



City Research Online

City St George's, University of London

Citation: Jørgensen, P. & Shah, A. (2024). Grothendieck groups of d-exangulated categories and a modified Caldero-Chapoton map. *Journal of Pure and Applied Algebra*, 228(5), 107587. doi: 10.1016/j.jpaa.2023.107587

This is the published version of the paper.

This version of the publication may differ from the final published version. To cite this item please consult the publisher's version.

Permanent repository link: <https://openaccess.city.ac.uk/id/eprint/35899/>

Link to published version: <https://doi.org/10.1016/j.jpaa.2023.107587>

Copyright and Reuse: Copyright and Moral Rights remain with the author(s) and/or copyright holders. Copies of full items can be used for personal research or study, educational, or not-for-profit purposes without prior permission or charge, unless otherwise indicated, provided that the authors, title and full bibliographic details are credited, a hyperlink and/or URL is given for the original metadata page and the content is not changed in any way. For full details of reuse please refer to [City Research Online policy](#).



Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaaGrothendieck groups of d -exangulated categories and a modified Caldero-Chapoton map

Peter Jørgensen, Amit Shah *

Department of Mathematics, Aarhus University, Ny Munkegade 118, 8000 Aarhus C, Denmark

ARTICLE INFO

Article history:

Received 21 June 2021

Received in revised form 22 July 2023

Available online 19 December 2023

Communicated by J. Pevtsova

MSC:

Primary: 16E20; secondary: 13F60; 18E05; 18E10; 18G80

Keywords:

Caldero-Chapoton map

Grothendieck group

Index

 n -cluster tilting subcategory n -exangulated category

Rigid subcategory

ABSTRACT

A strong connection between cluster algebras and representation theory was established by the cluster category. Cluster characters, like the original Caldero-Chapoton map, are maps from certain triangulated categories to cluster algebras and they have generated much interest. Holm and Jørgensen constructed a modified Caldero-Chapoton map from a sufficiently nice triangulated category to a commutative ring, which is a generalised frieze under some conditions. In their construction, a quotient $K_0^{\text{sp}}(\mathcal{T})/M$ of a Grothendieck group of a cluster tilting subcategory $\mathcal{T}$ is used. In this article, we show that this quotient is the Grothendieck group of a certain extriangulated category, thereby exposing the significance of it and the relevance of extriangulated structures. We use this to define another modified Caldero-Chapoton map that recovers the one of Holm–Jørgensen.

We prove our results in a higher homological context. Suppose $\mathcal{S}$ is a $(d+2)$ -angulated category with subcategories $\mathcal{X} \subseteq \mathcal{T} \subseteq \mathcal{S}$, where $\mathcal{X}$ is functorially finite and $\mathcal{T}$ is $2d$ -cluster tilting, satisfying some mild conditions. We show there is an isomorphism between the Grothendieck group $K_0(\mathcal{S}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$ of the category $\mathcal{S}$, equipped with the d -exangulated structure induced by $\mathcal{X}$, and the quotient $K_0^{\text{sp}}(\mathcal{T})/N$, where N is the higher analogue of M above. When $\mathcal{X} = \mathcal{T}$ the isomorphism is induced by the higher index with respect to $\mathcal{T}$ introduced recently by Jørgensen. Thus, in the general case, we can understand the map taking an object in $\mathcal{S}$ to its K_0 -class in $K_0(\mathcal{S}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$ as a higher index with respect to the rigid subcategory $\mathcal{X}$.

© 2023 The Author(s). Published by Elsevier B.V. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>).

1. Introduction

Cluster algebras were introduced by Fomin and Zelevinsky in [24] and have since seen links to several different fields, such as integrable systems, Poisson geometry and particle physics. The cluster category associated to a hereditary algebra, defined in [13], is a categorification of the corresponding cluster algebra. This relationship is exhibited by the so-called *Caldero-Chapoton map*, which is a cluster character that

* Corresponding author.

E-mail addresses: peter.jorgensen@math.au.dk (P. Jørgensen), amit.shah@math.au.dk (A. Shah).

associates elements of the cluster algebra to certain types of objects of the cluster category; see, for example, [14], [15], [16], [18], [19], [20], [25], [30], [31], [36], [44], [46], [52].

Let us recall a version of the Caldero-Chapoton map equivalent to the original of [14, Sec. 3]. For unexplained notation, see Subsection 1.1 below. Let k be an algebraically closed field and let $\mathcal{C}$ be a k -linear, Hom-finite, Krull-Schmidt, triangulated category that is 2-Calabi-Yau. Denote by Σ the suspension functor of $\mathcal{C}$. The Caldero-Chapoton map depends on the choice of a cluster tilting object $T \in \mathcal{C}$, which we assume to be basic. Set $\mathcal{T} = \text{add } T$. Define the functor $G: \mathcal{C} \rightarrow \text{mod } \mathcal{T}$ by $G_T(C) = \mathcal{C}(T, \Sigma C)$. Since $\mathcal{T}$ is cluster tilting, for each object $C \in \mathcal{C}$ there is a triangle $T^0 \rightarrow T^1 \rightarrow C \rightarrow \Sigma T^0$, and we put $\text{ind}_{\mathcal{T}}(C) = [T^1] - [T^0]$, which is an element of $K_0^{\text{sp}}(\mathcal{T})$ known as the *index* (with respect to $\mathcal{T}$) of C . There is a group homomorphism $\bar{\varphi}: K_0(\text{mod } \mathcal{T}) \rightarrow K_0^{\text{sp}}(\mathcal{T})$, measuring how far the index is from being additive over triangles in $\mathcal{C}$ (see [35, Thm. 4.4], or Section 5). Put $A = \mathbb{Z}[x_{T'}, x_{T'}^{-1}]_{T' \in \text{ind } \mathcal{T}}$. Define maps $\varepsilon: K_0^{\text{sp}}(\mathcal{T}) \rightarrow A$, $\alpha: \text{obj}(\mathcal{C}) \rightarrow A$ and $\beta: K_0(\text{mod } \mathcal{T}) \rightarrow A$ as follows: $\varepsilon([T']) = x_{T'}$ for $T' \in \text{ind } \mathcal{T}$; $\alpha = \varepsilon \circ \text{ind}_{\mathcal{T}}$; and $\beta = \varepsilon \circ \bar{\varphi}$. Assume also that ε is “exponential” in the sense that $\varepsilon(0) = 1$ and $\varepsilon(e + f) = \varepsilon(e)\varepsilon(f)$. Then the formula of the Caldero-Chapoton map given in [36, 1.8] is

$$\rho(C) = \alpha(C) \sum_e \chi(\text{Gr}_e G_T(C)) \beta(e), \quad (1.1)$$

where $\text{Gr}_e(G_T(C))$ is the Grassmannian of submodules of $\mathcal{C}(T, \Sigma C)$ in $\text{mod } \mathcal{T}$ with K_0 -class e in $K_0(\text{mod } \mathcal{T})$, and χ is the Euler characteristic defined by étale cohomology with proper support.

From its initial context, the Caldero-Chapoton map has been studied and generalised. For example, in [31] an extension of the map was given dealing with the rigid subcategory case and where A is a general commutative ring. Let $\mathcal{X} = \text{add } R$ be a *rigid* subcategory (i.e. $\mathcal{C}(\mathcal{X}, \Sigma \mathcal{X}) = 0$) contained in the cluster tilting subcategory $\mathcal{T}$. In this generalisation, Holm and Jørgensen make the following substitutions; see Section 5 for more details.

(i) ε is replaced with a map $\bar{\varepsilon}: K_0^{\text{sp}}(\mathcal{T})/N \rightarrow A$, where

$$N = \left\langle [X] - [Y] \mid \begin{array}{l} T \rightarrow Y \rightarrow T^* \rightarrow \Sigma T \text{ and } T^* \rightarrow X \rightarrow T \rightarrow \Sigma T^* \\ \text{are exchange triangles with } T \in \text{ind } \mathcal{T} \setminus \text{ind } \mathcal{X} \end{array} \right\rangle.$$

(ii) α is replaced by $\bar{\varepsilon}Q \text{ind}_{\mathcal{T}}$, where $Q: K_0^{\text{sp}}(\mathcal{T}) \rightarrow K_0^{\text{sp}}(\mathcal{T})/N$ is the canonical surjection.

(iii) β is replaced by $\bar{\varepsilon}\nu$, where $\nu: K_0(\text{mod } \mathcal{X}) \rightarrow K_0^{\text{sp}}(\mathcal{T})/N$ is the unique homomorphism making

$$\begin{array}{ccc} K_0(\text{mod } \mathcal{T}) & \xrightarrow{\bar{\varphi}} & K_0^{\text{sp}}(\mathcal{T}) \\ \kappa \downarrow & & \downarrow Q \\ K_0(\text{mod } \mathcal{X}) & \xrightarrow{\nu} & K_0^{\text{sp}}(\mathcal{T})/N \end{array}$$

commute, in which κ is induced by the inclusion $\mathcal{X} \subseteq \mathcal{T}$.

(iv) G_T and $\text{Gr}_e(G_T(C))$ are replaced by similarly defined analogues G_R and $\text{Gr}_e(G_R(C))$, respectively.

With these adjustments, (1.1) is then called the *modified Caldero-Chapoton map* in [31]. Note that if $\mathcal{X} = \mathcal{T}$ is cluster tilting then $N = 0$ and the domains of ε and $\bar{\varepsilon}$ agree. Furthermore, [31, Thm. A] showed that the modified Caldero-Chapoton map ρ is a *generalised frieze* (see [30, Def. 3.4]) under suitable assumptions, thereby demonstrating this theory has particular implications in combinatorics. This map also recovers the combinatorial generalised friezes of [10]; see also [17].

Although $K_0^{\text{sp}}(\mathcal{T})/N$ is observably a correct group to consider as the domain for $\bar{\varepsilon}$, there has also been other considerable interest in this quotient. For instance, Palu proved that $K_0^{\text{sp}}(\mathcal{T})/N$ recovers the

Grothendieck group of the triangulated category $\mathcal{C}$ when $\mathcal{X} = 0$ (see [45, Thm. 10]). In this case, N is generated by relations coming from all pairs of exchange triangles for indecomposable objects in $\mathcal{T}$. These results were generalised to the higher angulated case by Fedele in [22]. However, there is not yet an explanation of the significance of $K_0^{\text{sp}}(\mathcal{T})/N$, even though it shows up in different contexts, indicating its importance and some deeper connection.

Motivated by this, and inspired by an approach taken in [43], in the present article we remedy this by giving an interpretation of $K_0^{\text{sp}}(\mathcal{T})/N$ as the Grothendieck group of a certain *extriangulated* category dependent on $\mathcal{X}$; see Theorem A. In particular, this shows that this group is indeed *the* correct group to use in the construction of the modified Caldero-Chapoton map and why. Conversely, the interest in $K_0^{\text{sp}}(\mathcal{T})/N$ and its appearance as the Grothendieck group of an extriangulated category illustrates the relevance of extriangulated and, more generally, higher exangulated structures. Although we prove our results in the more general setting of higher homological algebra, we will remain in the classical setting for the purposes of this introduction, which is also the setting of Section 5.

Since $\mathcal{C}$ is triangulated, we have that $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an extriangulated category, where $\mathbb{E} = \mathcal{C}(-, \Sigma -)$ is a biadditive functor and $\mathfrak{s}$ is the (canonical) realisation of $\mathbb{E}$ (see [41, Prop. 3.22] or Example 2.7). There is a subfunctor of $\mathbb{E}$ given as follows. Define $\mathbb{E}_{\mathcal{X}}$ on objects by

$$\mathbb{E}_{\mathcal{X}}(C, A) = \{ \delta \in \mathcal{C}(C, \Sigma A) \mid \delta \circ \gamma = 0 \text{ for all } \gamma: X \rightarrow C \text{ with } X \in \mathcal{X} \}.$$

Then, by equipping $\mathcal{C}$ with $\mathbb{E}_{\mathcal{X}}$ instead, we obtain another extriangulated structure $(\mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$ on $\mathcal{C}$, where $\mathfrak{s}_{\mathcal{X}}$ is the restriction of $\mathfrak{s}$ to $\mathbb{E}_{\mathcal{X}}$. The group $K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$ is then defined to be the quotient of $K_0^{\text{sp}}(\mathcal{C})$ by the subgroup generated by elements $[A] - [B] + [C]$, where $A \rightarrow B \rightarrow C \xrightarrow{\delta} \Sigma A$ is a triangle in $\mathcal{C}$ with $\delta \in \mathbb{E}_{\mathcal{X}}(C, A)$. See Section 2 for more details. As a special case of Theorem 4.18, we have:

Theorem A. *There is an isomorphism $K_0^{\text{sp}}(\mathcal{T})/N \cong K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$.*

In case $\mathcal{X} = \mathcal{T}$, the isomorphism $K_0(\mathcal{C}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}}) \rightarrow K_0^{\text{sp}}(\mathcal{T})$ is induced by the index $\text{ind}_{\mathcal{T}}: \text{obj}(\mathcal{C}) \rightarrow K_0^{\text{sp}}(\mathcal{T})$; see [43, Prop. 4.11]. The authors are very grateful to P.-G. Plamondon, who pointed out this result from [43] to the first author. This led to the development of Theorem 3.10, which is a higher dimensional version. This cluster tilting special case suggests that we can think of the surjection $Q_{\mathcal{X}}: K_0^{\text{sp}}(\mathcal{C}) \rightarrow K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$ as an index with respect to $\mathcal{X}$. Furthermore, there is also an additivity formula with error-term for the homomorphism $Q_{\mathcal{X}}$ (see [37]). Moreover, using $Q_{\mathcal{X}}$ and a map that also behaves like an error-term map under suitable conditions (see Remark 5.6), we give a new version of the Caldero-Chapoton map, which we call the $\mathcal{X}$ -Caldero-Chapoton map; see Definition 5.3. This recovers the modified Caldero-Chapoton map from [31].

This paper is organised as follows. In Section 2, we cover the definitions and results we will need on n -exangulated categories, including their Grothendieck groups and n -exangulated structures induced by subcategories. In Section 3 we show, for an Oppermann-Thomas cluster tilting subcategory $\mathcal{T}$ of a $(d+2)$ -angulated category $\mathcal{S}$, there is an isomorphism between the split Grothendieck group of $\mathcal{T}$ and the Grothendieck group of $\mathcal{S}$ with d -exangulated structure induced by $\mathcal{T}$. In Section 4 we prove the higher version of Theorem A. Lastly, in Section 5 we apply our results to the classical setting and define the $\mathcal{X}$ -Caldero-Chapoton map.

1.1. Conventions and notation

- (i) **Ab** denotes the category of all abelian groups.
- (ii) By an *additive subcategory*, we will mean a full subcategory that is closed under isomorphisms, direct sums and direct summands. Note that if $\mathcal{B}$ is an additive subcategory of $\mathcal{A}$ and $\mathcal{A}$ is idempotent complete, then $\mathcal{B}$ is also idempotent complete since it is closed under direct summands.

