv(n)}{dn} \right|_{n=1} \quad \text{with} \quad v(1) = 1. \quad (3.236)$$

On the other hand, if the region A is finite of length r , the trace is given by

$$\text{Tr}_A(\rho_A^n) = \varepsilon^{4\Delta_n^0} \langle \mathcal{T}(0) \mathcal{T}^\dagger(r) \rangle_n^0. \quad (3.237)$$

If the theory is critical, this formula can also be used to obtain the logarithmic growth of the entropies, which follows simply from power-law scaling of the two-point function. In CFT we have that

$$S_n(r) = \frac{c(n+1)}{6n} \log \frac{r}{\varepsilon}, \quad S(r) = \frac{c}{3} \log \frac{r}{\varepsilon}, \quad (3.238)$$

where c is the central charge. Crucially the pre-factor of the logarithmic terms in (3.238) is related to the dimension of the BPTF in a simple way. For the Rényi entropies, this prefactor is given by $-\frac{4\Delta_n^0}{1-n}$ whereas for the von Neumann entropy it can be obtained as

$$-\lim_{n \rightarrow 1} \frac{4\Delta_n^0}{1-n} = 4 \left. \frac{d\Delta_n^0}{dn} \right|_{n=1}. \quad (3.239)$$

All formulae above have been written for the case $\alpha = 0$. Assuming that the relationship between correlators of BPTFs and entanglement measures is preserved by a generalised $T\bar{T}$ -perturbation, it follows that if Δ_n^α has a different dependence on n than Δ_n^0 , then the coefficient of the log-term in the entanglement entropy and Rényi entropies will be modified. That is, both the function inside the limit and the limit itself in (3.239) will be different. Notice that for (3.239) to be correct, it is essential that $\Delta_1^0 = 0$, a condition that is a consequence of $\mathcal{T}$ becoming the identity field in the absence of replicas. The Δ -sum rule formulae presented originally proposed in [218] violated this essential property¹⁸.

As we have just discussed

$$S(r) = \lim_{n \rightarrow 1} S_n(r) = - \lim_{n \rightarrow 1} \frac{d}{dn} \text{Tr}_A \rho_A^n = - \lim_{n \rightarrow 1} \frac{d}{dn} \varepsilon^{4\Delta_n^\alpha} \langle \mathcal{T}(0) \mathcal{T}^\dagger(r) \rangle. \quad (3.240)$$

The two-point correlation function of twist fields can be then calculated using its spectral representation in terms of form factors:

$$\langle \mathcal{T}(0) \mathcal{T}^\dagger(r) \rangle_n^\alpha = \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} \frac{d\theta_1 \dots d\theta_k}{k! (2\pi)^k} |F_k^\mathcal{T}(\theta_1, \dots, \theta_k; n, \alpha)|^2 e^{-mr \sum_{i=1}^k \cosh \theta_i}. \quad (3.241)$$

¹⁸The original derivation in [218] contained some inaccuracies, which were later corrected in a subsequent revision following the appearance of our work.

As is common with these kinds of expansions, we can truncate the sum at the two-particle level and so consider the leading non-trivial contribution for large r , which for the Ising model comes from the two-particle form factors. We can write

$$\begin{aligned}\langle \mathcal{T}(0)\mathcal{T}^\dagger(r) \rangle_n^\alpha &= (\langle \mathcal{T} \rangle_n^\alpha)^2 + \sum_{i,j} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\theta_1 d\theta_2}{2!(2\pi)^2} |F_2^{ij}(\theta_{12}; n, \alpha)|^2 e^{-mr(\cosh \theta_1 + \cosh \theta_2)} \\ &= (\langle \mathcal{T} \rangle_n^\alpha)^2 \left(1 + \frac{n}{4\pi^2} \int_{-\infty}^{\infty} d\theta g(\theta; n, \alpha) K_0(2rm \cosh \frac{\theta}{2}) \right),\end{aligned}\quad (3.242)$$

where $K_0(z)$ is the modified Bessel function of the second kind and

$$(\langle \mathcal{T} \rangle_n^\alpha)^2 g(\theta; n, \alpha) = \sum_{j=1}^n |F_2^{1j}(\theta; n, \alpha)|^2 = \quad (3.243)$$

$$= |F_2^{11}(\theta; n, \alpha)|^2 + \sum_{j=1}^{n-1} |F_2^{11}(-\theta + 2\pi ij; n, \alpha)|^2. \quad (3.244)$$

The last identity above follows from the relations between form factors on different copies we presented earlier. Specialising again to the $\text{T}\bar{\text{T}}$ -deformation, we can use (3.223) to write

$$|F_2^{11}(\theta; n, \alpha)|^2 = \frac{(\langle \mathcal{T} \rangle_n^\alpha)^2 \sinh^2 \frac{\theta}{2n} \cos^2 \frac{\pi}{2n}}{n^2 \sinh^2 \left(\frac{i\pi - \theta}{2n} \right) \sinh^2 \left(\frac{i\pi + \theta}{2n} \right)} e^{\frac{\alpha m^2}{\pi} \frac{\theta}{n} \sinh \theta}, \quad (3.245)$$

and

$$\begin{aligned}|F_2^{11}(-\theta + 2\pi ij; n, \alpha)|^2 &= F_2^{11}(-\theta + 2\pi ij; n, \alpha) (F_2^{11})^*(-\theta - 2\pi ij; n, \alpha) = \\ &= -\frac{i\langle \mathcal{T} \rangle_n^\alpha \sinh \left(\frac{-\theta + 2\pi ij}{2n} \right) \cos \left(\frac{\pi}{2n} \right)}{n \sinh \left[\frac{i\pi + \theta - 2\pi ij}{2n} \right] \sinh \left[\frac{i\pi - \theta + 2\pi ij}{2n} \right]} e^{-\frac{\alpha m^2}{2\pi} (i\pi + \frac{\theta - 2\pi ij}{n}) \sinh(-\theta + 2\pi ij)} \times \\ &\times \frac{i\langle \mathcal{T} \rangle_n^\alpha \sinh \left(\frac{-\theta - 2\pi ij}{2n} \right) \cos \left(\frac{\pi}{2n} \right)}{n \sinh \left[\frac{-i\pi + \theta + 2\pi ij}{2n} \right] \sinh \left[\frac{-i\pi - \theta - 2\pi ij}{2n} \right]} e^{-\frac{\alpha m^2}{2\pi} (-i\pi + \frac{\theta + 2\pi ij}{n}) \sinh(-\theta - 2\pi ij)} = \\ &= \frac{(\langle \mathcal{T} \rangle_n^\alpha)^2 \sinh \left(\frac{-\theta + 2\pi ij}{2n} \right) \sinh \left(\frac{\theta + 2\pi ij}{2n} \right) \cos^2 \frac{\pi}{2n} e^{\frac{\alpha m^2}{\pi n} \theta \sinh \theta}}{n^2 \sinh \left(\frac{i\pi + \theta - 2\pi ij}{2n} \right) \sinh \left(\frac{i\pi - \theta + 2\pi ij}{2n} \right) \sinh \left(\frac{-i\pi + \theta + 2\pi ij}{2n} \right) \sinh \left(\frac{-i\pi - \theta - 2\pi ij}{2n} \right)}.\end{aligned}\quad (3.246)$$

The integral in (3.242) can be performed numerically for n integer but due to the presence of sums such as in (3.244) it does not make sense to consider real, non-integer values of n . These are however needed in order to compute the derivative and limit required in the computation of the von Neumann entropy. In [185] the analytic continuation to real positive $n \geq 1$ was performed using the cotangent trick, that is replacing the sum (3.244) by a contour integral in the variable j with poles at $j = 1, \dots, n-1$. The same idea can be used here. The upshot of the computation in [185] was that the sum has a special behaviour when $n \rightarrow 1$ and $\theta \rightarrow 0$ simultaneously which can be characterised by a δ -function centred at $\theta = 0$. The same behaviour occurs here since at $\theta = 0$ the exponential factor, which is the only factor that depends on α is just 1. Both for the perturbed and unperturbed Ising model, it is possible to perform the sum

$$n \sum_{j=1}^{n-1} |F_2^{11}(2\pi i j; n, \alpha)|^2 = \frac{n}{2} \quad \text{for } n > 1. \quad (3.247)$$

exactly. In fact, whether or not $\alpha = 0$ the value of the sum is the same because the $\sinh \theta$ in the exponential factor is vanishing for $\theta = 2\pi i j$. This result however reveals a non-analytic behaviour around $n = 1$. In fact for $n = 1$ the sum should be 0, but the formula above suggests that the analytic continuation from n large to $n = 1$ should give the value $1/2$ at $n = 1$. It was shown in [185] that the value $1/2$ at $n = 1$ and the non-analyticity at $\theta = 0$ are found for all diagonal IQFTs, even when they are interacting.

The similarity with the unperturbed case extends fully to the computation of the sum (3.244) by contour integration. The same reasoning presented in Appendix 3 of [185] applies here, crucially because the exponential factor in (3.246) does not depend on the index j which becomes the integration variable when using the cotangent trick. Therefore, we arrive to the interesting conclusion that *the next-to-leading order correction to the entanglement entropy of a large interval is exactly the same as in the unperturbed theory* and it **does not** depend on the perturbation, even if the leading order term coming from the VEV (the saturation value) appears to be perturbation-dependent. We have then the same result as in [185], namely

$$S(r; \alpha) - S(\infty, \alpha) = -\frac{1}{8} K_0(2mr) + \mathcal{O}(e^{-3mr}), \quad (3.248)$$

where we now introduced the α -dependence explicitly in the notation for $S(r; \alpha)$.

This conclusion is in principle not too surprising if we consider the physics of $\overline{\text{T}}\overline{\text{T}}$ perturbations. We know that for $\alpha > 0$ (< 0) we can interpret the effect of the perturbation as particles acquiring a positive (negative) length. This has strong effect on the short-distance (UV) physics of the theory. However, when distance is large compare to particle's width, this width should not play a major role and so the entropy correction remains unchanged. While this argument is qualitative in nature and not totally satisfactory, it supports the correctness of the result (3.248).

The results of the Δ -sum rule and of the general expansion of the two-point function both suggest that the leading contribution $S(\infty, \alpha)$ to the entanglement entropy is perturbation dependent, that is $\langle \mathcal{T} \rangle_n^\alpha \neq \langle \mathcal{T} \rangle_n^0$. This is also supported by the perturbative results obtained in [217] and those of [218]. So even if the next-to-leading order correction is α -independent, the leading contribution is not. As in the unperturbed case, the Rényi entropies acquire a less universal large-distance correction. In theories where the field $\mathcal{T}$ has a non-vanishing one-particle form factor¹⁹, the leading correction to the saturation of the Rényi entropies will be $\frac{n}{\pi} |F_1(n, \alpha)|^2$, which is very similar to the usual one (see e.g. [234, 235]), except that the one-particle form factor will be a function of α . If the leading correction comes from the two-particle form factor, as in the Ising model, then as our formula (3.242) shows, the integral will only be convergent for $\alpha < 0$ due to the presence of an exponential factor $\exp(\frac{\alpha m^2}{n\pi} \theta \sinh \theta)$ and will clearly depend on α .

3.2.5 Comparison with Previous Results in the Literature

We conclude this section with a discussion of the leading term $S(\infty, \alpha)$ in (3.248). This discussion is especially important because both in [217] and in [218] it is claimed that

¹⁹In this case, our conclusion for the von Neumann entropy above will still hold, since the one-particle form factor is vanishing as $n \rightarrow 1$.

$S(\infty, \alpha)$ is substantially different from the standard form (3.235) for the free fermion in the presence of a $\text{T}\bar{\text{T}}$ perturbation. In both papers it is found that in addition to a logarithmic term – just as in (3.235), albeit with a different coefficient – there are terms of other types. In [217] a term proportional to $\log^2 m\varepsilon$ is found for the Dirac fermion and the same type of term is found in [218] for the Ising field theory.

Our calculation based on the Δ -sum rule does not produce any of those additional, unusual terms, only a term proportional to $\log(m\varepsilon)$ as usual, but with a different coefficient that we compute below. In fact, any computation based on the assumption that the vacuum expectation value of $\mathcal{T}$ scales as a power of mass $m^{2\Delta_n^\alpha}$ with $\Delta_1^\alpha = 0$ will inevitably produce a result of this type. The reason why [218] obtain something different is not entirely clear to us. It is evident though that they assume a different type of scaling for the VEV. In fact the formula for Δ_n^α used there is not zero at $n = 1$. As mentioned before, this is inconsistent with the integral form they use for Δ_n^α , similar to our (3.224). Additionally, as we can see from (3.239) the equality between the limit and the derivative relies on the presence in Δ_n^α of a simple zero at $n = 1$. If this were not true, as it is the case in [218], the leading contribution to the von Neumann entropy would be divergent due to the denominator $1 - n$ of the Rényi entropies. In our case, using the formula (3.224), we have that the leading contribution to the entropy is given by

$$\begin{aligned} \lim_{n \rightarrow 1} \frac{\log(\varepsilon^{4\Delta_n^\alpha} (\langle \mathcal{T} \rangle_n^\alpha)^2)}{1 - n} &= \lim_{n \rightarrow 1} \frac{\log((m\varepsilon)^{4\Delta_n^\alpha} v(n, \alpha)^2)}{1 - n} = \\ &= \log(m\varepsilon) \lim_{n \rightarrow 1} \frac{4\Delta_n^\alpha}{1 - n} + \lim_{n \rightarrow 1} \frac{2 \log v(n, \alpha)}{1 - n}, \end{aligned} \quad (3.249)$$

where $v(n, \alpha)$ is the generalisation of the constant in (3.236) and satisfies $v(1, \alpha) = 1$. Since we do not know the explicit form of this function, we call

$$\lim_{n \rightarrow 1} \frac{2 \log v(n, \alpha)}{1 - n} = - \left. \frac{\partial}{\partial n} v(n, \alpha) \right|_{n=1} := U(\alpha), \quad (3.250)$$

As for the main term we have that

$$\lim_{n \rightarrow 1} \frac{4\Delta_n^\alpha}{1 - n} = -4 \left. \frac{\partial \Delta_n^\alpha}{\partial n} \right|_{n=1}. \quad (3.251)$$

where from (3.224)

$$-4 \left. \frac{\partial \Delta_n^\alpha}{\partial n} \right|_{n=1} = -\frac{1}{8} \int_{-\infty}^{\infty} d\theta \frac{\tanh^2 \frac{\theta}{2}}{\cosh^2 \frac{\theta}{2}} \left| \frac{\sin(\frac{\alpha}{2} \sinh \theta)}{\frac{\alpha}{2} \sinh \theta} \right| e^{\frac{\alpha\theta}{\pi} \sinh \theta} := -\frac{\hat{c}(\alpha)}{3}. \quad (3.252)$$

We can then write that, for the Ising field theory

$$S(r, \alpha) = -\frac{\hat{c}(\alpha)}{3} \log(m\varepsilon) + U(\alpha) - \frac{1}{8} K_0(2mr) + \mathcal{O}(e^{-3mr}) \quad \text{for } \alpha < 0, \quad (3.253)$$

with

$$\hat{c}(0) = \frac{3}{8} \int_{-\infty}^{\infty} d\theta \frac{\tanh^2 \frac{\theta}{2}}{\cosh^2 \frac{\theta}{2}} = \frac{1}{2}. \quad (3.254)$$

As a function of α , $\hat{c}(\alpha)$ is different from the one obtained in (3.173) or (3.198) using the c -theorem, even though both reduce to the correct value at $\alpha = 0$ and display very similar dependence in α as shown in Fig. 3.16 (this can be compared to Fig. 3.12 (right) and to Fig. 3.13.)

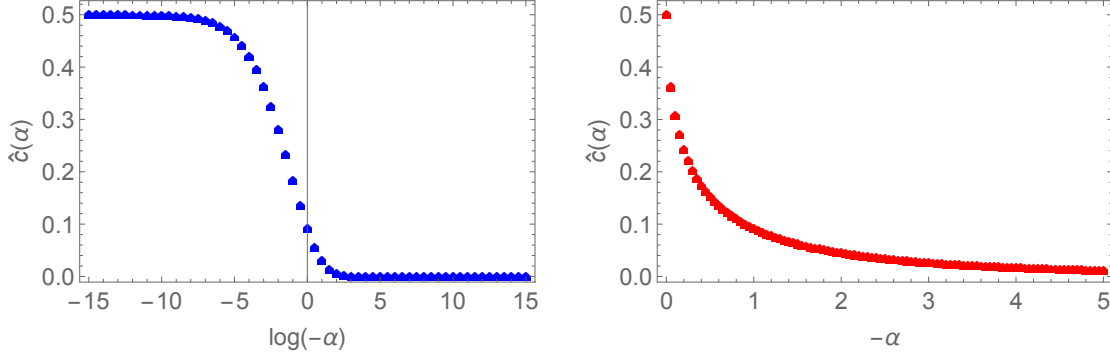


Figure 3.16: The function $\hat{c}(\alpha)$ for $\alpha < 0$ in the Ising field theory, as defined in (3.252). Although a distinct function from the function obtained from Zamolodchikov’s c -theorem, the share the same main features. As expected $\hat{c}(0) = 1/2$.

Our results do not mean that unusual terms such as $\log^2(m\varepsilon)$ are not possible at all, so they do not necessarily contradict [217]. Such terms could still be present. What our results show is that this would require the VEV of the BPTF to scale differently with the mass. VEVs are notoriously difficult to compute in QFT so this is a point that requires further investigation. A different scaling of the VEV is found in some simple unperturbed theories, like the non-compactified free boson (see [236, 237] for a detailed discussion), where the unusual scaling gives rise to a $\log \log(m\varepsilon)$ correction to the entropy. A similar type of correction appears naturally in computations of the symmetry resolved entanglement entropy [238, 239] but in those cases the relevant function is not the VEV but its Fourier transform in a certain variable. More precisely, in order to obtain a correction proportional to $\log^2(m\varepsilon)$ it would be necessary to assume that

$$\langle \mathcal{T} \rangle_n^\alpha = v(n, \alpha) m^{2\Delta_n^\alpha + \kappa(n, \alpha)(1-n) \log m}, \quad (3.255)$$

with $\kappa(1, \alpha) \neq 0$, so that the calculation (3.249) would give instead (leaving out the cut-off for now)

$$\begin{aligned} \lim_{n \rightarrow 1} \frac{\log(\langle \mathcal{T} \rangle_n^\alpha)^2}{1-n} &= \log m \lim_{n \rightarrow 1} \frac{4\Delta_n^\alpha + 2\kappa(n, \alpha)(1-n) \log m}{1-n} + \lim_{n \rightarrow 1} \frac{2 \log v(n, \alpha)}{1-n} = \\ &= -\frac{\hat{c}(\alpha)}{3} \log m + 2\kappa(1, \alpha) \log^2 m + U(\alpha), \end{aligned} \quad (3.256)$$

with $U(\alpha)$ defined as in (3.250). We can then reintroduce the cut-off by replacing $m \mapsto m\varepsilon$ which is equivalent to adding a non-universal constant to (3.256) as well as making the coefficient of the $\log m$ term non-universal (it will become cut-off dependent as well). This is because

$$\begin{aligned} &-\frac{\hat{c}(\alpha)}{3} \log(m\varepsilon) + 2\kappa(1, \alpha) \log^2(m\varepsilon) + U(\alpha) \\ &= \left(-\frac{\hat{c}(\alpha)}{3} + 2 \log \varepsilon \right) \log m + 2\kappa(1, \alpha) \log^2 m + U(\alpha) + d(\varepsilon), \end{aligned} \quad (3.257)$$

with

$$d(\varepsilon) = 2\kappa(1, \alpha) \log^2 \varepsilon - \frac{\hat{c}(\alpha)}{3} \log \varepsilon. \quad (3.258)$$

3.2.6 Some Thoughts

Let us take a moment to reflect on what we have learned so far, to summarise the insights gained, the specific points discovered, and the open challenges ahead. We found that the most general solution to the MFF equations (3.91) is

$$F_{\min}(\theta; \boldsymbol{\alpha}) = F_{\min}(\theta; \mathbf{0}) \varphi_{\boldsymbol{\alpha}}(\theta) C_{\boldsymbol{\beta}}(\theta), \quad (3.259)$$

where $F_{\min}(\theta; \mathbf{0})$ is the MFF of the undeformed theory, $\varphi_{\boldsymbol{\alpha}}(\theta)$ is given by (3.94) while the functions $C_{\boldsymbol{\beta}}(\theta)$ given in (3.95) are characterised by the presence of a set of free parameters $\boldsymbol{\beta}$, which are absent in the S -matrix. These function is a sort of CDD-factor of the MFF, in the sense that any solution $\varphi_{\boldsymbol{\alpha}}(\theta)$ can be multiplied by a function $C_{\boldsymbol{\beta}}(\theta)$ for any choice of parameters $\boldsymbol{\beta}$ and any choice of integers $s \in \mathcal{S}'$ and still give a solution to (3.259). The presence of factors like $C_{\boldsymbol{\beta}}(\theta)$ clearly changes the asymptotic properties of the MFF, hence also of higher particle form factors, where products of $F_{\min}(\theta; \boldsymbol{\alpha})$ are typically involved. According to the usual interpretation that each distinct solution to the form factor equations corresponds to a different local or semi-local field, it is tempting to interpret each choice of $\boldsymbol{\beta}$, that is, each choice of the MFF, as the choice of a different building block for the form factors of local and semi-local fields. However until now we were not able to fix the free parameters and we decided to consider the case $\boldsymbol{\beta} = \mathbf{0}$ and try to solve the kinematical residue equations without them.

We obtain equations whose solutions display an unusual dependence on the rapidities, typically double-exponential and oscillatory. Indeed, already the MFF contains a factor that can grow doubly-exponentially as a function of rapidity differences. There is an infinite set of possible deformations, corresponding to different parameter choices, but the leading behaviours are essentially determined by the term associated with the highest spin s^* entering the scattering matrix (2.43) and the minimal form factor (3.94), both characterised by the coupling α_{s^*} .

- For $\alpha_{s^*} > 0$, correlators diverge at short distances, i.e. in the UV limit. The UV regime is thus described by excitations of finite positive length, and the divergence reflects the fact that the deep UV is no longer accessible. A similar issue arises in the TBA framework, where for Gibbs ensembles the choice $\alpha_{s^*} > 0$ leads to convergence problems in the recursive solution of the equations. In addition, correlators also diverge at large distances unless a rapidity cut-off is introduced. Physically, even if r is large, sufficiently high momenta probe the finite size of excitations and spoil convergence. At low rapidities convergence is maintained. The natural cut-off is described by the Lambert W -function, which also appears in delay equations [207] and in the GHD description of $\overline{\text{T}}\overline{\text{T}}$ -perturbed models [146].
- For $\alpha_{s^*} < 0$, the form factor expansion converges rapidly at all scales, even faster than in the undeformed model. At short distances, correlators scale as power laws $r^{-4z^{\mathcal{O}}(\boldsymbol{\alpha})}$, with exponents $z^{\mathcal{O}}(\boldsymbol{\alpha}) \leq z^{\mathcal{O}}(\mathbf{0})$ that decrease with $-\alpha_{s^*}$ and vanish as

$$\lim_{\alpha_{s^*} \rightarrow -\infty} z_n^{\mathcal{O}}(\boldsymbol{\alpha}) = 0. \quad (3.260)$$

For the field μ this behaviour is confirmed both from the short-distance scaling and from the Δ -sum rule in the two-particle approximation. Zamolodchikov’s c -theorem also shows that the perturbed c -function is a rapidly decreasing function of α .

- The picture suggested is therefore that for $\alpha_{s*} < 0$ the perturbed IQFT flows between a UV regime characterised by finite, α -dependent critical exponents and an effective central charge, and a trivial IR fixed point dictated by the mass scale.

For the cases analysed we find power-law scaling at short distances, with exponents $z^{\mathcal{O}}(\alpha)$ that also display a non-monotonic, oscillatory dependence on α . As known from the literature, the UV fixed point cannot be a standard CFT, so the interpretation of these exponents and of the Δ ’s and c obtained from the sum rules remains an open question. For example, the two-particle estimate of $F_{\mu}(\alpha)$ decreases from $1/16$ to zero and resembles $z_2^{\mu}(\alpha)$, but the two functions are not identical. Moreover, including higher-particle contributions in the correlator $\langle \mu(0)\Theta(r) \rangle$ can introduce terms that are not positive or even real, since the form factors contain square roots of non-positive functions. Thus the Δ -sum rule result differs from the short-distance power in $\langle \mu(0)\mu(r) \rangle$, where all contributions must be positive.

In the Ising model, an additional feature is that the unperturbed form factors diverge for large rapidities, leading to sharp discontinuities in $z_2^{\mu}(\alpha)$, $z_2^{\varepsilon}(\alpha)$ exactly at $\alpha = 0$. This shows that not only are the $\alpha > 0$ and $\alpha < 0$ regimes distinct, but also that $\alpha = 0$ cannot be simply obtained as a limit from either side. These considerations suggest that both the Δ -sum rule and c -theorem require modifications in the presence of finite particle length. Our results for the c -function differ strongly from the TBA version: in particular, for $\alpha < 0$ the TBA c -function becomes complex in the UV, reflecting the Hagedorn transition [81], while the standard c -theorem forbids such behaviour because the correlator of Θ is positive-definite. A possible interpretation is that for $\alpha < 0$ particles acquire a negative length, in analogy with the hard rod model [146], in the sense that additional free space is effectively introduced between the particles, leading to a scattering delay proportional to α . In this picture, $\alpha > 0$ corresponds to a positive rod length and $\alpha < 0$ to a negative rod length, consistent with the scattering phase shift and the thermodynamic properties of the model. When the length is negative, i.e., there is a minimum length in time, the theory remains effectively free, which may provide an explanation for the power-law scaling observed in this regime.

Another unusual feature of our solutions is the appearance of square roots, reminiscent of finite-temperature and finite-volume form factors, which also involve square-root factors built from TBA quantities [240, 241, 242, 243]. This suggests a deeper connection between $T\bar{T}$ -deformed and thermal/finite-volume form factors that seems very attractive to us.

Moreover, analyticity issues remain, as the choice of minimal form factor is not unique. In standard IQFT the parameters β are fixed by analyticity, but in the $T\bar{T}$ context different choices can lead to non-analytic behaviour. Whether this has a physical interpretation or should be fixed by a “natural” prescription is still unclear.

Finally, we turn to the branch point twist field. Our result for its minimal form factor coincides exactly with that of [218], but some conclusions regarding the Δ -sum rule and the saturation value of the entropy in the Ising field theory differ. We attribute this discrepancy at least in part to an error in the evaluation of the Δ -sum rule in [218], as just discussed. The main conclusion is the following. Starting from the two-particle form factor solution (3.219) and assuming the Δ -sum rule provides a reliable characterisation of the scaling of the vacuum expectation value of the twist field, the von Neumann entropy

of the Ising field theory saturates to a value different from that of the undeformed theory. For a $\text{T}\bar{\text{T}}$ -perturbation with $\alpha < 0$, the saturation value reads

$$-\frac{\hat{c}(\alpha)}{3} \log(\varepsilon m) + U(\alpha), \quad (3.261)$$

where $\hat{c}(\alpha)$ and $U(\alpha)$ are α -dependent coefficients. In particular, $\hat{c}(\alpha)$ reduces to the central charge c of the UV undeformed CFT for $\alpha = 0$, and decreases in absolute value for $|\alpha| > 0$. The next-to-leading order correction to the von Neumann entropy remains unchanged compared to the undeformed theory, in agreement with [218]. In any case, we have not been able to uncover much about the entanglement properties. Unfortunately, we were not able to compute higher-particle form factors. Moreover, the Δ -sum rule, even for the twist fields, provides limited information, and as already discussed, its applicability is far from guaranteed. Undoubtedly, the main problem at the moment remains the minimal form factor. Without fully fixing it, we cannot even be certain about the higher-particle form factors or the general validity of the form factor bootstrap program. The next and final chapter will be entirely devoted to addressing this issue.

Chapter 4

Complete Minimal Form Factors of $T\bar{T}$ -deformed IQFTs

The first problem that needs to be addressed in order to correctly formulate the FFBP for $T\bar{T}$ -deformed IQFTs concerns the minimal form factor (MFF), i.e. the minimal solution of Watson’s equations (3.32). Indeed, we have not been able to derive the MFF from first principles: instead, we simply “guessed” a possible solution. This approach has been only partially successful. While we managed to capture the correct general structure, we did not obtain a unique solution. An infinite number of free parameters remain, and consequently the validity of the explicit expressions we found is, at best, questionable.

A very important results for us comes from [244]. In this paper by my collaborators it was in fact understood that even for standard IQFTs the MFF can be casted in the representation (3.259). In fact, even standard block-type S-matrices can be regarded as genuine, though *regular*, CDD-factors: they satisfy crossing and unitarity and have a well-behaved UV limit. Consider the sinh-Gordon S-matrix,

$$S(\theta) = \frac{\sinh \theta - i \cos \frac{\pi b}{2}}{\sinh \theta + i \cos \frac{\pi b}{2}}. \quad (4.1)$$

Expanding the logarithmic derivative at large rapidities and symmetrising under $\theta \rightarrow -\theta$, we can formally rewrite it in a generalised $T\bar{T}$ -like form,

$$S(\theta) = -\exp \left[-4i \sum_{s \in 2\mathbb{Z}^+ - 1} \frac{i^{s+1}}{s} \cos \frac{s\pi b}{2} \sinh(s\theta) \right] = S_0(\theta) \Phi_\alpha(\theta), \quad S_0(\theta) = -1, \quad (4.2)$$

where the non-vanishing couplings are

$$\alpha_s m^{2s} = \frac{4i^{s+1}}{s} \cos \frac{s\pi b}{2}, \quad s \in 2\mathbb{Z}^+ - 1. \quad (4.3)$$

The series in (4.2) should be understood as a formal expansion rather than as a convergent series; the corresponding object is defined non-perturbatively, and the series only serves as a convenient parametrisation of its structure and shows that the sinh-Gordon S-matrix can be interpreted as an Ising theory deformed by infinitely many irrelevant perturbations, with the couplings α_s encoding an infinite tower of parameters. At this point, it is useful to clarify the rôle played by the different representations of integrable S-matrices that will appear in the following, in order to avoid possible sources of confusion.

It is well known that many minimal building blocks, including the sinh-Gordon S-matrix, admit representations in terms of infinite products of Euler Gamma functions. These representations arise naturally in the explicit solution of the factorisation problem for the S-matrix: once unitarity, crossing symmetry and analyticity are imposed, and the positions of poles and zeros are fixed by the semiclassical bound-state spectrum, the resulting minimal solution is essentially determined and can be conveniently expressed as an infinite Gamma-product. This mechanism was first exhibited in the sine-Gordon model by Zamolodchikov [61]. Starting from such infinite-product expressions, it is then possible to obtain equivalent integral or exponential representations by employing standard integral formulas for $\log \Gamma$, such as the Malmsten representation. This step does not introduce new physical input, but provides a technically useful and more elegant reformulation of the same analytic structure. Approaches of this type were systematically developed by Karowski and Weisz [65], and play a central rôle in the construction of minimal form factors. Independently of this construction, one may also consider multiplicative deformations of a given S-matrix associated with irrelevant operators, such as those induced by $T\bar{T}$ deformations. These deformations are naturally formulated in terms of exponential representations involving an additive factorisation of $\log S(\theta)$, as in (2.43). Multiplicative deformations of a given S-matrix associated with irrelevant operators, such as those induced by $T\bar{T}$ deformations, provide yet another way of parametrising the same scattering data. These deformations are naturally formulated in terms of exponential representations involving an additive factorisation of $\log S(\theta)$, as in (2.43). While, in concrete examples, representations of this type can indeed be related to the analytic structure encoded by infinite Gamma-products, such a relation is highly non-trivial and does not follow directly from the existence of the latter. Establishing this connection requires an explicit reconstruction of the S-matrix from its singularity structure, rather than a mere inspection of its infinite-product form. Once the S-matrix is written in this form, one can proceed to construct the corresponding minimal form factor.

