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COMPUTATIONAL MODELLING OF WAVES IN HARBOURS
USING RAY METHODS

HOWARD NEIL SOUTHGATE

A thesis submitted for the degree of

Doctor of Philosophy

at The City University, Department of
Electrical, Electronic and Information Engineering

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HOWARD N SOUTHGATE

ABSTRACT

This thesis is concerned with the development of a new computational model for the prediction of wind-generated waves in harbours using a new type of ray tracing technique. The thesis includes a background to the study, an exposition of the theory of the new ray tracing technique, a description of the computational techniques adopted, a new method for interpreting ray diagrams, comparisons with analytical solutions, alternative numerical models and a physical model, and finally an account of the model's use in a commercial harbour design study.

Most alternative computational harbour modelling techniques are limited by computational effort to small harbour areas and/or long wavelengths. The new model is designed for the opposite case, of large harbour areas and short waves. The main theoretical problem is the construction of a ray system to represent breakwater diffraction. The conventional method of tracing rays radially from the breakwater tip is shown to break down in an unbounded area surrounding the geometric shadow boundary. A completely new ray system is adopted in this work which involves tracing rays from the shadow boundary rather than the breakwater tip. This method is shown to make good predictions of wave heights and directions everywhere except in a very small bounded area surrounding the breakwater tip.

Full numerical comparisons of the new ray method with the radial ray method and Sommerfeld's analytical solution are carried out. A computational model is then developed in which this new ray tracing technique for breakwater diffraction is combined with refraction due to arbitrarily varying depth profiles, and reflections from harbour boundaries with general plan shapes and reflection coefficients. The model also includes a new technique of interpreting ray diagrams to obtain wave heights and directions in a general manner, overcoming most of the problems associated with earlier methods. This new computational model is compared against a finite-element model for a simplified harbour layout, and then against a random-wave physical model of an actual harbour design in which the processes of wave diffraction, refraction and reflection are combined in a general manner. Finally, the computational ray model is used in parallel with a physical model in a commercial investigation, and an assessment of the merits of the computational model as a commercial tool is made.

List of Symbols

Listed below are the main symbols used in the thesis. Some other symbols are used locally and are defined in the text.

a	Breakwater gap width (m)
A	Wave amplitude (m)
b	Ray separation (m)
c	Wave phase velocity or celerity (ms^{-1})
c_g	Wave group velocity (ms^{-1})
d	Length of side of averaging square (m)
e	Base of natural logarithms ($= 2.71828 \dots$)
E	Wave energy flux constant (m^4s^{-1})
f	Weighting factor used in the ray averaging method
F_c	Fresnel integral
F_s	Fresnel integral
g	Acceleration due to gravity (ms^{-2})
G	Defined by Eq 3.3.5
G_-	Defined by Eq 3.3.8
G_+	Defined by Eq 3.3.9
h	Water depth (m)
i	$\sqrt{-1}$
I	Intensity function for radiating energy source ($\text{m}^{\frac{1}{2}}$)
k	Wavenumber (m^{-1})
n	Normal direction to harbour boundary
p	Ray arc length within grid triangle from entry point to u-ray starting point (m)
p_i	Defined in Fig 6.2 (m)
p_m	Defined in Fig 6.2 (m)
q	Length of ray in averaging square (m)
r	Radial polar co-ordinate (m)
R	Radius of curvature of ray (m)
R_o	Amplitude reflection coefficient
s	Path length along ray (m)
S	Wave phase (rad)
S_o	Wave phase at starting point of ray (rad)
t	Gradient of u-ray ($= \text{Cot}\alpha$)

T	Wave period (s)
u	Defined by Eq 3.2.8
v	Defined by Eq 3.2.9
w	Defined by Eq 5.2.3
W	Defined by Eq 4.12.2
W_u	Defined by Eq 4.7.3 (m^3)
x	Cartesian co-ordinate (m)
y	Cartesian co-ordinate (m)
y_o	Distance along shadow boundary from breakwater tip to u-ray starting point (m)
α	Angle between u-ray direction and shadow boundary (rad)
β	Angle between ray and harbour boundary (rad)
γ	Ray divergence
δ	Angular polar co-ordinate (rad)
ϵ	Defined by Eq 5.5.4 and Eq 5.5.5 (m^{-1})
ζ	Angle between ray and averaging line (rad)
η	Complex wave amplitude (m)
θ	Angular polar co-ordinate in primed co-ordinate system, Eq 3.4.7 (rad)
θ_o	Angle between breakwater and shadow boundary (rad)
θ_1, θ_2	Angles between breakwaters and gap line in breakwater gap problem (rad)
λ	Wavelength (m)
μ	Angle between direction of celerity gradient and forward ray direction (rad)
ν	Dummy argument
ξ	Complex reflection coefficient
π	3.14159 ...
ρ	Dummy argument
$\sum$	Summation sign
ϕ	Angle between ray direction and x-axis (rad)
χ	Angle between ray direction and grid side (rad)
φ	Phase of G (rad)
Ψ	Phase part of complex reflection coefficient (rad)
ω	Angular wave frequency (s^{-1})
∇	Two-dimensional horizontal gradient operator
$\Delta\alpha$	Increment in α between two u-rays (rad)

Subscripts

u	u-wave or u-ray quantities
u=0	Quantities calculated along the shadow boundary (where $u = 0$)
u,somm	u-wave quantities calculated from the Sommerfeld solution
u,ray	u-wave quantities calculated from the u-ray method
av	Average value of wave quantity calculated in averaging square

Superscripts

'	Quantities defined in primed co-ordinate system (Section 4.7)
---	---

Nomenclature

Wave celerity	Wave phase velocity
Prototype (or proto)	Refers to natural, as opposed to physical model, dimensions
u-wave, u-ray etc	Diffacted incident wave, ray etc
v-wave, v-ray etc	Diffacted reflected wave, ray etc

1 INTRODUCTION

Initial design studies are crucial to the successful and cost-effective construction, operation and maintenance of harbours. All harbours have specific design problems associated with their location and purpose, but some are common to most harbours. Amongst the most important are:

- 1) Choice of overall layout.
- 2) Extensions of existing piers and breakwaters.
- 3) Structural design of walls and breakwaters.
- 4) Siltation and dredging.
- 5) Strength of moorings and fenders.
- 6) Motions of moored ships, and ships under way.

These problems are essentially technical ones, concerning the interaction of water with structures and the seabed. In addition, there are problems concerned with the economic operation of the harbour such as:

- 1) An estimation of the average annual 'downtime' (the proportion of time during a year when wind and wave conditions are sufficiently bad that the harbour has to close).
- 2) Operation of a 'tidal window'. In some circumstances it is more cost-effective to allow shipping in and out of the harbour for a limited period of time around high water than to dredge the harbour and approach channels to a sufficient depth to allow passage at all states of the tide.

All of these problems of harbour design, both technical and economic, require knowledge of wave conditions inside the harbour, and so the reliable prediction of wave conditions must form a central part of any harbour design study.

In terms of the physical processes involved, the prediction of waves in harbours presents a particularly complex problem. The

theory of the propagation of surface water waves indicates the refractive index of the medium is a function of water depth. Since most harbours have more-or-less random spatial variations of water depth (except perhaps in well-maintained dredged areas), the medium is randomly inhomogeneous and will cause waves to refract in an irregular manner. However, refraction theory alone would not be adequate to deal with this problem since strong diffraction effects are also present. These are of two types: 'external' diffraction caused by the breakwaters at the harbour entrance (and any other surface-piercing obstacles within the harbour), and 'internal' diffraction occurring where refraction theory breaks down due to the creation of unrealistic ray patterns such as areas of crossing rays or 'caustics'. Furthermore, in an enclosed area such as a harbour the reflection of waves from harbour boundaries will be important. Reflections can either be total, such as from a vertical-sided smooth wall, or partial, such as from a sloping rubble-mounded structure.

A typical harbour therefore presents a problem of combined wave refraction, diffraction and reflection in an arbitrarily dispersive medium. In this regard the problem is considerably more complex than many problems of wave propagation in other areas of physics. In optics, acoustics, ultrasonics, electromagnetism or atomic physics, it is typical to have a diffraction-only problem in a medium of constant refractive index. Refraction and reflection, if they occur, are often at the boundaries of two different media each with constant refractive properties, and it is possible to treat the refraction/reflection problem at this interface as a separate study from the main diffraction problem. This is not always possible in the water wave harbour problem, where refraction, diffraction and reflection need to be combined in an integrated manner.

As well as these basic aspects of wave propagation, which are common to many areas of physics, there are others specific to surface water waves. The most important of these are the losses of wave energy due to frictional interaction with the seabed and the breaking of waves. If strong currents are present a Doppler

shifting of wave frequencies will occur and there will be an enhancement of energy dissipation by seabed friction. The waves will, in turn, act on the currents, mainly by the release of wave momentum during the breaking process. Strong winds will generate waves, although this effect is usually negligible in a relatively small enclosed area such as a harbour, in comparison with waves travelling through the entrance. Overtopping of breakwaters can be another source of wave energy during severe storms.

In the past the surest means of investigating wave behaviour in a harbour has been to construct a scaled physical model of the harbour in a wave basin. Recently, more and more use has been made of computational models of harbour wave disturbance, either as an alternative to or in conjunction with physical models. However, because of the complexity of the physics involved, no one mathematical approach has been found to give a comprehensive description of all the wave processes that take place in a general manner for a reasonable computing effort. Present-day computational models are therefore designed to cater for some physical processes in detail and/or to be relevant to particular broad types of harbour layout (but still to be generally applicable within each layout grouping). The practice of using both computational and physical models in a thorough commercial harbour design study is probably the most cost-effective means at present. The speed and flexibility of computational models can be exploited in investigating a wide range of options initially, while a physical model can then be used to study a relatively small number of the best of these options in greater detail. If necessary, the physical model can combine a wave study with an investigation of other design aspects such as ship movement or siltation.

The work presented in this thesis is concerned with a computational modelling technique based on a new type of mathematical approach. It is designed to be applicable to a range of harbour layouts and wave conditions that alternative computational models cannot or are poorly suited to describe. In

order to put this work in context, a brief description of these alternative modelling techniques is given below.

The earliest approaches did not attempt to combine the refraction and diffraction processes but treated them separately, using one or the other depending on which process was judged to be dominant. Sometimes these separate refraction and diffraction calculations were combined in an ad hoc manner. One approach was simply to consider the diffraction effect of the breakwaters at the harbour entrance. Most harbours fall into one of two general categories: the 'single breakwater' type in which the harbour is mainly protected by a single breakwater arm, and the 'small breakwater gap' type in which there are two breakwater arms with a relatively small gap between them. Both types of problem have been investigated earlier in other areas of physics, and the water wave diffraction models simply used a translation of results derived in these other disciplines to surface water wave theory. The first attempt to use such a 'borrowed' theory was by Penney and Price (1944 and 1952) during the design of the Mulberry harbour in the Second World War. They used the analytical solution to the problem of diffraction of light by a plane screen, formulated by Sommerfeld in 1896 (Sommerfeld (1896 and 1964)). In the water wave analogy, a single breakwater arm was regarded as playing the same role as the plane screen in the optics problem. A similar adaptation has been used for a small breakwater gap for which a solution was first discovered in acoustics by Rayleigh (1897) for the special case of a small gap to wavelength ratio between collinear screens. More recent work, outlined in Chapter 2, has investigated the cases of general gap widths and angled breakwaters.

The first types of investigation of wave refraction involved ray tracing techniques. Initially this was done graphically using a superimposed grid on a chart of the coastal area of interest. Series of closely spaced rays would be traced from deep water to the coastline. The construction of the rays was carried out in small spatial steps, at the end of each of which the new ray direction and the refraction and shoaling coefficients were

calculated. The whole process was repeated for different incident wave periods and directions. Since the nineteen-sixties this rather tedious process has been computerised, giving greater accuracy and far greater speed (Skovgaard et al (1975), Abernethy and Gilbert (1975), Brampton (1977)). This procedure is best used in coastal, rather than harbour, problems where there are no (or insignificant) diffracting obstacles.

Attempts to combine the refraction and diffraction processes in an integrated manner began with the development of the Mild-Slope equation (or Refraction-Diffraction equation, Eq 2.2.1) in the early nineteen-seventies. This equation is a generalisation of the Helmholtz equation in diffraction-only problems, to situations of combined refraction and diffraction on seabeds with gentle slopes. The Mild-Slope equation in its time-independent form is an elliptic equation, for which the solution procedure involves a discretisation of the harbour area and the specification of appropriate boundary conditions. Amongst the first models developed to solve this equation were those of Ito and Tanimoto (1973) using finite differences and the time-dependent form, Berkhoff (1973 and 1976) using finite differences and the time-independent form, and Bettess and Zienkiewicz (1977) using finite elements and the time-independent form. These models were found to be successful for a number of idealised test cases, although for general applications some difficulties were encountered with specifying boundary conditions for partial reflections and radiation from the harbour entrance. The main drawback, however, was a computational one. In order to resolve the wave profile a minimum number of grid points per wavelength (about five or six) were needed, and the computational effort required limited the practical use of these models to harbours whose dimensions were a relatively small multiple of the wavelengths considered. In practice these models were therefore best applied to small harbours and/or long-period waves.

A number of other approaches have since been tried. The Boussinesq equations (Peregrine (1967)) incorporate refraction and diffraction in a general manner and also include non-linear terms,

but as with the Mild-Slope equation, models based on these equations (e.g. Abbott et al (1978)) are subject to the computational restriction of relatively small ratios of harbour dimensions to wavelengths. A further limitation is that the theory breaks down at very short wavelengths. Another technique which has received a lot of attention in recent years is the parabolic approximation. This involves splitting the wave field into forward and back scattered parts and solving only for the forward scattered wave field. The technique had previously been used in optics (Corones (1975)) and acoustics (McDaniel (1975)), and was first applied to surface water waves by Radder (1979) and Lozano and Liu (1980). Southgate (1986) contains a review of the rapid developments in this modelling technique since these early investigations. Although the same type of computational restriction concerning the grid size to wavelength ratio applies in the parabolic method, the restriction is less strong compared with the other methods, and furthermore the computational effort increases less strongly with the total number of grid points. However, the absence of back-scattered reflected waves is a major drawback in harbour problems, and the technique is mainly used in coastal problems where the reflected wave is negligible or absent. Copeland (1985) has devised a technique of splitting the Mild-Slope equation into two first-order equations, but again, for reasons of computational effort, the method is best applied to harbours of small area and/or large wavelengths.

The method adopted in this thesis is based on a ray tracing technique and is designed to cater for large harbour areas and short wavelengths, the cases where the alternative methods are either not theoretically suitable or are prohibitively costly in computing effort. In principle, ray methods are well suited to this type of problem because they solve separately for the wave amplitude and phase, and therefore there is no restriction of a minimum number of grid points per wavelength. Much larger grid sizes can be used, resulting in a greatly reduced computing effort. However, ray methods are strictly applicable when diffraction effects are absent, and this clearly presents a difficulty when considering their use for harbour problems. It is

probably for this reason that very little work has been done previously to combine refraction and diffraction in a ray tracing method. It appears that only Larsen (1977 and 1978) has constructed a usable harbour model based on ray tracing. This model treats the harbour as consisting of a number of flat areas joined by boundaries with discontinuities in depth, so that refraction is treated as a series of localised problems rather than as a continuous process.

In contrast, the method developed in this thesis treats the depth variations continuously, using a superimposed grid, and the breakwater diffraction is modelled by introducing additional sets of rays. It is shown in Chapters 2 and 4 that both the single breakwater and small breakwater gap problems are capable of correct representation in the far-field (of the radial distance from the harbour entrance) by this method. The single breakwater problem presents a particular difficulty in that there exists a discontinuity in the incident wave field between the shelter provided by the breakwater and the area directly exposed to the incident wave. Much of the theoretical development in this thesis is concerned with constructing a new ray system for this problem, the difficulty being overcome by choosing the rays to originate along the line of discontinuity rather than from the tip of the breakwater as in previous methods. Diffraction of waves by obstacles within the harbour can be treated in a similar way. Wave reflections from harbour boundaries are incorporated by the use of line segments, with an associated reflection coefficient, within each grid square. A new method of deducing wave amplitudes from knowledge of ray paths is developed. This method has the important advantages of taking into account intersecting wave trains, and of producing wave amplitude results in a regular array over the harbour area. A further advantage is that the effects of caustics and other unrealistic ray patterns are smoothed, mimicking to some extent the actual process of internal diffraction. The ray model is tested against alternative ray methods, a finite-element model, and a physical model of an actual harbour design, and is then used in conjunction with a physical model in a commercial investigation.

The thesis is structured as follows. Chapter 2 contains a description of the general theory of ray tracing and its application to diffraction problems. Chapter 3 develops in some detail the analytical solution to the single breakwater diffraction problem, which forms the basis for the subsequent development of the new ray method. The theory of the new method for the single breakwater problem is described in Chapter 4, and the computational techniques required for its implementation in a computer model are given in Chapter 5. Chapter 6 contains the theory and testing of the new ray averaging technique for obtaining wave amplitudes from ray paths. In Chapter 7 an alternative ray method is developed and a comparison is made between it and the original method. The better of these methods is then compared with a finite-element model on flat and sloping seabeds. In Chapter 8 the verification of the computational model is taken a stage further by a comparison with results from a physical model of an actual harbour design. In Chapter 9 the computational model is used in a commercial investigation alongside a physical model, and an assessment of its merits as a commercial tool is made. Finally, the main findings and conclusions from the whole work are summarised in Chapter 10.

2.1 Introduction - Use of Ray Tracing Methods in Harbour Modelling

A review of various types of numerical model for combined refraction and diffraction of water waves has been given in Chapter 1. Most of these methods require a discretisation of the area to be studied and, in order to resolve the wave profile, also require a minimum number of grid points per wavelength, usually about 5 or 6. This can make the data preparation, computing time and storage, and effort in analysis of results unacceptably high in practical harbour design studies where the dimensions of the harbour are large compared with the wavelengths.

The principal virtue of a ray tracing model is that there is no requirement for a minimum number of grid points per wavelength. Much larger grid sizes can therefore be used, making ray tracing models much more computationally efficient. The main drawback of ray tracing models is their inability to incorporate diffraction in a general manner. If, however, it can be demonstrated that it is possible to include the diffraction effects of breakwaters, ray tracing methods will provide a very useful computational technique for evaluating wave disturbance in large, outer harbour areas. It is shown in this work that diffraction caused by two kinds of breakwater arrangements can be modelled by a ray tracing technique. These arrangements are (1) a small gap between two straight breakwaters and (2) a single, straight, semi-infinite breakwater. The entrances to most outer harbours approximate well to one of these types.

In Chapters 3 and 4 a new method of representing wave diffraction by a single straight breakwater is presented. As a preliminary to this work, the present chapter discusses the general theory of ray tracing and its application to diffraction problems. In Section 2.2 the standard ray equations are derived and briefly discussed. Section 2.3 then describes how some types of diffraction can be incorporated into a ray tracing model. The breakwater gap problem is then used as an illustration of representing diffraction by ray tracing in Section 2.4.

2.2 The Ray Formalism

The work throughout this thesis is based on linear, small-amplitude surface water wave theory. The propagation of such waves over a sea bed of gently varying depth can be described by solutions to the Mild Slope Equation:

$$\nabla \cdot (cc_g \nabla \eta) + \frac{\omega^2 c_g}{c} \eta = 0 \quad 2.2.1$$

In this equation c is the wave phase velocity (also referred to as 'wave celerity' in this work), c_g the group velocity, ω the angular wave frequency, η the complex wave amplitude, and ∇ the two-dimensional horizontal gradient operator. Formal derivations of the Mild Slope Equation are given in Berkhoff (1976) and Smith and Sprinks (1975). Eq 2.2.1 is of the elliptic type, and a direct numerical solution of the equation requires boundary conditions to be specified along the whole boundary of the domain under study.

To obtain the refraction approximation, the substitution:

$$\eta = Ae^{iS} \quad 2.2.2$$

is made in Eq 2.2.1, where A is the wave amplitude and S the wave phase. Equating real and imaginary parts leads to the refraction equations:

$$(\nabla S)^2 = k^2 \text{ Eiconal equation} \quad 2.2.3$$

$$\nabla \cdot (cc_g A^2 \nabla S) = 0 \text{ Transport equation} \quad 2.2.4$$

Eq 2.2.4 is exact, but Eq 2.2.3 requires the following conditions at any location to be true:

$$\frac{\nabla^2 A}{k^2 A} \ll 1 \quad 2.2.5$$

$$\text{and } \left| \frac{\nabla A}{kA} \right|, \left| \frac{\nabla h}{kh} \right| \ll 1 \quad 2.2.6$$

where h is the water depth and k is the wavenumber ($= \omega/c$). A derivation and discussion of conditions 2.2.5 and 2.2.6 is given in Jonsson (1979).

By applying Gauss' Divergence Theorem to an area between neighbouring rays, Eq 2.2.4 can be expressed as the conservation of energy flux between two rays:

$$A^2 c_g b = E \quad 2.2.7$$

where E is a constant between a given pair of rays, and b is the ray separation. Integration of Eq 2.2.3 gives:

$$S = \int_{\text{ray}} k \, ds + S_0 \quad 2.2.8$$

where s is distance directed along a ray and S_0 is the phase at the ray's starting point. Eq 2.2.3 determines the wave kinematics (the paths taken by the rays) while Eq 2.2.4, or equivalently Eq 2.2.7, determines the wave dynamics (the variation of wave amplitude along the ray paths).

Eq 2.2.7 and Eq 2.2.8 can alternatively be expressed as a set of ordinary differential equations along the ray paths:

$$\frac{dS}{ds} = k \quad 2.2.9$$

$$\frac{d}{ds} (A^2 c_g b) = 0 \quad 2.2.10$$

Therefore it is seen that this transformation has the effect of changing from an Eulerian viewpoint to a Lagrangian one. The problem presented by Eq 2.2.7 and Eq 2.2.8 is an initial value problem for each ray. This means that each ray's direction, energy flux (E in Eq 2.2.7) and phase at the ray's starting point (S_0 in Eq 2.2.8) need to be specified. Thereafter, the path of each ray, and the wave amplitude along each ray, are completely determined by Eq 2.2.7 and 2.2.8. The wave field throughout the studied domain is thus uniquely determined. The procedure in a computational model based on ray tracing involves the tracing of

sets of closely-spaced rays over the sea area of interest, each set of rays representing one monochromatic and mono-directional wave. At any inshore location the field of one wave is computed from Eq 2.2.7 and Eq 2.2.8 and, if there is more than one wave, the total field is obtained from the linear superposition of the component waves. One of the most important properties of the ray tracing method from the point of view of its use as a computational model is that no boundary conditions need be specified. This is of particular importance in coastal and harbour modelling at open sea 'side' boundaries where, generally, wave conditions are not known.

The physical processes represented in this ray tracing method are refraction and shoaling. Further physical processes can be incorporated provided their effects on wave propagation can be adequately expressed within the ray formalism. Two such processes are bottom friction (Bretschneider and Reid (1954), Southgate (1987)) and reflections from harbour boundaries (Larsen (1978)). The information about such additional physical processes can be incorporated into a ray formalism either through the initial conditions for each ray (as is the case for wave reflections) or through a progressive modification of the wave field along each ray path (bottom friction).

The situation is different with diffraction. Since, by definition, diffraction occurs where the ray formalism of the incident wave breaks down (ie where conditions 2.2.5 and 2.2.6 are not satisfied), diffraction cannot be represented within the ray formalism of the incident wave. In some cases, however, the effect of diffraction is to create an additional wave which can be represented by constructing additional set(s) of rays, known as diffracted rays. Since diffraction does not have the effect of altering the wave field progressively along a ray path, information about diffraction effects can be introduced into a ray formalism only through the initial conditions of these diffracted rays. A ray tracing representation of diffraction is therefore limited to those diffraction processes which can be represented to a good approximation by the construction of such a set of diffracted rays.

2.3 Representation of Diffraction in a Ray Tracing Model

To identify those diffraction processes which can be modelled by ray tracing, it is useful to distinguish between two general causes of diffraction. The resulting diffraction processes are called Internal and External Diffraction.

Internal Diffraction - Diffraction in regions in the open sea where refraction theory predicts that conditions 2.2.5 and 2.2.6 are violated, for example near caustics and ray crossings.

External Diffraction - Diffraction in regions where refraction theory predicts that a surface obstacle (such as a breakwater) causes a discontinuity in wave amplitude and therefore that conditions 2.2.5 and 2.2.6 are violated.

The concept of external diffraction is a more general one than edge diffraction as used by Young (1845). External diffraction refers only to the fact that surface obstacles cause diffraction and makes no reference to the mechanism of the diffraction processes. Edge diffraction postulates a mechanism for the diffraction process, namely that the edges of the diffracting obstacles act as apparent energy sources for the diffracted wave.

In practical problems, a ray tracing method cannot be used to model internal diffraction. This is because, firstly, the locations of areas of internal diffraction cannot be predicted in advance and, secondly, because diffraction in these areas generally redistributes wave energy in a manner different from that of a local energy source (see below). Errors arising from internal diffraction can, however, be reduced by averaging the effects of rays over each element of the depth grid (discussed in Chapter 6) and by considering a spectrum of incident periods and directions.

In contrast, external diffraction resulting from some simple breakwater arrangements can be represented by ray tracing. More particularly, this can be done for those external diffraction processes in which the far-field behaviour of the diffracted wave

on a flat sea bed appears to be the result of a radiating energy source. In such cases the far-field wave has the form:

$$Ae^{iS} = \frac{I}{r^{1/2}} e^{ikr} \quad 2.3.1$$

where r is the radial co-ordinate measured from a suitably chosen origin and I is an intensity function varying with angle but independent of r . It can be seen by substitution that Eq 2.3.1 is identical to the refraction equations Eq 2.2.7 and Eq 2.2.8 for a set of diverging rays for which $b = r\Delta\theta$ where $\Delta\theta$ is the angle between two rays. This shows that a ray tracing method can be used to model the far-field of the diffracted wave. Making the above substitution results in a relation between I and E for each ray:

$$E = I^2 c_{go} \Delta\theta \quad 2.3.2$$

where c_{go} is the group velocity at the apparent energy source.

External diffraction processes in which the far-field form of the diffracted wave is given by Eq 2.3.1 can therefore be modelled by a ray tracing method. The problem now is: given a particular external diffraction problem (say, an arrangement of breakwaters in a harbour), how can it be determined whether the external diffraction processes is of the form given by Eq 2.3.1, and, if it is, what is the correct system of diverging rays to represent the diffraction process? There is no simple answer to the first part; each diffraction problem has to be considered separately.

However, it is shown here that diffraction caused by two simple breakwater arrangements, namely a gap between two straight breakwaters, and a single, straight, semi-infinite breakwater, can be represented by ray tracing. The entrances to many harbours approximate well to one of these two cases.

The determination of the correct ray system involves making a tentative but physically plausible initial choice of ray starting points and directions. The initial conditions for the diffracted rays can then be obtained by matching the far-field form of the

exact diffraction solution (obtained analytically or by a standard numerical method) with the wave field predicted by the ray method. The justification of any tentative ray system is that it should predict the flat-bed diffraction solution in the far-field of kr in all directions, where r is measured from the vicinity of the surface feature which causes diffraction. Once such a set of diffracted rays has been constructed, the ray tracing technique can be applied to an arbitrary bathymetry using the initial conditions obtained from the flat-bed diffraction problem. It has to be assumed that no other diffracting obstacles or significant refraction effects occur in the near field of kr , although this restriction is unlikely to be important in most practical applications involving short waves and large harbour areas.

2.4 The Small Breakwater Gap Problem

The small breakwater gap problem provides a simple illustration of choosing a diffracted ray system and determining the initial conditions for the rays. There are three variables which determine the gap geometry: the gap length, a , and the orientations θ_1 , θ_2 of the two breakwaters with the gap line (Fig 2.1). In this problem, the far-field wave behaviour appears to be that of a wave coming from a point source of energy in the gap. The set of diffracted rays therefore consists of rays emanating radially from the gap (precisely where or how distributed within the gap does not matter since this will make negligible difference in the far-field). The value of E for each ray is found from Eq 2.2.7, and the starting phase S_0 is identical for each ray and is chosen arbitrarily.

The choice of ray system and the determination of the initial conditions (given the flat-bed diffraction solution) are therefore very simple. The major difficulty with this problem is the determination of the flat-bed diffraction solution in the general case of arbitrary values of a , θ_1 and θ_2 , and breakwater reflectivity. A number of special cases have been investigated, in particular the case where two perfectly reflecting breakwaters lie in the same straight line. An analytical solution for this problem has been derived by Carr and Stelzreide (1952) using the

theoretical work of Morse and Rubenstein (1938). A numerical integral equation method has been developed by Gilbert and Brampton (1985) based on an idea in Lamb (1932). This method is probably easier to use computationally than the analytical solution which involves evaluating series of Mathieu functions. Numerical solutions to other simple breakwater gap layouts have been obtained using integral equation techniques. Memos (1980) has derived an integral equation for the case where the lines of the two breakwaters meet at the tip of one of the breakwaters. Porter (1979) and Smallman and Porter (1985) have used integral equations to solve a number of other breakwater gap layouts, such as a gap between two collinear breakwaters with protecting spurs and a gap between two thick collinear breakwaters.

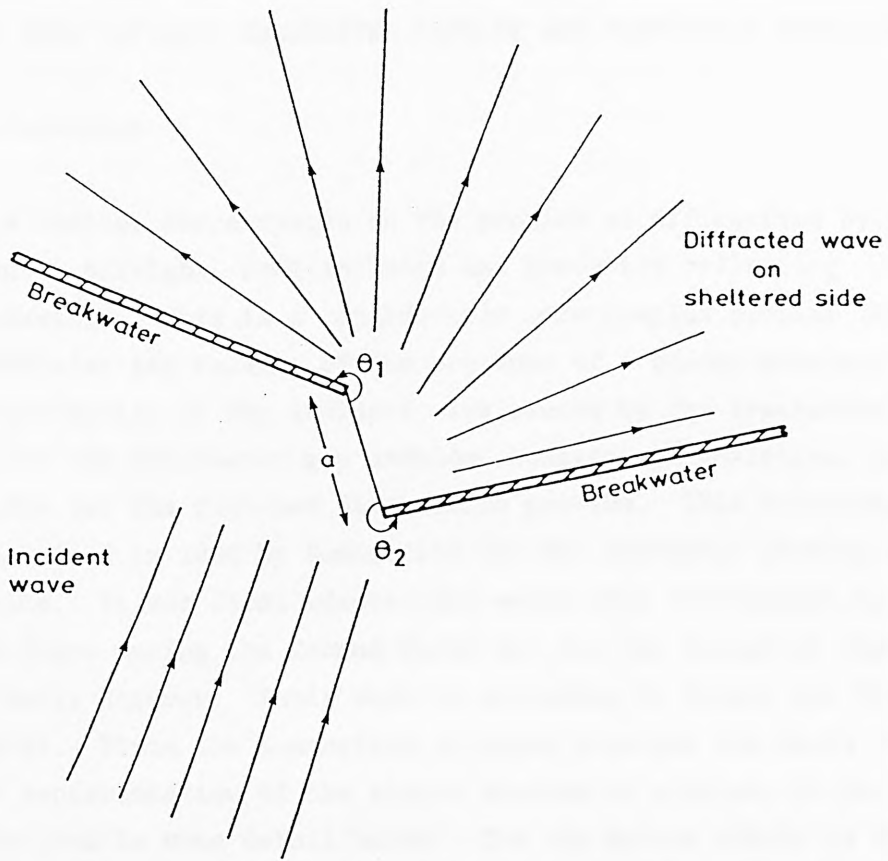


Fig 2-1 Geometry of the Breakwater Gap Problem

3 THE SEMI-INFINITE BREAKWATER PROBLEM AND SOMMERFELD SOLUTION

3.1 Introduction

This chapter concentrates on the problem of diffraction by a single, straight, semi-infinite and perfectly reflecting breakwater. This is a considerably more complex problem than the breakwater gap because of the presence of a shadow boundary (the discontinuity in the incident wave caused by the breakwater). Unlike the breakwater gap problem, however, an analytical solution exists for the flat-bed diffraction problem. This solution was determined in 1896 by Sommerfeld for the analogous problem in optics. It was first adapted for water-wave diffraction by Penney and Price during the Second World War for the design of the Mulberry Harbour. Their work is described in Penney and Price (1952). Since the Sommerfeld solution provides the basis for the ray representation of the single breakwater problem, it is described in some detail below. The ray method itself is then presented in Chapter 4.

3.2 The Sommerfeld Solution

Fig 3.1 shows the geometry of the problem. A single, straight, semi-infinite, and perfectly reflecting breakwater lies in an infinite sea area of constant water depth. Regular sinusoidal waves are incident from a direction θ_0 as defined in Fig 3.1. Radial co-ordinates (r, δ) are chosen with the origin at the breakwater tip and $\delta = 0$ line along the breakwater.

The governing equation applied throughout the whole domain is the Helmholtz Equation:

$$\nabla^2 \eta + k^2 \eta = 0 \quad 3.2.1$$

The boundary condition for perfect wave reflection applies along both sides of the breakwater:

$$\frac{\partial \eta}{\partial n} = 0 \quad 3.2.2$$

where n is the perpendicular direction outwards from the breakwater. To specify the boundary condition at infinity, the domain is split into three regions, the sheltered region, the transmission region and the reflection region as shown in Fig 3.1. In these regions the total wave field, η , is split into incident, reflected and diffracted parts (η_i , η_r , and η_d respectively).

Sheltered region	$\eta = \eta_d$	
Transmission region	$\eta = \eta_i + \eta_d$	3.2.3
Reflection region	$\eta = \eta_i + \eta_r + \eta_d$	

The incident and reflected waves in the transmission and reflection regions are given by:

$$\eta_i = e^{ikr \cos(\delta - \theta_o)} \quad 3.2.4$$

$$\eta_r = e^{ikr \cos(\delta + \theta_o)} \quad 3.2.5$$

The problem remains of specifying the boundary condition at infinity for the diffracted wave η_d . In keeping with the physical picture of external diffraction, the diffracted wave has the form of an outgoing cylindrical wave (Eq 2.3.1) in the far-field of kr . Sommerfeld (1964) has shown that this condition can be written in the form:

$$\lim_{r \rightarrow \infty} r^{1/2} \left(\frac{\partial \eta_d}{\partial r} - ik\eta_d \right) = 0 \quad 3.2.6$$

as can be seen by substitution of Eq 2.2.2 and Eq 2.3.1 in Eq 3.2.6. Eq 3.2.6 therefore represents the boundary condition for the diffracted wave along the whole infinite boundary, and is commonly known as the Sommerfeld radiation condition.

A well-defined boundary value problem has now been set up, with the governing equation given by Eq 3.2.1, and boundary values by Eq 3.2.2 and Eq 3.2.6. The solution is derived in Sommerfeld (1896) and Sommerfeld (1964).

$$Ae^{iS} = e^{ikr \cos(\delta - \theta_o)} \frac{1-i}{2} \int_{-\infty}^u e^{i\pi v^2/2} dv + e^{ikr \cos(\delta + \theta_o)} \frac{1-i}{2} \int_{-\infty}^v e^{i\pi v^2/2} dv \quad 3.2.7$$

where:

$$u = 2\left(\frac{kr}{\pi}\right)^{1/2} \sin \frac{1}{2}(\delta - \theta_o) \quad 3.2.8$$

$$v = 2\left(\frac{kr}{\pi}\right)^{1/2} \sin \frac{1}{2}(\delta + \theta_o) \quad 3.2.9$$

Lines of constant u are parabolas with the shadow boundary as axis and focus at the breakwater tip (Fig 3.2). Lines of constant v are parabolas with the axis along the reflection boundary and focus at the breakwater tip. The integrals in Eq 3.2.7 are related to Fresnel integrals.

3.3 Alternative Form and Asymptotic Expansion of the Sommerfeld Solution

An alternative form of the Sommerfeld solution and an asymptotic expansion is required for the following chapter. Using the relations:

$$\frac{1-i}{2} = \frac{1}{\sqrt{2}} e^{-i\pi/4} \quad 3.3.1$$

$$kr \cos(\delta - \theta_o) = kr - \frac{\pi u^2}{2} \text{ (from Eq 3.2.8)} \quad 3.3.2$$

$$kr \cos(\delta + \theta_o) = kr + \frac{\pi v^2}{2} \text{ (from Eq 3.2.9)} \quad 3.3.3$$

the Sommerfeld solution (Eq 3.2.7) can be rewritten as:

$$Ae^{iS} = e^{i(kr + \pi/4)} [G(u) + G(v)] \quad 3.3.4$$

where:

$$G(\rho) = \frac{1}{\sqrt{2}} e^{-i\pi(1+\rho^2)/2} \int_{-\infty}^{\rho} e^{i\pi v^2/2} dv \quad 3.3.5$$

and ρ is a dummy variable to be substituted by u or v . Provided ρ is negative, the asymptotic expansion of $G(\rho)$ can be found by writing $\int_{-\infty}^{\rho} e^{i\pi v^2/2} dv$ as $\int_{-\infty}^{\rho} (i\pi v e^{i\pi v^2/2}) \cdot 1/i\pi v dv$ and integrating by parts. The result is:

$$G(\rho) \sim -\frac{1}{\sqrt{2\pi\rho}} \left[1 - \frac{i}{\pi\rho^2} - \frac{3}{\pi^2\rho^4} + \dots \right] \quad 3.3.6$$

For positive ρ , the singularity at $\rho = 0$ in the range of integration can be avoided by rewriting the integral as:

$$\begin{aligned} \int_{-\infty}^{\rho} e^{i\pi v^2/2} dv &= \int_{-\infty}^{\infty} e^{i\pi v^2/2} dv + \int_{\infty}^{\rho} e^{i\pi v^2/2} dv \\ &= 1 + i + \int_{\infty}^{\rho} e^{i\pi v^2/2} dv \end{aligned} \quad 3.3.7$$

The integral $\int_{\infty}^{\rho} e^{i\pi v^2/2} dv$ has an identical asymptotic expansion, given by Eq 3.3.5 and Eq 3.3.6.

This asymptotic expansion is required later for both negative and positive values of u and v . It is therefore necessary to express the Sommerfeld solution in terms of the quantities $G_-(\rho)$ and $G_+(\rho)$ defined below.

$$G_-(\rho) = \frac{1}{\sqrt{2}} e^{-i\pi(1+\rho^2)/2} \int_{-\infty}^{\rho} e^{i\pi v^2/2} dv \text{ used only for } \rho \leq 0 \quad 3.3.8$$

$$G_+(\rho) = \frac{1}{\sqrt{2}} e^{-i\pi(1+\rho^2)/2} \int_{+\infty}^{\rho} e^{i\pi v^2/2} dv \text{ used only for } \rho \geq 0 \quad 3.3.9$$

Using these quantities G_- and G_+ , the Sommerfeld solution needs to be split into three parts corresponding to the sheltered, transmission and reflection regions where u and v take different signs. Substituting Eq 3.3.8 and Eq 3.3.9 into Eq 3.3.4 the final form of the Sommerfeld solution is determined:

Sheltered Region (u negative, v negative)

$$Ae^{iS} = e^{i(kr+\pi/4)} [G_-(u) + G_-(v)] \quad 3.3.10$$

Transmission Region (u positive, v negative)

$$Ae^{iS} = e^{ikr \cos(\delta-\theta_o)} + e^{i(kr+\pi/4)} [G_+(u) + G_-(v)] \quad 3.3.11$$

Reflection Region (u positive, v positive)

$$Ae^{iS} = e^{ikr \cos(\delta-\theta_o)} + e^{ikr \cos(\delta+\theta_o)} + e^{i(kr+\pi/4)} [G_+(u) + G_+(v)] \quad 3.3.12$$

In this form, the incident and reflected waves can be seen explicitly. The diffracted wave is also seen to consist of two terms representing the diffracted incident wave (which, for brevity, will be denoted subsequently as the 'u-wave') and the diffracted reflected wave (denoted as 'v-wave'). The expressions 'u-ray' and 'v-ray' are similarly defined.

3.4 Wave Amplitudes, Phases and Directions from the Sommerfeld Solution

In this section, expressions for the amplitude, phase and direction of the u-wave field in the shelter, and u-wave field plus incident wave field in the unsheltered area, are derived. A similar method can be employed for deriving expressions for the v-wave field and reflected wave field.

From Eq 3.3.4 and Eq 3.3.5, considering the u-wave field only,

$$A_u e^{iS_u} = e^{i(kr+\pi/4)} \cdot \frac{1}{\sqrt{2}} e^{-i\pi(1+u^2)/2} \int_{-\infty}^u e^{i\pi v^2/2} dv \quad 3.4.1$$

$$= e^{i(kr+\pi/4)} \cdot \frac{1}{\sqrt{2}} e^{-i\pi(1+u^2)/2} \left[\pm \frac{1+i}{2} + F_c + iF_s \right] \quad 3.4.2$$

where F_c and F_s are Fresnel integrals, defined by

$$F_c = \int_{-\infty}^u \cos \frac{\pi v^2}{2} dv \quad 3.4.3$$

$$F_s = \int_{-\infty}^u \sin \frac{\pi v^2}{2} dv \quad 3.4.4$$

and the first term in the square bracket in Eq 3.4.2 is positive for negative u (and hence negative F_c and F_s , corresponding to the sheltered area), and negative for positive u (and hence positive F_c and F_s , corresponding to the unsheltered area).

From Eq 3.4.2, the wave amplitude and phase are,

$$A_u = \frac{1}{\sqrt{2}} [(\pm \frac{1}{2} + F_c)^2 + (\pm \frac{1}{2} + F_s)^2]^{\frac{1}{2}} \quad 3.4.5$$

$$S_u = kr - \frac{\pi}{2} (\frac{1}{2} + u^2) + \tan^{-1} \left[\frac{\pm 1 + 2F_s}{\pm 1 + 2F_c} \right] \quad 3.4.6$$

in which the $\pm$ is to be interpreted as $+$ for negative u and $-$ for positive u .

Determination of wave direction requires a lengthier derivation. It is usually more useful to have wave directions determined relative to the shadow boundary, so the axes are rotated with the y -axis lying along the shadow boundary. r, θ are the polar coordinates in the new system (Fig 7.1). θ is related to δ and θ_o by

$$\theta = \frac{\pi}{2} + \delta - \theta_o \quad 3.4.7$$

u expressed in terms of r, θ is,

$$u = 2 \left(\frac{kr}{\pi} \right)^{\frac{1}{2}} \sin \left(\frac{\theta}{2} - \frac{\pi}{4} \right) \quad 3.4.8$$

The wave direction is determined by calculation of the spatial derivatives of the phase function, $(\partial S_u / \partial y)$ and $(\partial S_u / \partial x)$. From Eq 3.3.4 and Eq 3.3.5, S_u can be written as

$$S_u = kr + \frac{\pi}{4} + \varphi(u) \quad 3.4.9$$

where

$$\varphi(u) = \text{Phase } [G(u)] \quad 3.4.10$$

Differentiating Eq 3.4.9 and expressing u in terms of the radial coordinates r and θ ,

$$dS_u = k \, dr + \frac{d\varphi}{du} \left[\left(\frac{\partial u}{\partial r} \right)_\theta dr + \left(\frac{\partial u}{\partial \theta} \right)_r d\theta \right] \quad 3.4.11$$

The required partial differentials are evaluated as:

$$\left(\frac{\partial u}{\partial r} \right)_\theta = \left(\frac{k}{\pi r} \right)^{1/2} \sin \left(\frac{\theta}{2} - \frac{\pi}{4} \right) \quad 3.4.12$$

$$\left(\frac{\partial u}{\partial \theta} \right)_r = \left(\frac{kr}{\pi} \right)^{1/2} \cos \left(\frac{\theta}{2} - \frac{\pi}{4} \right) \quad 3.4.13$$

$$\left(\frac{\partial r}{\partial x} \right)_y = \cos \theta \quad 3.4.14$$

$$\left(\frac{\partial r}{\partial y} \right)_x = \sin \theta \quad 3.4.15$$

$$\left(\frac{\partial \theta}{\partial x} \right)_y = - \frac{\sin \theta}{r} \quad 3.4.16$$

$$\left(\frac{\partial \theta}{\partial y} \right)_x = \frac{\cos \theta}{r} \quad 3.4.17$$

from which,

$$\left(\frac{\partial S_u}{\partial y} \right)_x = k \sin \theta + \frac{d\varphi}{du} \left(\frac{k}{\pi r} \right)^{1/2} \cos \left(\frac{\theta}{2} + \frac{\pi}{4} \right) \quad 3.4.18$$

$$\left(\frac{\partial S_u}{\partial x} \right)_y = k \cos \theta - \frac{d\varphi}{du} \left(\frac{k}{\pi r} \right)^{1/2} \sin \left(\frac{\theta}{2} + \frac{\pi}{4} \right) \quad 3.4.19$$

The phase derivatives $\partial S_u / \partial x$ and $\partial S_u / \partial y$ have now been calculated apart from the term $d\varphi/du$. This term is found in the following way:

In general, G can be written as

$$G(u) = A(u) e^{i\varphi(u)} \quad 3.4.20$$

Differentiating Eq 3.4.20 with respect to u gives

$$\frac{dG}{du} = \frac{dA}{du} e^{i\varphi} + iAe^{i\varphi} \frac{d\varphi}{du} \quad 3.4.21$$

Differentiating Eq 3.3.5 with respect to u gives

$$\begin{aligned} \frac{dG}{du} &= -\frac{i\pi u}{\sqrt{2}} e^{-i\pi(1+u^2)/2} \int_{-\infty}^u e^{i\pi v^2/s} dv \\ &\quad + \frac{1}{\sqrt{2}} e^{-i\pi(1+u^2)/2} \cdot e^{i\pi u^2/2} \end{aligned} \quad 3.4.22$$

$$= -i\pi u A e^{i\varphi} - \frac{i}{\sqrt{2}} \quad 3.4.23$$

Equating real and imaginary parts of Eq 3.4.21 and Eq 3.4.23 gives

$$\frac{dA}{du} = -\frac{\sin \varphi}{\sqrt{2}} \quad 3.4.24$$

and

$$\frac{d\varphi}{du} = -\pi u - \frac{\cos \varphi}{\sqrt{2} A} \quad 3.4.25$$

Eq 3.4.25 can be evaluated using values of A given by Eq 3.4.5 and φ given by Eq 3.4.6 and Eq 3.4.9.

The phase derivatives $\partial S_u / \partial y$ and $\partial S_u / \partial x$ can now be determined from Eq 3.4.18 and Eq 3.4.19. Knowing these phase derivatives, it is possible to find the wave direction.

In general, the equation of a line of constant phase is

$S_u(x,y) = \text{constant}$, and the gradient of this line at any point is

$$\frac{dy}{dx} = - \frac{(\partial S_u / \partial x)_y}{(\partial S_u / \partial y)_x}$$

3.4.26

This is the wave crest direction. The wave ray direction is normal to this line.

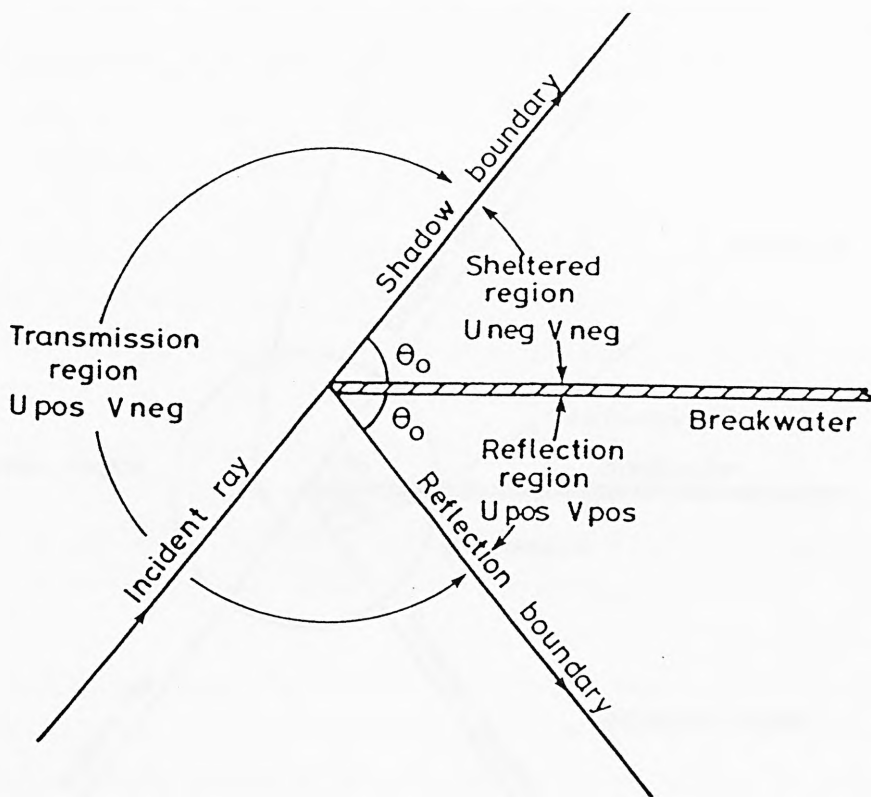


Fig 3-1 Geometry of the Semi-Infinite Breakwater Problem

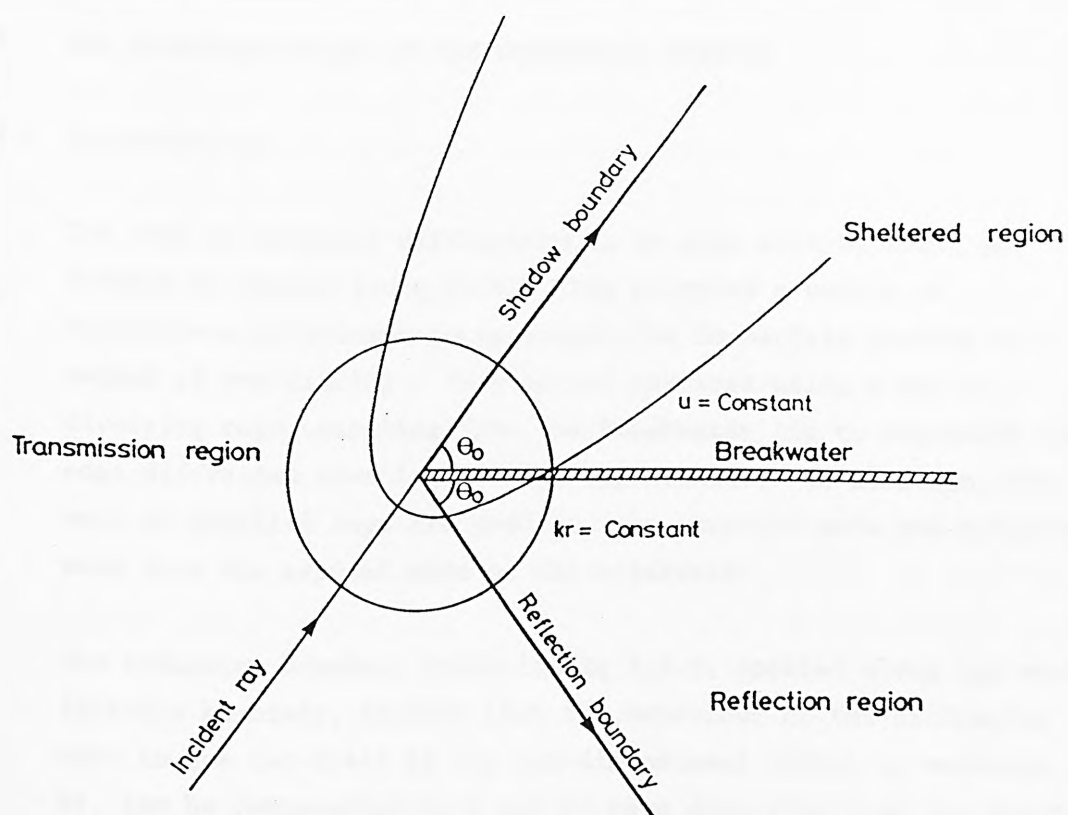


Fig 3:2 Nearfield and Farfield Regions of u and kr

4 RAY REPRESENTATIONS OF THE SOMMERFELD PROBLEM

4.1 Introduction

The idea of external diffraction as an edge effect, first put forward by Thomas Young (1845), has prompted a number of researchers to attempt to represent the Sommerfeld problem by a method of ray tracing. This method involves using a set of diverging rays emanating from the breakwater tip to represent the edge-diffracted wave (the radial ray method). In addition, two sets of parallel rays are used for the incident wave and reflected wave from the exposed side of the breakwater.

The radiation boundary condition Eq 3.2.6, applied along the whole infinite boundary, implies that the behaviour of the diffracted wave in the far-field of the non-dimensional radial co-ordinate, kr , can be represented by a set of rays diverging from the finite region of the domain. In such a representation, the diffracted wave amplitude and phase would be determined from the refraction equations Eq 2.2.7 and Eq 2.2.8, given appropriate values of E and S_0 . Therefore, it is anticipated that the ray tracing method will give accurate solutions in the far-field of kr in all directions, breaking down only in the near-field of kr , a bounded finite region centred on the breakwater tip.

However, I show in this chapter that this is not the case when applied to a set of diverging rays from the breakwater tip. The radial ray method is found to fail, not only in a bounded area close to the breakwater tip, but also in an unbounded area either side of the shadow boundary (strictly speaking, within the limiting parabola centred on the shadow boundary shown in Fig 3.2). It should be pointed out that identical considerations apply to diffraction of the wave reflected from the exposed side of the breakwater. In this instance, the ray representation breaks down within a limiting parabola centred on the reflection boundary (Fig 3.2).

