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Frequency-dependent mass, elastic and geometric stiffness matrices of an axially loaded Timoshenko-Ehrenfest beam with applications

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Abstract

Earlier research on the development of explicit algebraic expressions for the elements of the frequency-dependent mass, elastic and geometric stiffness matrices for free vibration analysis was carried out on Bernoulli-Euler, Timoshenko-Ehrenfest and axially loaded Bernoulli-Euler beams. Seeking solution for the correspondingly more difficult problem for an axially loaded Timoshenko-Ehrenfest beam seemed too difficult at the time when these earlier developments took place. Now, with the experience and knowledge gained, the difficulty is overcome in part by enhanced application of symbolic computing. Thus, the explicit algebraic expressions for the elements of the frequency-dependent mass, elastic and geometric stiffness matrices of an axially loaded Timoshenko-Ehrenfest beam are derived from first principles. The equivalency of these matrices when added altogether, with the dynamic stiffness matrix is ensured. The derived matrices are then applied using the Wittrick-Williams algorithm as a solution technique to investigate the free vibration characteristics of some illustrative examples. The results are discussed, and significant conclusions are drawn. The proposed method preserves the exactness of results in the same way as the dynamic stiffness method, but importantly, it opens the possibility of including damping in free vibration and response analysis when using exact methods such as the dynamic stiffness method.

Key words: axially-loaded Timoshenko-Ehrenfest beams, free vibration, frequency-dependent mass, elastic and geometric stiffness matrices, Wittrick-Williams algorithm, dynamic stiffness method

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1. Introduction

It is well recognised that beams are essential components of many practical structures such as pylons, cranes, bridges, towers, antennae and multi-storey and multi-bay buildings, amongst others. An understanding of the free vibration behaviour of beams is thus important to achieve an efficient, reliable and robust design of these structures, particularly when dynamic loads are encountered. In this respect, more than half a century ago, Przemieniecki [1, 2] developed the frequency dependent mass and stiffness matrices of beams and provided series expansions of these matrices by retaining two frequency dependent terms to investigate their free vibration characteristics accurately. Several investigators have carried forward Przemieniecki's original idea relying, by and large, on the power series expansion of the mass and stiffness matrices [3-6]. These investigations provided significant contributions to the literature because they achieved much better accuracy than the conventional finite element method when carrying out the free vibration analysis of beams and frameworks. However, all these investigations [3-6] including Przemieniecki's original work [1, 2] still have a serious limitation in that the frequency-dependent mass and stiffness matrices derived by the authors in series form had to be truncated at some point when arriving at the solution, which makes the results somewhat approximate or rather less accurate. This is a disadvantage when performing free vibration analysis of beams and frameworks, particularly in the high frequency range, which is a necessary requirement in the statistical energy analysis (SEA) method because the modal density in the high frequency range is usually very high [7, 8]. The need for accurate and reliable high frequency vibration analysis for beams and frameworks is also great in energy flow analysis [9, 10].

In his recent investigations, the author of this paper attempted to circumvent the limitation of the earlier research on the developments of frequency-dependent mass, elastic and geometric stiffness matrices. In the first instance, he published a paper [11] which presented explicit algebraic expressions for the elements of the frequency-dependent mass and stiffness matrices of a Bernoulli-Euler beam in an exact sense by including all the terms of the infinite series implicitly, to investigate the free vibration characteristics of beams and frameworks. His research was greatly facilitated by symbolic computation [12, 13]. Later he extended this work [11] to a Timoshenko-Ehrenfest beam [14] and then to an axially loaded Bernoulli-Euler beam [15], presenting explicit algebraic expressions for the elements of the frequency dependent mass, elastic and geometric stiffness matrices which accounted for all the terms of the infinite series. References [11, 14, 15] are recent papers which reviewed earlier publications and highlighted the progressive developments of the frequency-dependent mass, elastic and geometric stiffness matrices of beams. These reviews are not repeated here for brevity. However, from a historical perspective, some comments regarding the progress made in the advancements of beam theories in general, are warranted.