Once the S-matrix is written in the form (4.2), one can construct a minimal form factor representation. Using the integral representation, an equivalent $T\bar{T}$ -like representation can also be obtained [244]:

$$\begin{aligned} \log F_{\min}^{\text{shG}}(\theta) = & \frac{1}{2} \log 2 + \log \sinh \frac{\theta}{2} - \frac{1}{2\pi} (\theta - i\pi) \log \frac{i \cos \frac{\pi b}{2} + \sinh \theta}{i \cos \frac{\pi b}{2} - \sinh \theta} \\ & - \frac{1}{4} \log \left[(-\cosh \theta + \sin \frac{\pi b}{2})(-\cosh \theta - \sin \frac{\pi b}{2}) \right] - \frac{b}{4} \log \frac{-\cosh \theta + \sin \frac{\pi b}{2}}{-\cosh \theta - \sin \frac{\pi b}{2}} \\ & - \frac{i}{4\pi} \left[\text{Li}_2(i e^{-\theta - i\frac{\pi}{2}b}) - \text{Li}_2(-i e^{-\theta - i\frac{\pi}{2}b}) + \text{Li}_2(i e^{-\theta + i\frac{\pi}{2}b}) - \text{Li}_2(-i e^{-\theta + i\frac{\pi}{2}b}) + (\theta \rightarrow -\theta) \right]. \end{aligned} \quad (4.4)$$

The structure of this MFF is transparent:

- The term $\log \sinh \frac{\theta}{2}$ reproduces the Ising MFF and isolates $S_0 = -1$.
- The term $-\frac{i(\theta - i\pi)}{2\pi} \log \Phi_{\alpha}^{\text{shG}}(\theta)$ corresponds to the logarithmic structure of the $T\bar{T}$ -like S-matrix with the parameters α_s defined in (4.3).
- The remaining terms, as shown in [244] can be collected in $C_{\beta}^{\text{shG}}(\theta)$, the term we were not able to fix previously. This contribution ensures analyticity and proper

asymptotics: unphysical poles get cancelled from the logarithmic term and the large-rapidity behaviour is tamed, guaranteeing convergence of form factor expansions and cluster decomposition.

Thus, the sinh-Gordon example demonstrates how the minimal form factor arises from a subtle interplay of CDD-factors, analyticity, asymptotics, and the infinite tower of α_s couplings. It provides a concrete laboratory to understand what features are required for physically consistent solutions in more general $T\bar{T}$ -like deformations, where the correct choice of parameters analogous to $C_\beta^{\text{shG}}(\theta)$ is far from obvious. The fact that an infinite number of deformations is required to describe a well-behaved IQFT is in fact quite natural. Recall that the operators generating the generalised $T\bar{T}$ flows (2.10) are constructed directly from the conserved currents of a massive integrable field theory, and that there is a one-to-one correspondence between each such current and a conserved charge of the model. As a consequence, reconstructing a given interacting IQFT starting from a free fermionic theory necessarily requires access to the full integrable structure, namely to the complete set of infinitely many conserved charges. This viewpoint can be usefully interpreted in geometric terms [167, 150]. One may regard the space of integrable quantum field theories as endowed with a connection determined by the conserved charges, and the sequence of $T\bar{T}$ -like deformations as providing a way to transport a theory along this space. In this sense, reproducing a fully consistent IQFT requires accounting for the entire tower of deformations, much as parallel transport depends on the full connection rather than on a finite subset of its components.

4.1 A Straightforward Extension to Theories with Boundaries

The analysis carried out in [244] can be immediately extended to the case with boundaries. This is precisely what my collaborators and I have done in [15]. We already introduced the basics new scattering amplitude $R_a(\theta)$, i.e. the reflection amplitude off the boundary (see Section 1.5). Unitarity and crossing relations lead to the constraints of cross-unitarity [83, 88]:

$$R_a(\theta)R_a(-\theta) = 1 \quad \text{and} \quad R_a(\theta)R_{\bar{a}}(\theta + i\pi) = S_{aa}(2\theta), \quad (4.5)$$

where $\bar{a}$ is the particle conjugate to a . As we can see, reflection amplitudes are related to the scattering phase. Thus, when this changes, like in the presence of irrelevant perturbations, we expect $R_a(\theta)$ to also change. In the presence of stable bound states [86, 87] there are additional requirements for the functions $R_a(\theta)$ in the form of boundary bootstrap equations. These take the form:

$$R_a(\theta + i\eta_{ac}^b)R_b(\theta + i\eta_{bc}^a)S_{ab}(\theta + i\eta_{bc}^a + i\eta_{ac}^b) = R_c(\theta), \quad (4.6)$$

where η_{ab}^c are values related to the position of the poles of the scattering matrix. If the S -matrix $S_{ab}(\theta)$ has a pole at $\theta = iu_{ab}^c$ corresponding to the formation of the bound state c in the process $a + b \mapsto c$, then $\eta_{ab}^c := \pi - u_{ab}^c$. Systematic solutions to these equations have been famously constructed for affine Toda field theories [245, 246, 247].

This program can be further extended by considering the possibility of a “dynamical” boundary, namely a boundary which can be excited to a different state by particle collision. This is associated with a pole of the reflection amplitudes themselves. This possibility was first put forward in [248]. Notably, it is fully compatible with integrability. If we label

the type of boundary by capital letters, then we can see this as the process $a + A \mapsto B$. In this case, boundary reflection amplitudes acquire an extra index $R_a^A(\theta)$ and there are additional boundary bootstrap equations

$$R_a^A(\theta) = R_a^B(\theta) S_{ab}(\theta + i\eta_{aA}^B) S_{ab}(\theta - i\eta_{aA}^B), \quad (4.7)$$

where $\theta = i\eta_{aA}^B$ is a pole of the amplitude $R_a^A(\theta)$. The solutions to these equations in Toda field theories were studied in great detail in [249]. Once solutions to these equations have been found, they can be employed as input data in the study of the thermodynamic properties of massive boundary IQFTs as done in [250, 251]. They can also be employed in the context of computing correlation functions and their building blocks (form factors). This may be done either by employing the boundary state as proposed in [83] or by developing a form factor program for boundary IQFTs, as done in [252] and employed for example in [253, 254, 255, 256, 257, 258, 259].

4.1.1 Reflection Amplitudes, Irrelevant Perturbations and the Boundary Form Factor Bootstrap Program

Consider a diagonal theory, i.e. with no back-scattering¹. Then the deformation takes the form

$$S_{ab}^\alpha(\theta) := S_{ab}(\theta) \Phi_{ab}^\alpha(\theta) \quad \text{with} \quad \log \Phi_{ab}^\alpha(\theta) = -i \sum_{s \in \mathcal{S}} m_a^s m_b^s \alpha_s \sinh(s\theta), \quad (4.8)$$

where $S_{ab}(\theta)$ is the original S -matrix associated to the process $a + b \mapsto a + b$, with a, b particle quantum numbers. The masses of these particles are denoted by m_a, m_b and α_s are couplings such that the combinations $m_a^s m_b^s \alpha_s$ are dimensionless. Let $R_a(\theta)$ be a reflection amplitude which preserves integrability and has no pole leading to excited boundary states. In this case the only relevant equations for $R_a(\theta)$ are (4.5). We will now promote $R_a(\theta)$ to $R_a^\alpha(\theta)$ to denote the deformed solution to equations (4.5) corresponding to the deformed S -matrix (2.43). We expect that

$$R_a^\alpha(\theta) = R_a(\theta) \Lambda_a^\alpha(\theta), \quad (4.9)$$

for some function $\Lambda_a^\alpha(\theta)$ which satisfies

$$\Lambda_a^\alpha(\theta) \Lambda_a^\alpha(-\theta) = 1 \quad \text{and} \quad \Lambda_a^\alpha(\theta) \Lambda_a^\alpha(\theta + i\pi) = \Phi_{aa}^\alpha(2\theta). \quad (4.10)$$

It is very easy to see that these equations are solved by

$$\Lambda_a^\alpha(\theta) = \sqrt{\Phi_{aa}^\alpha(2\theta)}. \quad (4.11)$$

This is the standard type of solution, namely a $2\pi i$ periodic, odd function of 2θ and it agrees with the boundary scattering factor found in [260] (see also [261] for some recent advancements). It should be emphasised that the square-root relation between the reflection factor and $S(2\theta)$ in (4.11) is valid **only** for CDD-factors of $\text{T}\bar{\text{T}}$ -type. For block-type S -matrices, such as the sinh-Gordon model (1.62), this relation does not hold for the exact coupling-dependent CDD-factor written in closed form. It becomes meaningful only after rewriting the latter in its formal $\text{T}\bar{\text{T}}$ -like representation (4.2), namely as an infinite

¹In this section on theories with boundaries we will, for the first and only time, briefly consider the case with more than one particle species. For this reason, we slightly modify our notation to accommodate the presence of particle species indices.

product (or asymptotic expansion) of $T\bar{T}$ -type CDD factors, as discussed at the beginning of this Chapter. Equation (4.11) gives the universal change of the reflection amplitudes in boundary IQFTs after a generalised $T\bar{T}$ -perturbation. The solutions for $\Lambda_a^\alpha(\theta)$ can however be more general than this. While the factor $\sqrt{\Phi_{aa}^\alpha(2\theta)}$ needs to be there, any function of the type $\sinh(k\theta)$ with k odd can be added to the exponent, providing a new solution to (4.5). In general we have

$$\Lambda_a^\alpha(\theta) = \sqrt{\Phi_{aa}^\alpha(2\theta)} \exp \left[-i \sum_{k \in \mathbb{Z}} \gamma_k m^{2(2k+1)} \sinh((2k+1)\theta) \right]. \quad (4.12)$$

The fact that the modification is not unique for a given S -matrix is expected since in general there are multiple boundary conditions allowed for one single scattering amplitude². From here onwards we will consider the simplest solution only (with $\gamma_i = 0$). The question of how the reflection amplitudes are deformed under irrelevant perturbations has already been discussed in the literature a few years ago [260] and then employed to develop a generalised boundary thermodynamic Bethe ansatz. Here we proceed instead to discuss the form factor program in the presence of boundaries.

There are two viewpoints we may take. If the boundary is located at the origin of time it can be represented by a boundary state in the Ghoshal-Zamolodchikov sense [94]. In this case matrix elements of local fields may be computed in terms of the matrix elements obtained in the absence of a boundary, assuming these are known via the standard form factor program [6, 10]. This can be achieved by expanding the boundary state in terms of bilinears of the Zamolodchikov-Fadeev algebra (see Eq. (1.85)). It is also possible to think of the boundary as located in space, say at the origin. In this case, one can derive a set of modified form factor equations for the matrix elements of local fields which now must take into account scattering processes off the boundary. These equations were presented in [252]. For our purposes, we are only interested in the equations for the one-particle form factors of a local field $\mathcal{O}$, namely:

$$F_a^\mathcal{O}(\theta) = R_a(\theta) F_a^\mathcal{O}(-\theta) \quad \text{and} \quad F_a^\mathcal{O}(\theta) = R_a(i\pi - \theta) F_a^\mathcal{O}(2\pi i - \theta), \quad (4.13)$$

where

$$F_a^\mathcal{O}(\theta) := \langle 0 | \mathcal{O}(0) | \theta \rangle_a, \quad (4.14)$$

with $|\theta\rangle_a$ an in-state containing a single particle of species a and $|0\rangle$ the vacuum state. Due to breaking of translation invariance, the one-particle form factor is rapidity dependent, even for spinless fields. Thus, it is the simplest non-trivial form factor that may be computed and a building block for higher particle form factors.

The solution procedure for equations (4.13) was presented in [252] and follows the usual construction. Starting with an integral representation for $R_a(\theta)$, a corresponding integral representation of $F_a^\mathcal{O}(\theta)$ can be found in line with the usual procedure [6] highlighted in Subsection 3.1.2. The minimal solution to the equations, without poles in the physical strip, is entirely determined by $R_a(\theta)$ whereas the pole structure of $F_a^\mathcal{O}(\theta)$ is

²Many interesting examples are known. The simplest cases are the Ising field theory, where a family of distinct boundary conditions exist parametrised by the boundary magnetic field [83] as we saw in Section 1.5. Similarly, the sinh-Gordon model admits a two-parameter family of solutions, as found in [262]. We will discuss these two models in a moment.

related to the operator $\mathcal{O}$.

Let $r_a^{\min}(\theta)$ be a minimal solution of (4.13). What would be its deformed version in the presence of a generalised TT perturbation? Following [12, 14, 218] we observe that the equations (4.13) are linear and factorised. Therefore, we expect that the modified minimal form factor to be of the form

$$r_a^{\min}(\theta; \boldsymbol{\alpha}) = r_a^{\min}(\theta) \varphi_a^{\boldsymbol{\alpha}}(\theta), \quad (4.15)$$

with

$$\varphi_a^{\boldsymbol{\alpha}}(\theta) = \Lambda_a^{\boldsymbol{\alpha}}(\theta) \varphi_a^{\boldsymbol{\alpha}}(-\theta) \quad \text{and} \quad \Lambda_a^{\boldsymbol{\alpha}}(\theta) \varphi_a^{\boldsymbol{\alpha}}(\theta) = \varphi_a^{\boldsymbol{\alpha}}(2\pi i - \theta), \quad (4.16)$$

which is solved by

$$\log \varphi_a^{\boldsymbol{\alpha}}(\theta) = \frac{2\theta - i\pi}{2\pi} i \log \Lambda_a^{\boldsymbol{\alpha}}(\theta) = \frac{2\theta - i\pi}{4\pi} i \log \Phi_{aa}^{\boldsymbol{\alpha}}(2\theta). \quad (4.17)$$

However, there is a larger family of solutions. Indeed, the exponent above, can be modified by a sum of $\cosh(k\theta)$ functions with $k \in \mathbb{Z}$, and still satisfy all requirements. We therefore find that the most general minimal solution to (4.13) is

$$r_a^{\min}(\theta; \boldsymbol{\alpha}, \boldsymbol{\beta}) = r_a^{\min}(\theta) \varphi_a^{\boldsymbol{\alpha}}(\theta) C_a^{\boldsymbol{\beta}}(\theta), \quad (4.18)$$

with $\varphi_a^{\boldsymbol{\alpha}}(\theta)$ given by (4.17) and

$$\log C_a^{\boldsymbol{\beta}}(\theta) = \sum_{s \in S'} \beta_s m_a^{2s} \cosh(s\theta). \quad (4.19)$$

Thus, the solution is parametrised by parameters $\boldsymbol{\alpha}$, which are determined by the deformation of the S -matrix, and $\boldsymbol{\beta}$ which can in principle be freely chosen.

The presence of free parameters in the minimal form factor is an issue that we already encountered in the case of theories without boundaries, as discussed in Chapter 3, and whose meaning is, for the time being, not fully understood. In standard IQFTs, however, the parameters $\boldsymbol{\beta}$ are completely fixed by means of an integral representation, which is always well defined provided that the functions display a good asymptotic behaviour, namely that they decay sufficiently fast at large rapidities. In the following, we will focus on the case of the Sinh-Gordon model, which, as in the previous section, serves both as a fundamental building block and as a representative example for any other case one may wish to address.

4.1.2 A New Representation for the Boundary Minimal Form Factors

The sinh-Gordon model with Dirichlet boundary conditions was one of the examples considered in [252] and later in [254]. This particular choice of boundary conditions has the advantage that the minimal form factor $r_a^{\min}(\theta)$ coincides with the one-particle form factor, that is, there are no additional poles to be included, which makes calculations particularly simple. In this case non-vanishing form factors associated with odd particle numbers can be identified as corresponding to the operator $\partial_x \phi$, where ϕ is the sinh-Gordon field. More generally, as reported in [262], there is a two-parameter family of

solutions for the reflection amplitudes of the sinh-Gordon model. They can be written in terms of fundamental blocks $(x)_\theta$, $[x]_\theta$ as

$$R(\theta, E, F) = \left(\frac{1}{2}\right)_\theta \left(\frac{2+B}{4}\right)_\theta \left(1 - \frac{B}{4}\right)_\theta \left[\frac{E-1}{2}\right]_\theta \left[\frac{F-1}{2}\right]_\theta. \quad (4.20)$$

Recalling here the definition of the fundamental blocks³ we gave in Chapter 1

$$(x)_\theta := \frac{\sinh \frac{1}{2}(\theta + i\pi x)}{\sinh \frac{1}{2}(\theta - i\pi x)}, \quad [x]_\theta = -(x)_\theta(1-x)_\theta = \frac{\tanh \frac{1}{2}(\theta + i\pi x)}{\tanh \frac{1}{2}(\theta - i\pi x)}. \quad (4.21)$$

The sinh-Gordon two-body scattering matrix is simply

$$S(\theta) = \left[-\frac{B}{2}\right]_\theta = \frac{\sinh \theta - i \sin \frac{\pi B}{2}}{\sinh \theta + i \sin \frac{\pi B}{2}}, \quad (4.22)$$

with $B \in [0, 2]$ the coupling constant. Here, for the first time, we adopt an alternative parametrisation of the coupling, using the capital letter B instead of the small b introduced in (1.62). The two are related simply by $b = B - 1$. The simplest version of (4.20) is obtained by removing the F -dependent factor and by setting $E = 0$, while introducing an overall minus sign (this is due to the particular definition of our $[x]_\theta$ symbol). This corresponds to Dirichlet boundary conditions that fix the boundary field to 0. In that special case, the amplitude (4.20) reduces to

$$R(\theta) = -\left(-\frac{1}{2}\right)_\theta \left(\frac{2+B}{4}\right)_\theta \left(1 - \frac{B}{4}\right)_\theta. \quad (4.23)$$

This choice also cancels out the pole of the reflection amplitude at $\theta = \frac{i\pi}{2}$ that is present in (4.20) due to the block $(\frac{1}{2})_\theta$. A special property of this amplitude is that for $B = 0$ it reduces to $R(\theta) = 1$ which corresponds to a free boson solution (the sinh-Gordon S -matrix reduces to 1 for $B = 0$). The minimal form factor solution corresponding to this free boson case is proportional to $\sinh \theta$. In [252], the minimal form factor solution corresponding to (4.23) was given as

$$r^{\min}(\theta) = \frac{\sinh \theta}{i + \sinh \theta} u(\vartheta, B) \quad (4.24)$$

with

$$u(\vartheta, B) = \exp \left[-2 \int_0^\infty \frac{dx}{x} \left(\cos \frac{\vartheta x}{\pi} - 1 \right) \frac{\cosh \frac{x}{2}}{\sinh^2 x} \left(\sinh \frac{x B}{4} + \sinh \left(1 - \frac{B}{2} \right) \frac{x}{2} + \sinh \frac{x}{2} \right) \right], \quad (4.25)$$

where $\vartheta = \frac{i\pi}{2} - \theta$. The normalisation is chosen so that $u(0, B) = 1$. We have that

$$u(\vartheta, 0) = \exp \left[-2 \int_0^\infty \frac{dx}{x} \left(\cos \frac{\vartheta x}{\pi} - 1 \right) \frac{\cosh \frac{x}{2}}{\sinh^2 x} \left(2 \sinh \frac{x}{2} \right) \right] = -\frac{i}{2}(i + \sinh \theta), \quad (4.26)$$

so that for $B = 0$ we recover the free boson solution $r^{\min}(\theta) = -\frac{i}{2} \sinh \theta$. For our purposes however, it is interesting to emphasise the connection with free fermions instead.

³Note that there is a minus sign difference between the definitions of $[x]_\theta$ in [252] and ours (the same as in [244]). Compared to [262] the blocks $(x)_\theta$ differ by a factor 1/2 in the definition of x .

The sinh-Gordon model is a fermionic theory, in the sense that $S(\theta = 0) = -1$ and if we factor out this -1 from the S -matrix $[-B/2]_\theta$ what remains can be seen as a CDD factor. This means that the sinh-Gordon S -matrix is of the type (2.43) with a CDD factor given by a sum over all odd integers and coefficients $m^{2s}\alpha_s$ which are functions of B as we can see from (4.2). Hence, according to our derivation, the minimal form factor (4.25) should also admit a representation of the type (4.18) with (4.17). This is indeed the case even in the presence of boundaries. Now, let

$$\rho(\vartheta, B) := \log u(\vartheta, B). \quad (4.27)$$

then, the derivative w.r.t. ϑ is,

$$\rho'(\vartheta, B) = h(\vartheta, B) + h(\vartheta, 2 - B) + g(\vartheta), \quad (4.28)$$

with

$$h(\vartheta, B) = \frac{2}{\pi} \int_0^\infty dx \frac{\sin \frac{\vartheta x}{\pi} \cosh \frac{x}{2} \sinh \frac{xB}{4}}{\sinh^2 x} \quad \text{and} \quad g(\vartheta) = \frac{1}{\pi} \int_0^\infty dx \frac{\sin \frac{\vartheta x}{\pi}}{\sinh x}. \quad (4.29)$$

We have that $g(\vartheta)$ can be easily integrated to $g(\vartheta) = \frac{1}{2} \tanh \frac{\vartheta}{2}$, while $h(\vartheta, B)$ can be computed using contour integration, along the same lines of the computations presented in [244]. For example, we have the integral

$$I(a, b) = \int_{-\infty}^\infty dx \frac{e^{(b+ia)x}}{\sinh^2 x} = i\pi(b+ia) \frac{1 + e^{i\pi(b+ia)}}{1 - e^{i\pi(b+ia)}}, \quad (4.30)$$

from where it follows

$$\begin{aligned} h(\vartheta, B) &= \frac{1}{8\pi i} \left[I\left(\frac{\vartheta}{\pi}, \frac{2+B}{4}\right) + I\left(\frac{\vartheta}{\pi}, \frac{-2+B}{4}\right) - I\left(\frac{\vartheta}{\pi}, \frac{2-B}{4}\right) - I\left(\frac{\vartheta}{\pi}, \frac{-2-B}{4}\right) \right. \\ &\quad \left. - I\left(-\frac{\vartheta}{\pi}, \frac{2+B}{4}\right) - I\left(-\frac{\vartheta}{\pi}, \frac{-2+B}{4}\right) + I\left(-\frac{\vartheta}{\pi}, \frac{2-B}{4}\right) + I\left(-\frac{\vartheta}{\pi}, \frac{-2-B}{4}\right) \right] \\ &= \frac{4\vartheta \sin \frac{\pi B}{2} - 4\pi \sin \frac{\pi B}{4} \sinh \vartheta + \pi B \sinh(2\vartheta)}{4\pi(\cosh(2\vartheta) + \cos \frac{\pi B}{2})}. \end{aligned} \quad (4.31)$$

Integrating gives

$$\begin{aligned} \rho(\vartheta, B) &= \log a(B) + \log \cosh \frac{\vartheta}{2} - \frac{i\vartheta}{2\pi} \log \left[\frac{\sin \frac{B\pi}{2} + i \sinh(2\vartheta)}{\sin \frac{B\pi}{2} - i \sinh(2\vartheta)} \right] \\ &\quad + \frac{1}{2} \log \left[\cos \frac{B\pi}{4} + \cosh \vartheta \right] + \frac{1}{4} \log \left[\frac{\cosh \vartheta + \sin \frac{B\pi}{4}}{\cosh \vartheta - \sin \frac{B\pi}{4}} \right] - \frac{B}{8} \log \left[\frac{\cosh(2\vartheta) - \cos \frac{\pi B}{4}}{\cosh(2\vartheta) + \cos \frac{\pi B}{4}} \right] \\ &\quad + \frac{i}{4\pi} \left(\text{Li}_2(-ie^{-\vartheta - \frac{i\pi B}{4}}) + \text{Li}_2(ie^{-\vartheta - \frac{i\pi B}{4}}) - \text{Li}_2(-ie^{-\vartheta + \frac{i\pi B}{4}}) - \text{Li}_2(ie^{-\vartheta + \frac{i\pi B}{4}}) + \vartheta \mapsto -\vartheta \right) \\ &\quad - \frac{i}{4\pi} \left(\text{Li}_2(-e^{-\vartheta - \frac{i\pi B}{4}}) + \text{Li}_2(e^{-\vartheta - \frac{i\pi B}{4}}) - \text{Li}_2(-e^{-\vartheta + \frac{i\pi B}{4}}) - \text{Li}_2(e^{-\vartheta + \frac{i\pi B}{4}}) + \vartheta \mapsto -\vartheta \right). \end{aligned} \quad (4.32)$$

Here $\log a(B)$ is an integration constant which can be fixed by asymptotic requirements. Requiring that $\rho(0, B) = 0$ and after some simplifications, we obtain

$$\begin{aligned}\log a(B) &= -\frac{1}{2} \log \left(\frac{\sin \frac{\pi(2+B)}{8}}{\sin \frac{\pi(2-B)}{8}} \right) - \log \left(2 \cos \frac{B\pi}{8} \right) + \frac{B}{4} \log \tan \frac{B\pi}{4} \\ &\quad + \frac{i}{2\pi} \left(\text{Li}_2(-e^{\frac{i\pi B}{2}}) - \text{Li}_2(e^{\frac{i\pi B}{2}}) \right) - \frac{i\pi(B-1)}{8}.\end{aligned}\quad (4.33)$$

It is interesting to consider the various contributions to (4.32):

- The contribution

$$\log \cosh \frac{\vartheta}{2} = \log \cos \frac{1}{2} \left(\frac{i\pi}{2} - \theta \right), \quad (4.34)$$

is such that when taking exponential of $\rho(\vartheta, B)$ it gives a factor $\cosh \frac{1}{2} \left(\frac{i\pi}{2} - \theta \right)$ in the minimal form factor that combines with the prefactor $\frac{\sinh \theta}{\sinh \theta + i}$ in (4.24) to give

$$r_{\text{fixed}}(\theta) = -\frac{i}{2} \frac{\sinh \theta}{\cosh \frac{1}{2} \left(\frac{i\pi}{2} - \theta \right)}, \quad (4.35)$$

which is the minimal form factor corresponding to the Ising model with reflection amplitude

$$R^0(\theta) = \left(-\frac{1}{2} \right)_\theta. \quad (4.36)$$

This is known as the fixed boundary condition of the Ising model, and corresponds to the limit of infinite boundary magnetic field, as discussed in [83, 252].

- The contribution

$$-\frac{i\vartheta}{2\pi} \log \left[\frac{\sin \frac{B\pi}{2} + i \sinh(2\vartheta)}{\sin \frac{B\pi}{2} - i \sinh(2\vartheta)} \right] = \frac{2\theta - i\pi}{4\pi} i \log(-S(2\theta)) = \frac{2\theta - i\pi}{4\pi} i \log \Phi^{\text{shG}}(2\theta), \quad (4.37)$$

where $\Phi^{\text{shG}}(\theta)$ is minus the scattering matrix of the sinh-Gordon model, which can be seen as a CDD factor. Hence, sinh-Gordon emerges as a perturbation of the Ising field theory.

- The remaining terms in (4.32) add up to an even function of θ which admits a formal expansion as a sum of $\cosh(s\theta)$ functions with s integer, both odd and even.

In summary, the minimal form factor $r^{\min}(\theta)$ introduced in (4.24) can be rewritten as

$$r^{\min}(\theta) = a(B) r_{\text{fixed}}(\theta) e^{\frac{2\theta - i\pi}{4\pi} i \log(-S(2\theta))} C^\beta(\theta), \quad (4.38)$$

with $r_{\text{fixed}}(\theta)$ given by (4.35), $S(\theta)$ the sinh-Gordon S -matrix and $C^\beta(\theta)$ given by

$$\begin{aligned}
\log C^\beta(\theta) = & \quad (4.39) \\
& \frac{1}{2} \log \left[\cos \frac{B\pi}{4} + \cosh \vartheta \right] + \frac{1}{4} \log \left[\frac{\cosh \vartheta + \sin \frac{B\pi}{4}}{\cosh \vartheta - \sin \frac{B\pi}{4}} \right] - \frac{B}{8} \log \left[\frac{\cosh(2\vartheta) - \cos \frac{\pi B}{4}}{\cosh(2\vartheta) + \cos \frac{\pi B}{4}} \right] \\
& + \frac{i}{4\pi} \left(\text{Li}_2(-ie^{-\vartheta - \frac{i\pi B}{4}}) + \text{Li}_2(ie^{-\vartheta - \frac{i\pi B}{4}}) - \text{Li}_2(-ie^{-\vartheta + \frac{i\pi B}{4}}) - \text{Li}_2(ie^{-\vartheta + \frac{i\pi B}{4}}) + \vartheta \mapsto -\vartheta \right) \\
& - \frac{i}{4\pi} \left(\text{Li}_2(-e^{-\vartheta - \frac{i\pi B}{4}}) + \text{Li}_2(e^{-\vartheta - \frac{i\pi B}{4}}) - \text{Li}_2(-e^{-\vartheta + \frac{i\pi B}{4}}) - \text{Li}_2(e^{-\vartheta + \frac{i\pi B}{4}}) + \vartheta \mapsto -\vartheta \right) ,
\end{aligned}$$

with $\vartheta = \frac{i\pi}{2} - \theta$ as before. As indicated by (4.19), this function admits a formal expansion in terms of $\cosh(\ell\theta)$ functions. We can show that

$$\begin{aligned}
\log C^\beta(\theta) = & -\frac{1}{2} \log 2 - \sum_{\ell=1}^{\infty} \frac{1}{\ell} \cos \left(\frac{B}{4} \ell \pi \right) \cosh(\ell\theta) \\
& - \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{2\ell+1} \sin \left(\frac{B}{4} (2\ell+1)\pi \right) \cosh((2\ell+1)\theta) \\
& + \frac{B}{2} \sum_{\ell=0}^{\infty} \frac{1}{2\ell+1} \cos \left(\frac{B}{4} (2\ell+1)\pi \right) \cosh(2(2\ell+1)\theta) \\
& - \sum_{\ell=0}^{\infty} \frac{1}{\pi(2\ell+1)^2} \sin \left(\frac{B}{2} (2\ell+1)\pi \right) \cosh(2(2\ell+1)\theta) .
\end{aligned} \quad (4.40)$$

Comparing to (4.19) we identify the coefficients (we take the mass scale $m = 1$)

$$\begin{aligned}
\beta_0 &= -\frac{1}{2} \log(2) , \\
\beta_{2\ell+1} &= -\frac{\cos \left(\frac{B}{4} (2\ell+1)\pi \right) + (-1)^\ell \sin \left(\frac{B}{4} (2\ell+1)\pi \right)}{2\ell+1} , \\
\beta_{4\ell} &= -\frac{\cos(B\ell\pi)}{4\ell} , \\
\beta_{4\ell+2} &= -\frac{\cos \left(\frac{B}{2} (2\ell+1)\pi \right) + B \cos \left(\frac{B}{4} (2\ell+1)\pi \right)}{2(2\ell+1)} - \frac{1}{\pi(2\ell+1)^2} \sin \left(\frac{B}{2} (2\ell+1)\pi \right) .
\end{aligned} \quad (4.41)$$

4.1.3 More General Boundary Conditions

We have just seen that the simplest boundary condition/reflection amplitude (4.23) in the sinh-Gordon theory admits a new interpretation. It can be seen, at the level of the minimal form factor at least, as resulting from irreversibly perturbing the boundary Ising model with fixed boundary conditions. The fixed boundary condition corresponds to taking the magnetic field h , which parametrises all Ising boundary conditions, to infinity. Indeed, the most general reflection amplitude in the Ising model (1.95) can be rewritten in terms of the blocks as

$$R_\kappa(\theta) = -[\kappa]_\theta \left(-\frac{1}{2} \right)_\theta , \quad (4.42)$$

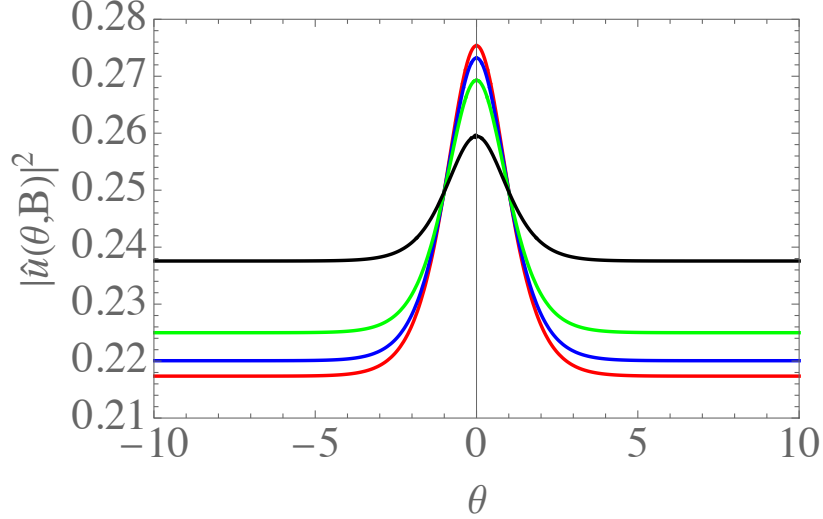


Figure 4.1: The absolute value squared of the function $\hat{u}(\theta, B) := \frac{u(\vartheta, B)}{i + \sinh \theta}$ with $u(\vartheta, B) = \exp \rho(\vartheta, B)$ evaluated numerically from (4.32). The colours correspond to different values of B : $B = 1$ (red), $B = 0.7$ (blue), $B = 0.5$ (green) and $B = 0.2$ (black).

where κ is related to the magnetic field h as $\sin(\pi\kappa) = 1 - \frac{h^2}{2m^2}$. The two “simplest” boundary conditions known as “free” and “fixed”, rewritten in blocks are

- $h = 0$ ($\kappa = \frac{1}{2}$) with reflection amplitude

$$R_{\frac{1}{2}}(\theta) = \left(\frac{1}{2} \right)_{\theta}, \quad (4.43)$$

This is the *free boundary condition*. As in the general case, this reflection amplitude has a pole at $\theta = \frac{i\pi}{2}$. This pole is dynamical, i.e. it changes position when changing the value of h .