This difficulty is well-known and a number of methods have been put forward to overcome it. However, in all cases, the means used have involved abandoning a pure ray-tracing approach and re-introducing some diffraction terms into the governing equations. Amongst these methods can be mentioned parabolic equation techniques (Radder (1979), Lozano and Liu (1980), Southgate (1986)), asymptotic methods (Liu, Lozano and Pantazaras (1979), Liu (1982)) and a hybrid ray-tracing/parabolic method (Kriegsmann and Larsen (1978)). Although these methods are perfectly valid, they do not shed any light on the difficulty of representing the Sommerfeld problem by a pure ray-tracing method. As I show later, the solution of this difficulty is not only of theoretical interest, but of practical importance in computational modelling of combined refraction-diffraction in areas whose dimensions are large relative to the wavelength.

The problems associated with the radial ray method are described in more detail in Section 4.2. In Sections 4.3 onwards I develop a new ray representation which overcomes these problems by a different choice of ray system.

4.2 The Radial Ray Method

In this section it is shown how the problem with the radial ray method arises and in the following sections a method is presented for its resolution. This method forms the theoretical basis of the computational model described later in the work.

The radiation condition implies that the diffracted wave can be represented in the far-field of kr by a set of diverging rays starting from within the finite domain. The simplest and most natural ray system obeying this requirement is one which comprises a set of rays emanating radially from the breakwater tip. Furthermore, this system is strongly suggested by Young's idea of edge diffraction (Young (1845)), that the breakwater tip acts as an apparent energy source for the diffracted wave. Given this ray system, it is next necessary to determine the quantities E and S_0 (defined in Eq 2.2.7 and Eq 2.2.8) for each ray. Once E and S_0

have been found for each ray, the amplitude and phase of the diffracted wave over the whole domain are determined from Eq 2.2.7 and Eq 2.2.8.

The method of finding E and S_o for each ray is to compare the far-field asymptotic form of the Sommerfeld solution for the diffracted wave, with the form predicted by the refraction equations. The sheltered zone (zone S in Fig 3.1) is considered first. In this zone, the asymptotic wave amplitude from the Sommerfeld solution for the u-wave is given by Eq 3.3.6 and Eq 3.2.8, reproduced below.

$$A_{u,somm} = \frac{1}{\sqrt{2\pi u}} \quad 4.2.1$$

$$\text{where } u = 2\left(\frac{kr}{\pi}\right)^{\frac{1}{2}} \sin \frac{1}{2} (\delta - \theta_o) \quad 4.2.2$$

In the same zone, the radial ray method predicts a wave amplitude from Eq 2.2.7, and using $b = r\Delta\theta$:

$$E = A_{u,ray}^2 c_g r \Delta\theta \quad 4.2.3$$

Putting $A_{u,somm} = A_{u,ray}$, an expression for E in the sheltered zone is obtained from Eq 4.2.1, Eq 4.2.2 and Eq 4.2.3.

$$E = \frac{c_g \Delta\theta}{4\pi k \sin \frac{1}{2} (\delta - \theta_o)} \text{ (sheltered zone)} \quad 4.2.4$$

In the sheltered zone, the asymptotic wave phase from the Sommerfeld solution for the u-wave is given by Eq 3.3.4 and Eq 3.3.6.

$$S_{u,somm} = kr + \pi/4 \quad 4.2.5$$

From Eq 2.2.8, the ray method predicts a wave phase in the sheltered zone:

$$S_{u,ray} = kr + S_o \quad 4.2.6$$

and therefore, by putting $S_{u,somm} = S_{u,ray}$,

$$S_0 = \pi/4 \text{ (sheltered zone)} \quad 4.2.7$$

Performing similar calculations for the unsheltered zone, it is found that:

$$E = \frac{c_g \Delta\theta}{4\pi k \sin \frac{1}{2} (\delta - \theta_0)} \text{ (unsheltered zone)} \quad 4.2.8$$

$$S_0 = -3\pi/4 \text{ (unsheltered zone)} \quad 4.2.9$$

Therefore, it is seen that the quantity E has a δ dependence and tends to infinity as the ray direction becomes closer to the shadow boundary. S_0 is constant for each ray apart from a discontinuous change of π on each side of the shadow boundary. The unrealistically high E values indicate that the method breaks down close to the shadow boundary. Furthermore, on the sheltered side of the shadow boundary, the wave amplitude will decay as $r^{1/2}$ (Eq 4.2.3) while on the unsheltered side the incident wave does not decay at all. Thus a discontinuity in total wave amplitude will be predicted across the shadow boundary, indicating a failure of the method around the shadow boundary.

Mathematically, the reason for the failure stems from the asymptotic expansion of the Sommerfeld solution for the u -wave (Eq 3.3.6). This expansion is in terms of, not the radial co-ordinate kr , but the quantity u defined by Eq 3.2.8. The ray tracing method will therefore fail in the region where the quantity u is small. This region is the area inside a limiting parabola with axis along the shadow boundary and focus at the breakwater tip (Fig 3.2).

It is noted that the presence of an unbounded nearfield region in u is not in conflict with the Sommerfeld radiation boundary condition. This boundary condition states that the wave field everywhere along the infinite boundary behaves like a cylindrical

outgoing wave, implying that a ray type of solution using diverging rays exists. An unbounded nearfield region (where the ray solution breaks down) is therefore allowed provided that as $r \rightarrow \infty$ the proportion of the distant circular boundary intersected by this region tends to zero. Equivalently, it can be stated that an unbounded nearfield region is allowed provided that the width of the region is proportional to r^n at large r where $n < 1$. For the present case where the nearfield region is defined by the parabola is Eq 3.2.8, the width of the parabola is proportional to $r^{\frac{1}{2}}$ at large r . In this way, the Sommerfeld radiation boundary condition is satisfied everywhere along the infinite boundary, apart from a singular point where it is intersected by the shadow boundary.

4.3 A New Representation - The u-ray Method

It has been shown that the attempt to represent the problem of diffraction by a semi-infinite breakwater using radial rays fails in the nearfield of u . However, the Sommerfeld solution shows that the wave crests in this region are regular and long-crested indicating that a ray solution is valid in this region, breaking down only close to the breakwater tip. The choice of a system of radial rays emanating from the breakwater tip is incorrect for this region, however natural it may be or however strongly suggested by a physical picture. The problem now is to determine a correct ray system, the only constraints supplied by the radiation boundary condition being that the rays should be diverging and start within, and travel outwards from, the finite domain.

The structure of this new ray system can be deduced from the general form of the Sommerfeld solution for the u -wave. Travelling outwards along a radial line from the breakwater tip at a large angle to the shadow boundary, this solution predicts a relatively rapid wave amplitude decay. For radial lines closer to the shadow boundary, this decay is slower. For a radial line along the shadow boundary itself, there is no decay, a wave amplitude of half the incident wave amplitude being predicted along the whole length of the shadow boundary. Since no amplitude

decay occurs along the shadow boundary but decay occurs to either side, the shadow boundary can be correctly modelled in a diverging ray system (which implies amplitude decay does occur along the rays) only if it forms the starting points for the rays itself. A further indication for this choice of ray system is provided by the fact that the phase of the u-wave changes discontinuously by π across the shadow boundary. This could not be introduced into a ray system in which rays crossed the shadow boundary (ie in which rays did not start from the shadow boundary).

In this new system, therefore, rays start from points along the shadow boundary in prescribed directions and with prescribed values of E and S_0 (the values of these three parameters being determinable functions of distance along the shadow boundary from the breakwater tip). Separate sets of rays travel from each side of the shadow boundary, respectively into the sheltered and unsheltered areas (Fig 4.1). Completely identical arguments apply to the diffracted reflected wave (v-wave) in which the ray system consists of sets of diverging rays starting from points on the reflection boundary. Physically, the new system corresponds to a different picture to Young's idea of edge diffraction. Here, it is the discontinuity in the wave field caused by the breakwater which acts as an apparent energy source for the diffracted wave, rather than the breakwater itself. It is possible to visualise the creation of the diffracted wave as energy "falling off" the truncated incident wave crest into the shelter.

At this stage, this proposal for a new ray system is a tentative one; the justification of the system lies in how well it predicts the Sommerfeld solution in the far-field of kr in all directions. The next task in this process is to relate the initial ray direction and values of E and S_0 to the position of the starting point of each ray on the shadow boundary. This is considered in the following sections.

4.4 Determination of u-ray Directions

The purpose of this section is to establish the relation between the direction of each u-ray and the distance of its starting point

on the shadow boundary from the breakwater tip. The method is to evaluate the spatial derivatives of the wave phase function from the Sommerfeld solution. Knowledge of these spatial phase derivatives will allow the u-ray directions to be calculated exactly. The method is a special case of that developed in Section 3.4, and is reproduced here using the simplification brought about by calculating quantities on the shadow boundary.

The method will be developed here for the u-ray system travelling into the shelter; the method for the u-ray system in the unsheltered area is very similar and is outlined afterwards. The starting point is the Sommerfeld solution for the u-wave in the shelter:

$$A_u e^{iS_u} = e^{i(kr+\pi/4)} G_-(u) \quad 4.4.1$$

The phase at any point in the sheltered zone is:

$$S_u = kr + \pi/4 + \varphi(u) \quad 4.4.2$$

$$\text{where } \varphi(u) = \text{Phase } [G_-(u)] \quad 4.4.3$$

Cartesian axes are chosen with the y-axis along the shadow boundary and the origin at the breakwater tip (Fig 4.2).

Differentiating Eq 4.4.2, and expressing u in terms of the radial co-ordinates r and θ ,

$$dS_u = kdr + \frac{d\varphi}{du} \left[\left(\frac{\partial u}{\partial r} \right)_{\theta} dr + \left(\frac{\partial u}{\partial \theta} \right)_r d\theta \right] \quad 4.4.4$$

Differentiating again with respect to x and y separately, and evaluating quantities along the shadow boundary (where $u = 0$, $\theta = \pi/2$ and $r = y$),

$$\left(\frac{\partial u}{\partial r} \right)_{\theta(u=0)} = 0 \quad 4.4.5$$

$$\left(\frac{\partial u}{\partial \theta} \right)_r(u=0) = \left(\frac{kr}{\pi} \right)^{1/2} \quad 4.4.6$$

$$\left(\frac{\partial \varphi}{\partial x}\right)_{y(u=0)} = -\frac{1}{r} \quad 4.4.7$$

$$\left(\frac{\partial \varphi}{\partial y}\right)_{x(u=0)} = 0 \quad 4.4.8$$

$$\left(\frac{\partial r}{\partial x}\right)_{y(u=0)} = 0 \quad 4.4.9$$

$$\left(\frac{\partial r}{\partial y}\right)_{x(u=0)} = 1 \quad 4.4.10$$

from which

$$\left(\frac{\partial S_u}{\partial y}\right)_{x(u=0)} = k \quad 4.4.11$$

$$\text{and } \left(\frac{\partial S_u}{\partial x}\right)_{y(u=0)} = -\left(\frac{d\varphi}{du}\right)_{(u=0)} \left(\frac{k}{\pi y}\right)^{1/2} \quad 4.4.12$$

The phase derivatives $\partial S_u / \partial x$ and $\partial S_u / \partial y$ have now been calculated apart from the term $(d\varphi/du)_{(u=0)}$. This term is found in the following way:

Putting $\rho = 0$ in Eq 3.3.8, and dropping the subscript in G_* for clarity,

$$G(0) = \frac{1}{\sqrt{2}} e^{-i\pi/2} \cdot \frac{1+i}{2} = \frac{1}{2} e^{-i\pi/4} \quad 4.4.13$$

Differentiating Eq 3.3.8 with respect to u ,

$$\begin{aligned} \frac{dG}{du} = & -\frac{i\pi u}{\sqrt{2}} e^{-i\pi(1+u^2)/2} \int_{-\infty}^{\infty} e^{i\pi\rho^2/2} d\rho + \\ & \frac{1}{\sqrt{2}} e^{-i\pi(1+u^2)/2} e^{i\pi u^2/2} \end{aligned} \quad 4.4.14$$

$$\text{and therefore } \frac{dG}{du} (u=0) = \frac{1}{\sqrt{2}} e^{-i\pi/2} \quad 4.4.15$$

In general, $G(u)$ can be written as:

$$G(u) = A(u) e^{i\varphi(u)} \quad 4.4.16$$

$$\text{and therefore } G(0) = A(0) e^{i\varphi(0)} \quad 4.4.17$$

Comparison of Eq 4.4.17 and Eq 4.4.13 shows that:

$$A(0) = \frac{1}{2} \text{ and } \varphi(0) = -\pi/4 \quad 4.4.18$$

Differentiating Eq 4.4.16:

$$\frac{dG}{du} = \frac{dA}{du} e^{i\varphi} + iAe^{i\varphi} \frac{d\varphi}{du} \quad 4.4.19$$

$$\text{and } \frac{dG}{du} (u=0) = \frac{dA}{du} (u=0) e^{i\varphi(0)} + iA(0) e^{i\varphi(0)} \frac{d\varphi}{du} (u=0) \quad 4.4.20$$

Substituting with Eq 4.4.18 gives:

$$\frac{dG}{du} (u=0) = \frac{dA}{du} (u=0) e^{-i\pi/4} + \frac{i}{2} e^{-i\pi/4} \frac{d\varphi}{du} (u=0) \quad 4.4.21$$

Comparison with Eq 4.4.15 finally yields:

$$\frac{dA}{du} (u=0) = \frac{1}{2} \text{ and } \frac{d\varphi}{du} (u=0) = -1 \quad 4.4.22$$

This completes the calculations needed for the phase derivatives along the shadow boundary:

$$\left(\frac{\partial S_u}{\partial x} \right)_{y(u=0)} = \left(\frac{k}{\pi y_0} \right)^{1/2} \quad 4.4.23$$

$$\left(\frac{\partial S_u}{\partial y} \right)_{x(u=0)} = k \quad 4.4.24$$

in which y_0 is the distance of the u-ray starting point on the shadow boundary from the breakwater tip.

4.5 The Ray Equation

Knowing the phase derivatives along the shadow boundary, it is now possible to find the directions of the u-rays and hence the equations satisfied by the family of u-rays.

In general, the equation of a line of constant phase is $S_u(x, y) =$

constant, and the gradient of this line at any point is:

$$\frac{dy}{dx} = - \frac{(\partial S_u / \partial x)_y}{(\partial S_u / \partial y)_x} \quad 4.5.1$$

The gradient of the constant-phase line on the shadow boundary is therefore:

$$\frac{dy}{dx}_{(u=0)} = - \frac{(\partial S_u / \partial x)_{y(u=0)}}{(\partial S_u / \partial y)_{x(u=0)}} = - \frac{1}{(\pi k y_0)^{1/2}} \quad 4.5.2$$

This can be expressed alternatively as the gradient of the u-ray, since the ray is normal to the constant-phase line. Defining:

$$t = \cot \alpha \quad 4.5.3$$

as the gradient of the u-ray, where α is the angle between the forward direction of the u-ray and the outward direction of the shadow boundary (Fig 4.1),

$$\frac{dy}{dx}_{(u=0)} = \tan (\pi - \alpha) = - \tan \alpha \quad 4.5.4$$

and therefore from Eq 4.5.2,

$$t = (\pi k y_0)^{1/2} \quad 4.5.5$$

The equation satisfied by the u-ray is therefore,

$$y = tx + y_0 \quad \text{or} \quad 4.5.6$$

$$y = tx + \frac{t^2}{\pi k} \quad 4.5.7$$

with t taking positive values.

The family of u-rays is obtained by varying the value of the parameter t .

4.6 The Equation of the Ray Envelope

In the following section, it will be necessary to know the points where neighbouring u-rays meet, after projection 'backwards' from the shadow boundary. In other words, it will be necessary to know the equation of the envelope formed by backward projection of the u-rays. This ray envelope is found by solving simultaneously the equations $F(t) = 0$ and $dF(t)/dt = 0$ where $F(t) = 0$ is the ray equation, Eq 4.5.7. These equations are:

$$y = tx + \frac{t^2}{\pi k} \text{ and } 2t + \pi kx = 0 \quad 4.6.1$$

from which

$$x = -\frac{2t}{\pi k} \text{ and } y = -\frac{t^2}{\pi k} \quad 4.6.2$$

This is the equation of a parabola passing through the breakwater tip and is shown in Fig 4.2.

4.7 Determination of E and S_0

The relation between the u-ray directions and their starting points on the shadow boundary has now been determined (Eq 4.5.5). The next step is to determine the initial conditions E and S_0 for the u-rays. These quantities are found by comparing the wave field predicted by the u-ray system with the asymptotic form of the Sommerfeld solution in the far-field of u, ie outside the limiting parabola in Fig 3.2.

The quantity E will be determined first. New cartesian axes (x', y') are defined with the origin at a point $(-2t/\pi k, -t^2/\pi k)$ on the u-ray envelope and with axes parallel to the original axes (Fig 4.1). Quantities measured in this co-ordinate system are primed. Eq 2.2.7 is now applied to two closely spaced rays meeting in the vicinity of the point $(-2t/\pi k, -t^2/\pi k)$,

$$A_u^2 c_g r' \Delta\alpha = E_u(t) \quad 4.7.1$$

in which $\Delta\alpha$ is the angle between the rays, and $E_u(t)$ expresses the fact that the energy flux is constant between a given pair of rays but varies from one ray pair to another. c_g and $\Delta\alpha$ are constants and may be incorporated into the right-hand side of Eq 4.7.1.

Furthermore, any function of t can be introduced into the right-hand side by suitably altering the unknown function $E_u(t)$.

Introducing the quantity $1-\cos(\delta'-\theta_0)$ [which is equal to $1-t(1+t^2)^{-1/2}$ since $\alpha = \theta_0 - \delta'$] into the denominator gives:

$$A_u^2 = \frac{W_u(t)}{r' [1-\cos(\delta'-\theta_0)]} \quad 4.7.2$$

in which $W_u(t)$ is the "altered" form of $E_u(t)$:

$$W_u(t) = \frac{E_u(t) (1-\cos\alpha)}{c_g \Delta\alpha} \quad 4.7.3$$

since $\cos\alpha = \cos(\delta'-\theta_0)$. Eq 4.7.2 can be compared with the far-field form of the Sommerfeld solution. From Eq 4.2.1 and Eq 4.2.2 this is:

$$A_{u,somm}^2 \sim \frac{1}{4\pi k r [1-\cos(\delta-\theta_0)]} \quad 4.7.4$$

In the asymptotic limit of u , $r' \rightarrow r$ and $\delta' \rightarrow \delta$, and thus Eq 4.7.2 and Eq 4.7.4 have the same form in r and δ . Comparison of the two determines $W_u(t)$,

$$W_u(t) = \frac{1}{4\pi k} \text{ in the asymptotic limit of } u \quad 4.7.5$$

Since the aim is to have a ray representation in the near-field of u as well as the far-field, it is now postulated that, in keeping with the refraction equation, Eq 2.2.7, energy flux is conserved everywhere between neighbouring rays.

$$\text{Therefore } W_u(t) = \frac{1}{4\pi k} \text{ everywhere} \quad 4.7.6$$

From Eq 4.7.3, $E_u(t)$ can be found,

$$E_u(t) = \frac{c_g \Delta\alpha}{4\pi k (1-\cos\alpha)} \quad 4.7.7$$

and from Eq 4.7.2 the wave amplitude at any point along a ray is:

$$A_u^2 = \frac{1}{4\pi k r' (1-\cos\alpha)} \quad 4.7.8$$

It is more convenient to have a formula for A_u in terms of s and α rather than r' and α , where s is the distance along a u-ray from the shadow boundary. Reference to Fig 4.2 shows that:

$$r' = s + \frac{2t}{\pi k} (1+t^2)^{\frac{1}{2}} = s + \frac{2 \cos\alpha}{\pi k \sin^2\alpha} \quad 4.7.9$$

and therefore,

$$A_u^2 = \frac{1 + \cos\alpha}{4(\pi k s \sin^2\alpha + 2 \cos\alpha)} \quad 4.7.10$$

This procedure has determined an expression for E for each u-ray. A similar expression for S_o needs to be found. Generally, the phase at a point on a u-ray can be written as:

$$S_u = ks + S_o \quad 4.7.11$$

where S_o is the phase of the u-ray at its starting point on the shadow boundary. S_o is found by matching the far field asymptotic form of the Sommerfeld solution for S_u with the values of S_u predicted by the ray method. The Sommerfeld value of S_u in the far field of u is given by Eq 4.2.5.

$$S_{u,somm} = kr + \pi/4 \quad 4.7.12$$

Matching with Eq 4.7.11 gives

$$S_o = k(r-s) + \pi/4 \quad 4.7.13$$

From the geometrical relations between r , s , y_o , α and $\theta_o - \delta$

(Fig 7.1), this equation can be written as

$$S_o = \frac{ky_o (\sin \alpha - \sin (\theta_o - \delta))}{\sin (\alpha - (\theta_o - \delta))} + \frac{\pi}{4}$$

Using a trigonometric identity, this becomes

$$S_o = ky_o \frac{\cos \frac{1}{2} (\alpha + (\theta_o - \delta))}{\cos \frac{1}{2} (\alpha - (\theta_o - \delta))} + \frac{\pi}{4} \quad 4.7.14$$

In the far field of u, $\alpha \approx \theta_o - \delta$ and therefore from Eq 4.7.11 and Eq 4.7.14,

$$S_u = ks + ky_o \cos \alpha + \frac{\pi}{4} \text{ in the sheltered zone} \quad 4.7.15$$

4.8 u-rays in the Unsheltered Zone

The method for determining the u-ray system in the unsheltered zone is very similar to that for the sheltered zone. Here, the method will be briefly outlined, mentioning differences in results from the sheltered zone where these occur.

The Sommerfeld solution for the u-wave in the unsheltered zone is:

$$A_u e^{iS_u} = e^{i(kr + \pi/4)} G_+(u) \quad 4.8.1$$

Identical arguments to those in the previous sections are followed. These lead to a different value of $d\varphi/du_{(u=0)}$. In the evaluation of this quantity,

$$G(o) = \frac{1}{\sqrt{2}} e^{-i\pi/2} \left(-\frac{i+1}{2}\right) = \frac{1}{2} e^{3i\pi/4} \quad 4.8.2$$

$$\text{and therefore } \varphi(o) = \frac{3\pi}{4} \quad 4.8.3$$

This results in a value of $d\varphi/du_{(u=0)} = +1$ and hence that:

$$t = -(\pi k y_0)^{\frac{1}{2}} \text{ in the unsheltered zone}$$

4.8.4

The ray equation is the same as Eq 4.5.7 (with t always taking negative values), and the equation of the ray envelope is the negative branch of the parabola in Eq 4.6.2. This gives rise to a u-ray system which is symmetrical about the shadow boundary. The expressions for E and A (Eq 4.7.7 and Eq 4.7.10) are unchanged, but the initial phase is altered by π (compare Eq 4.2.9 for the unsheltered zone with Eq 4.2.7 for the sheltered zone). In the unsheltered zone, therefore:

$$S_u = ks + ky_0 \cos \alpha - \frac{3\pi}{4} \text{ in the unsheltered zone}$$

4.8.5

4.9 Radial u-rays

The ray system just described involves tracing rays covering the range 90° either side of the shadow boundary (except when the breakwater intervenes, which will occur with incident wave directions, θ_0 , less than 90°). The u-wave however exists throughout the whole 360° range. In the region not covered by shadow boundary rays, the u-wave can be represented by radial rays from the breakwater tip since in this region the asymptotic limit in u is always reached quickly. Values of E and S_0 for u-rays in this region are therefore given by Eq 4.2.4 and Eq 4.2.7, or Eq 4.2.8 and Eq 4.2.9.

4.10 Representation of the v-wave

A set of v-rays can be constructed starting from the reflection boundary to represent diffraction of the wave reflected from the exposed side of the breakwater. The method is entirely analogous to the u-ray method. Again, outside the region covered by reflection-boundary rays, the v-wave can be represented by radial rays from the breakwater tip. Fig 4.3 shows the complete u-ray and v-ray system.

4.11 Wave Amplitudes and Phases along the Shadow Boundary

The new ray system has now been set up and been shown to predict correctly the Sommerfeld solution in the far-field of u (considering the u -wave only). This has been a formal process which could have been applied to any system of diverging rays. The real test of the ray system lies in its prediction of the Sommerfeld solution around the shadow boundary in the near field of u but the far-field of kr (Fig 3.2). This test is carried out by comparing the wave amplitude and phase predicted by the ray method (Eq 4.7.10 and Eq 4.7.14) with the Sommerfeld solution.

Along the shadow boundary itself (at the centre of the 'difficult' region) both the ray method and the Sommerfeld solution reduce to simple expressions. Considering the u -wave only, the Sommerfeld solution predicts a wave amplitude of 0.5 everywhere along the shadow boundary. The wave amplitude predicted by the ray method is found by setting $s = 0$ in Eq 4.7.10,

$$A_{u=0} = \left(\frac{1 + \cos \alpha}{8 \cos \alpha} \right)^{\frac{1}{2}} \text{ along shadow boundary} \quad 4.11.1$$

Using $t = \cot \alpha$, this can be rewritten as:

$$A_{u=0} = \left(\frac{1}{8t[(1+t^2)^{\frac{1}{2}} - t]} \right)^{\frac{1}{2}} \quad 4.11.2$$

Expanding the term in the denominator binomially for large t ,

$$A_{u=0} \approx \frac{1}{2} \left(\frac{1}{1 - 1/(4t^2)} \right)^{\frac{1}{2}} \quad 4.11.3$$

and expanding binomially again,

$$A_{u=0} \approx \frac{1}{2} \left(1 + \frac{1}{16\pi^2 (y_0/\lambda)} \right) \text{ along the shadow boundary} \quad 4.11.4$$

in which Eq 4.5.5 has been used to express the variation of $A_{u=0}$ with distance per wavelength (y_0/λ) along the shadow boundary.

The term $1/16\pi^2 (y_0/\lambda)$ gives a measure of the error of the ray

method from the correct value of 0.5. It can be seen that the ray method converges quickly to the correct value, remarkably so for a method which is strictly valid only in the far-field of kr . At a distance of one wavelength from the breakwater tip, for instance, the ray method error is only 0.6%. Fig 4.4 shows a plot of $A_{u=0}$ against distance along the shadow boundary, and indicates the rapid approach of $A_{u=0}$ to the correct wave amplitude.

A similar comparison can be made for wave phases along the shadow boundary. The Sommerfeld value for the wave phase at a point on the shadow boundary can be determined from Eq 4.4.2, Eq 4.4.3 and Eq 4.4.13 in which $u = 0$. This gives

$$S_{u,\text{sommm}} = ky_0 \text{ along the shadow boundary} \quad 4.11.5$$

The ray method value can be found by putting $s = 0$ in Eq 4.7.15

$$S_{u,\text{ray}} = ky_0 \cos \alpha + \frac{\pi}{4} \text{ along the shadow boundary} \quad 4.11.6$$

In contrast to the wave amplitudes, it can be seen that the wave phases for the Sommerfeld and ray methods do not match along the shadow boundary. If Eq 4.7.15 is used, therefore, wave phases close to the shadow boundary will be incorrectly predicted. Normally, this would not matter in practical problems since the wave amplitude and direction are the important quantities in almost all applications. The phase will be important, however, if there is interference with other phase-locked wave trains. This is the case in this problem. In the shelter there is the v-wave, and in the unsheltered area there are the v-wave and incident wave. Since the v-wave has the same type of phase error, the relative phases of the u and v waves will be generally well predicted. However, around the shadow boundary in the unsheltered zone, poor total wave amplitude predictions will be made because the relative phases of the u-waves with the incident wave will be incorrect. Of course, further from the shadow boundary, in the far field of u, predictions should be good.

This presents a serious difficulty since it is precisely in the region where improvements to the conventional radial ray method are required (the near field of u) that errors due to the incorrect u-wave phases will be worst. A means of improving results in this region would be to use a different initial phase for the u-rays. These could be set equal to the exact Sommerfeld phase for the u-field on the shadow boundary (the starting phases for the v-waves would be similarly adjusted). If this were done, total wave amplitudes around the shadow boundary would be improved, but the phase error would appear in the far field. However, in the unsheltered area, away from the shadow boundary, the incident wave will be very much greater than the u and v-waves, and the total amplitude error will be small. This point is investigated quantitatively in Chapter 7, and these conclusions are borne out.

It is recommended, therefore, that the starting phases for the u-rays should be the exact Sommerfeld values on the shadow boundary. These are determined as follows. For u-rays travelling into the shelter, S_o is given by

$$S_o = ky_o + \frac{\pi}{4} + \varphi(o) \quad 4.11.7$$

From Eq 4.4.18 and Eq 4.5.5, this reduces to

$$S_o = \frac{t^2}{\pi} \quad 4.11.8$$

and therefore

$$S_u = ks + \frac{t^2}{\pi} \text{ in the sheltered zone} \quad 4.11.9$$

Following similar arguments for the unsheltered zone,

$$S_u = ks + \frac{t^2}{\pi} + \pi \text{ in the unsheltered zone} \quad 4.11.10$$

4.12 Wave Amplitudes at Small Values of α

Algebraic Manipulation of Eq 4.7.10 gives the following expression for the u-ray amplitude in terms of u and α ,

$$A_u^2 = \frac{1 + \sec\alpha}{8[1 + W^2 + |W| (1 + W^2 \sec^2\alpha)^{\frac{1}{2}}]} \quad 4.12.1$$

$$\text{where } W = \frac{\pi u}{2} \quad 4.12.2$$

For large u or α , Eq 4.12.1 reduces to the far-field asymptotic form of the Sommerfeld wave amplitude, Eq 4.2.1, which is independent of α . The case of small u has been investigated in the previous section. In the present section the behaviour of Eq 4.12.1 at small α is investigated. Putting $\sec\alpha = 1$ in Eq 4.12.1 gives,

$$A_u^2 = \frac{1}{4[1 + W^2 + |W| (1 + W^2)^{\frac{1}{2}}]} \quad 4.12.3$$

Table 4.1 shows a comparison of wave heights calculated using Eq 4.12.3 and using the full Sommerfeld equation, Eq 3.4.5, for a range of values of u. Eq 4.12.3 gives the exact Sommerfeld value at $u = 0$ and $u = \infty$, and the worst difference at intermediate values of u is less than 2.8%. Eq 4.12.3 is therefore a useful approximation for hand calculations of diffracted wave amplitudes at all values of u and α using a pocket calculator.

4.13 Combined Refraction and Diffraction

The ray method has now been shown to predict the Sommerfeld solution correctly in the far-field of kr in all directions. The importance of the ray method, however, lies in its use in an inhomogeneous medium of arbitrary varying depth, ie in which refraction (and, generally speaking, internal diffraction) takes place as well as external diffraction by the breakwater. In principle, the extension to varying depth presents no difficulty. Standard ray tracing routines, involving a discretisation of the

area to be modelled, can be used (eg Brampton (1977)). However, there will be differences in the values of E and S_o of each u-ray from those given in Eq 4.7.7 and Eq 4.11.9 for the flat-bed case.

These differences arise from the fact that when varying depths are present, the shadow boundary itself will undergo refraction. The local incident wave conditions at a u-ray starting point will therefore differ from those in the flat-bed case. It is assumed that, in keeping with Young's idea of external diffraction as a local effect, it is the local incident wave conditions in the vicinity of the u-ray starting point which determine the starting values of the u-ray.

Firstly, by analogy with Eq 4.11.7, the quantity S_o is taken to be the integral of ky_o along the shadow boundary from the breakwater tip to the u-ray starting point, ie:

$$S_o = \int_0^{y_o} k \, dy_o \text{ in the sheltered zone} \quad 4.13.1$$

$$\text{and } S_o = \int_0^{y_o} k \, dy_o + \pi \text{ in the unsheltered zone} \quad 4.13.2$$

The starting direction of the u-ray (α) is defined as the angle between the local forward direction of the shadow boundary and the initial u-ray direction. By analogy with the flat-bed equation Eq 4.5.5, its value is given by:

$$t = \cot \alpha = (\pi \int_0^{y_o} k \, dy_o)^{1/2} \quad 4.13.3$$

The quantity E is modified from its flat-bed value by the factor A_1^2/A_o^2 where the subscript i denotes local values on the shadow boundary and the subscript o denotes values at the breakwater tip. From Eq 2.2.7,

$$\frac{A_1^2}{A_o^2} = \frac{c_{go} \, b_o}{c_{gi} \, b_i} \quad 4.13.4$$

and therefore the varying-depth expression for E_u is (from Eq 4.7.7 and Eq 4.13.4):

$$E_u = \frac{c_{go} b_o \Delta\alpha}{4\pi k b_i (1-\cos\alpha)} \quad 4.13.5$$

$ U $	A (Somm)	A (Algebraic)	% Diff
0.0	0.500000	0.500000	0.00
0.1	0.452481	0.459571	-1.57
0.2	0.409803	0.418415	-2.10
0.3	0.371700	0.378722	-1.89
0.4	0.337834	0.342046	-1.25
0.5	0.307828	0.309165	-0.43
0.6	0.281296	0.280238	0.38
0.7	0.257860	0.255046	1.09
0.8	0.237160	0.233196	1.67
0.9	0.218864	0.214242	2.11
1.0	0.202672	0.197760	2.42
1.1	0.188316	0.183370	2.63
1.2	0.175558	0.170745	2.74
1.3	0.164189	0.159614	2.79
1.4	0.154030	0.149747	2.78
1.5	0.144922	0.140956	2.74
1.6	0.136730	0.133086	2.67
1.7	0.129339	0.126007	2.58
1.8	0.122648	0.119612	2.48
1.9	0.116571	0.113809	2.37
2.0	0.111033	0.108524	2.26
2.1	0.105971	0.103692	2.15
2.2	0.101331	0.099260	2.04
2.3	0.097064	0.095180	1.94
2.4	0.093130	0.091415	1.84
2.5	0.089493	0.087930	1.75
2.6	0.086121	0.084695	1.66
2.7	0.082989	0.081685	1.57
2.8	0.080071	0.078877	1.49
2.9	0.077348	0.076253	1.42
3.0	0.074801	0.073795	1.35
3.1	0.072414	0.071488	1.28
3.2	0.070173	0.069319	1.22
3.3	0.068064	0.067276	1.16
3.4	0.066077	0.065349	1.10
3.5	0.064202	0.063527	1.05
3.6	0.062430	0.061803	1.00
3.7	0.060752	0.060169	0.96
3.8	0.059161	0.058619	0.91
3.9	0.057650	0.057146	0.88
4.0	0.056215	0.055744	0.84
4.1	0.054849	0.054409	0.80
4.2	0.053547	0.053135	0.77
4.3	0.052306	0.051920	0.74
4.4	0.051120	0.050758	0.71
4.5	0.049987	0.049647	0.68
4.6	0.048903	0.048583	0.65
4.7	0.047864	0.047564	0.63
4.8	0.046869	0.046586	0.61
4.9	0.045914	0.045647	0.58
5.0	0.044998	0.044745	0.56

TABLE 4.1

Comparison of Eq 3.4.5 and Eq 4.12.3

(Cont'd overleaf)

$ U $	A (Somm)	A (Algebraic)	% Diff
5.1	0.044117	0.043878	0.54
5.2	0.043270	0.043043	0.52
5.3	0.042454	0.042240	0.50
5.4	0.041669	0.041466	0.49
5.5	0.040912	0.040720	0.47
5.6	0.040182	0.040000	0.46
5.7	0.039478	0.039304	0.44
5.8	0.038798	0.038633	0.43
5.9	0.038141	0.037984	0.41
6.0	0.037506	0.037356	0.40
6.1	0.036891	0.036749	0.39
6.2	0.036297	0.036161	0.38
6.3	0.035721	0.035591	0.36
6.4	0.035163	0.035039	0.35
6.5	0.034623	0.034504	0.34
6.6	0.034098	0.033985	0.33
6.7	0.033590	0.033481	0.32
6.8	0.033096	0.032992	0.32
6.9	0.032617	0.032517	0.31
7.0	0.032151	0.032055	0.30
7.1	0.031698	0.031606	0.29
7.2	0.031258	0.031170	0.28
7.3	0.030830	0.030745	0.27
7.4	0.030414	0.030332	0.27
7.5	0.030008	0.029930	0.26
7.6	0.029613	0.029538	0.25
7.7	0.029229	0.029156	0.25
7.8	0.028854	0.028785	0.24
7.9	0.028489	0.028422	0.24
8.0	0.028133	0.028068	0.23

$$A \text{ (Somm)} = \frac{1}{\sqrt{2}} [(\frac{1}{2} - F_c)^2 + (\frac{1}{2} - F_s)^2]^{\frac{1}{2}} \quad \text{Eq 3.4.5}$$

$$A \text{ (Algebraic)} = \frac{1}{2[1 + W^2 + |W| (1 + W^2)^{\frac{1}{2}}]^{\frac{1}{2}}} \quad \text{Eq 4.12.3}$$

$$W = \frac{\pi u}{2}$$

$$\% \text{ Diff} = \frac{A \text{ (Somm)} - A \text{ (Algebraic)}}{A \text{ (Somm)}} \times 100\%$$

TABLE 4.1 (Cont'd)

Comparison of Eq 3.4.5 and Eq 4.12.3

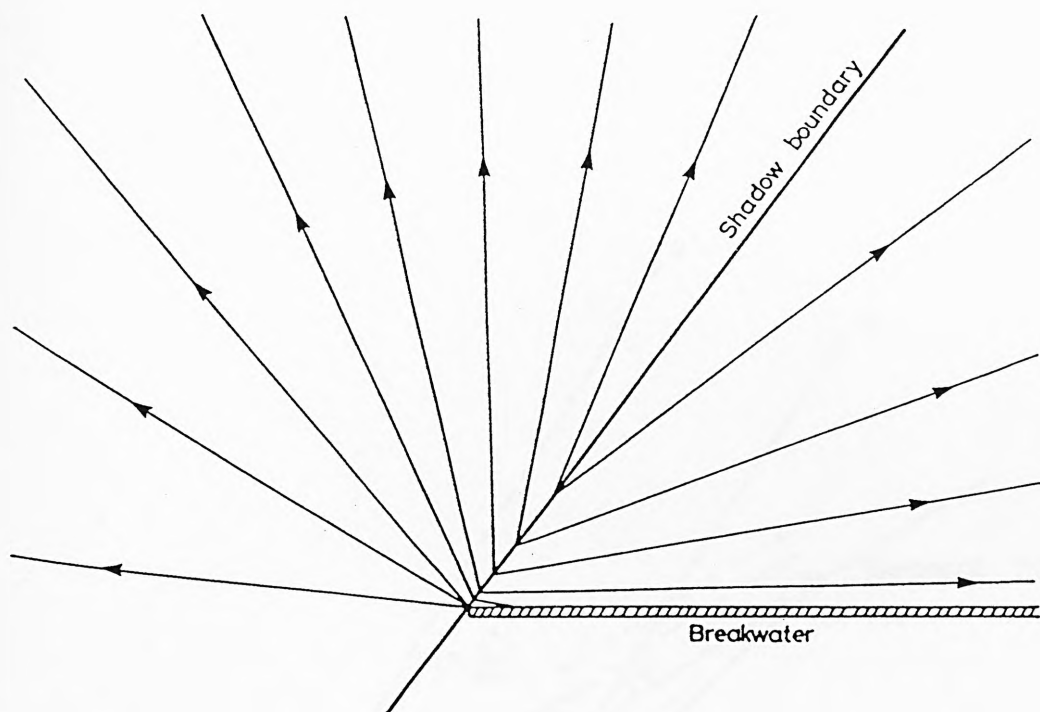


Fig 4.1 U-ray System for Diffraction around a Semi-Infinite Breakwater on a Flat Seabed

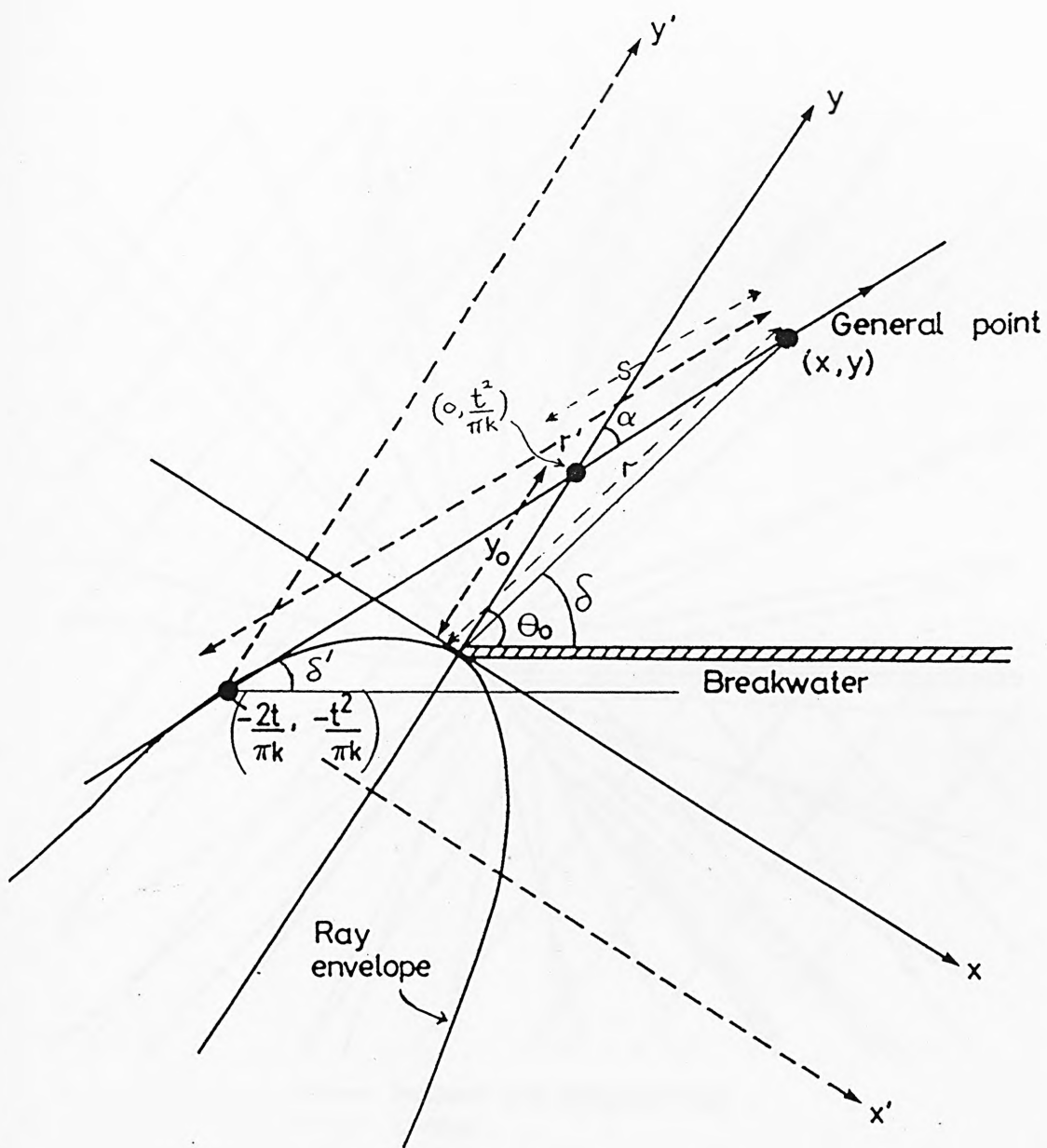


Fig 4-2 Definition of Quantities in the u-ray System

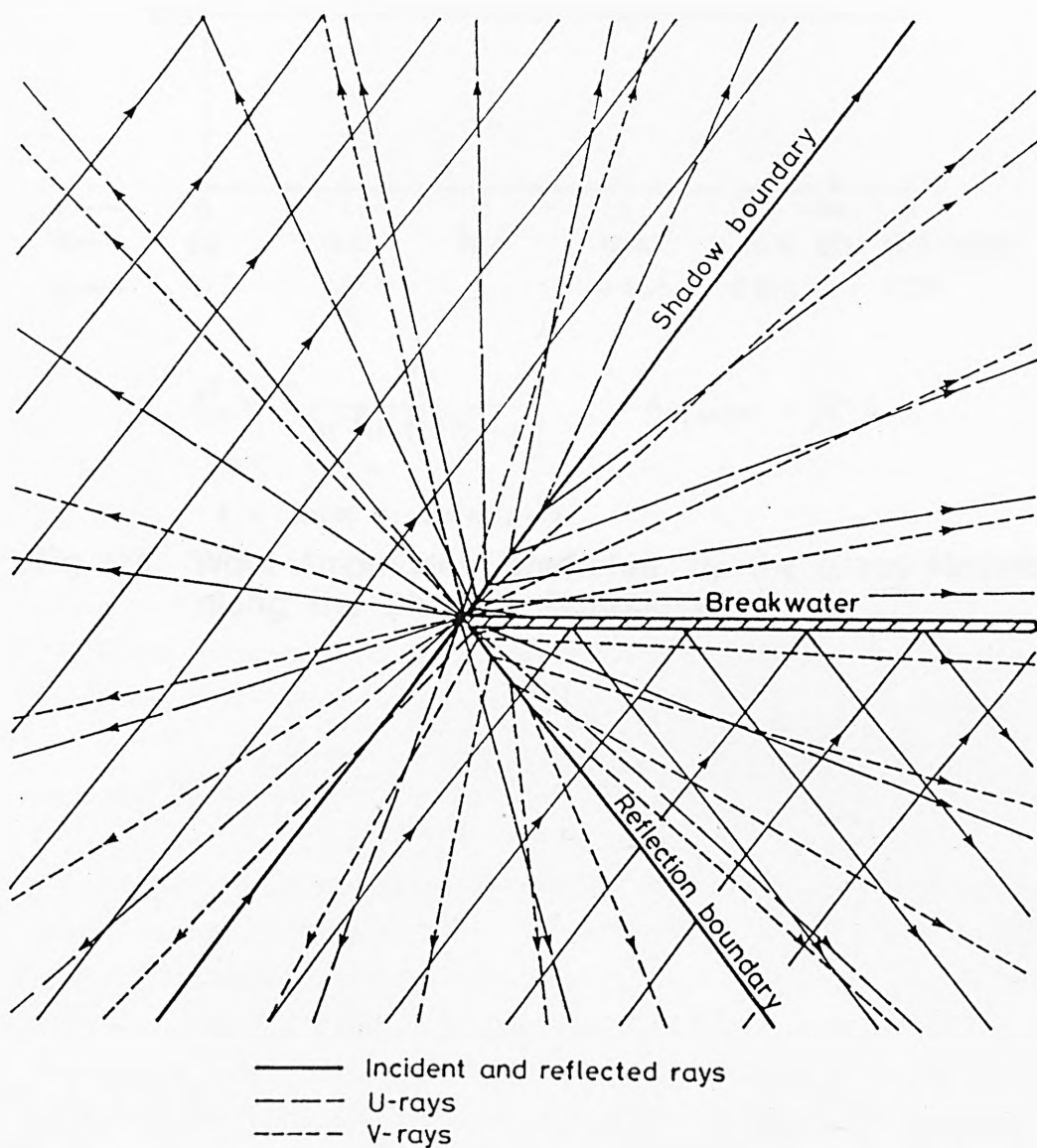
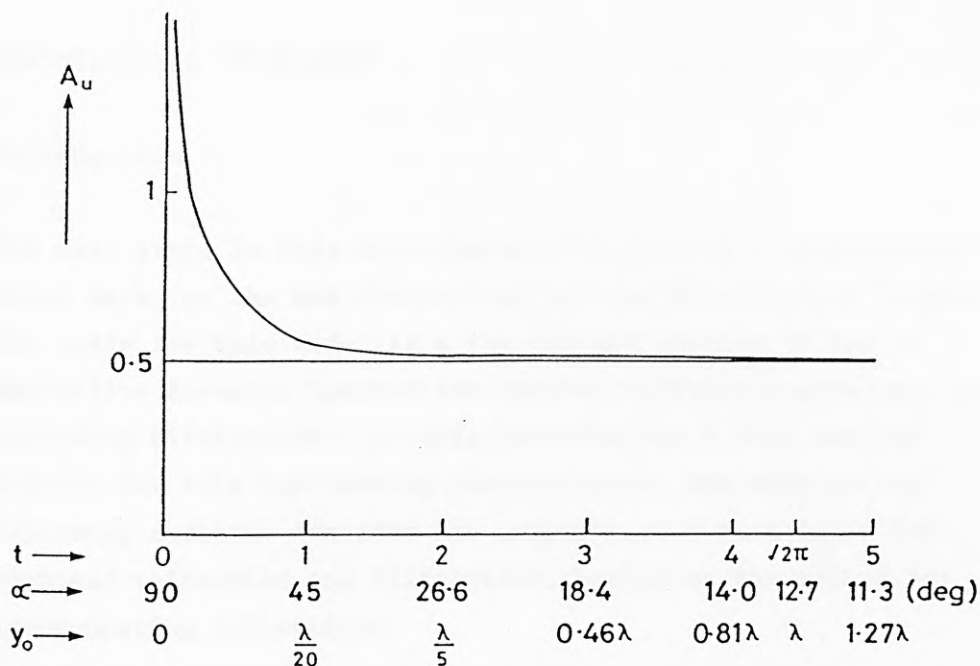


Fig 4.3 Complete u-ray and v-ray Systems



$$A_u^2 = \frac{1}{8t \left[(1+t^2)^{1/2} - t \right]} \quad A_{u, \text{somm}} = 0.5$$

$$t = \cot \alpha = (\pi k y_0)^{1/2}$$

Fig 4.4 Wave Amplitudes predicted by the u-ray Method along the Shadow Boundary

5 COMPUTATIONAL TECHNIQUES

5.1 Introduction

The next stage in this work has been to develop a computational model based on the new theoretical methods described in Chapter 4. The basis for this model is a ray tracing program in use at Hydraulics Research Limited for coastal refraction problems, not including diffraction. In this introduction I will outline briefly how this ray tracing program works, and then in the following sections describe the computational techniques for combined refraction and diffraction, including the method for incorporating reflections.

The coastal refraction program is essentially a ray tracing algorithm and it does not for instance calculate wave heights or refraction coefficients. The method involves imposing a rectangular grid over the sea area under study, with each grid element divided into two right-angled triangles. A depth value is assigned to each grid intersection point, and these values are converted to wave celerities (for a given incident wave period) using the dispersion relation,

$$\omega^2 = g k \tanh kh \qquad 5.1.1$$

The assumption is now made that within each triangle the celerity varies linearly between the three vertices of the triangle. This is a good approximation provided the celerity differences and grid sides are not too large. In practical problems this difficulty can always be overcome by introducing smaller element sizes. The effect of the constant-celerity assumption is that each ray path across a triangle is a circular arc, with the centre of the circle lying in the extrapolated celerity plane on the line where the celerity is equal to zero. Knowledge of the celerity plane and the co-ordinates and direction of a ray as it enters a triangle provides sufficient information to determine the ray arc across the triangle. In turn, this means that the co-ordinates and direction of the ray as it leaves the triangle can be calculated. Details of this calculation are given in Brampton (1977).

Sometimes the circular arcs may have radii which are so large that it is necessary to replace the arc with a straight line. A zero celerity gradient in a triangle, for instance, will cause straight-line rays. The refraction program contains two segments to deal with circular-arc and straight-line rays. Once the exit position and direction in a triangle have been found, they become the entry values for the next triangle, and the process is repeated. In this way the ray is traced across the grid system until the grid edge or shoreline is reached.

5.2 Determination of u-ray Starting Points and Directions. - Equations

The new computational model for combined refraction and diffraction is structured so that before the main body of ray tracing is performed, a single shadow-boundary ray is traced from the breakwater tip, and the starting co-ordinates and directions of u-rays are calculated at various intervals along the shadow-boundary ray. In this section the method of calculation of these u-ray starting points and directions is described. The program structure for these calculations is then described in Section 5.3.

On a constant-depth sea bed, u-ray starting points and directions are determined from Eq 4.5.5. It is now necessary to determine these values on a varying-depth sea bed over which the shadow boundary will refract. These values can be determined exactly, subject to the approximation of constant celerity slope in each grid triangle. This calculation is best considered in two stages:

- (a) calculation of the co-ordinates of a u-ray starting point (P) and the angle at P between the shadow boundary and the x-axis, in terms of the arc length p from the entry point of the triangle to P.
- (b) calculation of p in terms of the given relative distance $\int k dy_0$ from the breakwater tip to P.

- (a) The first step is a simple geometric problem. Reference to Fig 5.1a shows that:

Anticlockwise Rays

Clockwise Rays

$$\begin{aligned}
 \phi &= \phi_s + p/R & \phi &= \phi_s - p/R \\
 x &= x_c + R \sin \phi & x &= x_c - R \sin \phi \\
 y &= y_c - R \cos \phi & y &= y_c + R \sin \phi
 \end{aligned}
 \tag{5.2.1}$$

If the rays are straight the formulas are simply:

$$\begin{aligned}
 \phi &= \phi_s \\
 x &= x_s + p \cos \phi \\
 y &= y_s + p \sin \phi
 \end{aligned}
 \tag{5.2.2}$$

It is necessary to distinguish between clockwise and anticlockwise ray paths. This is achieved in the program by considering the sign of the product $(x_c - x_s)(\phi_s - \pi)$. If this product is negative the ray is travelling clockwise; if positive, anticlockwise.