- (iii) Let $\mathcal{A}$ be an additive category. For a subcategory $\mathcal{B}$ of $\mathcal{A}$ that is closed under finite direct sums, $[\mathcal{B}]$ denotes the two-sided ideal of $\mathcal{A}$ consisting of morphisms factoring through $\mathcal{B}$.

Fix a skeleton $\mathcal{A}^{\text{skel}}$ of $\mathcal{A}$. We denote by $\text{ind } \mathcal{A}$ the class of objects of $\mathcal{A}^{\text{skel}}$ consisting only of indecomposable objects. Furthermore, if $\mathcal{B} \subseteq \mathcal{A}$ is an additive subcategory, we assume that skeletons are chosen in a compatible way, i.e. $\mathcal{B}^{\text{skel}} \subseteq \mathcal{A}^{\text{skel}}$, so in particular $\text{ind } \mathcal{B} \subseteq \text{ind } \mathcal{A}$.

- (iv) For a field k , we denote by $\text{Mod } k$ (respectively, $\text{mod } k$) the category of all (respectively, finite-dimensional) k -vector spaces.
- (v) Suppose $\mathcal{C}$ is a skeletally small, k -linear category. We denote by $\text{Mod } \mathcal{C}$ the category of all k -linear, contravariant functors $\mathcal{C} \rightarrow \text{Mod } k$. The category $\text{Mod } \mathcal{C}$ is k -linear and abelian (see, for example, [48, Thm. 10.1.3] or [47, Sec. 3, Thm. 4.2]).

A functor $M \in \text{Mod } \mathcal{C}$ is called *finitely presented* if there is an exact sequence $\mathcal{C}(-, A) \rightarrow \mathcal{C}(-, B) \rightarrow M \rightarrow 0$ for some $A, B \in \mathcal{C}$; see [6, p. 155]. (These functors have also been called ‘coherent’ in e.g. [1] and [33]. However, there is a more general notion of a coherent functor, see [23, Def. B.5]. This notion and what we call finitely presented agree when, for example, $\mathcal{C}$ is also idempotent complete and has weak kernels. See [23, App. B] and the references therein for more details.) We denote by $\text{mod } \mathcal{C}$ the full subcategory of $\text{Mod } \mathcal{C}$ consisting of all the finitely presented k -linear, contravariant functors $\mathcal{C} \rightarrow \text{Mod } k$. The category $\text{mod } \mathcal{C}$ is also k -linear. If $\mathcal{C}$ is also idempotent complete and has weak kernels (e.g. $\mathcal{C}$ is a contravariantly finite, additive subcategory of an idempotent complete, triangulated category, see [33, Def. 2.9]), then $\text{mod } \mathcal{C}$ is abelian and the inclusion functor $\text{mod } \mathcal{C} \rightarrow \text{Mod } \mathcal{C}$ is exact. See [2, Sec. III.2]; see also [1, Sec. 2] and [4].

Finally, $\text{fl } \mathcal{C}$ denotes the full subcategory of $\text{Mod } \mathcal{C}$ consisting of objects of *finite length* (see [38, Sec. 5]).

- (vi) Let $\mathcal{S}$ be a k -linear category and suppose $\mathcal{D}$ is a skeletally small, additive subcategory of $\mathcal{S}$. We let $F_{\mathcal{D}}: \mathcal{S} \rightarrow \text{mod } \mathcal{D}$ denote the covariant restricted Yoneda functor defined on objects by sending $X \in \mathcal{S}$ to $\mathcal{S}(-, X)|_{\mathcal{D}}$. Denote by $\mathcal{D}^{\perp_0} := \text{Ker } F_{\mathcal{D}}$ the full subcategory of $\mathcal{S}$ consisting of objects X for which $\mathcal{S}(\mathcal{D}, X) = 0$.

2. n -exangulated categories

2.1. The definition

Let $n \geq 1$ be a positive integer. Exact, abelian and triangulated categories (see [49], [27] and [51], respectively) form the core of homological algebra. Indeed, in Sections 3 through 5 of this paper we are in a setting where there is some ambient triangulated category; see Setup 3.3. These ideas have been generalised to higher dimensions by Geiss–Keller–Oppermann in [26], where $(n+2)$ -angulated categories are introduced, and by Jasso in [34], where n -abelian and n -exact categories are defined. In a parallel direction, the theories of exact and triangulated categories have been unified with the introduction of *extriangulated* categories by Nakaoka–Palu in [41].

Naturally, the counterpart in higher homological algebra of extriangulated categories has also been developed—namely, n -exangulated categories as introduced by Herschend–Liu–Nakaoka in [29]. In this subsection, we briefly recall the definition of an n -exangulated category (see Definition 2.6). For more details, we refer the reader to [29, Sec. 2]. We also recall how one can view an $(n+2)$ -angulated category as an n -exangulated category (see Example 2.7).

Let $\mathcal{C}$ be an additive category equipped with a biadditive functor $\mathbb{E}: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Ab}$. A prototypical example when $n = 1$ of an n -exangulated category (or an extriangulated category in the sense of [41]) is an exact category. In this case, $\mathbb{E}$ models the functor Ext^1 , which motivates the following terminology.

Definition 2.1.

- (i) For any $A, C \in \mathcal{C}$, any element of $\mathbb{E}(C, A)$ is called an *extension*.
- (ii) Suppose $\delta \in \mathbb{E}(C, A)$ is an extension, and let $a: A \rightarrow A'$ and $c: C' \rightarrow C$ be arbitrary morphisms in $\mathcal{C}$. We define $a_*\delta := \mathbb{E}(C, a)(\delta) \in \mathbb{E}(C, A')$ and $c^*\delta := \mathbb{E}(c, A)(\delta) \in \mathbb{E}(C', A)$.
- (iii) Suppose $\delta \in \mathbb{E}(C, A)$ and $\eta \in \mathbb{E}(D, B)$ are extensions. A *morphism of extensions* $\delta \rightarrow \eta$ is a pair (a, c) of morphisms $a: A \rightarrow B$ and $c: C \rightarrow D$ in $\mathcal{C}$, such that $a_*\delta = c^*\eta$.

Let $\delta \in \mathbb{E}(C, A)$ be an extension. By the Yoneda Lemma, there is a natural transformation $\delta^\sharp: \mathcal{C}(A, -) \Rightarrow \mathbb{E}(C, -)$, given by $(\delta^\sharp)_B(a) = a_*\delta$ for each object $B \in \mathcal{C}$ and each morphism $a: A \rightarrow B$. Dually, there is also a natural transformation $\delta_\sharp: \mathcal{C}(-, C) \Rightarrow \mathbb{E}(-, A)$.

We denote the *category of (co)complexes* in $\mathcal{C}$ by $\mathcal{C}_C$. Note that we use cohomological numbering throughout. Since we will be interested in equipping categories with certain classes of complexes with $n+2$ terms, it is helpful to introduce the following: the full subcategory of complexes concentrated in (cohomological) degrees $0, 1, \dots, n, n+1$ will be denoted by $\mathcal{C}_C^n$. Since a morphism $f^\bullet: X^\bullet \rightarrow Y^\bullet$ in $\mathcal{C}_C^n$ will be of the form $(\dots, 0, 0, f^0, f^1, \dots, f^{n+1}, 0, 0, \dots)$, we will simply write $f^\bullet = (f^0, \dots, f^{n+1})$. Moreover, we will restrict our attention to objects of $\mathcal{C}_C^n$ that also have a connection with an extension.

Definition 2.2. Let $X^\bullet \in \mathcal{C}_C^n$ and $\delta \in \mathbb{E}(X^{n+1}, X^0)$. The pair $\langle X^\bullet, \delta \rangle$ is an *n-exangle* if

$$\mathcal{C}(-, X^0) \xrightarrow{\mathcal{C}(-, d_X^0)} \mathcal{C}(-, X^1) \xrightarrow{\mathcal{C}(-, d_X^1)} \dots \xrightarrow{\mathcal{C}(-, d_X^n)} \mathcal{C}(-, X^{n+1}) \xrightarrow{\delta_\sharp} \mathbb{E}(-, X^0)$$

and

$$\mathcal{C}(X^{n+1}, -) \xrightarrow{\mathcal{C}(d_X^n, -)} \mathcal{C}(X^n, -) \xrightarrow{\mathcal{C}(d_X^{n-1}, -)} \dots \xrightarrow{\mathcal{C}(d_X^0, -)} \mathcal{C}(X^0, -) \xrightarrow{\delta^\sharp} \mathbb{E}(X^{n+1}, -)$$

are exact sequences of functors.

Definition 2.3. Fix objects A, C in $\mathcal{C}$. The (not necessarily full) subcategory $\mathcal{C}_{(A,C)}^n$ of $\mathcal{C}_C^n$ is defined as follows: an object of $\mathcal{C}_{(A,C)}^n$ is a complex $X^\bullet \in \mathcal{C}_C^n$ for which $X^0 = A$ and $X^{n+1} = C$; and a morphism $f^\bullet: X^\bullet \rightarrow Y^\bullet$ in $\mathcal{C}_{(A,C)}^n$ is a commutative diagram

$$\begin{array}{ccccccc} A & \longrightarrow & X^1 & \longrightarrow & \dots & \longrightarrow & X^n & \longrightarrow & C \\ \parallel & & \downarrow f^1 & & & & \downarrow f^n & & \parallel \\ A & \longrightarrow & Y^1 & \longrightarrow & \dots & \longrightarrow & Y^n & \longrightarrow & C \end{array}$$

in $\mathcal{C}$.

For $A, C \in \mathcal{C}$ and $X^\bullet, Y^\bullet \in \mathcal{C}_{(A,C)}^n$, the usual notion of a *homotopy* between morphisms in $\mathcal{C}_C$ gives an equivalence relation on $\mathcal{C}_{(A,C)}^n(X^\bullet, Y^\bullet)$, which we denote by $\sim$. Thus, we can form a specialised homotopy category, denoted $\mathcal{K}_{(A,C)}^n$, which has the same objects as $\mathcal{C}_{(A,C)}^n$ and, for $X^\bullet, Y^\bullet \in \mathcal{K}_{(A,C)}^n$, we set $\mathcal{K}_{(A,C)}^n(X^\bullet, Y^\bullet) := \mathcal{C}_{(A,C)}^n(X^\bullet, Y^\bullet)/\sim$. Furthermore, we call $f^\bullet \in \mathcal{C}_{(A,C)}^n(X^\bullet, Y^\bullet)$ a *homotopy equivalence* if its image in $\mathcal{K}_{(A,C)}^n(X^\bullet, Y^\bullet)$ is an isomorphism. In this case, the complexes $X^\bullet$ and $Y^\bullet$ are said to be *homotopy equivalent*, and the homotopy equivalence class of $X^\bullet$ in $\mathcal{C}_{(A,C)}^n$ is denoted $[X^\bullet]$. This is *not* necessarily the usual homotopy equivalence class of $X^\bullet$ in $\mathcal{C}_C^n$; see [29, Rem. 2.18].

Just as Ext^1 groups in an abelian category can be viewed as equivalence classes of short exact sequences, in an *n-exangulated* category homotopy equivalence classes will be associated to extensions.

Definition 2.4. Let $\mathfrak{s}$ be a correspondence, which assigns to each $\delta \in \mathbb{E}(C, A)$, for each $A, C \in \mathcal{C}$, a homotopy equivalence class $\mathfrak{s}(\delta) = [X^\bullet]$ for some $X^\bullet \in \mathcal{C}_{(A, C)}^n$. Then $\mathfrak{s}$ is called an *exact realisation of $\mathbb{E}$* if the following three conditions are met.

- (R0) For each $\delta \in \mathbb{E}(C, A), \eta \in \mathbb{E}(D, B)$ such that $\mathfrak{s}(\delta) = [X^\bullet], \mathfrak{s}(\eta) = [Y^\bullet]$, and for each morphism $(a, c): \delta \rightarrow \eta$ of extensions, there exists a morphism $f^\bullet = (f^0, \dots, f^{n+1}) \in \mathcal{C}_{\mathcal{C}}^n(X^\bullet, Y^\bullet)$ realising (a, c) , i.e. with $f^0 = a$ and $f^{n+1} = c$.
- (R1) If $\mathfrak{s}(\delta) = [X^\bullet]$, then $\langle X^\bullet, \delta \rangle$ is an n -exangle.
- (R2) Let $A \in \mathcal{C}$. Consider the group identity elements ${}_A 0_0 \in \mathbb{E}(0, A)$ and ${}_0 0_A \in \mathbb{E}(A, 0)$. Then

$$\mathfrak{s}({}_A 0_0) = [A \xrightarrow{1_A} A \longrightarrow 0 \longrightarrow \dots \longrightarrow 0]$$

and

$$\mathfrak{s}({}_0 0_A) = [0 \longrightarrow \dots \longrightarrow 0 \longrightarrow A \xrightarrow{1_A} A].$$

If $\mathfrak{s}$ is an exact realisation of $\mathbb{E}$ and

$$\mathfrak{s}(\delta) = [X^\bullet] = [X^0 \xrightarrow{d_X^0} X^1 \longrightarrow \dots \longrightarrow X^n \xrightarrow{d_X^n} X^{n+1}],$$

then we call d_X^0 an $\mathfrak{s}$ -inflation and d_X^n an $\mathfrak{s}$ -deflation.

In order to simplify the statement of the main definition in this subsection, we recall the following.

Definition 2.5. [29, Def. 2.27] The *mapping cone* $M^\bullet := \text{MC}(f)^\bullet$ of a morphism $f^\bullet \in \mathcal{C}_{\mathcal{C}}^n(X^\bullet, Y^\bullet)$ with $f_0 = 1_A$ for $X_0 = A = Y_0$ is the complex

$$X^1 \xrightarrow{d_M^0} X^2 \oplus Y^1 \xrightarrow{d_M^1} X^3 \oplus Y^2 \xrightarrow{d_M^2} \dots \xrightarrow{d_M^{n-1}} X^{n+1} \oplus Y^n \xrightarrow{d_M^n} Y^{n+1}$$

in $\mathcal{C}_{\mathcal{C}}^n$, where $d_M^0 := \begin{pmatrix} -d_X^1 \\ f^1 \end{pmatrix}$,

$$d_M^i := \begin{pmatrix} -d_X^{i+1} & 0 \\ f^{i+1} & d_Y^i \end{pmatrix}$$

for $i \in \{1, \dots, n-1\}$, and $d_M^n := (f^{n+1} \quad d_Y^n)$.

With this terminology in place, we can define an n -exangulated category.

Definition 2.6. An *n -exangulated category* is a triplet $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$, where $\mathcal{C}$ is an additive category, $\mathbb{E}: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Ab}$ is a biadditive functor, $\mathfrak{s}$ is an exact realisation of $\mathbb{E}$, and where the following conditions are satisfied.