- $h \rightarrow \infty$ ($\kappa \rightarrow i\infty - \frac{\pi}{2}$), with reflection amplitude

$$R_{i\infty - \frac{\pi}{2}}(\theta) = \left(-\frac{1}{2} \right)_{\theta}. \quad (4.44)$$

This is the so-called *fixed boundary condition* and it is the simplest configuration possible. It corresponds to moving the pole away from the physical strip. Notice that in this case the factor $-\left[\kappa\right]_{\theta} \rightarrow 1$ while in the free case it is non trivial.

It is not difficult to generalise the construction of the minimal form factor (4.38) to the case of generic reflection amplitudes (4.20). One viewpoint is to consider the Ising field theory with generic boundary conditions itself as a “perturbation” of the Ising field theory with fixed boundary conditions. Again, this is meant in the sense of how we compute the minimal form factor. This viewpoint allows us to both generalise the Ising and the sinh-Gordon results, to more general boundary conditions. Due to space limitations, the treatment of minimal form factors in the sinh-Gordon model with more general boundary conditions is presented in Appendix E.

4.2 Complete Minimal Form Factors for Irrelevant Deformations of IQFTs

We have finally reached the last work presented in this thesis. In [16] the problem of uniquely determining the minimal form factor for a $T\bar{T}$ -deformed IQFT is addressed and solved. We have seen how the representation (3.259), originally derived in [12, 13], is in fact completely general and not merely a feature of $T\bar{T}$ -deformed IQFTs. Let us briefly review the problem.

In the case of a deformed model with S -matrix (2.43), a two-particle form factor should satisfy

$$F_2^\mathcal{O}(\theta; \alpha) = S_\alpha(\theta) F_2^\mathcal{O}(-\theta; \alpha) = F_2^\mathcal{O}(2\pi i - \theta; \alpha) . \quad (4.45)$$

Typically there is at least one other requirement, which specifies the residue of the form factor at the kinematic pole $\theta = i\pi$. However, if we limit our attention to solutions without poles, then we need to consider only the two equations (4.45). Entire solutions to these equations are known as *minimal form factors* (MFFs) and in the deformed theory we will denote these as $F_{\min}(\theta; \alpha)$. The same equations with $S_\alpha(\theta)$ replaced by $S(\theta)$ are satisfied by the MFF of the undeformed theory, which we will denote as $F_{\min}(\theta)$:

$$F_{\min}(\theta) = S(\theta) F_{\min}(-\theta) = F_{\min}(2\pi i - \theta) . \quad (4.46)$$

This means that the MFFs $F_{\min}(\theta; \alpha)$ and $F_{\min}(\theta)$ are proportional to each other through a function $\mathcal{D}_\alpha(\theta)$ which satisfies

$$F_{\min}(\theta; \alpha) = \mathcal{D}_\alpha(\theta) F_{\min}(\theta) \implies \mathcal{D}_\alpha(\theta) = \Phi_\alpha(\theta) \mathcal{D}_\alpha(-\theta) = \mathcal{D}_\alpha(2\pi i - \theta) . \quad (4.47)$$

Note that each of the first equalities in (4.45–4.47) define a Riemann-Hilbert problem for the corresponding functions, with $S_\alpha(\theta)$, $S(\theta)$ and $\Phi_\alpha(\theta)$ playing the role of “jump functions” [176]. This can be used to explicitly write the solution as an integral representation involving the logarithm of the jump function. In fact, given the S -matrix of a massive, UV complete IQFT, it was shown in [6] that the MFF is uniquely fixed by the combined requirements of analyticity in the physical strip $\text{Im}(\theta) \in [0, \pi]$ and (at most) exponential growth for θ large (see subsection 3.1.2). The integral representation assumes a particularly simple form in terms of the Fourier transform of the S -matrix phase as we can see from (3.40) and (3.41). Here we present two different and complementary ways to obtain a *complete* MFF for $T\bar{T}$ -deformed IQFTs.

The parameters β_s are fixed by analyticity and asymptotic requirements, but also, crucially, by the requirement that the resulting MFF is a smooth function of the perturbation parameters α_s . For any set of parameters α we find that the function $\mathcal{D}_\alpha(\theta)$ is given by

$$\mathcal{D}_\alpha(\theta) = \mathcal{D}_\alpha(i\pi) \prod_{s \in 2\mathbb{Z}^+ - 1} \left[-\frac{1}{2} e^{\frac{\theta - i\pi}{2} \sinh(s\theta) - \sum_{n=1}^{s-1} \frac{\cosh(n\theta)}{s-n} + c_s} \left(2i \sinh \frac{\theta}{2} \right)^{-\cosh(s\theta)} \right]^{\frac{\alpha_s}{\pi}} , \quad (4.48)$$

where c_s is a constant (4.105). This formula can be easily generalised to other interesting cases, such as the branch point twist fields (see (F.43) in Appendix F.3) and to the

boundary case just discussed in Section 4.1, where the deformation of the one-particle form factor is given by $\sqrt{\mathcal{D}_\alpha(2\theta)}$. Note that in (4.48) and hereafter, we set the mass scale to $m = 1$ to lighten the formulas.

First, we give a heuristic derivation of our result, showing how generalised pairs $\delta(\theta)$ - $\rho(\theta)$ can be constructed without relying on the existence of a Fourier transform $g(t)$. Then we present an alternative but equivalent derivation of the same result, by regularising the standard contour-integral representation of the MFF proposed in the foundational paper [6] by Karowski and Weisz. Finally, we discuss some properties of the correlation functions that might be obtained building on this new minimal form factor.

4.2.1 A Heuristic Derivation: MFFs and their Fourier Transforms

In this section, we are going to produce a generalisation of the integral representation of MFFs, capable of accounting for the presence of exponential CDD factors in the S -matrix. In fact, the representation (3.41) relies on the possibility of expressing the scattering phase in integral form (3.40). This is not possible if the theory we are considering is a $\text{T}\bar{\text{T}}$ -deformed IQFT, whose S -matrix is of the form (2.43) with a finite number of deformation parameters. Our goal is then to knead (3.40) and (3.41) into a new, more general form. Note that the manipulations we will be performing here are formal and that we will be quite cavalier about convergence issues. As such, the content of this section is to be taken as a heuristic derivation of our main results.

Integral Representations, Asymptotic Behavior of the S -Matrix and Temperedness

Let us consider an interacting factorised scattering theory with diagonal S -matrix,

$$S(\theta) = \epsilon e^{i\delta(\theta)} . \quad (4.49)$$

The quantity $\delta(\theta)$ is the *phase shift* and is normalised so that

$$\delta(0) = 0 . \quad (4.50)$$

Consequently, the remaining constant ϵ encodes the value of the S -matrix at zero rapidity difference $S(0) = \epsilon$. For $\epsilon = \pm 1$ we have the usual bosonic/fermionic statistics. However, more general theories exist, such as the Federbush model [49, 183, 184] where $S(0)$ is a generic phase (in this example, there are also two particle types). If our factorised scattering theory is a UV complete IQFT, then the phase shift can be Fourier transformed as in (3.40). The converse is however not true. A good counterexample is the “bosonic” sinh-Gordon model studied in [161] whose S -matrix has a well defined Fourier transform despite the theory lacking UV completion. More general cases have been discussed in [109]. The meaning of this statement is made clearer below.

We now discuss in more detail the conditions for existence of the Fourier transform of the phase shift, that is the notion of *tempered distribution* [263, 264]. We show that the possibility of defining the phase shift as a Fourier transform is related to its nice decaying properties at large rapidities and that $\text{T}\bar{\text{T}}$ -perturbed S -matrices are precisely of the type where this condition is violated.

Let f be a function on $\mathbb{R}$ and consider $\mathcal{D}(\mathbb{R})$, the class of test functions on $\mathbb{R}$ and $\mathcal{D}'(\mathbb{R})$, the class of distributions on $\mathbb{R}$, i.e. the set of linear functionals on $\mathcal{D}(\mathbb{R})$. Explicitly, given $f : \mathbb{R} \rightarrow \mathbb{R}$ and $\varphi \in \mathcal{D}(\mathbb{R})$ it is possible to define a distribution⁴

$$\langle f, \varphi \rangle = \int_{-\infty}^{\infty} f(x) \varphi(x) dx \in \mathcal{D}'(\mathbb{R}). \quad (4.51)$$

Suppose that $\delta(\theta)$ is a smooth function satisfying the asymptotic condition

$$\lim_{|\theta| \rightarrow \infty} \delta(\theta) = 0. \quad (4.52)$$

This means that this function and all its derivatives decay faster than any inverse polynomial, i.e.,

$$\forall n \in \mathbb{N}, \quad \sup_{\theta \in \mathbb{R}} |\theta^k \delta^{(n)}(\theta)| < \infty \quad \text{for all } k \in \mathbb{N}. \quad (4.53)$$

Therefore, $\delta(\theta)$ belongs to the Schwartz class $\mathcal{S}(\mathbb{R})$. This implies that its Fourier transform

$$g(t) := \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} e^{it\theta} \delta(\theta) \quad (4.54)$$

exists as an ordinary Lebesgue integral and defines a function $g(t) \in \mathcal{S}'(\mathbb{R}) \subseteq \mathcal{D}'(\mathbb{R})$, the dual to the Schwartz class, i.e. the class of *tempered distributions*. In particular, we also have the inversion formula

$$\delta(\theta) = \int_{-\infty}^{\infty} dt e^{-it\theta} g(t), \quad (4.55)$$

with absolute convergence of the integral due to rapid decay of $g(t)$. In summary, if $\delta(\theta) \in L^1(\mathbb{R})$, meaning that its absolute value is Lebesgue-integrable, we can define its Fourier transform, which will also have nice decaying properties by construction. This is what happens for standard phase shifts. Now, consider a phase shift coming from a TT̄-deformation: $\delta(\theta) \sim \sinh(s\theta)$. It still defines a distribution in the sense of $\mathcal{D}'(\mathbb{R})$ (class of distributions in $\mathbb{R}$), i.e., for instance, it acts continuously on test functions with compact support:

$$\varphi(\theta) \in C_c^\infty(\mathbb{R}) \quad \implies \quad \langle \sinh, \varphi \rangle = \int_{-\infty}^{\infty} \sinh(s\theta) \varphi(\theta) d\theta \quad \text{is finite.} \quad (4.56)$$

However if we consider the real line, $\sinh(s\theta)$ grows exponentially as $|\theta| \rightarrow \infty$:

$$|\sinh(s\theta)| \sim \frac{1}{2} e^{s|\theta|}, \quad (4.57)$$

and therefore it does not satisfy any polynomial bound of the form (4.53) which is a necessary condition for tempered distributions. Consequently, $\delta(\theta)$ in this case is a distribution, but it is not tempered; thus, its Fourier transform is not well defined within the classical theory of tempered distributions. Indeed, the subclass of test functions $\mathcal{E}(\mathbb{R}) \subset \mathcal{D}(\mathbb{R})$ necessary to make an integral like (4.56) convergent, can only be of two kinds: the class of test functions with compact support $\mathcal{E}(\mathbb{R}) = C_c^\infty(\mathbb{R})$ as written above or a subclass with a quicker exponential decay than $\exp s|\theta|$:

$$\mathcal{E}_s(\mathbb{R}) := \left\{ \varphi \in C^\infty(\mathbb{R}) \mid \exists s > 0 \text{ such that } \sup_{\theta \in \mathbb{R}} |\varphi(\theta) e^{s|\theta|}| < \infty \right\} \subset \mathcal{D}(\mathbb{R}). \quad (4.58)$$

⁴Note that this symbols $\mathcal{D}$ and $\mathcal{D}'$ have nothing to do with the function $\mathcal{D}_\alpha(\theta)$ defined earlier (4.47). Similarly, the symbols $\mathcal{S}$ and $\mathcal{S}'$ around equation (4.54), are distinct from the set of spins $\mathcal{S}$ in (2.43) and (3.95).

Let us now apply these ideas to the integral representation of a generic S -matrix. Consider then the scattering phase $\delta(\theta)$ and suppose it can be written in terms of its Fourier transform as

$$\delta(\theta) = \frac{i}{2} \int_{-\infty}^{\infty} \frac{dt}{t} g(t) e^{\frac{it\theta}{\pi}} . \quad (4.59)$$

Since the phase shifts are taken to be real functions of the rapidity difference, the function $g(t)$ must be even: $g(-t) = g(t)$. We can then rewrite the Fourier transform in the form (3.40)

$$\delta(\theta) = - \int_0^{\infty} \frac{dt}{t} g(t) \sin \frac{t\theta}{\pi} , \quad (4.60)$$

which is the representation given earlier (3.40). In order for this Fourier transform to be a proper function – rather than a distribution – we must demand that $g(t)$ approaches constant values at the integral bounds

$$g(t) = g^{(0)} + O(t^2) , \quad \lim_{|t| \rightarrow \infty} g(t) = g^{(\infty)} < \infty . \quad (4.61)$$

Note that we assumed $g(t)$ to be Taylor expandable around $t = 0$ and to possess only isolated singularities. The constants $g^{(0)}$ and $g^{(\infty)}$ may vanish. We can think of $g(t)$ as a function on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ which means that it must be either a constant or a meromorphic function. In both cases, we can use a Mittag-Leffler expansion [265] to write

$$g(t) = g^{(0)} - \sum_{n=1}^{\infty} \frac{t^2}{\pi p(n)} \frac{g^{(n)}}{t^2 + p(n)^2} , \quad g^{(n)} = \pm 2\pi i \operatorname{Res}_{t=\pm ip(n)} [g(t)] , \quad (4.62)$$

for some set of poles $\{\pm ip(n) \mid n = 1, \dots, \infty\}$. We can use the physical properties of the phase shifts to specify the positions of these poles. In particular, we want $\delta(\theta)$ to be $2\pi i$ -periodic. We see that (4.62) and the residue theorem – which we can use since the integrand of (4.59) decays as t^{-1} as $|t| \rightarrow \infty$ – imply

$$\delta(\theta) = \text{const.} + \sum_{n=1}^{\infty} \frac{g^{(n)}}{2p(n)} e^{-\frac{p(n)\theta}{\pi}} , \quad \forall \theta > 0. \quad (4.63)$$

If we want this expression to be $2\pi i$ -periodic, then we need to fix

$$p(n) = n\pi . \quad (4.64)$$

In conclusion, the function $g(t)$ must take the following form

$$g(t) = g^{(0)} - \sum_{n=1}^{\infty} \frac{t^2}{n\pi^2} \frac{g^{(n)}}{t^2 + n^2\pi^2} . \quad (4.65)$$

The Logarithm of the MFF and its Fourier Transform

Given the integral representation (4.60), we introduce the new function

$$\rho(\theta) = -\frac{1}{4} \int_{-\infty}^{\infty} \frac{dt}{t} \frac{g(t) - g^{(0)}}{\sinh t} e^{\frac{t(i\pi - \theta)}{i\pi}} + \frac{g^{(0)} - 1 + \epsilon}{2} \log \left(-i \sinh \frac{\theta}{2} \right) . \quad (4.66)$$

Note that the Fourier integral converges as long as $0 \leq \text{Im } \theta \leq 2\pi$. We immediately verify the identities⁵

$$\rho(i\pi - \theta) = \rho(i\pi + \theta) , \quad (4.68a)$$

$$\rho(\theta) - \rho(-\theta) = i\delta(\theta) + \frac{i\pi(1-\epsilon)}{2} \text{sign}(\theta) . \quad (4.68b)$$

Consequently, we can take this function to be the logarithm of the MFF

$$\log F_{\min}(\theta) = \rho(\theta) \implies \begin{cases} F_{\min}(\theta) = F_{\min}(2\pi i - \theta) , \\ F_{\min}(\theta) = S(\theta)F_{\min}(-\theta) . \end{cases} \quad (4.69)$$

With some manipulations⁶ we can rewrite $\rho(\theta)$ as follows

$$\rho(\theta) = \rho(i\pi) + \int_0^\infty \frac{dt}{t} g(t) \frac{\sin^2\left(\frac{t(i\pi-\theta)}{2\pi}\right)}{\sinh t} - \frac{1-\epsilon}{2} \log\left(-i \sinh \frac{\theta}{2}\right) , \quad (4.70)$$

where $\rho(i\pi)$ is a constant that can be absorbed into the normalisation of the MFF.

Using the Mittag-Leffler expansion (4.65) in the integral representation (4.59), we can write

$$\begin{aligned} \delta(\theta) &= -g^{(0)} \int_0^\infty \frac{dt}{t} \sin \frac{t\theta}{\pi} + \sum_{n=1}^\infty \frac{g^{(n)}}{n\pi} \frac{1}{2\pi i} \int_{-\infty}^\infty dt \frac{t e^{\frac{it\theta}{\pi}}}{t^2 + n^2\pi^2} \\ &= \left[-\frac{\pi g^{(0)}}{2} + \frac{1}{2\pi} \sum_{n=1}^\infty \frac{g^{(n)}}{n} e^{-n|\theta|} \right] \text{sign}(\theta) . \end{aligned} \quad (4.71)$$

Similarly, putting together the integral representation (4.66) and the expansion (4.65), we get

$$\rho(\theta) = \frac{i}{2\pi} \sum_{n=1}^\infty \frac{g^{(n)}}{n} \frac{1}{2\pi i} \int_{-\infty}^\infty dt \frac{t}{t^2 + n^2\pi^2} \frac{e^{\frac{t(i\pi-\theta)}{i\pi}}}{\sinh t} + \frac{g^{(0)} - 1 + \epsilon}{2} \log\left(-i \sinh \frac{\theta}{2}\right) . \quad (4.72)$$

The integral can be computed by residue theorem, with contributions from a double pole at $t = i\pi n$ and simple poles at $t = i\pi m$ with $m \neq n$. This gives⁷

$$\begin{aligned} \frac{1}{2\pi i} \int_{-\infty}^\infty dt \frac{t}{t^2 + n^2\pi^2} \frac{e^{\frac{t(i\pi-\theta)}{i\pi}}}{\sinh t} &= \sum_{m=1}^\infty \text{Res}_{t=i\pi m} \left[\frac{t}{t^2 + n^2\pi^2} \frac{e^{\frac{t(i\pi-\theta)}{i\pi}}}{\sinh t} \right] \\ &= i \frac{2n\pi \left(\frac{\theta-i\pi}{\pi}\right) - 1}{4\pi n} e^{-n\theta} + \frac{i}{\pi} \sum_{\substack{m=1 \\ m \neq n}}^\infty \frac{m e^{-m\theta}}{n^2 - m^2} . \end{aligned} \quad (4.73)$$

⁵For equation (4.68b), we can use the known integral $\int_{-\infty}^\infty \frac{dt}{t} \sin \frac{t\theta}{\pi} = \pi \text{sign}(\theta)$ and

$$\log(-i \sinh \frac{\theta}{2}) - \log(i \sinh \frac{\theta}{2}) = -i\pi \text{sign}(\theta) . \quad (4.67)$$

⁶We use another known integral: $\int_{-\infty}^\infty \frac{dt}{t} \frac{\sin^2(tx)}{\sinh t} = \frac{1}{2} \log \cosh(\pi x)$.

⁷Note that this integral needs regularisation: we suppose that θ has a vanishingly small imaginary part.

For generic θ the formula above generalised in a simple way, which can be captured by introducing the $\text{sign}(\theta)$ function in the appropriate place, giving the final result:

$$\begin{aligned} \rho(\theta) = & -\frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{g^{(n)}}{n} \left[\frac{2n\pi \text{sign}(\theta) \left(\frac{\theta-i\pi}{\pi} \right) - 1}{4n} e^{-n|\theta|} + \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \frac{m e^{-m|\theta|}}{n^2 - m^2} \right] \\ & + \frac{g^{(0)} - 1 + \epsilon}{2} \log \left(-i \sinh \frac{\theta}{2} \right) . \end{aligned} \quad (4.74)$$

Let us compute the difference $\rho(\theta) - \rho(-\theta)$ and check that (4.68b) is satisfied. Thanks to the relation (4.67) we have that the last term in (4.74) produces the first term in (4.71). Furthermore, the sum over m in (4.74) is symmetric under $\theta \rightarrow -\theta$, so it does not contribute and, finally, the remaining term in (4.74) yields the sum over n in (4.71):

$$\frac{2n\pi \text{sign}(\theta) \left(\frac{\theta-i\pi}{\pi} \right) - 1}{4n} e^{-n|\theta|} - \frac{2n\pi \text{sign}(-\theta) \left(\frac{-\theta-i\pi}{\pi} \right) - 1}{4n} e^{-n|-\theta|} = -i\pi \text{sign}(\theta) e^{-n|\theta|} . \quad (4.75)$$

Recovering a $\text{T}\bar{\text{T}}$ -Like Representation

So far, the contents of this section provide a rederivation of known results. We now come to the new conceptual step which will allow us to use these results to construct a pair $\delta(\theta)$ - $\rho(\theta)$ which corresponds to IQFTs perturbed by $\text{T}\bar{\text{T}}$. The key observation is that, for the purpose of the above identities, the only property that matters is the parity of the functions $e^{-n|\theta|}$ and $\text{sign}(\theta)\theta e^{-n|\theta|}$, that is, that the former is even under $\theta \rightarrow -\theta$, while the latter is odd. This means that we could replace these functions with others having the same parity properties and the MFF equations (4.68) would still hold. We will now exploit this property by investigating what the most general choices for these functions can be. We perform the substitutions

$$e^{-n|\theta|} \text{sign}(\theta) \longrightarrow f_n(\theta) , \quad e^{-n|\theta|} \theta \text{sign}(\theta) \longrightarrow g_n(\theta) , \quad e^{-n|\theta|} \longrightarrow h_n(\theta) , \quad (4.76)$$

in the series expansions (4.71) and (4.74). For simplicity, we will also take $g^{(0)} = 0$ and $\epsilon = 1$ so that the term $\log(-i \sinh \frac{\theta}{2})$ in (4.74) drops out and we have an S -matrix which is a “pure” $\text{T}\bar{\text{T}}$ -deformation. We then have

$$\begin{aligned} \delta(\theta) &= \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{g^{(n)}}{n} f_n(\theta) , \\ \rho(\theta) &= -\frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{g^{(n)}}{n} \left[-\frac{\pi i}{2} f_n(\theta) + \frac{1}{2} g_n(\theta) - \frac{1}{4n} h_n(\theta) + \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \frac{m}{n^2 - m^2} h_m(\theta) \right] . \end{aligned} \quad (4.77)$$

We want these two quantities to be, respectively, a phase shift and the logarithm of the corresponding MFF. Clearly, only some functions $f_n(\theta)$, $g_n(\theta)$ and $h_n(\theta)$ will work. We must have

$$\begin{aligned} \delta(\theta) + \delta(-\theta) &= 0 , & \text{unitarity of the } S\text{-matrix} , \\ \delta(\theta) - \delta(i\pi - \theta) &= 0 , & \text{crossing symmetry of the } S\text{-matrix} , \\ \rho(\theta) - \rho(-\theta) &= i\delta(\theta) , & \text{Watson's equation} , \\ \rho(i\pi + \theta) - \rho(\pi i - \theta) &= 0 , & \text{crossing symmetry of the form factor} . \end{aligned} \quad (4.78)$$

We then see immediately that the functions $f_n(\theta)$ have to satisfy the conditions

$$f_n(\theta + i\pi) = f_n(-\theta) = -f_n(\theta) = -f_n(\theta - i\pi) . \quad (4.79)$$

Watson's equation tells us that $g_n(\theta)$ and $h_n(\theta)$ have to be even

$$g_n(-\theta) = g_n(\theta) , \quad h_n(-\theta) = h_n(\theta) , \quad (4.80)$$

and, finally, the crossing symmetry of the form factor yields the following identities⁸

$$g_n(\theta + i\pi) - g_n(\theta - i\pi) = 2\pi i f_n(\theta + i\pi) , \quad h_n(\theta + i\pi) = h_n(\theta - i\pi) . \quad (4.81)$$

If we require that $\{f_n(\theta)\}_{n \in \mathbb{Z}^+}$, $\{g_n(\theta)\}_{n \in \mathbb{Z}^+}$ and $\{h_n(\theta)\}_{n \in \mathbb{Z}^+}$ each constitute an infinite, countable set of independent, smooth functions that satisfy the above requirements, then we can almost uniquely fix the solution to be

$$f_{2n-1}(\theta) = -\sinh((2n-1)\theta) , \quad g_{2n-1}(\theta) = -\theta \sinh((2n-1)\theta) , \quad h_n(\theta) = \cosh(n\theta) , \quad (4.82)$$

and

$$f_{2n}(\theta) = 0 , \quad g_{2n}(\theta) = 0 . \quad (4.83)$$

With “almost uniquely” we mean to say that - since the equations (4.79)-(4.81) are homogeneous - we still have the freedom to rescale each of the functions $f_{2n-1}(\theta)$, $g_{2n-1}(\theta)$ and $h_n(\theta)$ by an arbitrary constant (in θ) factor. For $f_{2n-1}(\theta)$ and $g_{2n-1}(\theta)$ this factor has to be the same, due to the first equation in (4.80), and it can be reabsorbed into the coefficients $g^{(n)}$. However for the functions $h_n(\theta)$, it is indeed an arbitrary choice. We can even choose it to vanish for any n . This fact is not surprising: it is the usual indeterminacy of the cosh terms that was already remarked in previous works [12, 13, 244]. Here, the choice we make (4.82), (4.83) is guided by the requirement that the functions (4.77) agree, for large $|\theta|$, with (4.71), (4.74) when stripped of their diverging behavior. So, for example, we ask $f_{2n-1}(\theta) \underset{|\theta| \rightarrow \infty}{\sim} \text{sign}(\theta) e^{-(2n-1)|\theta|} + \text{“diverging terms”}$ and we see that $f_{2n-1}(\theta) = -\sinh((2n-1)\theta)$ fits the bill. Notice that this also means that

$$g^{(2n)} = 0 . \quad (4.84)$$

Thus, we have found that the following functions (we now add an index α to include the parameter dependence)

$$\begin{aligned} \delta_\alpha(\theta) &= - \sum_{n=1}^{\infty} \alpha_{2n-1} \sinh((2n-1)\theta) , \\ \rho_\alpha(\theta) &= \frac{i\pi - \theta}{2\pi} \delta(\theta) + \sum_{n=1}^{\infty} \alpha_{2n-1} \left[\frac{\cosh((2n-1)\theta)}{4\pi(2n-1)} - \frac{1}{\pi} \sum_{\substack{m=1 \\ m \neq 2n-1}}^{\infty} \frac{m \cosh(m\theta)}{(2n-1)^2 - m^2} \right] , \\ \alpha_{2n-1} &= \frac{1}{2\pi} \frac{g^{(2n-1)}}{2n-1} , \end{aligned} \quad (4.85)$$

constitute a valid $\delta(\theta)$ - $\rho(\theta)$ pair. Notice that the form of the phase shift is exactly that of the TT-deformation (2.43). However, the MFF is different from that found in [12, 13],

⁸Here we are assuming that each term in the series - also in the double series - appearing in the equations (4.78) has to vanish *independently*.

in the sense that the coefficients of the $\cosh(n\theta)$ terms are not arbitrary, but related to the same α_n appearing in $\delta(\theta)$. This form complies instead with the expression for the sinh-Gordon MFF derived in [244]. In fact, (4.82) is not the only possible set of smooth functions we could choose. Setting $h_n(\theta) = 0$ for all indices $n \in \mathbb{Z}$ is also an alternative choice that satisfies all the conditions. This choice reproduces the results of [12, 13], when all free parameters are set to zero. We argue that the MFF in (4.85) is a more natural choice, for at least two reasons. First of all, by construction, it reproduces the MFFs obtained from the integral representation for UV complete IQFTs.

Secondly, as a function of $\eta = \frac{\theta - i\pi}{\pi}$, the logarithm of the MFF in (4.85) has more desirable asymptotic properties than most of the possible solutions found in [12, 13]. Let us look at a simple case to illustrate this fact.

4.2.2 The $\overline{\text{T}}\overline{\text{T}}$ -Deformed Ising Field Theory

Let us take the simplest possible factorised scattering theory, a massive free fermion, with S -matrix $S(\theta) = -1$. Let us perform a basic $\overline{\text{T}}\overline{\text{T}}$ deformation of this theory. Then the scattering phase becomes

$$\delta_\alpha^{\text{Ising}}(\theta) = -\alpha \sinh \theta, \quad \epsilon \equiv S(0) = -1. \quad (4.86)$$

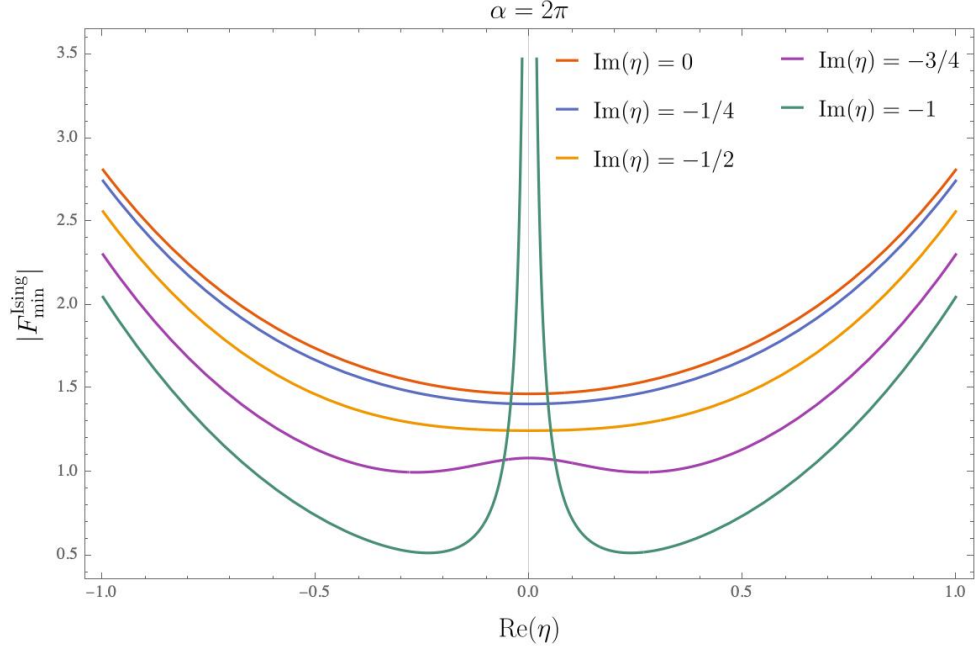
From the general formula (4.85), we find the logarithm of the MFF to be

$$\begin{aligned} \rho_\alpha^{\text{Ising}}(\theta) &= \log \left(-i \sinh \frac{\theta}{2} \right) - \frac{i\pi - \theta}{2\pi} \alpha \sinh \theta + \frac{\alpha}{4\pi} \cosh \theta - \frac{\alpha}{\pi} \sum_{m=2}^{\infty} \frac{m \cosh(m\theta)}{1 - m^2} \\ &= \log \left(-i \sinh \frac{\theta}{2} \right) - \frac{i\pi - \theta}{2\pi} \alpha \sinh \theta - \frac{\alpha}{2\pi} - \frac{\alpha}{2\pi} \cosh \theta \log(-4 \sinh^2 \frac{\theta}{2}) \\ &= \log \left(\cosh \frac{\pi\eta}{2} \right) - \frac{\alpha}{2\pi} \left[1 + \pi\eta \sinh(\pi\eta) - \cosh(\pi\eta) \log \left(4 \cosh^2 \frac{\pi\eta}{2} \right) \right], \end{aligned} \quad (4.87)$$

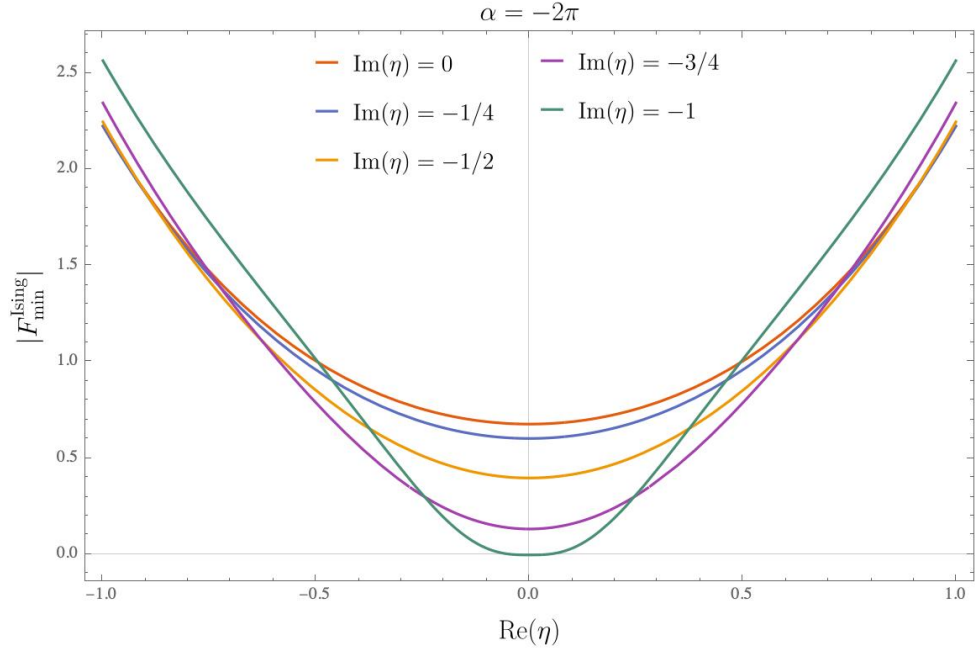
where we now reinstated the “Ising term”⁹ $\log(-i \sinh \frac{\theta}{2})$, since $\epsilon = -1$. We introduced the variable $\eta = \frac{\theta - i\pi}{\pi}$. This function decays nicely for $|\theta| \rightarrow \infty$ in the whole strip $0 \leq \text{Im} \theta \leq 2\pi$, for both α positive and negative, as can be seen in Figure 4.2. Figures 4.3 and 4.4 show a comparison of the MFF obtained from (4.87) with that derived in [12, 13], for, respectively, $\text{Im} \theta = i\pi$ and $\text{Im} \theta = 0$. We notice how the MFF we obtained here preserves the large-rapidity behavior of the unperturbed ($\alpha = 0$) theory, while the one obtained in [12, 13] agrees with the $\alpha = 0$ situation only in the vicinity of $\theta = 0$.