- (b) The second step involves consideration of the local celerity plane. Separate formulae are derived for a circular-arc ray and a straight-line ray.

Circular-arc ray. - w is the given relative path length from the triangle's entry point to the required u-ray starting point. The subscript f denotes quantities at the u-ray starting point, and i denotes quantities at the triangle entry point.

$$\text{Thus } w = \int_i^f k \, ds
 \tag{5.2.3}$$

Reference to Fig 5.1b shows that:

$$c = \pm \nabla c \, R \sin \mu
 \tag{5.2.4}$$

where c is the wave celerity, and μ is the angle between the direction of the celerity gradient and the forward ray direction,

$$\text{and } ds = \pm R d\mu$$

5.2.5

where + is for an anticlockwise ray and - for a clockwise ray.

$$\text{Therefore, } w = \int_i^f \frac{2\pi}{cT} ds$$

$$= \frac{2\pi}{\nabla c T} \int_i^f \frac{d\mu}{\sin \mu}$$

$$= \frac{2\pi}{\nabla c T} \ln \frac{\tan \mu_f / 2}{\tan \mu_i / 2} \quad 5.2.6$$

where T is the wave period.

$$\text{Solving for } \mu_f, \mu_f = 2 \tan^{-1} \left(\tan \frac{\mu_i}{2} \exp (w \nabla c T / 2\pi) \right) \quad 5.2.7$$

$$\text{and then } p = R |\mu_f - \mu_i| \quad 5.2.8$$

Straight-line ray. - The wave celerity at a distance s along a ray is given by:

$$c = c_i + \nabla c s \cos \mu_i \quad 5.2.9$$

$$\text{Therefore, } w = \int_i^f \frac{2\pi}{cT} ds$$

$$= \frac{2\pi}{\nabla c T \cos \mu_i} \int_i^f \frac{dc}{c}$$

$$= \frac{2\pi}{TVc \cos \mu_i} \ln \left(1 + \frac{\nabla c p \cos \mu_i}{c_i} \right) \quad 5.2.10$$

$$\text{from which } p = \frac{c_i}{\nabla c \cos \mu_i} (\exp (w TVc \cos \mu_i / 2\pi) - 1) \quad 5.2.11$$

For zero or very small values of $\nabla c \cos \mu_i$ this formula reduces to:

$$p = \frac{wTc_i}{2\pi}$$

5.2.12

5.3 Determination of u-ray Starting Points and Directions. - Program Structure

In order to determine u-ray starting points and directions it is necessary to have some parameter which is incremented for successive u-rays. The most convenient parameter is the angle α . The increment $\Delta\alpha$ is read in as an input datum, and the co-ordinates of u-ray starting points are calculated for values of α equal to $\pi/2 - \Delta\alpha/2$, $\pi/2 - 3\Delta\alpha/2$, $\pi/2 - 5\Delta\alpha/2$ etc. Two further quantities need to be input to the program: the maximum value of $\int k dy_0$ along the shadow boundary at which u-rays are required to be traced, and an estimated number of u-ray starting points. The algorithm then carries out the following procedure.

- (a) The value of α for the first u-ray starting point is converted to the corresponding relative distance ($\int k dy_0$) along the shadow boundary using Eq 4.13.3.
- (b) The shadow boundary is traced from the breakwater tip to the edge of the grid triangle containing the breakwater tip. The value of $\int k dy_0$ for this segment of the shadow boundary is calculated. Comparison of this value with the required value for the u-ray starting point will determine if the starting point lies within the triangle.
- (c) If the starting point does lie within the triangle, its co-ordinates and starting direction are calculated and stored. The angle α is then incremented and the procedure repeated until a starting point is reached which lies outside the triangle.
- (d) When this occurs, the path of the shadow boundary in the next triangle is traced, and the starting points and directions in the new triangle are calculated and stored. The process stops either when the specified maximum $\int k dy_0$ has been exceeded or

when the shadow boundary hits the edge of the grid system or the shoreline.

It has been found that having $\Delta\alpha$ as the incremental parameter works well for the middle ranges of α , but for α near 90° the u-ray starting points are so close to the breakwater tip as to be almost indistinguishable from it, and for small α (large distances along the shadow boundary) the rays become so widely separated that errors will arise through insufficient ray density. Fig 5.2 shows how the u-ray starting points are spread out along the shadow boundary in a typical case. In a modification to this scheme, u-rays between $\alpha = 90^\circ$ and $\alpha = 60^\circ$ are regarded as starting from the breakwater tip where they can be incorporated in with the v-rays. This saving in ray tracing can then be used to increase the density of rays further along the shadow boundary. This is described in the following section.

5.4 Modified u-ray Tracing Scheme

To demonstrate how u-ray starting points become widely spaced at large distances along the shadow boundary, it is necessary to determine the initial separation between neighbouring u-rays, b , in terms of the angular separation $\Delta\alpha$. Reference to Fig 4.2 shows that for a constant-depth sea bed,

$$b = r' \Delta\alpha = \frac{Tc \Delta\alpha \cos \alpha}{\pi^2 \sin^2 \alpha} \quad 5.4.1$$

The same formula applies for a varying-depth sea bed using local values of c and α . The initial ray divergence (equal to $\Delta\alpha$ on a constant-depth sea bed) is modified by the curvature of the shadow boundary and the local celerity gradient. By considering all the separate cases of different shadow boundary and u-ray curvatures, the ray divergence on a varying depth sea bed is determined as,

$$\gamma = \Delta\alpha + \Delta\phi \sin \alpha (\sin \alpha + \cos \alpha \cot \mu) \quad 5.4.2$$

for α measured anticlockwise from the shadow boundary, and

$$\gamma = \Delta\alpha + \Delta\phi \sin \alpha (-\sin \alpha + \cos \alpha \cot \mu) \quad 5.4.3$$

for α measured clockwise from the shadow boundary. $\Delta\phi$ is the change in direction of the shadow boundary between neighbouring u-ray starting points, and μ is the angle between the direction of the celerity gradient and the local shadow boundary ray direction.

In the original u-ray scheme, the initial ray divergence ($\Delta\alpha$) is the same for each ray on a constant-depth sea bed but the initial ray separation varies with α . As α becomes small, the initial ray separation increases rapidly due to the $\cos \alpha / \sin^2 \alpha$ term. In the modified u-ray scheme, in order to have rays at a more uniform density at small α they are sent out at a constant increment in ray separation.

This increment is:

$$\frac{\cos \alpha}{\sin^2 \alpha} \Delta\alpha = \Delta(\operatorname{cosec} \alpha) \quad 5.4.4$$

In the modified scheme, therefore, the initial ray separation and divergence are given by:

$$b = \frac{T_c}{\pi^2} \Delta(\operatorname{cosec} \alpha) \quad 5.4.5$$

$$\text{and } \gamma = \frac{\Delta(\operatorname{cosec} \alpha) \sin^2 \alpha}{\cos \alpha} \quad 5.4.6$$

It is interesting to note that the increments in path lengths along the shadow boundary for the two schemes are:

$$\Delta w = \frac{2 \cos \alpha}{\sin^3 \alpha} \Delta\alpha \text{ original scheme} \quad 5.4.7$$

$$\Delta w = \frac{2}{\sin \alpha} \Delta(\operatorname{cosec} \alpha) \text{ modified scheme} \quad 5.4.8$$

In the modified scheme the increment in path length increases with increasing w due to the $1/\sin \alpha$ term, but not as rapidly as the original scheme where the increment is proportional to $\cos \alpha / \sin^3 \alpha$.

The value of α at which the program changes from the original scheme to the modified scheme occurs when the initial ray separation is the same in both schemes, ie when:

$$\Delta(\text{cosec}\alpha) = \frac{\cos\alpha}{\sin^2\alpha} \Delta\alpha \quad 5.4.9$$

The value of α at which the changeover occurs therefore is the solution of this equation (a quadratic in $\cos\alpha$) for α . For angles greater than the changeover value the original scheme is used, for smaller angles, the modified scheme. $\Delta(\text{cosec}\alpha)$ and $\Delta\alpha$ are input data.

5.5 Wave Reflection - Introduction

New methods of incorporating waves reflected from the outside of the main breakwaters and from walls within the harbour are described in the following three sections. Both total and partial reflection are considered. Before describing these methods, the mathematical formulation of wave reflection is outlined briefly.

The boundary condition for a totally reflecting boundary is:

$$\frac{\partial\eta}{\partial n} = 0 \quad 5.5.1$$

where n is measured normal to the boundary. This condition, together with the law of reflection,

$$\text{Angle of incidence} = \text{Angle of reflection} \quad 5.5.2$$

leads to a reflection coefficient ξ (in general a complex quantity):

$$\xi = 1 \quad 5.5.3$$

The boundary condition at a partially reflecting boundary is:

$$\frac{\partial\eta}{\partial n} + \epsilon \eta = 0 \quad 5.5.4$$

In general ϵ depends on the amplitude of the total wave. However, in order to use this condition in a linear model it has to be assumed that ϵ is independent of the total wave amplitude. Making this assumption, it is found that for an incident angle β ,

$$\epsilon = \frac{1 - \xi}{1 + \xi} i k \cos \beta \quad 5.5.5$$

5.6 Representation of Internal Harbour Boundaries

In the computational model, two types of harbour boundary are represented:

- (a) harbour walls which are either totally or partially reflecting;
- (b) gently sloping beach areas which are regarded as total absorbers of wave energy.

The first type of boundary is represented by a series of straight-line segments in the refraction grid rectangles (Fig 5.3). The co-ordinates of the points where the boundary lines intersect the grid, and the reflection coefficient ξ (in the form of two real numbers R_0 and Ψ where $\xi = R_0 e^{i\Psi}$) for each boundary segment are read as input data and stored. The second type of boundary is represented by a minimum depth (converted to a wave celerity within the program) at which the wave is regarded as having broken.

Within the model, the following procedure is carried out to determine if a ray has encountered a harbour boundary. Firstly, once the celerities at the entry and exit points of a ray in a triangle have been found, it can be readily determined if the ray has crossed the minimum celerity value. If it has, the ray is stopped. If it has not, the program checks whether the grid rectangle contains a harbour wall segment and, if it does, whether the path of the ray crosses it. If this is the case, the ray is reflected and its associated energy is modified by the reflection coefficient of that wall segment. Finally, if the cumulative

reflection coefficient falls below a certain value, the ray is stopped to avoid tracing of rays with negligible wave energy.

5.7 Representation of Partially Reflecting Breakwaters

The Sommerfeld solution, and the u-ray system, requires that the breakwater is perfectly reflecting. There is no simple modification to the Sommerfeld solution for the case of a partially reflecting breakwater. Although the behaviour of the reflected incident wave can be accounted for exactly by a reflection coefficient ξ , the same is not rigorously true for the diffracted reflected wave (v-wave). However, if the expression for the v-wave, Eqs 3.3.10-3.3.12, (for a perfectly reflecting breakwater) is multiplied by ξ , the assumptions made can be shown not to exceed the assumptions in representing the v-wave field by rays. This is readily demonstrated since, within a ray formalism, the wave energy at any point on a ray is directly proportional to the quantity E specified in the initial conditions for the ray. For v-rays, E itself is directly proportional to the local reflected wave energy on the reflection boundary, and this contains the energy reflection coefficient ξ^2 as a multiplicative factor.

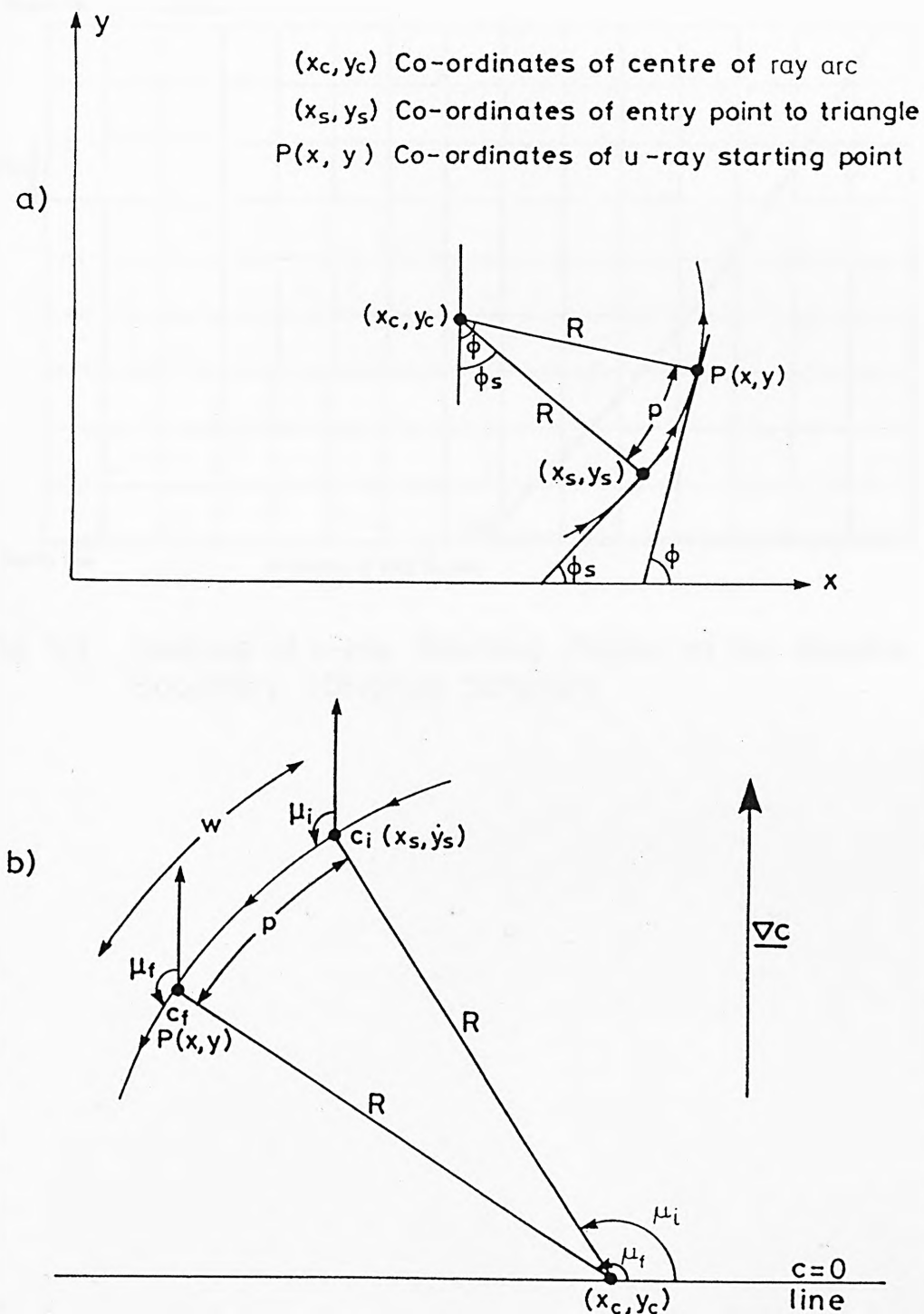


Fig 5.1 Determination of u-ray Starting Points on a Refracted Shadow Boundary

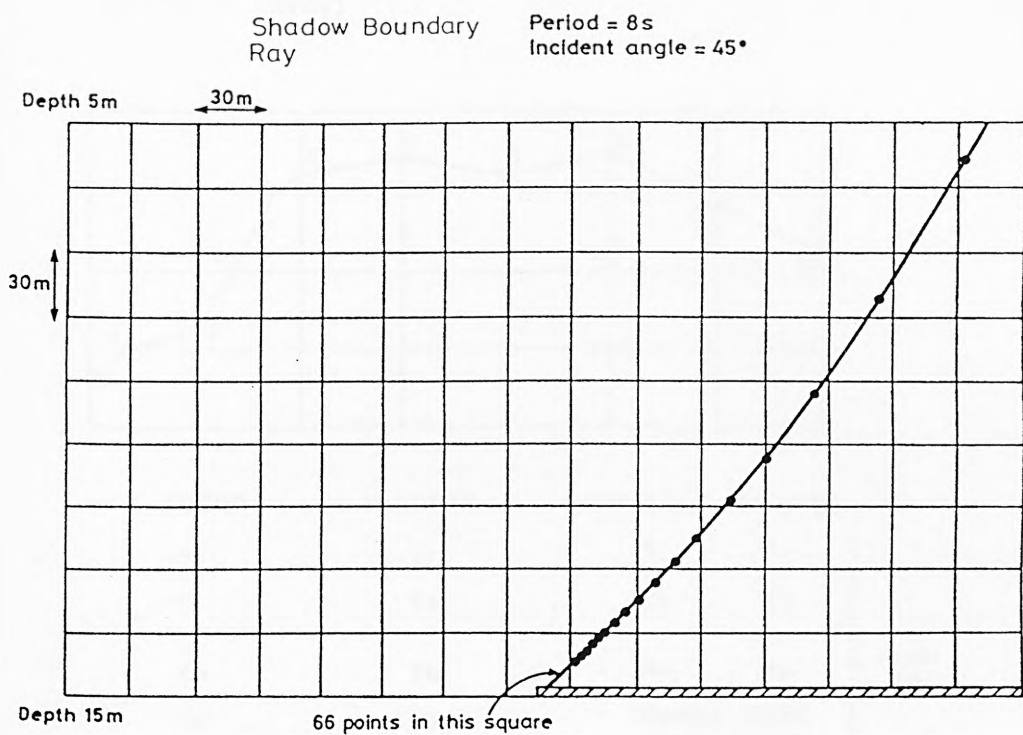


Fig 5.2 Spacing of u-ray Starting Points on the Shadow Boundary (Original Scheme)

Part of a Refraction Grid (with subdivision into triangles omitted for clarity)

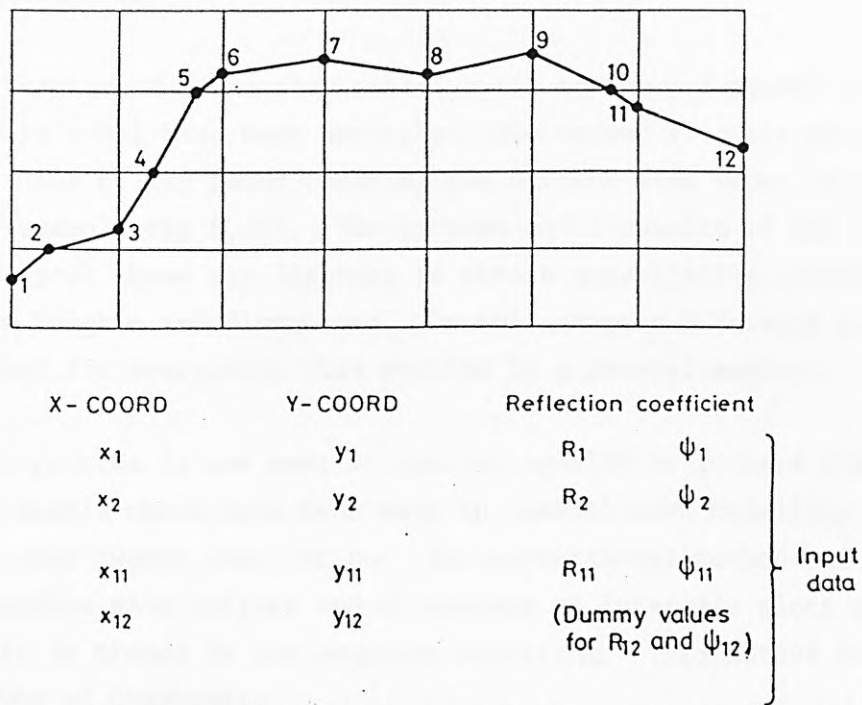


Fig 5-3 Representation of a Harbour Boundary by the Line-Segment Method

6.1 Introduction

In Chapters 2-5 the theoretical basis and computational procedure of the model have been described. The output from the model will be a set of ray paths covering the harbour area under study (see for example Fig 8.26). The problem still remains of how to interpret these ray diagrams to obtain quantitative information on wave heights and directions. In this chapter I develop a new method for overcoming this problem in a general manner.

This problem is now new; it applies equally to forward tracking ray models which have been used in coastal wave modelling during the past twenty years or so. The conventional method has been to determine wave heights and directions at intervals along each ray as it is traced in the computer algorithm. This method has a number of drawbacks:

1. There is no interpretation of predicted wave heights in areas where refraction theory breaks down, such as near caustics and dead areas.
2. Results are distributed more-or-less randomly over the study area. It is usually preferable to have results in a regular spatial array covering the study area.
3. It is not possible to account for the intersection of two or more wavetrains. In coastal problems this is not a major drawback since the wave field is usually dominated by the shoreward travelling wave. In harbour applications, however, intersecting waves can result from interference of incident and diffracted waves or from direct and reflected waves. Intersecting waves are therefore common occurrences.

The new method described in this chapter overcomes each of these problems. It is based on a spatial averaging of rays, in which the area under study is divided into a number of small subdomains and average values of the wave parameters are calculated in each of these subdomains. The subdomains are chosen to be the square

elements of a grid overlain on the studied area; the results will therefore be in the form of an array of regularly spaced values. The computational process is greatly simplified if this grid is chosen to coincide with the grid holding depth information, or if each element is composed of a simple multiple of depth grid elements. In the rest of this chapter it will be assumed that the depth grid and spatial averaging grid are identical, but it should be remembered that they are distinct conceptually.

It is possible to perform a spatial averaging of rays over a line instead of a square. The theories of the two methods are very similar, but the line-averaging method has the disadvantage of breaking down when rays make an acute angle with the line. Previous line-averaging methods (Berkhoff et al, 1982) have only considered the case where the relative phases of intersecting wave trains are random.

In Section 6.2 I develop the theory of the square-averaging method. The following two sections analyse the sensitivity of the predicted wave amplitude to the square size and ray density respectively. Upper limits on the square size per wavelength and lower limits on the ray density are obtained.

6.2 Theory of Ray Averaging

The new ray averaging technique is based on the spatial averaging of rays passing through a square area or across a line. The term 'with phasing' will be used to denote the case where the relative phases of separate wave trains crossing an averaging square or line are taken into account. Similarly, the term 'without phasing' will be used when separate wave trains are treated as having random relative phases. In both the square and line-averaging methods the idea of a 'region of influence' of a ray is used. If one imagines two lines drawn on either side of a ray midway between it and its neighbouring rays, the region of influence of the ray is the area between these two lines (see Fig 6.1).

In this section, the theory of the square-averaging method is developed. Square-averaging with phasing and without phasing are

considered separately. The arguments used in the development of the line- and square-averaging methods are identical. Therefore, formulae for the line-averaging method will simply be quoted where the corresponding formulae for square-averaging have been obtained.

6.2.1 Square-averaging with phasing

Consider several wave trains crossing an averaging square. The law of superposition of waves combined with the formula for weighted averages gives the following expression for the average wave amplitude and phase in the square.

$$A_{av} e^{iS_{av}} = \left(\frac{f_1 A_1 e^{iS_1} + f_2 A_2 e^{iS_2} + \dots}{f_1 + f_2 + \dots} \right)_1 + \left(\frac{f_1 A_1 e^{iS_1} + f_2 A_2 e^{iS_2} + \dots}{f_1 + f_2 + \dots} \right)_2 + \dots \quad 6.2.1$$

where

A_{av} is the average wave amplitude in the square.

S_{av} is the average wave phase in the square.

Brackets denote separate wave trains.

Terms in each bracket denote rays in a particular wave train.

A_1, A_2 etc. are the average wave amplitudes of rays calculated from Eq 2.2.7

S_1, S_2 etc. are the average wave phases of rays calculated from Eq 2.2.8.

f_1, f_2 etc. are the factors by which the wave amplitudes of each ray are weighted.

The basic assumption is now made that the wave amplitude of each ray is weighted in proportion to the region of influence of the ray in the averaging square. In other words, the f 's are chosen to be

$$f = \frac{qb}{d^2}$$

6.2.2

where

q is the length of the ray in the averaging square

b is the average separation of the ray from its neighbouring rays in the averaging square

d is the length of the side of the square

As shown later, this choice of weighting factor gives a desired smoothing behaviour near ray crossings and caustics, and allows a computationally efficient means of implementing the averaging technique.

qb represents the area of the region of influence of the ray in the averaging square (Fig 6.1). Substituting Eq 6.2.2 for the f's and Eq 2.2.7 for the A's in Eq 6.2.1

$$A_{av} e^{iS_{av}} = \frac{1}{d^2} \left(\frac{q_1 (b_1 E_1 / c_{g1})^{1/2} e^{iS_1} + q_2 (b_2 E_2 / c_{g2})^{1/2} e^{iS_2} + \dots}{q_1 b_1 / d^2 + q_2 b_2 / d^2 + \dots} \right)_1$$

6.2.3

$$+ \frac{1}{d^2} \left(\frac{q_1 (b_1 E_1 / c_{g1})^{1/2} e^{iS_1} + q_2 (b_2 E_2 / c_{g2})^{1/2} e^{iS_2} + \dots}{q_1 b_1 / d^2 + q_2 b_2 / d^2 + \dots} \right)_2 + \frac{1}{d^2} (\dots)_3 + \dots$$

For sets of rays which cover the whole averaging square the denominators in each set of brackets ($\sum qb/d^2$) are unity. Thus, each expression for the wave amplitude and phase contribution of a ray contains quantities which relate to that ray only, and not to the other rays in its wave train. The average wave amplitude and phase in an averaging square can therefore be written as

$$A_{av} e^{iS_{av}} = \frac{1}{d^2} \sum q \left(\frac{bE}{c_g} \right)^{1/2} e^{iS} \quad \text{for all rays} \quad 6.2.4$$

Precisely the same arguments are used in deriving the formula for the average wave amplitude and phase for sets of rays crossing an averaging line. The only difference is that the f's are given by (see Fig 6.1).

$$f = \frac{b}{d \sin \zeta} \quad 6.2.5$$

where

ζ is the angle between the ray and the averaging line.

d is the length of the averaging line.

This leads to the following formula for the average wave amplitude and phase:

$$A_{av} e^{iS_{av}} = \frac{1}{d} \sum \frac{1}{\sin \zeta} \left(\frac{bE}{c_g} \right)^{1/2} e^{iS} \quad \text{for all rays} \quad 6.2.6$$

It can be seen that this method will break down when rays make a very acute angle (small ζ) with the averaging line.

Each quantity on the right-hand side of Eq 6.2.4 (and Eq 6.2.6) can be readily determined in a ray plotting algorithm. An average group velocity, c_g , can be found for each averaging square from the expression for linear small-amplitude waves,

$$c_g = \frac{c}{2} \left(1 + \frac{2kh}{\sinh(2kh)} \right) \quad 6.2.7$$

The energy flux constant, E , is specified in the initial conditions of each ray. The length of the ray, q , and the phase, S , follow from the determination of the ray path. The ray separation, b , can be found by evaluating an ordinary differential equation along each ray (Gilbert and Abernethy (1975), Munk and Arthur (1952)).

Average values of b and S between the ray's entry and exit points in the averaging square are used in Eq 6.2.4. As discussed in Section 6.3, this method of determining S does place an upper limit on the square size per wavelength.

It is of considerable importance computationally that the expression for the amplitude and phase contributions of a ray should be independent of other rays in its wave train. If this is the case only two result arrays (containing the real and imaginary parts of the wave field in each averaging square) are needed. If the expression for a ray's contribution did depend on terms

involving other rays in its wave train, separate result arrays would be needed for each wave train in the problem. As well as complicating the computing algorithm, this could lead to a high demand on computing storage particularly in harbour problems where, in addition to the incident wave, there may be several diffracted and reflected wave trains.

Since A_{av} is inversely proportional to the area of the averaging square (d^2) in Eq 6.2.4, it follows that the wave field will be predicted correctly by Eq 6.2.4 when an averaging square is only partially covered by a set of rays. In other words if a set of rays covers a fraction F of an averaging square, the predicted amplitude will be F times the value obtained if the whole square had been covered. This property is essential in harbour problems where diffraction of waves at the entrance is by a single, semi-infinite, breakwater. In these types of problems, squares intersected by the shadow boundary will be partially covered by the set of incident rays and by the two sets of u-rays.

6.2.2 Caustics and ray crossings

An interesting feature of the square- and line-averaging methods is the position of the ray separation factor, b , in Eq 6.2.4 and Eq 6.2.6. It appears in the numerator and not, as might be expected from Eq 2.2.7, in the denominator. Therefore, near caustics and ray crossings as $b \rightarrow 0$ the unrealistically high wave amplitudes of single-ray methods are not obtained. The explanation is easy to see if the amplitude contribution of a ray to an averaging square is written as

$$A = \frac{qb}{d^2} \left(\frac{E}{bc_g} \right)^{1/2} \quad 6.2.8$$

As $b \rightarrow 0$, the condition of conservation of energy flux (second term on the right) causes the amplitude to tend to infinity as $b^{-1/2}$. However, the region of influence of the ray (first term on the

right) tends to zero as b . This latter effect is therefore the stronger.

This smoothing of caustics and ray crossings is similar in its effect to the actual diffraction process that always occurs around such areas, despite there being no attempt to model the mechanism of the diffraction process. Good, but not highly accurate, amplitude results can be expected where caustic features are contained within averaging squares. Larger scale areas of 'internal' diffraction cannot be expected to be modelled correctly, though even in these cases qualitative accuracy is often obtained (see Chapter 8 and Berkhoff et al (1982)). Errors caused by caustics and ray crossings are often further reduced by carrying out a series of runs covering a spectrum of incident wave periods and directions. The accuracy of the method is examined quantitatively in Sections 6.3 and 6.4.

6.2.3 Square averaging without phasing

For several intersecting wave trains with random relative phases the law of superposition in an averaging square is

$$A_{av}^2 = \left(\frac{f_1 A_1 + f_2 A_2 + \dots}{f_1 + f_2 + \dots} \right)_1^2 + \left(\frac{f_1 A_1 + f_2 A_2 + \dots}{f_1 + f_2 + \dots} \right)_2^2 + (\dots)_3^2 + \dots \quad 6.2.9$$

The same basic assumption as before is made that the wave amplitude contribution of a ray is weighted in proportion to the region of influence of the ray in the averaging square.

Substituting Eq 6.2.2 for the f 's in Eq 6.2.9:

$$A_{av}^2 = \frac{1}{d^4} \left(\frac{q_1 b_1 A_1 + q_2 b_2 A_2 + \dots}{q_1 b_1 / d^2 + q_2 b_2 / d^2 + \dots} \right)_1^2 + \frac{1}{d^4} \left(\frac{q_1 b_1 A_1 + q_2 b_2 A_2 + \dots}{q_1 b_1 / d^2 + q_2 b_2 / d^2 + \dots} \right)_2^2 + \frac{1}{d^4} (\dots)_3^2 + \dots \quad 6.2.10$$

Because each bracket is squared, the wave amplitude contribution of each ray is dependent on terms involving other rays in its wave train. This is in contrast to Eq 6.2.3 where phasing was included. As discussed in Section 6.2.1, it is desirable computationally that a ray's contribution should be independent of other rays in its wave train. A formula with this computational requirement can be derived if it is assumed that the variation in amplitude between rays in each wave train in an averaging square is negligible. In other words, it is assumed that for each wave train in an averaging square:

$$A_1 \approx A_2 \approx A_3 \approx \dots \quad 6.2.11$$

The condition that rays in each wave train cover the whole of the averaging squares is also needed.

$$\sum \frac{q_b}{d^2} = 1 \text{ for each wave train} \quad 6.2.12$$

Using Eq 6.2.11 and Eq 6.2.12, Eq 6.2.10 can be written as

$$A_{av}^2 = \frac{1}{d^2} (q_1 b_1 A_1^2 + q_2 b_2 A_2^2 + \dots) + \frac{1}{d^2} (q_1 b_1 A_1^2 + q_2 b_2 A_2^2 + \dots) + \frac{1}{d^2} (\dots) + \dots \quad 6.2.13$$

In this formula, the contribution of each ray is now independent of other rays in its wave train. Substituting Eq 2.2.7 for the A's:

$$A_{av} = \frac{1}{d} \left(\sum \frac{qE}{c_g} \right)^{1/2} \text{ for all rays} \quad 6.2.14$$

The corresponding formula for sets of rays crossing an averaging line is derived in an identical way:

$$A_{av} = \frac{1}{\sqrt{d}} \left(\sum \frac{E}{c_g \sin \zeta} \right)^{1/2} \quad \text{for all rays} \quad 6.2.15$$

Since A_{av} is not inversely proportional to the area of the averaging square in Eq 6.2.14 (or to the length of the averaging line in Eq 6.2.15) it follows that the wave field will not be predicted correctly when an averaging square is only partially covered by a set of rays. Two situations where partial covering of squares could occur are outlined below. In neither case, however, will the incorrect prediction of wave amplitude be important.

- 1 In problems involving diffraction by a single breakwater, the averaging squares intersected by the shadow boundary are partially covered by three sets of rays (see Fig 4.2). However, since these sets of rays are phase-locked, it is essential that square averaging with phasing should be used.
- 2 Irregularities or sharp boundaries in an obstacle will produce reflected waves with truncated crests in a ray method. The sets of rays at the end of these crests will only partially cover the squares through which they pass. The errors that will be introduced by this will, however, be masked by the more important (and unavoidable) error that diffraction which is always present at truncated wave crests will not be modelled.

In most practical coastal and harbour problems it is envisaged that square averaging without phasing is used for the interference of reflected waves with direct waves. It is usually possible to obtain a reasonable estimate of the amplitude reflection coefficient of a boundary, either by model tests or the use of semi-empirical formulae (for example, Miche (1951), Madsen and White (1976)). However, the phase change undergone on reflection is often unknown except for the simplest types of boundary (such as a straight vertical wall). The inclusion of the relative

phases of direct and reflected waves gives an interference pattern with large spatial variations in the total wave amplitude. Obviously, the phase change undergone on reflection would need to be known with considerable accuracy for such predicted wave amplitudes to be realistic. Square-averaging without phasing, on the other hand, would tend to give a smooth spatial variation of the total wave amplitude.

6.3 Sensitivity of Square Averaging to Size of Squares

In the square-averaging formula with phasing, Eq 6.2.4, the average of the phases of each ray at the entry and exit points in an averaging square is used. Fig 6.2 shows the simple case of one set of parallel rays with no refraction crossing an averaging square. It can be seen that the mean phase of each ray is different from the correct value at the centre of the square. The larger the square size relative to the wavelength, the greater will be the error introduced by taking the mean phase of each ray. It is therefore important to determine the maximum limit of square size per wavelength for use in practical problems. Clearly, this consideration does not apply to square-averaging without phasing.

This square size sensitivity analysis is carried out for three cases:

- 1 A single set of parallel rays with no refraction (Fig 6.2).
- 2 Several intersecting sets of rays (parallel and diverging) and no refraction (Fig 4.2).
- 3 As (1), but with refraction present.

It was thought that an improvement could be made to the calculation of the mean phase by extrapolating a ray's path on to the circumcircle of each averaging square (Fig 6.9). An average of the phases at the intersections of the ray with the circumcircle would be used. This idea proved to be very successful and to increase considerably the maximum limit on the

square size per wavelength. The method, and the results obtained from it, are described in Section 6.3.4.

6.3.1 One set of parallel rays. No refraction

It is possible to obtain a formula for the error in wave amplitude for a single set of parallel rays at infinite ray density. This formula is in terms of the length per wavelength of the sides of the squares (d/λ) and the angle between the rays and the sides of the squares (χ). Fig 6.2 shows one set of rays crossing an averaging square at an arbitrary angle χ . Because of symmetry, values of χ between 0° and 45° only need be considered,

For N rays crossing an averaging square, the average wave amplitude and phase in the square are

$$A_{av} e^{iS_{av}} = \left(A e^{ik[(p_m + (p_1 - p_m)) \frac{q_1 b}{d^2}] + A e^{ik[(p_m - (p_1 - p_m)) \frac{q_1 b}{d^2}]} \right. \\ \left. + (A e^{ik[p_m + (p_2 - p_m)] \frac{q_2 b}{d^2}] + A e^{ik[p_m - (p_2 - p_m)] \frac{q_2 b}{d^2}]} + \dots \text{to } N/2 \text{ terms} \right) \quad 6.3.1$$

In this equation:

- 1 Each wave amplitude A has been weighted in proportion to the ray's region of influence in the averaging square (factor qb/d^2).
- 2 Corresponding rays either side of the centre ray have been paired together.
- 3 p_m and p_1, p_2 , etc. (denoted by p_i) are defined in Fig 6.2.

Evaluating the pair of terms in each bracket and taking the limit as $N \rightarrow \infty$:

$$A_{av} e^{iS_{av}} = \lim_{N \rightarrow \infty} A e^{ikp_m} \frac{2b}{d^2} \sum_{i=1}^{N/2} q_i \cos k(p_i - p_m) \quad 6.3.2$$

Eq 6.3.2 shows that the averaging method predicts the phase correctly (this is obvious from the symmetry about the centre ray) but the amplitude A is altered from its correct value by the factor.

$$\begin{aligned} \frac{A_{av}}{A} &= \lim_{N \rightarrow \infty} \frac{2b}{d^2} \sum_{i=1}^{N/2} q_i \cos k(p_i - p_m) \\ &= \frac{2}{d^2} \int_{\text{extreme ray}}^{\text{centre ray}} q \cos k(p - p_m) db \end{aligned} \quad 6.3.3$$

This integral can be evaluated in terms of the size per wavelength of the averaging square (d/λ) and the ray angle (χ) . The result is:

$$\begin{aligned} \frac{A_{av}}{A} &= \frac{2}{\pi(d/\lambda)^2} \left(\frac{(d/\lambda) \sin(2\theta \sin \chi)}{2 \sin \chi (\cos^2 \chi - \sin^2 \chi)} \right. \\ &\quad \left. + \frac{2 \sin \chi \cos \chi \sin[\theta(\cos \chi + \sin \chi)] \sin[\theta(\cos \chi - \sin \chi)]}{\pi(\cos^2 \chi - \sin^2 \chi)^2} \right) \end{aligned} \quad 6.3.4$$

where

$$\theta = \frac{\pi(d/\lambda)(\cos \chi - \sin \chi)}{2 \cos \chi} \quad 6.3.5$$

Using Eq 6.3.4, A_{av}/A has been calculated for values of d/λ from 0.25 to 2.5 at intervals of 0.25 and for values of χ from 0° to 45° at 1° intervals. For each value of d/λ , A_{av}/A shows a single minimum with respect to χ , increasing to 1 at $\alpha=0^\circ$ and 45° . Table 6.1 lists the minimum A_{av}/A (and the corresponding percentage error) for each value of d/λ and the χ at which it occurred. It can be seen that as d/λ is increased there is a systematic decrease in A_{av}/A from the correct value of 1. This error trend is to be expected because, as errors in determining the mean phase

of each ray increase, the effect is that rays in the same wave train will combine in an increasingly random phase manner.

These results indicate that to keep amplitude errors within 5 per cent, the square size per wavelength should not be greater than about 0.75.

6.3.2 Several sets of rays. No refraction

In the first part of this section, the error in the total wave amplitude for several intersecting plane wave trains in an averaging square is considered theoretically. Results from the computational ray model run at different square sizes per wavelength are then presented.

Consider n sets of rays crossing an averaging square in which the correct wave amplitude of each wave train is A_i and the calculated amplitude has an error δA_i . It has already been shown that there is no error in the phase S_i for sets of straight parallel rays. From linear superposition, the total amplitude error, δA , can be determined as

$$\delta A = \left(\sum_{i=1}^n \sum_{j=1}^n \delta A_i \delta A_j \cos(S_i - S_j) \right)^{1/2} \quad 6.3.6$$

This can be expressed as a fraction of the correct total wave amplitude, A .

$$\frac{\delta A}{A} = \left(\frac{\sum_{i=1}^n \sum_{j=1}^n \frac{\delta A_i}{A_i} \frac{\delta A_j}{A_j} (A_i A_j \cos(S_i - S_j))}{\sum_{i=1}^n \sum_{j=1}^n A_i A_j \cos(S_i - S_j)} \right)^{1/2} \quad 6.3.7$$

If the waves destructively interfere to give a total wave amplitude close to zero, it becomes meaningless to give δA as a fraction of A . For small A , it is more meaningful to express δA

as a fraction of the total random-phase amplitude $A_{rp} = \sqrt{\sum_{i=1}^n A_i^2}$,

provided no two of the waves are phase-locked. This fractional error can be written as

$$\frac{\delta A}{A_{rp}} = \left(\frac{\sum_{i=1}^n \delta A_i^2}{\sum_{i=1}^n A_i^2} + \frac{\sum_{i=1}^n \sum_{j=1, j \neq i}^n \frac{\delta A_i}{A_i} \frac{\delta A_j}{A_j} (A_i A_j \cos(S_i - S_j))}{\sum_{i=1}^n A_i^2} \right)^{1/2} \quad 6.3.8$$

Eq 6.3.7 is used for $A > A_{rp}$ and Eq 6.3.8 for $A < A_{rp}$. The condition $A < A_{rp}$ can be expressed alternatively as

$$\sum_{i=1}^n \sum_{j=1, j \neq i}^n A_i A_j \cos(S_i - S_j) < 0 \quad 6.3.9$$

Some qualitative conclusions can be drawn from Eq 6.3.7 and Eq 6.3.8 about the total fractional amplitude error in comparison with the worst fractional amplitude error of the individual wave trains $(\delta A_i / A_i)_{\max}$.

- 1 Eq 6.3.7 has the form of a weighted average of $(\delta A_i / A_i)(\delta A_j / A_j)$. If the 'weights' are all positive, $\delta A / A$ is clearly no worse than $(\delta A_i / A_i)_{\max}$. Although Eq 6.3.7 is used when the wave trains tend to be in phase, it is possible that some of the 'weights' may be negative and combine in such a way to give a fractional amplitude error greater than $(\delta A_i / A_i)_{\max}$.
- 2 Eq 6.3.8 is used for the case where the waves tend to be out of phase and the inequality 6.3.9 holds. Usually, therefore, the second term on the right of Eq 6.3.8 will be negative. If this is so, then $\delta A / A_{rp}$ is less than $(\delta A_i / A_i)_{\max}$. However, there is a small possibility that the second term on the right of Eq 6.3.8 could be positive and cause $\delta A / A_{rp}$ to be greater than $(\delta A_i / A_i)_{\max}$.

The problem of diffraction around a semi-infinite breakwater on a constant-depth sea bed was chosen for the computational ray model investigation. The semi-infinite breakwater problem is well suited for this purpose because it involves the interference of several wave trains (Fig 4.2) and because it is possible to obtain at any point the exact total wave amplitude predicted by the ray method.

A grid of 15 by 9 squares was used with the breakwater lying along one of the long boundaries with the tip at the centre of the boundary (Fig 6.3). Separate runs were performed with square sizes per wavelength (d/λ) of 0.25 to 2.5 at 0.25 intervals. Different d/λ were obtained by altering the wavelength. A ray density of about 20 rays per grid side ensured that inadequate density of rays did not contribute significantly to the total error. The exact total wave amplitudes given by the ray method were calculated at the centres of the averaging squares in order that a comparison with the model results could be made.

Results are presented at four averaging squares, each in a different wave regime. Squares 14,2 and 7,8 are well into the sheltered and open zones respectively, and squares 11,5 and 11,7 are close to the shadow boundary on either side of it (Fig 6.3). Figures 6.4 to 6.7 show the exact ray method amplitudes and the model amplitudes at each d/λ in each of the four squares. Tables 6.2 to 6.5 show the model fractional amplitude error, $\delta A/A$ (expressed as a percentage) and the theoretical fractional error, Eq 6.3.4, for the wave train that crosses at the worst angle. Also shown in these Figures and Tables are results using the circumcircle phase correction discussed in Section 6.3.4.

The following observations are made on the results:

- 1 Square 14,2 (Fig 6.4 and Table 6.2). In this square there are two sets of diverging rays (representing the diffracted incident and diffracted reflected wave trains) almost in phase and travelling very nearly in the same direction. It would be expected from Eq 6.3.7 that $\delta A/A$ would be very close to the individual fractional amplitude errors. This is seen to be the case. The fact that the rays are diverging rather than parallel seems to have little effect on $\delta A/A$.
- 2 Square 7,8 (Fig 6.5 and Table 6.3). The incident wave has a far larger amplitude than the diffracted waves and also has the largest fractional amplitude error. It would be expected that the total fractional amplitude error would be very

similar to the fractional amplitude error of the incident wave alone. Table 6.3 shows this to be so.

- 3 Square 1,5 (Fig 6.6 and Table 6.4). This square lies in the sheltered zone close to the shadow boundary. In this region there are two sets of rays (u-rays and v-rays) nearly in phase but travelling in significantly different directions (between about 4° and 13° depending on the wavelength). The worst fractional amplitude error occurs for the v-wave for all the wavelengths considered. Because of the similar phases but different $\delta A_i/A_i$ of the two waves, Eq 6.3.7 predicts that the total fractional amplitude error, $\delta A/A$, will be somewhat less than the worst $\delta A_i/A_i$. The model results in Table 6.4 agree with this prediction.
- 4 Square 11,7 (Fig 6.7 and Table 6.5). There are three wave trains crossing this square: the incident wave, the u-wave (out of phase with the incident wave by close to π) and the v-wave (nearly in phase with the incident wave). The incident and u-waves are phase-locked and therefore Eq 6.3.7 should be used in evaluating $\delta A/A$. When one wave train is out of phase with the others, it is not immediately clear from Eq 6.3.7 whether $\delta A/A$ should be greater or less than the worst $\delta A_i/A_i$ (which, in this case, is for the v-wave). Table 6.5 shows that $\delta A/A$ is somewhat less.

These results are in agreement with the theoretical conclusion that the total fractional amplitude error for several intersecting wave trains will only rarely be greater than the worst fractional amplitude error among the individual wave trains.

6.3.3 One set of parallel rays. Refraction present

The semi-infinite breakwater problem cannot be used for testing the grid size sensitivity on a varying depth sea bed (i.e. with refraction) since it is not possible to obtain exact ray method values at the centres of the averaging squares. Instead, the simpler problem of a single plane wave train propagating over a parallel-contoured sea bed is considered. A 15 by 9 grid of

square elements (identical to the grid in the previous section) is used with a linear variation of depth from 15m at the offshore boundary to 5m at the inshore boundary. In this problem, exact results at the centres of the averaging squares can be obtained from Snell's law of refraction and Eq 6.2.7 and Eq 2.2.7.

The same values of grid size per wavelength (d/λ) as those in the previous section were tested ($d/\lambda=0.25$ to 2.5 at 0.25 intervals) but with an incident wave angle of 45° . Since it was necessary that rays should undergo the same amount of refraction in each run with different d/λ , the variation of wavelength over the whole grid had to be the same for each of these runs. This meant that d/λ could only be changed by altering the square size. The incident wave period was chosen so that the wavelength which gave the desired value of d/λ occurred at a depth of 10m (which is at the centre of the fifth row of squares). A ray density of 20 rays per square side was used.

Fig 6.8 shows the model wave amplitudes (averaged over all squares in the fifth row) for each d/λ . The percentage error relative to the amplitude determined from Snell's law is shown in Table 6.6. Also shown in this table is the theoretical percentage error determined from Eq 6.3.4 assuming a flat sea bed and a ray angle equal to that at the 10m depth contour in the model. Table 6.8 shows that the model errors are only slightly greater than the theoretical errors. These small additional errors, resulting from the presence of curved rays, can be accounted for in two ways.

- 1 The locus of the mid-points of a set of curved rays crossing a square will be different from the locus of the mid-points of a set of straight rays.
- 2 The average of the phases of a curved ray at the entry and exit points in a square will be different from the phase at the mid-point.

6.3.4 Circumcircle phase correction

It has been shown that, by taking the mean phase between the entry and exit points of rays crossing an averaging square, the maximum allowable square size is about 0.75 wavelength. This limit gives errors in wave amplitude of around 5 per cent for the worst ray angles. Although this limit is considerably larger than those in finite element or finite difference models (maximum element size about 0.2 wavelength) it will still lead to high data preparation time, computing storage and run time for short waves in large sea areas. The sensitivity analysis has shown that by far the largest contribution to this error has arisen from the simple geometrical consideration that the entry and exit points in a square were used to determine the mean phase of each ray. The influences of intersecting sets of rays and refraction have been shown to be much smaller.

The idea of the circumcircle phase correction is to extrapolate a ray's path onto the circumcircle of the averaging square. The intersections of the ray with the circumcircle are then used to determine the mean phase. It is assumed that the extrapolated ray paths between the entry/exit points on the square and the entry/exit points on the circumcircle are straight. The determination of the intersections of the ray with the circumcircle then reduces to the simple problem of finding the intersections of two known straight lines with a known circle. This process does, of course, add somewhat to the run time of plotting each ray, but this time penalty will be greatly outweighed by the potential savings in computing time using larger grid elements.

The circumcircle method removes completely the error arising from taking the mean phase between the entry and exit points on a square. Consider a single set of straight, parallel rays crossing an averaging square with their paths extrapolated to the circumcircle (Fig 6.9). It is obvious that the mean phase of each ray determined by the circumcircle method is exactly the value at the centre of the averaging square, whatever the ray angle and square size. An analysis for infinite ray density, corresponding

to that in Section 6.3.1, therefore shows zero error in amplitude for all ray angles and square sizes.

It remains to see how this idea works in practice. The tests in Section 6.3.2 (semi-infinite breakwater on a horizontal sea bed) and Section 6.3.3 (single wave train on parallel contoured sea bed) were repeated using the circumcircle method. Figs 6.4 to 6.8 and Tables 6.2 to 6.6 show the results. A big improvement is obtained over the corresponding results without the circumcircle phase correction. At the largest square size per wavelength ($d/\lambda=2.5$), the worst amplitude error amongst the squares considered is about 4 per cent with the circumcircle method. The other squares show much smaller errors. Without the circumcircle phase correction, the averaging method breaks down completely giving errors of over 40 per cent in the worst cases. Depending on the required accuracy in any problem, even larger square sizes than 2.5 wavelengths could be used. It was found that the extra computation required by the circumcircle method led to an increase in run time of 17 per cent but this, of course, will be greatly offset in practical problems by the savings in using larger square sizes.

6.4 Sensitivity of Square Averaging to Ray Density

The accuracy of the square averaging method clearly depends on the number of rays in each wave train crossing a square. This applies to both square averaging with phasing and square averaging without phasing. A particularly large amplitude error resulting from insufficient ray density occurs in the simple example shown in Fig 6.10. A single set of straight parallel rays crosses an averaging square parallel to two sides of the square. The maximum error occurs when two rays just graze the two sides of the square, as shown in Fig 6.10. The amplitude error for this case can be calculated from Eq 6.2.4 (with phasing) and Eq 6.2.14 (without phasing) for a general ray density of n rays per square side. These amplitude errors, expressed as a fraction, are:

$$\begin{array}{ll} \frac{1}{n} & \text{with phasing} \\ 1 - (1 - \frac{1}{n})^{\frac{1}{2}} & \text{without phasing} \end{array}$$

This shows that larger errors are given by the method of averaging with phasing.

This example, of course, is an extreme case and smaller errors can be expected in practice. The semi-infinite breakwater problem again provides a good test of the sensitivity to ray density when there are several intersecting wave trains. The same model set-up as before (Section 6.3.2) was used, with an incident wave angle of 60° (Fig 6.3) and square size per wavelength of 0.25. Separate runs were performed for ray densities (expressed as the number of rays, n , per grid side) of 1 to 6 at intervals of 1, and 6 to 12 at intervals of 2. The angular separation of the diverging diffracted rays was chosen so that their density was roughly equal to the density of the parallel incident rays at a distance of one and a half wavelengths from the breakwater tip. Results in averaging squares at about this distance from the breakwater tip were used for comparison with the exact ray method values.

Table 6.7 shows the percentage amplitude errors in ten of these averaging squares (Fig 6.3) for each of the ray densities. Two general trends can be seen from the table:

- 1 For a particular ray density, there is considerable variation in amplitude error from square to square. This becomes more marked at the lower ray densities.
- 2 For a particular averaging square, the amplitude error tends to increase unsystematically as the ray density is decreased.

These trends are to be expected. As the ray density is decreased, it becomes a question of chance as to whether rays cross a square in such a way as to produce a small or large amplitude error. It would appear that a ray density as low as four or five rays per square side does not generally give errors greater than 5 per cent.

$\frac{d}{\lambda}$	Minimum A_{av}/A	% Error of Min. A_{av}/A	χ at Minimum A_{av}/A (degrees)
0.25	0.992	-0.8	19
0.5	0.970	-3.0	19
0.75	0.934	-6.6	19
1.0	0.887	-11.3	20
1.25	0.829	-17.1	20
1.5	0.765	-23.5	21
1.75	0.694	-30.6	21
2.0	0.620	-38.0	22
2.25	0.544	-45.6	22
2.5	0.468	-53.2	23

TABLE 6.1

Theoretical square-averaging amplitudes for a single parallel set of rays. Shows the minimum value of A_{av}/A (Eq 6.3.4) and the corresponding percentage error and ray angle (χ) for different square sizes per wavelength (d/λ).

$\frac{d}{\lambda}$	% Error Theoretical Worst wave train	% Error Square Averaging	% Error square averaging with phase correction
0.25	-0.7	-0.4	0.4
0.5	-2.8	-2.7	0
0.75	-6.1	-5.8	0
1.0	-10.4	-9.7	0
1.25	-15.3	-15.0	0
1.5	-20.6	-20.0	0
1.75	-26.0	-26.5	0
2.0	-31.4	-31.6	0
2.25	-36.6	-36.7	0
2.5	-41.7	-42.4	0

TABLE 6.2

Figure 14.2. Semi-infinite breakwater, flat sea bed. Percentage error of square-averaging amplitude relative to exact ray method amplitude at different square sizes per wavelength (d/λ). Worst ray angle ($\chi = 14^\circ$) occurs for both the u-wave and the v-wave.