- (EA1) The classes of $\mathfrak{s}$ -inflations and of $\mathfrak{s}$ -deflations are each closed under composition.
- (EA2) Suppose there are $\delta \in \mathbb{E}(D, A)$ and $c \in \mathcal{C}(C, D)$, with $\mathfrak{s}(c^*\delta) = [X^\bullet]$ and $\mathfrak{s}(\delta) = [Y^\bullet]$ for some $X^\bullet, Y^\bullet \in \mathcal{C}_{\mathcal{C}}^n$. Then there exists a morphism $f^\bullet = (1_A, f^1, \dots, f^n, c): X^\bullet \rightarrow Y^\bullet$ realising $(1_A, c): c^*\delta \rightarrow \delta$, i.e. a commutative diagram

$$\begin{array}{ccccccc} c^*\delta & & A = X^0 & \xrightarrow{d_X^0} & X^1 & \longrightarrow & \dots \longrightarrow X^n \longrightarrow X^{n+1} = C \\ (1_A, c) \downarrow & & \parallel & & \downarrow f^1 & & \downarrow f^n & & \downarrow c \\ \delta & & A = Y^0 & \longrightarrow & Y^1 & \longrightarrow & \dots \longrightarrow Y^n \longrightarrow Y^{n+1} = D, \end{array}$$

such that $\mathfrak{s}((d_X^0)_*\delta) = [\text{MC}(f)^\bullet]$.

(EA2)^{op} Dual of (EA2).

If $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an n -exangulated category and $\mathfrak{s}(\delta) = [X^\bullet]$ for some extension $\delta \in \mathbb{E}(C, A)$ and $X^\bullet \in \mathcal{C}_{(A, C)}^n$, then $X^\bullet$ is said to be an $\mathfrak{s}$ -conflation and $\langle X^\bullet, \delta \rangle$ is called an $\mathfrak{s}$ -distinguished n -exangle.

We close this subsection by recalling how an $(n+2)$ -angulated category carries an n -exangulated structure.

Example 2.7. Suppose $(\mathcal{C}, \Sigma^n, \diamond)$ is an $(n+2)$ -angulated category (in the sense of [26]). Thus, Σ^n is an automorphism of $\mathcal{C}$ and $\diamond$ is a collection of $(n+2)$ -angles. Using Σ^n , we can define a biadditive functor $\mathbb{E}: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{Ab}$ as follows. For objects $X^0, X^{n+1} \in \mathcal{C}$ put $\mathbb{E}(X^{n+1}, X^0) := \mathcal{C}(X^{n+1}, \Sigma^n X^0)$ and, for any $f: Y^{n+1} \rightarrow X^{n+1}$ and $g: X^0 \rightarrow Y^0$, the morphism $\mathbb{E}(f, g): \mathbb{E}(X^{n+1}, X^0) \rightarrow \mathbb{E}(Y^{n+1}, Y^0)$ is given by $\mathbb{E}(f, g)(\delta) := (\Sigma^n g) \circ \delta \circ f$.

Now let $\delta \in \mathbb{E}(X^{n+1}, X^0)$ be some extension. Since $\mathcal{C}$ is an $(n+2)$ -angulated category, there is an $(n+2)$ -angle of the form $X^0 \rightarrow \cdots \rightarrow X^{n+1} \xrightarrow{\delta} \Sigma^n X^0$. Setting $\mathfrak{s}(\delta) = [X^0 \rightarrow \cdots \rightarrow X^{n+1}]$, it can then be checked that this assignment gives an exact realisation of $\mathbb{E}$. Moreover, it can also be shown that $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an n -exangulated category; see [29, Subsec. 4.2] for more details.

2.2. An n -exangulated structure induced by a subcategory

Suppose $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an n -exangulated category and let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$. We first recall how to obtain an n -exangulated category $(\mathcal{C}, \mathbb{E}_{\mathcal{D}}, \mathfrak{s}_{\mathcal{D}})$ using the relative theory developed in [29]. Our main goal in this subsection is to show that $(\mathcal{C}, \mathbb{E}_{\mathcal{D}}, \mathfrak{s}_{\mathcal{D}})$ is an n -exangulated subcategory (in the sense of [28]) of $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$.

Definition 2.8. [29, Def. 3.7, Def. 3.10] Denote by $\mathbf{Set}$ the category of sets. Suppose $\mathbb{F}: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{Set}$ is a functor.

- (i) If, for all $A, A', C, C' \in \mathcal{C}$, $a \in \mathcal{C}(A, A')$ and $c \in \mathcal{C}(C', C)$, we have $\mathbb{F}(C, A) \subseteq \mathbb{E}(C, A)$ and $\mathbb{F}(c, a) = \mathbb{E}(c, a)|_{\mathbb{F}(C, A)}$, then $\mathbb{F}$ is called a *subfunctor of $\mathbb{E}$* . In this case, we write $\mathbb{F} \subseteq \mathbb{E}$.
- (ii) For any subfunctor $\mathbb{F}$ of $\mathbb{E}$, if $\mathbb{F}(C, A)$ is a subgroup of $\mathbb{E}(C, A)$ for all $A, C \in \mathcal{C}$, then we say $\mathbb{F}$ is an *additive subfunctor of $\mathbb{E}$* . Note that, in this case, we deduce $\mathbb{F}: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{Ab}$ is a biadditive functor.
- (iii) Suppose $\mathbb{F}$ is an additive subfunctor of $\mathbb{E}$, and let $\mathfrak{s}|_{\mathbb{F}}$ denote the restriction of $\mathfrak{s}$ to $\mathbb{F}$. We say that $\mathbb{F}$ is *closed on the right* if, for every $\mathfrak{s}|_{\mathbb{F}}$ -conflation $X^\bullet$, and every $Z \in \mathcal{C}$,

$$\mathbb{F}(Z, X^0) \xrightarrow{\mathbb{F}(Z, d_X^0)} \mathbb{F}(Z, X^1) \xrightarrow{\mathbb{F}(Z, d_X^1)} \mathbb{F}(Z, X^2)$$

is exact. Dually, one defines what is meant by $\mathbb{F}$ is *closed on the left*.

It has been shown that being closed on the right is equivalent to being closed on the left for an additive subfunctor $\mathbb{F}$ of $\mathbb{E}$; see [29, Lem. 3.15]. Therefore, $\mathbb{F}$ is simply said to be *closed* if it is closed on one side.

The definition of $\mathbb{E}_{\mathcal{D}}$ in the next proposition comes from [29, Def. 3.18], and the proof of the result itself follows from [29, Sec. 3].

Proposition 2.9. *The assignment $\mathbb{E}_{\mathcal{D}}$ from $\mathcal{C}^{\text{op}} \times \mathcal{C}$ to $\mathbf{Set}$ given by*

$$\mathbb{E}_{\mathcal{D}}(X^{n+1}, X^0) := \{ \delta \in \mathbb{E}(X^{n+1}, X^0) \mid (\delta_{\#})_D = 0 \text{ for every } D \in \mathcal{D} \}$$

is a closed subfunctor of $\mathbb{E}$. The restriction $\mathfrak{s}_{\mathcal{D}} := \mathfrak{s}|_{\mathbb{E}_{\mathcal{D}}}$ of the realisation $\mathfrak{s}$ to $\mathbb{E}_{\mathcal{D}}$ is an exact realisation of $\mathbb{E}_{\mathcal{D}}$. Moreover, the triplet $(\mathcal{C}, \mathbb{E}_{\mathcal{D}}, \mathfrak{s}_{\mathcal{D}})$ is an n -exangulated category.

Proof. The first two statements follow from [29, Prop. 3.19] and [29, Claim 3.9], respectively. Then [29, Prop. 3.16] implies that $(\mathcal{C}, \mathbb{E}_{\mathcal{D}}, \mathfrak{s}_{\mathcal{D}})$ is an n -exangulated category. ■

The next notion captures what it means for a functor to preserve an n -exangulated structure. It will allow us to show that the n -exangulated structure induced by $\mathcal{D}$ is compatible with the original n -exangulated structure of $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$. For a covariant functor $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{C}'$, we denote by $\mathcal{F}^{\text{op}}: \mathcal{C}^{\text{op}} \rightarrow (\mathcal{C}')^{\text{op}}$ the induced *opposite* functor.

Definition 2.10. [8, Def. 2.32] Suppose $(\mathcal{C}', \mathbb{E}', \mathfrak{s}')$ is also an n -exangulated category. An n -exangulated functor from $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ to $(\mathcal{C}', \mathbb{E}', \mathfrak{s}')$ is a pair $(\mathcal{F}, \Gamma)$, where $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{C}'$ is an additive covariant functor and

$$\Gamma = \{\Gamma_{(C,A)}\}_{(C,A) \in \mathcal{C}^{\text{op}} \times \mathcal{C}}: \mathbb{E} \Longrightarrow \mathbb{E}'(\mathcal{F}^{\text{op}}-, \mathcal{F}-)$$

is a natural transformation of functors from $\mathcal{C}^{\text{op}} \times \mathcal{C}$ to $\mathbf{Ab}$, such that $\mathfrak{s}'(\Gamma_{(X^{n+1}, X^0)}(\delta)) = [\mathcal{F}X^{\bullet}]$ whenever $\mathfrak{s}(\delta) = [X^{\bullet}]$.

Definition 2.11. [28, Def. 3.7] Suppose $(\mathcal{S}, \mathbb{E}', \mathfrak{s}')$ is an n -exangulated category, where $\mathcal{S}$ is a full subcategory of $\mathcal{C}$ that is closed under isomorphisms, and let $\iota: \mathcal{S} \rightarrow \mathcal{C}$ be the inclusion functor. Then $(\mathcal{S}, \mathbb{E}', \mathfrak{s}')$ is called an n -exangulated subcategory of $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ if there is an n -exangulated functor (ι, Γ) , where $\Gamma_{(C,A)}: \mathbb{E}'(C, A) \rightarrow \mathbb{E}(C, A)$ is an inclusion of abelian groups for all $A, C \in \mathcal{S}$.

We are now in a position to give the main result of this subsection.

Theorem 2.12. Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$. The triplet $(\mathcal{C}, \mathbb{E}_{\mathcal{D}}, \mathfrak{s}_{\mathcal{D}})$ is an n -exangulated subcategory of $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$.

Proof. Since $\mathbb{E}_{\mathcal{D}} \subseteq \mathbb{E}$ and $\mathfrak{s}_{\mathcal{D}}$ is a restriction of $\mathfrak{s}$, it is clear that the identity functor $\mathbb{1}_{\mathcal{C}}$, with the natural transformation $\Gamma: \mathbb{E}_{\mathcal{D}} \Rightarrow \mathbb{E}(\mathbb{1}_{\mathcal{C}}^{\text{op}}-, \mathbb{1}_{\mathcal{C}}-) = \mathbb{E}$ given by $\Gamma_{(C,A)}(\delta) = \delta$ for each $A, C \in \mathcal{C}$ and each $\delta \in \mathbb{E}_{\mathcal{D}}(C, A)$, is an n -exangulated functor. ■

Remark 2.13. Suppose $(\mathcal{C}, \Sigma^n, \Diamond)$ is an $(n+2)$ -angulated category, and write $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ for its n -exangulated structure; see Example 2.7 above. Let $\mathcal{D}$ be an additive subcategory of $\mathcal{C}$. It is immediate that, for any $\delta \in \mathcal{C}(X^{n+1}, \Sigma^n X^0)$, we have $\delta \in \mathbb{E}_{\mathcal{D}}(X^{n+1}, X^0)$ if and only if $F_{\mathcal{D}}(\delta) = 0$ (see (vi) in Subsection 1.1), i.e.

$$\mathbb{E}_{\mathcal{D}}(X^{n+1}, X^0) = \{ \delta \in \mathcal{C}(X^{n+1}, \Sigma^n X^0) \mid F_{\mathcal{D}}(\delta) = 0 \}.$$

Furthermore, since $\mathcal{D}^{\perp_0}$ is closed under finite direct sums, we may consider the ideal $[\mathcal{D}^{\perp_0}]$ of $\mathcal{C}$. The subfunctor $\mathbb{E}_{\mathcal{D}}$ of $\mathbb{E}$ satisfies $[\mathcal{D}^{\perp_0}](X^{n+1}, \Sigma^n X^0) \subseteq \mathbb{E}_{\mathcal{D}}(X^{n+1}, X^0)$.

We note that if $n = 1$ (so that $\mathcal{C}$ is a triangulated category), $\mathcal{D}$ is extension-closed in $\mathcal{C}$ and each object of $\mathcal{C}$ admits a minimal right $\mathcal{D}$ -approximation, then one can show this inclusion is an equality using the triangulated Wakamatsu Lemma [12, Lem. 1.1].

2.3. The Grothendieck group of an n -exangulated category

In classical settings, the notion of a Grothendieck group has been used previously to give connections between certain subcategories of a category and subgroups of the ambient category's Grothendieck group; see [40], [50]. More recently, some of these results have been extended to the n -exangulated cases; see [9], [28], [53]. In contrast, we will look at how the Grothendieck group theory changes as we tweak the n -exangulated structure on a fixed category and we will also be considering certain quotients of split Grothendieck groups.

The first construction we need is well-known; see, for example [5, Ch. VII].

Definition 2.14. Let $\mathcal{A}$ be a skeletally small additive category. An isomorphism class of an object $A \in \mathcal{A}$ will be denoted as usual by $[A]$. Denote by $\mathcal{F}(\mathcal{A})$ the free abelian group on the isomorphism classes of objects in $\mathcal{A}$. The *split Grothendieck group* $K_0^{\text{sp}}(\mathcal{A})$ of $\mathcal{A}$ is the quotient of $\mathcal{F}(\mathcal{A})$ by the subgroup $\langle [A] - [A \oplus B] + [B] \mid A, B \in \mathcal{A} \rangle$.

Since we will be working modulo relations coming from n -exangles, it is convenient to introduce the following notation.

Definition 2.15. Let $\mathcal{A}$ be a skeletally small additive category. For a complex

$$X^\bullet: X^0 \longrightarrow X^1 \longrightarrow \cdots \longrightarrow X^n \longrightarrow X^{n+1}$$

in $\mathcal{C}_{\mathcal{A}}^n$, we define the associated *Euler relation* to be the alternating sum

$$\chi(X^\bullet) := \sum_{i=0}^{n+1} (-1)^i [X^i] = [X^0] - [X^1] + \cdots + (-1)^n [X^n] + (-1)^{n+1} [X^{n+1}],$$

of isomorphism classes, viewed as an element of $\mathcal{F}(\mathcal{A})$.

Remark 2.16. For the convenience of the reader, we make the following observation that we use several times in the remainder of the article:

$$\begin{aligned} (-1)^{n+1} \chi(X^\bullet) &= [X^{n+1}] - [X^n] + \cdots + (-1)^n [X^1] + (-1)^{n+1} [X^0] \\ &= \sum_{i=0}^{n+1} (-1)^i [X^{n+1-i}]. \end{aligned}$$

We can now state the main definition of this subsection.

Definition 2.17. [28, Def. 4.1] Let $n \geq 1$ be a positive integer. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be a skeletally small n -exangulated category. As before, let $\mathcal{F}(\mathcal{C})$ be the free abelian group on the isomorphism classes of objects in $\mathcal{C}$. Let $\mathcal{G}(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ denote the subgroup of $\mathcal{F}(\mathcal{C})$ generated by the subset

$$\begin{cases} \{ \chi(X^\bullet) \mid X^\bullet \text{ is an } \mathfrak{s}\text{-conflation in } (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \} & \text{if } n \text{ is odd} \\ \{ \chi(X^\bullet) \mid X^\bullet \text{ is an } \mathfrak{s}\text{-conflation in } (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \} \cup \{ [0] \} & \text{if } n \text{ is even.} \end{cases}$$

The *Grothendieck group* of $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is the quotient $K_0(\mathcal{C}, \mathbb{E}, \mathfrak{s}) := \mathcal{F}(\mathcal{C}) / \mathcal{G}(\mathcal{C}, \mathbb{E}, \mathfrak{s})$.

Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an n -exangulated category and let $\mathcal{D} \subseteq \mathcal{C}$ be a full subcategory. We conclude this subsection by giving a description of the Grothendieck group of $(\mathcal{C}, \mathbb{E}_{\mathcal{D}}, \mathfrak{s}_{\mathcal{D}})$ as a quotient of the split Grothendieck group $K_0^{\text{sp}}(\mathcal{C})$.

Definition 2.18. Define the subgroup

$$\mathcal{I}_{\mathcal{D}} := \langle \chi(X^\bullet) \mid X^\bullet \text{ is an } \mathfrak{s}_{\mathcal{D}}\text{-conflation in } (\mathcal{C}, \mathbb{E}_{\mathcal{D}}, \mathfrak{s}_{\mathcal{D}}) \rangle$$

of the split Grothendieck group $K_0^{\text{sp}}(\mathcal{C})$. Then we have $K_0(\mathcal{C}, \mathbb{E}_{\mathcal{D}}, \mathfrak{s}_{\mathcal{D}}) \cong K_0^{\text{sp}}(\mathcal{C}) / \mathcal{I}_{\mathcal{D}}$, and we identify these groups. Furthermore, let us denote by $Q_{\mathcal{D}}$ the canonical surjection $K_0^{\text{sp}}(\mathcal{C}) \rightarrow K_0^{\text{sp}}(\mathcal{C}) / \mathcal{I}_{\mathcal{D}}$.

There are the two extremal cases, which we quickly discuss now.

- (a) If $\mathcal{D} = 0$, then $\mathbb{E}_{\mathcal{D}} = \mathbb{E}_0 = \mathbb{E}$ and so $(\mathcal{C}, \mathbb{E}_0, \mathfrak{s}_0) = (\mathcal{C}, \mathbb{E}, \mathfrak{s})$. In addition, $\mathcal{I}_{\mathcal{D}} = \mathcal{I}_0$ is generated by $\chi(X^\bullet)$ for each $\mathfrak{s}$ -conflation $X^\bullet$ in $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$.
- (b) If $\mathcal{D} = \mathcal{C}$, then $\mathbb{E}_{\mathcal{D}}(X^{n+1}, X^0) = 0$ for any $X^0, X^{n+1} \in \mathcal{C}$, and so $\mathcal{I}_{\mathcal{D}} = \mathcal{I}_{\mathcal{C}}$ is generated by the Euler relations associated to the $\mathfrak{s}$ -conflations of the form

$$0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow A \xrightarrow{1_A} A \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0,$$

where $A \in \mathcal{C}$ and the identity morphism 1_A may be in any position; see [28, Prop. 2.14]. Thus, $\mathcal{I}_{\mathcal{C}} = \langle [0] \rangle$ is the trivial subgroup of $K_0^{\text{sp}}(\mathcal{C})$, and hence $K_0^{\text{sp}}(\mathcal{C})/\mathcal{I}_{\mathcal{C}} = K_0^{\text{sp}}(\mathcal{C})$.

3. Cluster tilting subcategories and the $(d+2)$ -angulated index

For a triangulated category $\mathcal{C}$ and cluster tilting subcategory $\mathcal{T} \subseteq \mathcal{C}$, a triangulated index was introduced by Palu in [44], giving a way to associate an element of the split Grothendieck group $K_0^{\text{sp}}(\mathcal{T})$ to an object of $\mathcal{C}$. A higher analogue was subsequently introduced in [35]; see Definition 3.5. We show that this gives an isomorphism between the split Grothendieck group of $\mathcal{T}$ and the Grothendieck group of the d -exangulated structure on $\mathcal{C}$ induced by $\mathcal{T}$.

First, we recall some essential definitions.

Definition 3.1. [32], [33, Sec. 3] Let $n \geq 2$ be a positive integer. Let $\mathcal{C}$ be an idempotent complete, triangulated category. Suppose $\mathcal{T}$ is an additive subcategory of $\mathcal{C}$. We say that $\mathcal{T}$ is an *n -cluster tilting subcategory of $\mathcal{C}$* if:

- (i) $\mathcal{T}$ is functorially finite; and
- (ii) $\mathcal{T} = \{ X \in \mathcal{C} \mid \text{Ext}_{\mathcal{C}}^i(\mathcal{T}, X) = 0 \text{ for all } 1 \leq i \leq n-1 \}$
 $= \{ X \in \mathcal{C} \mid \text{Ext}_{\mathcal{C}}^i(X, \mathcal{T}) = 0 \text{ for all } 1 \leq i \leq n-1 \}.$

An interesting characterisation of n -cluster tilting subcategories is given in [7, Thm. 5.3].

Definition 3.2. [22, Def. 6.2], [42, Def. 5.3] Let $d \geq 1$ be a positive integer. Let $\mathcal{S}$ be a $(d+2)$ -angulated category. Suppose $\mathcal{T}$ is an additive subcategory of $\mathcal{S}$. We say that $\mathcal{T}$ is an *Oppermann-Thomas cluster tilting subcategory of $\mathcal{S}$* if:

- (i) $\mathcal{S}(\mathcal{T}, \Sigma^d \mathcal{T}) = 0$; and
- (ii) for each $S \in \mathcal{S}$ there is a $(d+2)$ -angle $T^0 \longrightarrow \cdots \longrightarrow T^d \longrightarrow S \longrightarrow \Sigma^d T^0$ in $\mathcal{S}$ with $T^i \in \mathcal{T}$ for all $0 \leq i \leq d$.

Note that (i) and (ii) in Definition 3.2 imply $\mathcal{T}$ is functorially finite.

Setup 3.3. For the remainder of Sections 3 and 4, we fix the following. Let k be a field, and let d, n be positive integers with $n = 2d$. We assume $\mathcal{C}$ is a skeletally small, k -linear, Krull-Schmidt, Hom-finite triangulated category with suspension functor Σ . We suppose there are subcategories $\mathcal{T} \subseteq \mathcal{S}$ in $\mathcal{C}$, where $\mathcal{T}$ is n -cluster tilting in $\mathcal{C}$, and $\mathcal{S}$ is d -cluster tilting in $\mathcal{C}$ and $\Sigma^d \mathcal{S} \subseteq \mathcal{S}$.

Remark 3.4.

- (i) As noted in [35, Rem. 5.2], Setup 3.3 implies: $\mathcal{S}$ has the structure of a $(d+2)$ -angulated category following the “standard construction” given by Geiss–Keller–Oppermann (see [26, Thm. 1]), with

d -suspension functor Σ^d , which is the restriction to $\mathcal{S}$ of the d 'th power of Σ ; and that $\mathcal{T}$ is an Oppermann-Thomas cluster tilting subcategory of $\mathcal{S}$ by [42, Thm. 5.25].

- (ii) We denote by $(\mathcal{S}, \mathbb{E}, \mathfrak{s})$ the d -exangulated structure of $\mathcal{S}$ (see Example 2.7).
- (iii) The subcategory $\mathcal{T}$ induces a d -exangulated structure on $\mathcal{S}$ as in Proposition 2.9, which we denote by $(\mathcal{S}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}})$. Furthermore, $(\mathcal{S}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}})$ is a d -exangulated subcategory of $(\mathcal{S}, \mathbb{E}, \mathfrak{s})$ by Theorem 2.12. Setup 3.3 implies $\mathcal{T}$ itself is an additive subcategory of $\mathcal{S}$, so, as in Remark 2.13, we have

$$\mathbb{E}_{\mathcal{T}}(X^{d+1}, X^0) = \{ \delta \in \mathcal{S}(X^{d+1}, \Sigma^d X^0) \mid F_{\mathcal{T}}(\delta) = 0 \}.$$

With this Setup, we can then recall the higher index with respect to $\mathcal{T}$.

Definition 3.5. [35, Def. B] For $S \in \mathcal{S}$, let

$$T_S^0 \xrightarrow{\tau^0} \cdots \xrightarrow{\tau^{d-1}} T_S^d \longrightarrow S \longrightarrow \Sigma^d T_S^0 \quad (3.1)$$

be a $(d+2)$ -angle in $\mathcal{S}$ with $T_S^i \in \mathcal{T}$ and τ^i in the radical of $\mathcal{S}$ for each i . The $(d+2)$ -angulated index of S with respect to $\mathcal{T}$ is

$$\text{ind}_{\mathcal{T}}^{\diamond}(S) = \sum_{i=0}^d (-1)^i [T_S^{d-i}] = [T_S^d] - [T_S^{d-1}] + \cdots + (-1)^{d-1} [T_S^1] + (-1)^d [T_S^0],$$

viewed as an element of $K_0^{\text{sp}}(\mathcal{T})$.

Remark 3.6. We note the following.

- (i) Let $T_S^{\bullet}$ denote the $\mathfrak{s}$ -conflation corresponding to the $(d+2)$ -angle (3.1), i.e. $T_S^{\bullet}$ is the complex $T_S^0 \xrightarrow{\tau^0} \cdots \xrightarrow{\tau^{d-1}} T_S^d \longrightarrow S$ in $\mathcal{C}_S^d$. Then notice that, as elements of $K_0^{\text{sp}}(\mathcal{S})$, we have the equality $\text{ind}_{\mathcal{T}}^{\diamond}(S) = [S] - (-1)^{d+1} \chi(T_S^{\bullet})$, using Definition 2.15 and Remark 2.16.
- (ii) For any $T \in \mathcal{T}$, we have $\text{ind}_{\mathcal{T}}^{\diamond}(T) = [T]$ as $\mathcal{S}$ admits the $(d+2)$ -angle

$$0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow T \xrightarrow{1_T} T \longrightarrow \Sigma^d 0.$$

In order to show that the $(d+2)$ -angulated index induces group homomorphisms between Grothendieck groups, we use the following result from [35], which shows this index is additive on $(d+2)$ -angles in $\mathcal{S}$ up to an error term. Recall that $F_{\mathcal{T}}: \mathcal{S} \rightarrow \text{mod } \mathcal{T}$ is given by $F_{\mathcal{T}}(S) = \mathcal{S}(-, S)|_{\mathcal{T}}$ on objects (see (vi) in Subsection 1.1).

Theorem 3.7. [35, Thm. C] There is a homomorphism $\theta: K_0(\text{mod } \mathcal{T}) \rightarrow K_0^{\text{sp}}(\mathcal{T})$, such that if $S^0 \longrightarrow \cdots \longrightarrow S^d \longrightarrow S^{d+1} \xrightarrow{\gamma} \Sigma^d S^0$ is a $(d+2)$ -angle in $\mathcal{S}$, then

$$\sum_{i=0}^{d+1} (-1)^i \text{ind}_{\mathcal{T}}^{\diamond}(S^{d+1-i}) = \theta([\text{Im } F_{\mathcal{T}}(\gamma)]).$$

The first homomorphism obtained via the higher index is from the split Grothendieck group of $\mathcal{S}$ to that of $\mathcal{T}$.

Proposition 3.8. The $(d+2)$ -angulated index $\text{ind}_{\mathcal{T}}^{\diamond}$ induces a group homomorphism $K_0^{\text{sp}}(\mathcal{S}) \rightarrow K_0^{\text{sp}}(\mathcal{T})$, which we also denote by $\text{ind}_{\mathcal{T}}^{\diamond}$.

Proof. By [35, Rem. 5.4], $\text{ind}_{\mathcal{T}}^{\diamond}(S)$ is well-defined and depends only on the isomorphism class of S in $\mathcal{S}$. Therefore, by extending linearly, it induces a group homomorphism $\mathcal{F}(\mathcal{S}) \rightarrow K_0^{\text{sp}}(\mathcal{T})$. Let $A, B \in \mathcal{S}$ be arbitrary. Then $\mathcal{S}$ admits the $(d+2)$ -angle

$$0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow A \xrightarrow{\begin{pmatrix} 1_A \\ 0 \end{pmatrix}} A \oplus B \xrightarrow{\begin{pmatrix} 0 & 1_B \end{pmatrix}} B \xrightarrow{0} \Sigma^d 0,$$

which yields $\text{ind}_{\mathcal{T}}^{\diamond}(B) - \text{ind}_{\mathcal{T}}^{\diamond}(A \oplus B) + \text{ind}_{\mathcal{T}}^{\diamond}(A) = \theta([\text{Im } 0]) = 0$ by Theorem 3.7. Hence, we obtain a well-defined group homomorphism $\text{ind}_{\mathcal{T}}^{\diamond}: K_0^{\text{sp}}(\mathcal{S}) \rightarrow K_0^{\text{sp}}(\mathcal{T})$. ■

Notation 3.9. Suppose $\mathcal{D}$ is a full subcategory of the $(d+2)$ -angulated category $\mathcal{S}$. Recall that there is, thus, a d -exangulated subcategory $(\mathcal{S}, \mathbb{E}_{\mathcal{D}}, \mathfrak{s}_{\mathcal{D}})$ of $(\mathcal{S}, \mathbb{E}, \mathfrak{s})$ by Theorem 2.12. Let $\delta \in \mathbb{E}_{\mathcal{D}}(X^{d+1}, X^0) \subseteq \mathcal{S}(X^{d+1}, \Sigma^d X^0)$. If $\mathfrak{s}_{\mathcal{D}}(\delta) = [X^0 \rightarrow \cdots \rightarrow X^{d+1}]$ such that $X^0 \rightarrow \cdots \rightarrow X^{d+1}$ completes to a $(d+2)$ -angle $X^0 \rightarrow \cdots \rightarrow X^{d+1} \rightarrow \Sigma^d X^0$ in $\mathcal{S}$, then we emphasise this by depicting the $\mathfrak{s}_{\mathcal{D}}$ -distinguished d -exangle $\langle X^{\bullet}, \delta \rangle$ as

$$X^0 \longrightarrow X^1 \longrightarrow \cdots \longrightarrow X^d \longrightarrow X^{d+1} \dashrightarrow^{\delta} \Sigma^d X^0,$$

to simultaneously indicate that $\langle X^{\bullet}, \delta \rangle$ is an $\mathfrak{s}_{\mathcal{D}}$ -distinguished d -exangle and that the whole diagram is a $(d+2)$ -angle in $\mathcal{S}$. Although this notation is used for exangulated categories more generally, we will reserve it specifically for the type of situation just described.

The authors thank P.-G. Plamondon for bringing [43, Prop. 4.11] to the attention of the first author. Our main result of this section is the following higher version of [43, Prop. 4.11], which was the starting point for the present article.