This difference in the behaviour of the deformation of the MFF is more profound than appears at a cursory glance. The expression found in [12, 13] cannot be considered a small deformation of the massive free fermion MFF! This was already remarked in [12, 13], where it was observed how the deformation leads to a radical difference in the correlation functions compared to the undeformed theory. In contrast, the expression (4.87), can be rightfully considered a well-behaved 1-parameter (α) deformation of the massive free fermion MFF. This is immediately clear from the plot of $e^{\rho_\alpha^{\text{Ising}}(\theta)}$ in Figure 4.3. From the plot in Figure 4.4, we see that something peculiar happens for $\alpha > 0$: the minimal form factor seems to change its $\theta \sim 0$ behaviour when α crosses the value $\alpha_c = \pi$. In fact,

⁹Notice that, thanks to this term, the minimal form factor $F_{\min}^{\text{Ising}}(\theta; \alpha) = \exp \rho_\alpha^{\text{Ising}}$ diverges as $\text{Re} \theta \rightarrow \pm\infty$. This is a characteristic behaviour of the massive free fermion. In non-free theories, this diverging behaviour will be tempered by the presence of interactions.

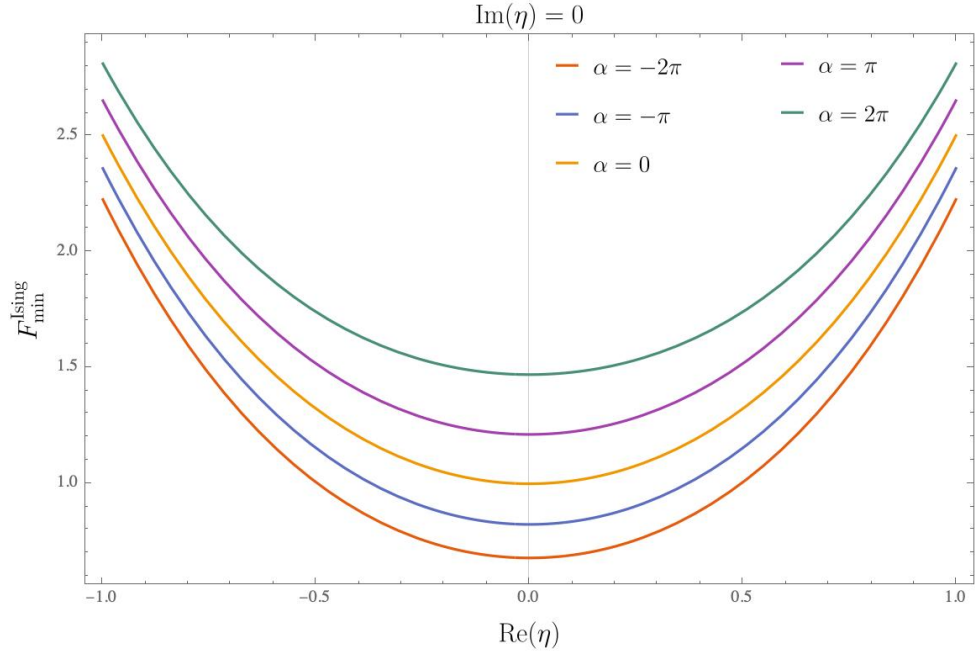


(a) MFF for $T\bar{T}$ -perturbed massive free fermion with the inclusion of the cosh terms, with $\alpha = 2\pi$.

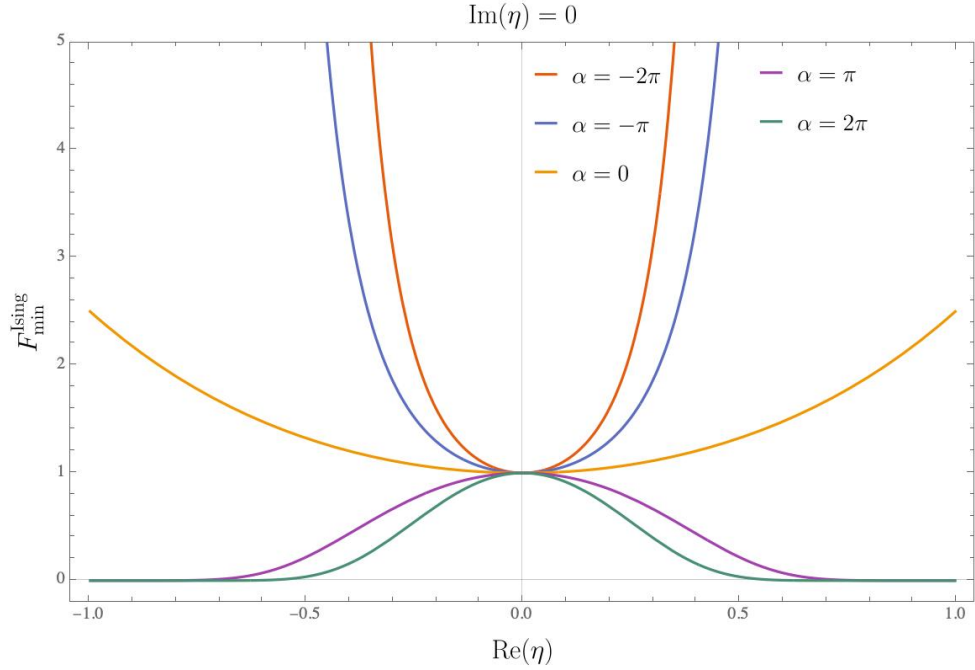


(b) MFF for $T\bar{T}$ -perturbed massive free fermion with the inclusion of the cosh terms, with $\alpha = -2\pi$.

Figure 4.2: Comparison of MFFs with the inclusion of cosh terms, for α positive and negative, on different lines $\text{Im}\eta = 0, -\frac{1}{4}, -\frac{1}{2}, -\frac{3}{4}, -1$.

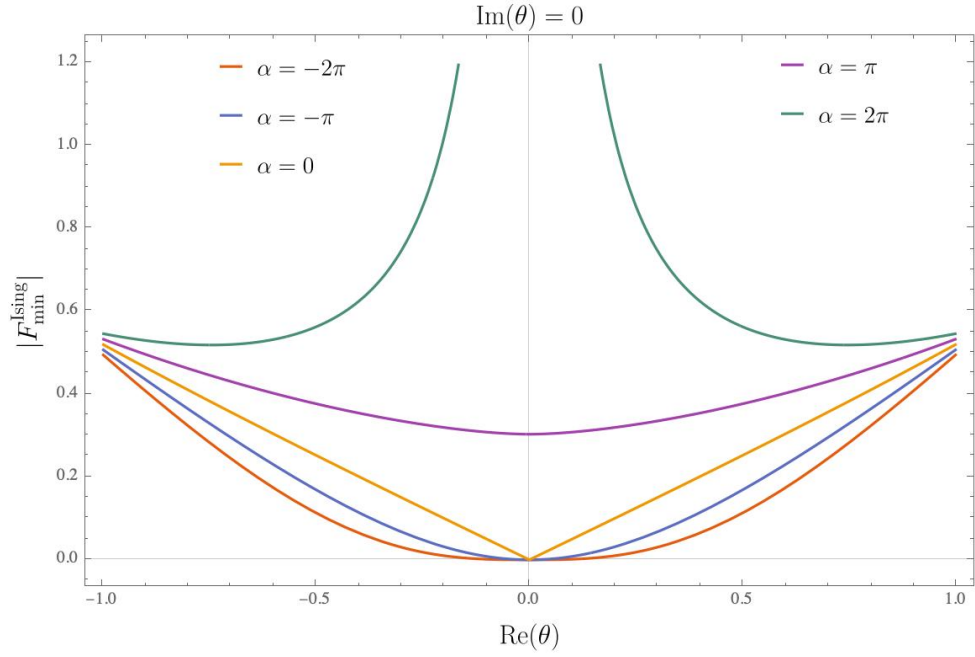


(a) MFF for $\overline{T}\overline{T}$ -perturbed massive free fermion with the inclusion of the cosh terms, as a function of $\eta \in \mathbb{R}$

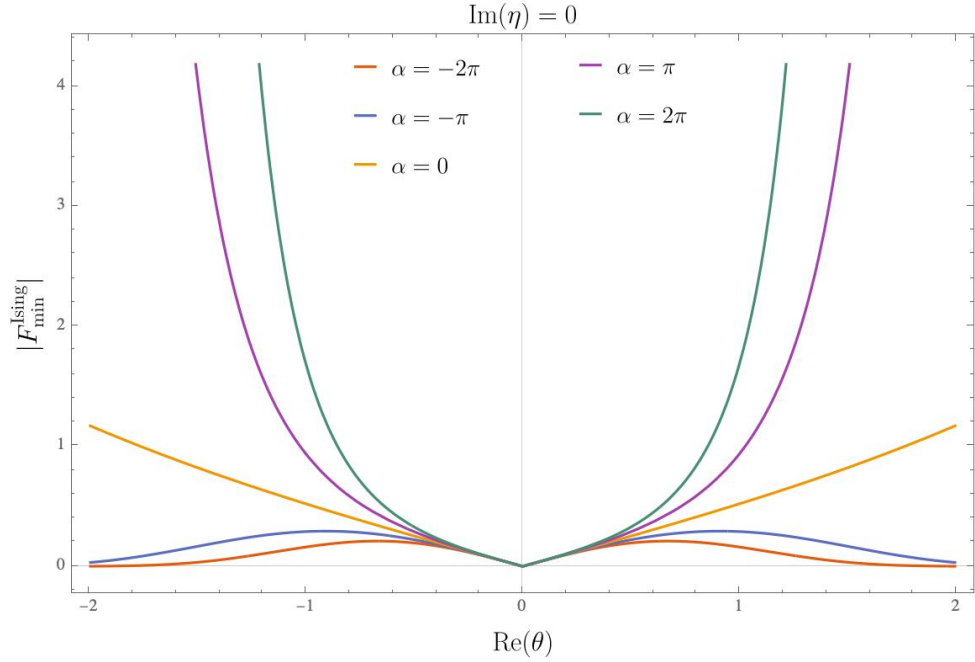


(b) MFF for $\overline{T}\overline{T}$ -perturbed massive free fermion without the inclusion of the cosh terms, as a function of $\eta \in \mathbb{R}$

Figure 4.3: Comparison of MFFs with and without the inclusion of cosh terms on the line $\eta \in \mathbb{R}$ (i.e. $\text{Im}\theta = i\pi$).



(a) MFF for $T\bar{T}$ -perturbed massive free fermion with the inclusion of the cosh terms, as a function of $\theta \in \mathbb{R}$



(b) MFF for $T\bar{T}$ -perturbed massive free fermion without the inclusion of the cosh terms (see [12, 13]), as a function of $\theta \in \mathbb{R}$

Figure 4.4: Comparison of MFFs with and without the inclusion of cosh terms on the line $\theta \in \mathbb{R}$.

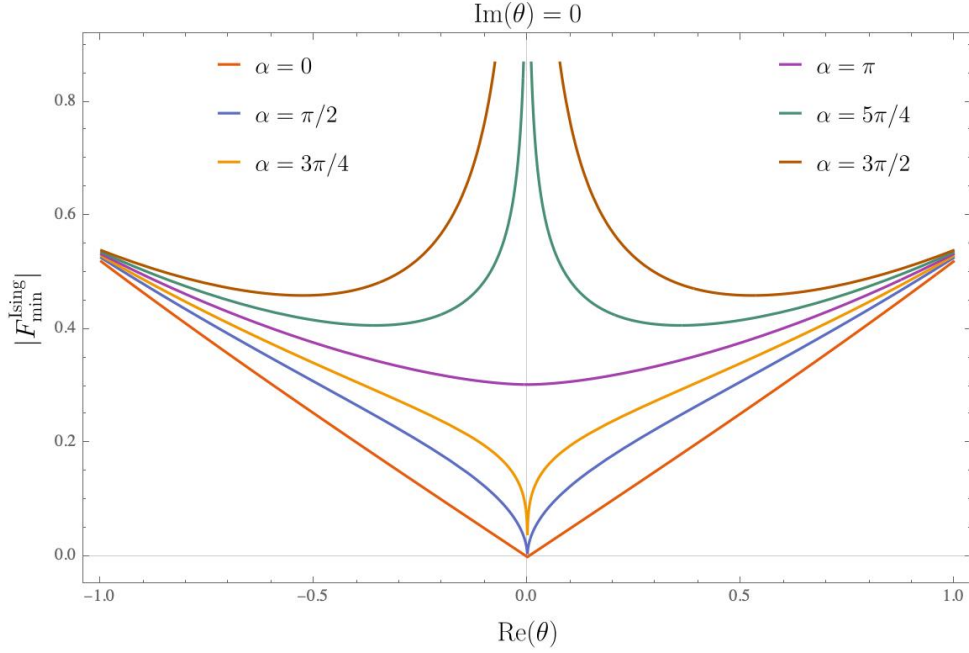


Figure 4.5: Plots of the MFF obtained from (4.87) as a function of $\theta \in \mathbb{R}$ for various values of $\alpha > 0$.

a more careful analysis confirms this observation. Let us consider the behaviour of the MFF $F_{\min}^{\text{Ising}}(\theta; \alpha) := e^{\rho_{\alpha}^{\text{Ising}}(\theta)}$ in the vicinity of $\eta = 0$ and $\theta = 0$.

$$\begin{aligned} F_{\min}^{\text{Ising}}(\theta; \alpha) &\underset{\eta \rightarrow 0}{\sim} e^{-\frac{\alpha(1-2\log 2)}{2\pi}} + O(\eta^2), \\ F_{\min}^{\text{Ising}}(\theta; \alpha) &\underset{\theta \rightarrow 0}{\sim} \frac{1}{2}(-\theta^2)^{\frac{1}{2}-\frac{\alpha}{2\pi}} e^{-\frac{\alpha}{2\pi}} \left[1 + \frac{\alpha}{2}(-\theta^2)^{\frac{1}{2}} + O(\theta^2) \right]. \end{aligned} \quad (4.88)$$

We immediately see that the behaviour at $\eta \rightarrow 0$ is qualitatively the same for any value of α : the MFF is a finite, α -dependent constant. On the other hand, we see that at $\theta = 0$, the behaviour of the MFF radically depends on the value of α : if $\alpha < \pi$, it vanishes (as happens for the undeformed case), while for $\alpha > \pi$ it diverges (reaching a constant in the limit case $\alpha = \alpha_c = \pi$). The situation is portrayed in Figure 4.5.

Notice that the critical value $\alpha_c = \pi$ is theory-dependent. In this case, it results from the fact that the MFF of the Ising field theory is $-i \sinh \frac{\theta}{2}$, as we see for instance in equation (4.87). The general statement is that if the leading asymptotic behaviour of the MFF of the unperturbed theory is:

$$F_{\min}(\theta) \underset{\theta \rightarrow 0}{\sim} \theta^{\kappa}, \quad (4.89)$$

for some constant $\kappa \in \mathbb{R}$, then, the MFF of the $\text{T}\bar{\text{T}}$ -perturbed theory will have the leading asymptotics

$$F_{\min}(\theta; \alpha) \underset{\theta \rightarrow 0}{\sim} \theta^{\kappa - \frac{\alpha}{\pi}}, \quad (4.90)$$

so that there is a critical value $\alpha_c = \kappa\pi$ above which the MFF develops a pole of order $\frac{\alpha}{\pi} - \kappa$ at $\theta = 0$. In the Ising field theory we simply have $\kappa = 1$. The more general case of a single spin- $2n - 1$ irrelevant perturbation is considered in Appendix F.

4.2.3 Alternative Construction from the Karowski-Weiss Approach

In subsection 3.1.2 we saw in detail the standard construction [6]. What happens if we try to implement it in our case? That is, consider an S -matrix of the type (2.43) and let $\mathcal{D}_\alpha(\theta)$ be the deformation of the corresponding MFF¹⁰, as defined by the equations (4.47) whose general solution we wrote in (3.259). The main property of the CDD factor $\Phi_\alpha(\theta)$ is that, at large rapidities it does not tend to a constant. Therefore in general we have $\mathcal{D}_\alpha(\theta) \sim \exp(\exp s^*|\theta|)$ for $\text{Re}(\theta) \rightarrow \infty$ and s^* the largest spin in the sum (2.43). This means that if we follow again the Karowski-Weisz construction we find the following asymptotics of the function $w(R + iy)$ defined in (3.36)

$$\left| \frac{\log \mathcal{D}_\alpha(R + iy)}{\sinh^2 \frac{1}{2}(R - \theta + iy)} \right| \sim \frac{e^{s^*|R|}}{e^{|R|}} \xrightarrow{R \rightarrow \infty} e^{(s^*-1)|R|}. \quad (4.91)$$

So we can no longer argue that the vertical contributions to the integral are vanishing and this is also the reason why these kind of solution was excluded from their construction (3.29). We can think of two ways, possibly related to each other, of solving this problem: the first consists in finding an appropriate choice of contour such that the exponential growth of the function on the real axis can be mitigated. The second comes from the freedom given by the equations in (4.45) regarding the choice of the MFF. As noticed in the previous chapter, we can always multiply a given solution by some function $C_\beta(\theta)$ given by (3.95). Let us call this modified solution $\tilde{\mathcal{D}}_\alpha(\theta)$. An appropriate choice of this function, which means a choice of the parameters β , will ensure that

$$\left| \frac{\log \tilde{\mathcal{D}}_\alpha(R + iy)}{\sinh^2 \frac{1}{2}(R - \theta + iy)} \right| = o(1) \quad \text{as } R \rightarrow \infty, \quad (4.92)$$

The additional parameters β , if chosen wisely, can enforce the double-exponential asymptotics of the MFF as we saw for instance in (4.4) in the case of the sinh-Gordon model. The idea is therefore to regularise the usual integral representation by imposing this asymptotics.

Notice also that in general, given an integrand $I(z, \theta) = \frac{\log S(z)}{\sinh^2 \frac{1}{2}(z - \theta)}$ over a contour such that $\log(S(z))$ is analytic everywhere inside and on the contour and z_n is a pole of the denominator, we have that the residue at the pole is given by

$$\text{Res}(I(z, \theta), z_n) = 4\partial_z [\log S(z)]_{z=z_n}. \quad (4.93)$$

As observed previously, the poles are exactly at $z_n = \theta + 2\pi in$, for $n = 0, 1, 2, \dots$. If we consider again the contour C above, including the pole at $z_0 = \theta$ we get

$$\oint_C dz I(z, \theta) = 4\partial_z [\log S(z)]_{z_0}. \quad (4.94)$$

Instead of deforming the contour we will see in the next section that imposing the asymptotics above is sufficient to find a closed solution that fixes the MFF completely. Not coincidentally the term that regularises the integral representation is proportional to contributions of the form (4.94).

¹⁰In (3.259) we wrote the most general solution to these equations, namely $\mathcal{D}_\alpha(\theta) = \varphi_\alpha(\theta)C_\beta(\theta)$.

A Solution

The main problem in defining a complete MFF for a generalised $\text{T}\bar{\text{T}}$ -deformation comes from the distinct properties of the CDD-factors. We cannot expand them in terms of a Fourier series, therefore we need to start from an integral representation of the type (3.39). In practice, it is more convenient to start from an equivalent representation which involves $\log S(z)$ explicitly. This kind of representation appears in [266]:

$$\rho(\theta) = \frac{1}{i\pi} \cosh^2 \frac{\theta}{2} \int_0^\infty dt \frac{\tanh \frac{t}{2} \ln S(t)}{\cosh t - \cosh \theta}, \quad \text{for } 0 \leq \text{Im}(\theta) \leq 2\pi. \quad (4.95)$$

The integrand reproduces exactly the same pole structure of (3.34) but it can also be shown to be equivalent to (3.41) once we invert the Fourier transform. Let us consider $\Phi_\alpha(\theta)$ as defined in (2.43). We have

$$\begin{aligned} \log \mathcal{D}_\alpha(\theta) &= \frac{1}{i\pi} \cosh^2 \frac{\theta}{2} \int_0^\infty dt \frac{\tanh \frac{t}{2} \ln \Phi_\alpha(t)}{\cosh t - \cosh \theta} \\ &= -\frac{1}{\pi} \cosh^2 \frac{\theta}{2} \sum_{s \in \mathcal{S}} \int_0^\infty dt \frac{\alpha_s \tanh \frac{t}{2} \sinh st}{\cosh t - \cosh \theta}. \end{aligned} \quad (4.96)$$

This integral is not convergent because of the asymptotics of the $\sinh(st)$ function, however it can be easily regularised by replacing $\sinh(st)$ by $-e^{-st}$. This amounts to the regularisation prescription

$$\log \mathcal{D}_\alpha(\theta) = \frac{1}{i\pi} \cosh^2 \frac{\theta}{2} \int_0^\infty dt \frac{\tanh \frac{t}{2}}{\cosh t - \cosh \theta} \left(\ln \Phi_\alpha(t) - \partial_t \ln \hat{\Phi}_\alpha(t) \right), \quad (4.97)$$

where

$$\ln \hat{\Phi}_\alpha(\theta) = -i \sum_{s \in 2\mathbb{Z}^+ - 1} \frac{\alpha_s}{s} \sinh(s\theta). \quad (4.98)$$

Let us see explicitly how this prescription leads to the same formulae found in the previous section. For $|\text{Re}(\theta)| < \text{Re}(t)$ the integrand can be expanded as a *long-wave expansion* in θ [151, 266] giving

$$\frac{\cosh^2 \frac{\theta}{2} \tanh \frac{t}{2}}{\cosh t - \cosh \theta} = \sum_{m=1}^\infty [\cosh(m\theta) + (-1)^{m+1}] e^{-mt}. \quad (4.99)$$

Consider next the case of a single perturbation of spin s . We get then

$$\log \mathcal{D}_{\alpha_s}(\theta) = -\frac{\alpha_s}{\pi} \sum_{m=1}^\infty [\cosh k\theta + (-1)^{m+1}] \int_0^\infty dt e^{-mt} \sinh st. \quad (4.100)$$

We are left with the computation of a simple integral

$$\int_0^\infty dt e^{-mt} \sinh st = \frac{s}{m^2 - s^2}, \quad (4.101)$$

which is only well defined away from the simple pole at $m = s$. This means that we need to regularise the infinite sum in k of (4.99) by subtracting the pole. We can do this by modifying (4.101) to

$$\frac{s}{m^2 - s^2} - \frac{m}{m^2 - s^2} = -\frac{1}{m + s}, \quad (4.102)$$

which is equivalent to the prescription (4.97) for each term in the sum in s . The regularised version reads

$$\log \mathcal{D}_{\alpha_s}(\theta) = \frac{\alpha_s}{\pi} \sum_{m=1}^{\infty} \frac{\cosh(m\theta) + (-1)^{m+1}}{m+s}. \quad (4.103)$$

Let us consider carefully the various contributions to this sum, taking care to compare them to the result of Appendix F. First, we have the constant

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m+s} &= - \sum_{m=s+1}^{\infty} \frac{(-1)^{m+1}}{m} \\ &= - \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} + \sum_{m=1}^s \frac{(-1)^{m+1}}{m} \\ &= -\log 2 + \frac{1}{s} + c_s, \end{aligned} \quad (4.104)$$

where c_s ¹¹ and s is odd. is a constant given by

$$c_{2s-1} = \sum_{m=1}^{2s-2} \frac{(-1)^{m+1}}{m} \quad \text{and} \quad \hat{c}_{2s-1} = \sum_{m=1}^{2s-2} \frac{1}{m}, \quad (4.105)$$

Then we have the sum involving $\cosh(m\theta)$. It is useful to consider the two terms in (4.102) separately and exclude the problematic value $m = s$. We compute

$$\sum_{\substack{m=1 \\ m \neq s}}^{\infty} \frac{s \cosh(m\theta)}{m^2 - s^2} = \sum_{m=1}^{s-1} \frac{s \cosh(m\theta)}{m^2 - s^2} + \sum_{m=s+1}^{\infty} \frac{s \cosh(m\theta)}{m^2 - s^2}. \quad (4.106)$$

The infinite sum can be computed to

$$\sum_{m=s+1}^{\infty} \frac{s \cosh(m\theta)}{m^2 - s^2} = - \sum_{m=1}^{s-1} \frac{s \cosh(m\theta)}{m^2 - s^2} + \frac{1}{2s} + \frac{\cosh(s\theta)}{4s} - \frac{\theta - i\pi}{2} \sinh(s\theta), \quad (4.107)$$

so that

$$\sum_{\substack{m=1 \\ m \neq s}}^{\infty} \frac{s \cosh(m\theta)}{m^2 - s^2} = \frac{1}{2s} + \frac{\cosh(s\theta)}{4s} - \frac{\theta - i\pi}{2} \sinh(s\theta). \quad (4.108)$$

The other sum in (4.102) was computed in (F.5). In addition, we have the $m = s$ term coming from the $\cosh(m\theta)$ sum in (4.103) which is simply $-\frac{\cosh(s\theta)}{2s}$ and cancels with contributions from the sums above. Putting everything together with $s = 2n - 1$ we get

$$\begin{aligned} \log \mathcal{D}_{\alpha_{2n-1}}(\theta) &= \frac{\theta - i\pi}{2\pi} \alpha_{2n-1} \sinh((2n-1)\theta) + \frac{\alpha_{2n-1}}{\pi} (-\log 2 + c_{2n-1}) \\ &\quad - \frac{\alpha_{2n-1}}{\pi} \sum_{m=1}^{2n-2} \frac{\cosh(m\theta)}{2n-1-m} - \frac{\alpha_{2n-1}}{2\pi} \cosh((2n-1)\theta) \log \left(-4 \sinh^2 \frac{\theta}{2} \right). \end{aligned} \quad (4.109)$$

¹¹See the alternative derivation in Appendix F for the details.

Compared to the general solution (F.6) without the Ising term, we see that they are identical up to normalisation constants. These are not important in applications, since the MFF is ultimately normalised by its value at $i\pi$. Generalising to any number of $\text{T}\bar{\text{T}}$ perturbations and employing the standard normalisation we can write the general formula (4.48). If we compare this solution to the general formula (3.259) we wrote in [12, 13], we have that the β -dependent part in (3.95) is now fully fixed to

$$C_\beta(\theta) = C_\beta(i\pi) \prod_{s \in 2\mathbb{Z}^+ - 1} \left[-\frac{1}{2} e^{c_s - \sum_{m=1}^{s-1} \frac{\cosh(m\theta)}{s-m}} \left(2i \sinh \frac{\theta}{2} \right)^{-\cosh(s\theta)} \right]^{\frac{\alpha s}{\pi}}. \quad (4.110)$$

and $\varphi_\alpha(\theta)$ is the same function as in (3.94). This construction can be easily extended to branch point twist fields, which satisfy slightly different form factor equations [185, 14]. We discuss this case in Appendix F.3.

4.2.4 Correlation Functions and their Asymptotic Properties

One of the main uses of the form factor program is for the computation of correlation functions of local fields in the ground state. Such correlators can be expanded in terms of integrals of the absolute value squared of the form factors. In the previous chapter, following our papers [12, 13] we argued that the correlation functions resulting from a MFF where the function $\mathcal{D}_\alpha(\theta)$ is chosen as $\mathcal{D}_\alpha(\theta) = e^{\frac{\theta - i\pi}{2\pi} \alpha \sinh \theta}$ will contain integrals whose integrand contains the absolute value of this function squared, that is, $e^{\frac{\alpha\theta}{\pi} \sinh \theta}$. This function grows extremely rapidly for $|\theta|$ large and $\alpha > 0$ and decays extremely rapidly if $\alpha < 0$. Therefore, there are very different physical properties in these two regimes, a feature that is (qualitatively) in line with what is expected from the TBA analysis [106].

Consider instead the new function (4.48). For simplicity, let us take $s = 1$ and $\alpha_1 = \alpha$ so that we have a single $\text{T}\bar{\text{T}}$ perturbation and a single perturbation parameter. The absolute value squared gives

$$|\mathcal{D}_\alpha(\theta)|^2 = |\mathcal{D}_\alpha(i\pi)|^2 \left[\frac{1}{4} e^{\theta \sinh \theta} \left(2 \sinh \frac{\theta}{2} \right)^{-2 \cosh \theta} \right]^{\frac{\alpha}{\pi}}, \quad (4.111)$$

and for $|\theta|$ large we have

$$|\mathcal{D}_\alpha(\theta)|^2 \sim |\mathcal{D}_\alpha(i\pi)|^2 \left[\frac{1}{4} e^{\frac{\theta}{2} e^\theta} e^{-\frac{\theta}{2} e^\theta} \right]^{\frac{\alpha}{\pi}} = |\mathcal{D}_\alpha(i\pi)|^2 4^{-\frac{\alpha}{\pi}}, \quad (4.112)$$

which is a constant. This is a radically different asymptotics compared to what we described previously. This means that the leading asymptotics of the MFF is dictated by the asymptotics of the MFF of the unperturbed theory, and little can be said about the short-distance scaling of correlators if we do not know the full two-particle form factor. However, from the asymptotics we no longer expect to see a marked difference between the α positive and negative cases. There is, however, an important difference in the behaviour of the minimal form factor depending on whether $\alpha > \pi$ or $\alpha < \pi$. In particular, if we consider a $\text{T}\bar{\text{T}}$ -perturbation of the Ising field theory so that

$$\begin{aligned} |F_{\min}^{\text{Ising}}(\theta; \alpha)|^2 &= |F_{\min}^{\text{Ising}}(i\pi; \alpha)|^2 \sinh^2 \frac{\theta}{2} \left[\frac{1}{4} e^{\theta \sinh \theta} \left| 2 \sinh \frac{\theta}{2} \right|^{-2 \cosh \theta} \right]^{\frac{\alpha}{\pi}} \\ &= |F_{\min}(i\pi; \alpha)|^2 4^{-\frac{\alpha}{\pi}} e^{\frac{\alpha\theta}{\pi} \sinh \theta} \left| 2 \sinh \frac{\theta}{2} \right|^{2(1 - \frac{\alpha}{\pi} \cosh \theta)}, \end{aligned} \quad (4.113)$$

we see that for $\alpha < \pi$ the function has a zero at $\theta = 0$ while for $\alpha > \pi$ it has an (unphysical) pole at $\theta = 0$. We see these properties numerically in Fig. 4.5. The presence of a zero at $\theta = 0$ is common with many other IQFTs such as the Ising and sinh-Gordon models. However, the presence of a pole is unphysical and is an indication that the critical value $\alpha_c = \pi$ represents a transition point between two quite different theories. Interestingly, this kind of transition also occurs for UV-complete models and can be seen for instance when comparing the MFF of the sinh-Gordon model [170] with the MFF of the Lee-Yang theory [267]. Indeed, the S -matrices of these two theories are very similar, yet the MFF of the Lee-Yang model with the standard construction would also have a zero at $\theta = 0$. A resolution of this problem was put forward in [267] which consists of multiplying this “natural” MFF by a function that cancels the pole, while still producing a MFF that is consistent with all axioms and asymptotic requirements.