$\frac{d}{\lambda}$	% Error Theoretical Worst wave train	% Error Square Averaging	% Error square averaging with phase correction
0.25	-0.5	-0.5	0.1
0.5	-2.1	-2.0	0.1
0.75	-4.7	-4.8	-0.1
1.0	-8.2	-7.9	0.1
1.25	-12.7	-12.9	0
1.5	-17.9	-17.8	-0.1
1.75	-23.8	-23.5	0.2
2.0	-30.3	-30.5	-0.3
2.25	-37.1	-36.5	0.1
2.5	-44.4	-44.9	0.1

TABLE 6.3

Sqaure 7, 8. Semi-infinite breakwater, flat sea bed. Percentage error of square-averaging amplitude relative to exact ray method amplitude at different square sizes per wavelength (d/λ). Worst ray angle ($\chi = 30^\circ$) occurs for the incident wave.

$\frac{d}{\lambda}$	% Error Theoretical Worst wave train	% Error Square Averaging	% Error square averaging with phase correction
0.25	-0.4	-0.2	0
0.5	-1.5	-1.0	0.2
0.75	-3.3	-2.1	0.4
1.0	-5.8	-4.0	0.9
1.25	-8.9	-6.4	1.6
1.5	-12.7	-9.1	1.9
1.75	-17.0	-12.7	2.4
2.0	-21.8	-17.2	2.9
2.25	-27.1	-21.6	3.5
2.5	-32.7	-26.6	4.1

TABLE 6.4

Square 11,5. Semi-infinite breakwater, flat sea bed. Percentage error of square-averaging amplitude relative to exact ray method amplitude at different square sizes per wavelength (d/λ). Worst ray angle ($\chi = 34^\circ$) occurs for the v-wave.

$\frac{d}{\lambda}$	% Error Theoretical Worst wave train	% Error Square Averaging	% Error square averaging with phase correction
0.25	-0.7	0.2	0.5
0.5	-2.7	-1.5	0.2
0.75	-6.0	-4.0	-0.4
1.0	-10.4	-7.5	-0.4
1.25	-15.9	-11.5	-0.8
1.5	-22.3	-16.4	-1.0
1.75	-29.2	-22.2	-1.2
2.0	-36.7	-28.5	-1.4
2.25	-44.5	-35.1	-1.6
2.5	-52.3	-42.5	-1.8

TABLE 6.5

Square 11,5. Semi-infinite breakwater, flat sea bed. Percentage error of square-averaging amplitude relative to exact ray method amplitude at different square sizes per wavelength (d/λ). Worst ray angle ($\chi = 25^\circ$) occurs for the v-wave.

$\frac{d}{\lambda}$	% Error Theoretical Worst wave train	% Error Square Averaging	% Error square averaging with phase correction
0.25	-0.3	-1.0	-0.7
0.5	-1.1	-1.9	-0.7
0.75	-2.5	-3.5	-0.7
1.0	-4.4	-5.2	-0.7
1.25	-6.8	-7.8	-0.8
1.5	-9.7	-10.8	-0.8
1.75	-13.1	-14.1	-0.8
2.0	-16.9	-18.2	-0.9
2.25	-21.1	-22.4	-0.9
2.5	-25.6	-27.0	-0.9

TABLE 6.6

Average of squares on fifth row. Single wave train, parallel contoured sea bed. Percentage error of square-averaging amplitude relative to exact ray method amplitude at different square sizes per wavelength (d/λ). Ray angle for flat bed theoretical errors is the ray angle at the 10m depth contour in the model ($\chi = 36^\circ$).

PERCENTAGE AMPLITUDE ERROR

Square										
Co-ordinates	Ray	Ray	Ray	Ray	Ray	Ray	Ray	Ray	Ray	Ray
Density	Density	Density	Density	Density	Density	Density	Density	Density	Density	Density
= 12	= 10	= 8	= 6	= 5	= 4	= 3	= 2	= 1		
14 2	0	0	-0.4	-1.1	-1.1	-1.9	-6.9	-1.9	12.6	
13 4	0.3	0	0	-1.2	-0.6	-0.6	3.0	-2.1	30.0	
12 5	0.2	0.2	-0.5	-1.4	-1.1	-1.6	4.1	-1.1	65.9	
11 5	0.2	0	-0.9	-1.3	-3.7	-6.1	-10.7	-17.0	-69.1	
10 6	-0.4	-0.7	-1.3	-1.6	-2.4	1.1	3.6	-18.8	-42.2	
9 6	-0.7	-1.3	0	-0.2	2.6	-3.8	-4.9	9.6	-27.6	
8 6	-0.5	-0.5	-0.5	-0.6	-0.1	-0.7	-0.5	-0.4	-1.2	
6 6	-0.6	-0.5	-0.6	-0.3	0.8	-0.4	-5.4	0.8	9.6	
4 5	-0.7	-0.7	-0.7	-0.7	0.7	-0.7	1.9	-0.9	8.7	
3 3	-0.5	-0.5	-0.5	-0.6	-1.8	-0.5	-0.7	-0.4	15.0	

TABLE 6.7

Semi-infinite breakwater, flat sea bed. Percentage error of square-averaging amplitude relative to exact ray method amplitude at different ray densities.

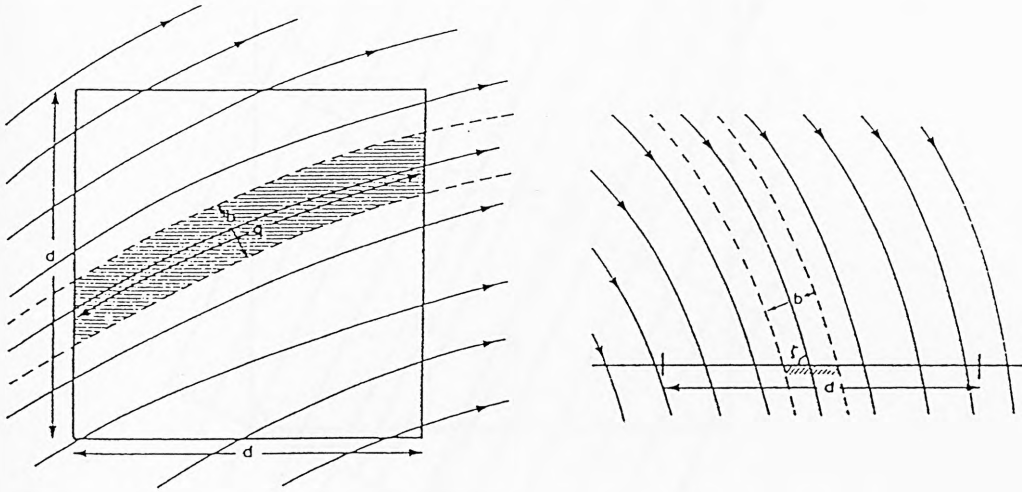


Fig 6.1 Square-Averaging (left) and Line-Averaging. The shaded area between the dashed lines is the region of influence of one ray

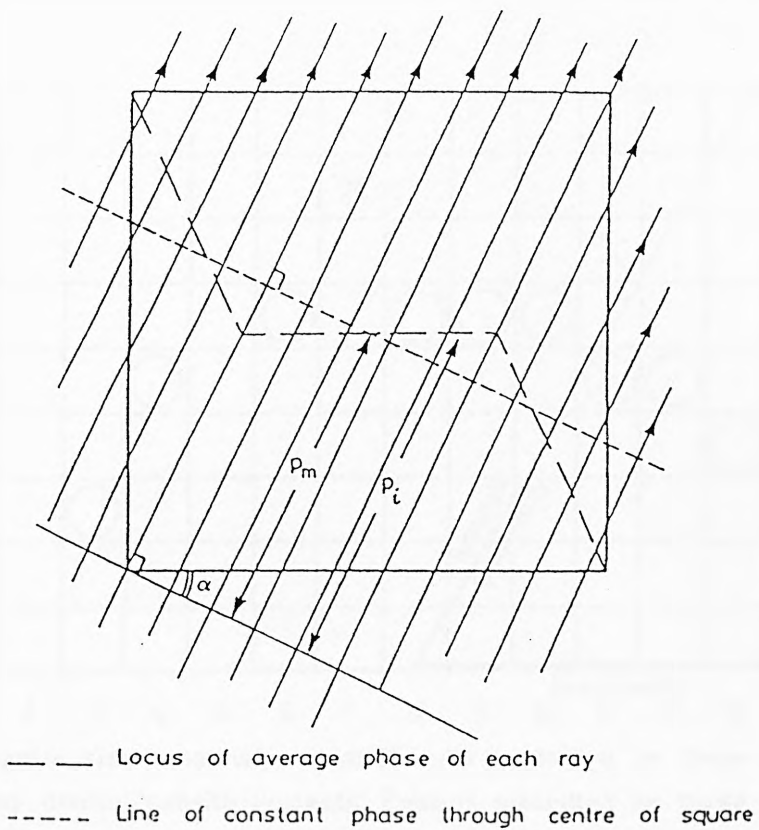
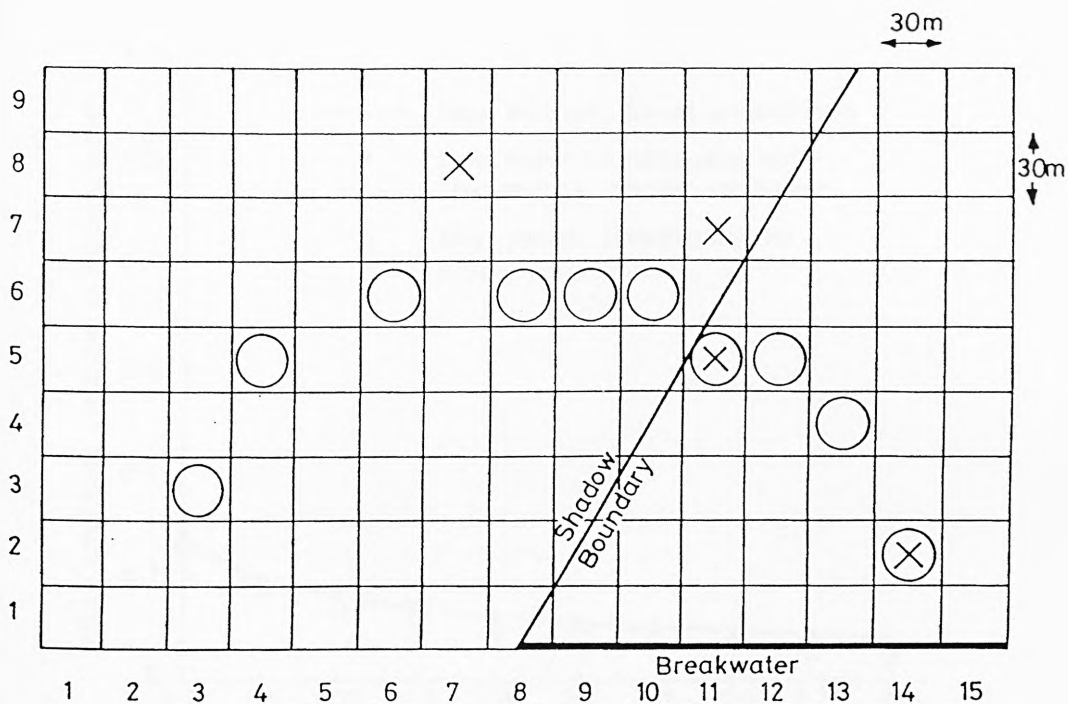


Fig 6·2 Average Phases of a Set of Parallel Rays crossing an Averaging Square



x Square size sensitivity tests. Results presented for these squares
 ○ Ray density sensitivity tests. Results presented for these squares

Fig 6.3 Ray Model Grid for Tests with a semi-Infinite Breakwater on a Flat Seabed. Incident Angle = 60°

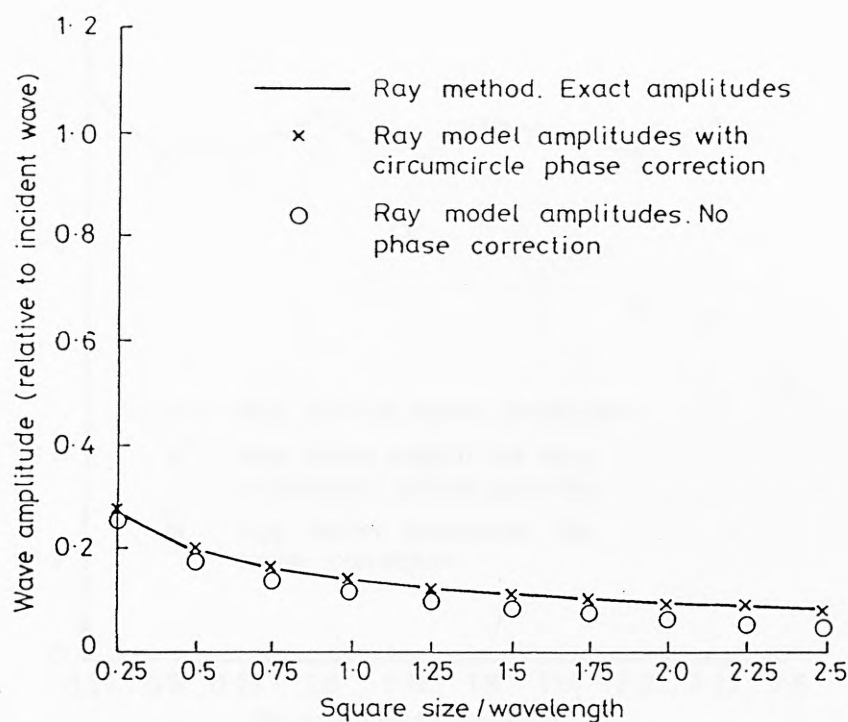


Fig 6.4 Variation of Wave Amplitude with Square Size per Wavelength (d/λ). Semi-Infinite Breakwater. Flat Seabed. Square 14, 2

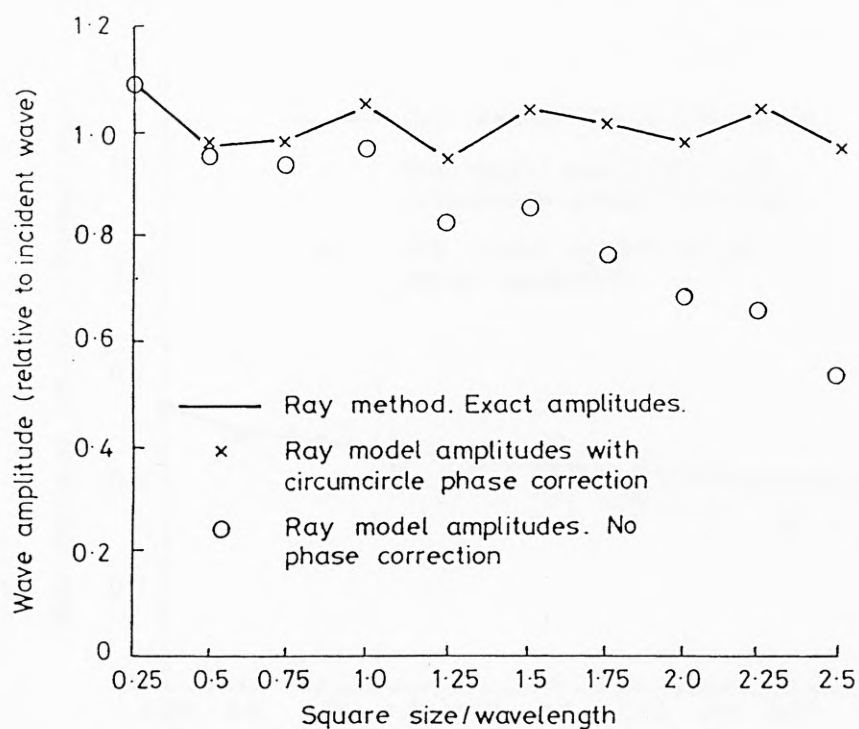


Fig 6.5 Variation of Wave Amplitude with Square Size per Wavelength (d/λ). Semi-Infinite Breakwater. Flat Seabed. Square 7, 8

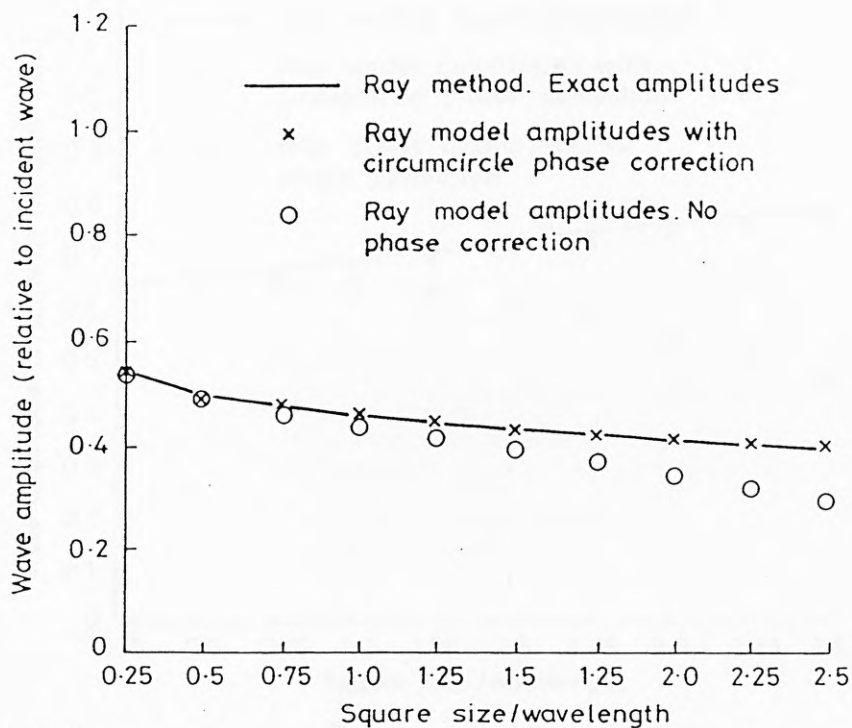


Fig6-6 Variation of Wave Amplitude with Square Size per Wavelength (d/λ). Semi-Infinite Breakwater. Flat Seabed. Square 11, 5

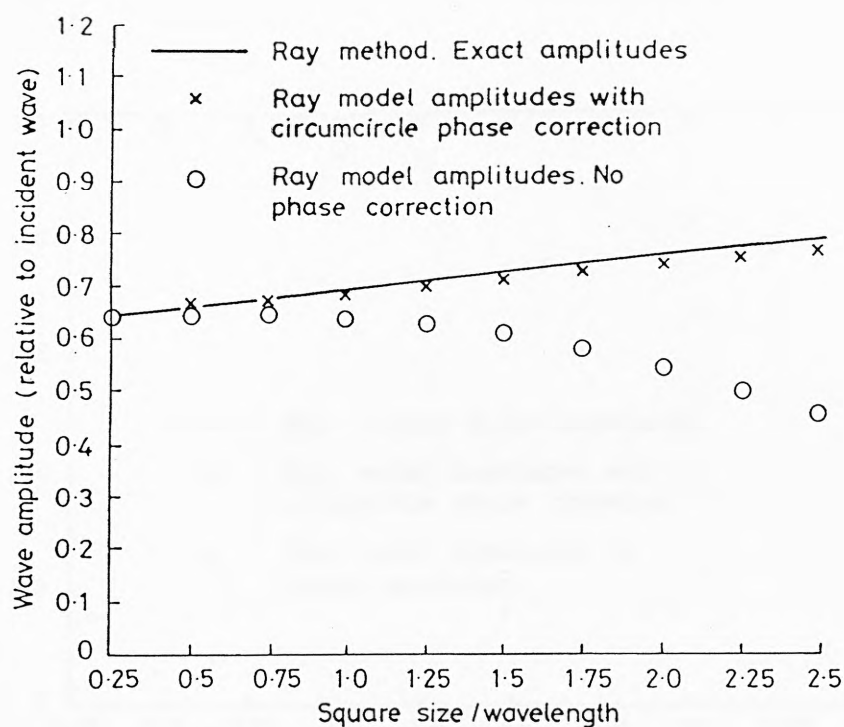


Fig 6.7 Variation of Wave Amplitude with Square Size per Wavelength (d/λ). Semi-Infinite Breakwater. Flat Seabed. Square 11,7

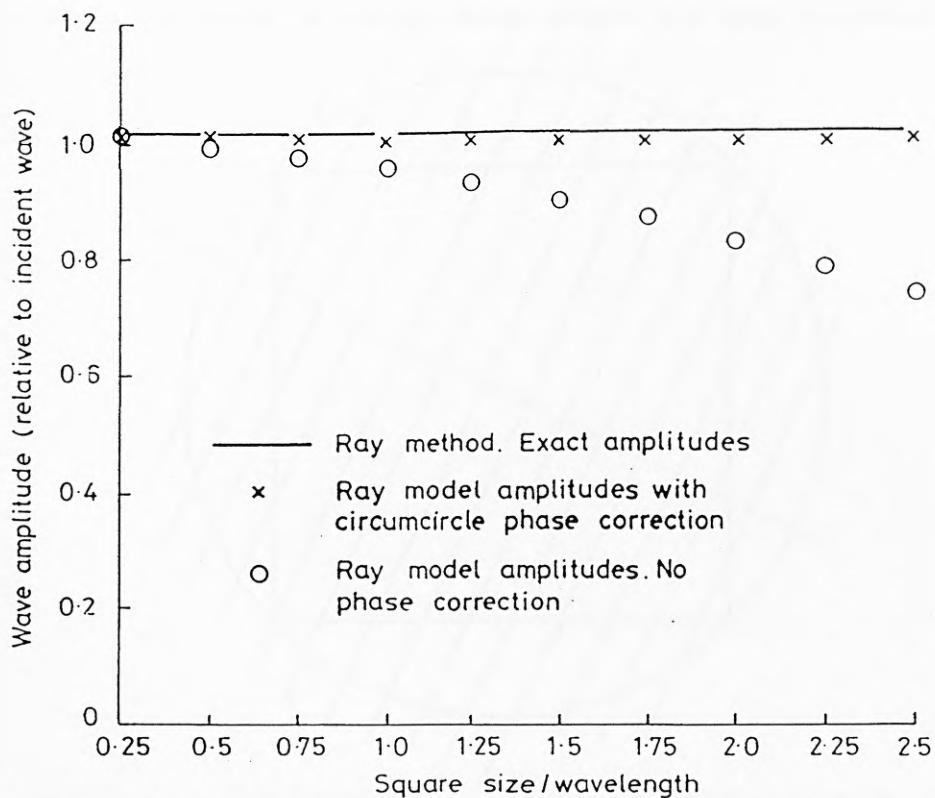


Fig 6-8 Variation of Wave Amplitude with Square Size per Wavelength (d/λ), Single Plane Wave Train. Parallel Contoured Seabed. Average of Amplitudes in Squares in Fifth Row.

----- Locus of average phase of each ray using circumcircle method

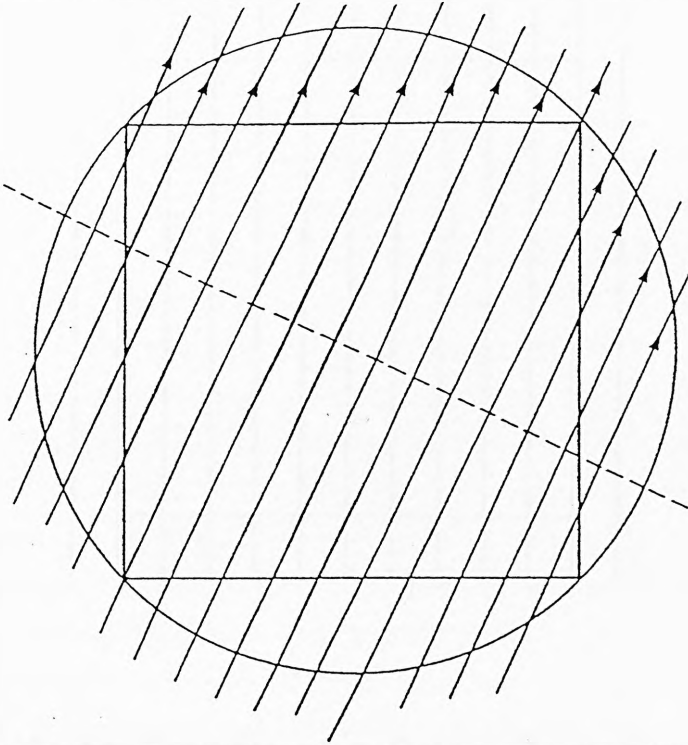


Fig 6-9 Average Phases of a Set of Parallel Rays crossing an Averaging Square with the Circumcircle Phase Correction.

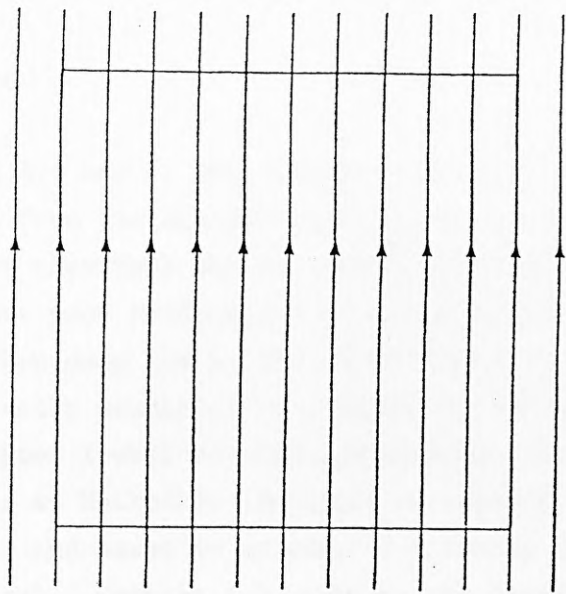


Fig 6-10 An Example of a Set of Rays crossing an Averaging Square giving a Large Ray Density Error.

7.1 Introduction

In Chapters 3,4 and 5, the theory of the ray tracing method using u-rays sent from the shadow boundary was described. It was shown in Chapter 4 that this method (referred to as Method A in this chapter) gave good predictions of wave amplitude and direction on the shadow boundary and in the far-field of u, but wave phases were incorrectly predicted by a factor of $\pi/4$ at worst. In the present chapter I will develop and examine a new ray system (referred to as Method B), designed to improve the prediction of wave phases, and based on an idea of R Stacey (private communication). Section 7.2 contains the theory of this new method. In Section 7.3, the u-wave amplitudes, phases and directions predicted by both methods and the radial ray method are compared with the Sommerfeld solution in the near, far and intermediate regions of both u and kr. In Section 7.4 the best of these methods is then compared with the Sommerfeld solution and a finite-element model. In this comparison the full incident, u and v ray system, with the ray averaging technique, is used (i.e. the full computational model based on the ray method is used). Finally in Section 7.5 the ray model and finite-element model are compared on a varying-depth seabed.

7.2 Theory of Ray Method B

The directions of the u-rays in the original method (Method A) were chosen so that they matched the exact Sommerfeld directions on the shadow boundary. The idea behind the present method (Method B) is to choose the ray directions in such a way that the wave phases are matched with the Sommerfeld solution on the shadow boundary and in the far-field of u. This method therefore is designed to improve the prediction of wave phase, but possibly at the expense of the good predictions of amplitude and direction achieved by Method A. A comprehensive comparison of the two methods and the radial ray method with the Sommerfeld solution is given in the following section.

The phase predicted by the Sommerfeld solution in the far-field of u is $kr + \pi/4$ (Eq 4.2.5) while that predicted by the ray method is $k(y_0 + s)$ (see Fig 7.1). By equating these two expressions, it can be ensured that the asymptotic wave phase is predicted correctly,

$$kr + \pi/4 = k(y_0 + s) \quad (r, s \rightarrow \infty) \quad 7.2.1$$

$$\text{or } r = y_0 + s - \frac{\pi}{4k} \quad (r, s \rightarrow \infty) \quad 7.2.2$$

The imposition of this condition determines the ray system, and therefore determines the relationships between the starting angle, α , and the distance of the ray starting point along the shadow boundary, y_0 . Reference to Fig 7.1 shows that the following geometrical relationship is satisfied,

$$r^2 = s^2 + y_0^2 + 2sy_0 \cos\alpha \quad 7.2.3$$

Eliminating r between Eq 7.2.2 and 7.2.3 gives,

$$\frac{\pi^2}{16k^2s} + \frac{\pi y_0}{2ks} + 2y_0 - 2y_0 \cos\alpha - \frac{\pi}{2k} = 0 \quad (\text{as } s \rightarrow \infty) \quad 7.2.4$$

As $s \rightarrow \infty$, the first two terms tend to zero, and therefore the relation between y_0 and α is given by

$$2y_0 - 2y_0 \cos\alpha - \frac{\pi}{2k} = 0 \quad 7.2.5$$

$$\text{or } \cos\alpha = 1 - \frac{\pi}{4ky_0} \quad 7.2.6$$

The ray system based on Eq 7.2.6 is shown in Fig 7.2. Note that, unlike Method A, a part of the area 90° either side of the shadow boundary is not covered by rays. The uncovered area is for

$$y_0 < \frac{\pi}{4k} \quad 7.2.7$$

corresponding to the condition $\text{Cos}\alpha < 0$

The ray equation is

$$y = x \cot\alpha + \frac{\pi}{4k(1 - \text{Cos}\alpha)} \quad 7.2.8$$

The determination of wave amplitude follows the same procedure as Sections 4.6 and 4.7. Firstly, the equation of the envelope formed by backward projection of the u-rays is calculated. This is found by solving simultaneously the equations $F(\alpha) = 0$ and $dF/d\alpha = 0$ where $F(\alpha) = 0$ is the ray equation, Eq 7.2.8, and $dF/d\alpha = 0$ is

$$y = -\frac{x}{\text{Sin}^2\alpha} - \frac{\pi \text{Sin}\alpha}{4k(1 - \text{Cos}\alpha)^2} \quad 7.2.9$$

Solving Eq 7.2.8 and Eq 7.2.9 simultaneously for y and x gives

$$y = \frac{\pi(\text{Sin}^2\alpha - \text{Cos}\alpha)}{4k(1 - \text{Cos}\alpha)(1 + \text{Cos}\alpha \text{Sin}\alpha)} \quad 7.2.10$$

$$\text{and } x = -\frac{\pi \text{Sin}\alpha (1 + \text{Cos}\alpha + \text{Sin}\alpha)}{4k(1 - \text{Cos}\alpha)(1 + \text{Cos}\alpha \text{Sin}\alpha)} \quad 7.2.11$$

Identical arguments to those in Section 4.7 are used to derive the expression for wave amplitude at a point on a u-ray,

$$A_u^2 = \frac{1}{4\pi k r' (1 - \text{Cos}\alpha)} \quad 7.2.12$$

Some straightforward algebra shows that r' is given by

$$r' = s + \frac{\pi(1 + \text{Sin}\alpha + \text{Cos}\alpha)}{4k(1 - \text{Cos}\alpha)(1 + \text{Cos}\alpha \text{Sin}\alpha)} \quad 7.2.13$$

The wave phase is simply given by

$$S_u = k(s + y_o) \quad \text{sheltered zone} \quad 7.2.14$$

$$\text{and } S_u = k(s + y_o) + \pi \quad \text{unsheltered zone} \quad 7.2.15$$

7.3 Comparison of Method A, Method B, Radial Ray Method and the Sommerfeld solution

7.3.1 Introduction

In this section u-wave amplitudes, phases and directions predicted by ray methods A and B and a radial ray method are compared with the Sommerfeld solution in the near, far and intermediate regions of u and kr . The incident wave and u-wave are included in this comparison; the v-wave is omitted. The omission of the v-wave means that the Sommerfeld wave amplitudes are constant along lines of constant u . These lines are parabolas with axis along the shadow boundary and focus at the breakwater tip. In the sheltered region, the half-parabola takes a negative values of u , while in the unsheltered region the half-parabola takes a positive value of u , implying that the superposition of the incident wave and u-wave is being considered. For these comparisons the incident wave direction is normal to the breakwater. All distances are non-dimensionalised by dividing by the wavelength.

The areas under consideration are shown in Figs 7.3 and 7.4. In Fig 7.3 the area extends five wavelengths behind the breakwater and ten wavelengths laterally. Fig 7.4 shows the area within one wavelength of the breakwater tip in more detail. Parabolas of constant u are drawn at values of u equal to 0, 0.1, 0.5, 1.0, 2.0, 3.0, and 5.0 on Fig 7.3, and 0, 0.1, 0.3, 0.5, 0.7, and 1.0 on Fig 7.4 (with corresponding negative values of u in the sheltered region). Values of the wave parameters using the two ray methods, a method based on radial rays from the breakwater tip, and the Sommerfeld solution have been calculated for a range of pairs of u and y values, marked by circles on the parabolas in Figs 7.3 and 7.4. Wave amplitudes, phases and directions at these points are tabulated in Tables 7.1 to 7.5. These results are discussed below.

7.3.2 Wave Amplitudes (Tables 7.1 and 7.2)

Radial ray amplitudes are well predicted at large u , becoming progressively worse at smaller u . At $u = -1$, the difference is

about 11%, improving to about 2% at $u = -2$. Depending on the required accuracy in any problem, the practical limiting value of u probably lies between these two values. This means that there is a sizeable area surrounding the shadow boundary where the radial ray method breaks down. Agreement with the analytical solution in the unsheltered region is generally somewhat poorer because of the poor prediction of phase in the far field.

Wave amplitudes predicted by ray method A appear to be very good almost everywhere, at all values of u . The only region of poor agreement appears to be around the breakwater tip (within about $1/10$ wavelength). This is unlikely to be important in practice, especially since the model is designed as a far field (in kr) model. As noted in Section 4.11, the agreement on the shadow boundary is particularly good. Typical errors in the intermediate region are around 2-3% (at $u = -0.5$, $y = 1.0$ the error is 2.3%). In the unsheltered area the amplitude predictions are less good because of the error in predicting the phase, but even here the worst errors in the intermediate region are around 8% (this value occurs at $u = 0.5$, $y = 1.0$).

Amplitude predictions for method B are not as good, although they do converge to the Sommerfeld value in the far field of u . In the intermediate region at $u = -0.5$, $y = 1.0$, the differences from the Sommerfeld solution are 36%, while on the shadow boundary method B underpredicts by about 50%. The performance in the unsheltered region is generally somewhat better than method A. In the intermediate region errors are around 4%, due largely to the better prediction of wave phase. At $u = 0.5$, $y = 1.0$ the difference from the Sommerfeld solution is 3.7%.

A possible adaptation to Method B, with the aim of improving wave amplitude predictions, would be to impose a wave amplitude of 0.5 everywhere along the shadow boundary instead of using the conservation of energy flux condition, which results in Eq 7.2.12. Predictions of wave amplitude using such a method show good agreement close to the shadow boundary, but with poorer results in the intermediate region, similar to results using the original method B. The main objection to this adaptation to method B,

however, is that the imposition of the wave amplitude along the shadow boundary means that the conservation of energy flux has been abandoned. It is necessary to retain conservation of energy flux when the method is extended to account for wave refraction in the varying depth case.

7.3.3 Wave Phases (Tables 7.3 and 7.4)

In both the radial ray method and ray method A the phases are initially exact along the shadow boundary (by design) but become progressively worse until in the far field of u they are different from the Sommerfeld value by 45° . In the unsheltered area, this affects the total wave amplitude since the relative phases of the diffracted and incident waves are taken into account when calculating the total wave amplitude.

Ray method B performs much better, giving exact values along the shadow boundary (by design) and in the far field. In the intermediate region differences are typically no worse than 2 or 3 degrees.

7.3.4 Wave Directions (Table 7.5)

The radial ray method predicts wave directions accurately for large u , but the predictions become progressively poorer as u is decreased. Along the shadow boundary at $y = 1.0$, the difference from the Sommerfeld value is 12.7° , and in the intermediate region at $u = -0.5$, $y = 1.0$ the difference is 6.5° .

Ray method A predicts wave directions very accurately everywhere, being exact (by design) on the shadow boundary and in the far field of u . The differences in the intermediate region are no worse than 2 or 3 degrees; at $u = -0.5$, $y = 1.0$ the difference is 0.9° . Very close to the breakwater tip, around one tenth of a wavelength, the differences become greater. Values for method A are consistently greater than the Sommerfeld values showing that the "bending" of rays takes place, but these results would indicate that the effect is small and that the u -rays are a good approximation to the actual wave front orthogonals.

Method B ray direction are poorer. They agree with the Sommerfeld values at large u (and are asymptotically correct) but the convergence is quite slow, and in the intermediate region differences of 10° to 15° occur. At $u = -0.5$ and $y = 1.0$, the difference from the Sommerfeld value is 12.9° .

7.3.5 Overall Conclusions

The radial ray method breaks down completely around the shadow boundary in its prediction of amplitude and is also significantly incorrect in its prediction of phase and direction in this region. Ray method A gives good predictions of amplitude and direction everywhere (except within about one tenth of a wavelength of the breakwater tip), but the far-field (in u) phases differ from the Sommerfeld values by 45° . Ray method B gives generally poor predictions of wave height and direction in the intermediate u -regions, although they tend to the correct values in the far-field u limit. Phases are well predicted everywhere.

It can be concluded that ray method A is significantly better than the other methods in all respects except in the prediction of phase. In most practical problems, however, the phase is not usually a quantity of interest. A further point in favour of ray method A is that the full area is covered, 90° either side of the shadow boundary, with a smooth transition to radial rays beyond that.

7.4 Comparison of the Ray Model with the Sommerfeld Solution and Finite-Element Model

In this section a comparison is made of the full ray model based on method A with the Sommerfeld solution and a finite-element model based on a direct solution of the Mild-Slope Equation, Eq 2.2.1 (Bettess and Zienkiewicz (1977)). The aim is to demonstrate the accuracies of both types of model on a horizontal seabed and then, in the following section, to compare their predictions on a sloping seabed.

A rectangular area 450m by 270m is used in these tests, with the breakwater lying along one of the long boundaries with its tip at the centre of the boundary (Fig 7.8, but ignore the depth contours in this figure for these tests). The area covered by the finite-element model is slightly smaller, extending to 225m in the direction normal to the breakwater. A constant water depth of 15m is used. Square elements with side lengths of 30m are used in the ray model, and 8-noded square elements with a side length of 37.5m in the finite element model. All tests were carried out with a wave period of 16s, chosen to be large enough to allow sufficient nodes per wavelengths in the finite-element model, but small enough that the far field is reached in order to test the ray model. This combination of period and depth gives a wavelength of 186.5m. Separate tests were performed for three incident wave directions (45° , 90° and 135°), and the results are shown in the form of contour plots of wave amplitude in Figs 7.5, 7.6 and 7.7.

The agreement between the ray model and Sommerfeld solution is within about 5% everywhere. The finite-element model shows some instabilities near the downwave grid edge particularly in the 90° case. Away from this grid edge near the shadow boundary, agreement with the Sommerfeld solution is within 5%.

7.5 Comparison of the Ray Model and Finite-Element Model on a Sloping Seabed

In this section the ray model is tested in a situation where refraction and shoaling are combined with diffraction. This is done by using the same grids as before (in both models) but with a seabed that slopes from 15m at the breakwater to just under 6m at the far boundary (in the ray model grid) with no lateral variation. Fig 7.8 shows the depth contours. As for the tests in the previous section, a single period, 16s, was considered, along with three directions, 45° , 90° and 135° . Results are presented as contours of wave amplitude in Figs 7.9, 7.10 and 7.11.

Agreement between the two models is around 5% in most areas. The only area of greater disagreement is at the downwave grid edge around the shadow boundary where instabilities occur in the finite-element model.

It is interesting to compare Figs 7.9, 7.10 and 7.11 with Figs 7.5, 7.6 and 7.7 to note the differences in wave amplitude caused by the inclusion of refraction and shoaling. For the incident angles of 45° and 135° it can be anticipated that the effects of refraction will be to decrease incident wave amplitude, while increases will occur due to shoaling. For these angles, shoaling outweighs refraction and a net increase of about 5% occurs at the downwave grid edge in the unsheltered area. In the 90° case there is no refraction of the incident wave, and a larger increase in wave amplitude due to shoaling alone would be expected. An increase of 15-20% occurs at the downwave grid edge.

For the diffracted rays, both shoaling and refraction will tend to increase wave amplitudes along comparable rays, but refraction also shifts the paths of the rays. For the 135° case, refraction causes the shadow boundary to be bent into the shelter, thereby adding to the general increase in wave amplitude. For the 45° case, refraction causes the shadow boundary to bend away from the shelter acting against the trend of higher wave amplitudes so that the increase is smaller. For the 90° case, there is no shift of the shadow boundary, and a general increase in wave height is seen. Well into the shelter for all incident angles, refraction causes the wave to bend away from the breakwater resulting in a slight decrease in wave amplitude.

In summary, therefore, the ray model predicts wave amplitudes on a horizontal seabed to an accuracy of about 5% at worst of the exact Sommerfeld solution. The finite-element model shows similar accuracy. When the ray model is used on a sloping seabed an agreement of about 5% is obtained with the finite-element model.

The situations modelled here have been idealised. In real applications, other wave phenomena such as internal diffraction will be important. In order to investigate how the ray model performs in a typical real harbour, comparisons have been made between the ray model and a scaled physical model of such a harbour. These investigations are described in the following chapter.

u	y (wavelengths)	x (wavelengths)	WAVE AMPLITUDE COEFFICIENTS			
			Radial Ray Method	Ray Method A	Ray Method B	Sommerfeld Solution
-5.0	1.0	7.18	0.045	0.045	0.044	0.045
	3.0	8.75		0.045	0.044	
	5.0	10.08		0.045	0.044	
-3.0	1.0	3.09	0.075	0.075	0.070	0.075
	3.0	4.31		0.074	0.070	
	5.0	5.25		0.074	0.070	
-2.0	1.0	1.73	0.113	0.110	0.098	0.111
	3.0	2.65		0.109	0.098	
	5.0	3.32		0.109	0.098	
-1.0	3.0	1.25	0.225	0.199	0.152	0.203
	5.0	1.60		0.199	0.151	
-0.5	3.0	0.62	0.450	0.311	0.194	0.308
	5.0	0.79		0.310	0.192	
-0.1	3.0	0.12	2.251	0.461	0.231	0.452
	5.0	0.16		0.460	0.228	
0 (Sheltered)	3.0	0	∞	0.501	0.240	0.500
	5.0	0		0.501	0.237	
0.1	3.0	-0.12	1.713	0.541	0.771	0.535
	5.0	-0.16		0.541	0.774	
0.5	3.0	-0.62	0.926	0.744	0.875	0.807
	5.0	-0.79		0.745	0.875	
1.0	3.0	-1.25	1.170	1.044	1.091	1.122
	5.0	-1.60		1.045	1.090	
2.0	1.0	-1.73	0.924	0.890	0.929	0.919
	3.0	-2.65		0.891	0.928	
	5.0	-3.32		0.891	0.928	
3.0	1.0	-3.09	1.054	1.004	1.049	1.052
	3.0	-4.31		1.008	1.049	
	5.0	-5.25		1.009	1.049	
5.0	1.0	-7.18	1.032	1.001	1.031	1.032
	3.0	-8.75		1.002	1.031	
	5.0	-10.08		1.003	1.031	

TABLE 7.1

Comparison of Ray Method Amplitude Coefficients with the Sommerfeld solution. Large area (Fig 7.3).

u	y (wavelengths)	x (wavelengths)	WAVE AMPLITUDE COEFFICIENTS			
			Radial Ray Method	Ray Method A	Ray Method B	Sommerfeld Solution
-1.0	0.1	0.335	0.225	0.215	-	0.203
	0.3	0.461		0.208	0.149	
	0.5	0.559		0.205	0.151	
	0.7	0.642		0.203	0.152	
	1.0	0.750		0.202	0.152	
-0.5	0.1	0.128	0.450	0.346	-	0.308
	0.3	0.203		0.325	0.192	
	0.5	0.258		0.319	0.195	
	0.7	0.302		0.317	0.196	
	1.0	0.359		0.315	0.196	
-0.1	0.1	0.023	2.251	0.493	-	0.452
	0.3	0.039		0.472	0.237	
	0.5	0.050		0.467	0.239	
	0.7	0.059		0.465	0.239	
	1.0	0.071		0.463	0.237	
0 (Sheltered)	0.1	0	∞	0.528	-	0.5
	0.3	0		0.510	0.250	
	0.5	0		0.506	0.251	
	0.7	0		0.504	0.249	
	1.0	0		0.503	0.247	
0.1	0.1	-0.023	1.713	0.509	-	0.553
	0.3	-0.039		0.530	0.766	
	0.5	-0.050		0.534	0.763	
	0.7	-0.059		0.536	0.764	
	1.0	-0.071		0.538	0.765	
0.5	0.1	-0.128	0.926	0.704	-	0.807
	0.3	-0.203		0.727	0.882	
	0.5	-0.258		0.734	0.878	
	0.7	-0.302		0.737	0.876	
	1.0	-0.359		0.740	0.876	
1.0	0.1	-0.335	1.170	1.027	-	1.122
	0.3	-0.461		1.033	1.094	
	0.5	-0.559		1.036	1.094	
	0.7	-0.642		1.038	1.093	
	1.0	-0.750		1.040	1.093	

TABLE 7.2

Comparison of Ray Method Amplitude Coefficients with the Sommerfeld solution. Small area (Fig 7.4).

u	y (wavelengths)	x (wavelengths)	WAVE PHASES (DEGREES)			
			Radial Ray Method	Ray Method A	Ray Method B	Sommerfeld Solution
-5.0	1.0	7.18	135.0	90.3	134.5	134.3
	3.0	8.75	135.0	91.4	134.4	134.3
	5.0	10.08	135.0	92.5	134.4	134.3
-3.0	1.0	3.09	135.0	91.3	133.5	133.0
	3.0	4.31	135.0	93.7	133.2	133.0
	5.0	5.25	135.0	95.1	133.1	133.0
-2.0	1.0	1.73	45.0	3.0	41.6	40.7
	3.0	2.65	45.0	5.7	41.0	40.7
	5.0	3.32	45.0	6.7	40.8	40.7
-1.0	3.0	1.25	135.0	97.3	123.4	122.6
	5.0	1.60	135.0	97.7	123.3	122.6
-0.5	3.0	0.62	67.5	29.2	45.3	44.0
	5.0	0.79	67.5	29.3	45.2	44.0
-0.1	3.0	0.12	45.9	3.3	6.9	6.2
	5.0	0.16	45.9	3.3	6.9	6.3
0 (Sheltered)	3.0	0	45.0	0	0	0
	5.0	0	45.0	0	0	0
0.1	3.0	-0.12	272.8	357.2	357.9	354.8
	5.0	-0.16	272.8	357.1	358.0	354.8
0.5	3.0	-0.62	286.6	348.2	351.0	344.6
	5.0	-0.79	286.6	348.2	351.1	344.6
1.0	3.0	-1.25	335.9	349.1	353.3	351.2
	5.0	-1.60	335.9	349.1	353.3	351.2
2.0	1.0	-1.73	289.7	359.6	356.0	355.5
	3.0	-2.65	289.7	359.4	356.0	355.5
	5.0	-3.32	289.7	359.3	356.0	355.5
3.0	1.0	-3.09	336.9	355.7	357.2	357.0
	3.0	-4.31	336.9	355.8	357.2	357.0
	5.0	-5.25	336.9	355.8	357.2	357.0
5.0	1.0	-7.18	337.1	357.4	358.3	358.2
	3.0	-8.75	337.1	357.4	358.3	358.2
	5.0	-10.08	337.1	357.4	358.3	358.2

TABLE 7.3

Comparison of Ray Method Phases with the Sommerfeld solution. Large area (Fig 7.3).

u	y (wavelengths)	x (wavelengths)	WAVE PHASES (DEGREES)			
			Radial Ray Method	Ray Method A	Ray Method B	Sommerfeld Solution
-1.0	0.1	0.335	171.0	127.1	-	158.6
	0.3	0.461	243.0	201.2	233.9	230.6
	0.5	0.559	315.0	274.4	305.1	302.6
	0.7	0.642	27.0	347.1	16.7	14.6
	1.0	0.750	135.0	95.8	124.3	124.3
-0.5	0.1	0.128	103.5	61.5	-	80.0
	0.3	0.203	175.5	135.5	155.7	152.0
	0.5	0.258	247.5	208.1	226.7	224.0
	0.7	0.302	319.5	280.4	298.3	296.0
	1.0	0.359	67.5	28.7	45.9	44.0
-0.1	0.1	0.023	81.9	38.9	-	42.3
	0.3	0.039	117.9	111.2	115.6	114.3
	0.5	0.050	153.9	183.2	187.3	186.3
	0.7	0.059	225.9	255.3	259.1	258.3
	1.0	0.071	297.9	3.3	7.0	6.3
0 (Sheltered)	0.1	0	81.0	36.0	-	36.0
	0.3	0	117.0	108.0	108.0	108.0
	0.5	0	225.0	180.0	180.0	180.0
	0.7	0	297.0	252.0	252.0	252.0
	1.0	0	45.0	0	0	0
0.1	0.1	-0.023	308.8	33.2	-	30.8
	0.3	-0.039	20.8	105.2	105.7	102.8
	0.5	-0.050	92.8	177.2	177.7	174.8
	0.7	-0.059	164.8	249.2	249.8	246.8
	1.0	-0.071	272.8	357.2	357.8	354.8
0.5	0.1	-0.128	322.6	23.8	-	20.6
	0.3	-0.203	34.6	96.1	98.7	92.6
	0.5	-0.258	106.6	168.1	170.7	164.6
	0.7	-0.302	178.6	240.2	242.7	236.6
	1.0	-0.359	286.6	348.2	350.7	344.6
1.0	0.1	-0.335	11.9	23.9	-	27.2
	0.3	-0.461	83.9	96.4	30.6	99.2
	0.5	-0.559	155.9	168.6	173.5	171.2
	0.7	-0.642	227.9	240.8	245.4	243.2
	1.0	-0.750	335.9	348.9	353.4	351.2

TABLE 7.4

Comparison of Ray Method Phases with the Sommerfeld solution. Small area (Fig 7.4).

u	y (wavelengths)	x (wavelengths)	WAVE RAY DIRECTION (DEGREES)			
			Radial Ray Method	Ray Method A	Ray Method B	Sommerfeld Solution
-5.0	1.0	7.18	7.9	7.9	6.8	7.9
	3.0	8.75	18.9	18.9	17.9	18.9
	5.0	10.08	26.4	26.3	25.4	26.4
-3.0	1.0	3.09	17.9	17.8	15.0	17.8
	3.0	4.31	34.8	34.6	32.4	34.7
	5.0	5.25	43.6	43.3	41.4	43.5
-2.0	1.0	1.73	30.0	29.6	24.4	29.5
	3.0	2.65	48.6	48.0	44.3	48.2
	5.0	3.32	56.4	55.9	52.9	56.1
-1.0	3.0	1.25	67.4	65.6	59.3	65.5
	5.0	1.60	72.2	70.8	65.9	70.8
-0.5	3.0	0.62	78.4	75.0	66.9	74.5
	5.0	0.79	81.0	78.3	72.0	77.9
-0.1	3.0	0.12	87.7	81.4	72.2	81.1
	5.0	0.16	88.2	83.3	76.2	83.1
0 (Sheltered)	3.0	0	90.0	82.6	73.4	82.6
	5.0	0	90.0	84.3	77.2	84.3

-1.0	0.1	0.335	16.6	16.0	-	12.2
	0.3	0.461	33.1	31.4	15.6	29.2
	0.5	0.559	41.8	39.8	26.3	38.3
	0.7	0.642	47.5	45.3	33.3	44.3
	1.0	0.750	53.1	50.9	40.6	50.3
-0.5	0.1	0.128	38.0	32.0	-	24.6
	0.3	0.203	55.9	48.9	23.9	45.6
	0.5	0.258	62.7	56.3	36.5	54.2
	0.7	0.302	66.6	60.7	43.9	59.1
	1.0	0.359	70.3	64.9	50.9	63.8
-0.1	0.1	0.023	77.3	50.2	-	47.1
	0.3	0.039	82.6	64.3	33.0	63.1
	0.5	0.050	84.3	69.6	46.1	68.7
	0.7	0.059	85.2	72.5	53.0	71.9
	1.0	0.071	86.0	75.3	59.1	74.7
0 (Sheltered)	0.1	0	90.0	54.6	-	54.6
	0.3	0	90.0	67.7	35.7	67.7
	0.5	0	90.0	72.3	48.6	72.3
	0.7	0	90.0	74.9	55.2	74.9
	1.0	0	90.0	77.3	61.0	77.3

TABLE 7.5

Comparison of Ray Method Wave Directions with the Sommerfeld solution. Large (top) and Small (bottom) areas. Results presented for sheltered region only.