Theorem 3.10. *The $(d+2)$ -angulated index $\text{ind}_{\mathcal{T}}^{\diamond}$ induces an isomorphism $K_0(\mathcal{S}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}}) \rightarrow K_0^{\text{sp}}(\mathcal{T})$, which we also denote by $\text{ind}_{\mathcal{T}}^{\diamond}$. The inverse $L: K_0^{\text{sp}}(\mathcal{T}) \rightarrow K_0(\mathcal{S}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}})$ is defined on generators by $L([T]) = [T]$ for each $T \in \mathcal{T}$.*

Proof. Recall that $K_0(\mathcal{S}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}}) = K_0^{\text{sp}}(\mathcal{S})/\mathcal{I}_{\mathcal{T}}$; see Definition 2.18. We first show $\mathcal{I}_{\mathcal{T}}$ is contained in the kernel of $\text{ind}_{\mathcal{T}}^{\diamond}: K_0^{\text{sp}}(\mathcal{S}) \rightarrow K_0^{\text{sp}}(\mathcal{T})$ from Proposition 3.8. Thus, let

$$S^0 \longrightarrow \cdots \longrightarrow S^d \longrightarrow S^{d+1} \dashrightarrow^{\gamma} \Sigma^d S^0$$

be an $\mathfrak{s}_{\mathcal{T}}$ -distinguished d -exangle in $(\mathcal{S}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}})$. Thus, $F_{\mathcal{T}}(\gamma) = 0$ by Remark 3.4(iii), and we have

$$\begin{aligned} \sum_{i=0}^{d+1} (-1)^i \text{ind}_{\mathcal{T}}^{\diamond}(S^i) &= (-1)^{d+1} \sum_{i=0}^{d+1} (-1)^i \text{ind}_{\mathcal{T}}^{\diamond}(S^{d+1-i}) && \text{(see Remark 2.16)} \\ &= (-1)^{d+1} \theta([\text{Im } F_{\mathcal{T}}(\gamma)]) && \text{by Theorem 3.7} \\ &= 0. \end{aligned}$$

Hence, $\text{ind}_{\mathcal{T}}^{\diamond}$ induces a group homomorphism $K_0(\mathcal{S}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}}) \rightarrow K_0^{\text{sp}}(\mathcal{T})$, which we again denote by $\text{ind}_{\mathcal{T}}^{\diamond}$.

It remains to show that this is an isomorphism, and we do this by constructing an inverse of $\text{ind}_{\mathcal{T}}^{\diamond}$. Define a group homomorphism $L: K_0^{\text{sp}}(\mathcal{T}) \rightarrow K_0(\mathcal{S}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}})$ on generators by $L([T]) = [T]$ for each $T \in \mathcal{T}$ and extend linearly. By Remark 3.6(ii), we see that $\text{ind}_{\mathcal{T}}^{\diamond} \circ L$ is the identity on $K_0^{\text{sp}}(\mathcal{T})$. On the other hand, for $S \in \mathcal{S}$, there is a $(d+2)$ -angle of the form (3.1) as $\mathcal{T}$ is Oppermann-Thomas cluster tilting. By dropping

trivial summands from the morphisms τ^i , one can assume each τ^i is radical. As $\mathcal{T}$ is Oppermann-Thomas cluster tilting, we have $\mathcal{S}(\mathcal{T}, \Sigma^d \mathcal{T}) = 0$. Thus, in $K_0(\mathcal{S}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}})$, we see

$$\begin{aligned} [S] &= - \sum_{i=1}^{d+1} (-1)^i [T^{d+1-i}] \\ &= \sum_{i=1}^{d+1} (-1)^{i-1} [T^{d+1-i}] \\ &= L \left(\sum_{i=0}^d (-1)^i [T^{d-i}] \right) \\ &= L(\text{ind}_{\mathcal{T}}^{\diamond}([S])). \end{aligned}$$

This shows that $L \circ \text{ind}_{\mathcal{T}}^{\diamond}$ is the identity on $K_0(\mathcal{S}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}})$, and hence $\text{ind}_{\mathcal{T}}^{\diamond}: K_0(\mathcal{S}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}}) \rightarrow K_0^{\text{sp}}(\mathcal{T})$ is indeed an isomorphism. ■

4. The quotient $K_0^{\text{sp}}(\mathcal{T})/N_{\mathcal{X}}$ as a Grothendieck group

Throughout this section, we still have the base assumptions laid out in Setup 3.3. In the coming subsections, we specialise a further three times as detailed below in order to prove our main result, namely, Theorem 4.18. Relevant definitions are recalled in the corresponding subsection.

Setup 4.1 (for Subsection 4.1). In addition to Setup 3.3, suppose: $\mathcal{S}$ is $2d$ -Calabi-Yau (see Definition 4.6); $\mathcal{X}$ is a functorially finite, additive subcategory of $\mathcal{T}$; and all objects in $\text{ind } \mathcal{T} \setminus \text{ind } \mathcal{X}$ have a *mutation* (see Definition 4.9).

Setup 4.2 (for Subsection 4.2). In addition to Setups 3.3 and 4.1, we assume that, for each $T \in \text{ind } \mathcal{T} \setminus \text{ind } \mathcal{X}$, the image $\text{Im } F_{\mathcal{T}}(\gamma^T)$ (see (vi) in Subsection 1.1) is isomorphic to the simple module $\overline{S}_T = \mathcal{T}(-, T)/\text{rad}_{\mathcal{T}}(-, T)$ in $\text{Mod } \mathcal{T}$ (see (4.5)), where γ^T is as in the exchange $(d+2)$ -angle (4.4).

Setup 4.3 (for Subsection 4.3). In addition to Setups 3.3, 4.1 and 4.2, we assume that for any $S \in \mathcal{S}$, the functor $F_{\mathcal{T}}(S) = \mathcal{S}(-, S)|_{\mathcal{T}}$ (see (vi) in Subsection 1.1) has finite length in $\text{mod } \mathcal{T}$, i.e. $F_{\mathcal{T}}(S) \in \text{fl } \mathcal{T}$.

Remark 4.4. We note these Setups are satisfied in several cases of interest.

- (i) For example, Setup 4.1 is satisfied if $\mathcal{T} = \text{add } T$ for some Oppermann-Thomas cluster tilting object $T \in \mathcal{S}$, and $\mathcal{X} \subseteq \mathcal{T}$ is chosen as needed for Setup 4.1. In particular, when $d > 1$, we note that not *any* choice of a functorially finite, additive subcategory $\mathcal{X} \subseteq \mathcal{T}$ will result in each indecomposable in $\mathcal{T} \setminus \mathcal{X}$ having a mutation, so one has to be careful when choosing $\mathcal{X}$; see [42, Rem. 5.8].
- (ii) Setup 4.2 holds if e.g. $d = 1$ and $\mathcal{T}$ is part of a *cluster structure* in the sense of [11, p. 1039].
- (iii) The additional condition in Setup 4.3—namely, that $F_{\mathcal{T}}(S)$ has finite length—is satisfied if, for example, $\mathcal{T}$ is *locally bounded*, i.e. Hom-finite and for each indecomposable object $T \in \mathcal{T}$, there are only finitely many objects $T' \in \text{ind } \mathcal{T}$ such that $\mathcal{T}(T, T') \neq 0$ or $\mathcal{T}(T', T) \neq 0$. See [36, Rem. 5.4]. (Note that we are using ‘locally bounded’ in the sense of [39], and not as in [21].)

In this section we will establish the following (under the appropriate assumptions as described above).

Theorem 4.5. *There is a commutative diagram*

$$\begin{array}{ccccc}
 K_0^{\text{sp}}(\mathcal{S}) & \xrightarrow{Q_0} & K_0(\mathcal{S}, \mathbb{E}, \mathfrak{s}) & & \\
 \downarrow Q_{\mathcal{T}} & \searrow Q_{\mathcal{X}} & \uparrow & & \\
 K_0(\mathcal{S}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}}) & \xrightarrow{\tilde{Q}} & K_0(\mathcal{S}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}}) & & \\
 \uparrow L \cong \text{ind}_{\mathcal{T}}^{\mathcal{O}} & & \uparrow G_{\mathcal{X}} \cong & & \\
 K_0^{\text{sp}}(\mathcal{T}) & \xrightarrow{Q_{N_{\mathcal{X}}}} & K_0^{\text{sp}}(\mathcal{T})/N_{\mathcal{X}} & &
 \end{array}
 \quad (4.1)$$

of abelian groups, with the indicated surjections and isomorphisms. In particular, there is an isomorphism $K_0^{\text{sp}}(\mathcal{T})/N_{\mathcal{X}} \cong K_0(\mathcal{S}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$.

Some of the categories and homomorphisms in (4.1) have already appeared in earlier sections, and others we introduce in this section. That L and $\text{ind}_{\mathcal{T}}^{\mathcal{O}}$ are mutually inverse follows from Theorem 3.10.

The upper half of (4.1) exists and commutes by the following formal considerations. Recall that in Definition 2.18, for each full subcategory $\mathcal{D}$ of an n -exangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$, we defined a subgroup $\mathcal{I}_{\mathcal{D}} \leq K_0^{\text{sp}}(\mathcal{C})$ and the canonical epimorphism

$$Q_{\mathcal{D}}: K_0^{\text{sp}}(\mathcal{C}) \rightarrow K_0^{\text{sp}}(\mathcal{C})/\mathcal{I}_{\mathcal{D}} = K_0(\mathcal{C}, \mathbb{E}_{\mathcal{D}}, \mathfrak{s}_{\mathcal{D}}).$$

Since $0 \subseteq \mathcal{X} \subseteq \mathcal{T} \subseteq \mathcal{S}$, we have that $0 = \mathcal{I}_{\mathcal{X}} \leq \mathcal{I}_{\mathcal{T}} \leq \mathcal{I}_{\mathcal{S}} \leq \mathcal{I}_0$ as subgroups of $K_0^{\text{sp}}(\mathcal{S})$. Thus, applications of the universal property of the quotient yield a commutative diagram

$$\begin{array}{ccc}
 K_0^{\text{sp}}(\mathcal{S}) & \xrightarrow{Q_0} & K_0(\mathcal{S}, \mathbb{E}, \mathfrak{s}) \\
 \downarrow Q_{\mathcal{T}} & \searrow Q_{\mathcal{X}} & \uparrow \\
 K_0(\mathcal{S}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}}) & \xrightarrow{\tilde{Q}} & K_0(\mathcal{S}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})
 \end{array}
 \quad (4.2)$$

of surjective abelian group homomorphisms. See Subsection 2.3 for more details.

Accordingly, the primary objective of this section is to establish the existence of the lower half of (4.1) and to show $G_{\mathcal{X}}$ is an isomorphism. We do this in Corollary 4.15 and Theorem 4.18, respectively.

4.1. Mutation pairs and the subgroup $N_{\mathcal{X}}$

We are in the situation of Setup 4.1. In this subsection we will define the subgroup $N_{\mathcal{X}}$ and show the existence of the homomorphism $G_{\mathcal{X}}$ appearing in (4.1).

Definition 4.6. Recall that k is an algebraically closed field. Let $D(-) = \text{Hom}_k(-, k)$ denote the standard k -duality. Then $\mathcal{S}$ is *2d-Calabi-Yau* if, for all $S, S' \in \mathcal{S}$, there are isomorphisms $\mathcal{S}(S, S') \cong D \mathcal{S}(S', (\Sigma^d)^2 S)$, which are natural in each argument.

Remark 4.7. Note that under Setup 4.1, we have that $\mathcal{T}$ is n -cluster tilting in $\mathcal{C}$ and hence rigid (see Definition 3.1), so the subcategory $\mathcal{X} \subseteq \mathcal{T}$ is also rigid. In addition, our situation meets all the conditions of [30, Setup 1.1], and hence $\text{Mod } \mathcal{X}$ and $\text{mod } \mathcal{X}$ share the same simple objects and finite length objects. In particular, $\text{fl } \mathcal{X}$ is abelian and the inclusions $\text{fl } \mathcal{X} \rightarrow \text{mod } \mathcal{X}$ and $\text{fl } \mathcal{X} \rightarrow \text{Mod } \mathcal{X}$ are exact. Similar statements hold for $\mathcal{T}$. See [30, 1.8].

Definition 4.8. [35, (0.3)] A pair (S, S^*) of objects in $\mathcal{S}$ is said to be an *exchange pair* if $\dim_k \mathcal{S}(S, \Sigma^d S^*) = 1$.

The following terminology we introduce is motivated by [42, Subsec. 1.3].

Definition 4.9. Let T be an object of $\text{ind } \mathcal{T}$, and suppose T^* lies in $\text{ind } \mathcal{S}$ with $T^* \not\cong T$. We call (T, T^*) a *mutation pair* if it is an exchange pair (see Definition 4.8) and $\text{add}((\text{ind } \mathcal{T} \setminus \{T\}) \cup \{T^*\})$ is again an Oppermann-Thomas cluster tilting subcategory of $\mathcal{S}$ (see Definition 3.2). That is, replacing T by T^* yields another Oppermann-Thomas cluster tilting subcategory of $\mathcal{S}$. In this case, we say T^* is a *mutation* of T .

For a mutation pair (T, T^*) , there is a pair of distinguished $(d+2)$ -angles that will play a key role in this whole section. A description of these can be given just as in [42].

Proposition 4.10. Suppose $T \in \text{ind } \mathcal{T}$ has a mutation $T^* \in \text{ind } \mathcal{S}$. Then, up to (non-unique) isomorphism, there are exchange $(d+2)$ -angles

$$Y^0 \longrightarrow Y^1 \longrightarrow \cdots \longrightarrow Y^d \longrightarrow Y^{d+1} \xrightarrow{\gamma^{T^*}} \Sigma^d Y^0, \quad (4.3)$$

$$X^0 \longrightarrow X^1 \longrightarrow \cdots \longrightarrow X^d \longrightarrow X^{d+1} \xrightarrow{\gamma^T} \Sigma^d X^0 \quad (4.4)$$

in $\mathcal{S}$ that satisfy the following.

- (i) $Y^0 = X^{d+1} = T$ and $Y^{d+1} = X^0 = T^*$.
- (ii) γ^{T^*}, γ^T are both non-zero.
- (iii) $Y^1 \rightarrow Y^2, Y^2 \rightarrow Y^3, \dots, Y^d \rightarrow Y^{d+1}$ and $X^0 \rightarrow X^1, X^1 \rightarrow X^2, \dots, X^{d-1} \rightarrow X^d$ are radical morphisms.
- (iv) For all $1 \leq i \leq d$, we have $X^i, Y^i \in \text{add}(\text{ind } \mathcal{T} \setminus \{T\}) \subseteq \mathcal{T}$.
- (v) $Y^0 \rightarrow Y^1$ is a minimal left $\text{add}(\text{ind } \mathcal{T} \setminus \{T\})$ -approximation.
- (vi) $X^d \rightarrow X^{d+1}$ is a minimal right $\text{add}(\text{ind } \mathcal{T} \setminus \{T\})$ -approximation.

Moreover, T^* is unique up to isomorphism.

Proof. As $\dim_k \mathcal{S}(T, \Sigma^d T^*) = 1$, we also have $\dim_k \mathcal{S}(T^*, \Sigma^d T) = 1$ since $\mathcal{S}$ is $2d$ -Calabi-Yau. This gives the existence of (4.3) and (4.4) satisfying (i) and (ii). One can then follow the proof of (1) $\Rightarrow$ (2) $\Rightarrow$ (3) of [42, Thm. 5.7] at our level of generality. ■

Remark 4.11.