Conclusion

The objective of this thesis was the adaptation of the Form Factor Bootstrap Program (FFBP) to integrable quantum field theories (IQFTs) deformed by generalised $T\bar{T}$ -type perturbations. We can affirm that the results obtained are encouraging, although not yet fully conclusive. Indeed, so far the only object that we have been able to compute exactly is the minimal form factor (MFF). Nevertheless, this represents a solid starting point for the extension of the FFBP to deformed IQFTs.

Although the basic structure of this MFF has been identified in previous work [12, 13, 14, 244, 15] for various types of fields and theories, uniqueness of the solution has been established only recently [16]. We find that the MFF of a $T\bar{T}$ -perturbed IQFT can be uniquely determined by requiring a convergent integral representation, which may be expressed as a regularised version of the usual integral representation. The expression that emerges from this construction has a smooth behaviour in the limit when the perturbation parameter(s) tend to zero and reproduces the MFF of standard IQFTs (such as the sinh-Gordon model) when the number of irrelevant perturbations is infinite and the couplings are suitably chosen (see Appendix F.2 where the sinh-Gordon example is considered). The formula is

$$F_{\min}(\theta; \boldsymbol{\alpha}) = F_{\min}(\theta) \mathcal{D}_{\boldsymbol{\alpha}}(\theta), \quad (4.114)$$

where $F_{\min}(\theta)$ is the MFF of the unperturbed theory and $\mathcal{D}_{\boldsymbol{\alpha}}(\theta)$ is the function (4.48), which is generalised to (F.43) for branch point twist fields [185, 14, 218] and to $\sqrt{\mathcal{D}_{\boldsymbol{\alpha}}(2\theta)}$ for the one-particle MFF of boundary IQFTs [15].

Although our analysis provides a way to fix the MFF which is reminiscent of the traditional construction of Karowski-Weisz [6], it does not make the construction of higher particle solutions any easier. In fact, the correct prescription for computing such solutions is still not fully understood. Solving by plugging our new MFF into the standard ansatz for models with a single particle (like sinh-Gordon [170]) and imposing factorisation (as we previously did) leads to solutions which are not only plagued by square-root factors (thus, points of nonanalyticity), similar to [12, 13], but also lead to a partial cancellation of the MFF in the form factor ansatz (this is discussed in some detail in Appendix F.2). This second issue was not present in our earlier work because we had set the function $C_{\beta}(\theta) = 1$ in (3.259) and worked with the simplest possible form of the MFF.

This cancellation suggests that it may not be possible for the kinematic residue equation as it is, the standard form factor ansatz with our complete MFF, and the assumption of factorisation into a $\boldsymbol{\alpha}$ -dependent and an undeformed part, to all hold simultaneously. As we already discussed in Chapter 3, the locality properties of the observables are crucial, and specifically the kinematic residue equation is underpinned by the notion of locality (or semi-locality). The work [268] sheds some light on the validity of the standard form

factor program. Using formal algebraic techniques, local observables for IQFTs are built by infinite expansions whose coefficients are a sort of generalisation of the usual form factors, coinciding with them for certain regions of the arguments. Local observables are characterised by the analyticity properties of their expansion coefficients. These coefficients must satisfy a set of axioms that correspond exactly to the usual form factor equations. Interestingly, in the conclusion of [268] it is pointed out how this result could be “employed to prove *non*-existence of local observables in the sense of a *no-go theorem*” by showing that for some S -matrices it is impossible to satisfy all the “axioms”.

The CDD-factor (3.86) is proposed as an example of such an S -matrix. Indeed the *non*-existence of local observables in these theories was widely conjectured [165, 162], based mainly on the failure of known algebraic and form factor methods, but this has never been rigorously established. It would be interesting to understand if our constructive approach can be used to prove this statement more rigorously.

As shown in Appendix F.2 a factorised ansatz [12, 13], where the form factor is a product of the unperturbed solution times a function of the perturbation parameter, seems extremely natural for $T\bar{T}$ -perturbed theories but is easily shown not to produce the correct solutions when considering more standard IQFTs, that is, when considering the case of infinitely many irrelevant perturbations. We have checked this explicitly for the sinh-Gordon model, where a factorised ansatz gives solutions that are distinct from those known in the literature [170, 26] and contain square roots, as already discussed (see Appendix F.2).

Therefore, the main result, for now, is a MFF that has good asymptotic and reduction properties. By this we mean that growth with the rapidity variable θ is exponential at most and dictated by the unperturbed MFF. “Reduction” is meant in two senses. First, that the perturbed MFF reduces to the unperturbed one when all $\alpha_s = 0$. Second, it reduces to a new functional representation of known MFFs of IQFTs when the number of α_s parameters is infinite and they take precise, theory-dependent values [244]. A special feature of our solution is that, in the simple case of a single perturbation, say by $T\bar{T}$, there exists a critical value $\alpha = \alpha_c$ above which the MFF develops a pole of order $\frac{\alpha}{\pi} - \kappa$ at $\theta = 0$, where κ is a property of the unperturbed MFF. The existence of a “transition” in the physical properties of the model seems reminiscent of what is observed in the thermodynamic Bethe ansatz analysis (see e.g. [106, 153]), in particular the emergence of a Hagedorn transition at finite temperature. However, there are notable differences between the two situations and we think that they are not related. For one, the Hagedorn transition occurs for *any* negative value of α (with our prescriptions), whereas for our MFF the “transition” happens at a specific, positive value of α . Secondly, it is well known that the Hagedorn transition is particular to the pure $T\bar{T}$ -perturbation and absent for higher spins (unless one modifies the driving term of the TBA and considers specific generalised Gibbs ensembles, as discussed in [202]). In our case, however, the critical point is present for any finite number of perturbations.

The presence of a zero at $\theta = 0$ is common to many IQFTs with a fermionic S -matrix. We find this also in the sinh-Gordon model, for instance [244], since the “Ising factor” $\sinh \frac{\theta}{2}$ is also present as a prefactor, even if the full form factor is much more complicated than (4.113). As discussed in [244] (see also Eq. (4.2)), the S -matrix of the sinh-Gordon model admits a formal representation of the type (2.43), with

$$\alpha_s = \frac{4i^{s+1}}{s} \cos \frac{sb\pi}{2}, \quad \text{with } s \in 2\mathbb{Z}^+ - 1, \quad (4.115)$$

where b is a coupling constant, a characteristic of the model. In this representation, the unperturbed theory is the Ising model, with its MFF given just by $F_{\min}(\theta, \mathbf{0}) = -i \sinh \frac{\theta}{2}$. This means that for $\theta = 0$ the sinh-Gordon model MFF has a zero of order given by

$$1 - \sum_{s \in 2\mathbb{Z}^+ - 1} \frac{4i^{s+1}}{\pi s} \cos \frac{sb\pi}{2} = \begin{cases} 2 & \text{for } b \in \mathbb{R} \text{ and } 0 < |b| < 1 \\ 1 & \text{for } b \in \mathbb{R} \text{ and } |b| = 1 \\ 0 & \text{for } b \in \mathbb{R} \text{ and } |b| > 1 \\ f(\theta_0) & \text{for } b \in \mathbb{C} \text{ and } b = \frac{2i\theta_0}{\pi} \end{cases} \quad (4.116)$$

These values reveal some interesting properties of standard IQFTs:

- The case $0 < |b| < 1$ is the natural range of coupling values for the sinh-Gordon model. There is a zero at $\theta = 0$, which is well known from the standard MFF representation [170].
- For $b = \pm 1$ the sinh-Gordon S -matrix reduces to 1, that is, it is the free-boson limit. In this limit the MFF of the “unperturbed” theory is no longer that of Ising but that of a free boson, which is also 1. In other words, for $b = \pm 1$ the power one of $\sinh \frac{\theta}{2}$ is absent, and we have neither a zero nor a pole at $\theta = 0$.
- For $|b| > 1$ we exit the allowed values in the sinh-Gordon theory and enter a regime which contains in particular the S -matrix of the Lee-Yang model [187]. The latter corresponds to the choice $|b| = 5/3$ and is also a fermionic theory, so the power (??) is present in the MFF, which is 0 at $\theta = 0$. The limit $\theta \rightarrow 0$ is, in fact, a little tricky, giving the indeterminate 0^0 . The known MFF of the Lee-Yang model [267] contains a modification that avoids this singularity. In [267] a MFF was proposed which contains an additional multiplicative factor $\frac{\cosh \theta - 1}{\cosh \theta - \cos \frac{\pi}{3}}$. This has the double effect of canceling the singularity at $\theta = 0$ and of introducing a bound state pole, which is necessary in this theory.
- Finally, for imaginary b , we recover the roaming trajectories model, a theory whose RG-flow is known to visit the vicinity of infinitely many critical points corresponding to the unitary minimal models of CFT [269]. In this case the cosine in the sum above becomes $\cosh(s\theta_0)$, giving a divergent sum. We can however, still make sense of (??) if we appeal to a property of the model, namely symmetry under $\theta_0 \mapsto -\theta_0$. Based on this property, we can argue that we might replace $\cosh(s\theta_0) \mapsto e^{-s|\theta_0|}$. A similar kind of argument was also employed in the form factor calculations presented in [270] and termed CPT symmetrisation. This step allows us to obtain a sum which is convergent and gives power

$$f(\theta_0) = \begin{cases} 2 & \text{for } |\theta_0| \text{ finite,} \\ 1 & \text{for } |\theta_0| \rightarrow \infty. \end{cases} \quad (4.117)$$

Therefore the roaming trajectories model has a MFF with a zero at the origin. The order of the zero is just 1 in the $|\theta_0| \rightarrow \infty$ limit, which corresponds to the Ising field theory.

Equation (4.116) shows that, for UV-complete fermionic IQFTs, the allowed S -matrices must satisfy strict constraints leading to positive powers in the absence of bound state poles and zero or negative powers in the presence of bound states. It would be interesting to investigate whether one could start from these constraints and work backwards to systematically classify all UV complete IQFTs, similar to what was done via thermodynamic

Bethe ansatz in [113].

Clearly, the main goal remains the computation of full correlation functions from the full form factors, if they exist. The fact that the observables are almost certainly non-local does not *a priori* exclude the possibility that form factors can be computed, and we cannot end this thesis without some proposals on how to pursue this program.

The FFBP relies on a few but fundamental concepts: locality, analyticity, and integrable structure. If at least one of these is not satisfied, the result is not guaranteed. In particular, we have shown that the Karowski–Weisz procedure for constructing the MFF can be recovered through a regularisation method. This is far from surprising in the context we are interested in, namely that of a theory with a minimal fundamental length, which can be interpreted essentially as a QFT on curved space-time [115, 100, 102].

On curved backgrounds it is not possible to define a tempered distribution in the same way as in flat space, and further regularisations are required. A simple example is given by the definition of the Dirac delta distribution in curved space-time.

Recall that in flat space the Dirac delta can be characterised as

$$\int_{\mathbb{R}^d} d^d x \delta^{(d)}(x) f(x) = f(0), \quad (4.118)$$

for every test function $f \in \mathcal{S}(\mathbb{R}^d)$. On a curved manifold $(\mathcal{M}, g)$, however, one must take into account the non-trivial volume element. The natural generalisation is

$$\int_{\mathcal{M}} d^d x \sqrt{|g(x)|} \delta_g(x, y) f(x) = f(y), \quad (4.119)$$

where $\delta_g(x, y)$ denotes the covariant delta distribution with respect to the metric g .

The distribution $\delta_g(x, y)$ cannot be introduced directly as in flat space but must instead be constructed through a regularisation procedure. Several techniques are commonly employed in quantum field theory on curved space-times. One possibility is to use the heat-kernel representation, where the delta emerges as the $t \rightarrow 0^+$ limit of the solution to the diffusion equation governed by the Laplace–Beltrami operator [271]. Another approach, known as point-splitting or Hadamard subtraction, is based on separating coincident points along a geodesic and subtracting the universal short-distance singularities before taking the coincidence limit [272]. Alternatively, in a small neighbourhood of a point one can employ Riemann normal coordinates, where the metric is approximately flat, and define the delta distribution perturbatively in the curvature. Other schemes, can also be employed depending on the physical context. All these methods share the same conceptual feature: a regulator is introduced to define the distribution at short distances, and only after suitable subtractions can one safely remove it. This perspective connects naturally with the Peierls quantisation scheme that we have already introduced in Chapter 0 (see Eq. (0.7)), this time upgraded to curved space-time. In this framework the Poisson bracket of two classical observables A and B is defined as

$$\{A, B\}_{\text{Peierls}} = \int_{\mathcal{M}} d^d x \sqrt{|g(x)|} \int_{\mathcal{M}} d^d y \sqrt{|g(y)|} \frac{\delta B}{\delta \phi(y)} \Delta(x, y) \frac{\delta A}{\delta \phi(x)}, \quad (4.120)$$

where $\Delta(x, y) = G_{\text{ret}}(x, y) - G_{\text{adv}}(x, y)$ is the causal Green’s function of the covariant wave operator, constructed using regularised distributions as in (4.119). Quantisation is then obtained by promoting $\{A, B\}_{\text{Peierls}}$ to the commutator $-i[A, B]$.

Hence, the need for regularisation in defining distributions on curved space-time is not just a technicality: it is essential for the very definition of observables and their algebraic

relations, and it provides a natural motivation for the regularisation methods we employed in the FFBP for theories with a minimal length scale.

The challenge now is to adapt the kinematical residue equation (3.27) to this set-up. The annihilation pole is a direct consequence of the LSZ-reduction formulas (see Appendix A) for which general results in curved space-time are actually available [273, 274, 275, 271]. Moreover, the most striking and troublesome problem of having curved backgrounds in a QFT is absent here: the theory is integrable and therefore there is no particle production. This fact should greatly simplify the LSZ-reduction formalism in our case. Anyway, this still seems to be a very challenging task, because, as can be seen from the form of the MFF in (4.48), we no longer expect to have a standard kinematical pole. In fact, the zeros (or poles) of (4.48) are both rapidity- and α -dependent. Again, this is exactly what one would expect from a theory on a dynamical background [100, 102]. Furthermore, in a curved background the notion of rapidity itself is not well defined, so is this even the best way to look at things?

Another possibility, in principle, comes from the original method used by Smirnov to obtain the form factors of the sine-Gordon model [7, 8, 9, 10]. The idea is to fully exploit the QISM to explicitly construct the soliton and antisoliton creation operators, from which the form factors can then be obtained. In this sense, one could attempt to build an algebra similar to the one we proposed in (2.71), but using the standard QISM procedure. In fact, the classical integrable structure of the $T\bar{T}$ -deformed sine-Gordon model was derived in [102], and having the deformed classical Lax pair should, in principle, allow us to attempt the QISM construction. The difficulty, however, is that the sine-Gordon model is already highly non-linear, and the deformation introduces additional non-linearity even at the classical level, exponentially increasing the complexity of the problem.

Appendix A

LSZ Reduction Formulas

In this section we develop the LSZ reduction formalism [34, 276]. This is a standard topic in QFT textbooks but for us it is particularly relevant because the form factor axioms can be derived from them. The problem is the computation of S-matrix elements between on-shell states in generic interacting field theories. In Section 1.1 we already outlined the main picture: on a macroscopic scale, interaction time is extremely small and therefore there is no point in following the time evolution during a scattering event. We consider *asymptotic states* (1.1.1): long before the collision, well-separated wave-packets evolve independently and freely. These incoming states build a Fock space, the in-space associated to free fields. Then the collision process happens and scattering, absorption and creation of particles occur (the permitted processes are dictated by the symmetries). Long after the collision, free wave packets separate, representing the outgoing states. These are again described by free particles, i.e. the modes of a free out-field. Although the form of the interaction (until it is local) is not relevant for the construction, just to give an idea, we consider a self-interacting scalar field

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4, \quad (\text{A.1})$$

that satisfies the equation of motion $(\square + m^2)\phi(x) = \frac{\lambda}{3!}\phi^3(x)$, the so-called local “interpolating field” of Section 1.1. In the context of canonical quantization we can construct an in-field (out-field)

$$\phi_{\text{in}}(x) = \frac{1}{2\pi} \int \frac{dp}{\sqrt{2p^0}} (a(p)e^{-ip\cdot x} + a^\dagger(p)e^{ip\cdot x}), \quad p^0 = \sqrt{(p^1)^2 + m^2}, \quad (\text{A.2})$$

that satisfies the K-G equation $(\square + m^2)\phi_{\text{in}}(x) = 0$ (same for the out field) and when acting on states (add ref to particle states) annihilates (creates) particles and creates (annihilates) anti-particles. The operators in (A.2) define the usual CCR algebra

$$[a(p), a_{\text{in}}(q)] = [a^\dagger(p), a^\dagger(q)] = 0, \quad [a(p), a_{\text{in}}^\dagger(q)] = 2p^0 2\pi\delta(q - p), \quad (\text{A.3})$$

and the Fock space generated by the action of the vacuum defined by these operators coincide with the space of the asymptotic initial states. We can then invert the relation (A.2) to rewrite the CCR operators in terms of the fields

$$\sqrt{2p^0}a_{\text{in}}(p) = \int dx \langle e^{-ip \cdot x}, \phi_{\text{in}}(x) \rangle = i \int dx (\dot{\phi}_{\text{in}}(x) - ip^0 \phi_{\text{in}}(x)) e^{ip \cdot x}, \quad (\text{A.4})$$

$$\sqrt{2p^0}a_{\text{in}}^\dagger(p) = - \int dx \langle e^{ip \cdot x}, \phi_{\text{in}}(x) \rangle = -i \int dx (\dot{\phi}_{\text{in}}(x) + ip^0 \phi_{\text{in}}(x)) e^{-ip \cdot x},$$

(again, same for the out-field) where

$$\langle f, g \rangle \equiv i \int dx (f^* \partial_0 g - \partial_0 f^* g) \quad (\text{A.5})$$

is the inner product preserved in the space of solutions of the K-G equation. Remember also the normalization of the 1-particle states

$$|p\rangle = \sqrt{2p^0} a^\dagger(p) |0\rangle, \quad \langle q | p \rangle = (2p^0)(2\pi) \delta(p - q). \quad (\text{A.6})$$

Here **Axiom III** (1.8) enters into play. *The matrix elements of the interacting field between asymptotic states coincide except for a normalization constant, with the matrix elements of the free fields:*

$$\phi_{\text{in}}(x) = \frac{1}{\sqrt{Z}} \lim_{t \rightarrow -\infty} \phi(x), \quad \phi_{\text{out}}(x) = \frac{1}{\sqrt{Z}} \lim_{t \rightarrow +\infty} \phi(x). \quad (\text{A.7})$$

Again, it has to be clear that this assumption is taken to be valid **only** for the matrix elements and not from an operatorial point of view (weak operator convergence). Now we want to consider a transition from a n -particle state $|p_1, p_2, \dots, p_n\rangle$ to a m -particle state $|q_1, q_2, \dots, q_m\rangle$:

$$\langle q_1, q_2, \dots, q_m | p_1, p_2, \dots, p_n \rangle, \quad (\text{A.8})$$

where the out-states are created by the action of $a_{\text{out}}^\dagger(q_i)$ and the in-states by the action of $a_{\text{in}}^\dagger(p_j)$. From this fact follows that we can rewrite the amplitude in terms of the in/out fields from (A.5)

$$\begin{aligned} \langle q_1, q_2, \dots, q_m | p_1, p_2, \dots, p_n \rangle &= \sqrt{2p_1^0} \langle q_1, q_2, \dots, q_m | a_{\text{in}}^\dagger(p_1) | p_2, \dots, p_n \rangle \\ &= -i \int dx_1 e^{-ip_1 \cdot x_1} \langle q_1, q_2, \dots, q_m | (\dot{\phi}_{\text{in}}(x_1) + ip_1^0 \phi_{\text{in}}(x_1)) | p_2, \dots, p_n \rangle. \end{aligned} \quad (\text{A.9})$$

And now, substituting the free-fields with the asymptotic limit of the interacting ones (A.7) we get

$$\begin{aligned} \langle q_1, q_2, \dots, q_m | p_1, p_2, \dots, p_n \rangle &= -\frac{i}{\sqrt{Z}} \lim_{t_1 \rightarrow -\infty} \int dx_1 e^{-ip_1 \cdot x_1} \\ &\quad \times \langle q_1, q_2, \dots, q_m | (\dot{\phi}(x_1) + ip_1^0 \phi(x_1)) | p_2, \dots, p_n \rangle. \end{aligned} \quad (\text{A.10})$$

Now applying the identity

$$\int_{-\infty}^{\infty} dt' \frac{df(t')}{dt'} = \lim_{t \rightarrow \infty} f(t) - \lim_{t \rightarrow -\infty} f(t) \quad (\text{A.11})$$

to $e^{-ip_1 \cdot x_1} (\dot{\phi}(x_1) + ip_1^0 \phi(x_1))$ we get

$$\begin{aligned}
\langle q_1, q_2, \dots, q_m | p_1, p_2, \dots, p_n \rangle = & \\
& - \frac{i}{\sqrt{Z}} \lim_{t_1 \rightarrow +\infty} \int dx_1 e^{-ip_1 \cdot x_1} \langle q_1, q_2, \dots, q_m | (\dot{\phi}(x_1) + ip_1^0 \phi(x_1)) | p_2, \dots, p_n \rangle \\
& + \frac{i}{\sqrt{Z}} \int d^2 x_1 e^{-ip_1 \cdot x_1} \langle q_1, q_2, \dots, q_m | (\ddot{\phi}(x_1) + (p_1^0)^2 \phi(x_1)) | p_2, \dots, p_n \rangle.
\end{aligned}$$

We can clearly identify in the first term the asymptotic field ϕ_{out} while for the second term we can use the fact that the asymptotic states are on-shell and therefore $(p_1^0)^2 = (p_1^1)^2 + m^2$:

$$\begin{aligned}
\langle q_1, q_2, \dots, q_m | p_1, p_2, \dots, p_n \rangle = & \langle q_1, q_2, \dots, q_m | a_{\text{out}}^\dagger(p_1) | p_2, \dots, p_n \rangle \\
& + \frac{i}{\sqrt{Z}} \int d^2 x_1 e^{-ip_1 \cdot x_1} \langle q_1, q_2, \dots, q_m | (\ddot{\phi}(x_1) + ((p_1^1)^2 + m^2) \phi(x_1)) | p_2, \dots, p_n \rangle.
\end{aligned} \tag{A.12}$$

Therefore the first term is non-zero only if there is a $q_i = p_1$. This means that the moments of the initial asymptotic state remain unaltered in the final state and therefore the first term represent a “disconnected” contribution in the scattering process. In the end, if we are looking at the amplitude of connected processes we can ignore this term. Now, integrating by parts in the second term

$$\int d^2 x_1 e^{-ip_1 \cdot x_1} (p_1^1)^2 \phi(x_1) = - \int d^2 x_1 (\nabla^2 e^{-ip_1 \cdot x_1}) \phi(x_1) = - \int d^2 x_1 e^{-ip_1 \cdot x_1} \nabla^2 \phi(x_1), \tag{A.13}$$

$$\begin{aligned}
\langle q_1, q_2, \dots, q_m | p_1, p_2, \dots, p_n \rangle = & \frac{i}{\sqrt{Z}} \int d^2 x_1 e^{-ip_1 \cdot x_1} (\square_{x_1} + m^2) \langle q_1, q_2, \dots, q_m | \phi(x_1) | p_2, \dots, p_n \rangle \\
& + \text{disconnected terms} \dots
\end{aligned} \tag{A.14}$$

We removed the first particle of the in-state by substituting it with the field operator $\phi(x_1)$. Now we can do the same with the second particle in the in-state:

$$\begin{aligned}
\langle q_1, q_2, \dots, q_m | p_1, p_2, \dots, p_n \rangle = & \\
& - i \int dx_2 e^{-ip_2 \cdot x_2} \langle q_1, q_2, \dots, q_m | \phi(x_1) (\dot{\phi}(x_2) + ip_2^0 \phi(x_2)) | p_3, \dots, p_n \rangle \\
& - \frac{i}{\sqrt{Z}} \int dx_2 e^{-ip_2 \cdot x_2} \langle q_1, q_2, \dots, q_m | T \left[\phi(x_1) (\ddot{\phi}(x_2) + ip_2^0 \phi(x_2)) \right] | p_3, \dots, p_n \rangle,
\end{aligned} \tag{A.15}$$

where T here indicates a time-ordering, that in fact can be used repeatedly to separate the disconnected contributions. Iterating this procedure n -times we get in the end, without disconnected processes

$$\begin{aligned}
\langle q_1, q_2, \dots, q_m | p_1, p_2, \dots, p_n \rangle = & \left(\frac{i}{\sqrt{Z}} \right)^n \prod_{i=1}^n \int d^2 x_i e^{-ip_i \cdot x_i} \\
& \times (\square_{x_i} + m^2) \langle q_1, q_2, \dots, q_m | T [\phi(x_1) \phi(x_2) \dots \phi(x_n)] | \Omega \rangle,
\end{aligned} \tag{A.16}$$

where $|\Omega\rangle$ represents the Fock space vacuum state. Clearly we can do the same thing for the out-state. The final result are the famous LSZ-reduction formulas [34]

$$\begin{aligned} \langle q_1, q_2, \dots, q_m | p_1, p_2, \dots, p_n \rangle = & \\ \left(\frac{i}{\sqrt{Z}} \right)^{n+m} \prod_{i=1}^n \prod_{j=1}^m \int d^2 x_i e^{-ip_i \cdot x_i} (\square_{x_i} + m^2) \int d^2 y_j e^{iq_j \cdot y_j} (\square_{y_j} + m^2) & \\ \times \langle \Omega | T[\phi(x_1) \phi(x_2) \cdots \phi(x_n) \phi(y_1) \phi(y_2) \cdots \phi(y_m)] | \Omega \rangle. & \quad (\text{A.17}) \end{aligned}$$

that relates S-matrix elements, i.e. products between asymptotic states, and connected correlation functions in the vacuum.

Appendix B

Expectation Value of $T\bar{T}$ in Two-Dimensional Quantum Field Theories

Consider a two-dimensional field theory of stress-energy tensor $T_{\mu\nu}$. Already in [190] the expectation value of the descendant field $T\bar{T}$ in the Sine-Gordon model was calculated to be

$$\langle T\bar{T} \rangle = -\langle \Theta \rangle^2. \quad (\text{B.1})$$

The foundational work by A. B. Zamolodchikov [90] generalizes this result to generic non-integrable two-dimensional field theories under quite general assumptions. In particular, the expectation value in a cylinder of infinite length (Matsubara representation at finite temperature $\sim 1/R$) is given by

$$\langle T\bar{T} \rangle = \langle T \rangle \langle \bar{T} \rangle - \langle \Theta \rangle \langle \Theta \rangle. \quad (\text{B.2})$$

Clearly, when the radial coordinate $R \rightarrow \infty$ (equivalently $T \rightarrow 0$), the translation symmetry gets renewed and the expectation values of the chiral components go to zero and we return to (B.1). Following [90] we prove (B.2) making some very general assumptions; some of them concern the “local dynamics” of the field theory (L) and others concern global geometrical aspects (G). We assume:

1. (L) *Local symmetry under rotations and translations.* It implies that the stress-energy tensor is symmetric $T_{\mu\nu} = T_{\nu\mu}$ and that it satisfies the continuity equation $\partial_\mu T^{\mu\nu} = 0$, or in terms of the chiral components

$$\bar{\partial}T(z) = \partial\Theta(z), \quad \partial\bar{T}(z) = \bar{\partial}\Theta(z). \quad (\text{B.3})$$

This natural assumption implies that the physical quantities depend only on relative space-time coordinates, as one would expect from a local theory. Indeed, this assumption is already taken into account writing an OPE for the observables of a given model

$$\mathcal{O}_i(z)\mathcal{O}_j(z') = \sum_k C_{ij}^k(z-z')\mathcal{O}_k(z'), \quad (\text{B.4})$$

where indeed the structure constants depend only on separations.

2. (G) *Global symmetry under translations.* For every local field $\mathcal{O}_i(z)$, its expectation value in the vacuum $\langle \mathcal{O}_i(z) \rangle$ is a space-time independent constant. Follows from (B.4) that also two-point functions depend only on separations

$$\langle \mathcal{O}_i(z)\mathcal{O}_j(z') \rangle = G_{ij}(z-z'). \quad (\text{B.5})$$

3. (G) *Infinite separations.* We assume it exists at least one direction, i.e. an Euclidean vector $e = (e, \bar{e})$, such that for any pair of fields $\mathcal{O}_i$ and $\mathcal{O}_j$ we have

$$\lim_{t \rightarrow \infty} \langle \mathcal{O}_i(z + et) \mathcal{O}_j(z') \rangle = \langle \mathcal{O}_i \rangle \langle \mathcal{O}_j \rangle. \quad (\text{B.6})$$

Together with the assumption 2, this condition imply that the underlying geometry is the one of an infinite plane or of a infinitely long cylinder.

4. (L) *CFT limit at short distances.* The short distance behavior is assumed to be described by a conformal field theory. This assumption is necessary to make unambiguous the definition of the composite operator $T\bar{T}$, which is essentially defined up to total derivative terms.

Now, consider the following two-point functions combination

$$\mathcal{C} = \langle T(z) \bar{T}(z') \rangle - \langle \Theta(z) \Theta(z') \rangle. \quad (\text{B.7})$$

By using (B.3), it is easy to see that $\bar{\partial}\mathcal{C} = \partial\mathcal{C} = 0$ and therefore this combination is a constant, independent from coordinates. Now, taking the limit, as in (B.5), we find that, for large separations

$$\mathcal{C} = \langle T \rangle \langle \bar{T} \rangle - \langle \Theta \rangle \langle \Theta \rangle. \quad (\text{B.8})$$

Now, what we need to prove is that $\mathcal{C}$ coincides with the expectation value of the composite field $T\bar{T}$. We expect to obtain this object from the OPE of $T(z)\bar{T}(z')$ in the limit $z \rightarrow z'$, but the main problem comes precisely from the divergencies of the OPE itself. We will see that the term subtracted in (B.7) is exactly needed to cancel the problematic singular terms. The continuity equations imply that

$$\begin{aligned} \bar{\partial} [T(z) \bar{T}(z') - \Theta(z) \Theta(z')] &= \\ &= (\partial + \partial') \Theta(z) \bar{T}(z') - (\bar{\partial} + \bar{\partial}') \Theta(z) \Theta(z') + \text{total derivatives} \end{aligned} \quad (\text{B.9})$$

and similarly

$$\begin{aligned} \partial [T(z) \bar{T}(z') - \Theta(z) \Theta(z')] &= \\ &= (\partial + \partial') T(z) \bar{T}(z') - (\bar{\partial} + \bar{\partial}') T(z) \Theta(z') + \text{total derivatives}. \end{aligned} \quad (\text{B.10})$$

If we then substitute the operator product expansions

$$T(z) \Theta(z') = \sum_i A_i(z - z') \mathcal{O}_i(z'), \quad \Theta(z) \bar{T}(z') = \sum_i B_i(z - z') \mathcal{O}_i(z'), \quad (\text{B.11})$$

$$\Theta(z) \Theta(z') = \sum_i C_i(z - z') \mathcal{O}_i(z'), \quad T(z) \bar{T}(z') = \sum_i D_i(z - z') \mathcal{O}_i(z'), \quad (\text{B.12})$$

where the sums involve a complete set of local fields $\{\mathcal{O}_i\}$. Now (B.9) and (B.10) read

$$\begin{aligned} \sum_i \bar{\partial} F_i(z - z') \mathcal{O}_i(z') &= \\ &= \sum_i (B_i(z - z') \partial' \mathcal{O}_i(z') - C_i(z - z') \bar{\partial}' \mathcal{O}_i(z')) , \end{aligned} \quad (\text{B.13})$$

and

$$\sum_i \partial F_i(z - z') \mathcal{O}_i(z') = \sum_i (D_i(z - z') \partial' \mathcal{O}_i(z') - A_i(z - z') \bar{\partial}' \mathcal{O}_i(z')) , \quad (\text{B.14})$$

where

$$F_i(z - z') = D_i(z - z') - C_i(z - z') . \quad (\text{B.15})$$

The equations above involve only derivatives of local fields and therefore any operator $\mathcal{O}_i$ in the expansion

$$T(z) \bar{T}(z') - \Theta(z) \Theta(z') = \sum_i F_i(z - z') \mathcal{O}_i(z') \quad (\text{B.16})$$

can only possess a constant coefficient F_i unless it is by itself a descendant of another local operator. Therefore, the OPE (B.16) can be rewritten as

$$T(z) \bar{T}(z') - \Theta(z) \Theta(z') = \mathcal{O}_{T\bar{T}}(z') + \text{derivative terms} , \quad (\text{B.17})$$

for some local operator $\mathcal{O}_{T\bar{T}}$. Then, from (B.17) we can define the composite field $T\bar{T}$ as

$$T\bar{T}(z) := \mathcal{O}_{T\bar{T}} , \quad (\text{B.18})$$

and (B.2) follows. As discussed previously there is still a necessity to understand better the nature of the limit $z \rightarrow z'$ that is taken in (B.16). This problem is related to the short-distance nature of the theory that is furnished through assumption 4. We refer for the original paper for a detailed discussion [90] but the main idea is to take into consideration the case of a CFT perturbed by some relevant field Φ_Δ of dimensions (Δ, Δ) with $\Delta < 1$. In this context the field $T\bar{T}$ of dimension (2,2), can be easily recognized as a descendent of the identity field, $T\bar{T} = L_{-2} \bar{L}_{-2} I$. The problem arises when certain “resonance conditions”, between our composite field and the other fields $\mathcal{O}_i$ in the theory are met. To evade this problem the author impose the following “non-resonance” assumption:

if the dimendions Δ_i of the fields $\mathcal{O}_i$ in the CFT limit satisfy the condition

$$\Delta_i - 2 + n(1 - \Delta) \neq 0 , \quad \text{for } n = 1, 2, 3, \dots ; \quad (\text{B.19})$$

with the exception of $\Delta_i = \Delta + 1$ (the dimension of $\partial \bar{\partial} \Phi_\Delta$). There will be no resonance-induced ambiguities in the definition of (B.18).