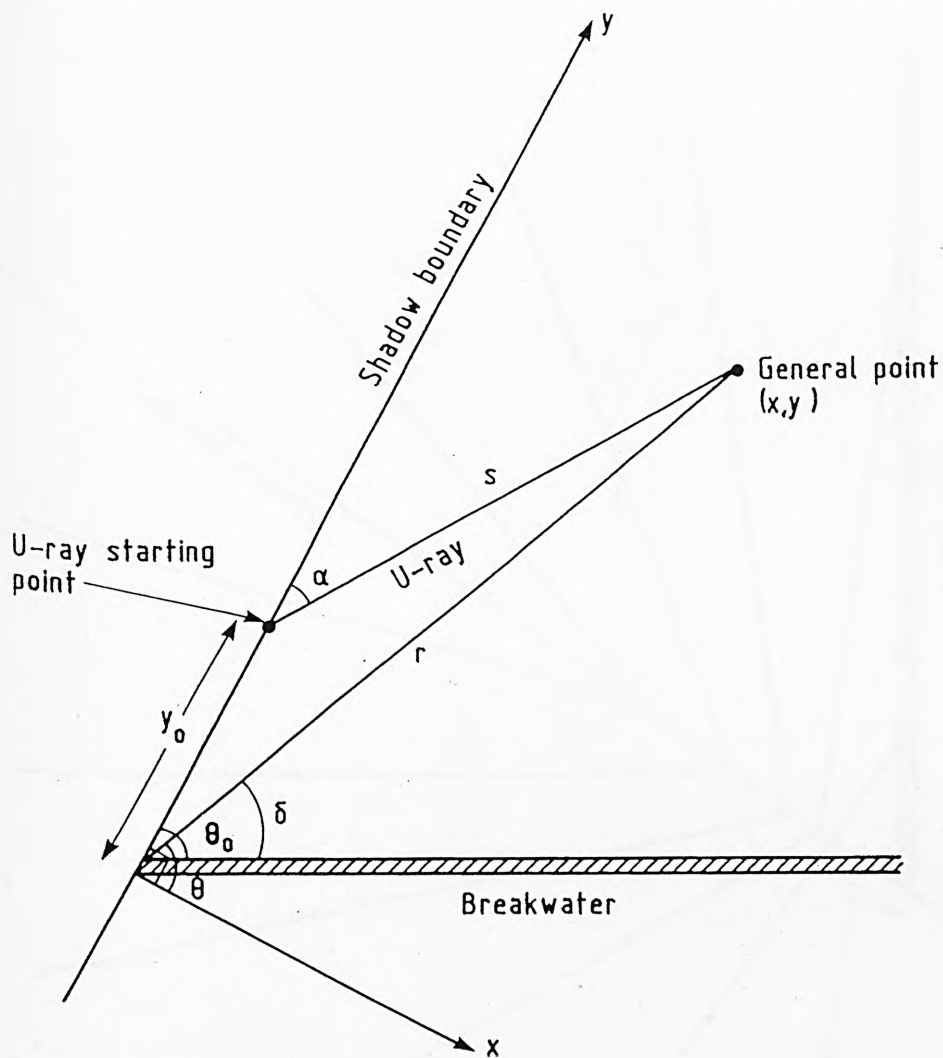


Fig 7.1 Geometrical relationship of y_0 , r , s and α

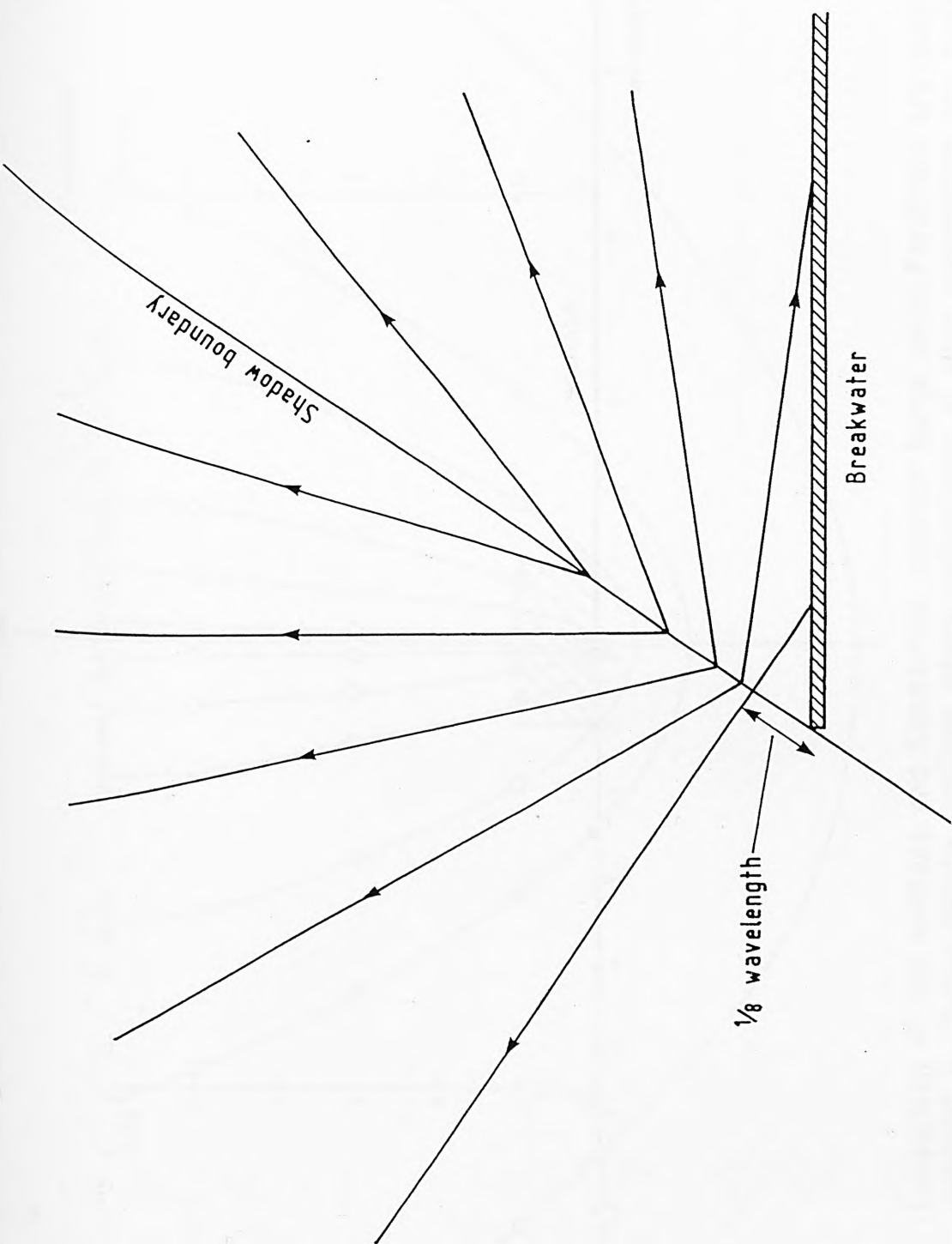


Fig 7.2 U-ray system for diffraction around a semi-infinite breakwater using ray method B

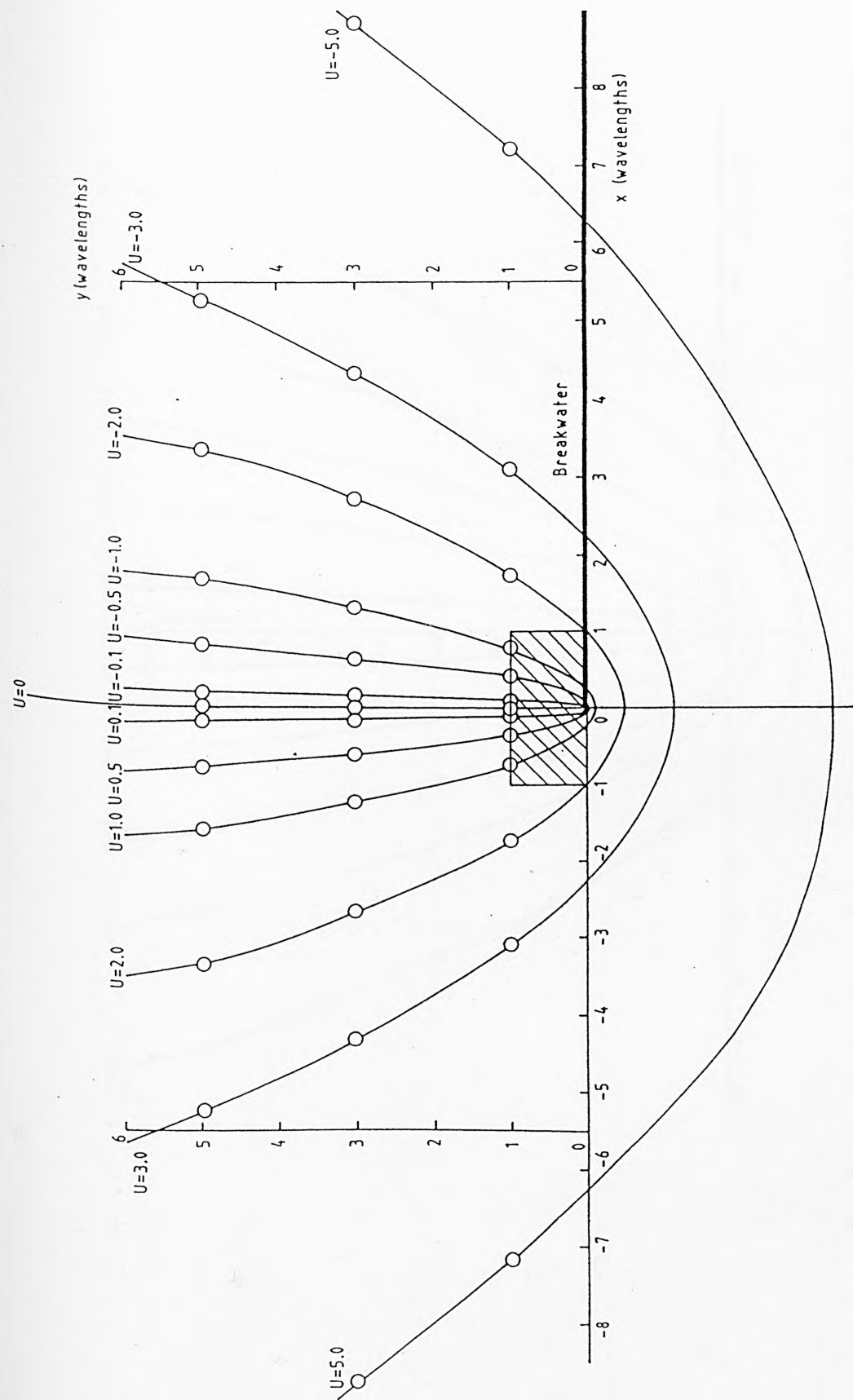


Fig 7.3 Comparison of ray methods and Sommerfeld solution (large area). Parabolas are lines of constant U . Circles denote locations where calculations of wave parameters are made (see Tables 7.1-7.5). Shaded area is enlarged in Fig 7.4

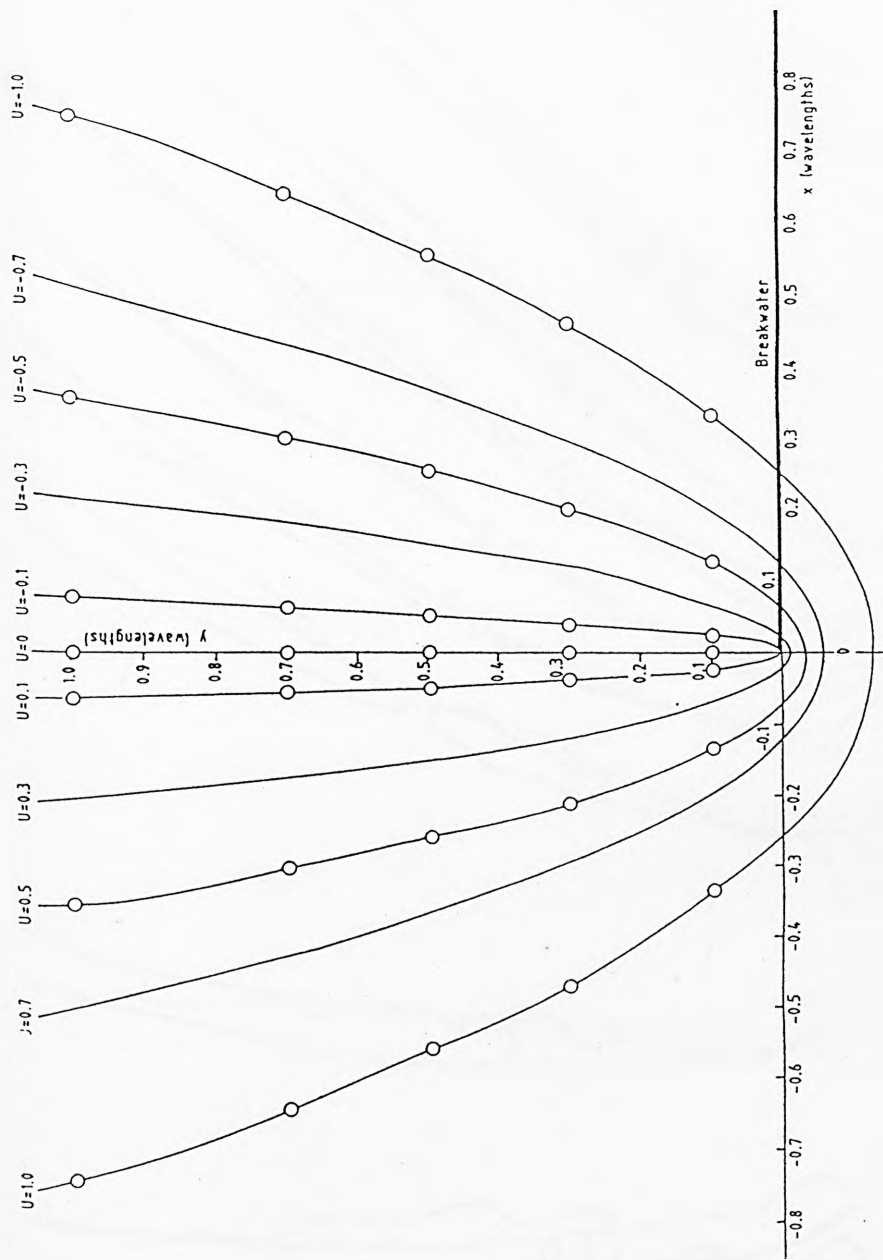


Fig 7.4 Comparison of ray methods and Sommerfeld solution (small area). Parabolas are lines of constant U. Circles denote locations where calculations of wave parameters are made (see Tables. 7.1-7.5)

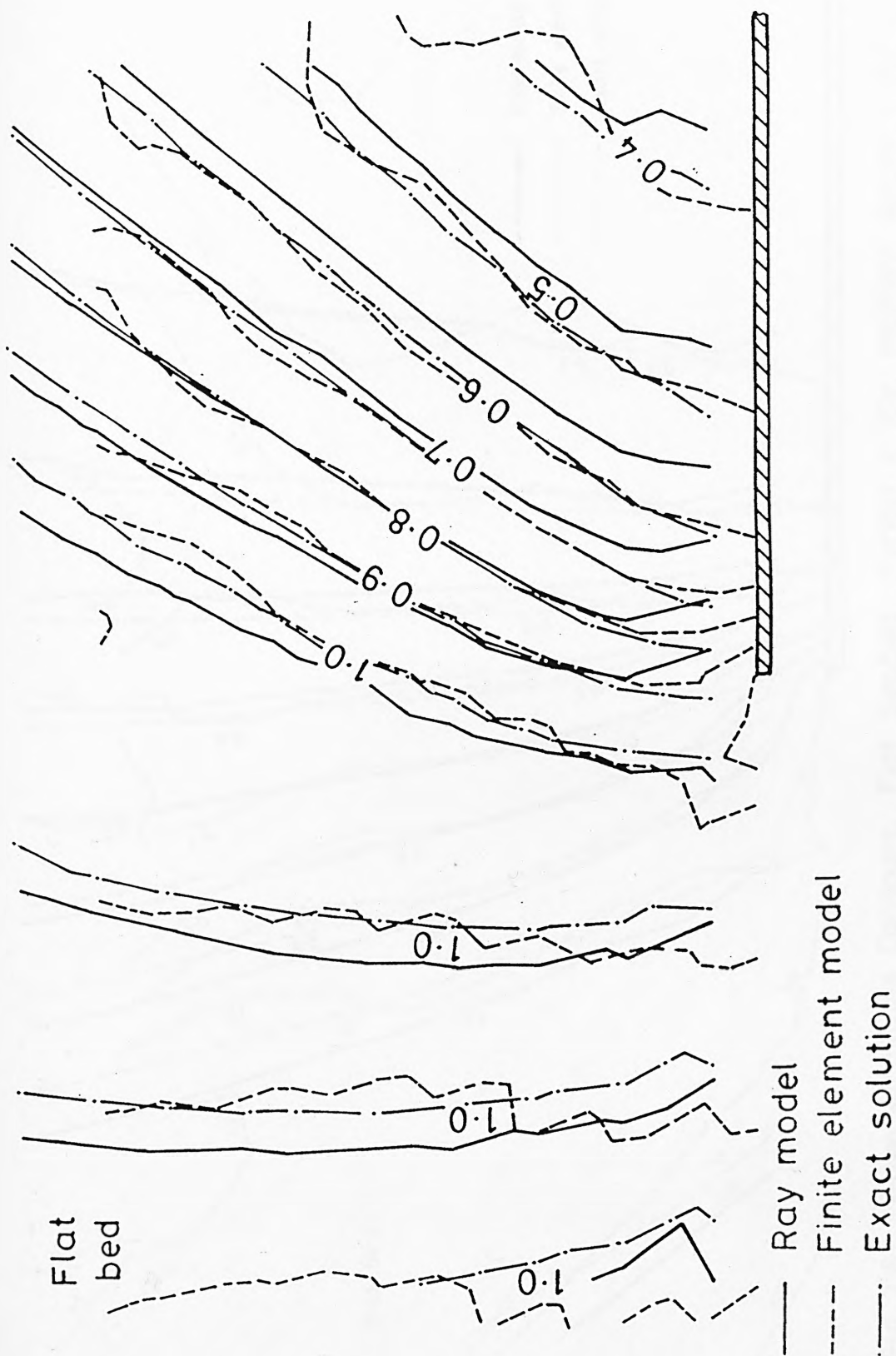


Fig 7.5 Wave Amplitude Contours. Flat Seabed. Period = 16s, Incident Angle = 45°

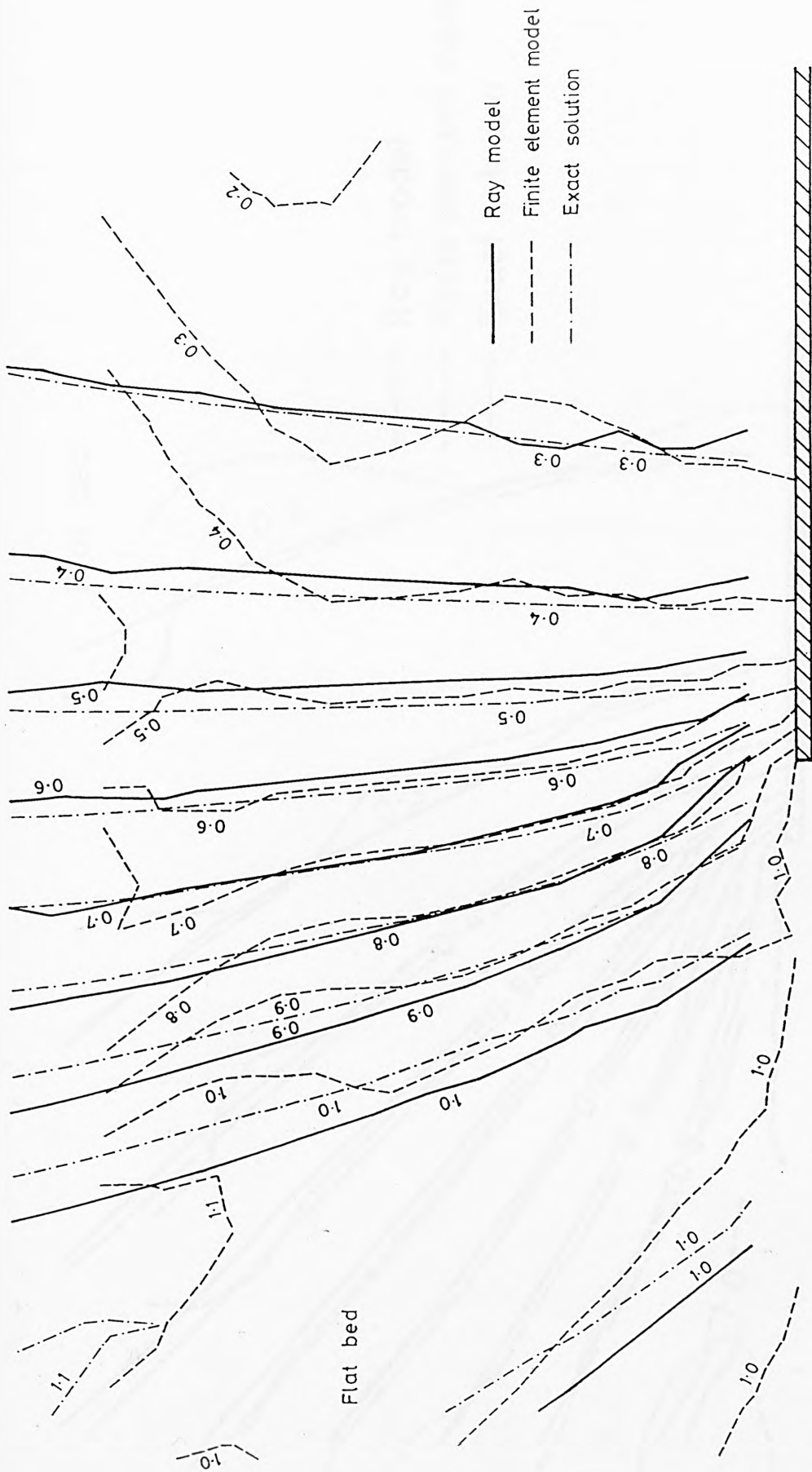


Fig 7-6 Wave Amplitude Contours. Flat Seabed. Period = 16s, Incident Angle = 90°

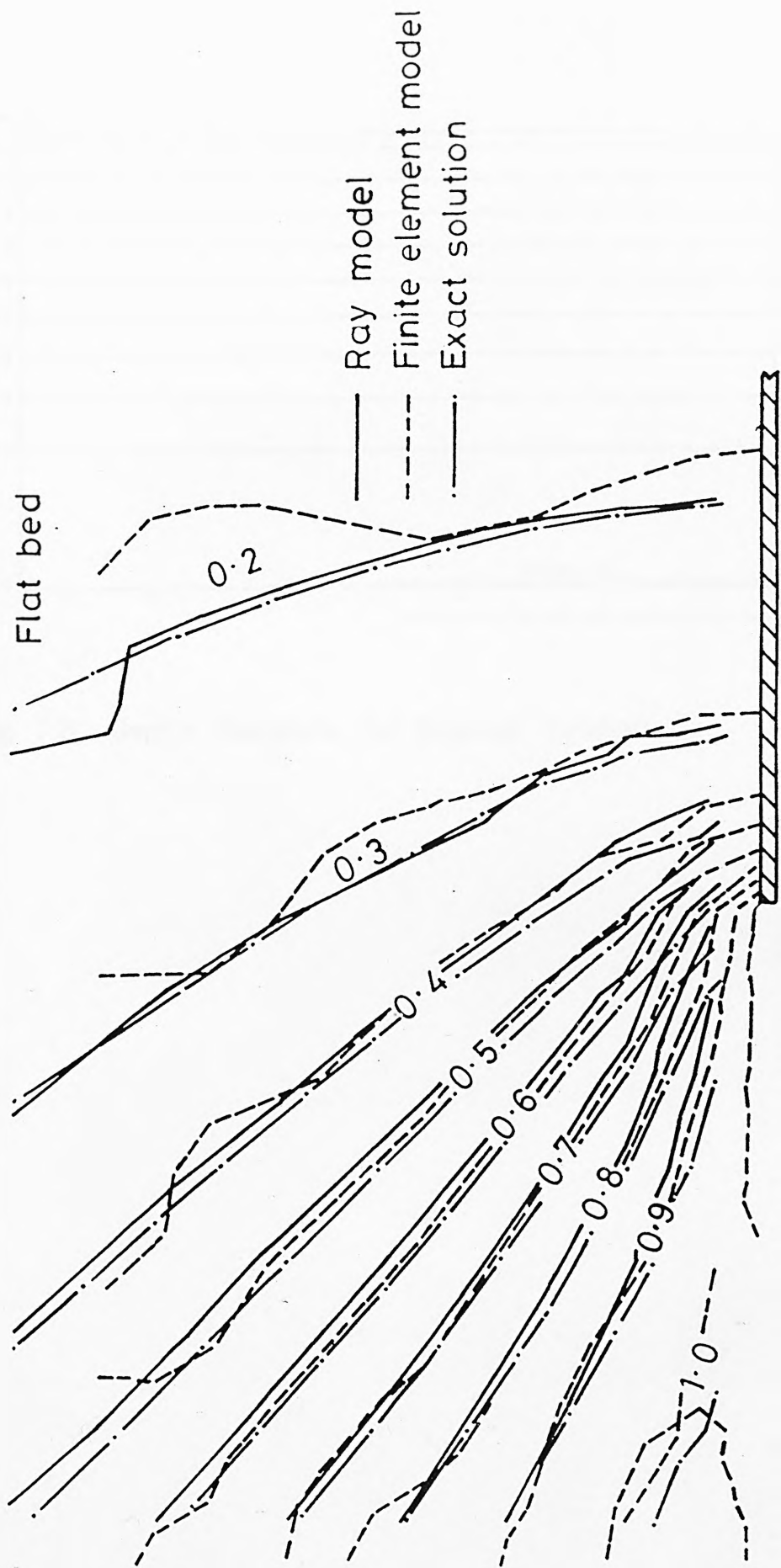


Fig 7.7 Wave Amplitude Contours. Flat Seabed. Period = 16s, Incident Angle = 135°

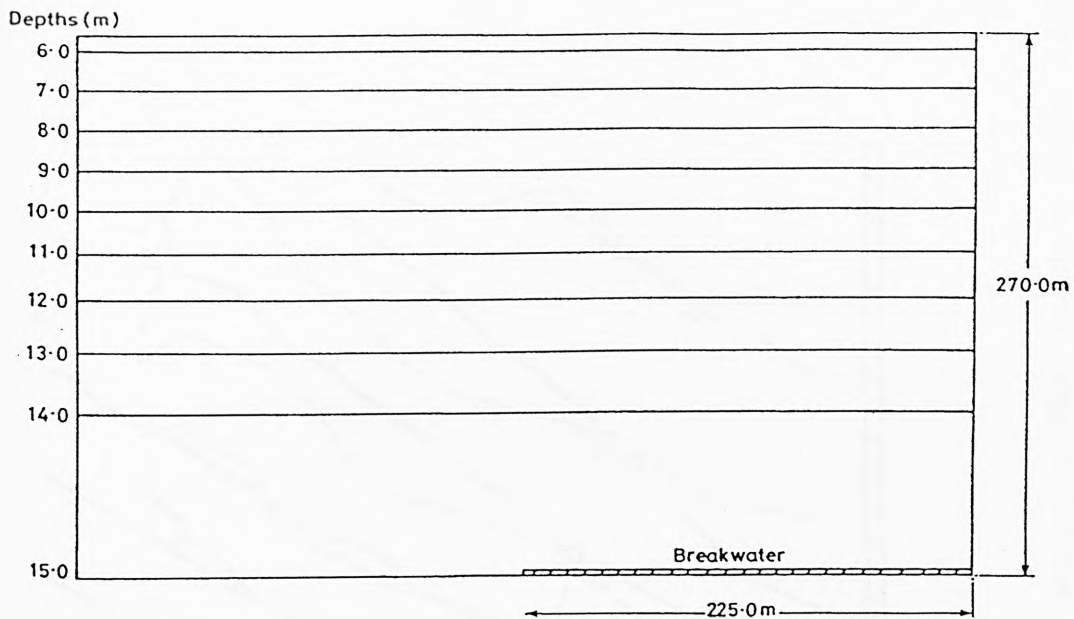


Fig 7.8 Depth Contours for Sloping Seabed

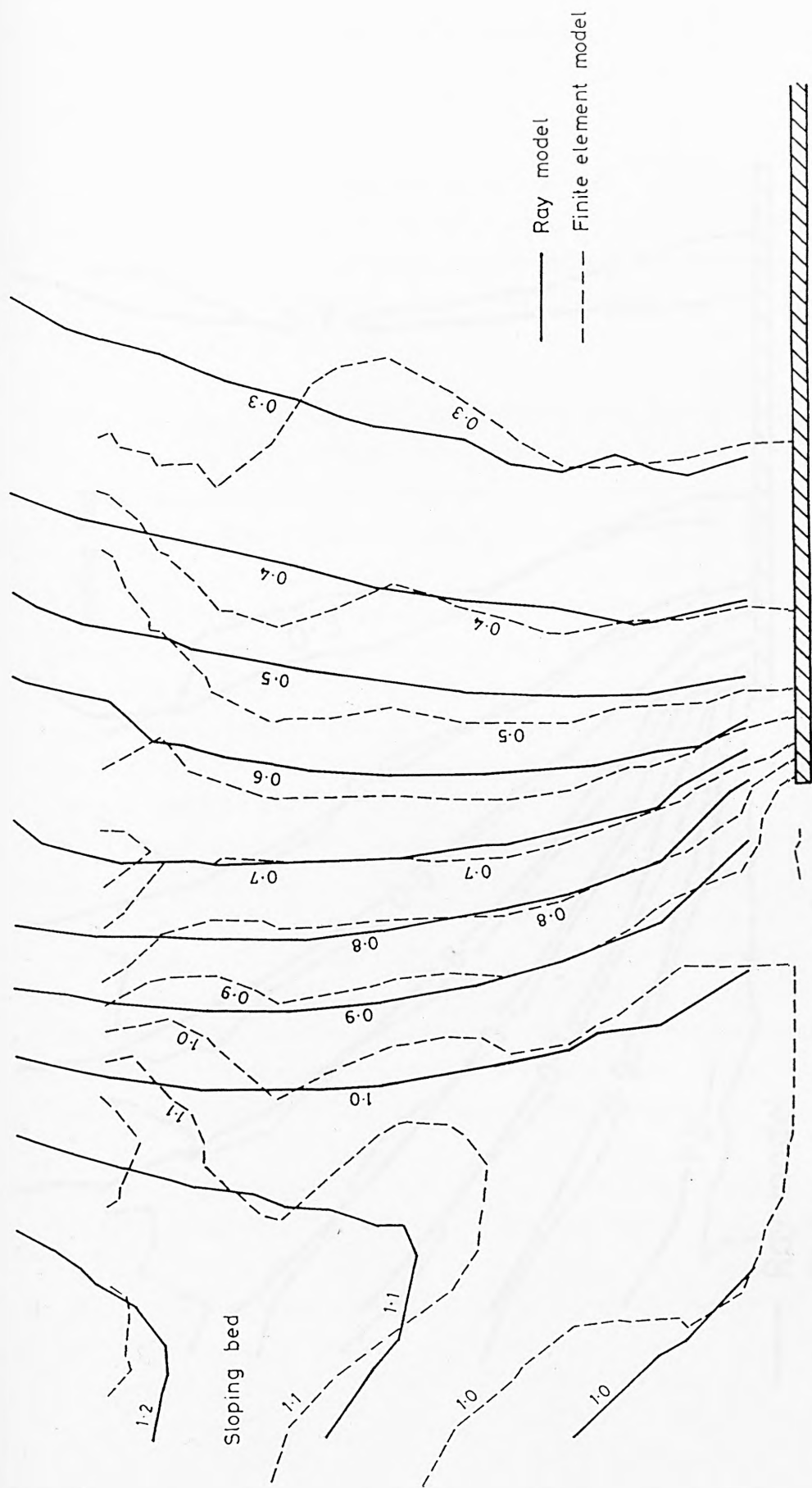


Fig 7.10 Wave Amplitude Contours. Sloping Seabed. Period = 16s, Incident Angle = 90°

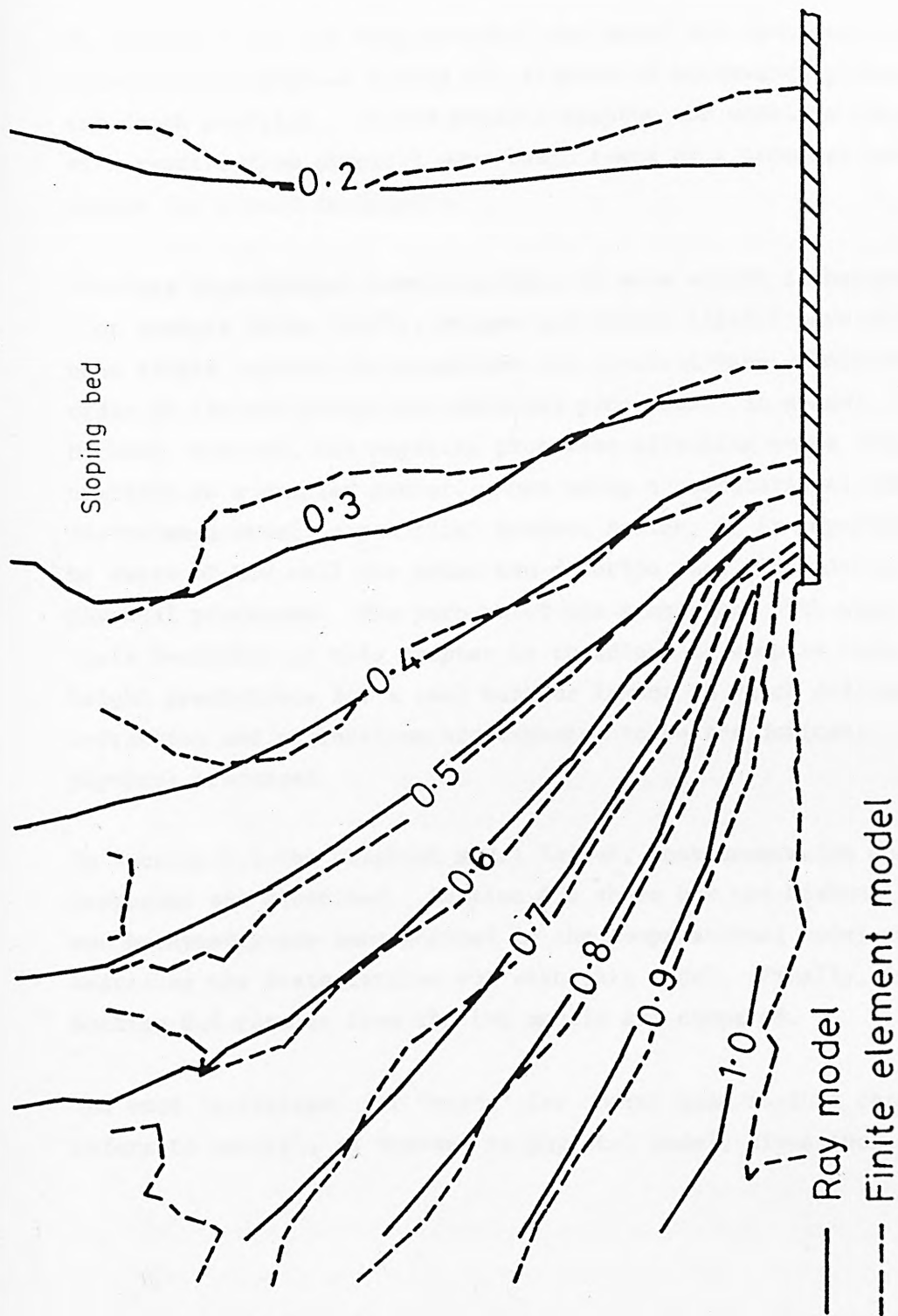


Fig 7.11 Wave Amplitude Contours. Sloping Seabed. Period = 16s, Incident Angle = 135°

8.1 Introduction

In Chapter 7 the new computational ray model was compared with alternative numerical models for simplified breakwater geometries and depth profiles. In the present chapter the model is compared with results from physical wave basin tests of a proposed harbour design for a port in Nigeria.

Previous experimental investigations of wave action in harbours (for example Harms (1979), Putnam and Arthur (1948)) have mainly used simple layouts, bathymetries and incident wave conditions in order to isolate particular physical processes. In a real harbour, however, the physical processes affecting waves are combined in a complex manner. When using a computational wave disturbance model in practical harbour design, it is important to be aware of how well the model can describe this combination of physical processes. The purpose of the comparison with wave basin tests described in this chapter is therefore to compare wave height predictions for a real harbour layout in which diffraction, refraction and reflections are expected to be the dominant physical processes.

In Section 8.2 the physical model layout, instrumentation and test programme are described. Section 8.3 shows how the harbour layout and bathymetry are incorporated in the computational model and describes the tests carried out with this model. Finally, in Section 8.4 results from the two models are compared.

The word 'prototype' (or 'proto' for short) used in this chapter refers to natural, as opposed to physical model, dimensions.

8.2 Physical Model

8.2.1 Experimental details

(a) Harbour layout

The rectangular basin used in this study measured 49m x 37.5m, and the model scale (undistorted) was 1 to 120. Fig 8.1 shows the model layout. A 15m-long random wave generator was aligned at an angle of 2 degrees to the shoreline to produce a representative wave direction for the swells that occur at the site. The natural seabed contouring was moulded in concrete from the tip of the main breakwater to the shoreline with an approach slope to this natural contouring from the wave-maker. Two dredged channels (-10.25m Chart Datum) were represented in the harbour area. Tests were performed at high water (+1.65m Chart Datum) in both the physical model and ray model. A high water level was chosen since the highest waves were expected at the site under these conditions.

The main breakwater was about 30m long (3.6km proto) and it extended from the shoreline in an arc out to a distance of 22m (2.6km proto) offshore. A smaller straight spur, 9m long (1.1km proto) was situated at a distance of 10m (1.2km proto) from the root of the main breakwater. A photograph of the model layout is shown in Fig 8.31.

(b) Breakwaters and harbour boundaries

The main breakwater in the model was constructed with a concrete core with stones grouted on top. Over most of its length the breakwater crest level was sufficient to prevent overtopping without modification. Side slopes were 1:2 on the sea side and 1:1.5 on the harbour side. The portion of the breakwater directly in front of the wave-maker was overlain by a layer of shingle and a polythene sheet which was held in position by a further shingle layer. This prevented transmission through and overtopping of the

breakwater. In plan, the head of the main breakwater was modified with two bricks at the apex and shingle on either side (Fig 8.2). It was found that wave heights in the physical model were insensitive to the method of construction of the tip of the breakwater.

The boundary at the shoreline and on the exposed side of the spur represented the natural beach slope of 1 in 7 (see Fig 8.1). The boundary on the sheltered side of the spur consisted of a steeper concrete slope with a cemented stone overlay.

(c) Instrumentation

The wave-maker was driven by an electro-hydraulic system with the electrical input being supplied by a wave spectrum synthesizer. The synthesizer allowed the generation of a repeating cycle of random waves with a specific wave frequency spectrum chosen by the experimenter. Measurements of wave heights were made using twin-wire probes (shown in Fig 8.31). Each probe had a pair of thin wires separated by 12.5mm and measured the water surface elevation by means of the electrical resistance between the wires.

(d) Wave spectrum

The wave frequency spectrum used represented swell wave conditions expected at the site with a return period of 100 years. The range of frequencies covered was 0.04 Hz proto (period = 25.0s) to 0.06815 Hz proto (period = 14.67s).

8.2.2 Method of performing physical model experiments

(a) Preliminary tests

Some preliminary tests were carried out with mono-frequency waves (period = 22.0s proto) to establish the general wave

characteristics of the wave-maker and harbour layout. The waves had a very non-linear appearance with short steep crests and long troughs. The presence of strong non-linear effects was borne out by the non-linear behaviour of the coefficients of wave heights inside the harbour. Large variations in these coefficients occurred for different incident wave amplitudes.

A spectral analysis of the incident wave revealed a large amount of energy at the second harmonic frequency. A typical result is shown in Fig 8.3. It is likely that a strong interaction developed between the second harmonic and the primary component. For example, second order interactions would produce components at the difference frequency which in this case equals the primary frequency. Such effects may explain the non-linear behaviour of wave height coefficients for positions inside the harbour.

Further preliminary tests were carried out to determine the variation in incident wave height along a wavefront for the wave frequency spectrum to be used in the main tests. For these preliminary tests pairs of probes were positioned along a line tangential to the tip of the main breakwater as described in Section 8.2.2c. The results showed a large increase in incident wave energy at a distance of about 2.5m (300m proto) from the breakwater tip. The possibility arose that some of this energy might diffract (by internal diffraction) away from the incident wave direction and into the harbour. To prevent this happening, a wave divider wall consisting of a 'dry' brick wall was built perpendicular to the wave-maker at a distance of 2.22m (266m proto) from the breakwater tip (Fig 8.1). The wave divider wall was sufficiently far from the breakwater tip for it not to interfere significantly with the wave diffraction process at the tip. The resulting wave height coefficients inside the harbour were similar to those found without the wave divider wall.

(b) Recording wave heights in the harbour

Fifteen wave probes were positioned in the harbour as shown in Fig 8.1. The probe positions were chosen to cover the whole harbour area and they included positions of particular interest such as ship berthing positions (Positions 11 and 13), and positions near or on channels and close to the harbour boundaries (Positions 4, 7, 8, 10, 14 and 15). All the probe positions corresponded to the centres of grid squares used in the computational ray model.

Values of the wave surface elevation at each probe in the harbour were sampled at regular intervals over a complete cycle of random waves. The signals were transmitted to a minicomputer for analysis. A fast Fourier transform program converted the surface elevation readings into spectral densities over a range of frequency bands. For the preliminary mono-frequency tests, 1024 surface elevation readings at a time interval of 0.2s were taken, giving a test duration of about 3m 25s. For the main frequency spectrum tests, 1024 surface elevation readings were taken at a time interval of 0.3035s giving a test duration of about 5m 10s. After smoothing, the Fourier transform program produced 32 frequency bands, each with a width of 0.0047 Hz.

(c) Recording incident wave heights

Pairs of probes were placed in the path of the incident wave, with the line joining each pair lying in the direction of wave travel. The probe separation was chosen to allow 'reflection' analysis to take place, ie determination of the incident and reflected wave spectra from the total spectra at the two probe positions. For these tests the probe pairs were positioned along the continuation of the line of the main breakwater at its tip (Fig 8.2). Along this line the influence of diffraction about the breakwater tip is small, thereby allowing determination of the incident wave close to the tip. Surface elevation readings were recorded from each

probe with the same incident conditions used for the harbour wave height tests.

(d) Main tests

The following tests were carried out with the breakwater at its original length:

1. Two harbour wave height tests
(synthesizer amplitude 130)
2. Two incident wave height tests
(synthesizer amplitude 130)
3. One harbour wave height test
(synthesizer amplitude 325)
4. One incident wave height test
(synthesizer amplitude 325)

The breakwater was then shortened by 1.94m (233m proto) and the breakwater head reconstructed to the original plan shape (Fig 8.2). The following tests were carried out:

1. Two harbour wave height tests
(synthesizer amplitude 130)
2. Two incident wave height tests
(synthesizer amplitude 130)

Two runs were performed for tests at synthesizer amplitude 130 in order to check the repeatability of the results. The positions of the probes used in the harbour tests are shown in Fig 8.1. The positions of the probe pairs used in the incident wave tests are shown in Fig 8.2a (original breakwater length) and Fig 8.2b (shortened breakwater length).

(e) Boundary reflection tests

Reflection analyses were performed to determine the reflection coefficients of the harbour boundaries in the model. These reflection coefficients were then used in the computational ray model.

It was expected that the reflection coefficients would vary with wave amplitude. In carrying out the reflection analyses, therefore, it was important to generate a wave amplitude close to the ones present at the boundaries in the harbour tests. The first task was therefore to determine an average significant wave height at each type of boundary. No reflection tests were carried out for the boundary on the sheltered side of the spur as this boundary was steeply sloping and the reflection coefficients can be expected to be close to unity (ie perfect reflection) for the frequencies generated. The correct significant wave height to use for the exposed side of the shingle covered section of the main breakwater would clearly be that generated by the wave-maker during the harbour tests. To obtain significant wave heights at the other boundaries, an array of probes was positioned close to the shoreline either side of the spur shown in Fig 8.4a. The wave-maker was set to generate the incident wave spectrum used in the harbour tests. The significant wave height at each probe and an average for each set of probes either side of the spur were calculated. The significant wave heights close to the main breakwater inside the harbour were assumed to be similar to those close to the sheltered part of the beach. Both the physical and computational ray model results showed that the significant wave heights close to the main breakwater were in fact slightly greater, but it was not thought that this would result in an important difference in the measured reflection coefficients of the main breakwater.

To measure the reflection coefficients, two 'dry' brick wave guides, 5m long, were built outwards at right angles to the

shoreline. Two probes were positioned between the wave guides on a line perpendicular to the shoreline. The wave synthesizer amplitude was adjusted (while maintaining the same spectral shape) to give a significant wave height at one of the probes as similar as possible to the average found in the probe array tests. A reflection analysis was then carried out. In this way, reflection coefficients were obtained for each frequency band for the exposed and sheltered parts of the shoreline. To determine reflection coefficients for the main breakwater, a section of the breakwater was positioned at the shoreward end of the wave guide area in front of the shoreline and a reflection analysis performed.

The reflection coefficients for the seaward side of the shingle covered part of the main breakwater were obtained by building dry brick wave guides outwards from the wave-maker with the shingle covered breakwater section positioned at the far end of the wave guides at the correct distance from the wave-maker. A wave reflection analysis was then carried out using two probes positioned in front of the breakwater within the wave guides.

8.2.3 Analysis of Data

(a) Repeatability of results

The repeat runs of the harbour tests (wave probes inside harbour) agreed well with the initial runs for both the original and shortened breakwater lengths. Good agreement was again obtained between the initial and repeat runs of the shortened breakwater length incident wave tests. The two runs for the original breakwater length incident wave tests did not show particularly good agreement. However, as the first run was carried out in freezing conditions it was thought that results for that run were unreliable. Hence only the second set of results were used to establish incident heights for the comparison with the computational

ray model. Since the other tests demonstrated repeatability it seemed reasonable to assume that the original breakwater length incident wave tests were also repeatable.

(b) Determination of incident wave heights

It was found during the preliminary tests that there were large variations in incident wave height at different points along a wave front. Even after the wave divider wall had been introduced in the model it was found that there was a considerable variation in incident wave height between the tip of the main breakwater and the wave divider. An averaging technique was required to obtain a representative value of the incident wave height. The method that was used is based on recognising that only the part of the incident wave close to the breakwater tip contributes significantly to the diffracted wave inside the harbour. The 'cut-off' point beyond which the contribution of the incident wave is insignificant depends on the wavelength and will therefore be different for each frequency band in the spectrum. The cut-off criterion was determined by considering the related problem of diffraction through a gap between two straight collinear breakwaters. Penney and Price (1952) have shown that for waves incident normally on a gap of one wavelength the effect of one breakwater on the diffracted field of the other is less than 2%. The effect of a variation in incident wave height, instead of the second breakwater, is unlikely to be worse.

This consideration however does not take into account any internal diffraction of the incident wave. The results show that, for the region between the main breakwater tip and wave divider, internal diffraction is negligible.

For the purpose of calculating an average incident wave height, the probe pairs used in measuring the incident wave were regarded as 'covering' a length of wave front. These lengths of wave front were bounded by the mid-points between

probe pairs, the main breakwater tip or the one wavelength limit from the tip, as appropriate. Measurements of the incident spectral density were used from those probe pairs whose 'covered' portion of wave front lay within the one wavelength limit. These measurements were weighted by the length of wave front covered, and an average was formed. Fig 8.5 shows the average spectral density at each frequency band in the incident wave spectrum for both the original and shortened breakwater lengths (synthesizer amplitude 130). It should be emphasized that the same signal was fed to the wave-maker for both layouts and that the differences in the incident wave spectrum arose from wave height variations along wave crests.

(c) Calculation of wave height coefficients

Wave height coefficients in the harbour were calculated for each frequency band in the spectrum by the formula

$$K = \left(\frac{\text{Spectral density at harbour probe position}}{\text{Spectral density of incident wave}} \right)^{\frac{1}{2}} \quad 8.2.1$$

An average of the initial and repeat runs of the harbour tests, and an average incident spectral density obtained by the method described in the previous section were used.

The significant wave height coefficient over the whole spectrum (ie the ratio of the significant wave height at a harbour position to that of the incident wave) was calculated by

$$K_s = \left(\frac{\sum \text{Spectral density at harbour probe position}}{\sum \text{Spectral density of incident wave}} \right)^{\frac{1}{2}} \quad 8.2.2$$

where the summations are taken over the frequency bands.

8.3 Computational Ray Model

8.3.1 Representation of the harbour layout

A depth grid composed of square elements with side lengths of 70m proto was constructed. The depth data is shown as a contour plot in Fig 8.6. In compiling this data it was important to represent the sides of the channels as accurately as possible as they have an important effect on the refraction of rays. The representation of the breakwaters and harbour boundaries is also shown in Fig 8.6.

The reflection coefficients of the harbour boundaries at each frequency band were measured experimentally (Section 8.2.2e). These reflection coefficients were plotted against period and a best straight line drawn (Figs 8.7 - 8.10). The reflection coefficients used in the ray model were obtained from these straight line fits.

The ray model contains the options of including the effects of wave phase, or of treating the relative phases as random, when a reflected wave is superimposed with a direct wave (Chapter 6). Including the effects of the wave phase results in an interference pattern with rapid spatial variation of wave height. It is envisaged that this option is only used when the phase change undergone on reflection is known accurately and when monochromatic waves are used. In most practical applications, including the present investigation, neither condition is generally true. Reflected waves were therefore treated as having random phases relative to the direct wave, and the procedure for combining the two waves is to add their energies. This procedure tends to give a smooth spatial variation of wave height.

When a ray undergoes multiple reflections, the ray path becomes increasingly inaccurate in representing the actual wave path and wave field. In this investigation, rays were stopped after they had been reflected three times. It was found that the wave energy

lost by doing this was negligible in comparison with the residual wave energy. Fig 8.26 shows a typical ray path diagram.

8.3.2 Layouts tested with the computational ray model

Using the original breakwater length, separate runs were performed for the seven frequency bands in the physical model wave spectrum. Results from these single frequency runs were later used to obtain significant wave height coefficients for the whole spectrum.

Two further runs were performed at the zero-crossing and median wave periods with the original breakwater length.

Three runs were performed at a period 22.4s proto with the original breakwater length to determine the separate effects of refraction and reflections. These runs had respectively:

1. the original harbour bathymetry and no reflections from the boundaries;
2. a constant depth harbour (12.05m proto which is the depth at high water at the main breakwater tip) with reflecting boundaries;
3. A constant depth harbour with no reflections from the boundaries;

A single run was performed with the shortened breakwater length at the median wave period. All runs were carried out with prototype dimensions.

8.4 Results

8.4.1 Comparison of physical model results at two synthesizer amplitudes

Table 8.1 shows the incident spectral densities of each frequency band measured at the six probe pair positions in the physical

model (Fig 8.2). Table 8.1a gives results for synthesizer amplitude 130 and Table 8.1b results for a higher synthesizer amplitude 325.

Table 8.2 shows significant wave height coefficients at the harbour probe positions calculated over the whole spectrum for the two synthesizer amplitudes. The agreement is reasonably close, apart from Positions 2 and 3. It appears that at these positions for the lower synthesizer amplitude, the combination of various non-linearities had the effect of creating an abnormally high wave. Further evidence for this is shown in the comparison of the original breakwater length and shortened breakwater length results, discussed in Section 8.4.3. At other probe positions, the variations in significant wave height coefficients between the two synthesizer amplitudes are generally considerably smaller than variations in wave height coefficient between frequency bands. This more linear behaviour indicates that the non-linear second harmonic effects described in Section 8.2.2a for mono-frequency waves were less important for random waves.

8.4.2 Comparison of results for individual frequency bands

Figs 8.11-8.25 show the physical and computational model wave height coefficients at each probe position in the harbour for the original breakwater length. These figures show plots of the wave height coefficients at each frequency band against the square root of wavelength. It can be anticipated that the effects of diffraction in the far field will result in a linear relationship between wave height and the square root of the wavelength. Thus, we would expect the figures to show this linear relationship, distorted somewhat by the effects of refraction, reflections and genuine non-linear effects. It is noticeable that at all the probe positions, the ray model results display this relationship very well. The physical model results show considerable scatter, although the trend is always towards larger wave heights at the lower wavelengths. It might be thought that this scatter was caused by reflected waves interfering with the straight diffraction contribution. However, the relatively long path

lengths for reflections make this explanation unlikely given the band width for components in the spectrum. This scatter of results can therefore be attributed to the strong influence of some non-linear wave effect although the exact effect is not known. Even so, the scatter is of the same order as that found in other experiments (Putnam and Arthur (1948), Harms (1979)) which were conducted under simpler conditions of harbour layout and incident wave.

When the individual frequency band wave heights are combined to give significant wave height coefficients (Table 8.3), it is found that at four positions (8,9,14,15) the ray model gives larger wave heights, while at the remaining eleven positions the physical model results are larger. This fact is also generally apparent from Figs 8.11-8.25. The four positions where the ray model gives larger wave heights are all in the harbour area sheltered by the spur where the influence of local refraction and reflection effects is strong. It is possible that at some of these positions internal diffraction on a scale larger than the grid size is taking place.