- (i) Suppose $T \in \text{ind } \mathcal{T}$ has a mutation T^* . Then T^* is unique up to isomorphism by Proposition 4.10. Thus, up to isomorphism, we speak of *the* mutation T^* of T . It also follows from the same result that $T^* \notin \mathcal{T}$.
- (ii) As noted in Remark 4.4(i), it is not guaranteed that every indecomposable object in $\mathcal{T}$ has a mutation; see [42, Rem. 5.8].

The next definition is motivated by [31, Def. 2.4], which deals with the triangulated version.

Definition 4.12. We define the subgroup

$$N_{\mathcal{X}} = \left\langle \sum_{i=1}^d (-1)^i [X^{d+1-i}] - \sum_{i=1}^d (-1)^i [Y^i] \mid \begin{array}{l} (4.3) \text{ and } (4.4) \text{ are exchange } (d+2)\text{-angles} \\ \text{with } T \in \text{ind } \mathcal{T} \setminus \text{ind } \mathcal{X} \end{array} \right\rangle$$

of $K_0^{\text{sp}}(\mathcal{T})$, and let

$$Q_{N_{\mathcal{X}}}: K_0^{\text{sp}}(\mathcal{T}) \rightarrow K_0^{\text{sp}}(\mathcal{T})/N_{\mathcal{X}}$$

denote the canonical surjection.

In Theorem 3.10 we defined $L: K_0^{\text{sp}}(\mathcal{T}) \rightarrow K_0(\mathcal{S}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}})$, which was shown to be the inverse of the $(d+2)$ -angulated index $\text{ind}_{\mathcal{T}}^{\circ}: K_0(\mathcal{S}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}}) \rightarrow K_0^{\text{sp}}(\mathcal{T})$. We will show the subgroup $N_{\mathcal{X}}$ vanishes under the composition $\tilde{Q}L: K_0^{\text{sp}}(\mathcal{T}) \rightarrow K_0(\mathcal{S}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$, in order to obtain a homomorphism $G_{\mathcal{X}}: K_0^{\text{sp}}(\mathcal{T})/N_{\mathcal{X}} \rightarrow K_0(\mathcal{S}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$. For this, we need the following key lemma.

Lemma 4.13. Suppose that (T, T^*) is a mutation pair with $T \in \text{ind } \mathcal{T} \setminus \text{ind } \mathcal{X}$. Let $\gamma^{T^*}: T^* \rightarrow \Sigma^d T$ and $\gamma^T: T \rightarrow \Sigma^d T^*$ be the morphisms in the exchange $(d+2)$ -angles (4.3) and (4.4). Then $F_{\mathcal{T}}(\gamma^{T^*}) = 0$ and $F_{\mathcal{X}}(\gamma^{T^*}) = 0 = F_{\mathcal{X}}(\gamma^T)$.

Proof. As $\mathcal{T}$ is Oppermann-Thomas cluster tilting and $\mathcal{X} \subseteq \text{add}(\text{ind } \mathcal{T} \setminus \{T\}) \subseteq \mathcal{T}$, we have that $\mathcal{S}(\mathcal{T}, \Sigma^d T) = 0 = \mathcal{S}(\mathcal{X}, \Sigma^d T)$. Thus, $F_{\mathcal{T}}(\gamma^{T^*})$ and $F_{\mathcal{X}}(\gamma^{T^*})$ both vanish. On the other hand, since replacing T by T^* in $\mathcal{T}$ yields another Oppermann-Thomas cluster tilting subcategory of $\mathcal{S}$ that also contains $\mathcal{X}$, we have $\mathcal{S}(\mathcal{X}, \Sigma^d T^*) = 0$. Hence, $F_{\mathcal{X}}(\gamma^T) = 0$ as well. ■

Proposition 4.14. Let $T \in \text{ind } \mathcal{T} \setminus \text{ind } \mathcal{X}$ and consider the exchange $(d+2)$ -angles (4.3) and (4.4). Then each generator

$$\sum_{i=1}^d (-1)^i [X^{d+1-i}] - \sum_{i=1}^d (-1)^i [Y^i]$$

of $N_{\mathcal{X}}$ vanishes under $\tilde{Q}L: K_0^{\text{sp}}(\mathcal{T}) \rightarrow K_0(\mathcal{S}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$.

Proof. By Proposition 4.10, we have that X^i, Y^i lie in $\mathcal{T}$ for $1 \leq i \leq d$. As $F_{\mathcal{X}}(\gamma^{T^*}) = F_{\mathcal{X}}(\gamma^T) = 0$ by Lemma 4.13, we see that

$$Y^0 \longrightarrow Y^1 \longrightarrow \dots \longrightarrow Y^d \longrightarrow Y^{d+1} \xrightarrow{\gamma^{T^*}} \Sigma^d Y^0,$$

$$X^0 \longrightarrow X^1 \longrightarrow \dots \longrightarrow X^d \longrightarrow X^{d+1} \xrightarrow{\gamma^T} \Sigma^d X^0$$

are $\mathfrak{s}_{\mathcal{X}}$ -distinguished d -exangles in the d -exangulated category $(\mathcal{S}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$. So $\chi(Y^\bullet) = \sum_{i=0}^{d+1} (-1)^i [Y^i]$ and $\chi(X^\bullet) = \sum_{i=0}^{d+1} (-1)^i [X^i]$ are both trivial in $K_0(\mathcal{S}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$. Then, since $Y^0 = X^{d+1} = T$ and $Y^{d+1} = X^0 = T^*$, we see that

$$\tilde{Q}L \left(\sum_{i=1}^d (-1)^i [X^{d+1-i}] - \sum_{i=1}^d (-1)^i [Y^i] \right)$$

$$\begin{aligned}
 &= \sum_{i=1}^d (-1)^i [X^{d+1-i}] - \sum_{i=1}^d (-1)^i [Y^i] \\
 &= \sum_{i=0}^{d+1} (-1)^i [X^{d+1-i}] - (-1)^0 [X^{d+1}] - (-1)^{d+1} [X^0] - \sum_{i=1}^d (-1)^i [Y^i] \\
 &= (-1)^{d+1} \chi(X^\bullet) - (-1)^0 [Y^0] - (-1)^{d+1} [Y^{d+1}] - \sum_{i=1}^d (-1)^i [Y^i] \\
 &= (-1)^{d+1} \chi(X^\bullet) - \chi(Y^\bullet) \\
 &= [0].
 \end{aligned}$$

■

As an immediate consequence, we have:

Corollary 4.15. *There is a homomorphism $G_{\mathcal{X}}: K_0^{\text{sp}}(\mathcal{T})/N_{\mathcal{X}} \rightarrow K_0(\mathcal{S}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$ making the following diagram commute.*

$$\begin{array}{ccc}
 K_0^{\text{sp}}(\mathcal{T}) & \xrightarrow{Q_{N_{\mathcal{X}}}} & K_0^{\text{sp}}(\mathcal{T})/N_{\mathcal{X}} \\
 L \downarrow & & \downarrow G_{\mathcal{X}} \\
 K_0(\mathcal{S}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}}) & \xrightarrow{\tilde{Q}} & K_0(\mathcal{S}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})
 \end{array}$$

The commuting square in Corollary 4.15 provides the lower half of diagram (4.1) in Theorem 4.5. We prove in Subsection 4.3 that $G_{\mathcal{X}}$ is actually an isomorphism under the conditions of Setup 4.3; see Theorem 4.18.

4.2. A formula for θ on simples

In this subsection we derive a formula for the homomorphism $\theta: K_0(\text{mod } \mathcal{T}) \rightarrow K_0^{\text{sp}}(\mathcal{T})$ from Theorem 3.7 on the simple objects in $\text{mod } \mathcal{T}$. Recall that we are in the situation of Setup 4.2. We remark that the condition of this setup is satisfied when $d = 1$ and $\mathcal{T}$ is part of a *cluster structure* in the sense of [11, p. 1039]. In the context of Setup 4.2 in general, there is (see [3, Prop. 2.3(b)]) a bijective correspondence:

$$\left\{ \begin{array}{c} \text{simple modules in } \text{Mod } \mathcal{T} \\ \text{up to isomorphism} \end{array} \right\} \longleftrightarrow \text{ind } \mathcal{T}$$

$$\overline{S}_T = \mathcal{T}(-, T) / \text{rad}_{\mathcal{T}}(-, T) \longleftrightarrow T. \tag{4.5}$$

It follows from Remark 4.7 that, for $T \in \text{ind } \mathcal{T} \setminus \text{ind } \mathcal{X}$, we have $\overline{S}_T \in \text{mod } \mathcal{T}$ and so $\theta([\overline{S}_T])$ makes sense. The next result is a generalisation of the triangulated version stated in [36, 1.5(ii)]. It plays an important role in the proof of Theorem 4.18, the main result of Section 4.

Proposition 4.16. *Suppose $T \in \text{ind } \mathcal{T} \setminus \text{ind } \mathcal{X}$. Then we have*

$$\theta([\overline{S}_T]) = \sum_{i=1}^d (-1)^i [X^{d+1-i}] - \sum_{i=1}^d (-1)^i [Y^i],$$

where $Y^\bullet, X^\bullet$ are the exchange $(d+2)$ -angles (4.3) and (4.4), respectively, for T . In particular, $\theta([\overline{S}_T]) \in N_{\mathcal{X}}$.

Proof. Let $T \in \text{ind } \mathcal{T} \setminus \text{ind } \mathcal{X}$, and denote by T^* its mutation. Then $F_{\mathcal{T}}(\gamma^{T^*}) = 0$ by Lemma 4.13. Thus, by Theorem 3.7, we have

$$\begin{aligned}
0 &= \theta([\operatorname{Im} F_{\mathcal{T}}(\gamma^{T^*})]) \\
&= \sum_{i=0}^{d+1} (-1)^i \operatorname{ind}_{\mathcal{T}}^{\hat{\circ}}(Y^{d+1-i}) \\
&= \sum_{i=1}^{d+1} (-1)^i [Y^{d+1-i}] + \operatorname{ind}_{\mathcal{T}}^{\hat{\circ}}(T^*).
\end{aligned}$$

Rearranging and multiplying by $(-1)^{d+1}$ yields

$$\begin{aligned}
(-1)^{d+1} \operatorname{ind}_{\mathcal{T}}^{\hat{\circ}}(T^*) &= - \sum_{i=1}^{d+1} (-1)^{d+1-i} [Y^{d+1-i}] \\
&= - \sum_{i=0}^d (-1)^i [Y^i] \\
&= - \left(\sum_{i=1}^d (-1)^i [Y^i] + [T] \right).
\end{aligned}$$

Then we see

$$\begin{aligned}
\theta([\overline{S}_T]) &= \theta([\operatorname{Im} F_{\mathcal{T}}(\gamma^T)]) && \text{by Setup 4.2} \\
&= \sum_{i=0}^{d+1} (-1)^i \operatorname{ind}_{\mathcal{T}}^{\hat{\circ}}(X^{d+1-i}) && \text{by Theorem 3.7} \\
&= [T] + \sum_{i=1}^d (-1)^i [X^{d+1-i}] + (-1)^{d+1} \operatorname{ind}_{\mathcal{T}}^{\hat{\circ}}(T^*) && \begin{array}{l} \text{since } X^{d+1} = T, X^0 = T^*, \\ \text{and } X^j \in \mathcal{T} \text{ for all } 1 \leq j \leq d \\ \text{by Proposition 4.10} \end{array} \\
&= [T] + \sum_{i=1}^d (-1)^i [X^{d+1-i}] - \left(\sum_{i=1}^d (-1)^i [Y^i] + [T] \right) && \text{from the observation above} \\
&= \sum_{i=1}^d (-1)^i [X^{d+1-i}] - \sum_{i=1}^d (-1)^i [Y^i]. && \blacksquare
\end{aligned}$$

4.3. $G_{\mathcal{X}}$ is an isomorphism

In this subsection we are in the case of Setup 4.3. We will show that the homomorphism

$$G_{\mathcal{X}}: K_0^{\operatorname{sp}}(\mathcal{T})/N_{\mathcal{X}} \rightarrow K_0(\mathcal{S}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}}),$$

which satisfies $\tilde{Q} \circ L = G_{\mathcal{X}} \circ Q_{N_{\mathcal{X}}}$ (see Corollary 4.15), is an isomorphism of abelian groups.

Remark 4.17. Setup 4.3 implies that, for any morphism $\gamma: S^{d+1} \rightarrow \Sigma^d S^0$ in $\mathcal{S}$, the (target of the) image $\operatorname{Im} F_{\mathcal{T}}(\gamma)$ also lies in $\operatorname{fl} \mathcal{T}$, because it is a subobject of the finite length object $F_{\mathcal{T}}(\Sigma^d S^0)$.

We can now prove our main result.

Theorem 4.18. $G_{\mathcal{X}}$ is an isomorphism.

Proof. Firstly, since $\tilde{Q}$ and L are surjective (see (4.2) and Theorem 3.10, respectively), the equality $G_{\mathcal{X}}Q_{N_{\mathcal{X}}} = \tilde{Q}L$ (see Corollary 4.15) implies $G_{\mathcal{X}}$ is surjective.

To show $G_{\mathcal{X}}$ is injective, let $g + N_{\mathcal{X}} \in K_0^{\text{sp}}(\mathcal{T})/N_{\mathcal{X}}$ be an element such that $G_{\mathcal{X}}(g + N_{\mathcal{X}}) = 0$ in $K_0(\mathcal{S}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$. We will show that $g + N_{\mathcal{X}}$ is trivial in $K_0^{\text{sp}}(\mathcal{T})/N_{\mathcal{X}}$, i.e. $g \in N_{\mathcal{X}}$. As $G_{\mathcal{X}}(g + N_{\mathcal{X}}) = 0$ and $G_{\mathcal{X}}Q_{N_{\mathcal{X}}} = \tilde{Q}L$, we have that

$$\tilde{Q}(L(g)) = G_{\mathcal{X}}Q_{N_{\mathcal{X}}}(g) = G_{\mathcal{X}}(g + N_{\mathcal{X}}) = 0.$$

Since $\tilde{Q}: K_0(\mathcal{S}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}}) \rightarrow K_0(\mathcal{S}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}}) = K_0^{\text{sp}}(\mathcal{S})/\mathcal{I}_{\mathcal{X}}$ is the canonical surjection induced by the containment $\mathcal{I}_{\mathcal{T}} \subseteq \mathcal{I}_{\mathcal{X}}$, we see that $L(g) \in \mathcal{I}_{\mathcal{X}} + \mathcal{I}_{\mathcal{T}}$ in $K_0(\mathcal{S}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}})$.