Appendix C

$T\bar{T}$ -deformed Theories as QFTs on Noncommutative Minkowski Space

In this appendix we review a formulation of quantum field theory on noncommutative Minkowski space proposed in [165] which, when specialized to two spacetime dimensions, naturally leads to wedge-local quantum fields and to a non-trivial factorizing S-matrix of the $T\bar{T}$ -type (2.43). This appendix provides an algebraic and geometric reinterpretation of the $T\bar{T}$ -deformed scattering data discussed in the main text, without assuming the existence of point-local observables. We work on two-dimensional Minkowski space $\mathbb{R}^{1,1}$ with coordinates $x = (x^0, x^1)$ and scalar product

$$p \cdot x = p^0 x^0 - p^1 x^1, \quad (\text{C.1})$$

and consider a real scalar particle of mass $m > 0$. The positive mass shell is

$$H_m^+ = \{p \in \mathbb{R}^{1,1} : p^2 = m^2, p^0 > 0\}, \quad (\text{C.2})$$

equipped with the standard Lorentz-invariant measure $d\mu(p)$. As in the rest of the thesis, it will be convenient to parametrize H_m^+ by the rapidity variable $\theta \in \mathbb{R}$,

$$p(\theta) = m(\cosh \theta, \sinh \theta), \quad d\mu(p) = d\theta. \quad (\text{C.3})$$

C.1 Noncommutative Minkowski space and deformed quantum fields

Noncommutative Minkowski space is modeled by promoting the spacetime coordinates to self-adjoint operators $\hat{x}^\mu$, $\mu = 0, 1$, satisfying the commutation relations

$$[\hat{x}^\mu, \hat{x}^\nu] = iQ^{\mu\nu}, \quad (\text{C.4})$$

where Q is a real antisymmetric 2×2 matrix. In two spacetime dimensions any such matrix is necessarily proportional to the Levi-Civita tensor. We therefore fix a reference noncommutativity parameter of the form

$$Q_1 = \alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \alpha \in \mathbb{R}. \quad (\text{C.5})$$

A central structural feature of the framework is that the noncommutativity parameter Q is not kept fixed under spacetime symmetries. Instead, one considers a family of

quantum fields $\phi(Q, x)$ labelled by Q and implements the action of the *untwisted* Poincaré group in such a way that Lorentz transformations act simultaneously on spacetime and on the parameter Q . In two dimensions, the Lorentz orbit of the reference matrix Q_1 reduces to the two-element set $\{Q_1, -Q_1\}$.¹ We now recall the construction of the deformed quantum fields. Let $\mathcal{H}$ denote the bosonic Fock space built over the one-particle Hilbert space $L^2(H_m^+, d\mu)$, and let $a^\dagger(p)$ and $a(p)$ be the standard creation and annihilation operators satisfying the canonical commutation relations

$$[a(p), a(q)] = 0, \quad [a(p), a^\dagger(q)] = \omega_p \delta(p - q) \mathbf{1}, \quad (\text{C.6})$$

with $\omega_p = p^0$. The free massive scalar field is given by

$$\phi_0(x) = \int d\mu(p) (e^{ip \cdot x} a^\dagger(p) + e^{-ip \cdot x} a(p)), \quad (\text{C.7})$$

understood as an operator-valued tempered distribution. The deformation associated with the noncommutativity parameter Q is implemented directly at the level of creation and annihilation operators by defining

$$a(Q, p) = e^{\frac{i}{2}pQP} a(p), \quad a^\dagger(Q, p) = e^{-\frac{i}{2}pQP} a^\dagger(p), \quad (\text{C.8})$$

where P_μ are the generators of spacetime translations on Fock space. By construction, the deformed creation operator is the Hilbert space adjoint of the deformed annihilation operator,

$$a^\dagger(Q, p) = (a(Q, p))^\dagger, \quad (\text{C.9})$$

a relation that follows from the antisymmetry of Q and the self-adjointness of the translation generators P_μ . The corresponding deformed field is then defined as

$$\phi(Q, x) = \int d\mu(p) (e^{ip \cdot x} a^\dagger(Q, p) + e^{-ip \cdot x} a(Q, p)), \quad (\text{C.10})$$

again in the sense of operator-valued distributions. The nontrivial structure induced by the deformation is already visible at the level of the exchange relations satisfied by the operators $a(Q, p)$ and $a^\dagger(Q, p)$. For arbitrary antisymmetric matrices Q, Q' and on-shell momenta p, p' , one finds

$$a(Q, p)a(Q', p') = e^{-\frac{i}{2}p(Q+Q')p'} a(Q', p')a(Q, p), \quad (\text{C.11})$$

$$a^\dagger(Q, p)a^\dagger(Q', p') = e^{-\frac{i}{2}p(Q+Q')p'} a^\dagger(Q', p')a^\dagger(Q, p), \quad (\text{C.12})$$

$$a(Q, p)a^\dagger(Q', p') = e^{\frac{i}{2}p(Q+Q')p'} a^\dagger(Q', p')a(Q, p) + \omega_p \delta(p - p') e^{\frac{i}{2}p(Q-Q')P}. \quad (\text{C.13})$$

Here we have introduced the bilinear form

$$pQp' \equiv p_\mu Q^{\mu\nu} p'_\nu, \quad (\text{C.14})$$

which controls all momentum-dependent phase factors appearing in the exchange relations. Since Q is antisymmetric, the quantity pQp' is antisymmetric under the exchange $p \leftrightarrow p'$ and vanishes when the two momenta coincide. All nontrivial effects of the deformation originate from this bilinear form. For $Q = 0$ this definition reduces identically to

¹Given a representation of the Lorentz group L on a space $\mathcal{X}$, the *orbit* of an element $x \in \mathcal{X}$ is the set $\mathcal{O}(x) = \{\Lambda \cdot x : \Lambda \in L\}$. In the present case, the Lorentz group acts on antisymmetric matrices by $Q \mapsto \Lambda Q \Lambda^T$, and in two spacetime dimensions this action maps Q_1 only to $\pm Q_1$.

the free field $\phi_0(x)$. Although the deformation enters only through unitary phase factors, the fields $\phi(Q, x)$ are no longer point-local for $Q \neq 0$. The violation of locality already appears at the level of the vacuum two-particle contribution of the commutator, which does not vanish for spacelike separated arguments. At the same time, the fields $\phi(Q, x)$ retain all other standard Wightman properties: they are well defined on the dense domain of finite particle number, transform covariantly under translations, and satisfy the Reeh-Schlieder property² [20]. This loss of pointlike locality will be shown in the following sections to admit a precise geometric interpretation in terms of wedge localization.

C.2 Wedge localization and geometric interpretation

The loss of pointlike locality exhibited by the deformed fields $\phi(Q, x)$ admits a precise geometric interpretation in terms of wedge regions of Minkowski space. In two spacetime dimensions this interpretation becomes particularly transparent, due to the simplicity of both the Lorentz group and the causal structure. We choose as reference wedge the right wedge

$$W_1 = \{x \in \mathbb{R}^{1,1} : x^1 > |x^0|\}, \quad (\text{C.15})$$

whose causal complement is the left wedge

$$W'_1 = -W_1 = \{x \in \mathbb{R}^{1,1} : x^1 < -|x^0|\}. \quad (\text{C.16})$$

In two dimensions every proper orthochronous Lorentz transformation is a boost in the x^1 -direction and therefore leaves W_1 invariant. As a consequence, the set of wedge regions with edge passing through the origin reduces to the two-element set

$$\mathcal{W}_0 = \{W_1, -W_1\}. \quad (\text{C.17})$$

A completely analogous simplification occurs for the noncommutativity parameter. Restricting attention to the reference matrix Q_1 introduced in the previous section, one finds that in two dimensions its Lorentz orbit consists of only two elements,

$$\mathcal{Q} = \{Q_1, -Q_1\}. \quad (\text{C.18})$$

This makes it possible to establish a canonical identification between wedge regions and noncommutativity parameters by setting

$$Q(W_1) = Q_1, \quad Q(-W_1) = -Q_1, \quad (\text{C.19})$$

which satisfies the compatibility condition

$$Q(W') = -Q(W). \quad (\text{C.20})$$

This relation mirrors the correspondence between a wedge and its causal complement and will play a crucial role in the localization properties of the associated fields. Using

²The Reeh-Schlieder property is necessary to ensure that the field theory is not degenerate, in the sense that local (or wedge-local) observables are capable of generating a dense set of physical states from the vacuum. Without this property, localization would lead to superselection sectors already at the kinematical level, preventing a meaningful scattering theory and obstructing the interpretation of the fields as generators of physical excitations.

the identification (C.19), the family of deformed fields $\phi(Q, x)$ can be reinterpreted as a family of wedge-indexed fields on ordinary Minkowski space. For $W \in \mathcal{W}_0$ we define

$$\phi_W(x) = \phi(Q(W), x), \quad (\text{C.21})$$

with smeared versions given by

$$\phi_W(f) = \int d^2x f(x) \phi_W(x), \quad (\text{C.22})$$

for test functions $f \in \mathcal{S}(\mathbb{R}^{1,1})$. These fields are not localized at points; however, they admit a natural interpretation as being localized in wedge regions. More precisely, the operator $\phi_W(f)$ may be regarded as localized in the region $W + \text{supp } f$. The crucial structural property is **wedge-local commutativity**. Let f and g be test functions such that

$$\text{supp } f \subset W_1, \quad \text{supp } g \subset -W_1. \quad (\text{C.23})$$

Then one has

$$[\phi_{W_1}(f), \phi_{-W_1}(g)] = 0 \quad (\text{C.24})$$

as an operator identity on the domain of finite particle number. The proof relies on the explicit form of the exchange relations satisfied by the deformed creation and annihilation operators and on analyticity properties of the Fourier transforms of f and g in the rapidity variable. In particular, the dependence of the deformation factors only on differences of rapidities allows one to shift integration contours in the complex rapidity plane, leading to an exact cancellation of the commutator. As a result, the family $\{\phi_{W_1}, \phi_{-W_1}\}$ defines a Poincaré-covariant, wedge-local quantum field theory. The weakening of localization from points to wedges is the key structural ingredient that permits the presence of interaction while preserving relativistic covariance and exact solvability in two dimensions.

C.3 Two-particle scattering and integrable structure in two dimensions

We now turn to the interpretation of the wedge-local theory constructed above as an interacting quantum field theory in two spacetime dimensions. As discussed in the previous sections, in two dimensions the theory involves precisely two wedge-local fields, ϕ_{W_1} and ϕ_{-W_1} , associated with the two noncommutativity parameters Q_1 and $-Q_1$. Their creation and annihilation operators satisfy exchange relations whose deformation factors depend only on the difference of rapidities. This feature is characteristic of factorizing quantum field theories and allows for a direct identification of the two-particle scattering function.

Restricting to the reference parameter Q_1 , the deformation phases appearing in the exchange relations are governed by the bilinear form pQ_1p' introduced in the previous section. In two spacetime dimensions this quantity admits a particularly simple expression. Writing

$$Q_1^{\mu\nu} = \alpha \varepsilon^{\mu\nu}, \quad \varepsilon^{01} = 1, \quad (\text{C.25})$$

and parametrizing on-shell momenta by rapidity according to $p(\theta) = m(\cosh \theta, \sinh \theta)$, one finds that $p(\theta_1)Q_1p(\theta_2)$ depends only on the difference of rapidities. A direct computation yields

$$p(\theta_1)Q_1p(\theta_2) = \alpha m^2 \sinh(\theta_1 - \theta_2). \quad (\text{C.26})$$

As a consequence, the exchange relations of the deformed creation operators take the form

$$a^\dagger(Q_1, \theta_1)a^\dagger(Q_1, \theta_2) = e^{i\alpha m^2 \sinh(\theta_1 - \theta_2)} a^\dagger(Q_1, \theta_2)a^\dagger(Q_1, \theta_1), \quad (\text{C.27})$$

with an analogous relation for the annihilation operators. This motivates the definition of a two-particle scattering function

$$S_2(\theta) = \exp(i\alpha m^2 \sinh \theta), \quad (\text{C.28})$$

which depends only on the rapidity difference. The function $S_2(\theta)$ is unitary for real θ and analytic in the physical strip, and therefore satisfies the standard consistency requirements for a two-particle S-matrix in relativistic integrable quantum field theory. In particular, the absence of particle production and the factorization of multiparticle scattering follow directly from the algebraic structure of the theory. From an algebraic point of view, the operators $a(Q_1, p)$ and $a^\dagger(Q_1, p)$ may be identified with generators of a ZF-algebra (1.45) with scattering function S_2 . The resulting scattering function coincides with the universal CDD factor induced by a TT-deformation, supporting the interpretation of the latter as a purely kinematical modification of localization rather than a dynamical interaction. In this interpretation, the wedge-local fields ϕ_{W_1} and ϕ_{-W_1} act as polarization-free generators [169]: they create single-particle states from the vacuum without inducing vacuum polarization, while their nontrivial exchange relations encode the interaction. The resulting S-matrix acts diagonally on asymptotic two-particle states and is given by multiplication with $S_2(\theta_1 - \theta_2)$. The theory is therefore interacting whenever $\alpha \neq 0$, despite being constructed on the Fock space of a free scalar particle. The interaction does not arise from a local interaction density in a Lagrangian, but is entirely encoded in the weakened localization properties of the wedge-local fields.

In summary, the two-dimensional specialization of quantum field theory on noncommutative Minkowski space yields a fully consistent example of an integrable relativistic quantum field theory with nontrivial scattering. The model provides a concrete realization of the idea that relaxing pointlike locality to wedge locality offers an alternative mechanism for introducing interaction while preserving relativistic covariance and exact solvability.

Appendix D

Correlation Functions and Cumulants for More General $T\bar{T}$ -Perturbations

In this Appendix we discuss briefly two cases: the case when $\alpha \neq 0$ and $\alpha_s \neq 0$ for some $s \neq 1$ (that is, there are two free parameters, a generalised $T\bar{T}$ -perturbation involving $T\bar{T}$ and a higher spin s operator), and the case when only a spin s term is present $\alpha_s \neq 0$ but also the parameter $\beta_s \neq 0$ with all other α s and β s equal zero.

D.1 Cut-off for a Generalised $T\bar{T}$ -Perturbation: $\alpha, \alpha_s \neq 0$

Consider the case when two parameters in the vector α are non-zero. We choose $\alpha_1 = \alpha$ and α_s with $s > 1$. The form of the cumulants is just a special case of the general formulae (3.158) and (3.159).

For $\alpha_s > 0$ and any values of α (positive or negative), we can again find (for large r) a rapidity cut-off below which convergence can be maintained. In this case, the condition (3.161) is replaced by

$$2r \cosh \frac{x}{2} > \frac{\alpha x}{\pi} \sinh x + \frac{\alpha_s x}{\pi} \sinh(sx), \quad (\text{D.1})$$

and for large x it is the term $\sinh(sx)$ that dominates, giving the following equation for the cut-off

$$\frac{2\pi r}{\alpha_s} = x e^{(s-\frac{1}{2})x}, \quad (\text{D.2})$$

which has solution

$$\Lambda = \frac{1}{s - \frac{1}{2}} W_0 \left(\frac{2\pi r(s - \frac{1}{2})}{\alpha_s} \right). \quad (\text{D.3})$$

In other words, the contribution of the parameter α_s with larger spin dominates over the lower one α and the above result holds whether α is positive or negative.

D.2 Cut-off for a Generalised $T\bar{T}$ -Perturbation: $\alpha_s, \beta_s \neq 0$

We now take a look at the cut-off for the two-particle cumulant in the case when only one higher spin charge is present $\alpha_s \neq 0$ but we allow for the parameter β_s in (3.95) to

be non-zero as well. In this case, convergence of the integral in the two-particle cumulant is achieved for values of rapidity x satisfying

$$2r \cosh \frac{x}{2} > \frac{\alpha_s x}{\pi} \sinh(sx) + \beta_s \cosh(sx). \quad (\text{D.4})$$

For large rapidities, the inequality above can be approximated by

$$\frac{2r\pi}{\alpha_s} > \left(x + \frac{\pi\beta_s}{\alpha_s}\right) e^{x(s-\frac{1}{2})}. \quad (\text{D.5})$$

The cut-off is the solution of this equation when inequality is replaced by equality and terms are rearranged so generate an equation whose solution may be expressed in terms of Lambert's function. This can be done by multiplying both sides of the equation by a factor

$$\frac{2r\pi}{\alpha_s} \left(s - \frac{1}{2}\right) e^{\frac{\pi\beta_s}{\alpha_s}(s-\frac{1}{2})} = \left(s - \frac{1}{2}\right) \left(x + \frac{\pi\beta_s}{\alpha_s}\right) e^{(x+\frac{\pi\beta_s}{\alpha_s})(s-\frac{1}{2})}, \quad (\text{D.6})$$

so that the solution is

$$\left(s - \frac{1}{2}\right) \left(\Lambda + \frac{\pi\beta_s}{\alpha_s}\right) = W_0 \left(\frac{2r\pi}{\alpha_s} \left(s - \frac{1}{2}\right) e^{\frac{\pi\beta_s}{\alpha_s}(s-\frac{1}{2})} \right), \quad (\text{D.7})$$

or

$$\Lambda = \frac{1}{s - \frac{1}{2}} W_0 \left(\frac{2r\pi}{\alpha_s} \left(s - \frac{1}{2}\right) e^{\frac{\pi\beta_s}{\alpha_s}(s-\frac{1}{2})} \right) - \frac{\pi\beta_s}{\alpha_s}, \quad (\text{D.8})$$

which reduces to (3.165) for $s = 1, \beta_s = 0, \alpha_s = \alpha$ and to (D.3) for $\beta_s = 0$.

Appendix E

A $\overline{\text{T}}\text{T}$ Picture of the Boundary Ising Model with Generic Boundaries

Let $r_\kappa(\theta)$ be the minimal form factor of the Ising field theory with generic boundary conditions, corresponding to the reflection amplitude (4.42). The minimal form factor should be a modified version of the solution for fixed boundary conditions such that

$$r_\kappa(\theta) = r_{\text{fixed}}(\theta)\varphi_\kappa(\theta), \quad (\text{E.1})$$

which implies

$$\varphi_\kappa(\theta) = -[\kappa]_\theta \varphi_\kappa(-\theta), \quad -[\kappa]_\theta \varphi_\kappa(\theta) = \varphi_\kappa(2\pi i - \theta). \quad (\text{E.2})$$

The task is now to compute the new function $\varphi_\kappa(\theta)$. Employing the standard integral representations that can be found in many places, such as [252], we have that

$$-[\kappa]_\theta = (\kappa)_\theta(1 - \kappa)_\theta = \exp \left\{ 2 \int_0^\infty \frac{dt}{t} \frac{\sinh \frac{t\theta}{i\pi}}{\sinh^2 t} (\sinh t\kappa + \sinh t(1 - \kappa)) \right\}, \quad (\text{E.3})$$

so we can write

$$\varphi_\kappa(\vartheta) = \exp \left\{ 2 \int_0^\infty \frac{dt}{t} \frac{(\sinh t\kappa + \sinh t(1 - \kappa)) \cosh \frac{t}{2}}{\sinh^2 t} \left(1 - \cos \frac{t\vartheta}{\pi} \right) \right\}, \quad (\text{E.4})$$

where we again use the variable $\vartheta = \frac{i\pi}{2} - \theta$. As before we take the logarithmic derivative

$$\begin{aligned} \frac{d}{d\vartheta} \log \varphi_\kappa(\vartheta) &= \frac{2}{\pi} \int_0^\infty \frac{dt}{t} \frac{(\sinh t\kappa + \sinh t(1 - \kappa)) \cosh \frac{t}{2}}{\sinh^2 t} \sin \frac{t\vartheta}{i\pi} \\ &= h(\vartheta, 4\kappa) + h(\vartheta, 4 - 4\kappa), \end{aligned} \quad (\text{E.5})$$

where $h(\vartheta, \kappa)$ is the same function defined earlier in (4.29). We can therefore use the same formula (4.31) to write

$$h(\vartheta, 4\kappa) = \frac{\vartheta \sin 2\pi\kappa - \pi \sin \pi\kappa \sinh \vartheta + \pi\kappa \sinh(2\vartheta)}{\pi(\cosh(2\vartheta) + \cos 2\pi\kappa)}. \quad (\text{E.6})$$

Until now everything is pretty much the same as in previous cases. However, the sum (E.5) simplifies greatly, so that after integration, we have simply

$$\log \varphi_\kappa(\vartheta) = \int d\vartheta [h(\vartheta, 4\kappa) + h(\vartheta, 4 - 4\kappa)] = \log (\cosh \vartheta + \sin \pi\kappa) + c. \quad (\text{E.7})$$

The constant is easily fixed to $c = -\log (1 + \sin \pi\kappa)$ so as to ensure that $\log \varphi_\kappa(0) = 0$. Therefore, writing everything back in terms of θ we get

$$\varphi_\kappa(\vartheta) = \frac{\sin \pi\kappa - i \sinh \theta}{1 + \sin \pi\kappa} = \frac{\sin \pi\kappa - i \sinh \theta}{\sin \pi\kappa + i \sinh \theta} \frac{\sin \pi\kappa + i \sinh \theta}{1 + \sin \pi\kappa}, \quad (\text{E.8})$$

or

$$\log \varphi_\kappa(\vartheta) = \log(-[\kappa]_\theta) + \log (\sin \pi\kappa + i \sinh \theta) - \log (1 + \sin \pi\kappa). \quad (\text{E.9})$$

This gives the generic factor that has to be added any time the boundary condition is changed. In particular, for $\kappa = \frac{1}{2}$ we can obtain the modification of the minimal form factor with fixed boundary conditions that corresponds to free boundary conditions in the Ising case. Since the factor $\varphi_\kappa(\vartheta)$ accounts for the contribution to the minimal form factor of a generic square block $-\kappa_\theta$ in the reflection amplitude, it can also be adapted to deal with the blocks $[\frac{F-1}{2}]_\theta$ and $[\frac{E-1}{2}]_\theta$ in the sinh-Gordon amplitude (4.20).

Appendix F

Technical Details and Supplementary Material to Section 4.2

F.1 The $\mathbb{T}\bar{\mathbb{T}}_{2n-1}$ -Deformed Ising Field Theory

Let us now consider a generalisation of the heuristic derivation of Section 4.2 to the case when the perturbation is a single spin- $(2n-1)$ irrelevant perturbation. That is:

$$\delta_{\alpha_{2n-1}}^{\text{Ising}}(\theta) = -\alpha_{2n-1} \sinh((2n-1)\theta), \quad \epsilon \equiv S(0) = -1. \quad (\text{F.1})$$

From the general formula (4.85), we find the logarithm of the MFF to be

$$\begin{aligned} \rho_{\alpha_{2n-1}}^{\text{Ising}}(\theta) &= \log\left(-i \sinh \frac{\theta}{2}\right) + \frac{\theta - i\pi}{2\pi} \alpha_{2n-1} \sinh((2n-1)\theta) + \frac{\alpha_{2n-1}}{4\pi(2n-1)} \cosh((2n-1)\theta) \\ &\quad - \frac{\alpha_{2n-1}}{\pi} \sum_{\substack{m=1 \\ m \neq 2n-1}}^{\infty} \frac{m \cosh(m\theta)}{(2n-1)^2 - m^2}. \end{aligned} \quad (\text{F.2})$$

The sum above can be computed as follows:

$$\sum_{\substack{m=1 \\ m \neq s}}^{\infty} \frac{m \cosh(m\theta)}{s^2 - m^2} = \sum_{m=1}^{s-1} \frac{m \cosh(m\theta)}{s^2 - m^2} + \sum_{m=s+1}^{\infty} \frac{m \cosh(m\theta)}{s^2 - m^2}, \quad (\text{F.3})$$

where the infinite sum is given by

$$\begin{aligned} \sum_{m=s+1}^{\infty} \frac{m \cosh(m\theta)}{s^2 - m^2} &= \frac{1}{2s} + \frac{1}{4s} \cosh(s\theta) \\ &\quad + \sum_{m=1}^{s-1} \frac{s \cosh(m\theta)}{s^2 - m^2} + \frac{1}{2} \cosh(s\theta) \log\left(-4 \sinh^2 \frac{\theta}{2}\right), \end{aligned} \quad (\text{F.4})$$

so that

$$\sum_{\substack{m=1 \\ m \neq s}}^{\infty} \frac{m}{s^2 - m^2} \cosh(m\theta) = \frac{1}{2s} + \frac{1}{4s} \cosh(s\theta) + \sum_{m=1}^{s-1} \frac{\cosh(m\theta)}{s - m} + \frac{1}{2} \cosh(s\theta) \log\left(-4 \sinh^2 \frac{\theta}{2}\right). \quad (\text{F.5})$$

Putting everything together we have that the logarithm of the minimal form factor is

$$\begin{aligned} \rho_{\alpha_{2n-1}}^{\text{Ising}}(\theta) = & \log\left(-i \sinh \frac{\theta}{2}\right) + \frac{\theta - i\pi}{2\pi} \alpha_{2n-1} \sinh((2n-1)\theta) - \frac{\alpha_{2n-1}}{2\pi(2n-1)} \\ & - \frac{\alpha_{2n-1}}{\pi} \sum_{m=1}^{2n-2} \frac{\cosh(m\theta)}{2n-1-m} \\ & - \frac{\alpha_{2n-1}}{2\pi} \cosh((2n-1)\theta) \log\left(-4 \sinh^2 \frac{\theta}{2}\right). \end{aligned} \quad (\text{F.6})$$

For the purpose of studying the asymptotic properties of the minimal form factor, it is useful to rewrite the function above in terms of the variable η , as in the previous subsection

$$\begin{aligned} \rho_{\alpha_{2n-1}}^{\text{Ising}}(\theta) = & \log\left(\cosh \frac{\pi\eta}{2}\right) - \frac{\alpha_{2n-1}}{2\pi} \left[\frac{1}{2n-1} + \pi\eta \sinh((2n-1)\pi\eta) \right] \\ & - \frac{\alpha_{2n-1}}{\pi} \sum_{m=1}^{2n-2} \frac{(-1)^m \cosh(m\pi\eta)}{2n-1-m} \\ & + \frac{\alpha_{2n-1}}{2\pi} \cosh((2n-1)\pi\eta) \log\left(4 \cosh^2 \frac{\pi\eta}{2}\right). \end{aligned} \quad (\text{F.7})$$

We then find that

$$\begin{aligned} F_{\min}^{\text{Ising}}(\theta; \alpha_{2n-1}) & \underset{\eta \rightarrow 0}{\sim} e^{-\frac{\alpha_{2n-1}(1-(2n-1)2(\log 2 + c_{2n-1}))}{2\pi(2n-1)}} + O(\eta^2), \\ F_{\min}^{\text{Ising}}(\theta; \alpha_{2n-1}) & \underset{\theta \rightarrow 0}{\sim} \theta^{1-\frac{\alpha_{2n-1}}{\pi}} e^{-\frac{\alpha_{2n-1}(1+2(2n-1)\hat{c}_{2n-1})}{2\pi(2n-1)}} \left[1 - \frac{i\alpha_{2n-1}\theta}{2} + O(\theta^2) \right], \end{aligned} \quad (\text{F.8})$$

where

$$c_{2n-1} = \sum_{m=1}^{2n-2} \frac{(-1)^{m+1}}{m} \quad \text{and} \quad \hat{c}_{2n-1} = \sum_{m=1}^{2n-2} \frac{1}{m}, \quad (\text{F.9})$$

which are very similar behaviours as found for the $n = 1$ case in (4.90). In particular, we find again that there is a stark change in the $\theta \rightarrow 0$ asymptotics depending on whether α_{2n-1} is greater or smaller than π . It is also easy to see that in the presence of several perturbations the critical value will correspond to solutions to $\pi\kappa - \sum_{s \in \mathcal{S}} \alpha_s = 0$ where κ is the value in (4.90).

F.2 Revisiting the Sinh-Gordon Model

F.2.1 Factorizing the Kinematical Pole

We already introduced the sinh-Gordon model. The two-body scattering matrix takes the simple form

$$S_{\alpha}(\theta) = \frac{\tanh \frac{1}{2} \left(\theta - \frac{i\pi B}{2} \right)}{\tanh \frac{1}{2} \left(\theta + \frac{i\pi B}{2} \right)}, \quad (\text{F.10})$$

where B is a coupling constant that takes values between 0 and 2. The value $B = 1$ is known as the self-dual point. Following the notation in our introduction we have called this S -matrix $S_\alpha(\theta)$. The reason for this is that there is an alternative way of thinking about this function, namely as an S -matrix of the type (2.43) where $S(\theta) = -1$, corresponding to the Ising field theory (Majorana fermion) and

$$\begin{aligned}\Phi_\alpha^{\text{shG}}(\theta) &= -\frac{\tanh \frac{1}{2}(\theta - \frac{i\pi B}{2})}{\tanh \frac{1}{2}(\theta + \frac{i\pi B}{2})} = \frac{\sinh \theta - i \cos \frac{b\pi}{2}}{\sinh \theta + i \cos \frac{b\pi}{2}} \\ &= \exp \left[-4i \sum_{s \in 2\mathbb{Z}^+ - 1} i^{s+1} \frac{\cos \frac{s\pi b}{2}}{s} \sinh(s\theta) \right],\end{aligned}\quad (\text{F.11})$$

where $b := B - 1$. This means in particular that

$$\alpha_{2n-1} = \frac{4(-1)^n}{2n-1} \cos \frac{(2n-1)\pi b}{2}, \quad (\text{F.12})$$

and plugging these values into (4.48) provides yet another way to represent the MFF of the sinh-Gordon theory:

$$\begin{aligned}\frac{F_{\min}^{\text{shG}}(\theta; \alpha)}{F_{\min}^{\text{shG}}(i\pi; \alpha)} &= -i \sinh \frac{\theta}{2} \\ &\times \prod_{n \in \mathbb{Z}^+} \left[-\frac{1}{2} e^{\frac{\theta - i\pi}{2} \sinh((2n-1)\theta) - \sum_{m=1}^{2n-2} \frac{\cosh(m\theta)}{2n-1-m} + c_{2n-1}} \left[2i \sinh \frac{\theta}{2} \right]^{-\cosh((2n-1)\theta)} \right]^{\frac{4(-1)^n \cos \frac{(2n-1)\pi b}{2}}{\pi(2n-1)}}.\end{aligned}\quad (\text{F.13})$$

Employing this representation of the sinh-Gordon scattering matrix, it is possible to show that its MFF [244] admits a representation of the type (3.259) where all values of α and β are fixed by the requirements of analyticity and asymptotics (see Eq. (4.4) and the discussion thereof.). The explicit form of this function is not needed for our current purposes. An important property of the MFF of sinh-Gordon is that

$$F_{\min}^{\text{shG}}(\theta; \alpha) F_{\min}^{\text{shG}}(\theta + i\pi; \alpha) = \frac{\sinh \theta}{\sinh \theta + i \cos \frac{\pi b}{2}}. \quad (\text{F.14})$$

Our aim is to find higher-particle form factors of the sinh-Gordon model by employing a factorised ansatz. We will start by separating the Ising, α and $\beta(\alpha)$ parts, using $F_{\min}^{\text{Ising}}(\theta) = -i \sinh \frac{\theta}{2}$ and¹

$$C_\beta^{\text{shG}}(\theta) C_\beta^{\text{shG}}(\theta + i\pi) = \frac{2}{\sqrt{\sinh^2 \theta + \cos^2 \frac{\pi b}{2}}} = -\frac{2i}{\sqrt{\Psi^{\text{shG}}(\theta) \Psi^{\text{shG}}(-\theta)}}, \quad (\text{F.15})$$

$$\varphi_\alpha^{\text{shG}}(\theta) \varphi_\alpha^{\text{shG}}(\theta + i\pi) = \sqrt{\Phi_\alpha^{\text{shG}}(\theta)} = \sqrt{-\frac{\sinh \theta - i \cos \frac{\pi b}{2}}{\sinh \theta + i \cos \frac{\pi b}{2}}}, \quad (\text{F.16})$$

¹From [244] it is interesting to note that only the terms involving β_s with s even contribute to obtaining the property (F.14). This excludes all terms involving dilogarithm functions. Note also that compared to [244] we included an extra multiplicative factor $\sqrt{2}$ into the definition of $C_\beta^{\text{shG}}(\theta)$.

where

$$\Psi^{\text{shG}}(\theta) = \sinh \theta + i \cos \frac{\pi b}{2}, \quad (\text{F.17})$$

with

$$\Psi^{\text{shG}}(\theta) \Phi_{\alpha}^{\text{shG}}(\theta) = \Psi^{\text{shG}}(-\theta). \quad (\text{F.18})$$

It is well known that the formula (F.14) plays a critical role in determining the higher-particle form factors. Writing it in the factorised form above allows us to separate the contributions related to the α parameters, $\beta = \beta(\alpha)$ parameters and the unperturbed theory, parameters which are inherited from the representation (3.95). We will now “track” these separate contributions into the structure of higher-particle form factors and investigate what they reveal about the form factors of $\text{T}\bar{\text{T}}$ -perturbed theories.