The effects of internal diffraction are not incorporated in the model other than by ray averaging. Generally, however, this fact would not seem to be too important since better agreement between the two models is obtained at positions in the shelter of the spur (where internal diffraction is likely to be more significant) than at positions between the tip of the spur and the main breakwater. At these latter positions and particularly at Positions 2 and 3, the physical model wave heights are considerably higher. It would seem that the combination of various non-linear effects in this region causes a general increase in wave height as well as a redistribution of energy between frequency bands.

The regularity of the ray model wave heights over the frequency range at all the probe positions suggests that a single run at a suitably chosen average frequency could give very similar results to the significant wave height coefficients calculated for the

whole spectrum. Runs at the peak period (22.40s proto) and zero-crossing period (19.43s proto) were found to give wave heights which were too high and too low respectively (see Table 8.4). This was because the peak period tends to occur at the lower frequency (higher period) end of the spectrum while the zero-crossing period gives undue weight to the higher frequencies. A run at the median frequency (20.02s proto) was shown to give very close agreement with the spectral results (Table 8.4). This finding indicates that even for complex harbour layouts such as the one studied here, a single run at the median frequency is sufficient, and considerable saving in computing effort is achieved. However, in some practical applications for which there are irregular offshore frequency spectra or particularly complex features of the harbour layout, it may be advisable to use a full frequency spectrum.

8.4.3 Comparison of results for original and shortened breakwater lengths

It would be expected that the shortened breakwater length would have the effect of generally increasing wave heights in the harbour. A comparison of the physical model significant wave height coefficients for the two breakwater lengths (Table 8.3 columns 1 and 4) shows this to be the case, apart from Positions 2,3,8 and 11 where decreases in the significant wave height coefficient for the shortened breakwater length are seen. This, at first sight, seems surprising since Positions 2 and 3 are relatively close to the main breakwater tip and the dominant wave effect would be expected to be diffraction by the main breakwater. However, it was at Positions 2 and 3 that the wave height versus frequency plots for the original breakwater lengths (Figs 8.13 and 8.14) showed the greatest deviation from the ray model results. It is concluded therefore that the various sources of non-linearities had the effect of creating an abnormally high wave at these two positions with the original breakwater length. Positions 8 and 11 are at the edge of a channel close to a boundary. Strong local refraction and reflection effects could

therefore be expected to account for the decrease in significant wave height with the shortened breakwater.

As discussed in Section 8.4.2 the ray model significant wave heights over the whole spectrum were very similar to wave heights from a single run at the median frequency (20.02s proto). These results are reproduced in columns 2 and 3 of Table 8.3. It was not thought necessary to perform individual runs to cover the whole spectrum for the shortened breakwater length, and that a single run at the median frequency (19.25s proto) would be satisfactory. Wave height coefficients for this run are shown in column 5 in Table 8.3. A comparison of column 5 with column 2 or 3 shows that the ray model gives an increase in significant wave height coefficient for the shortened breakwater length at all positions apart from Positions 8, 11 and 14. These positions are well into the shelter close to the harbour boundaries and channels. It can be expected that any increase in wave height at these positions due to the shortening of the main breakwater could be more than compensated by important local refraction and reflection effects.

The average increase in height predicted by the ray model for all the probe positions inside the harbour was 15% for the median frequency. This compares very well with an average increase of 15% in the significant wave height coefficients obtained from the physical model.

8.4.4 Separate effects of refraction and reflections

Table 8.5 shows the separate contributions made to the total wave height by refraction and reflections in the ray model with the original breakwater length. The four columns in Table 8.5 show wave height coefficients with a wave period of 22.40s. Columns 1 to 4 show respectively results for:-

1. Reflections and refraction
2. Refraction and no reflections

3. Reflections and no refraction
4. No reflections and no refraction

Column 4 contains the effects of diffraction around the main breakwater and spur only. Results in this column are those that would be obtained in a simple desk study of diffraction.

Comparison of columns 1 and 2 (or 3 and 4) show the effect of reflections. At all probe positions, and particularly in the shelter of the spur, the effect of reflections is very important. At Positions 11, 12 and 14 the reflected waves increase the wave height by almost four times.

Comparison of columns 1 and 3 (or 2 and 4) show the effect of refraction. Refraction can be seen to have an important effect at positions 4, 8, 9, 12 and 15 which are all close to channels or are reached by rays crossing channels. At position 12, the presence of refraction reduces the wave height by half for the case of no reflections.

It is noticeable that the effects of refraction at any position, firstly with reflections present (compare columns 1 and 3) and secondly with reflections absent (compare columns 2 and 4), are different. At Position 12 the effect of refraction when reflections are present is to decrease the wave height slightly. As stated above, however, when there are no reflections the effect of refraction is to reduce the wave height by half. This phenomenon is explained by the fact that reflected rays will themselves undergo refraction. Similar considerations apply to a comparison of the effects of reflections when refraction is present and absent.

Figs 8.27-8.30 give wave height contour plots over the whole harbour for the four cases considered.

(a) Synthesizer amplitude 130

Distance from end of bwtr (m)	Period (s)						
25.03	22.40	20.26	18.50	17.02	15.76	14.67	
0.62	1.997	4.276	3.489	2.332	2.075	1.416	0.765
0.92	1.812	4.442	3.297	2.927	2.485	1.350	1.029
1.22	2.687	6.470	4.793	4.856	4.565	2.215	1.285
1.52	2.615	6.932	6.812	6.587	5.549	2.100	1.356
1.82	2.187	7.523	7.110	6.794	5.044	1.648	1.065
2.12	2.696	7.232	7.742	7.826	5.165	1.520	0.877

(b) Synthesizer Amplitude 325

Distance from end of bwtr (m)	Period (s)						
25.03	22.40	20.26	18.50	17.02	15.76	14.67	
0.62	9.522	16.096	13.885	9.047	6.865	3.095	1.548
0.92	8.531	17.814	16.434	14.404	10.797	4.050	2.632
1.22	8.465	20.186	23.838	20.607	17.122	4.569	3.908
1.52	7.514	22.975	26.403	28.392	23.899	4.236	4.565
1.82	9.140	28.344	32.747	30.635	24.501	5.520	5.413
2.12	8.861	30.047	37.724	33.249	30.167	5.026	5.578

TABLE 8.1

Incident wave spectral densities for two synthesizer
amplitudes (original breakwater length)

Significant wave height coefficients

Probe position	Synthesizer amplitude	
	130	325
1	0.228	0.198
2	0.207	0.159
3	0.189	0.121
4	0.0808	0.0682
5	0.207	0.199
6	0.157	0.153
7	0.0860	0.100
8	0.0719	0.0607
9	0.0965	0.0890
10	0.134	0.122
11	0.0789	0.0692
12	0.0707	0.0708
13	0.0425	0.0472
14	0.0576	0.0589
15	0.0670	0.0656

TABLE 8.2

Physical model significant wave height coefficients for two synthesizer amplitudes (original breakwater length)

Original breakwater length				Shortened breakwater length	
Probe Position	Physical	Ray model	Ray model	Physical	Ray model
	model	significant	median	model	median
	significant	wave	period	significant	period
	wave	height	(20.02s)	wave	(19.25s)
	height	coefficient	wave	height	wave
	coefficient		height	coefficient	height
			coefficient		coefficient
1	0.228	0.210	0.210	0.331	0.282
2	0.207	0.133	0.130	0.180	0.158
3	0.189	0.0975	0.0910	0.145	0.111
4	0.0808	0.0628	0.0627	0.0954	0.0690
5	0.207	0.167	0.168	0.243	0.207
6	0.157	0.103	0.103	0.167	0.124
7	0.0860	0.0835	0.0861	0.121	0.0882
8	0.0719	0.123	0.122	0.0706	0.0990
9	0.0965	0.0991	0.0993	0.124	0.120
10	0.134	0.0989	0.0990	0.139	0.125
11	0.0789	0.0598	0.0598	0.0765	0.0491
12	0.0707	0.0570	0.0575	0.0841	0.0830
13	0.0425	0.0206	0.0200	0.0514	0.0240
14	0.0576	0.0613	0.0589	0.0720	0.0553
15	0.0670	0.0923	0.0902	0.105	0.110

TABLE 8.3

Physical model and ray model significant wave height coefficients

Probe position	Significant Wave height coefficient	Wave height coefficients		
		Peak period 22.40s	Median period 20.02s	Zero-crossing period 19.43s
1	0.210	0.228	0.210	0.203
2	0.133	0.138	0.130	0.127
3	0.0975	0.0965	0.0910	0.0892
4	0.0628	0.0673	0.0627	0.0614
5	0.167	0.182	0.168	0.164
6	0.103	0.111	0.103	0.101
7	0.0835	0.0894	0.0861	0.0838
8	0.123	0.133	0.122	0.119
9	0.0991	0.107	0.0993	0.0960
10	0.0989	0.108	0.0990	0.0967
11	0.0598	0.0661	0.0598	0.0580
12	0.0570	0.0607	0.0575	0.0552
13	0.0206	0.0230	0.0200	0.0191
14	0.0613	0.0702	0.0589	0.0571
15	0.0923	0.0998	0.0902	0.0889

TABLE 8.4

Ray model significant wave height coefficients and single
frequency runs (original breakwater length)

Ray model period = 22.40s

Probe Position	With reflections and refraction	With refraction no reflections	With reflections no refraction	No reflections no refraction
1	0.228	0.120	0.230	0.119
2	0.138	0.0724	0.161	0.0731
3	0.965	0.0743	0.103	0.0739
4	0.0673	0.0404	0.104	0.0593
5	0.182	0.108	0.181	0.107
6	0.111	0.0837	0.111	0.0824
7	0.0894	0.0695	0.0971	0.0839
8	0.133	0.0491	0.122	0.0558
9	0.107	0.0761	0.118	0.0839
10	0.108	0.0628	0.140	0.0650
11	0.0661	0.0172	0.0644	0.0175
12	0.0607	0.0175	0.0743	0.0338
13	0.0230	0.0084	0.0266	0.0092
14	0.0702	0.0178	0.0665	0.0170
15	0.0998	0.0453	0.134	0.0488

TABLE 8.5

Effects of refraction and reflections in the ray model

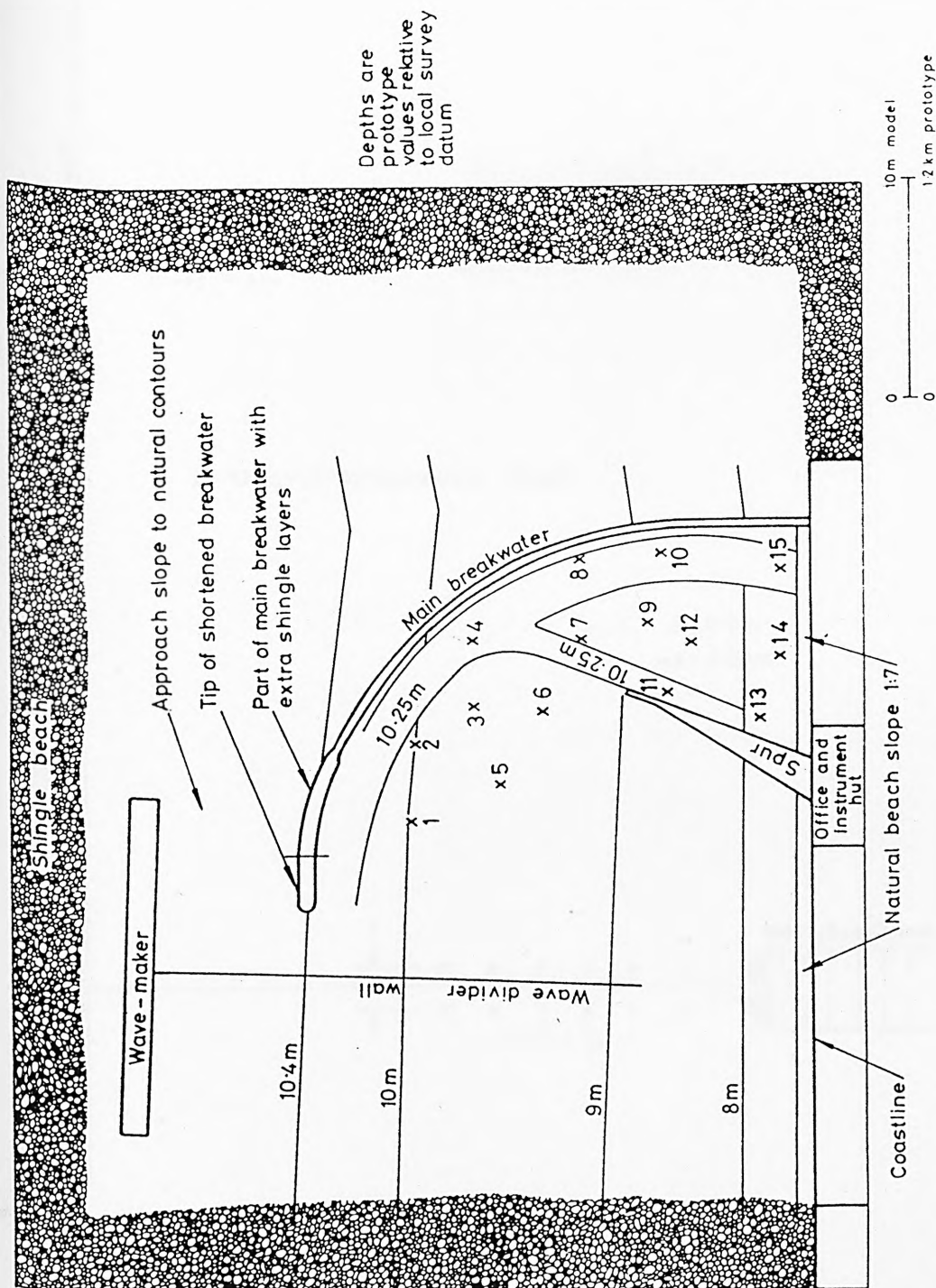
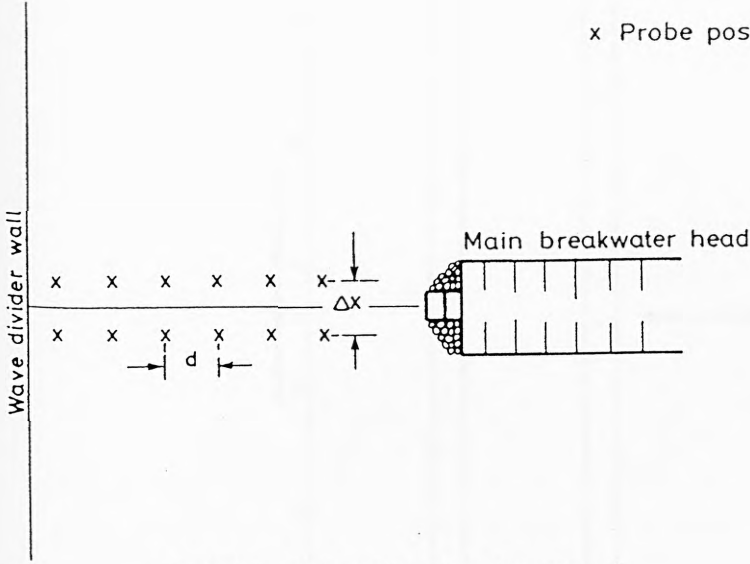


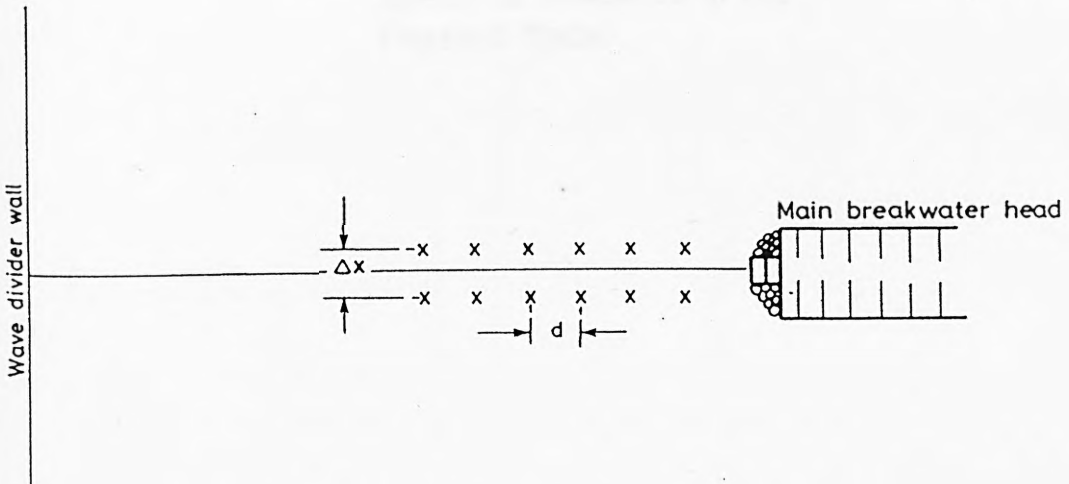
Fig 8.1 Physical Model Layout, Bathymetry and Probe Positions

x Probe positions



a) Original breakwater length

$d = 0.3\text{m}$
 $\Delta x = 0.29\text{m}$



b) Shortened breakwater length

Fig 8.2 Positions of Probes for Incident Wave Measurements

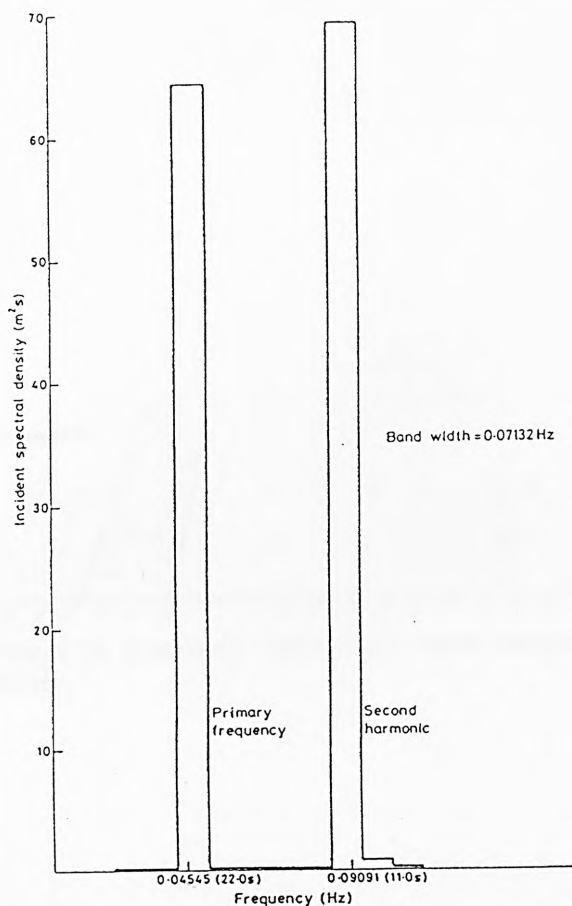
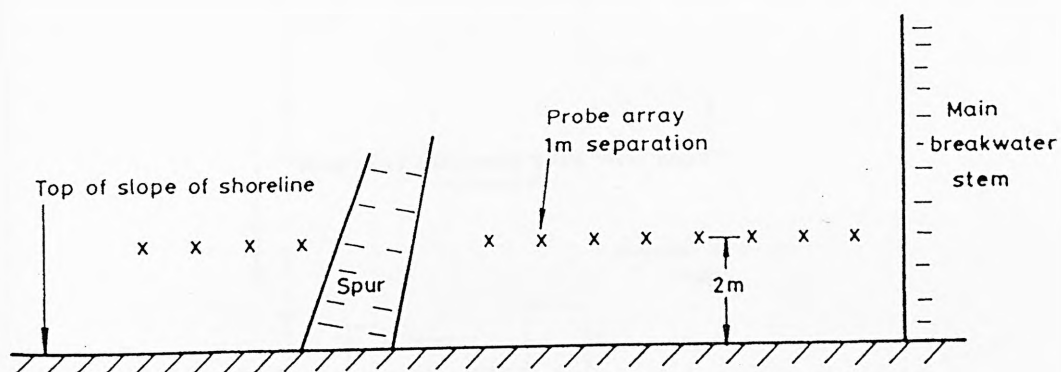
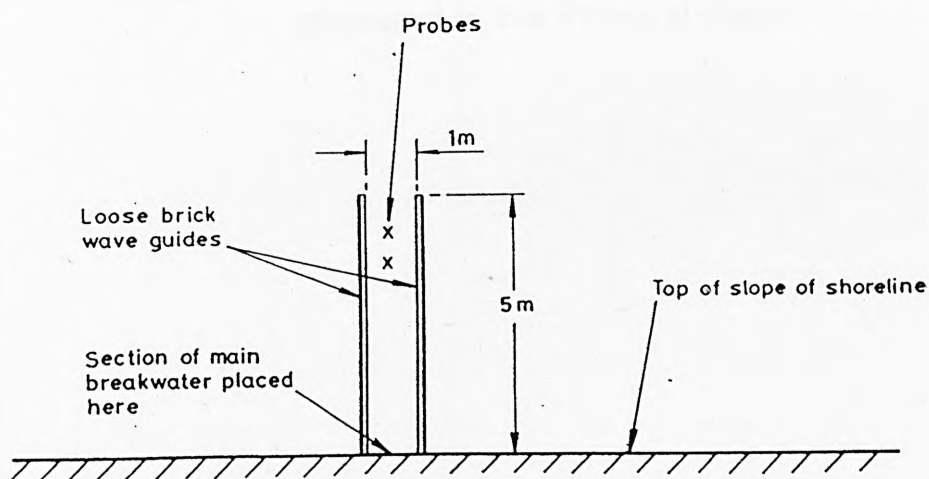


Fig 8.3 Single-Frequency Incident Wave Spectrum measured in the Physical Model



(a) Probe arrays to measure significant wave height at shoreline



(b) Experimental arrangement to measure reflection coefficients

Fig 8.4 Experimental Arrangements for Boundary Reflection Tests

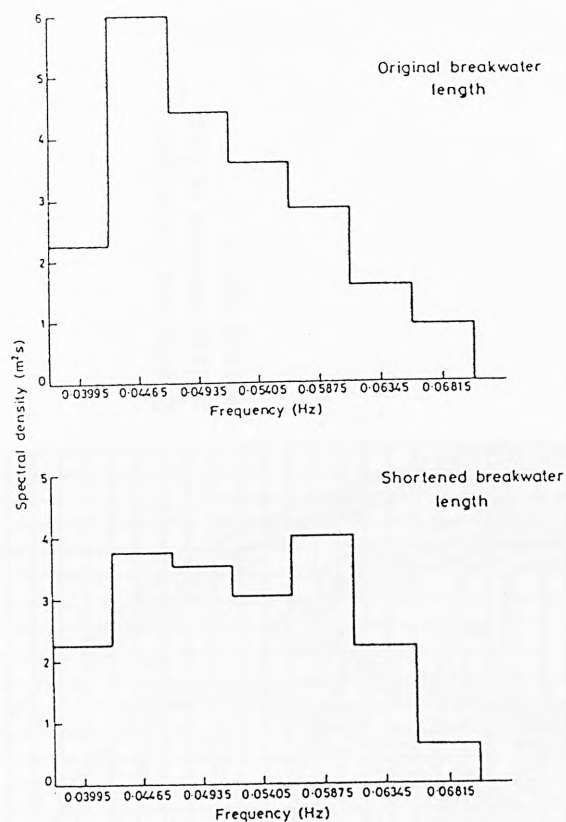
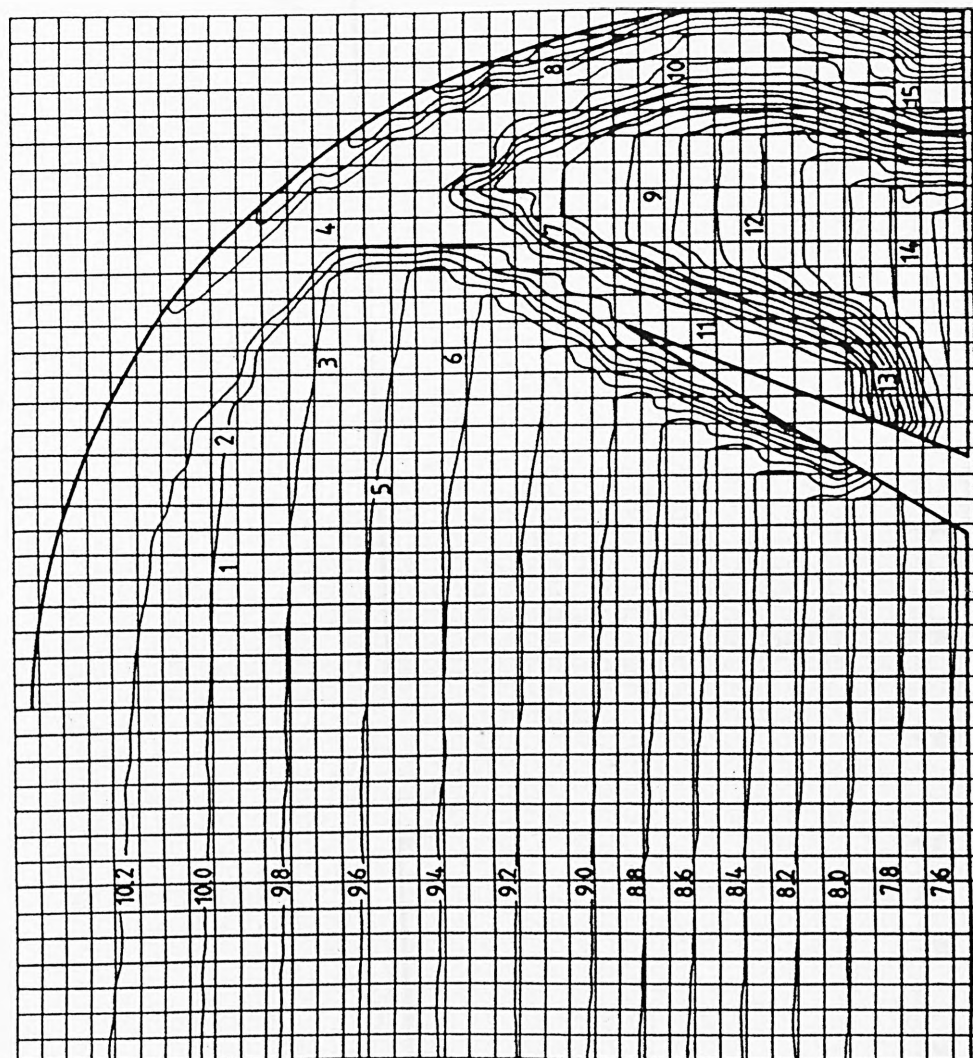


Fig 8.5 Average Incident Wave Spectra measured in the Physical Model



Depths are prototype
metres relative to local
survey datum

Fig 8.6 Ray Model Layout and Bathymetry showing the Physical
Model Probe Positions

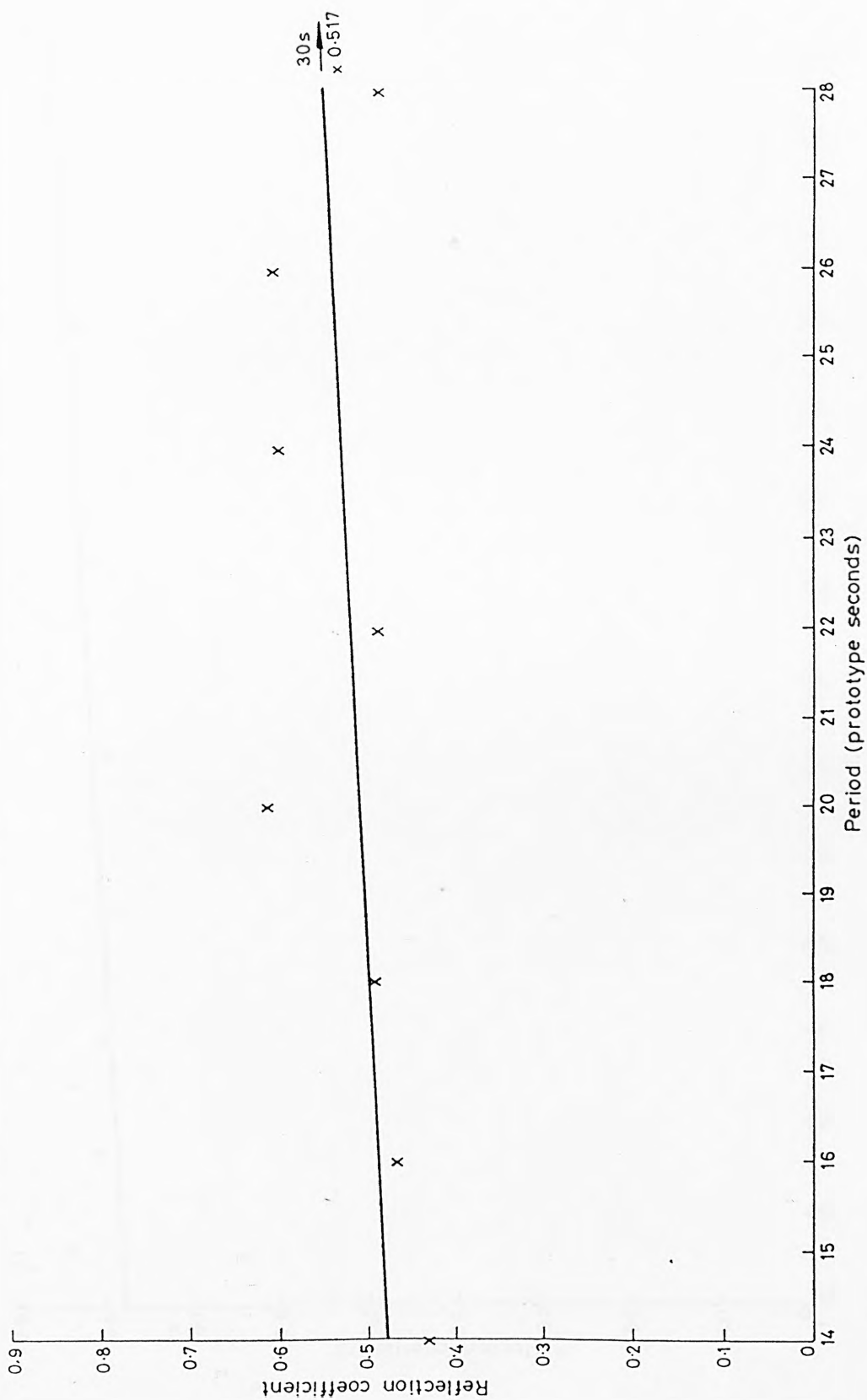


Fig 8.7 Physical Model Reflection Coefficients. Exposed Side of Main Breakwater

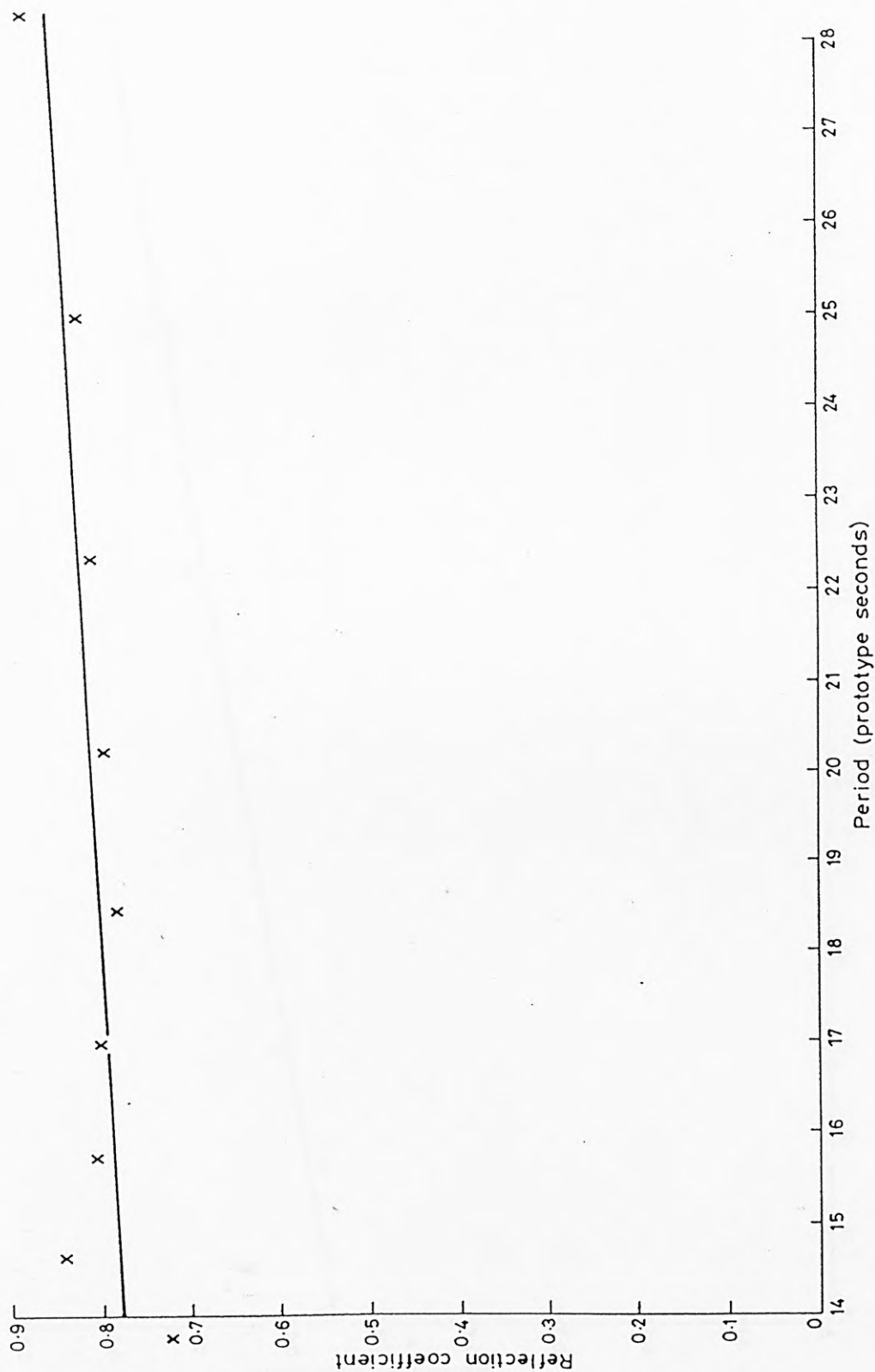


Fig 8.8 Physical Model Reflection Coefficients. Sheltered Side of Main Breakwater.

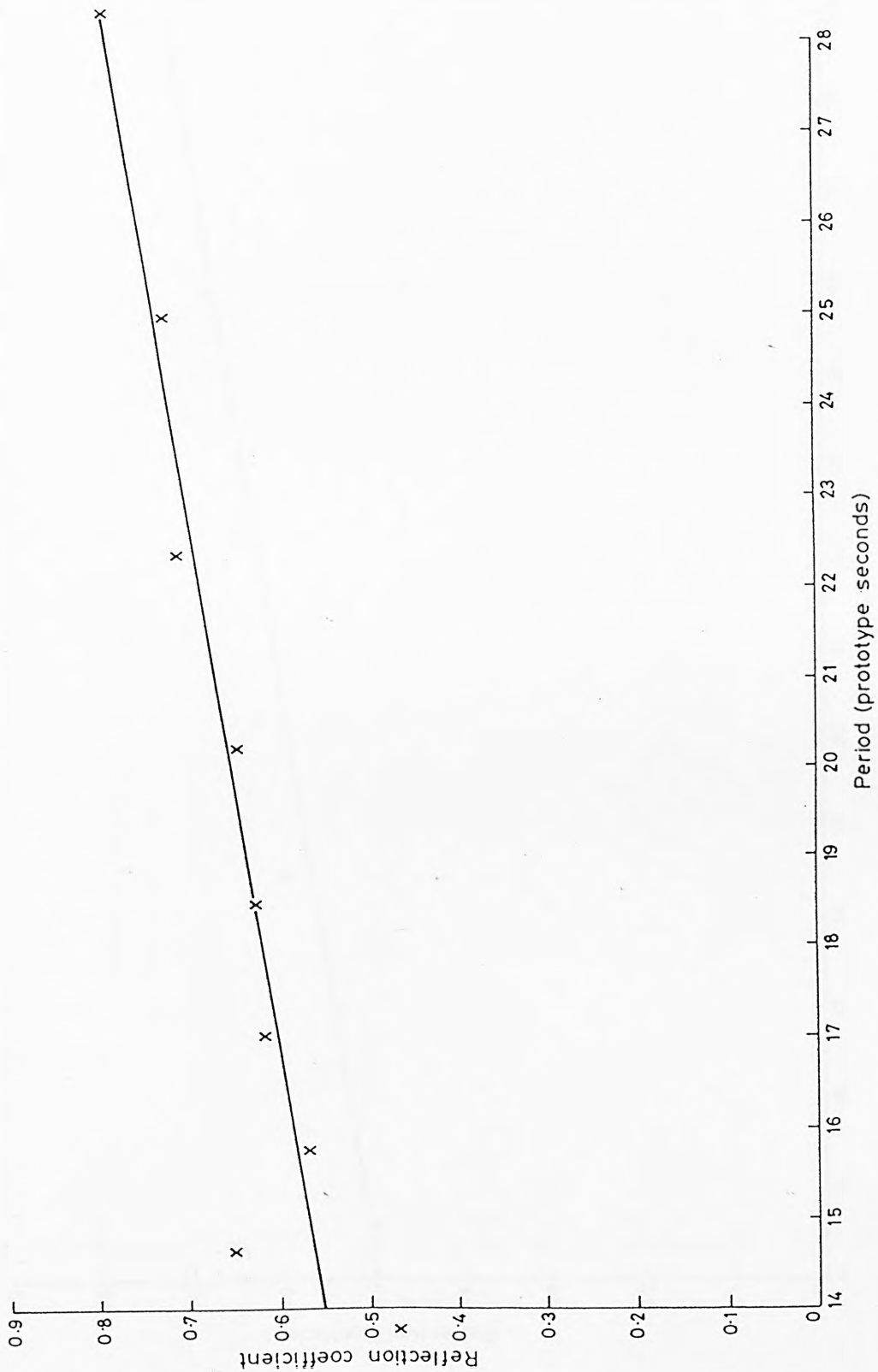


Fig 8.9 Physical Model Reflection Coefficients. Exposed Shoreline

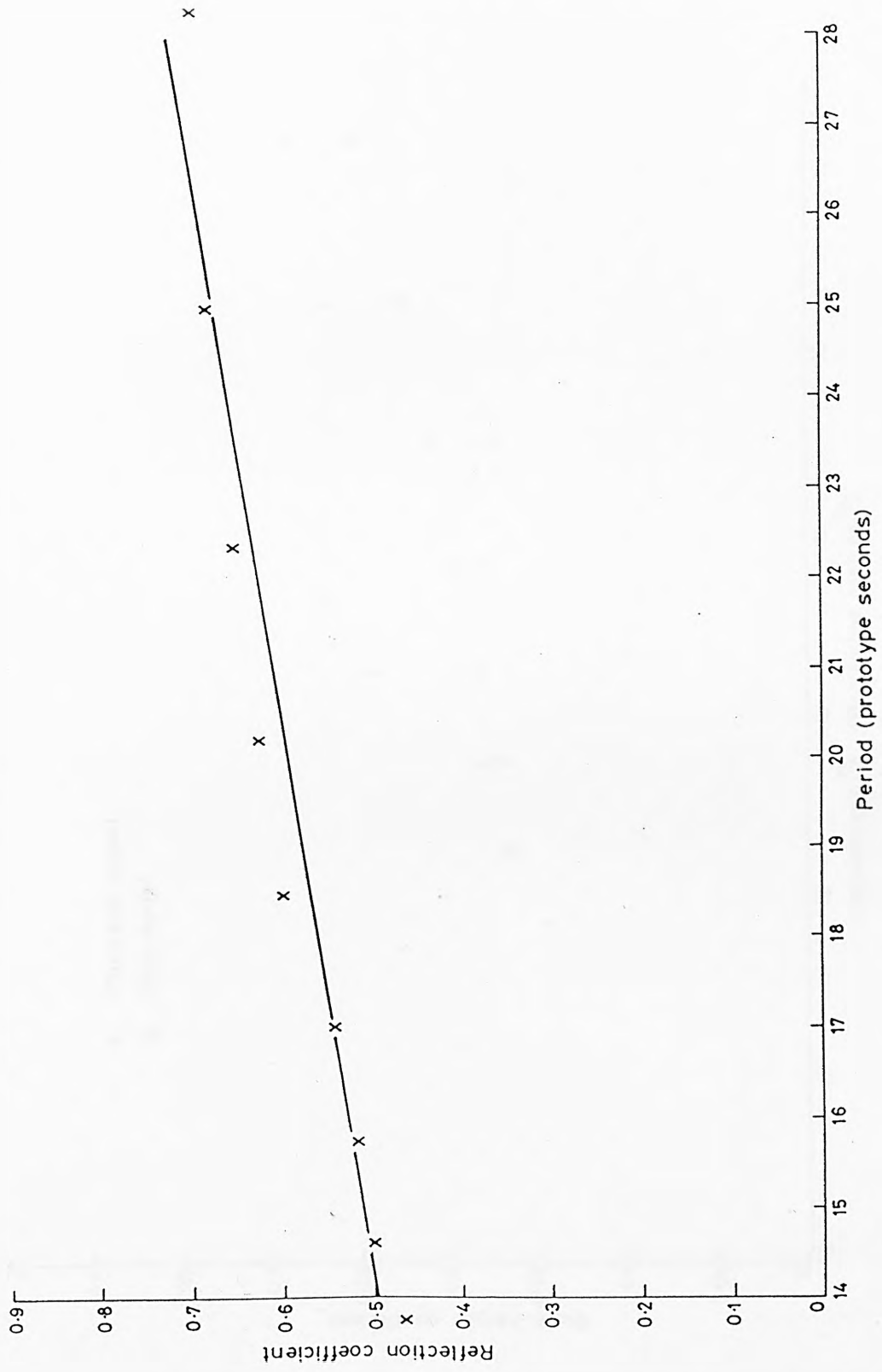


Fig 8.10 Physical Model Reflection Coefficients. Sheltered Shoreline.

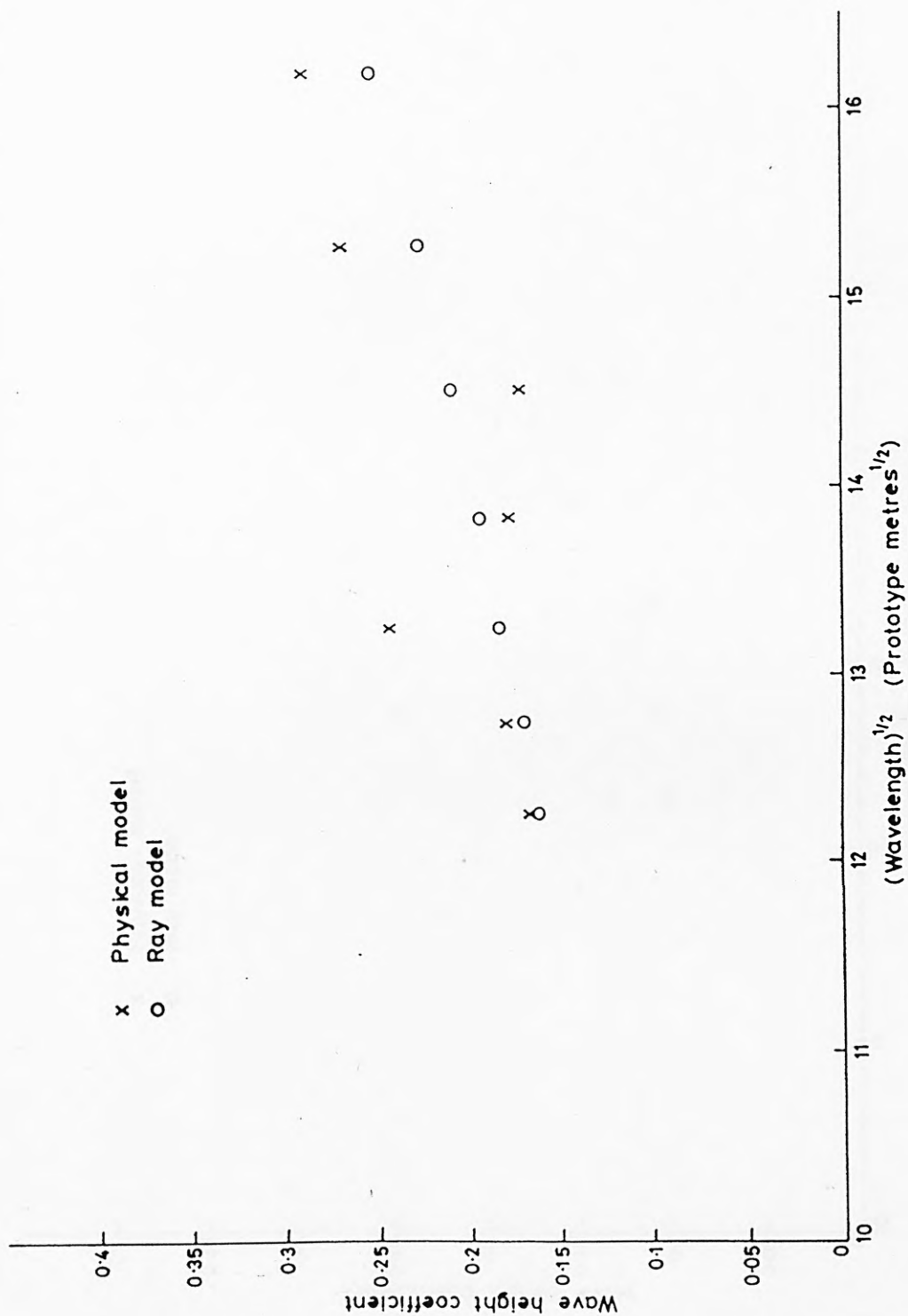


Fig 8.11 Wave Height Coefficients. Original Breakwater Length. Position 1

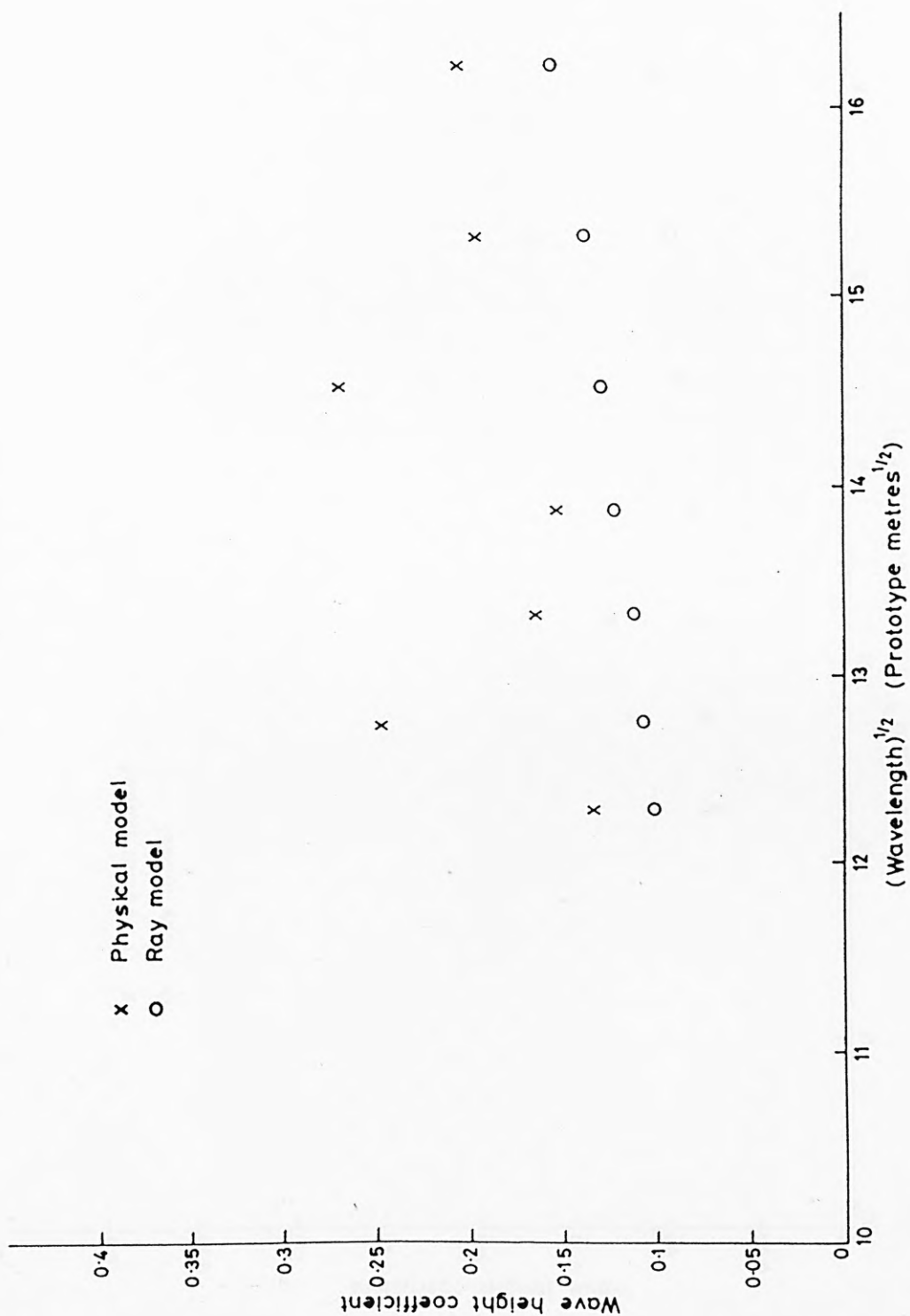


Fig 8.12 Wave Height Coefficients. Original Breakwater Length. Position 2

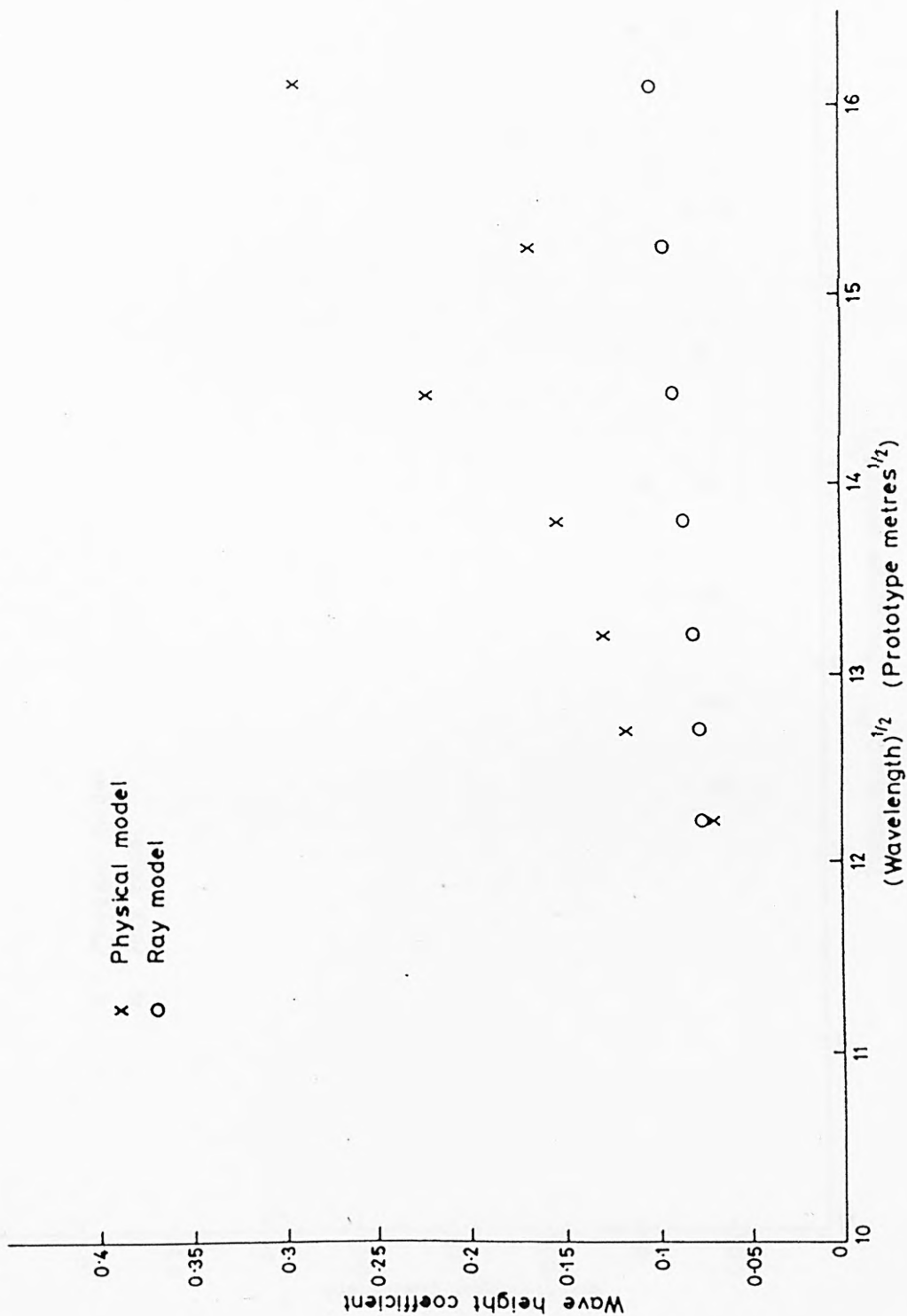


Fig 8.13 Wave Height Coefficients. Original Breakwater Length. Position 3.