Recall that $\mathcal{I}_{\mathcal{X}} \leq K_0^{\text{sp}}(\mathcal{S})$ (see Definition 2.18) is generated by elements of the form $\chi(S^{\bullet})$ (see Definition 2.15), where $\langle S^{\bullet}, \gamma \rangle$ is an $\mathfrak{s}_{\mathcal{X}}$ -distinguished d -exangle

$$S^0 \longrightarrow S^1 \longrightarrow \cdots \longrightarrow S^d \longrightarrow S^{d+1} \dashrightarrow^{\gamma} \Sigma^d S^0$$

in $(\mathcal{S}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$. Thus, for some integer $s \geq 1$, we have

$$L(g) = \sum_{j=1}^s (-1)^{a_j} \chi(S_j^{\bullet}) + \mathcal{I}_{\mathcal{T}}, \quad (4.6)$$

where, for each $1 \leq j \leq s$, there is an $\mathfrak{s}_{\mathcal{X}}$ -distinguished d -exangle

$$S_j^0 \longrightarrow S_j^1 \longrightarrow \cdots \longrightarrow S_j^d \longrightarrow S_j^{d+1} \dashrightarrow^{\gamma_j} \Sigma^d S_j^0,$$

and $a_j \in \{0, 1\}$. Hence,

$$\begin{aligned} g &= \text{ind}_{\mathcal{T}}^{\diamond}(L(g)) && \text{as } L^{-1} = \text{ind}_{\mathcal{T}}^{\diamond} \text{ by Theorem 3.10} \\ &= \text{ind}_{\mathcal{T}}^{\diamond} \left(\sum_{j=1}^s (-1)^{a_j} \chi(S_j^{\bullet}) + \mathcal{I}_{\mathcal{T}} \right) && \text{by (4.6)} \\ &= \sum_{j=1}^s (-1)^{a_j} \sum_{i=0}^{d+1} (-1)^i \text{ind}_{\mathcal{T}}^{\diamond}(S_j^i) && \text{(note } \text{ind}_{\mathcal{T}}^{\diamond}(\mathcal{I}_{\mathcal{T}}) = 0 \text{ since } \\ & && K_0(\mathcal{S}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}}) = K_0^{\text{sp}}(\mathcal{S})/\mathcal{I}_{\mathcal{T}}) \\ &= (-1)^{d+1} \sum_{j=1}^s (-1)^{a_j} \sum_{i=0}^{d+1} (-1)^i \text{ind}_{\mathcal{T}}^{\diamond}(S_j^{d+1-i}) && \text{(see Remark 2.16)} \\ &= (-1)^{d+1} \sum_{j=1}^s (-1)^{a_j} \theta([\text{Im } F_{\mathcal{T}}(\gamma_j)]) && \text{by Theorem 3.7.} \end{aligned}$$

With this observation, it suffices to show that $\theta([\text{Im } F_{\mathcal{T}}(\gamma_j)]) \in N_{\mathcal{X}}$ for each $1 \leq j \leq s$. Thus, fix $j \in \{1, \dots, s\}$. As $\text{Im } F_{\mathcal{T}}(\gamma_j)$ is an object in $\text{fl } \mathcal{T}$, it admits a composition series

$$0 = M_0 < M_1 < \cdots < M_{r-1} < M_r = \text{Im } F_{\mathcal{T}}(\gamma_j)$$

in $\text{Mod } \mathcal{T}$. Thus, for each $0 \leq l \leq r-1$, there exists $T_l \in \text{ind } \mathcal{T}$ such that $M_{l+1}/M_l = \overline{S}_{T_l}$ is simple.

We claim that $T_l \notin \text{ind } \mathcal{X}$ for all $0 \leq l \leq r-1$. Assume, for contradiction, that $T_l \in \text{ind } \mathcal{X}$ for some $l \in \{0, \dots, r-1\}$. Denote the simple objects in $\text{mod } \mathcal{X}$ by S_X for an object $X \in \text{ind } \mathcal{X}$. As shown in [31, 2.5], the inclusion $\iota: \mathcal{X} \rightarrow \mathcal{T}$ induces an exact functor $\iota^*: \text{fl } \mathcal{T} \rightarrow \text{fl } \mathcal{X}$, which, on simple objects, is given by

$$\iota^* \overline{S}_T = \begin{cases} S_T & \text{if } T \in \text{ind } \mathcal{X} \\ 0 & \text{else.} \end{cases}$$

In particular, we have $\iota^* M_{l+1}$ is a submodule of $\iota^* \text{Im } F_{\mathcal{T}}(\gamma_j) = \text{Im } F_{\mathcal{X}}(\gamma_j)$. But $F_{\mathcal{X}}(\gamma_j)$ is 0 since γ_j factors through $\mathcal{X}^{\perp_0}$, so $S_{T_l} = \iota^* \overline{S}_{T_l} = \iota^*(M_{l+1}/M_l) = 0$, which is a contradiction as S_{T_l} is simple. This proves the claim.

Using the composition series for $\text{Im } F_{\mathcal{T}}(\gamma_j)$, we have

$$\theta([\text{Im } F_{\mathcal{T}}(\gamma_j)]) = \theta([\overline{S}_{T_0}]) + \cdots + \theta([\overline{S}_{T_{r-1}}]).$$

For each $l \in \{0, \dots, r-1\}$ we have $\theta([\overline{S}_{T_l}]) \in N_{\mathcal{X}}$ by Proposition 4.16, and hence $\theta([\text{Im } F_{\mathcal{T}}(\gamma_j)])$ also lies in $N_{\mathcal{X}}$. Since $j \in \{1, \dots, s\}$ was arbitrary, we see that $g \in N_{\mathcal{X}}$. Hence, $G_{\mathcal{X}}$ is injective and, moreover, an isomorphism. ■

5. The $\mathcal{X}$ -Caldero-Chapoton map

In this section, we describe the connection between our results and the modified Caldero-Chapoton map defined by Holm and Jørgensen in [31].

Setup 5.1. In this section, we suppose the following.

- (i) k is an algebraically closed field.
- (ii) $\mathcal{C}$ is a skeletally small, k -linear, Krull-Schmidt, Hom-finite, 2-Calabi-Yau triangulated category.
- (iii) $\mathcal{X}$ is a functorially finite, additive subcategory of $\mathcal{C}$.
- (iv) $\mathcal{T}$ is a locally bounded, cluster tilting (i.e. 2-cluster tilting) subcategory of $\mathcal{C}$, which belongs to a cluster structure in the sense of [11, p. 1039].
- (v) $\mathcal{X} \subseteq \mathcal{T} \subseteq \mathcal{C}$.

Remark 5.2. Note that Setup 5.1 implies: Setup 3.3 is met with $d = 1$, $n = 2$ and $\mathcal{S} = \mathcal{C}$; Setups 4.1 and 4.2 hold as $\mathcal{T}$ is part of a cluster structure; and Setup 4.3 holds as $\mathcal{T}$ is assumed to be locally bounded. Thus, all results from Sections 3 and 4 apply. Furthermore, it also follows from $\mathcal{T}$ being part of a cluster structure that all indecomposables in $\mathcal{T}$ have a mutation.

Over [30] and [31], Holm and Jørgensen constructed a modified Caldero-Chapoton map, which generalised the original of [14] in two ways: it only depends on a rigid subcategory and can take values in a general commutative ring A . This modified Caldero-Chapoton map depends on several other functions, which we recall now following [31, Sec. 2].

In our setup, $d = 1$ so there are exchange triangles

$$T \longrightarrow Y \longrightarrow T^* \xrightarrow{\gamma^{T^*}} \Sigma T, \quad (5.1)$$

$$T^* \longrightarrow X \longrightarrow T \xrightarrow{\gamma^T} \Sigma T^* \quad (5.2)$$

for each $T \in \text{ind } \mathcal{T}$; see Subsection 4.1. Then the subgroup $N_{\mathcal{X}} \leq K_0^{\text{sp}}(\mathcal{T})$ from Definition 4.12 can now be written more simply as

$$N_{\mathcal{X}} = \left\langle [X] - [Y] \left| \begin{array}{l} \text{(5.1) and (5.2) are exchange triangles} \\ \text{with } T \in \text{ind } \mathcal{T} \setminus \text{ind } \mathcal{X} \end{array} \right. \right\rangle, \quad (5.3)$$

and we still have the quotient homomorphism $Q_{N_{\mathcal{X}}} : K_0^{\text{sp}}(\mathcal{T}) \rightarrow K_0^{\text{sp}}(\mathcal{T})/N_{\mathcal{X}}$.

Let $\iota: \mathcal{X} \rightarrow \mathcal{T}$ be the inclusion functor and $\iota^*: \text{fl } \mathcal{T} \rightarrow \text{fl } \mathcal{X}$ the induced exact functor, as in the proof of Theorem 4.18. Then ι^* induces a surjective homomorphism

$$\kappa: K_0(\text{fl } \mathcal{T}) \twoheadrightarrow K_0(\text{fl } \mathcal{X})$$

on the level of Grothendieck groups, given by

$$\kappa([\overline{S}_T]) = \begin{cases} [S_T] & \text{if } T \in \text{ind } \mathcal{X} \\ 0 & \text{if } T \in \text{ind } \mathcal{T} \setminus \text{ind } \mathcal{X}. \end{cases}$$

Furthermore, there is a homomorphism

$$\overline{\varphi}: K_0(\text{fl } \mathcal{T}) \rightarrow K_0^{\text{sp}}(\mathcal{T})$$

defined by

$$\overline{\varphi}([\overline{S}_T]) = [Y] - [X] \quad (5.4)$$

for each $T \in \text{ind } \mathcal{T}$, where X, Y are as in (5.1) and (5.2); see [36, 1.5(ii)]. Post-composing with $L: K_0^{\text{sp}}(\mathcal{T}) \rightarrow K_0(\mathcal{C}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}})$ (see e.g. Theorem 4.5) gives a homomorphism

$$\overline{\psi} := L\overline{\varphi}: K_0(\text{fl } \mathcal{T}) \rightarrow K_0(\mathcal{C}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}}),$$

such that

$$\overline{\psi}([\overline{S}_T]) = L([Y]) - L([X]) = L \text{ind}_{\mathcal{T}}^{\diamond}(Y) - L \text{ind}_{\mathcal{T}}^{\diamond}(X) = Q_{\mathcal{T}}([Y]) - Q_{\mathcal{T}}([X]).$$

Moreover, note that, for any $T \in \text{ind } \mathcal{T} \setminus \text{ind } \mathcal{X}$, we have

$$\begin{aligned} \tilde{Q}\overline{\psi}([\overline{S}_T]) &= \tilde{Q}L\overline{\varphi}([\overline{S}_T]) && \text{as } \overline{\psi} = L\overline{\varphi} \\ &= G_{\mathcal{X}}Q_{N_{\mathcal{X}}}\overline{\varphi}([\overline{S}_T]) && \text{by Theorem 4.5} \\ &= G_{\mathcal{X}}Q_{N_{\mathcal{X}}}([Y] - [X]) && \text{by (5.4)} \\ &= 0 && \text{using (5.3).} \end{aligned}$$

Therefore, $\tilde{Q}\overline{\psi}$ factors uniquely through κ , i.e. there is a unique homomorphism

$$\psi: K_0(\text{fl } \mathcal{X}) \rightarrow K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$$

making the following diagram commute.

$$\begin{array}{ccccc} & K_0(\text{fl } \mathcal{T}) & \xrightarrow{\kappa} & K_0(\text{fl } \mathcal{X}) & \\ & \swarrow \overline{\varphi} & & \downarrow \psi & \\ K_0^{\text{sp}}(\mathcal{T}) & & & & \\ & \searrow L & & \downarrow \tilde{Q} & \\ & K_0(\mathcal{C}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}}) & \xrightarrow{\tilde{Q}} & K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}}) & \end{array} \quad (5.5)$$

Definition 5.3. (cf. [31, 1.2]) Let A be a commutative ring. Suppose

$$\overline{\varepsilon}: K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}}) \rightarrow A$$

is a map, such that

$$\bar{\varepsilon}(0) = 1 \quad \text{and} \quad \bar{\varepsilon}(e + f) = \bar{\varepsilon}(e)\bar{\varepsilon}(f). \quad (5.6)$$

Define $\alpha: \text{obj}(\mathcal{C}) \rightarrow A$ and $\beta: K_0(\text{fl } \mathcal{X}) \rightarrow A$ as follows:

$$\alpha := \bar{\varepsilon} \circ Q_{\mathcal{X}} \quad \text{and} \quad \beta := \bar{\varepsilon} \circ \psi.$$

Let $C \in \text{obj}(\mathcal{C})$. Then the $\mathcal{X}$ -Caldero-Chapoton map is given by the formula

$$\rho(C) = \alpha(C) \sum_e \chi(\text{Gr}_e(F_{\mathcal{X}}(\Sigma C))) \beta(e),$$

where $\text{Gr}_e(F_{\mathcal{X}}(\Sigma C)) = \text{Gr}_e(\mathcal{C}(-, \Sigma C)|_{\mathcal{X}})$ is the Grassmannian of finite length submodules $M \subseteq \mathcal{C}(-, \Sigma C)|_{\mathcal{X}}$ in $\text{Mod } \mathcal{X}$ with $[M] = e \in K_0(\text{fl } \mathcal{X})$, and χ is the Euler characteristic defined by étale cohomology with proper support.

Remark 5.4.

- (i) It is not immediately apparent that the formula for ρ above is well-defined for all $C \in \text{obj}(\mathcal{C})$. However, if $F_{\mathcal{X}}(\Sigma C)$ has finite length, then the sum over e is finite and $\rho(C)$ gives an element of A . Recall that Setup 4.3 holds since we are in the situation of Setup 5.1 (see Remark 5.2). Thus, $F_{\mathcal{T}}(\Sigma C)$ has finite length for all $C \in \text{obj}(\mathcal{C})$, and so $\kappa(F_{\mathcal{T}}(\Sigma C)) = F_{\mathcal{X}}(\Sigma C)$ indeed has finite length.
- (ii) The definition of ρ is precisely how the original Caldero-Chapoton map is given in [14], but with the involved maps and functors adjusted relative to the functorially finite, rigid subcategory $\mathcal{X} \subseteq \mathcal{T}$. Indeed, replacing $Q_{\mathcal{X}}$, ψ and $\bar{\varepsilon}$, respectively, by $\text{ind}_{\mathcal{T}}$, $\bar{\varphi}$, and a map $\bar{\varepsilon}: K_0^{\text{sp}}(\mathcal{T}) \rightarrow A$ (still satisfying condition (5.6)), where A is the Laurent polynomial ring $\mathbb{Z}[x_T, x_T^{-1}]_{T \in \text{ind } \mathcal{T}}$, recovers the version of the Caldero-Chapoton map given in [36, 1.8].

Theorem 5.5. *The $\mathcal{X}$ -Caldero-Chapoton map defined above recovers the modified Caldero-Chapoton map of Holm-Jørgensen [31].*

Proof. To show this, let $\varepsilon: K_0^{\text{sp}}(\mathcal{T})/N_{\mathcal{X}} \rightarrow A$ be a map satisfying $\varepsilon(0) = 1$ and $\varepsilon(e + f) = \varepsilon(e)\varepsilon(f)$, as assumed in [31, Def. 2.8]. Define $\bar{\varepsilon} := \varepsilon G_{\mathcal{X}}^{-1}$. Then $\bar{\varepsilon}$ satisfies the requirements for the definition of the $\mathcal{X}$ -Caldero-Chapoton map.