F.2.2 $\text{T}\bar{\text{T}}$ -like Solution of the sinh-Gordon Model

Typically, the form factors of a field $\mathcal{O}$, defined in (3.99) have the structure

$$F_n^{\mathcal{O}}(\theta_1, \dots, \theta_n; \alpha) = H_n^{\mathcal{O}}(\alpha) Q_n^{\mathcal{O}}(\theta_1, \dots, \theta_n; \alpha) \prod_{i < j} \frac{F_{\min}(\theta_{ij}; \alpha)}{e^{\theta_i} + e^{\theta_j}}, \quad (\text{F.19})$$

where

$$Q_{n+2}^{\mathcal{O}}(\theta + i\pi, \theta, \theta_1, \dots, \theta_n; \alpha) = G_n(\theta, \theta_1, \dots, \theta_n; \alpha, \beta) Q_n^{\mathcal{O}}(\theta_1, \dots, \theta_n; \alpha). \quad (\text{F.20})$$

In the sinh-Gordon case we have the structure (normalisation constants can be absorbed into the equation for the constants $H_n^{\mathcal{O}}(\alpha)$)

$$\begin{aligned} G_n(\theta, \theta_1, \dots, \theta_n; \alpha) &:= i^n e^{(n+1)\theta} \prod_{j=1}^n e^{\theta_j} \left(\prod_{j=1}^n \sqrt{\Psi^{\text{shG}}(\theta - \theta_j) \Psi^{\text{shG}}(\theta_j - \theta)} \right) \\ &\times \left[\prod_{j=1}^n \Phi_{\alpha}^{\text{shG}}(\theta - \theta_j)^{-\frac{1}{2}} - \gamma_{\mathcal{O}}(-1)^n \prod_{j=1}^n \Phi_{\alpha}^{\text{shG}}(\theta - \theta_j)^{\frac{1}{2}} \right]. \end{aligned} \quad (\text{F.21})$$

Employing the property (F.18) and the fact that $\Phi_{\alpha}(\theta) = (\Phi_{\alpha}(-\theta))^{-1}$ (by unitarity) we can rewrite

$$G_n(\theta, \theta_1, \dots, \theta_n; \alpha, \beta) = i^n e^{(n+1)\theta} \prod_{j=1}^n e^{\theta_j} \left[\prod_{j=1}^n \Psi^{\text{shG}}(\theta - \theta_j) - \gamma_{\mathcal{O}}(-1)^n \prod_{j=1}^n \Psi^{\text{shG}}(\theta_j - \theta) \right]. \quad (\text{F.22})$$

These equations are equivalent to those presented in [170] and [26]. Proceeding in this way allows us also to see how the square roots present in the form (F.21) are absent when combining the α and β factors together in the sinh-Gordon model case. If we stick with the representation (F.21) though we have a neat separation into three different contributions: the factor $e^{(n+1)\theta} \prod_{j=1}^n e^{\theta_j}$ is the contribution from the Ising field theory, the next factor incorporates the β dependence and the factor in the second line incorporates the α dependence and, as expected, is of exactly the type found for generalised $\text{T}\bar{\text{T}}$ -perturbed theories [12, 14]. Mixing the α and β factors together with the Ising factors gives the standard solutions for the sinh-Gordon operator content. We will see in the following that instead, imposing factorization we end up with different solutions even though the S-matrix is still the sinh-Gordon one.

F.2.3 A Factorized Solution

Therefore, it is natural to try to find solutions to these equations using the same methodology as in [12, 14]. We make a solution ansatz stating that the sinh-Gordon form factors of some “local” field factorize as the form factors of an Ising field theory times a α -dependent part. Furthermore, contrary to our treatment in [12, 14] where we considered the case with all $\beta_i = 0$, in the sinh-Gordon model we have parameters β_i entering the MFF, so we need a more general ansatz. Again, we assume further factorisation into an α -dependent and β -dependent part. We write

$$Q_n^\mathcal{O}(\theta_1, \dots, \theta_n; \alpha) = Q_n^\mathcal{O}(\theta_1, \dots, \theta_n) \Theta_n^\mathcal{O}(\theta_1, \dots, \theta_n; \alpha) \Xi_n^\mathcal{O}(\theta_1, \dots, \theta_n; \beta(\alpha)), \quad (\text{F.23})$$

where

$$Q_{n+2}^\mathcal{O}(\theta + i\pi, \theta, \theta_1, \dots, \theta_n) = e^{(n+1)\theta} \left[\prod_{j=1}^n e^{\theta_j} \right] Q_n(\theta_1, \dots, \theta_n), \quad (\text{F.24})$$

which (up to normalisation) is the equation for the Ising form factors of the order field μ (for n even) and for the disorder field σ (for n odd) [182]. The solution to this equation, or rather, the full form factors are very simple and proportional to the product $\prod_{i < j} \tanh \frac{\theta_{ij}}{2}$ which automatically incorporates the denominator $e^{\theta_i} + e^{\theta_j}$ in (F.19). The next equation of interest is

$$\Theta_{n+2}^\mathcal{O}(\theta + i\pi, \theta, \theta_1, \dots, \theta_n; \alpha) = \left[\prod_{j=1}^n \Phi_\alpha(\theta - \theta_j)^{-\frac{1}{2}} - \gamma_\mathcal{O}(-1)^n \prod_{j=1}^n \Phi_\alpha(\theta - \theta_j)^{\frac{1}{2}} \right] \Theta_n^\mathcal{O}(\theta_1, \dots, \theta_n; \alpha), \quad (\text{F.25})$$

which satisfies the generalised $\text{T}\bar{\text{T}}$ form factor equation discussed and solved in Chapter 3. The general solution reads

$$\Theta_n^\mathcal{O}(\theta_1, \dots, \theta_n; \alpha) = \prod_{i=1}^n \sqrt{\frac{\prod_{j=1}^n S_\alpha(\theta_{ij})^{1/2} - \gamma_\mathcal{O} \prod_{j=1}^n S_\alpha(\theta_{ij})^{-1/2}}{\prod_{j=1}^n S(\theta_{ij})^{1/2} - \gamma_\mathcal{O} \prod_{j=1}^n S(\theta_{ij})^{-1/2}}}, \quad (\text{F.26})$$

as shown in (3.191). The β -dependent recursive equation instead reads

$$\Xi_{n+2}^\mathcal{O}(\theta + i\pi, \theta, \theta_1, \dots, \theta_n; \beta(\alpha)) = i^n \left[\prod_{j=1}^n \sqrt{\Psi^{\text{shG}}(\theta - \theta_j) \Psi^{\text{shG}}(\theta_j - \theta)} \right] \Xi_n^\mathcal{O}(\theta_1, \dots, \theta_n; \beta(\alpha)). \quad (\text{F.27})$$

Like in the Ising model and in its deformed version, we expect to have different sectors dictated by the $\mathbb{Z}_2$ symmetry: either “even” fields such as the order field μ and the energy field ε and “odd” fields, such as σ . Their counterparts in the sinh-Gordon model, which also possesses this symmetry, should at least be in the same symmetry sector, with the fundamental sinh-Gordon field ϕ in the “odd” sector and the trace of the stress-energy tensor in the “even” sector. Anyway, we notice that the factor of local commutativity does not enter equation (F.27). A solution to this equation is relatively easy to find, namely

$$\Xi_n^\mathcal{O}(\theta_1, \dots, \theta_n; \beta(\alpha)) = v_n \prod_{i=1}^n \prod_{j=1}^n \sqrt[4]{\Psi^{\text{shG}}(\theta_{ij})}, \quad (\text{F.28})$$

where v_n is a normalisation constant which satisfies

$$v_{n+2} = \cos \frac{\pi b}{2} v_n, \quad (\text{F.29})$$

which, depending on whether n is even or odd gives solutions

$$v_{2k} = \left(\cos \frac{\pi b}{2} \right)^k v_0 \quad \text{and} \quad v_{2k+1} = \left(\cos \frac{\pi b}{2} \right)^{k-1} v_1. \quad (\text{F.30})$$

In addition to this, if we specialise to $n = 2k + 1$ odd and $\gamma_{\mathcal{O}} = 1$, we get for the α -dependent part

$$\Theta_{2k+1}^{\mathcal{O}}(\theta_1, \dots, \theta_{2k+1}; \alpha) = \frac{1}{2^k \sqrt{2}} \prod_{i=1}^{2k+1} \sqrt{\prod_{j=1}^{2k+1} \Phi_{\alpha}^{\text{shG}}(\theta_{ij})^{1/2} + \prod_{j=1}^{2k+1} \Phi_{\alpha}^{\text{shG}}(\theta_{ij})^{-1/2}}, \quad (\text{F.31})$$

and similarly for n even and $\gamma_{\mathcal{O}} = -1$. This expression is rather complicated and there is no immediately obvious way to simplify it. In our example, we have in addition that $\Phi_{\alpha}^{\text{shG}}(\theta) \Phi_{\alpha}^{\text{shG}}(-\theta) = 1$ so we can rewrite the formula in several equivalent ways. For example,

$$\prod_{i=1}^n \sqrt{\prod_{j=1}^n \Phi_{\alpha}^{\text{shG}}(\theta_{ij})^{1/2} + \prod_{j=1}^n \Phi_{\alpha}^{\text{shG}}(\theta_{ji})^{1/2}} = \prod_{i=1}^n \sqrt{1 + \prod_{j=1}^n \Phi_{\alpha}^{\text{shG}}(\theta_{ji})}. \quad (\text{F.32})$$

It seems therefore that it is possible to consistently solve the form factor equations by assuming factorisation, but the resulting solutions are distinct from those previously found and contain square roots.

F.2.4 Another Solution

We could instead assume factorisation but separate only the Ising part from the rest. That is, instead of (F.23) we now have

$$Q_n^{\mathcal{O}}(\theta_1, \dots, \theta_n; \alpha) = Q_n^{\mathcal{O}}(\theta_1, \dots, \theta_n) \Gamma_n^{\mathcal{O}}(\theta_1, \dots, \theta_n; \alpha). \quad (\text{F.33})$$

If we specialise to the sector $\gamma_{\mu} = -1$ and with a sum over even particle numbers, we get

$$\Gamma_{2n+2}^{\mu}(\theta + i\pi, \theta, \theta_1, \dots, \theta_{2n}; \alpha) = \left[\prod_{j=1}^{2n} \Psi^{\text{shG}}(\theta - \theta_j) + \prod_{j=1}^{2n} \Psi^{\text{shG}}(\theta_j - \theta) \right] \Gamma_{2n}^{\mu}(\theta_1, \dots, \theta_{2n}; \alpha). \quad (\text{F.34})$$

To solve this equation, we just need the property $\Psi^{\text{shG}}(\theta_{ij} + i\pi) = \Psi^{\text{shG}}(\theta_{ji})$ and we find the solution

$$\Gamma_{2n}^{\mu}(\theta_1, \dots, \theta_{2n}; \alpha) = C_{2n} \frac{\prod_{i=1}^{2n} \sqrt{\prod_{j=1}^{2n} \Psi^{\text{shG}}(\theta_{ij}) + \prod_{j=1}^{2n} \Psi^{\text{shG}}(\theta_{ji})}}{\prod_{i < j} \sqrt[4]{\Psi^{\text{shG}}(\theta_{ij}) \Psi^{\text{shG}}(\theta_{ji})}}. \quad (\text{F.35})$$

with C_{2n} a constant. Once more the solution involves roots which do not cancel out and therefore it is distinct from the usual solutions for the sinh-Gordon model. In addition, the factor in the denominator can be shown to cancel part of the contribution coming from the product of MFFs. It seems clear that the Ising part of the MFF and the rest must be considered together in order to solve the kinematic residue equation.

F.3 Generalisation to Branch Point Twist Fields

In this Appendix we present a generalisation of the results of Subsection 4.2.3 to the case of branch point twist fields introduced in Subsection 3.2.3. For our purposes, the starting point is the integral representation

$$\log F_{\min}(\theta; n) = \frac{1}{i\pi n} \cosh^2 \frac{\theta}{2n} \int_0^\infty dt \frac{\tanh \frac{t}{2n} \ln S(t)}{\cosh \frac{t}{n} - \cosh \frac{\theta}{n}}, \quad \text{for } 0 \leq \text{Im}(\theta) \leq 2\pi n, \quad (\text{F.36})$$

which reduces to (4.95) for $n = 1$. Again, we can obtain this representation by inverting the Fourier transform of the S -matrix, that is, by expressing the kernel $g(t)$ in terms of the Fourier transform of $\ln S(t)$ in the usual representation (see [185]). As we did earlier, let us consider the deformation of the MFF resulting from a generalised $\text{T}\bar{\text{T}}$ perturbation in its regularised form

$$\log \mathcal{D}_\alpha(\theta; n) = \frac{1}{i\pi n} \cosh^2 \frac{\theta}{2n} \int_0^\infty dt \frac{\tanh \frac{t}{2n}}{\cosh \frac{t}{n} - \cosh \frac{\theta}{n}} \left(\ln \Phi_\alpha(t) - \partial_t \ln \hat{\Phi}_\alpha(\theta) \right), \quad (\text{F.37})$$

where $\hat{\Phi}_\alpha(\theta)$ is the function defined in (4.98). Expanding for small $|\theta|$ we can write

$$\frac{\cosh^2 \frac{\theta}{2n} \tanh \frac{t}{2n}}{\cosh \frac{t}{n} - \cosh \frac{\theta}{n}} = \sum_{m=1}^{\infty} \left[\cosh \frac{k\theta}{n} + (-1)^{m+1} \right] e^{-m \frac{t}{n}}, \quad (\text{F.38})$$

which when substituting in (F.37) gives the simple integral

$$\int_0^\infty dt e^{-t(\frac{m}{n} + s)} = -\frac{1}{\frac{m}{n} + s}. \quad (\text{F.39})$$

The resulting contribution for a given spin s is

$$\log \mathcal{D}_{\alpha_s}(\theta; n) = \frac{\alpha_s}{\pi} \sum_{m=1}^{\infty} \frac{\cosh\left(\frac{m\theta}{n}\right) + (-1)^{m+1}}{m + s n}. \quad (\text{F.40})$$

The sum can be performed as before and it gives a solution, with the structure (3.218) found in [218, 14], namely

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{\cosh \frac{m\theta}{n}}{m + s n} &= \frac{\theta - i\pi n}{2n} \sinh(s\theta) - \frac{1}{2} \log \left(-4 \sinh^2 \frac{\theta}{2n} \right) \cosh(s\theta) \\ &\quad - \sum_{m=0}^{ns-1} \frac{1}{ns - m} \cosh \frac{m\theta}{n}, \end{aligned} \quad (\text{F.41})$$

and the constant is

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m + s n} &= \sum_{m=sn+1}^{\infty} \frac{(-1)^{m+sn+1}}{m} = (-1)^{sn} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} + \frac{1}{sn} - (-1)^{sn} c_{sn} \\ &= (-s)^{sn} \log 2 + \frac{1}{sn} - (-1)^{sn} c_{sn}. \end{aligned} \quad (\text{F.42})$$

Here c_{sn} is the same constant defined in (F.9). In summary, normalising $\mathcal{D}_\alpha(\theta; n)$ by its value at $\theta = i\pi n$ we can write

$$\frac{\mathcal{D}_\alpha(\theta; n)}{\mathcal{D}_\alpha(i\pi n; n)} = \prod_{s \in 2\mathbb{Z}^+ - 1} \left[-\frac{1}{2^{(-1)^{sn}}} e^{\frac{\theta - i\pi n}{2n} \sinh(s\theta) - \sum_{m=1}^{ns-1} \frac{\cosh \frac{m\theta}{n}}{ns-m} + (-1)^{ns} c_{sn}} \left(2i \sinh \frac{\theta}{2n} \right)^{-\cosh(s\theta)} \right]^{\frac{\alpha s}{\pi}}. \quad (\text{F.43})$$

Setting $n = 1$ we recover the solution (4.48). The formulae above can be easily generalized to real n (as opposed to n integer) by introducing integer part symbols in some of the summation limits.

Bibliography

- [1] P. A. M. Dirac. Quantum theory of emission and absorption of radiation. *Proc. Roy. Soc. Lond. A*, 114:243, 1927.
- [2] G. de Santillana. *The Origins of Scientific Thought*. University of Chicago Press and New American Library, Chicago, 1961.
- [3] A. B. Zamolodchikov. Integrable field theory from conformal field theory. *Adv. Stud. Pure Math.*, 19:641–674, 1989.
- [4] C. N. Yang and C. P. Yang. Thermodynamics of one-dimensional system of bosons with repulsive delta function interaction. *J. Math. Phys.*, 10:1115–1122, 1969.
- [5] A. B. Zamolodchikov. Thermodynamic Bethe Ansatz in Relativistic Models. Scaling Three State Potts and Lee-yang Models. *Nucl. Phys. B*, 342:695–720, 1990.
- [6] M. Karowski and P. H. Weisz. Exact Form-Factors in (1+1)-Dimensional Field Theoretic Models with Soliton Behavior. *Nucl. Phys. B*, 139:455–476, 1978.
- [7] F. A. Smirnov. The Quantum Gelfand-Levitan-Marchenko Equations and Form-Factors in the sine-Gordon Model. *J. Phys. A*, 17:L873–L878, 1984.
- [8] F.A. Smirnov. Solution of Quantum Gelfand-levitan-marchenko Equations for the Sine-Gordon Model in the Soliton Sector for $\gamma = \pi / \text{Nu}$. *Theor. Math. Phys.*, 67:344–351, 1986.
- [9] F. A. Smirnov. A General Formula for Soliton Form-factors in the Quantum Sine-Gordon Model. *J. Phys. A*, 19:L575–L578, 1986.
- [10] F. A. Smirnov. *Form-factors in completely integrable models of quantum field theory*, volume 14. World Scientific, 1992.
- [11] W. Donnelly and S. B. Giddings. Observables, gravitational dressing, and obstructions to locality and subsystems. *Phys. Rev. D*, 94(10):104038, 2016.
- [12] O. A. Castro-Alvaredo, S. Negro and F. Sailis. Form Factors and Correlation Functions of $\text{T}\bar{\text{T}}$ -Deformed Integrable Quantum Field Theories, 2023.
- [13] O. A. Castro-Alvaredo, S. Negro and F. Sailis. Completing the Bootstrap Program for $\text{T}\bar{\text{T}}$ Deformed Massive Integrable Quantum Field Theories. *Journal of Physics A: Mathematical and Theoretical*, 57(26):265401, 2024.
- [14] O. A. Castro-Alvaredo, S. Negro and F. Sailis. Entanglement Entropy from Form Factors in $\text{T}\bar{\text{T}}$ Deformed Integrable Quantum Field Theories. *Journal of High Energy Physics*, 2023(129), 2023.

- [15] O. A. Castro-Alvaredo, S. Negro and F. Sailis. Boundary quantum field theories perturbed by $T\bar{T}$: Towards a form factor program. *Nucl. Phys. B*, 1017:116924, 2025.
- [16] F. Sailis, O. A. Castro-Alvaredo and S. Negro. Complete minimal form factors for irrelevant deformations of integrable quantum field theory. *Nucl. Physics B*, page 117131, 2025.
- [17] K. G. Wilson and J. B. Kogut. The Renormalization group and the epsilon expansion. *Phys. Rept.*, 12:75–199, 1974.
- [18] G. Mussardo. *Statistical Field Theory*. Oxford Graduate Texts. Oxford University Press, 3 2020.
- [19] R. E. Peierls. The Commutation laws of relativistic field theory. *Proc. Roy. Soc. Lond. A*, 214:143–157, 1952.
- [20] R. Haag. *Local Quantum Physics*. Theoretical and Mathematical Physics. Springer, Berlin, 1996.
- [21] A. S. Wightman. Quantum Field Theory in Terms of Vacuum Expectation Values. *Phys. Rev.*, 101:860–866, 1956.
- [22] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov. Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory. *Nucl. Phys. B*, 241:333–380, 1984.
- [23] P. Di Francesco, P. Mathieu and D. Senechal. *Conformal Field Theory*. Graduate Texts in Contemporary Physics. Springer-Verlag, New York, 1997.
- [24] G. Delfino, G. Mussardo and P. Simonetti. Non-integrable quantum field theories as perturbations of certain integrable models. *Nuclear Physics B*, 473(3):469–508, August 1996.
- [25] J. L. Cardy and G. Mussardo. Form-factors of Descendent Operators in Perturbed Conformal Field Theories. *Nucl. Phys. B*, 340:387–402, 1990.
- [26] A. Koubek and G. Mussardo. On the operator content of the sinh-Gordon model. *Phys. Lett. B*, 311:193–201, 1993.
- [27] O. A. Castro-Alvaredo and A. Fring. Identifying the operator content, the homogeneous sine-Gordon models. *Nucl. Phys. B*, 604:367–390, 2001.
- [28] G. Delfino and G. Niccoli. Isomorphism of critical and off-critical operator spaces in two-dimensional quantum field theory. *Nucl. Phys. B*, 799:364–378, 2008.
- [29] O. A. Castro-Alvaredo. *Bootstrap Methods in 1+1-Dimensional Quantum Field Theories: The Homogeneous Sine-Gordon Models*. PhD Thesis. PhD thesis, 2001. arXiv:0109212 [hep-th].
- [30] A. B. Zamolodchikov. Irreversibility of the Flux of the Renormalization Group in a 2D Field Theory. *JETP Lett.*, 43:730–732, 1986.
- [31] J. Callan and G. Curtis. Broken scale invariance in scalar field theory. *Phys. Rev. D*, 2:1541–1547, 1970.

- [32] K. Symanzik. Small distance behavior in field theory and power counting. *Commun. Math. Phys.*, 18:227–246, 1970.
- [33] G. Barton. *Introduction to dispersion techniques in field theory*. Lecture notes and supplements in physics. W.A. Benjamin, New York, 1965.
- [34] H. Lehmann, K. Symanzik and W. Zimmermann. On the formulation of quantized field theories — ii. *Il Nuovo Cimento (1955-1965)*, 6:319–333, 1957.
- [35] S. J. Parke. Absence of Particle Production and Factorization of the S Matrix in $(1+1)$ -dimensional Models. *Nucl. Phys. B*, 174:166–182, 1980.
- [36] A. Fasano and S. Marmi. *Analytical Mechanics: An Introduction*. OUP Oxford, 2006.
- [37] S. Novikov, S. V. and Manakov, L. P. Pitaevsky and V. E. Zakharov. *Theory of solitons. The inverse scattering method*. Springer-Verlag, Berlin, 1984.
- [38] V. E. Korepin, N. M. Bogoliubov and A. G. Izergin. *Quantum Inverse Scattering Method and Correlation Functions*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1993.
- [39] B. Sutherland. *Beautiful Models: 70 Years of Exactly Solved Quantum Many-Body Problems*. World Scientific, River Edge, NJ, 2005.
- [40] J. S. Caux and J. Mossel. Remarks on the notion of quantum integrability. *Journal of Statistical Mechanics: Theory and Experiment*, 2011(02):P02023, February 2011.
- [41] S. Coleman and J. Mandula. All Possible Symmetries of the S Matrix. *Phys. Rev.*, 159:1251–1256, Jul 1967.
- [42] A. M. Polyakov. Hidden Symmetry of the Two-Dimensional Chiral Fields. *Phys. Lett. B*, 72:224–226, 1977.
- [43] R. Shankar and E. Witten. The S Matrix of the Supersymmetric Nonlinear Sigma Model. *Phys. Rev. D*, 17:2134, 1978.
- [44] A. B. Zamolodchikov and A. B. Zamolodchikov. Factorized S -matrices in two dimensions as the exact solutions of certain relativistic quantum field theory models. *Annals of Physics*, 120(2):253–291, 1979.
- [45] D. Iagolnitzer. Factorization of the Multiparticle S Matrix in Two-Dimensional Space-Time Models. *Phys. Rev. D*, 18:1275, 1978.
- [46] C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura. Method for solving the korteweg-devries equation. *Phys. Rev. Lett.*, 19:1095–1097, Nov 1967.
- [47] C. N. Yang. Some exact results for the many body problems in one dimension with repulsive delta function interaction. *Phys. Rev. Lett.*, 19:1312–1314, 1967.
- [48] R. J. Baxter. Partition function of the eight vertex lattice model. *Annals Phys.*, 70:193–228, 1972.
- [49] P. Federbush. A Two-Dimensional Relativistic Field Theory. *Phys. Rev.*, 121:1247–1249, 1961.

- [50] B. Schroer, T. T. Truong and P. Weisz. Model Study of Nonleading Mass Singularities. 1. *Annals Phys.*, 102:156, 1976.
- [51] J. L. Miramontes. Hermitian analyticity versus real analyticity in two-dimensional factorized S matrix theories. *Phys. Lett. B*, 455:231–238, 1999.
- [52] O. A. Castro-Alvaredo, A. Fring, C. Korff and J. L. Miramontes. Thermodynamic bethe ansatz of the homogeneous sine-gordon models. *Nuclear Physics B*, 575(3):535–560, June 2000.
- [53] J. B. McGuire. Study of Exactly Soluble One-Dimensional N-Body Problems. *J. Math. Phys.*, 5(5):622–636, 1964.
- [54] C. N. Yang. S matrix for the one-dimensional N body problem with repulsive or attractive delta function interaction. *Phys. Rev.*, 168:1920–1923, 1968.
- [55] V. E. Zakharov and A. B. Shabat. Exact theory of two-dimensional selffocusing and one-dimensional selfmodulation of waves in nonlinear media. *Zh. Eksp. Teor. Fiz.*, 61:118–134, 1971.
- [56] L. D. Faddeev and L. A. Takhtajan. Essentially nonlinear one-dimensional model of the classical field theory. *Teor. Mat. Fiz.*, 21:160–174, 1974.
- [57] M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur. Nonlinear-Evolution Equations of Physical Significance. *Phys. Rev. Lett.*, 31:125–127, 1973.
- [58] D. W. Mclaughlin. Four Examples of the Inverse Method as a Canonical Transformation. *J. Math. Phys.*, 16:96–99, 1975.
- [59] S. R. Coleman. The Quantum Sine-Gordon Equation as the Massive Thirring Model. *Phys. Rev. D*, 11:2088, 1975.
- [60] V. E. Korepin and L. D. Faddeev. Quantization of Solitons. *Theor. Math. Phys.*, 25:1039–1049, 1975.
- [61] A. B. Zamolodchikov. Exact S Matrix of Quantum Sine-Gordon Solitons. *JETP Lett.*, 25:468, 1977.
- [62] A. B. Zamolodchikov. Exact Two Particle S Matrix of Quantum Sine-Gordon Solitons. *Pisma Zh. Eksp. Teor. Fiz.*, 25:499–502, 1977.
- [63] T. R. Klassen and E. Melzer. Purely Elastic Scattering Theories and their Ultraviolet Limits. *Nucl. Phys. B*, 338:485–528, 1990.
- [64] D. Bombardelli. S-matrices and integrability. *Journal of Physics A: Mathematical and Theoretical*, 49(32):323003, July 2016.
- [65] M. Karowski, H. J. Thun, T. T. Truong and P. H. Weisz. On the Uniqueness of a Purely Elastic S Matrix in (1+1)-Dimensions. *Phys. Lett. B*, 67:321–322, 1977.
- [66] P. H. Weisz. Exact Quantum Sine-Gordon Soliton Form-Factors. *Phys. Lett. B*, 67:179–182, 1977.
- [67] P. Mitra. Elasticity, Factorization and S Matrices in (1+1)-Dimensions. *Phys. Lett. B*, 72:62–64, 1977.

- [68] K. Hepp. Analyticity and cluster decomposition properties of the S-matrix in local relativistic quantum field theory. In *12th International Conference on High Energy Physics*, 1964.
- [69] K. Hepp. On the connection between wightman and lsz quantum field theory. In *8th Brandeis University Summer Institute in Theoretical Physics: Particle symmetries and axiomatic field theory*, pages 135–240, 1966.
- [70] P. G. O. Freund, T. R. Klassen and E. Melzer. S Matrices for Perturbations of Certain Conformal Field Theories. *Phys. Lett. B*, 229:243–247, 1989.
- [71] L. Castillejo, R. H. Dalitz and F. J. Dyson. Low’s scattering equation for the charged and neutral scalar theories. *Phys. Rev.*, 101:453–458, 1956.
- [72] F. J. Dyson. Meaning of the solutions of Low’s scattering equation. *Phys. Rev.*, 106(1):157–159, 1957.
- [73] A. E. Arinshtein, V. A. Fateev and A. B. Zamolodchikov. Quantum S matrix of the (1+1)-Dimensional Todd Chain. *Phys. Lett. B*, 87:389–392, 1979.
- [74] E. K. Sklyanin. Exact Quantization of the Sinh-gordon Model. *Nucl. Phys. B*, 326:719–736, 1989.
- [75] A. Bastianello, A. De Luca and G. Mussardo. Non relativistic limit of integrable qft and lieb–liniger models. *Journal of Statistical Mechanics: Theory and Experiment*, 2016(12):123104, December 2016.
- [76] A. Bastianello, B. Doyon, G. Watts and T. Yoshimura. Generalized hydrodynamics of classical integrable field theory: the sinh-gordon model. *SciPost Physics*, 4(6), June 2018.
- [77] A. Bastianello and L. Piroli. From the sinh-gordon field theory to the one-dimensional bose gas: exact local correlations and full counting statistics. *Journal of Statistical Mechanics: Theory and Experiment*, 2018(11):113104, November 2018.
- [78] Z. Hermon, E. Ben-Jacob and G. Schön. Charge solitons in one-dimensional arrays of serially coupled josephson junctions. *Physical Review B*, 54(2):1234–1245, July 1996.
- [79] A. B. Zamolodchikov. Integrals of Motion and S Matrix of the (Scaled) $T=T(c)$ Ising Model with Magnetic Field. *Int. J. Mod. Phys. A*, 4:4235, 1989.
- [80] R. Coldea, D. A. Tennant, E. M. Wheeler, E. Wawrzynska, D. Prabhakaran, M. Telling, K. Habicht, P. Smeibidl and K. Kiefer. Quantum Criticality in an Ising Chain: Experimental Evidence for Emergent $E(8)$ Symmetry. *Science*, 327:177–180, 2010.
- [81] R. Hagedorn. Statistical thermodynamics of strong interactions at high-energies. *Nuovo Cim. Suppl.*, 3:147–186, 1965.
- [82] B. Schroer, T. T. Truong and P. Weisz. Towards an Explicit Construction of the Sine-Gordon Field Theory. *Phys. Lett. B*, 63:422–424, 1976.