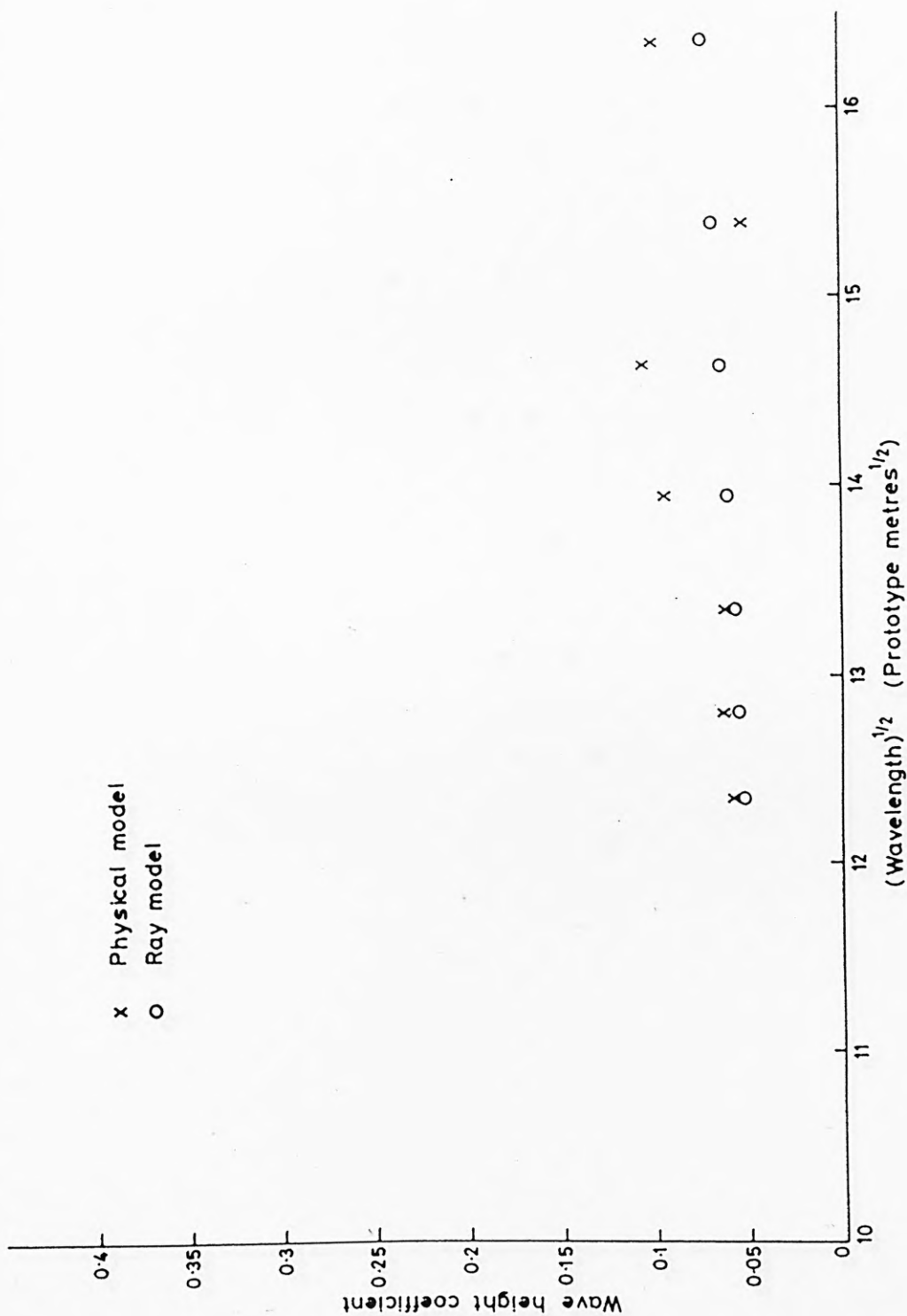


Fig 8.14 Wave Height Coefficients. Original Breakwater Length. Position 4.

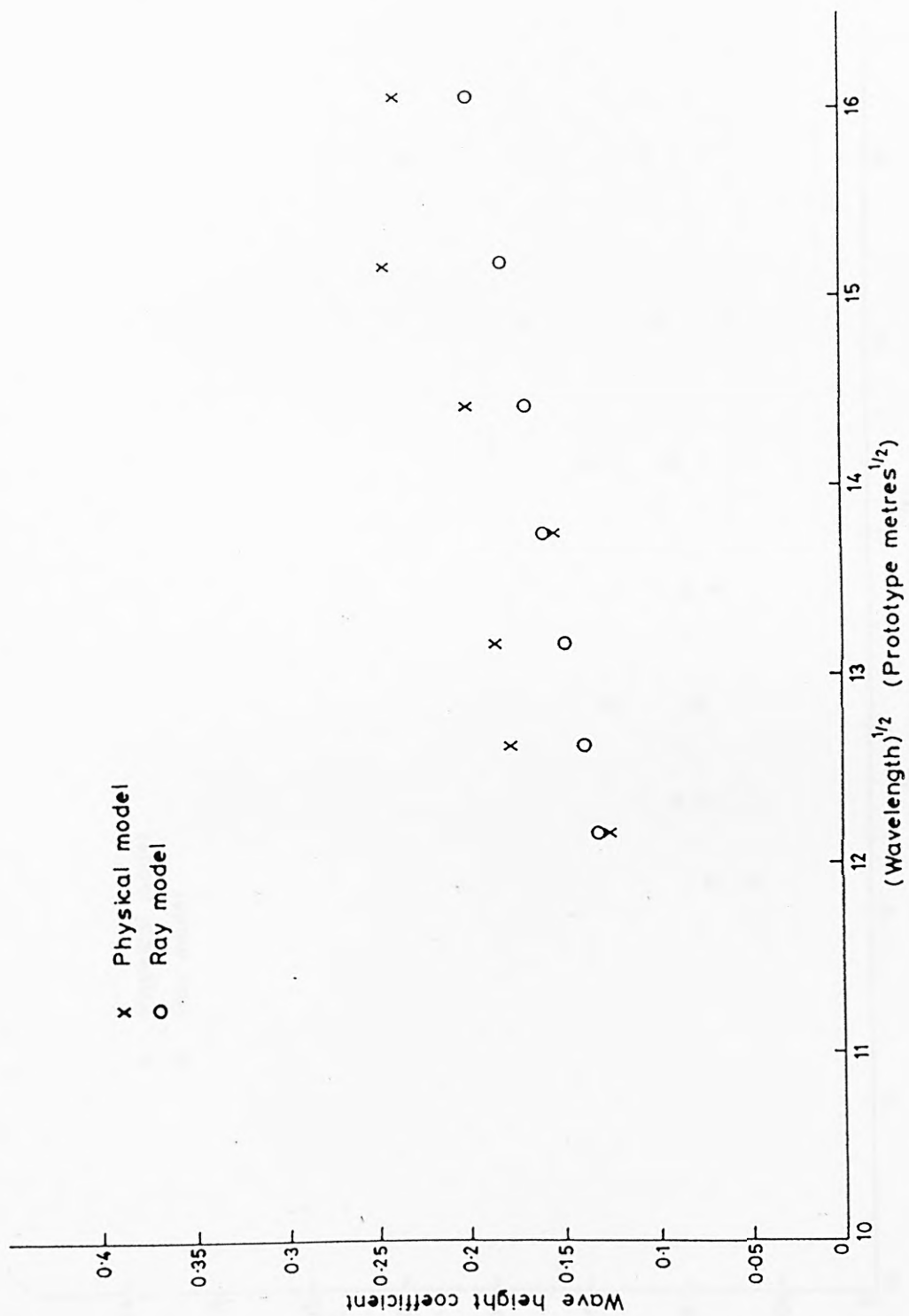


Fig 8.15 Wave Height Coefficients. Original Breakwater Length. Position 5.

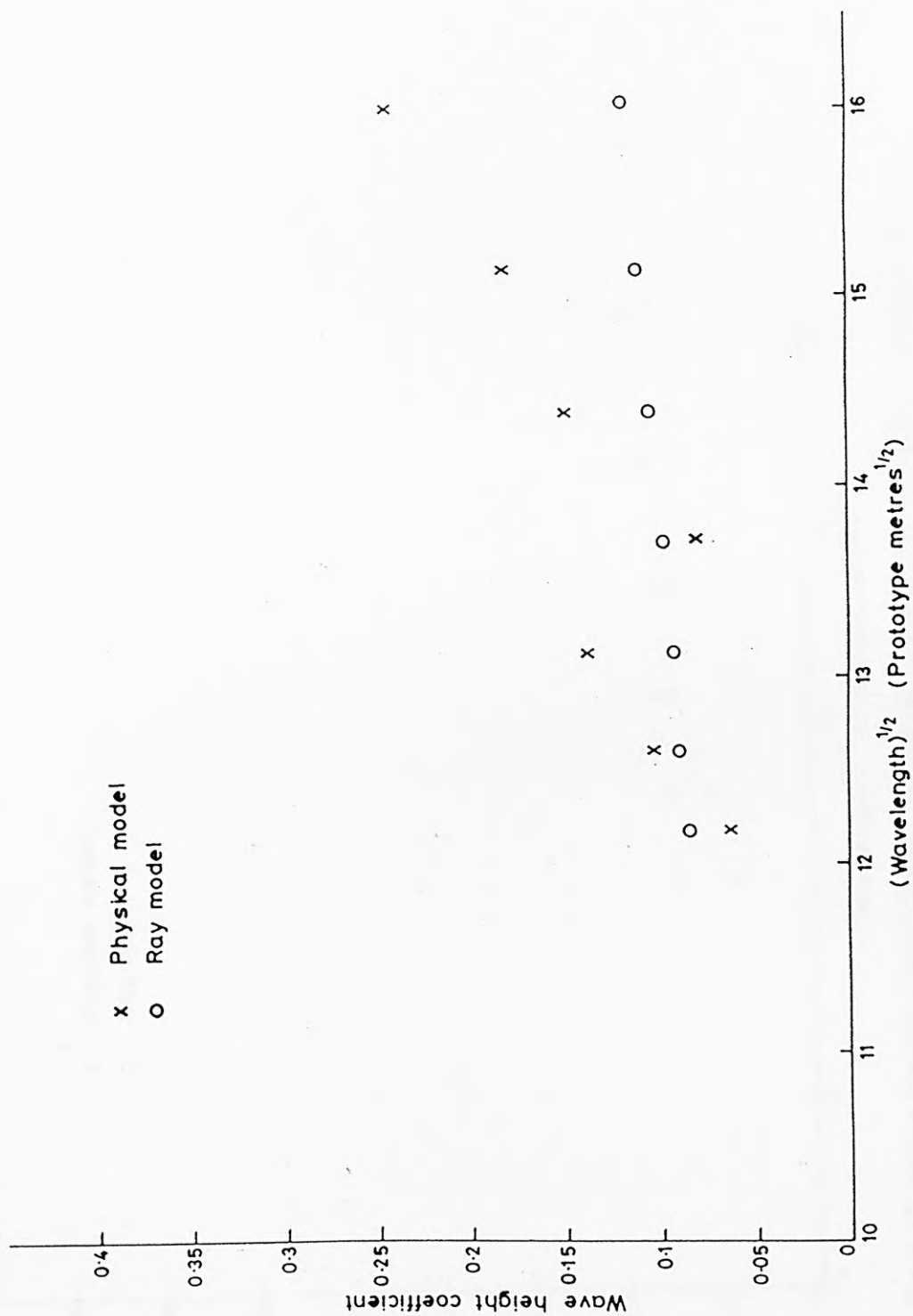


Fig 8.16 Wave Height Coefficients. Original Breakwater Length. Position 6

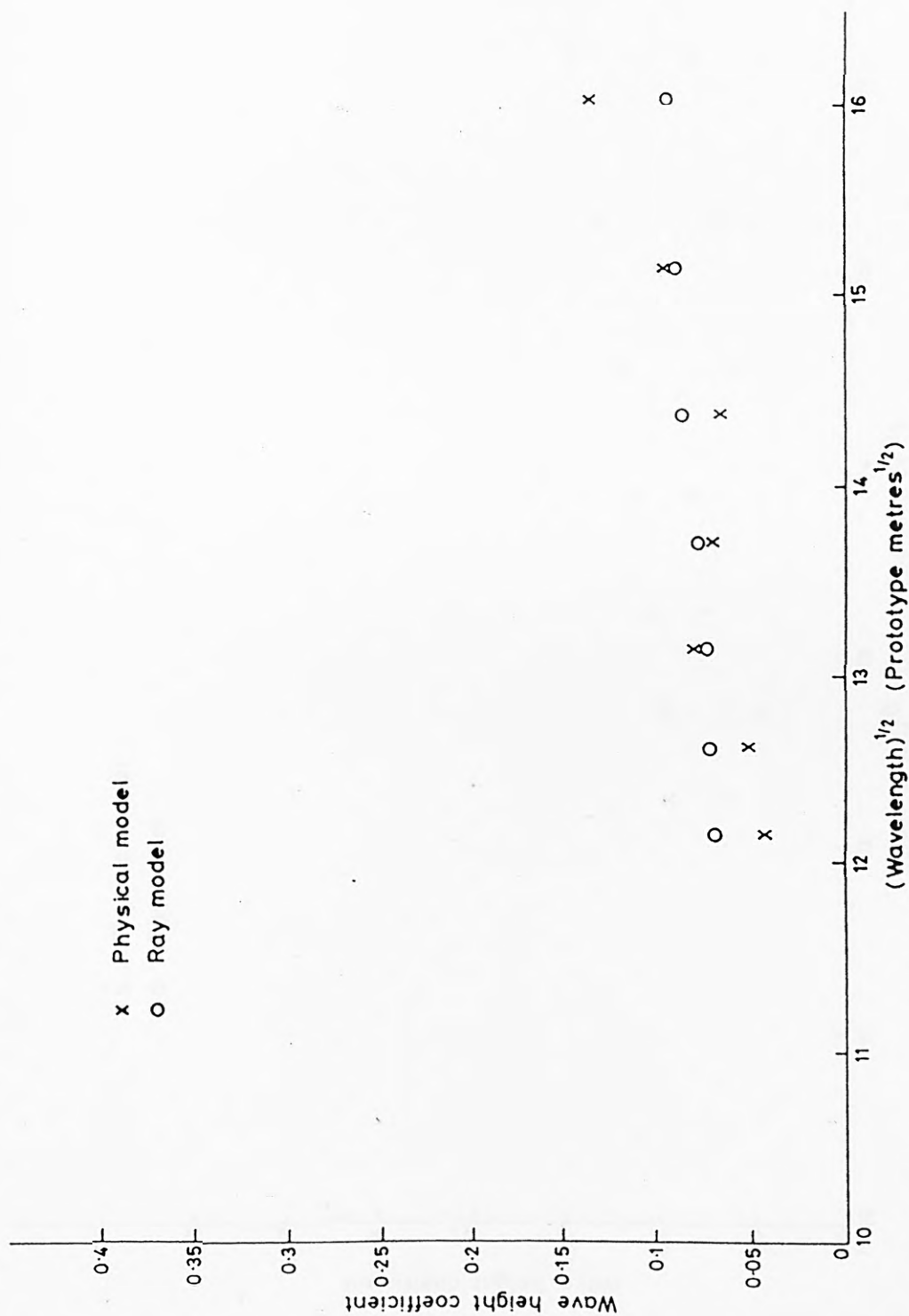


Fig 8.17 Wave Height Coefficients. Original Breakwater Length. Position 7

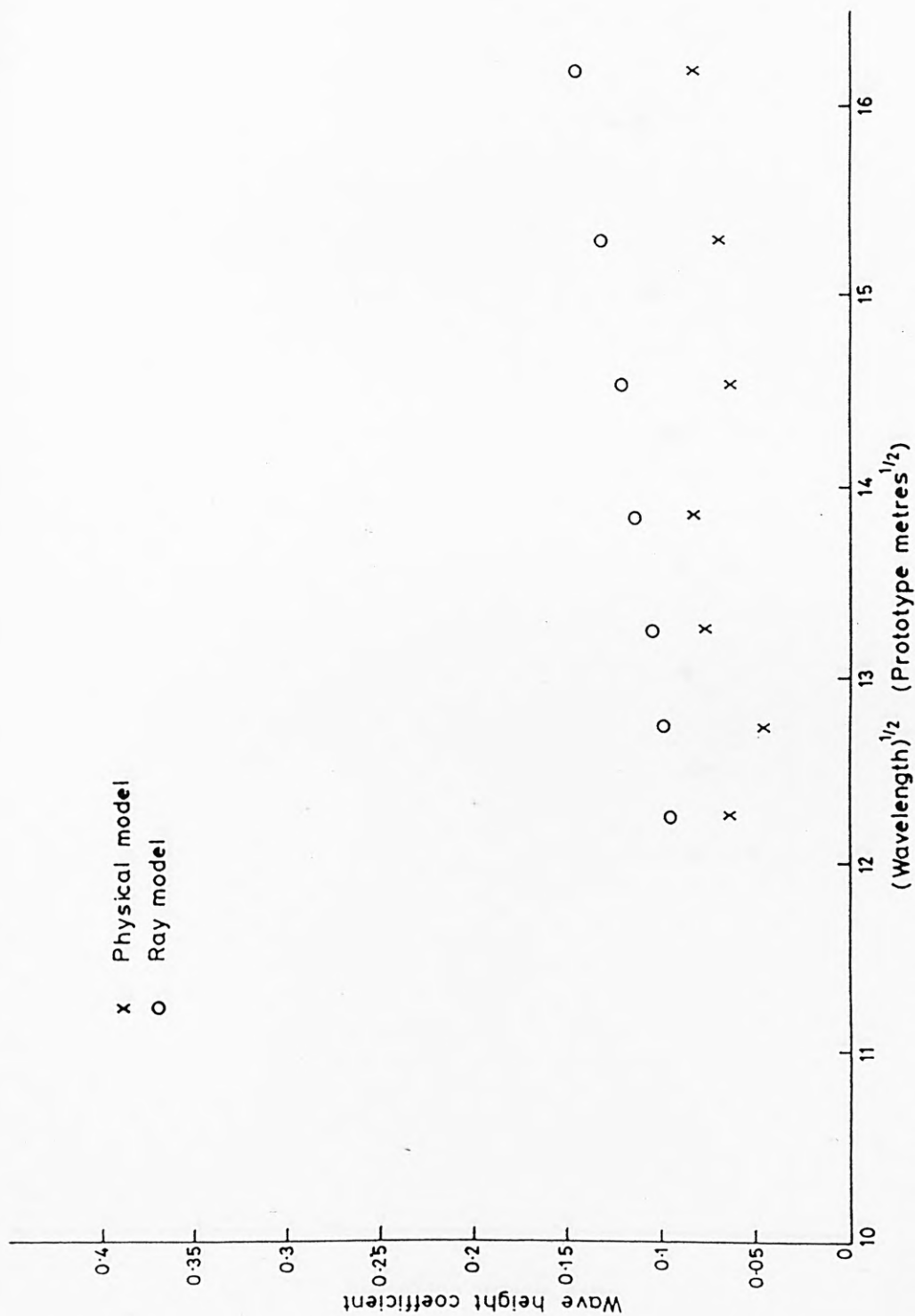


Fig 8.18 Wave Height Coefficients. Original Breakwater Length. Position 8

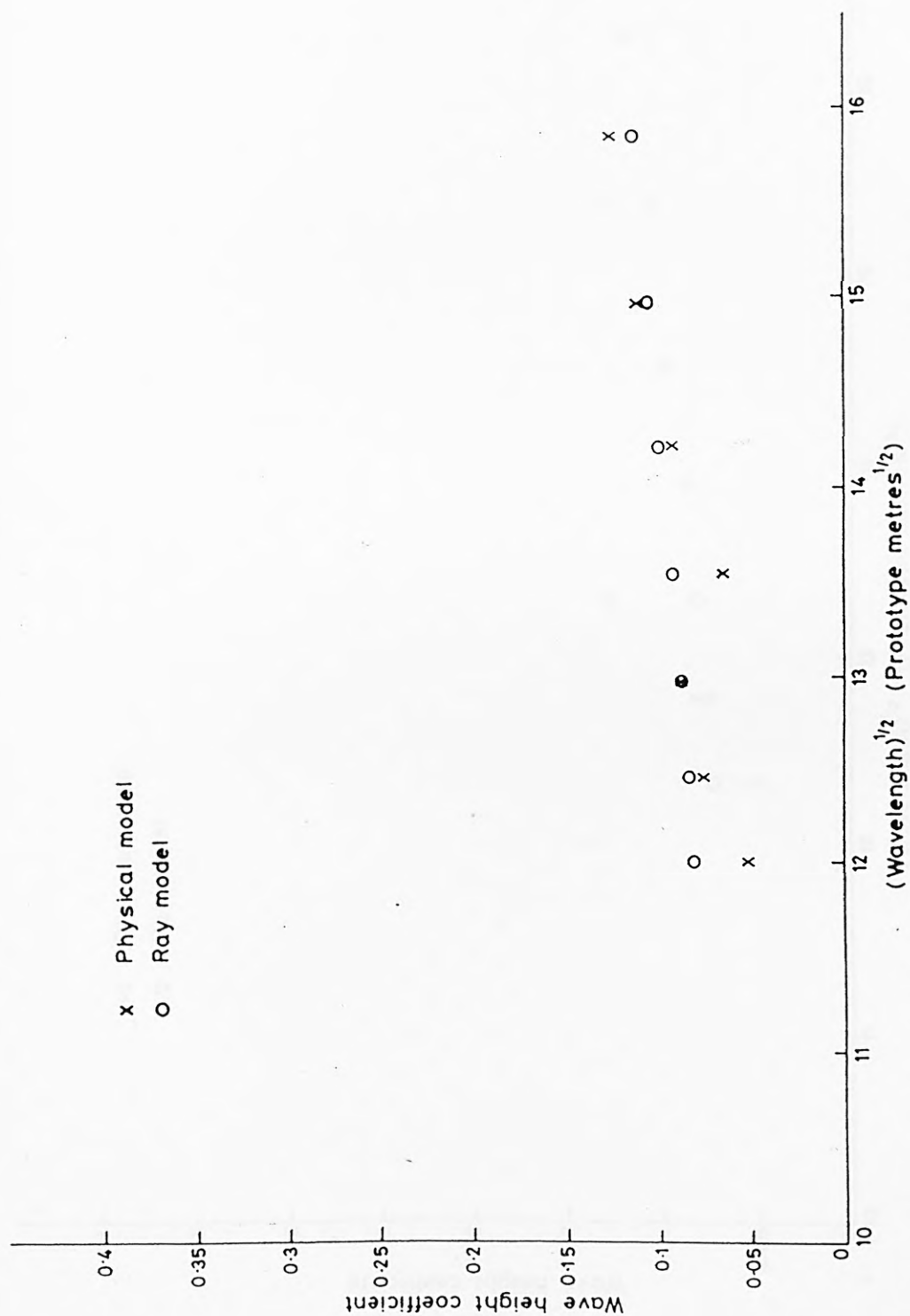


Fig 8.19 Wave Height Coefficients. Original Breakwater Length. Position 9

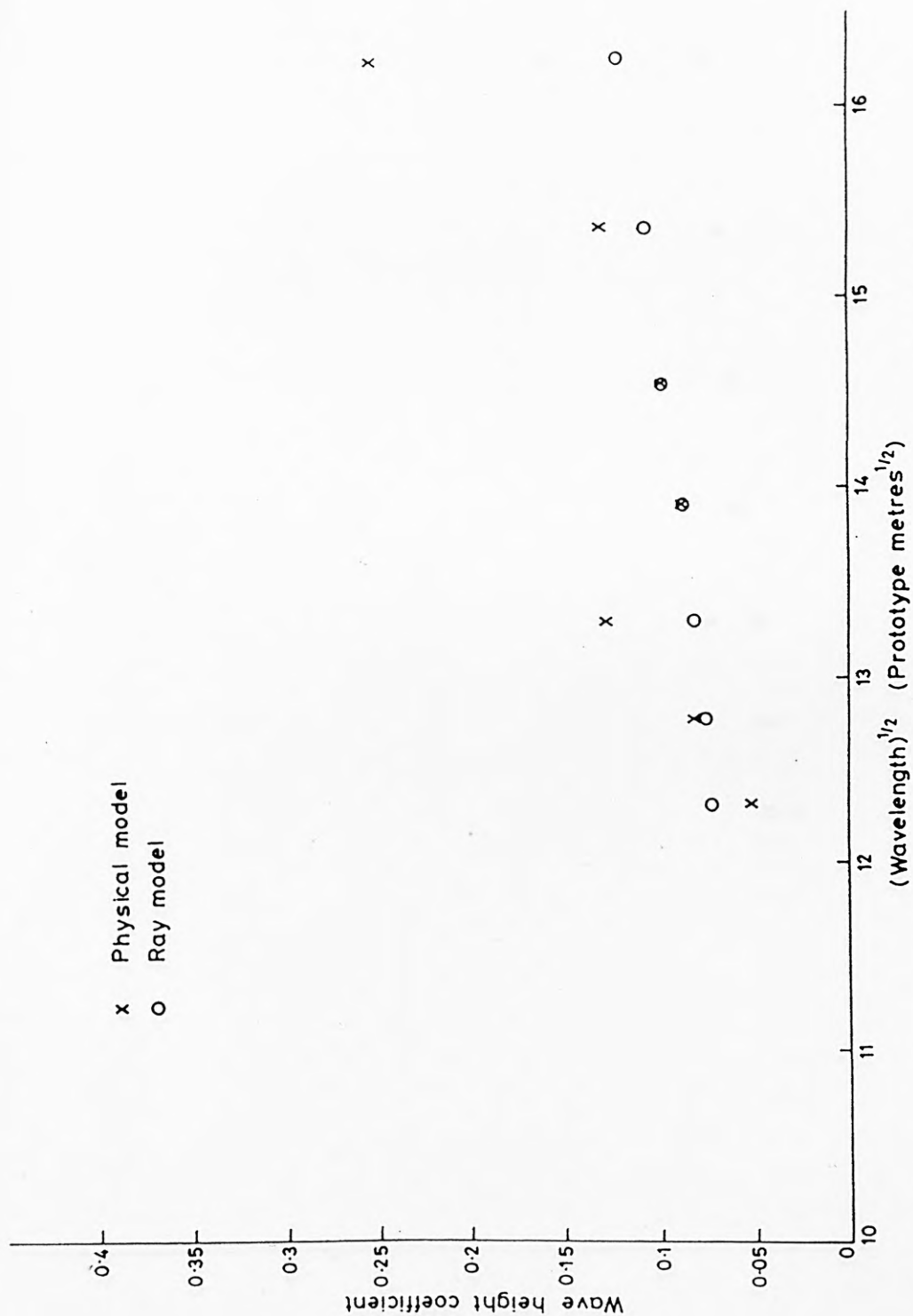


Fig 8.20 Wave Height Coefficients. Original Breakwater Length. Position 10

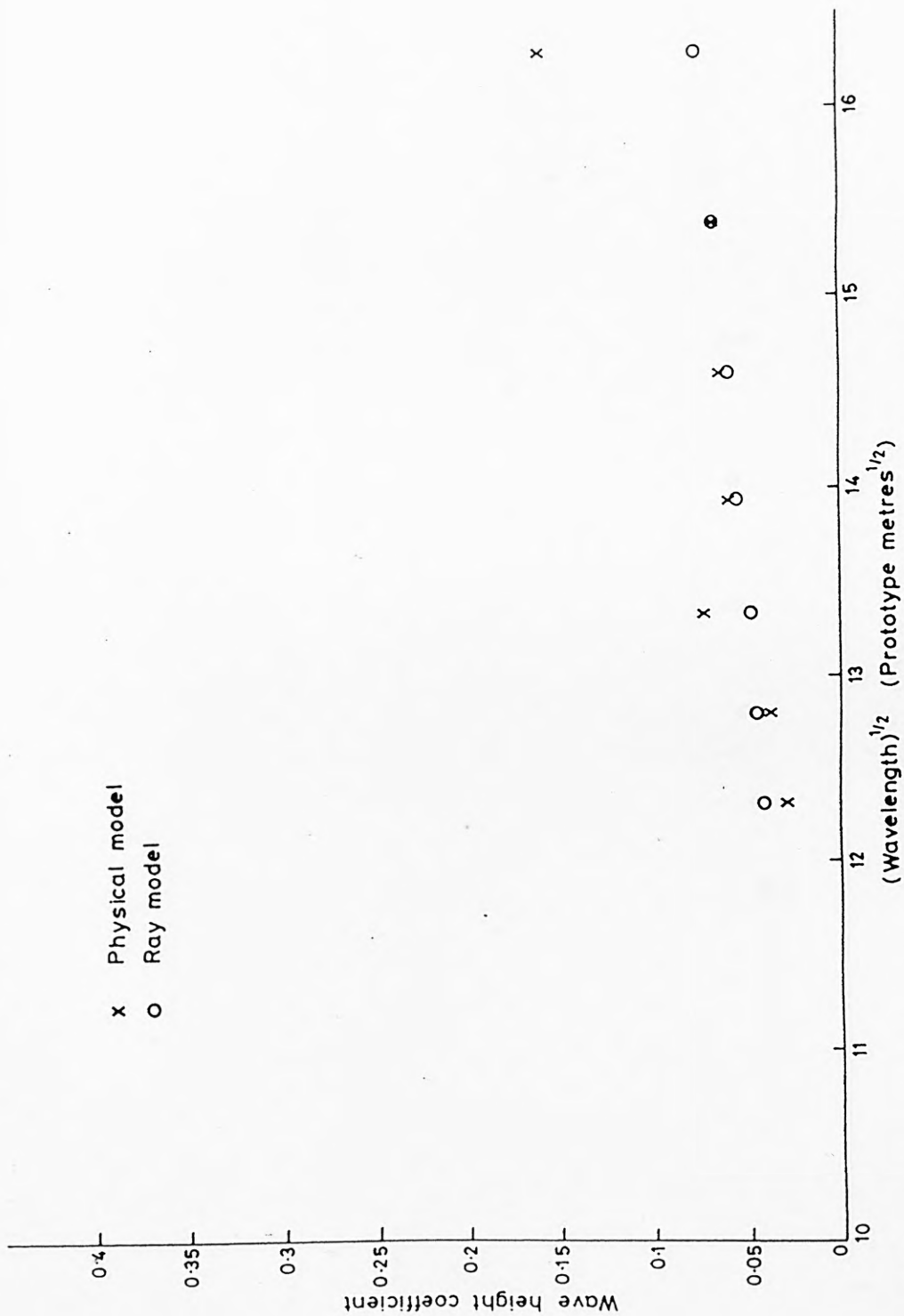


Fig 8.21 Wave Height Coefficients. Original Breakwater Length. Position 11

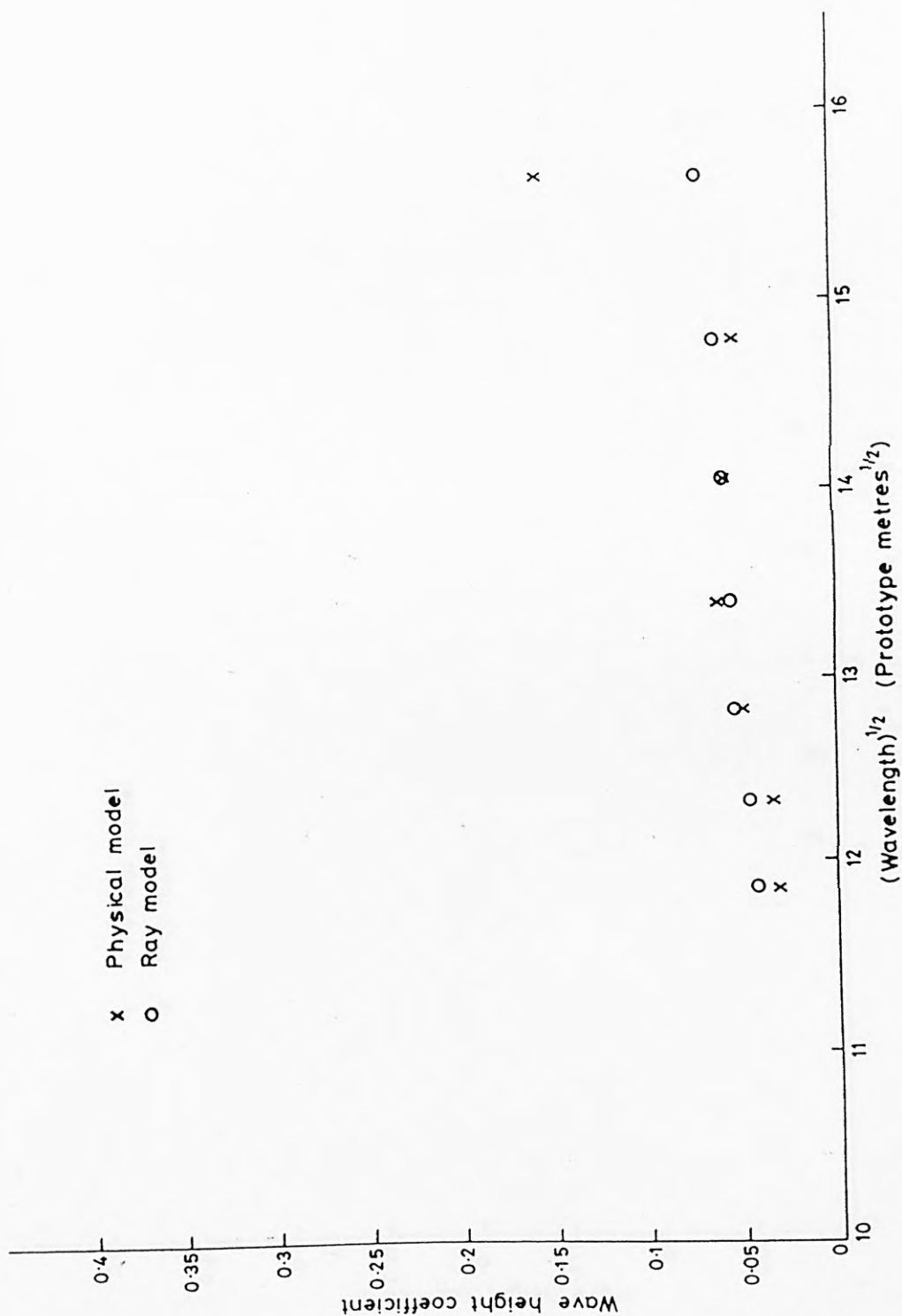


Fig 8.22 Wave Height Coefficients. Original Breakwater Length. Position 12

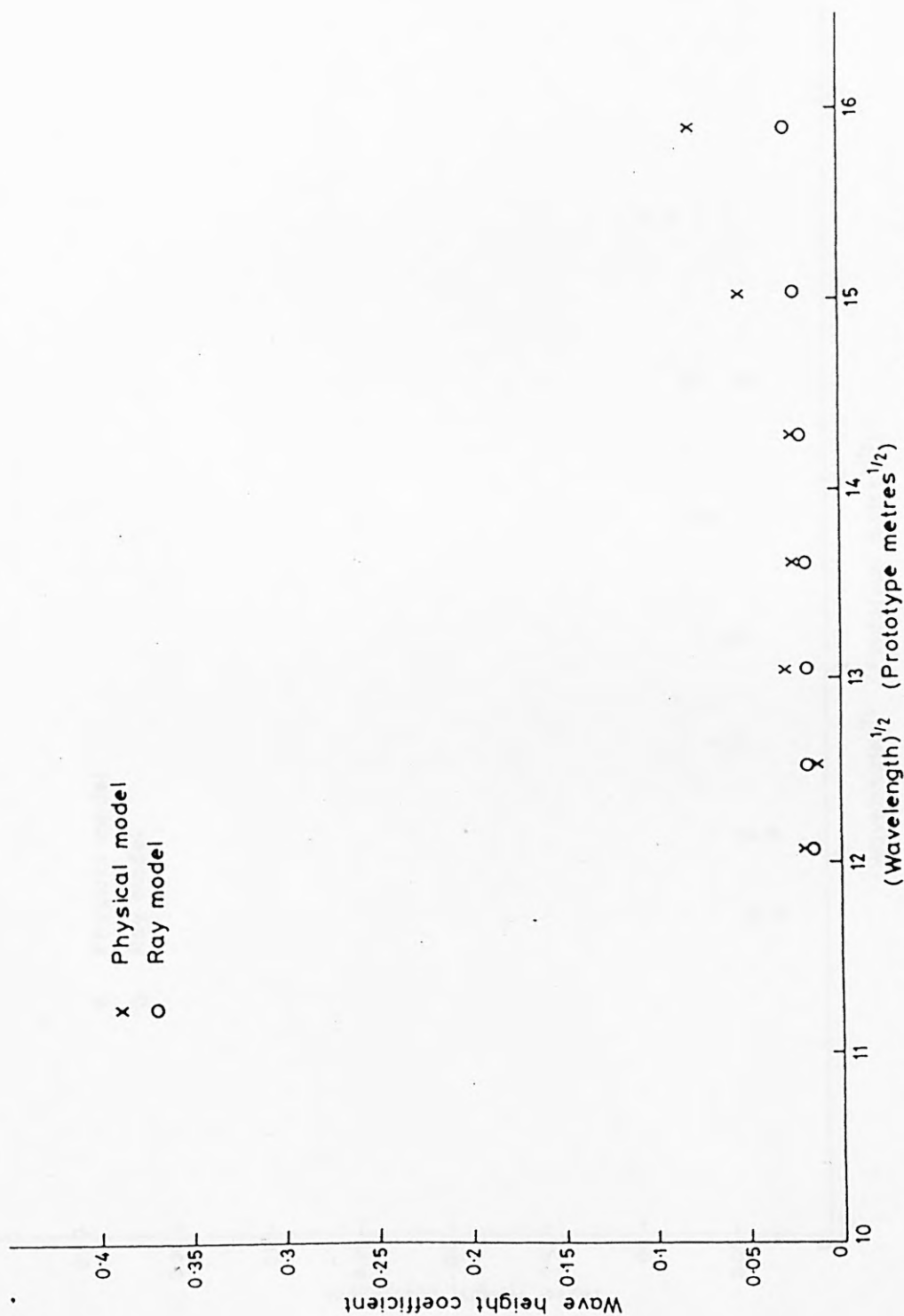


Fig 8. 23 Wave Height Coefficients. Original Breakwater Length. Position 13

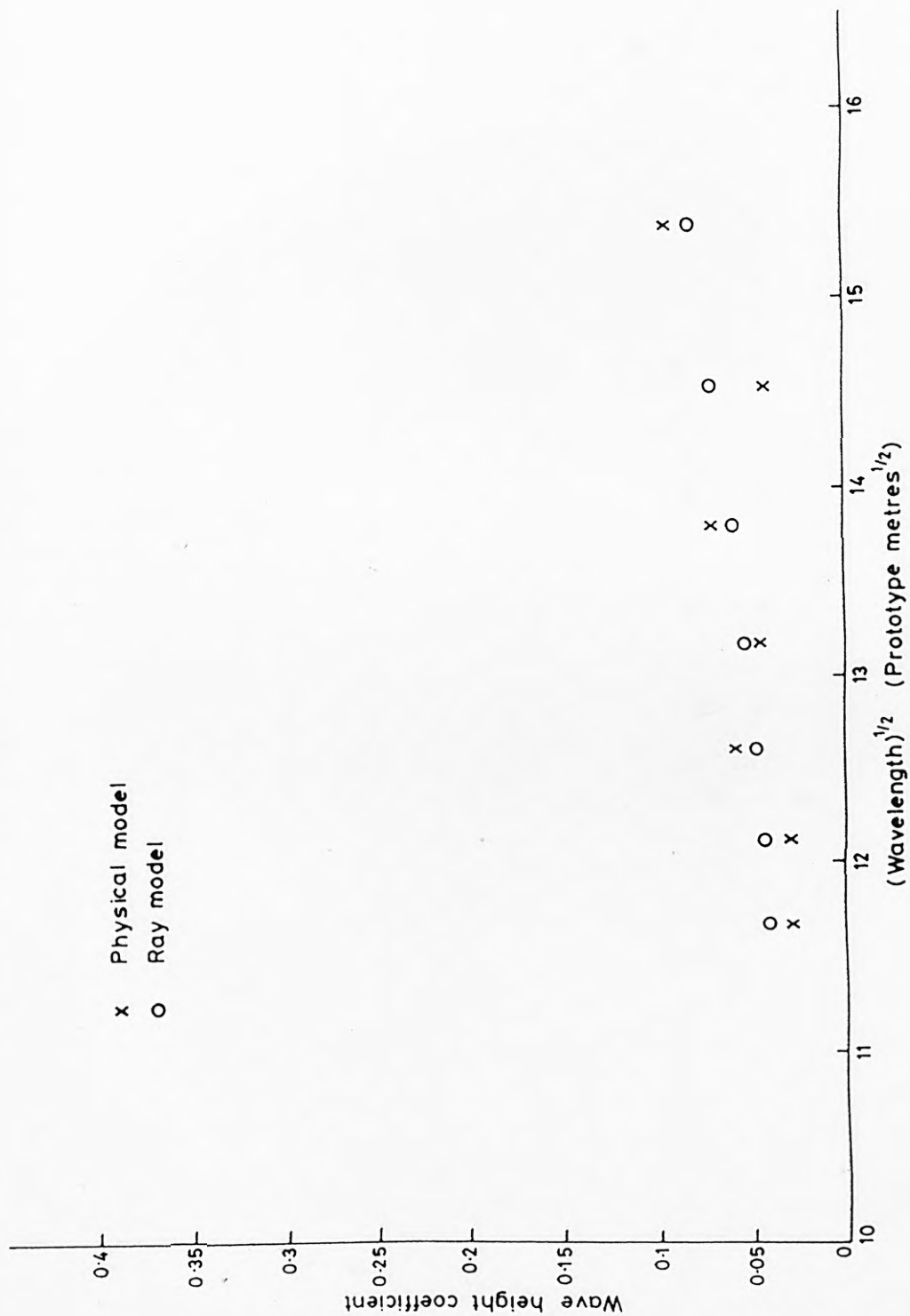


Fig 8.24 Wave Height Coefficients. Original Breakwater Length. Position 14

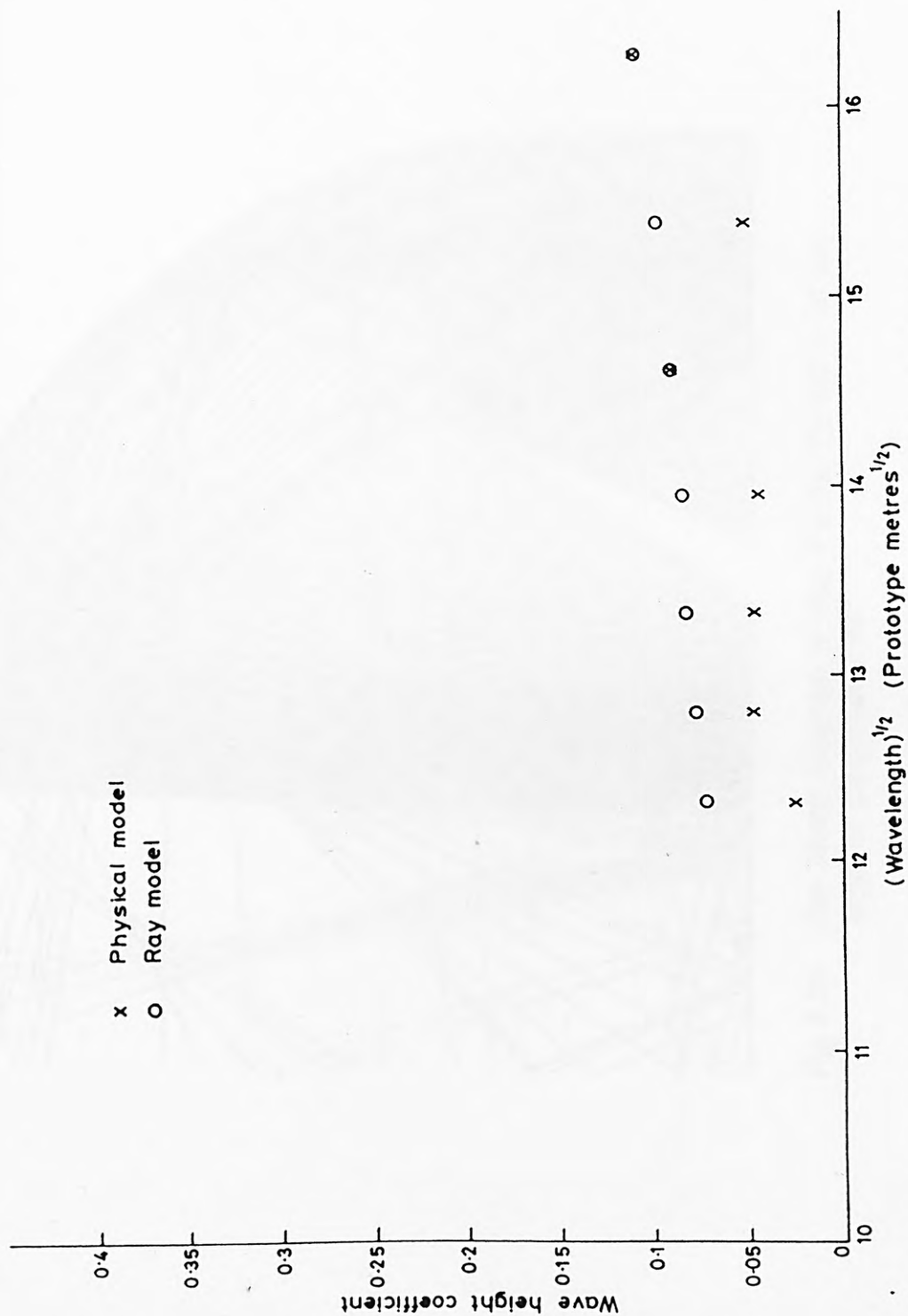


Fig 8. 25 Wave Height Coefficients. Original Breakwater Length. Position 15

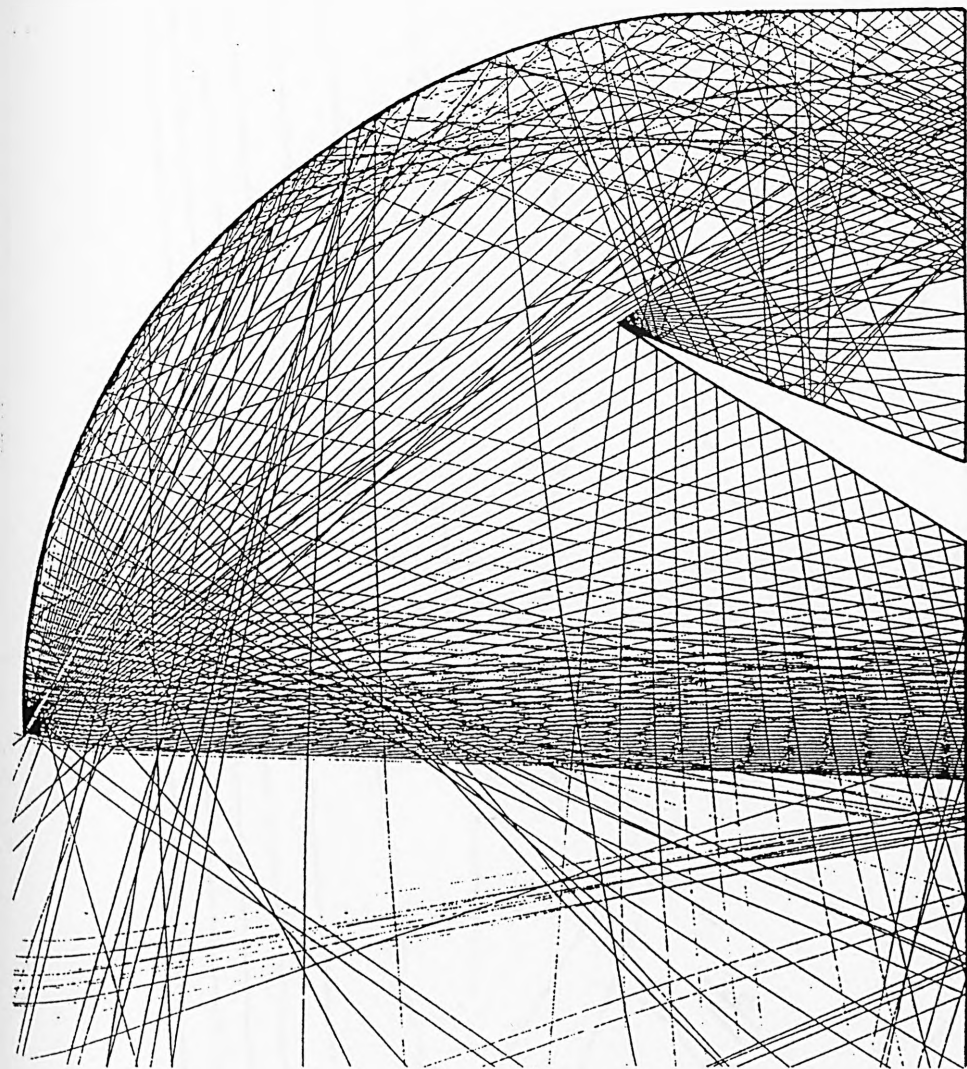


Fig 8.26 Ray Model Diagram of Ray Paths. Period = 22.4s,
Incident Direction = 92°