The map α is $\alpha = \bar{\varepsilon} \circ Q_{\mathcal{X}} = \varepsilon G_{\mathcal{X}}^{-1} Q_{\mathcal{X}} = \varepsilon Q_{N_{\mathcal{X}}} \text{ind}_{\mathcal{T}}^{\diamond}$, which is readily seen to agree with [31, Def. 2.8]. For $\beta = \bar{\varepsilon} \circ \psi = \varepsilon G_{\mathcal{X}}^{-1} \psi$, first note that in [31, Def. 2.8] the map is defined as $\varepsilon \varphi$, where $\varphi: K_0(\text{fl } \mathcal{X}) \rightarrow K_0^{\text{sp}}(\mathcal{T})/N_{\mathcal{X}}$ is the unique homomorphism satisfying $\varphi \kappa = Q_{N_{\mathcal{X}}} \bar{\varphi}$. Thus, it suffices to show that $G_{\mathcal{X}}^{-1} \psi = \varphi$. This does indeed hold:

$$\begin{aligned} G_{\mathcal{X}}^{-1} \psi \kappa &= G_{\mathcal{X}}^{-1} \tilde{Q} \bar{\psi} && \text{using (5.5)} \\ &= G_{\mathcal{X}}^{-1} \tilde{Q} L \bar{\varphi} && \text{since } \bar{\psi} = L \bar{\varphi} \\ &= Q_{N_{\mathcal{X}}} \bar{\varphi} && \text{using Theorem 4.5,} \end{aligned}$$

and hence the claim follows by uniqueness. ■

Remark 5.6. We conclude this section with a remark about how our considerations have a connection with recent work by the authors on indices with respect to rigid subcategories. The map $\psi: K_0(\text{fl } \mathcal{X}) \rightarrow K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$ obtained above in (5.5) relates to the homomorphism $\theta_{\mathcal{X}}: K_0(\text{mod } \mathcal{X}) \rightarrow K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$

defined recently in [37]. The map $\theta_{\mathcal{X}}$ is shown to measure how far the index with respect to $\mathcal{X}$ is from being additive on triangles.

The inclusion $\mathfrak{fl} \mathcal{X} \rightarrow \mathbf{mod} \mathcal{X}$ (see Remark 4.7) induces a homomorphism

$$\mu: K_0(\mathfrak{fl} \mathcal{X}) \rightarrow K_0(\mathbf{mod} \mathcal{X}).$$

Recall that $F_{\mathcal{T}}(C) \in \mathfrak{fl} \mathcal{T}$ for all $C \in \mathcal{C}$ (see Remark 5.4(i)). Consider the K_0 -classes $[F_{\mathcal{A}}(C)]_{\mathfrak{fl} \mathcal{A}}$ and $[F_{\mathcal{A}}(C)]_{\mathbf{mod} \mathcal{A}}$ in $K_0(\mathfrak{fl} \mathcal{A})$ and $K_0(\mathbf{mod} \mathcal{A})$, respectively, for $\mathcal{A} \in \{\mathcal{X}, \mathcal{T}\}$. Then we have

$$\begin{aligned} \psi([F_{\mathcal{X}}(C)]_{\mathfrak{fl} \mathcal{X}}) &= \psi \kappa([F_{\mathcal{T}}(C)]_{\mathfrak{fl} \mathcal{T}}) \\ &= \tilde{Q} \overline{\psi}([F_{\mathcal{T}}(C)]_{\mathfrak{fl} \mathcal{T}}) && \text{using (5.5)} \\ &= \tilde{Q} L(\mathrm{ind}_{\mathcal{T}}(C) + \mathrm{ind}_{\mathcal{T}}(\Sigma^{-1}C)) && \text{by [31, Lem. 2.10]} \\ &= Q_{\mathcal{X}}([C]) + Q_{\mathcal{X}}([\Sigma^{-1}C]) && \text{using Theorem 4.5,} \end{aligned} \quad (5.7)$$

viewing $[C]$ and $[\Sigma^{-1}C]$ as elements of $K_0^{\mathrm{sp}}(\mathcal{S})$. Note that the term in (5.7) is equal to $\theta_{\mathcal{X}}([F_{\mathcal{X}}(C)]_{\mathbf{mod} \mathcal{X}})$, where $\theta_{\mathcal{X}}: K_0(\mathbf{mod} \mathcal{X}) \rightarrow K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$ is the homomorphism of [37]. Therefore, we see that $\psi([F_{\mathcal{X}}(C)]_{\mathfrak{fl} \mathcal{X}}) = \theta_{\mathcal{X}} \mu([F_{\mathcal{X}}(C)]_{\mathfrak{fl} \mathcal{X}})$. In particular, this suggests ψ can also be seen as some measure of how far the index with respect to $\mathcal{X}$ defined in [37] is from being additive on triangles.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Peter Jørgensen reports financial support was provided by Danish National Research Foundation. Peter Jørgensen reports financial support was provided by Independent Research Fund Denmark. Peter Jørgensen reports financial support was provided by Aarhus University Research Foundation. Peter Jørgensen reports financial support was provided by Engineering and Physical Sciences Research Council. Amit Shah reports financial support was provided by London Mathematical Society. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

The authors would like to thank P.-G. Plamondon for informing the first author of [43, Prop. 4.11]. This motivated the present article. The authors also thank M. Gorsky, H. Nakaoka and Y. Palu for directing the authors to [23, App. B]. We are grateful to M. Pressland for finding an error in a previous version of the article, and to the referee for their comments.

This work was supported by a DNRF Chair from the Danish National Research Foundation (grant DNRF156), by a Research Project 2 from the Independent Research Fund Denmark (grant 1026-00050B), by the Aarhus University Research Foundation (grant AUFF-F-2020-7-16), and by the Engineering and Physical Sciences Research Council (grant EP/P016014/1). The second author is also grateful to the London Mathematical Society for funding through an Early Career Fellowship with support from Heilbronn Institute for Mathematical Research (grant ECF-1920-57).

References

- [1] M. Auslander, Coherent functors, in: Proc. Conf. Categorical Algebra, La Jolla, Calif., 1965, Springer, New York, 1966, pp. 189–231.
- [2] M. Auslander, Representation Dimension of Artin Algebras, Queen Mary College Mathematics Notes, 1971 (republished in [4]).

- [3] M. Auslander, Representation theory of Artin algebras. II, *Commun. Algebra* 1 (1974) 269–310.
- [4] M. Auslander, Selected Works of Maurice Auslander. Part 1, American Mathematical Society, Providence, RI, 1999, edited and with a foreword by Idun Reiten, Sverre O. Smalø, and Øyvind Solberg.
- [5] H. Bass, Algebraic K -Theory, W. A. Benjamin, Inc., New York-Amsterdam, 1968.
- [6] A. Beligiannis, On the Freyd categories of an additive category, *Homol. Homotopy Appl.* 2 (2000) 147–185.
- [7] A. Beligiannis, Relative homology, higher cluster-tilting theory and categorified Auslander-Iyama correspondence, *J. Algebra* 444 (2015) 367–503.
- [8] R. Bennett-Tennenhaus, A. Shah, Transport of structure in higher homological algebra, *J. Algebra* 574 (2021) 514–549.
- [9] P.A. Bergh, M. Thaulé, The Grothendieck group of an n -angulated category, *J. Pure Appl. Algebra* 218 (2) (2014) 354–366.
- [10] C. Bessenrodt, T. Holm, P. Jørgensen, Generalized frieze pattern determinants and higher angulations of polygons, *J. Comb. Theory, Ser. A* 123 (2014) 30–42.
- [11] A.B. Buan, O. Iyama, I. Reiten, J. Scott, Cluster structures for 2-Calabi-Yau categories and unipotent groups, *Compos. Math.* 145 (4) (2009) 1035–1079.
- [12] A.B. Buan, B.R. Marsh, From triangulated categories to module categories via localization II: calculus of fractions, *J. Lond. Math. Soc.* (2) 86 (1) (2012) 152–170.
- [13] A.B. Buan, B.R. Marsh, M. Reineke, I. Reiten, G. Todorov, Tilting theory and cluster combinatorics, *Adv. Math.* 204 (2) (2006) 572–618.
- [14] P. Caldero, F. Chapoton, Cluster algebras as Hall algebras of quiver representations, *Comment. Math. Helv.* 81 (3) (2006) 595–616.
- [15] P. Caldero, B. Keller, From triangulated categories to cluster algebras. II, *Ann. Sci. Éc. Norm. Supér.* (4) 39 (6) (2006) 983–1009.
- [16] P. Caldero, B. Keller, From triangulated categories to cluster algebras, *Invent. Math.* 172 (1) (2008) 169–211.
- [17] İ. Çanakçı, P. Jørgensen, Friezes, weak friezes, and T-paths, *Adv. Appl. Math.* 131 (2021) 102253.
- [18] L. Demonet, Categorification of skew-symmetrizable cluster algebras, *Algebr. Represent. Theory* 14 (6) (2011) 1087–1162.
- [19] H. Derksen, J. Weyman, A. Zelevinsky, Quivers with potentials and their representations. I. Mutations, *Sel. Math. New Ser.* 14 (1) (2008) 59–119.
- [20] H. Derksen, J. Weyman, A. Zelevinsky, Quivers with potentials and their representations II: applications to cluster algebras, *J. Am. Math. Soc.* 23 (3) (2010) 749–790.
- [21] P. Dowbor, A. Skowroński, On the representation type of locally bounded categories, *Tsukuba J. Math.* 10 (1) (1986) 63–72.
- [22] F. Fedele, Grothendieck groups of triangulated categories via cluster tilting subcategories, *Nagoya Math. J.* 244 (2021) 204–231.
- [23] L. Fiorot, n -quasi-abelian categories vs n -tilting torsion pairs: with an application to flops of higher relative dimension, *Doc. Math.* 26 (2021) 149–198.
- [24] S. Fomin, A. Zelevinsky, Cluster algebras. I. Foundations, *J. Am. Math. Soc.* 15 (2) (2002) 497–529.
- [25] C. Fu, B. Keller, On cluster algebras with coefficients and 2-Calabi-Yau categories, *Trans. Am. Math. Soc.* 362 (2) (2010) 859–895.
- [26] C. Geiss, B. Keller, S. Oppermann, n -angulated categories, *J. Reine Angew. Math.* 675 (2013) 101–120.
- [27] A. Grothendieck, Sur quelques points d’algèbre homologique, *Tohoku Math. J.* (2) 9 (1957) 119–221.
- [28] J. Haugland, The Grothendieck group of an n -exangulated category, *Appl. Categ. Struct.* 29 (3) (2021) 431–446.
- [29] M. Herschend, Y. Liu, H. Nakaoka, n -exangulated categories (I): definitions and fundamental properties, *J. Algebra* 570 (2021) 531–586.
- [30] T. Holm, P. Jørgensen, Generalized friezes and a modified Caldero-Chapoton map depending on a rigid object, *Nagoya Math. J.* 218 (2015) 101–124.
- [31] T. Holm, P. Jørgensen, Generalised friezes and a modified Caldero-Chapoton map depending on a rigid object, II, *Bull. Sci. Math.* 140 (4) (2016) 112–131.
- [32] O. Iyama, Maximal orthogonal subcategories of triangulated categories satisfying Serre duality, in: Representation Theory of Finite-Dimensional Algebras, in: Oberwolfach Rep., vol. 2, EMS Publishing House, Berlin, 2005, pp. 319–406.
- [33] O. Iyama, Y. Yoshino, Mutation in triangulated categories and rigid Cohen-Macaulay modules, *Invent. Math.* 172 (1) (2008) 117–168.
- [34] G. Jasso, n -abelian and n -exact categories, *Math. Z.* 283 (3–4) (2016) 703–759.
- [35] P. Jørgensen, Tropical friezes and the index in higher homological algebra, *Math. Proc. Camb. Philos. Soc.* 171 (1) (2021) 23–49.
- [36] P. Jørgensen, Y. Palu, A Caldero-Chapoton map for infinite clusters, *Trans. Am. Math. Soc.* 365 (3) (2013) 1125–1147.
- [37] P. Jørgensen, A. Shah, The index with respect to a rigid subcategory of a triangulated category, *Int. Math. Res. Not.* (2023), in press, <https://doi.org/10.1093/imrn/rnad130>.
- [38] H. Krause, Krull–Schmidt categories and projective covers, *Expo. Math.* 33 (4) (2015) 535–549.
- [39] H. Lenzing, I. Reiten, Hereditary Noetherian categories of positive Euler characteristic, *Math. Z.* 254 (1) (2006) 133–171.
- [40] H. Matsui, Classifying dense resolving and coresolving subcategories of exact categories via Grothendieck groups, *Algebr. Represent. Theory* 21 (3) (2018) 551–563.
- [41] H. Nakaoka, Y. Palu, Extriangulated categories, Hovey twin cotorsion pairs and model structures, *Cah. Topol. Géom. Différ. Catég.* 60 (2) (2019) 117–193.
- [42] S. Oppermann, H. Thomas, Higher-dimensional cluster combinatorics and representation theory, *J. Eur. Math. Soc.* 14 (6) (2012) 1679–1737.
- [43] A. Padrol, Y. Palu, V. Pilaud, P.-G. Plamondon, Associahedra for finite type cluster algebras and minimal relations between $\mathbf{g}$ -vectors, Preprint, <https://arxiv.org/abs/1906.06861>, 2019.
- [44] Y. Palu, Cluster characters for 2-Calabi-Yau triangulated categories, *Ann. Inst. Fourier (Grenoble)* 58 (6) (2008) 2221–2248.

- [45] Y. Palu, Grothendieck group and generalized mutation rule for 2-Calabi-Yau triangulated categories, *J. Pure Appl. Algebra* 213 (7) (2009) 1438–1449.
- [46] P.-G. Plamondon, Cluster characters for cluster categories with infinite-dimensional morphism spaces, *Adv. Math.* 227 (1) (2011) 1–39.
- [47] N. Popescu, *Abelian Categories with Applications to Rings and Modules*, London Mathematical Society Monographs, vol. 3, Academic Press, London-New York, 1973.
- [48] M. Prest, *Purity, Spectra and Localisation*, Encyclopedia of Mathematics and Its Applications, vol. 121, Cambridge Univ. Press, Cambridge, 2009.
- [49] D. Quillen, Higher algebraic K -theory. I, in: H. Bass (Ed.), *Algebraic K-Theory I. Proceedings of the Conference Held at the Seattle Research Center of Battelle Memorial Institute, August 28 – September 8, 1972*, Springer, Berlin, 1973, pp. 85–147.
- [50] R.W. Thomason, The classification of triangulated subcategories, *Compos. Math.* 105 (1) (1997) 1–27.
- [51] J.-L. Verdier, Des catégories dérivées des catégories abéliennes, *Astérisque* 239 (1996), xii+253, with a preface by Luc Illusie, edited and with a note by Georges Maltsiniotis.
- [52] Y. Zhou, B. Zhu, Cluster algebras arising from cluster tubes, *J. Lond. Math. Soc.* (2) 89 (3) (2014) 703–723.
- [53] B. Zhu, X. Zhuang, Grothendieck groups in extriangulated categories, *J. Algebra* 574 (2021) 206–232.