- [83] S. Ghoshal and A. B. Zamolodchikov. Boundary S matrix and boundary state in two-dimensional integrable quantum field theory. *Int. J. Mod. Phys. A*, 9:3841–3886, 1994. [Erratum: *Int.J.Mod.Phys.A* 9, 4353 (1994)].
- [84] H. Saleur. Lectures on non perturbative field theory and quantum impurity problems, 1998.
- [85] P. Dorey. Boundary integrable quantum field theories. In *Non-Perturbative QFT Methods and Their Applications*, page 167–172. World Scientific, May 2001.
- [86] I. V. Cherednik. Factorizing Particles on a Half Line and Root Systems. *Theor. Math. Phys.*, 61:977–983, 1984.
- [87] E. K. Sklyanin. Boundary Conditions for Integrable Quantum Systems. *J. Phys. A*, 21:2375–2389, 1988.
- [88] A. Fring and R. Koberle. Factorized scattering in the presence of reflecting boundaries. *Nucl. Phys. B*, 421:159–172, 1994.
- [89] A. Fring and R. Koberle. Affine Toda field theory in the presence of reflecting boundaries. *Nucl. Phys. B*, 419:647–664, 1994.
- [90] A. B. Zamolodchikov. Expectation value of composite field $T\bar{T}$ in two-dimensional quantum field theory. 2004. arXiv:0401146 [hep-th].
- [91] G. Delfino and G. Niccoli. Matrix elements of the $T\bar{T}$ operator in integrable quantum field theory. *Nuclear Physics B*, 707(3):381–404, February 2005.
- [92] G. Delfino and G. Niccoli. The composite operator $T\bar{T}$ in sinh-Gordon and a series of massive minimal models. *Journal of High Energy Physics*, 2006(05):035–035, May 2006.
- [93] D. J. Gross, J. Kruthoff, A. Rolph and E. Shaghoulian. Hamiltonian deformations in quantum mechanics, $T\bar{T}$, and the SYK model. *Phys. Rev. D*, 102(4):046019, 2020.
- [94] D. J. Gross, J. Kruthoff, A. Rolph and E. Shaghoulian. $T\bar{T}$ in AdS_2 and Quantum Mechanics. *Phys. Rev. D*, 101(2):026011, 2020.
- [95] F. Giordano, S. Negro and R. Tateo. The generalized born oscillator and the berry-keating hamiltonian. *Journal of High Energy Physics*, 2023(10), October 2023.
- [96] G. Bonelli, N. Doroud and M. Zhu. $T\bar{T}$ -deformations in closed form. *JHEP*, 06:149, 2018.
- [97] M. Taylor. TT deformations in general dimensions. 5 2018.
- [98] R. Conti, J. Romano and R. Tateo. Metric approach to a $T\bar{T}$ -like deformation in arbitrary dimensions. *JHEP*, 09:085, 2022.
- [99] T. Morone, S. Negro and R. Tateo. Gravity and $T\bar{T}$ flows in higher dimensions. *Nucl. Phys. B*, 1005:116605, 2024.
- [100] R. Conti, S. Negro and R. Tateo. The $T\bar{T}$ perturbation and its geometric interpretation. *JHEP*, 02:085, 2019.

- [101] S. Dubovsky, V. Gorbenko and G. Hernández-Chifflet. $T\bar{T}$ partition function from topological gravity. *JHEP*, 09:158, 2018.
- [102] R. Conti, L. Iannella, S. Negro and R. Tateo. Generalised Born-Infeld models, Lax operators and the $T\bar{T}$ perturbation. *JHEP*, 11:007, 2018.
- [103] S. Dubovsky, S. Negro and M. Porrati. Topological gauging and double current deformations. *Journal of High Energy Physics*, 2023(5), May 2023.
- [104] F. Aramini, N. Brizio, S. Negro and R. Tateo. Deforming the ODE/IM correspondence with $T\bar{T}$. *JHEP*, 03:084, 2023.
- [105] P. Dorey, C. Dunning, S. Negro and R. Tateo. Geometric aspects of the ODE/IM correspondence. *J. Phys. A*, 53(22):223001, 2020.
- [106] A. Cavaglià, S. Negro, I. M. Szécsényi and R. Tateo. $T\bar{T}$ -deformed 2D Quantum Field Theories. *JHEP*, 10:112, 2016.
- [107] R. Conti, S. Negro and R. Tateo. Conserved currents and $T\bar{T}_s$ irrelevant deformations of 2D integrable field theories. *JHEP*, 11:120, 2019.
- [108] G. Hernández-Chifflet, S. Negro and A. Sfondrini. Flow Equations for Generalized $T\bar{T}$ Deformations. *Phys. Rev. Lett.*, 124(20):200601, 2020.
- [109] G. Camilo, T. Fleury, M. Lencsés, S. Negro and A. B. Zamolodchikov. On factorizable S-matrices, generalized $T\bar{T}$, and the Hagedorn transition. *JHEP*, 10:062, 2021.
- [110] L. Córdova, S. Negro and F. I. Schaposnik Massolo. Thermodynamic Bethe Ansatz past turning points: the (elliptic) sinh-Gordon model. *JHEP*, 01:035, 2022.
- [111] A. LeClair. Thermodynamics of perturbations of some single particle field theories. *J. Phys. A*, 55(18):185401, 2022.
- [112] A. LeClair. deformation of the Ising model and its ultraviolet completion. *J. Stat. Mech.*, 2111:113104, 2021.
- [113] C. Ahn and A. LeClair. On the classification of UV completions of integrable $T\bar{T}$ deformations of CFT. *JHEP*, 08:179, 2022.
- [114] M. Guica. An integrable Lorentz-breaking deformation of two-dimensional CFTs. *SciPost Phys.*, 5(5):048, 2018.
- [115] J. Cardy. The $T\bar{T}$ deformation of quantum field theory as random geometry. *JHEP*, 10:186, 2018.
- [116] J. L. Cardy. $T\bar{T}$ deformation of correlation functions. *JHEP*, 12:160, 2019.
- [117] O. Aharony and T. Vaknin. The TT^* deformation at large central charge. *JHEP*, 05:166, 2018.
- [118] O. Aharony, S. Datta, A. Giveon, Y. Jiang and D. Kutasov. Modular invariance and uniqueness of $T\bar{T}$ deformed CFT. *JHEP*, 01:086, 2019.
- [119] M. Guica and R. Monten. Infinite pseudo-conformal symmetries of classical $T\bar{T}$, $J\bar{T}$ and JT_a - deformed CFTs. *SciPost Phys.*, 11:078, 2021.

- [120] M. Guica. $J\bar{T}$ -deformed CFTs as non-local CFTs. 10 2021.
- [121] M. Guica, R. Monten and I. Tsiaras. Classical and quantum symmetries of $T\bar{T}$ -deformed CFTs. 12 2022.
- [122] M. Baggio and A. Sfondrini. Strings on NS-NS Backgrounds as Integrable Deformations. *Phys. Rev. D*, 98(2):021902, 2018.
- [123] A. Dei and A. Sfondrini. Integrable S matrix, mirror TBA and spectrum for the stringy $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ WZW model. *JHEP*, 02:072, 2019.
- [124] S. Chakraborty, A. Giveon and D. Kutasov. $T\bar{T}$, $J\bar{T}$, $T\bar{J}$ and String Theory. *J. Phys. A*, 52(38):384003, 2019.
- [125] N. Callebaut, J. Kruthoff and H. Verlinde. $T\bar{T}$ deformed CFT as a non-critical string. *JHEP*, 04:084, 2020.
- [126] L. McGough, M. Mezei and H. Verlinde. Moving the CFT into the bulk with $T\bar{T}$. *JHEP*, 04:010, 2018.
- [127] A. Giveon, N. Itzhaki and D. Kutasov. $T\bar{T}$ and LST. *JHEP*, 07:122, 2017.
- [128] V. Gorbenko, E. Silverstein and G. Torroba. dS/dS and $T\bar{T}$. *JHEP*, 03:085, 2019.
- [129] P. Kraus, J. Liu and D. Marolf. Cutoff AdS_3 versus the $T\bar{T}$ deformation. *JHEP*, 07:027, 2018.
- [130] T. Hartman, J. Kruthoff, E. Shaghoulian and A. Tajdini. Holography at finite cutoff with a T^2 deformation. *JHEP*, 03:004, 2019.
- [131] M Guica and R. Monten. $T\bar{T}$ and the mirage of a bulk cutoff. *SciPost Phys.*, 10(2):024, 2021.
- [132] Y. Jiang. Expectation value of $T\bar{T}$ operator in curved spacetimes. *JHEP*, 02:094, 2020.
- [133] G. Jafari, A. Naseh and H. Zolfi. Path Integral Optimization for $T\bar{T}$ Deformation. *Phys. Rev. D*, 101(2):026007, 2020.
- [134] L. Griguolo, R. Panerai, J. Papalini and D. Seminara. Nonperturbative effects and resurgence in Jackiw-Teitelboim gravity at finite cutoff. *Phys. Rev. D*, 105(4):046015, 2022.
- [135] S. Dubovsky, V. Gorbenko and M. Mirbabayi. Asymptotic fragility, near AdS_2 holography and $T\bar{T}$. *JHEP*, 09:136, 2017.
- [136] A. J. Tolley. $T\bar{T}$ deformations, massive gravity and non-critical strings. *JHEP*, 06:050, 2020.
- [137] L. V. Iliesiu, J. Kruthoff, G. J. Turiaci and H. Verlinde. JT gravity at finite cutoff. *SciPost Phys.*, 9:023, 2020.
- [138] S. Okumura and K. Yoshida. $T\bar{T}$ -deformation and Liouville gravity. *Nucl. Phys. B*, 957:115083, 2020.

- [139] S. Ebert, C. Ferko, H. Y. Sun and Z. Sun. $T\bar{T}$ in JT Gravity and BF Gauge Theory. *SciPost Phys.*, 13(4):096, 2022.
- [140] M. Medenjak, G. Policastro and T. Yoshimura. $T\bar{T}$ -Deformed Conformal Field Theories out of Equilibrium. *Phys. Rev. Lett.*, 126(12):121601, 2021.
- [141] M. Medenjak, G. Policastro and T. Yoshimura. Thermal transport in $T\bar{T}$ -deformed conformal field theories: From integrability to holography. *Phys. Rev. D*, 103(6):066012, 2021.
- [142] T. Bargheer, N. Beisert and F. Loebbert. Boosting Nearest-Neighbour to Long-Range Integrable Spin Chains. *J. Stat. Mech.*, 0811:L11001, 2008.
- [143] T. Bargheer, N. Beisert and F. Loebbert. Long-Range Deformations for Integrable Spin Chains. *J. Phys. A*, 42:285205, 2009.
- [144] B. Pozsgay, Y. Jiang and G. Takács. $T\bar{T}$ -deformation and long range spin chains. *JHEP*, 03:092, 2020.
- [145] E. Marchetto, A. Sfondrini and Z. Yang. $T\bar{T}$ Deformations and Integrable Spin Chains. *Phys. Rev. Lett.*, 124(10):100601, 2020.
- [146] J. Cardy and B. Doyon. $T\bar{T}$ deformations and the width of fundamental particles. *JHEP*, 04:136, 2022.
- [147] B. Doyon, J. Durnin and T. Yoshimura. The Space of Integrable Systems from Generalised $T\bar{T}$ -Deformations. *SciPost Phys.*, 13(3):072, 2022.
- [148] B. Doyon, F. Hübner and T. Yoshimura. New Classical Integrable Systems from Generalized TT^- -Deformations. *Phys. Rev. Lett.*, 132(25):251602, 2024.
- [149] B. Doyon, F. Hübner and T. Yoshimura. Generalised $T\bar{T}$ -deformations of classical free particles. 12 2023.
- [150] F. A. Smirnov and A. B. Zamolodchikov. On space of integrable quantum field theories. *Nucl. Phys. B*, 915:363–383, 2017.
- [151] A. B. Zamolodchikov. From tricritical Ising to critical Ising by thermodynamic Bethe ansatz. *Nucl. Phys. B*, 358:524–546, 1991.
- [152] D. V. Volkov and V. P. Akulov. Is the Neutrino a Goldstone Particle? *Phys. Lett. B*, 46:109–110, 1973.
- [153] A. LeClair. $t\bar{T}$ -deformation of the ising model and its ultraviolet completion. *Journal of Statistical Mechanics: Theory and Experiment*, 2021(11):113104, November 2021.
- [154] Y. Jiang. A pedagogical review on solvable irrelevant deformations of 2d quantum field theory. *Communications in Theoretical Physics*, 73(5):057201, March 2021.
- [155] J. J. Atick and E. Witten. The hagedorn transition and the number of degrees of freedom of string theory. *Nuclear Physics B*, 310(2):291–334, 1988.
- [156] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas and M. Van Raamsdonk. The deconfinement and hagedorn phase transitions in weakly coupled large n gauge theories. *Comptes Rendus. Physique*, 5(9–10):945–954, November 2004.

- [157] P. Ginsparg and G. Moore. Lectures on 2d gravity and 2d string theory (tasi 1992), 1993.
- [158] S. Negro. Integrable structures in quantum field theory. *J. Phys. A*, 49(32):323006, 2016.
- [159] P. Christe and G. Mussardo. Elastic s Matrices in (1+1)-Dimensions and Toda Field Theories. *Int. J. Mod. Phys. A*, 5:4581–4628, 1990.
- [160] H. W. Braden, E. Corrigan, P. E. Dorey and R. Sasaki. Affine Toda Field Theory and Exact S Matrices. *Nucl. Phys. B*, 338:689–746, 1990.
- [161] G. Mussardo and P. Simon. Bosonic-type S-matrix, vacuum instability and CDD ambiguities. *Nuclear Physics B*, 578(3):527–551, July 2000.
- [162] S. Dubovsky, R. Flauger and V. Gorbenko. Solving the Simplest Theory of Quantum Gravity. *JHEP*, 09:133, 2012.
- [163] S. Dubovsky, V. Gorbenko and M. Mirbabayi. Natural tuning: towards a proof of concept. *Journal of High Energy Physics*, 2013(9), September 2013.
- [164] M. Caselle, D. Fioravanti, F. Gliozzi and R. Tateo. Quantisation of the effective string with TBA. *JHEP*, 07:071, 2013.
- [165] H. Grosse and G. Lechner. Wedge-local quantum fields and noncommutative minkowski space. *Journal of High Energy Physics*, 2007(11):012–012, November 2007.
- [166] S. Sotiriadis, D. Fioretto and G. Mussardo. Zamolodchikov–faddeev algebra and quantum quenches in integrable field theories. *Journal of Statistical Mechanics: Theory and Experiment*, 2012(02):P02017, February 2012.
- [167] S. Alazzawi. Deformations of fermionic quantum field theories and integrable models. *Letters in Mathematical Physics*, 103(1):37–58, June 2012.
- [168] H. Grosse. On the construction of möller operators for the nonlinear Schrödinger equation. *Phys. Lett. B*, 86:267–271, 1979.
- [169] D. Buchholz H. J. Borchers and B. Schroer. Polarization-free generators and the s-matrix. *Communications in Mathematical Physics*, 219:125–140, 2000.
- [170] A. Fring, G. Mussardo and P. Simonetti. Form-factors for integrable Lagrangian field theories, the sinh-Gordon theory. *Nucl. Phys. B*, 393:413–441, 1993.
- [171] H. M. Babujian, A. Fring, M. Karowski and A. Zapletal. Exact form-factors in integrable quantum field theories: The Sine-Gordon model. *Nucl. Phys. B*, 538:535–586, 1999.
- [172] K. K. Kozłowski. On convergence of form factor expansions in the infinite volume quantum sinh-gordon model in 1+1 dimensions. *Inventiones mathematicae*, 233(2):725–827, April 2023.
- [173] K. K. Kozłowski and A. Simon. On multipoint correlation functions in the sinh-gordon 1+1 dimensional quantum field theory, 2025.

- [174] K. M. Watson. Some general relations between the photoproduction and scattering of pi mesons. *Phys. Rev.*, 95:228–236, 1954.
- [175] G. Delfino. *Form Factors in Two-Dimensional Quantum Field Theory*. PhD thesis, SISSA, Trieste, 1994.
- [176] F. D. Gakhov. *Boundary value problems*. Dover Publications Inc. (New York), 2014.
- [177] B. Doyon. Lecture notes: Introduction to integrable quantum field theory, 2008.
- [178] J. B. Zuber. and C. Itzykson. Quantum Field Theory and the Two-Dimensional Ising Model. *Phys. Rev. D*, 15:2875, 1977.
- [179] B. Schroer and T. T. Truong. The Order / Disorder Quantum Field Operators Associated to the Two-dimensional Ising Model in the Continuum Limit. *Nucl. Phys. B*, 144:80–122, 1978.
- [180] O. Babelon and D. Bernard. From form-factors to correlation functions: The Ising model. *Phys. Lett. B*, 288:113–120, 1992.
- [181] B. Schroer. Modular localization and the bootstrap form-factor program. *Nucl. Phys. B*, 499:547–568, 1997.
- [182] V. P. Yurov and A. B. Zamolodchikov. Correlation functions of integrable 2-D models of relativistic field theory. Ising model. *Int. J. Mod. Phys. A*, 6:3419–3440, 1991.
- [183] O. A. Castro-Alvaredo and A. Fring. Mutually local fields from form factors. *International Journal of Modern Physics B*, 16(14n15):1915–1924, June 2002.
- [184] O. A. Castro-Alvaredo and A. Fring. Form factors from free fermionic fock fields, the federbush model. *Nucl. Phys. B*, 618(3):437–464, December 2001.
- [185] O. A. Castro-Alvaredo J. L. Cardy and B. Doyon. Form factors of branch-point twist fields in quantum integrable models and entanglement entropy. *J. Statist. Phys.*, 130:129–168, 2008.
- [186] B. Feigin and M. Lashkevich. Form factors of descendant operators: free field construction and reflection relations. *Journal of Physics A: Mathematical and Theoretical*, 42(30):304014, July 2009.
- [187] J. L. Cardy and G. Mussardo. S Matrix of the Yang-Lee Edge Singularity in Two-Dimensions. *Phys. Lett. B*, 225:275–278, 1989.
- [188] S. Lukyanov and A. B. Zamolodchikov. Exact expectation values of local fields in the quantum sine-gordon model. *Nuclear Physics B*, 493(3):571–587, June 1997.
- [189] V. Fateev, S. Lukyanov, A. B. Zamolodchikov and A. B. Zamolodchikov. Expectation values of local fields in the bullough-dodd model and integrable perturbed conformal field theories. *Nuclear Physics B*, 516(3):652–674, April 1998.
- [190] V. Fateev, D. Fradkin, S. Lukyanov, A. B. Zamolodchikov and A. B. Zamolodchikov. Expectation values of descendent fields in the sine-gordon model. *Nuclear Physics B*, 540(3):587–609, February 1999.

- [191] G. Delfino and G. Mussardo. The spin-spin correlation function in the two-dimensional ising model in a magnetic field at $t = t_c$. *Nuclear Physics B*, 455(3):724–758, September 1995.
- [192] G. Delfino, P. Simonetti and J.L. Cardy. Asymptotic factorisation of form factors in two-dimensional quantum field theory. *Physics Letters B*, 387(2):327–333, October 1996.
- [193] F. A. Smirnov. Reductions of the sine-Gordon model as a perturbation of minimal models of conformal field theory. *Nucl. Phys. B*, 337:156–180, 1990.
- [194] G. Mussardo and P. Simonetti. Stress - energy tensor and ultraviolet behavior in massive integrable quantum field theories. *Int. J. Mod. Phys. A*, 9:3307–3338, 1994.
- [195] S. Datta and Y. Jiang. $T\bar{T}$ deformed partition functions. *JHEP*, 08:106, 2018.
- [196] G. Delfino, G. Mussardo and P. Simonetti. Correlation functions along a massless flow. *Phys. Rev. D*, 51:6620–6624, 1995.
- [197] A. B. Zamolodchikov and A. B. Zamolodchikov. Massless factorized scattering and sigma models with topological terms. *Nucl. Phys. B*, 379:602–623, 1992.
- [198] R. Koberle, V. Kurak, V. and J. A. Swieca. Scattering Theory and $1/N$ Expansion in the Chiral Gross-Neveu Model. *Phys. Rev. D*, 20:897, 1979. [Erratum: Phys.Rev.D 20, 2638 (1979)].
- [199] L. Griguolo, R. Panerai, J. Papalini and D. Seminara. The phase diagram of $T\bar{T}$ -deformed Yang-Mills theory on the sphere. *JHEP*, 11:078, 2022.
- [200] L. Griguolo, R. Panerai, J. Papalini and D. Seminara. Exact $T\bar{T}$ deformation of two-dimensional Yang-Mills theory on the sphere. *JHEP*, 10:134, 2022.
- [201] L. Griguolo, R. Panerai, J. Papalini and D. Seminara. Exact $T\bar{T}$ Deformation of Two-Dimensional Maxwell Theory. *Phys. Rev. Lett.*, 128(22):221601, 2022.
- [202] R. Travaglino, M. Mazzoni and O. A. Castro-Alvaredo. Generalised hydrodynamics of $T\bar{T}$ -deformed integrable quantum field theories. *JHEP*, 08:090, 2024.
- [203] J. Mossel and J. S. Caux. Generalized TBA and generalized Gibbs. *J. Phys. A*, 45:255001, 2012.
- [204] H. M: Babujian and M. Karowski. Towards the construction of Wightman functions of integrable quantum field theories. *Int. J. Mod. Phys. A*, 19S2:34–49, 2004.
- [205] O. A. Castro-Alvaredo and B. Doyon. Bi-partite entanglement entropy in massive QFT with a boundary: The Ising model. *J. Statist. Phys.*, 134:105–145, 2009.
- [206] O. A. Castro-Alvaredo and M. Mazzoni. Two-point functions of composite twist fields in the Ising field theory. *J. Phys. A*, 56(12):124001, 2023.
- [207] S. R. Valluri, D. J. Jeffrey and R. M. Corless. Some applications of the Lambert W function to physics. *Can. J. Phys.*, 78:823–831, 2000.
- [208] R. B. Mann and T. Ohta. Exact solution for the metric and the motion of two bodies in (1+1)-dimensional gravity. *Phys. Rev. D*, 55:4723–4747, 1997.

- [209] W. Donnelly and V. Shyam. Entanglement entropy and $T\bar{T}$ deformation. *Phys. Rev. Lett.*, 121(13):131602, 2018.
- [210] B. Chen, L. Chen and P.X. Hao. Entanglement entropy in $T\bar{T}$ -deformed CFT. *Phys. Rev. D*, 98(8):086025, 2018.
- [211] H. S. Jeong, K. H. Kim and M. Nishida. Entanglement and Rényi entropy of multiple intervals in $T\bar{T}$ -deformed CFT and holography. *Phys. Rev. D*, 100(10):106015, 2019.
- [212] S. He and H. Shu. Correlation functions, entanglement and chaos in the $T\bar{T}/J\bar{T}$ -deformed CFTs. *JHEP*, 02:088, 2020.
- [213] K. Allameh, A. F. Astaneh and A. Hassanzadeh. Aspects of holographic entanglement entropy for TT^- -deformed CFTs. *Phys. Lett. B*, 826:136914, 2022.
- [214] X. Jiang, P. Wang, H. Wu and H. Yang. Timelike entanglement entropy and $T\bar{T}$ deformation. 2 2023.
- [215] D. Basu, Lavish and B. Paul. Entanglement negativity in $T\bar{T}$ -deformed CFTs. *Phys. Rev. D*, 107(12):126026, 2023.
- [216] S. Khoeini-Moghaddam, F. Omid and C. Paul. Aspects of Hyperscaling Violating Geometries at Finite Cutoff. *JHEP*, 02:121, 2021.
- [217] S. Ashkenazi, S. Chakraborty, Z. Ma and T. Shachar. Linear response of entanglement entropy to $T\bar{T}$ in massive QFTs. *JHEP*, 04:077, 2023.
- [218] J. Hou, M. He and Y. Jiang. $T\bar{T}$ -deformed entanglement entropy for IQFT. *JHEP*, 03:056, 2024.
- [219] P. Calabrese and J. L. Cardy. Entanglement entropy and quantum field theory. *J. Stat. Mech.*, 0406:P002, 2004.
- [220] Z. Bajnok, G. Linardopoulos, I. M. Szécsényi and I. Vona. Finite volume form factors in integrable theories. *JHEP*, 02:083, 2024.
- [221] L. Capizzi, C. De Fazio, M. Mazzoni, L. Santamaría-Sanz and O. A. Castro-Alvaredo. Symmetry resolved entanglement of excited states in quantum field theory. Part II. Numerics, interacting theories and higher dimensions. *JHEP*, 12:128, 2022.
- [222] O. A. Castro-Alvaredo and B. Doyon. Bi-partite entanglement entropy in massive 1+1-dimensional quantum field theories. *J. Phys. A*, 42:504006, 2009.
- [223] O. A. Castro-Alvaredo and A. Fring. Renormalization group flow with unstable particles. *Phys. Rev. D*, 63:021701, 2001.
- [224] L. J. Dixon, D. Friedan, E. J. Martinec and S. H. Shenker. The Conformal Field Theory of Orbifolds. *Nucl. Phys. B*, 282:13–73, 1987.
- [225] V. G. Knizhnik. Analytic Fields on Riemann Surfaces. 2. *Commun. Math. Phys.*, 112:567–590, 1987.
- [226] C. H. Bennett, H. J. Bernstein, S. Popescu and B. Schumacher. Concentrating partial entanglement by local operations. *Phys. Rev. A*, 53:2046–2052, 1996.

- [227] Jr. C. G. Callan and F. Wilczek. On geometric entropy. *Phys. Lett. B*, 333:55–61, 1994.
- [228] C. Holzhey, F. Larsen and F. Wilczek. Geometric and renormalized entropy in conformal field theory. *Nucl. Phys.*, B424:443–467, 1994.
- [229] G. Vidal, J. I. Latorre, E. Rico and A. Kitaev. Entanglement in quantum critical phenomena. *Phys. Rev. Lett.*, 90:227902, 2003.
- [230] B. Q. Jin and V. E. Korepin. Quantum Spin Chain, Toeplitz Determinants and the Fisher—Hartwig Conjecture. *J. Statist. Phys.*, 116(1):79–95, 2004.
- [231] J. I. Latorre, E. Rico and G. Vidal. Ground state entanglement in quantum spin chains. *Quant. Inf. Comput.*, 4(1):48–92, 2004.
- [232] J. I. Latorre, C. A. Lutken, E. Rico and G. Vidal. Fine grained entanglement loss along renormalization group flows. *Phys. Rev. A*, 71:034301, 2005.
- [233] M. B. Hastings. An area law for one-dimensional quantum systems. *J. Stat. Mech.*, 0708:P08024, 2007.
- [234] O. A. Castro-Alvaredo. Massive Corrections to Entanglement in Minimal E_8 Toda Field Theory. *SciPost Phys.*, 2(1):008, 2017.
- [235] D. Bianchini, O. A. Castro-Alvaredo and B. Doyon. Entanglement Entropy of Non-Unitary Integrable Quantum Field Theory. *Nucl. Phys. B*, 896:835–880, 2015.
- [236] O. Blondeau-Fournier, O. A. Castro-Alvaredo and B. Doyon. Universal scaling of the logarithmic negativity in massive quantum field theory. *J. Phys. A*, 49(12):125401, 2016.
- [237] O. A. Castro-Alvaredo D. Bianchini. Branch Point Twist Field Correlators in the Massive Free Boson Theory. *Nucl. Phys. B*, 913:879–911, 2016.
- [238] M. Goldstein and E. Sela. Symmetry-resolved entanglement in many-body systems. *Phys. Rev. Lett.*, 120(20):200602, 2018.
- [239] J. C. Xavier, F. C. Alcaraz and G. Sierra. Equipartition of the entanglement entropy. *Phys. Rev. B*, 98(4):041106, 2018.
- [240] B. Doyon. Finite-temperature form-factors in the free Majorana theory. *J. Stat. Mech.*, 0511:P11006, 2005.
- [241] B. Doyon. Finite-temperature form-factors: A Review. *SIGMA*, 3:011, 2007.
- [242] B. Pozsgay and G. Takacs. Form factors in finite volume. II. Disconnected terms and finite temperature correlators. *Nucl. Phys. B*, 788:209–251, 2008.
- [243] B. Pozsgay and G. Takacs. Form-factors in finite volume I: Form-factor bootstrap and truncated conformal space. *Nucl. Phys. B*, 788:167–208, 2008.
- [244] O. A. Castro-Alvaredo, S. Negro and I. M. Szécsényi. On the representation of minimal form factors in integrable quantum field theory. *Nucl. Phys. B*, 1000:116459, 2024.

- [245] E. Corrigan, P. E. Dorey and R. H. Rietdijk. Aspects of affine Toda field theory on a half line. *Prog. Theor. Phys. Suppl.*, 118:143–164, 1995.
- [246] P. Bowcock, E. Corrigan, P. E. Dorey and R. H. Rietdijk. Classically integrable boundary conditions for affine Toda field theories. *Nucl. Phys. B*, 445:469–500, 1995.
- [247] O. A. Castro-Alvaredo and A. Fring. Universal boundary reflection amplitudes. *Nucl. Phys. B*, 682:551–584, 2004.
- [248] A. Fring and R. Koberle. Boundary bound states in affine Toda field theory. *Int. J. Mod. Phys. A*, 10:739–752, 1995.
- [249] V. Riva. Boundary bootstrap principle in two-dimensional integrable quantum field theories. *Nucl. Phys. B*, 604:511–536.
- [250] A. LeClair, G. Mussardo, H. Saleur and S. Skorik. Boundary energy and boundary states in integrable quantum field theories. *Nucl. Phys. B*, 453:581–618, 1995.
- [251] Z. Bajnok, C. Rim and A. B. Zamolodchikov. Sinh-Gordon boundary TBA and boundary Liouville reflection amplitude. *Nucl. Phys. B*, 796:622–650, 2008.
- [252] Z. Bajnok, L. Palla and G. Takacs. On the boundary form-factor program. *Nucl. Phys. B*, 750:179–212, 2006.
- [253] O. A. Castro-Alvaredo. Form factors of boundary fields for $A(2)$ -affine Toda field theory. *J. Phys. A*, 41:194005, 2008.
- [254] O. A. Castro-Alvaredo. Boundary form factors of the sinh-Gordon model with Dirichlet boundary conditions at the self-dual point. *J. Phys. A*, 39:11901–11914, 2006.
- [255] M. Kormos and G. Takacs. Boundary form-factors in finite volume. *Nucl. Phys. B*, 803:277–298, 2008.
- [256] G. Takacs. Form-factors of boundary exponential operators in the sinh-Gordon model. *Nucl. Phys. B*, 801(3):187–206, 2008.
- [257] M. Lencses and G. Takacs. Breather boundary form factors in sine-Gordon theory. *Nucl. Phys. B*, 852:615–633, 2011.
- [258] Z. Bajnok and L. Hollo. On form factors of boundary changing operators. *Nucl. Phys. B*, 905:96–131, 2016.
- [259] L. Hollo, Z. B. Laczko, and Z. Bajnok. Explicit boundary form factors: The scaling Lee–Yang model. *Nucl. Phys. B*, 886:1029–1045, 2014.
- [260] Y. Jiang, F. Loebbert and D. Zhong. Irrelevant deformations with boundaries and defects. *J. Stat. Mech.*, 2204(4):043102, 2022.
- [261] N. Brizio, T. Morone and R. Tateo. A Note on $T\bar{T}$ Deformations and Boundaries. 2024. arXiv:2410.20947 [hep-th].
- [262] E. Corrigan and A. Taormina. Reflection factors and a two parameter family of boundary bound states in the Sinh-Gordon model. *J. Phys. A*, 33:8739–8754, 2000.

- [263] I.M. Gel'fand and G.E. Shilov. *Generalized Functions, Volume 1: Properties and Operations*. AMS Chelsea Publishing, 2016.
- [264] J. I. Richards and H. K. Youn. *Tempered distributions*, page 58–88. Cambridge University Press, 1990.
- [265] G. Mittag-Leffler. Sur la représentation analytique des fonctions monogènes uniformes: D'une variable indépendante. *Acta Math.*, 4:1–79, 1884.
- [266] M. R. Niedermaier. On the free field realization of form factors. 1995. arXiv:9503139 [hep-th].
- [267] A. B. Zamolodchikov. Two point correlation function in scaling Lee-Yang model. *Nucl. Phys. B*, 348:619–641, 1991.
- [268] H. Bostelmann and D. Cadamuro. Characterization of local observables in integrable quantum field theories. *Commun. Math. Phys.*, 337(3):1199–1240, 2015.
- [269] A. B. Zamolodchikov. Resonance factorized scattering and roaming trajectories. *J. Phys. A*, 39:12847–12862, 2006.
- [270] D. X. Horvath, P. E. Dorey and G. Takacs. Roaming form factors for the tricritical to critical Ising flow. *JHEP*, 07:051, 2016.
- [271] N. D. Birrell and P. C. W. Davies. *Quantum Fields in Curved Space*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, UK, 1982.
- [272] S. M. Christensen. Vacuum Expectation Value of the Stress Tensor in an Arbitrary Curved Background: The Covariant Point Separation Method. *Phys. Rev. D*, 14:2490–2501, 1976.
- [273] B. S. DeWitt. Quantum Field Theory in Curved Space-Time. *Phys. Rept.*, 19:295–357, 1975.
- [274] N. D. Birrell and J. G. Taylor. Analysis of Interacting Quantum Field Theory in Curved Space-time. *J. Math. Phys.*, 21:1740–1760, 1980.
- [275] N. D. Birrell. Interacting Quantum Field Theory in Curved Space-time. In *Oxford Conference on Quantum Gravity*, 6 1980.
- [276] C. Itzykson and J. B. Zuber. *Quantum Field Theory*. International Series In Pure and Applied Physics. McGraw-Hill, New York, 1980.