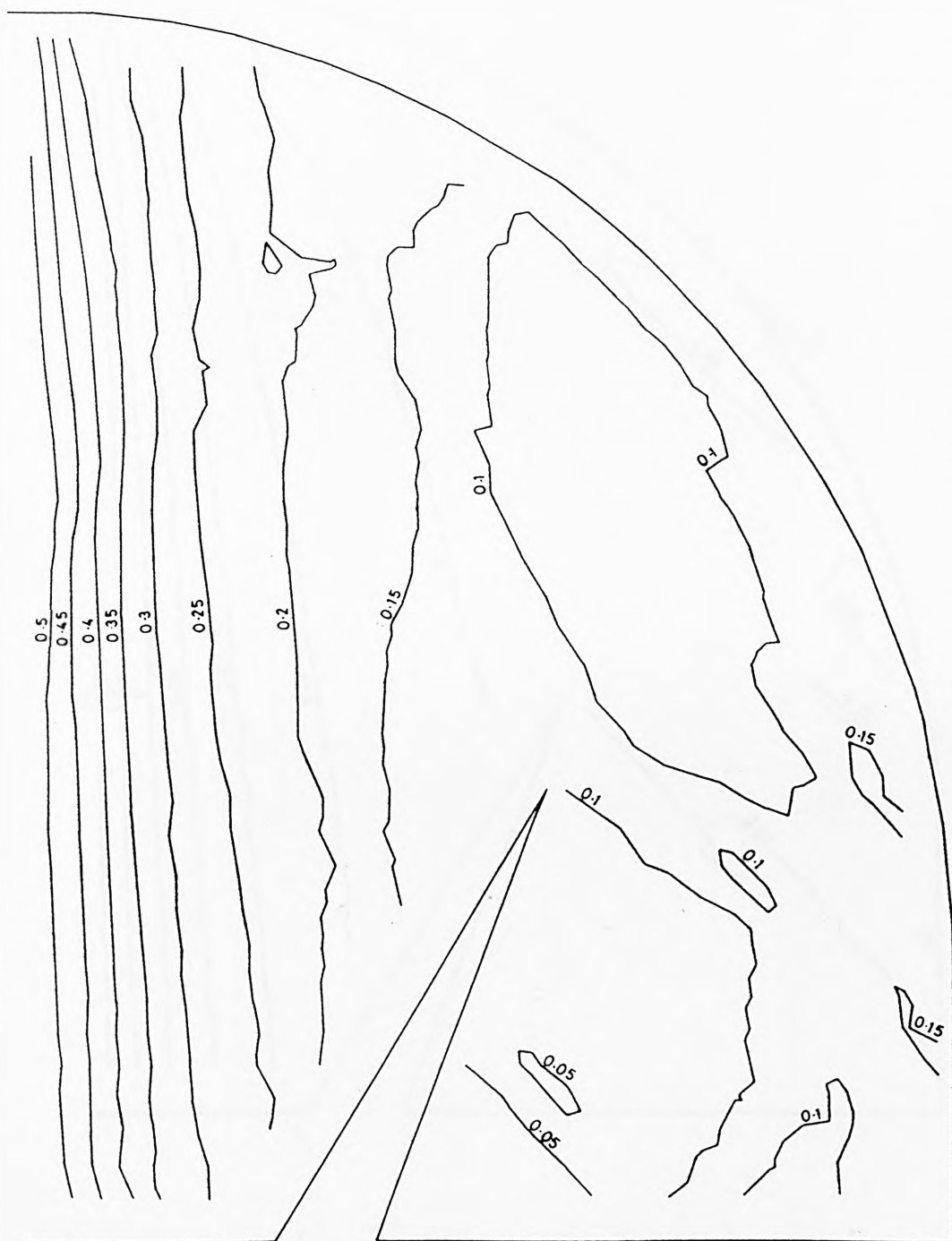


Fig 8.27 Ray Model Contours of Wave Height with Refraction and Reflections

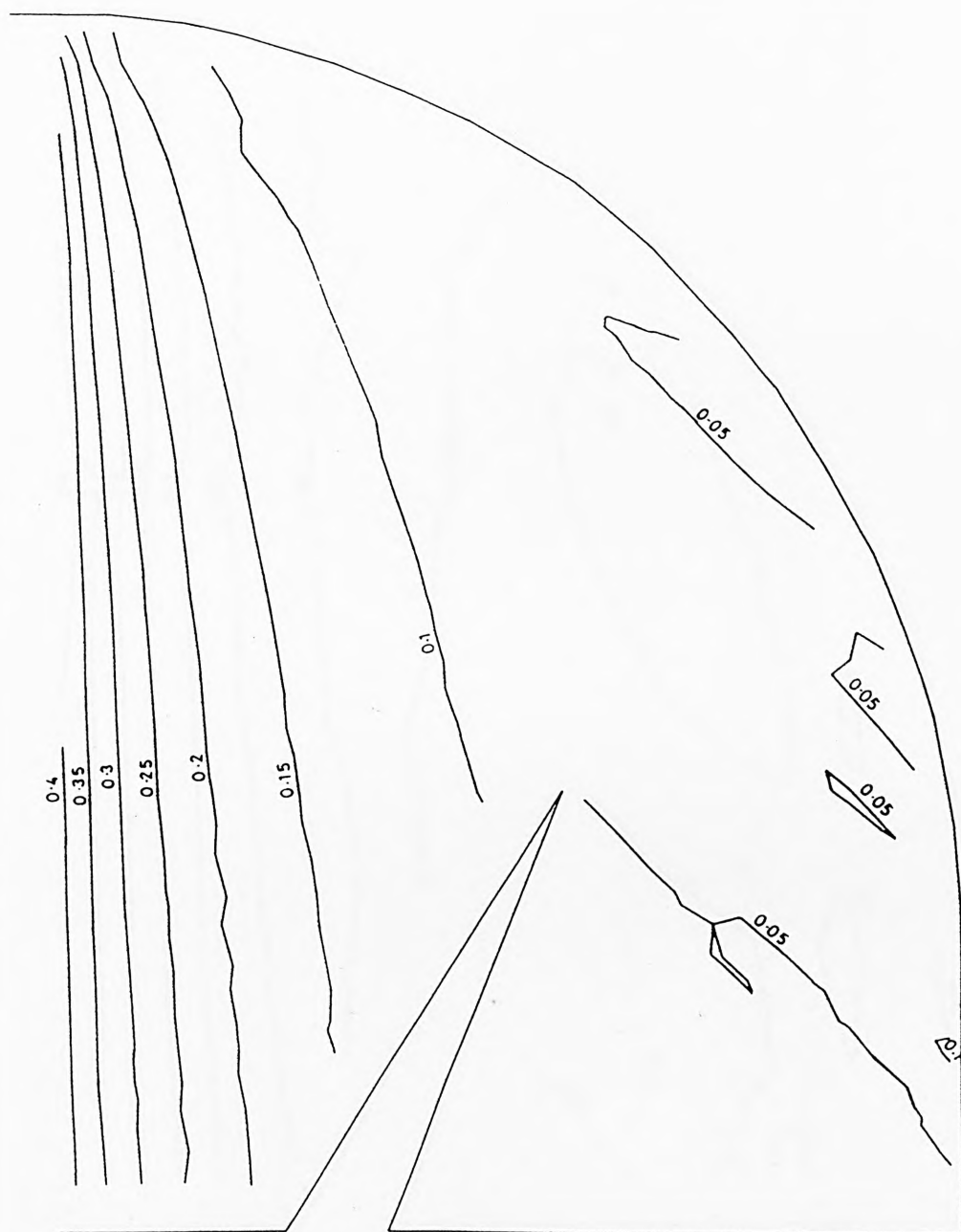


Fig 8.28 Ray Model Contours of Wave Height with Refraction and No Reflections

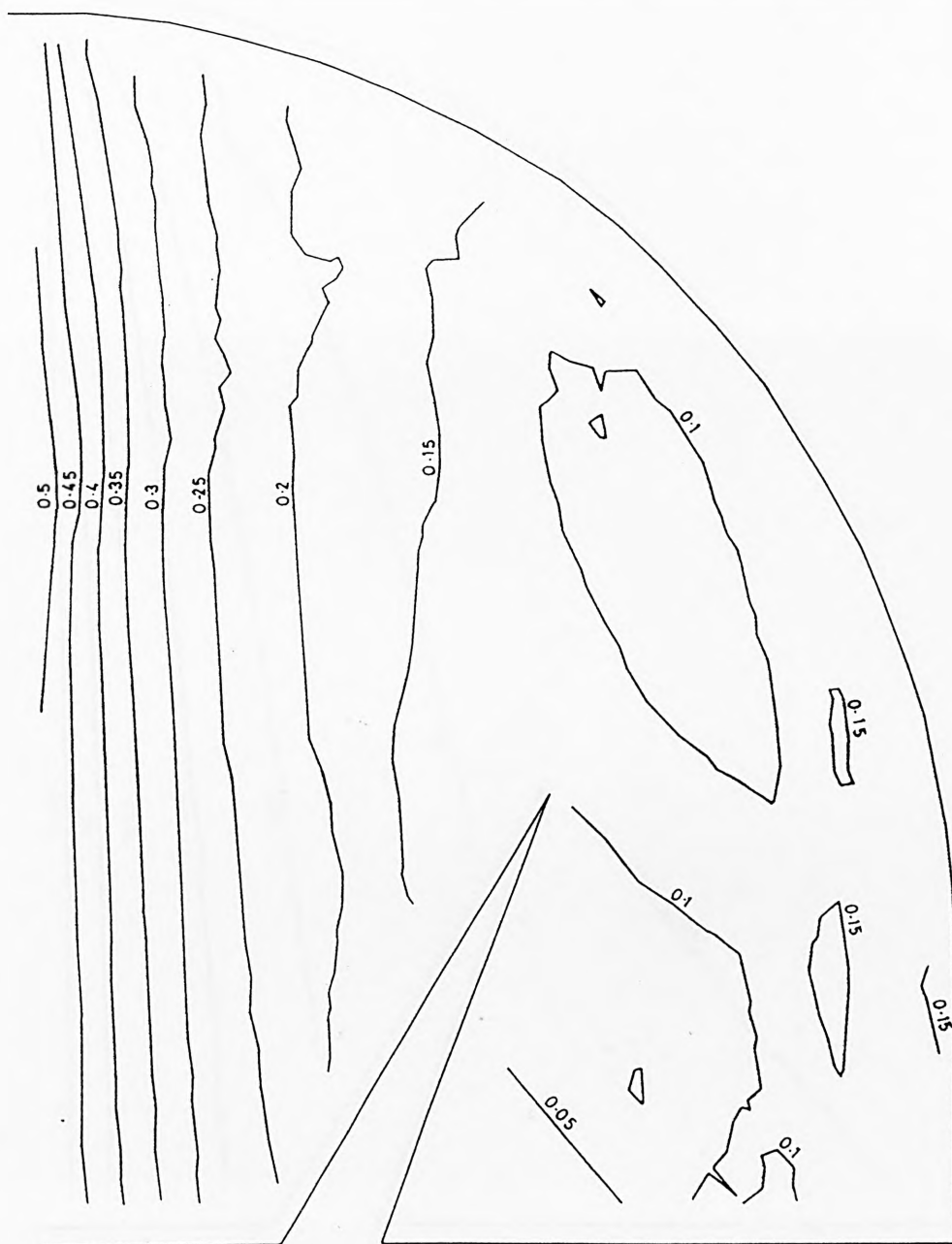


Fig 8.29 Contours of Wave Height with No Refraction (constant depth) and with Reflections

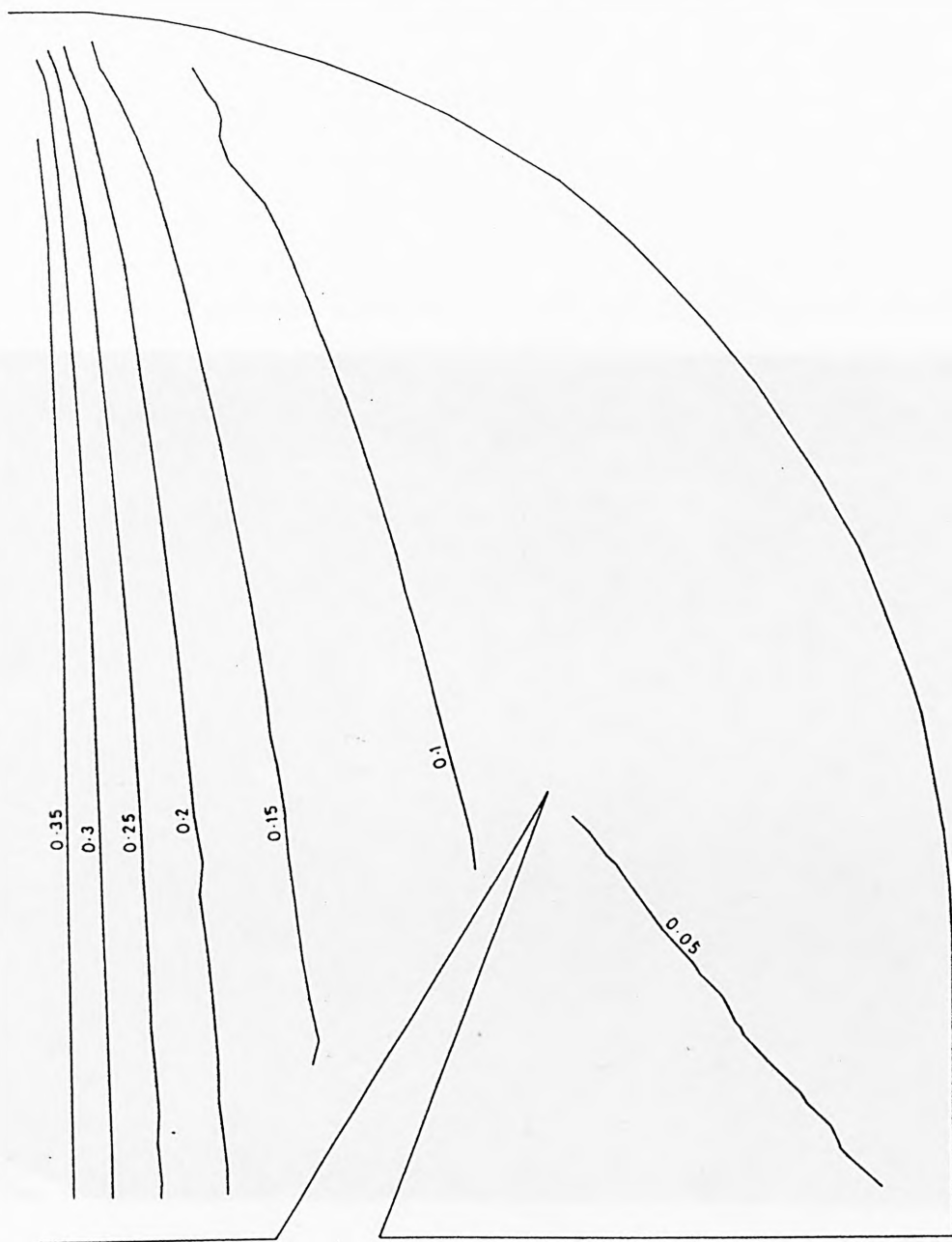


Fig 8.30 Contours of Wave Height with No Refraction (constant depth) and No Reflections



Fig 8.31 Photograph of Physical Model

9.1 Introduction

Chapters 1 to 8 have considered the theory, computational techniques and verification of the new ray model. In the present chapter an example is given of how the model can be used in practical problems of harbour design in a commercial environment. This application is to Torquay Harbour on the South Devon coast (Fig 1). Modifications to the harbour were required to make it more suitable for moored small boats. In this investigation, the ray model was used in conjunction with a physical model.

Before the study is described, the following section (section 9.2) discusses the general criteria that a computational harbour wave disturbance model should satisfy to be a useful tool in a commercial investigation. In Section 9.3, the background to the Torquay study is described, and in Section 9.4 it is shown how the ray model was calibrated and verified. In the remaining sections, the model tests used to investigate various modifications to the harbour are described.

9.2 Assessment of the Usefulness of a Computational Model in a Commercial Application

- a) Accuracy. A computational model should predict wave conditions to an accuracy required by the engineering problem. If a number of harbour schemes are being compared, relative accuracy rather than absolute accuracy is often required. This can be easier to achieve with the new ray model since unrealistic ray patterns (such as an excessive focussing or divergence) will often be common to the various schemes, and these effects will cancel when the schemes are compared. The ray model will be most accurate for large outer harbour areas, with few diffracting obstacles, relatively gentle depth variations, and harbour boundaries which do not cause multiple wave reflections.

- b) Computational Efficiency. The commercial use of a model usually involves the testing of many options, and therefore computational efficiency is an important requirement. A factor which often determines the computational speed of a model is that a minimum number of grid points is needed per wavelength. The new ray model does not have this requirement, and computational efficiency is not normally a problem.
- c) Generality. A useful commercial model should be capable of being used for many commonly-occurring types of harbour layout and not be restricted to special or simplified cases. The new ray model can be used for large, outer harbour areas where the entrances are of the single breakwater or breakwater gap type. In addition the model can be used for general bathymetries within the harbour, for layouts of any shape, for boundaries with different reflection coefficients, and for secondary diffracting obstacles within the harbour. In practice, the model can be applied to most designs of outer harbour areas.
- d) Flexibility. In most practical harbour design problems, a range of options needs to be investigated. These options can include different lengths and orientations of breakwaters, the effects of dredging or positioning of navigation channels within the harbour, and the effects of harbour boundaries with different reflection characteristics. In addition, a range of incident wave conditions and water levels usually needs to be investigated. A flexible model is therefore one in which these alterations to the model or data can be made quickly and accurately. The new ray model is a particularly flexible model. The lengths and orientations of breakwaters can easily be changed by the line-segment method of representing harbour boundaries (Section 5.6). Alterations to reflections coefficients can be done easily for same reason. Changes to the bathymetry can be made by suitable adjustments to the depth data. Alterations to incident wave

height, period and direction, and water level, involve simple changes to the input data.

9.3 Background to the Torquay Harbour Study

Torquay Harbour is situated on the northern coast of Tor Bay in South Devon (Fig 1). In order to expand facilities available to recreational boat users, it was proposed to construct a small-boat marina on the western side of the outer harbour (Fig 9.2).

Observations in the outer harbour during severe south-easterly gales showed that it would be necessary to improve protection from waves if pleasure craft were to be moored safely close together. Hydraulics Research Ltd was commissioned to carry out a joint physical and computational model study to determine the most cost-beneficial extensions to the main breakwaters, and any other measures to reduce wave action inside the harbour to acceptable levels.

The physical model was constructed to a scale of 1 to 100 in both vertical and horizontal planes (Fig 9.3). The wave generator was able to produce random-frequency wave conditions to any frequency spectrum supplied by the experimenter. Up to sixteen twin-wire wave probes were available to record wave heights and periods in the model harbour. In the computational ray model a grid of 11 by 13 square elements was used to cover the main outer harbour. Contours of water depth at Admiralty Chart Datum (CD) for the original bathymetry are shown in Fig 9.2 and Fig 9.4. All runs in this investigation were performed at a Mean High Water Springs water level (4.9m above CD). Breakwaters and harbour boundaries were represented by straight-line segments within grid squares (Fig 9.4).

Offshore wave conditions were determined by a standard method of hindcasting from locally recorded wind data. This method was calibrated and verified against records of wave height further along the coast at Budleigh Salterton (Fig 9.1). Standard statistical techniques were used to extrapolate the hindcast offshore wave conditions to various extreme storm conditions.

Storms with a return period of 1 in 1 year and 1 in 50 years from SE, S and SSW directions were studied in the model tests. Wave refraction calculations were then made to transform these design offshore wave conditions to the site of the wavemaker to provide input for the physical model. Input wave conditions for the ray model were obtained from probe measurements close to the harbour entrance in the physical model.

9.4 Calibration and Verification of the Computational Ray Model

At the outset of the study, the ray model was calibrated and verified against results from the physical model using the existing harbour layout. Single runs of the ray model at the median frequency of a spectrum were used to represent the whole spectrum (see Section 8.4.2). In the physical model, most probes were positioned at points corresponding to the element intersections in the ray model. Therefore, average wave heights over the four neighbouring grid squares were used for comparison with the physical model. In the calibration tests, incident wave heights and directions at the harbour entrance were measured in the physical model for south-easterly 1 in 50 years and 1 in 1 year return period waves.

The calibration tests showed that the best agreement between the models was obtained with the following factors in the ray model.

- a) A reflection coefficient of 0.85 on all boundaries other than Princess Parade.
- b) Rays should be stopped after they had undergone one reflection from the harbour boundaries.
- c) Any residual energy from the rays stopped after one reflection should be smoothed over the whole harbour and added to the wave energy already calculated in the harbour.

- d) The reflection coefficients for Princess Parade were determined by physical model flume tests using random waves. These reflections coefficients were

0.58 for 1 in 50 years wave from SE

0.50 for 1 in 1 year wave from SE

0.20 for 1 in 50 years wave from S and SSW

Table 9.1 shows the comparison between the computational and physical model results for the two storm conditions from the SE. Leaving aside the three positions where agreement is poor (0, 13 and 14), the average wave heights agree well.

The next stage after this calibration exercise was to introduce changes to the harbour layout to reduce wave activity. These alterations comprised changes to the bathymetry (dredging of the marina area), extensions to one of the breakwaters (Haldon Pier), and the effects of introducing rubble mound on part of the harbour boundary to try to reduce wave reflections. These investigations are considered in turn.

9.5 Dredging of the Marina Area

To allow sufficient underkeel clearance, it was proposed that the marina area should be dredged to a depth of 2.0m below CD. This change will alter the refraction pattern and therefore the wave conditions in the area. The ray model was used to determine the changes in wave height for four design offshore wave conditions:

- a) 1 in 50 years wave from SE
- b) 1 in 1 year wave from SE
- c) 1 in 50 years wave from S
- d) 1 in 50 years wave from SSW

The wave heights at nine probe positions are shown in Table 9.2. The differences between the two sets of results are very small. It was concluded therefore that dredging the marina area would

have no significant effect on wave disturbance in the harbour. This finding allowed the moulding of the marina area in the physical model to be removed at an early stage of testing, thereby shortening the time needed for the full programme of physical model tests.

9.6 Extensions to Haldon Pier

The principal means of providing additional shelter to the harbour was to extend Haldon Pier. The physical and mathematical models were used to determine by how much and in what direction this extension should be made. The plan was to run the ray model for a range of different extension lengths and directions. The most promising of these (judged on a cost-benefit basis) would then be investigated in more detail in the physical model. Nine cases were run in the ray model, listed below:

- a) Extension A30. Length 30m, orientation 281.5°N
- b) Extension A40. Length 40m, orientation 279°N
- c) Extension A50. Length 50m, orientation 277°N
- d) Extension B30. Length 30m, orientation 259°N
- e) Extension B40. Length 40m, orientation 259°N
- f) Extension B50. Length 50m, orientation 259°N
- g) Extension C40. Length 40m, orientation 250°N
- h) Extension D40. Length 40m, orientation 241°N
- i) Extension E40. Length 40m, orientation 232°N

The positions of extensions A and C are shown in Fig 9.2. For each extension, three tests were performed with 1 in 50 years incident wave conditions from three directions (SE, S and SSW), giving a total of 27 tests. All tests were done with the marina dredged to 2.0m below CD. Figs 9.5 and 9.6 show contours of significant wave height for the original and A40 extensions. Table 9.3 shows the full comparison of wave heights in the harbour for alignments A and B and the original breakwater length. As would be expected, the wave heights decrease as the extension is made longer. However, for all three incident wave conditions, the reduction in average wave height in going from an extension of 40m

to 50m is not as great as going from 30m to 40m. The most cost-beneficial length therefore appears to be 40m.

Since waves from the SE were much stronger than from the other directions, it was decided to test different orientations of the optimum 40m extension to try to reduce wave heights inside the harbour further. Since as a general rule greater shelter will be provided if a breakwater is more nearly perpendicular to the wave direction, three alignments (C, D and E) were tested at 9° intervals southwards from B. Alignment E was almost perpendicular to the south-easterly waves. Table 9.4 shows that some benefit is obtained in going from B to C, but there is only a very marginal difference (in fact, a slight increase) at the other orientations. Furthermore, for these orientations, considerably larger waves from the SSW were obtained, although the SE remained the worst direction. On the basis of these results, alignments A and C with a 40m extension length were tested in the physical model.

9.7 Tests with Rubble Mound under Princess Parade and on Haldon Pier

A possible means of decreasing wave action in the harbour was to reduce wave reflection from Princess Parade by introducing rubble mound on to the slope. Tests were performed with reduced reflection coefficients on three parts of Princess Parade. These were (Fig 9.2):

- a) Princess Pier to the semi-circular promenade
- b) The semi-circular promenade
- c) The semi-circular promenade to Old Fish Quay

The 1 in 50 years wave condition from the SE was used for these tests. It was estimated that rubble mound would reduce the reflection coefficient to 0.30 from the value of 0.58 measured in the flume experiments. It was found that rubble mound produces only a very slight decrease in wave activity, and therefore it was concluded that the beneficial effect would be negligible.

A further test was carried out on the A40 extension in which the effect of v-rays was removed by assigning a reflection coefficient of zero to the outer face of Haldon Pier. This resulted in an average 40% reduction in wave height in the harbour. This indicated that the construction of a rubble mounded tip to the Haldon Pier extension would be profitable (although the rubble mound would not remove the v-wave completely). Further tests in the physical model indicated that a rubble mound tip to Haldon Pier would reduce wave heights inside the harbour by about 30% compared with a vertical, perfectly reflecting structure.

50 year wave SE - existing harbour

Probe	Physical Model H_s (m)	Ray Model H_s (m)
0	0.72	1.26
1	0.72	0.68
2	0.83	0.79
3	0.93	0.86
4	0.90	0.93
5	0.83	0.75
6	1.08	1.01
7	0.83	0.86
8	0.86	1.11
9	0.65	0.83
10	0.79	0.83
13	0.68	1.04
14	0.68	1.04
15	0.79	0.72
17	0.75	0.75
20	0.68	0.61
23	0.75	0.75
25	0.86	0.68
28	0.66	0.83
Av of positions (except 0, 13, 14) is 0.81		Av of positions (except 0, 13, 14) is 0.81

TABLE 9.1

Comparison between Computational Ray Model and Physical Model

Probe	50 yr wave SE existing contours H_s (m)	50 yr wave SE dredged contours H_s (m)	1 yr wave SE existing contours H_s (m)	1 yr wave SE dredged contours H_s (m)
1	0.68	0.68	0.43	0.43
2	0.79	0.75	0.50	0.48
3	0.86	0.86	0.58	0.58
6	1.01	0.93	0.69	0.69
8	1.11	1.11	0.72	0.71
9	0.83	0.79	0.53	0.51
17	0.75	0.74	0.47	0.48
20	0.61	0.61	0.38	0.38
28	0.83	0.83	0.53	0.53
	Av = 0.83	Av = 0.81	Av = 0.54	Av = 0.53

Probe	50 yr wave S existing contours H_s (m)	50 yr wave S dredged contours H_s (m)	50 yr wave SSW existing contours H_s (m)	50 yr wave SSW dredged contours H_s (m)
1	0.18	0.18	0.18	0.18
2	0.53	0.53	0.19	0.19
3	0.60	0.60	0.48	0.48
6	0.79	0.81	0.20	0.22
8	0.51	0.51	0.20	0.20
9	0.44	0.45	0.70	0.74
17	0.23	0.23	0.38	0.38
20	0.21	0.21	0.21	0.21
28	0.29	0.29	0.54	0.54
	Av = 0.42	Av = 0.42	Av = 0.34	Av = 0.35

TABLE 9.2

Computational Ray Model results to show effect of dredging
with present breakwaters

Alignment A extensions (40m entrance)

Probe	Existing harbour H_S (m)	30m extension H_S (m)	40m extension H_S (m)	50m extension H_S (m)
1	0.68	0.43	0.39	0.39
2	0.79	0.54	0.41	0.38
3	0.86	0.79	0.68	0.61
6	1.01	0.72	0.57	0.43
8	1.11	0.54	0.47	0.39
9	0.83	0.57	0.66	0.72
17	0.75	0.65	0.61	0.56
20	0.61	0.57	0.57	0.54
28	0.83	0.68	0.61	0.57
	Av = 0.83	Av = 0.61	Av = 0.55	Av = 0.51

Alignments B extensions (20° S of alignment A40)

1	0.68	0.43	0.39	0.39
2	0.79	0.50	0.39	0.36
3	0.86	0.72	0.65	0.57
6	1.01	0.65	0.54	0.43
8	1.11	0.48	0.43	0.39
9	0.83	0.54	0.48	0.65
17	0.75	0.61	0.57	0.56
20	0.61	0.43	0.43	0.43
28	0.83	0.68	0.65	0.61
	Av = 0.83	Av = 0.56	Av = 0.50	Av = 0.49

TABLE 9.3

Computational Ray Model results - 50 years wave SE

Alignment A extensions (40m entrance)

Probe	Existing harbour H_S (m)	30m extension H_S (m)	40m extension H_S (m)	50m extension H_S (m)
1	0.18	0.11	0.09	0.08
2	0.53	0.29	0.09	0.08
3	0.60	0.38	0.20	0.15
6	0.79	0.26	0.16	0.09
8	0.51	0.15	0.10	0.09
9	0.44	0.10	0.20	0.16
17	0.23	0.18	0.15	0.15
20	0.21	0.17	0.15	0.15
28	0.29	0.21	0.18	0.15
	Av = 0.42	Av = 0.21	Av = 0.14	Av = 0.12

Alignments B extensions (20° S of alignment A40)

1	0.18	0.11	0.09	0.08
2	0.53	0.27	0.09	0.08
3	0.60	0.38	0.23	0.17
6	0.79	0.27	0.18	0.12
8	0.51	0.15	0.11	0.10
9	0.44	0.20	0.14	0.23
17	0.23	0.17	0.15	0.14
20	0.21	0.15	0.15	0.14
28	0.29	0.21	0.20	0.18
	Av = 0.42	Av = 0.21	Av = 0.15	Av = 0.14

TABLE 9.3 (Cont'd)

Computational Ray Model results - 50 years wave S

Alignment A extensions (40m entrance)

Probe	Existing harbour H_S (m)	30m extension H_S (m)	40m extension H_S (m)	50m extension H_S (m)
1	0.18	0.13	0.11	0.10
2	0.19	0.17	0.13	0.11
3	0.48	0.33	0.24	0.17
6	0.20	0.17	0.14	0.12
8	0.20	0.17	0.13	0.11
9	0.70	0.43	0.31	0.30
17	0.38	0.23	0.20	0.19
20	0.21	0.17	0.15	0.15
28	0.54	0.35	0.27	0.23
	Av = 0.34	Av = 0.24	Av = 0.19	Av = 0.16

Alignments B extensions (20° S of alignment A40)

1	0.18	0.15	0.13	0.11
2	0.19	0.20	0.15	0.13
3	0.48	0.38	0.29	0.22
6	0.20	0.20	0.16	0.13
8	0.20	0.19	0.15	0.13
9	0.70	0.50	0.38	0.30
17	0.38	0.28	0.23	0.20
20	0.21	0.20	0.17	0.15
28	0.54	0.42	0.34	0.29
	Av = 0.34	Av = 0.28	Av = 0.22	Av = 0.18

TABLE 9.3 (Cont'd)

Computational Ray Model results - 50 years wave SSW

Probe	Alignment C (29° S of A40)	Alignment D (38° S of A40)	Alignment E (47° S of A40)
50 year wave SE	H _s (m)	H _s (m)	H _s (m)
1	0.30	0.30	0.30
2	0.43	0.41	0.41
3	0.61	0.61	0.61
6	0.57	0.57	0.57
8	0.39	0.39	0.41
9	0.48	0.50	0.50
17	0.61	0.54	0.48
20	0.39	0.26	0.25
28	0.65	0.68	0.65
	Av = 0.49	Av = 0.47	Av = 0.46
50 year wave S			
1	0.08	0.06	0.06
2	0.09	0.10	0.13
3	0.26	0.27	0.30
6	0.18	0.21	0.26
8	0.10	0.10	0.12
9	0.15	0.18	0.21
17	0.15	0.13	0.12
20	0.12	0.06	0.06
28	0.21	0.21	0.18
	Av = 0.15	Av = 0.15	Av = 0.16
50 year wave SSW			
1	0.14	0.14	0.15
2	0.17	0.19	0.20
3	0.34	0.38	0.43
6	0.17	0.19	0.21
8	0.17	0.18	0.19
9	0.44	0.51	0.59
17	0.26	0.28	0.30
20	0.19	0.15	0.15
28	0.39	0.45	0.50
	Av = 0.25	Av = 0.27	Av = 0.30

TABLE 9.4

Computational Ray Model results - alignment tests for 40m extension

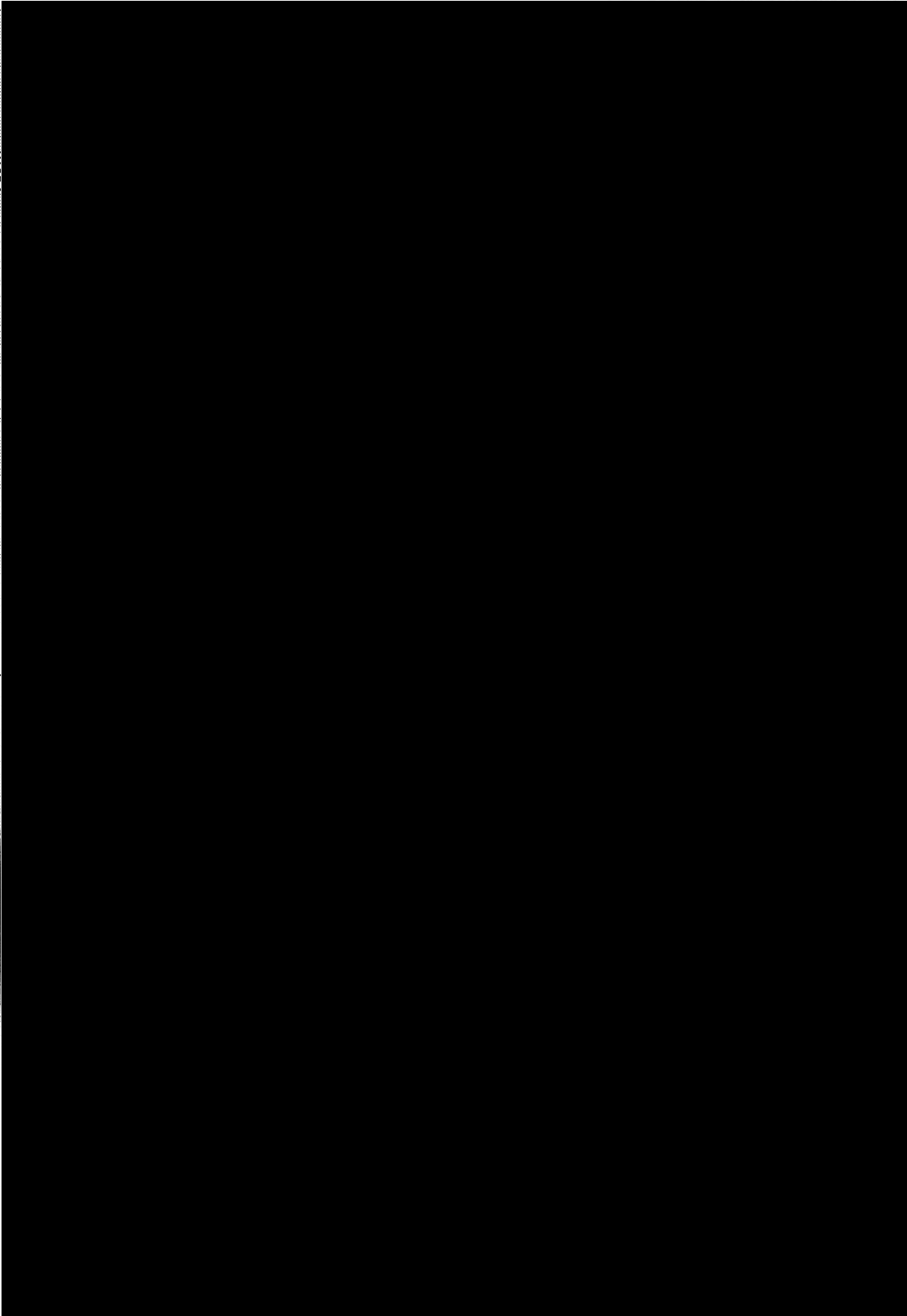


Fig 9.1 Location of Torquay Harbour

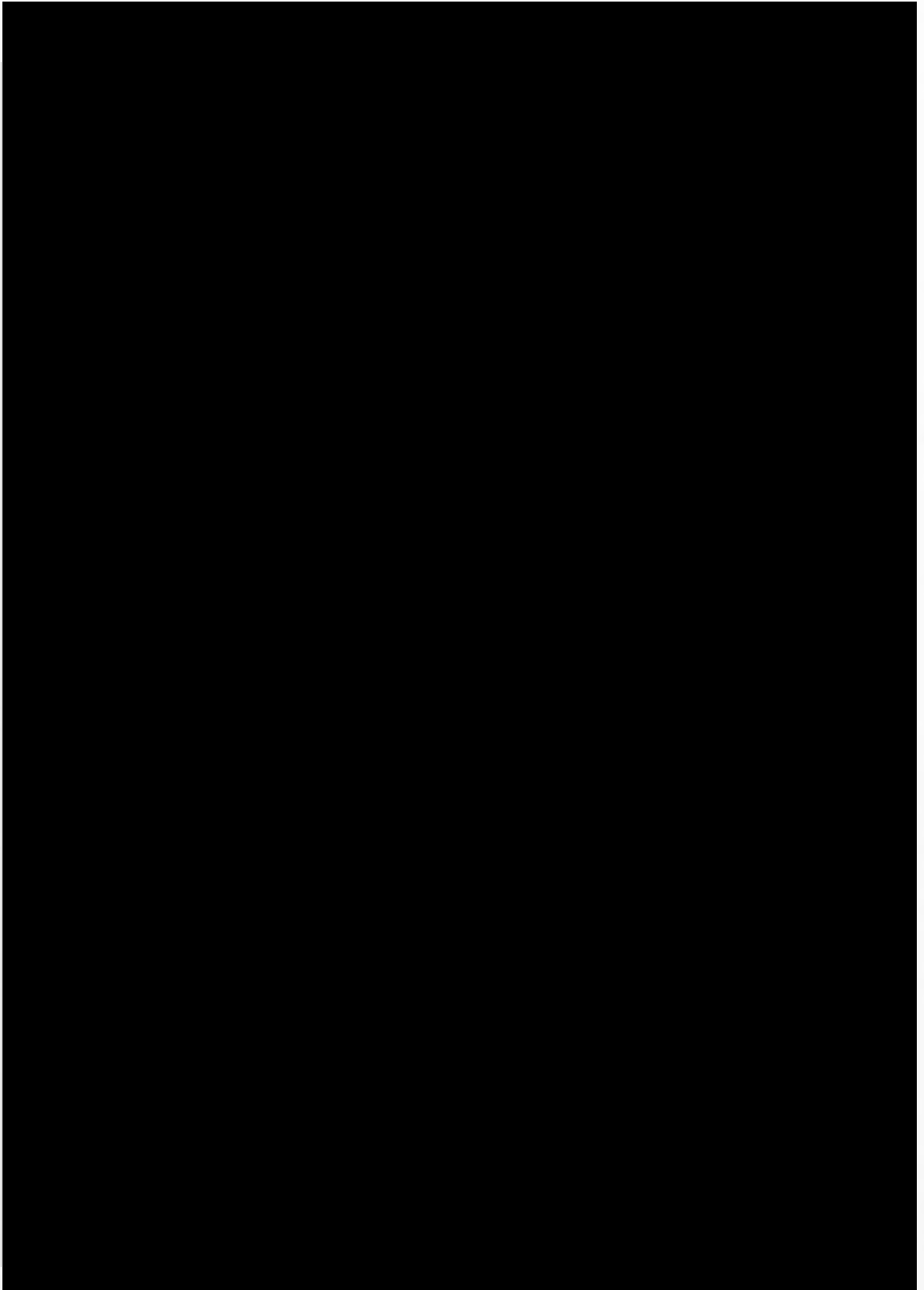


Fig 9.2 Layout of Torquay Harbour in the Physical Model with Original Depth Con tours and Probe Positions.

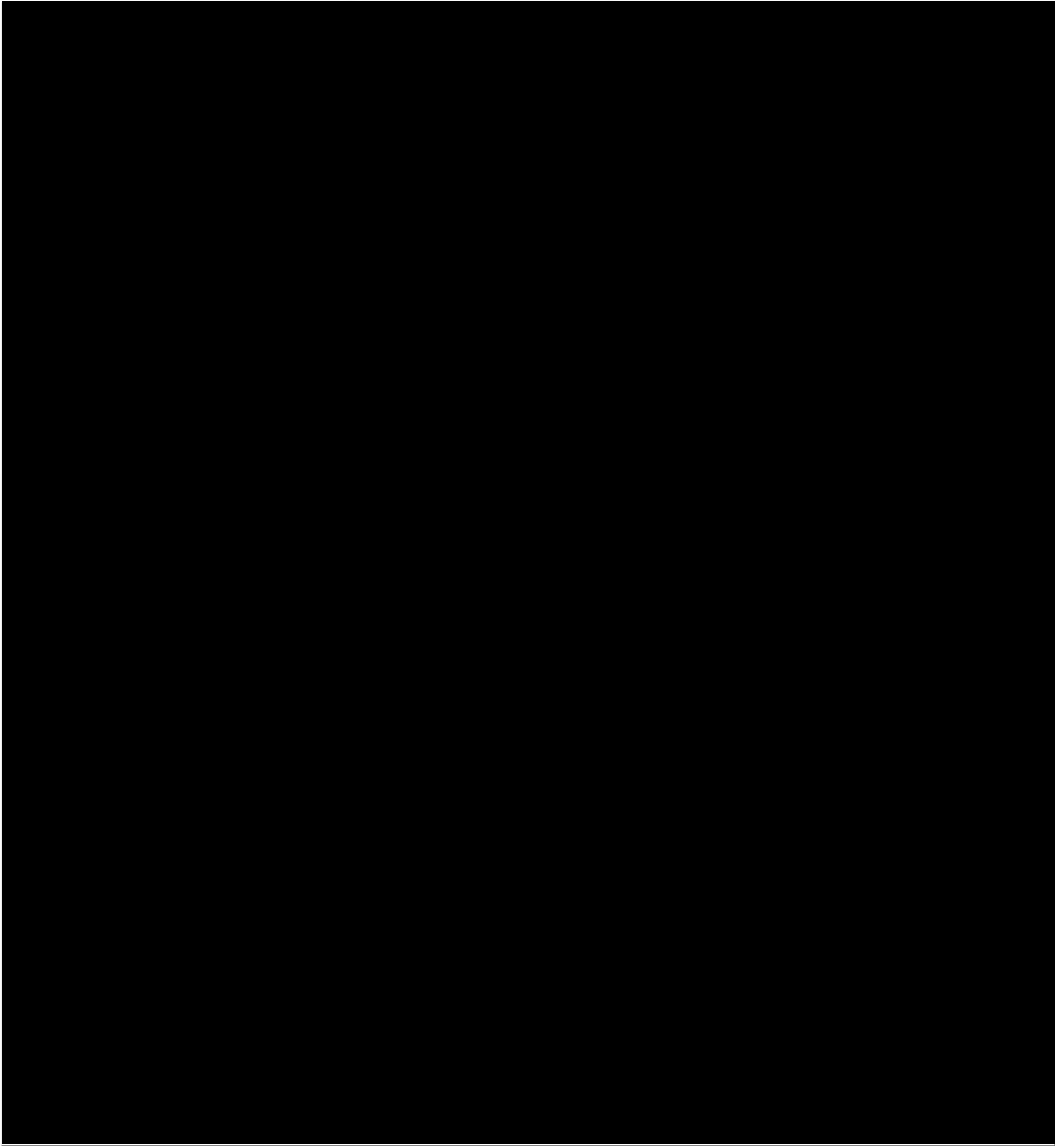


Fig 9.3 Overall Layout of Physical Model

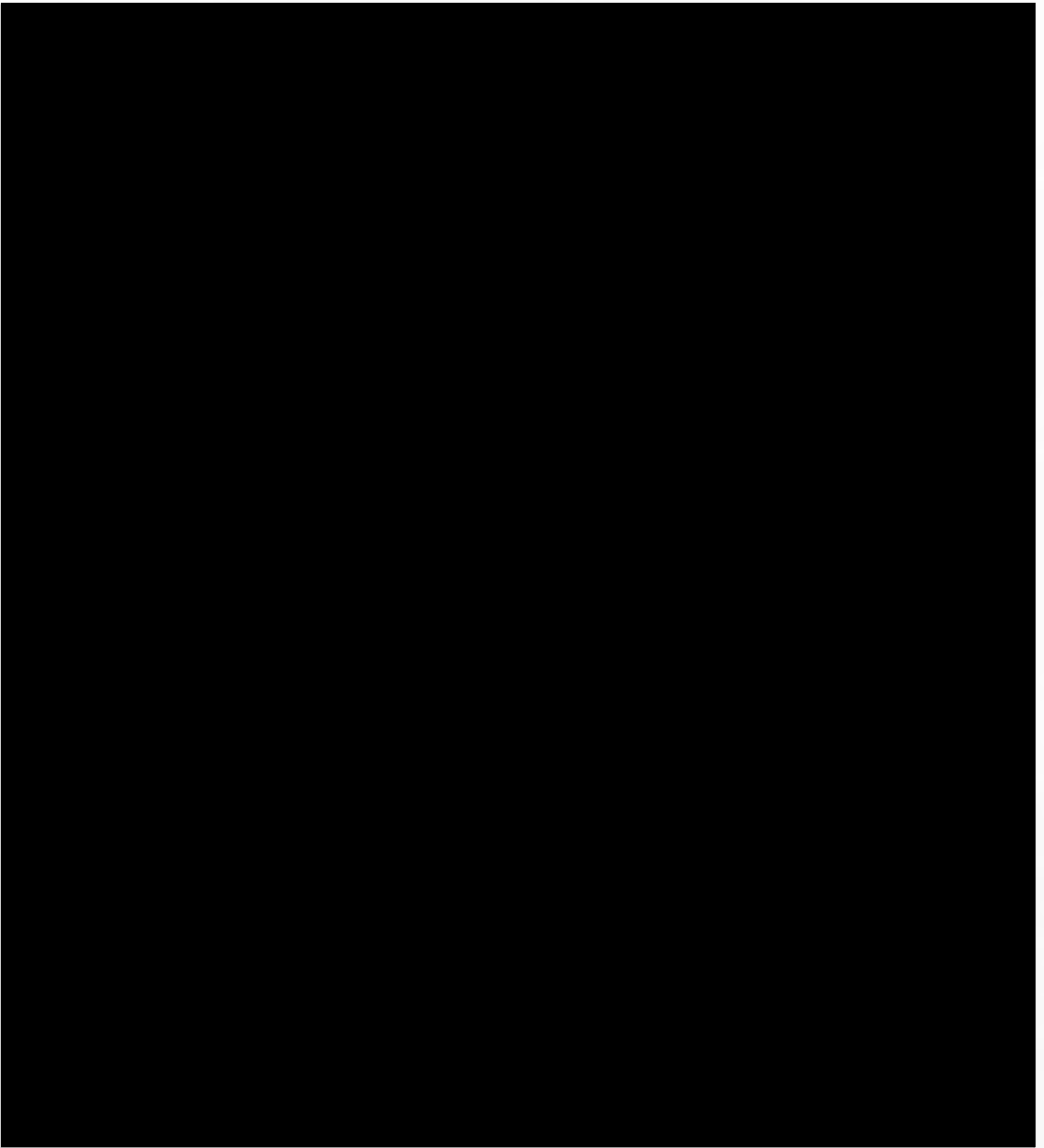


Fig 9.4 Ray Model Layout with Original Depth Contours
and Physical Model Probe Positions

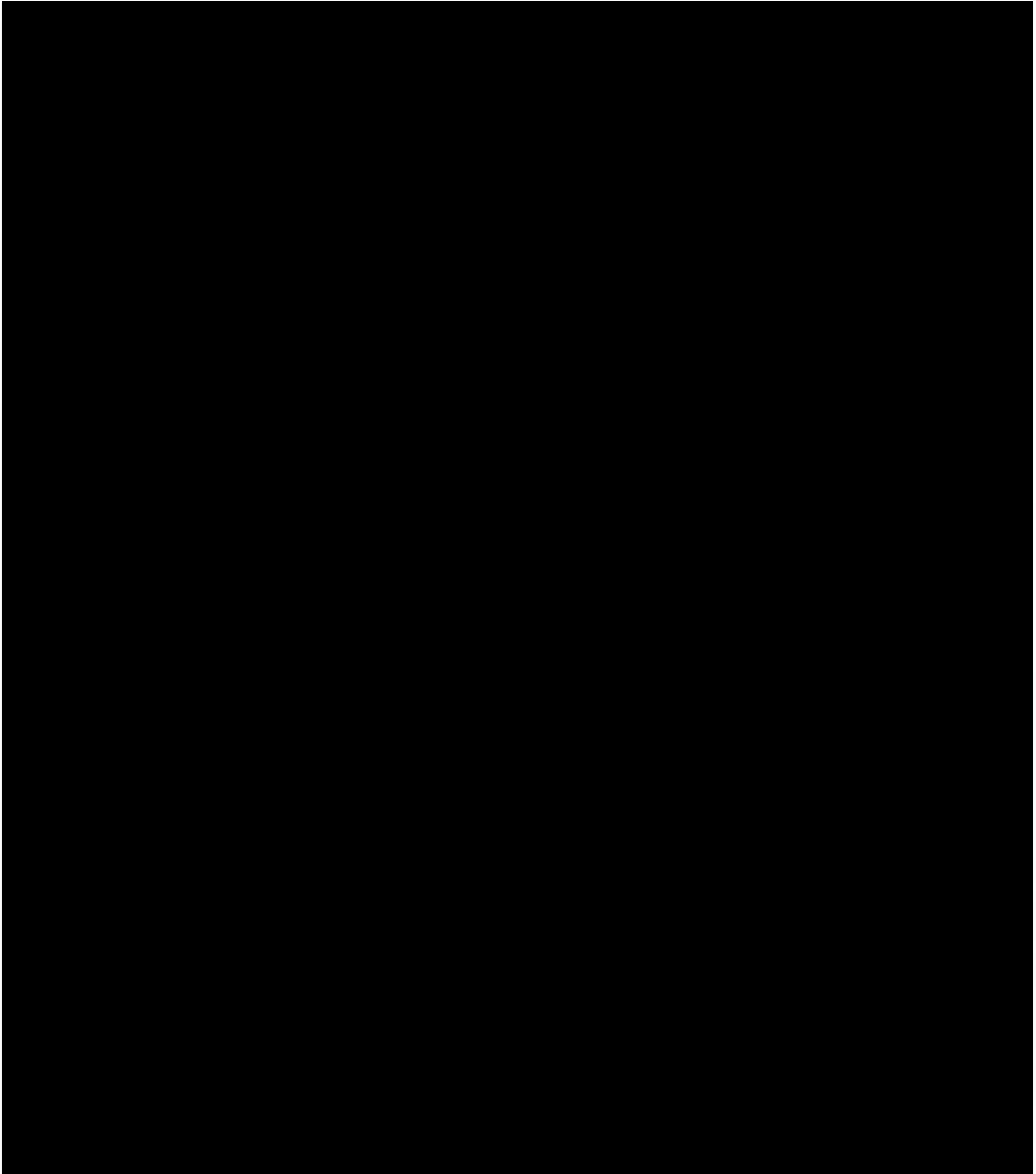


Fig 9.5 Ray Model Wave Height Contours (m). Original Breakwater Length. 1 in 50 Years Wave from SE



Fig 9.6 Ray Model Wave Height Contours (m) . A40 Break-water Extension. 1 in 50 Years Wave from SE.

A summary of the main findings and conclusions of each chapter is given below.

- 1) The prediction of wave activity in harbours is an essential part of almost all harbour design studies. In terms of reproducing physical processes, physical models are the most comprehensive means of investigating harbour wave disturbance, but computational models can provide a cheaper, quicker and more flexible alternative. These merits of computational models complement those of physical models, and the two are often best used in conjunction in a thorough harbour design study.

The physical processes affecting waves in harbours are complex, involving the refraction, diffraction and reflection of waves, and also wave energy dissipation due to seabed friction and breaking. Much recent computational model development has centred on combining the processes of refraction, diffraction and reflection in an integrated manner, but most models of this type are best suited for small harbour areas and/or long wavelengths. The computational model described and developed in this thesis is based on a new ray tracing technique and is designed for the opposite case, of large harbour areas and short waves.

- 2) The new ray tracing method incorporates the diffraction effect of breakwaters by the tracing of additional, suitably chosen sets of rays. It is shown that two types of breakwater layout at harbour entrances, the single breakwater and small breakwater gap, can be well represented by this method. The technique of constructing such a ray system is illustrated by the small breakwater gap problem which has a relatively simple ray system. Subsequent work in the thesis is concerned with the single breakwater problem which presents greater difficulties due to the presence of a shadow boundary separating the sheltered and unsheltered zones.

- 3) An analytical solution for the single breakwater problem, originally formulated by Sommerfeld (1896), is described in detail, and expressions for the wave amplitude, phase and direction at any point are determined. This analytical solution provides the basis for the construction of the new ray method.
- 4) Previous attempts at a ray representation of the single breakwater problem have used radial rays from the breakwater tip. It is shown that these methods fail in an unbounded area surrounding the shadow boundary. Previous methods of correcting this problem have involved the re-introduction of diffraction terms into the governing equations, while in contrast the present method retains a pure ray tracing technique and overcomes the problem by choosing the ray starting points and directions differently. The new ray system consists of sets of rays starting from the shadow boundary. The relationship between the starting angles of the rays and the starting distance along the shadow boundary is determined, and expressions for the wave amplitude and phase along rays are developed. The method is demonstrated to give good accuracy in predicting amplitudes not only in the far field of the quantity u (Eq 3.2.8), but also in the nearfield along the shadow boundary where $u = 0$. Moving outwards along this line the ray method amplitudes tend very rapidly to the correct value. At a distance of one wavelength from the breakwater tip the error is only 0.6%. Phases are in error by up to 45° , but this is of little practical significance unless there exist phase-locked intersecting wave trains.
- 5) The construction of the ray method is such that it is theoretically applicable to seabeds with a random spatial variation of depths, and in which refraction is therefore combined with diffraction. Some of the important programming techniques for the general case of combined refraction, diffraction and reflection are described in this chapter.

- 6) The outcome of the ray tracing method is to produce a series of ray paths over the gridded area of interest (e.g. Fig 8.26), and it is therefore necessary to have a means of obtaining wave amplitudes and directions from these ray paths. A new method of achieving this is described by averaging the effects of rays as they pass through each depth grid element or simple combination of elements. This method has advantages over alternative methods in that intersecting wave trains can be taken into account, unrealistic ray patterns such as caustics are 'smoothed over' mimicking to some extent the process of internal diffraction, and results are obtained in a regular array covering the harbour. Sensitivity tests have indicated that a maximum square size per wavelength of 2.5 gives errors in amplitude of at worst 4% (for the case where phasing of intersecting waves is taken into account). Investigation of the sensitivity of results with respect to ray density indicates amplitude errors no greater than about 5% for ray densities as low as four or five rays per square side.
- 7) Although it was found that the new ray system (method A) for the single breakwater problem gave good predictions of wave amplitude and direction, the wave phase was poorly predicted in the far field of u. An alternative ray system (method B) was constructed with the aim of removing this deficiency, but possibly at the expense of the good predictions of amplitude and direction achieved by method A. Comprehensive comparisons of the two methods, and one based on radial rays from the breakwater tip, were made with the analytical solution. It was found that method A gave very good predictions of amplitude and direction at all points (except within about one tenth of a wavelength of the breakwater tip), but that method B and the radial ray method performed considerably worse. Method B did give better predictions of phase. Since amplitude and direction (rather than phase) are the important quantities in practical applications, method A was the preferred choice. The computational model based on this method was tested against a finite-element model on flat

and sloping seabeds, with agreement to at worst about 5%. The ray model was about ten times faster computationally, a factor which should increase for larger grids.

- 8) The ray model was compared with measured wave heights from a random wave physical model of an actual harbour design, in which refraction, diffraction and reflection were all important processes. Agreement was generally good for design purposes throughout the harbour although the physical model displayed some variability in wave height across frequency bands, and at two locations there was evidence of strong non-linear effects. The importance of including diffraction, refraction and reflection in combination in the computational model was demonstrated by repeating the runs on a flat seabed (no refraction) and with zero reflection coefficients on the boundaries (no reflection).
- 9) The ray model was used in a commercial investigation in conjunction with a physical model, enabling an assessment of its merit as a commercial tool to be made. The ray model was able to investigate a large number of options initially, including extensions to and alignments of breakwaters, and variations to the bathymetry, for a variety of design incident wave conditions. The physical model could then be used to investigate the most promising options in more detail.

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