Euler and Bernoulli developed the earliest known beam theory in the eighteenth century [16, 17], which survived the test of time and is being used even to this day. The theory known as the Bernoulli-Euler or Euler-Bernoulli beam theory was enhanced about a century later by Lord Rayleigh [18] who included the effect of the rotatory inertia of the beam cross-section and improved the accuracy of results. Later, this relatively unknown Rayleigh-beam theory was overshadowed by the work of Timoshenko and Ehrenfest in the earlier part of the twentieth century [19, 20] when both the effects of rotatory inertia and shear deformation were incorporated in the Bernoulli-Euler beam theory. From these advancements made during the developments of beam theories, it is obvious that successive improvements in beam theories have taken place in stages, and the tendency of the early pioneers

appears to have been to proceed from easy to hard. This is perhaps how research evolves, and novel possibilities unfold and grow. The development of frequency-dependent mass, elastic and geometric stiffness matrices for beams in an exact sense is no exception to this trend. The desire to improve the theories for the frequency-dependent mass, elastic and geometric stiffness matrices for beams using frequency dependent exact shape functions has thus continued as is evident from the literature [11, 14, 15]. The first attempt on this research topic was made by applying the Bernoulli-Euler beam theory [11], then by using Timoshenko-Ehrenfest beam theory [14] and, most recently, by exploiting the theory for an axially loaded Bernoulli-Euler beam [15]. These publications demonstrated the equivalency between the dynamic stiffness matrix and the frequency-dependent mass, elastic and geometric stiffness matrices. Also, the past research reveals that as the complexity of the problem increases, the additional challenge to enhance the earlier theories also increases. This is probably true for all theoretical (and experimental) developments in any field. For instance, improving the original (classical) Bernoulli-Euler beam theory to the more advanced Timoshenko-Ehrenfest beam theory [19, 20] needed intensive efforts, which paved the way for future research. Thus, the quest for advanced theories from the relatively simplest to the most complex has continued progressively with time. In this context, the investigation carried out and reported in this paper which focuses on the development of frequency-dependent mass, elastic and geometric stiffness matrices of an axially loaded Timoshenko-Ehrenfest beam and then applies the theory for free vibration analysis is both timely and pertinent and can be viewed as a natural sequence of the progression to a new development. The research carried out in this paper could not be accomplished earlier because the scale of the problem was immense, and the available knowledge on the subject in the past was not mature. Thus, a continuing growth of knowledge and understanding has been necessary as a part of the process to initiate this research.

It would be instructive to enlighten the readers with the current developments in beam dynamics which have taken numerous turns since the periods of Bernoulli and Euler, Lord Rayleigh, and Timoshenko and Ehrenfest, amongst others. Foremost amongst these developments is a focus on the areas of wave propagation, functionally graded materials and micro and nano structures, amongst others. For instance, Zhang and Fan [21] investigated the propagation of bending and shear waves in an infinite Timoshenko-Ehrenfest beam supported by a periodic elastic foundation and they evaluated the dispersion characteristics of the waves by using the spectral element method. By contrast, Gawah et al. [22] analysed wave propagations in functionally graded carbon nanotube-reinforced composite beams resting on a viscoelastic foundation, by resorting to first order shear deformation theory. Lakhdar et al. [23] examined the free vibration behaviour of functionally graded beams using power, sigmoid and exponential laws, based on the Timoshenko-Ehrenfest theory. One of the interesting features of their work is that they addressed the effects of the porosity of materials on the results. On the other hand, Civalek et al. [24] applied first order shear deformation theory to study the forced vibration characteristics of a simply supported carbon nanotube-reinforced composite beam subjected to a harmonic point load at the mid-point of the beam. They observed that the distribution pattern of the reinforcement played an important role on the dynamic response of carbon nanotube-reinforced composite beams. Using an enhanced differential model of non-local elasticity theory, Demir and Civalek [25] studied the bending behaviour of micro and nano beams for different boundary conditions. Contrastingly, Numauoglu et al. [26] investigated the thermo-mechanical vibration characteristics of nanobeams by integrating the Timoshenko-Ehrenfest beam theory with the nonlocal elasticity theory. They solved the resulting eigenvalue problem by using the separation of variable method in conjunction with the finite element

method. The ongoing research on the advancement of beam theories described above in [21-26] gives many useful cross-references on the subject, which may help interested readers to develop the subject matter further.

The layout of the current paper is organised as follows. Following this Introduction section, section 2 gives the details of the derivation of the shape functions of an axially loaded Timoshenko-Ehrenfest beam. This is followed by sections 3, 4, and 5 which deal with the developments of frequency dependent mass, elastic and geometric matrices, respectively. Next, the application aspect of the theory is accentuated in section 6 in which the Wittrick-Williams algorithm [27] as a solution technique is briefly summarised in order to make the paper self-contained. Section 7 deals with numerical results and discussion, and finally principal conclusions are drawn in section 8.

2. Derivation of the frequency-dependent exact shape functions

The first step to develop the frequency-dependent mass, elastic and geometric stiffness matrices of an axially loaded Timoshenko-Ehrenfest beam is to derive its frequency-dependent shape functions, relating the bending displacement and bending rotation within the beam to its nodal bending displacements and bending rotations, when the beam is undergoing free natural vibration. Figure 1 shows the coordinate system and notations for an axially loaded uniform Timoshenko-Ehrenfest beam with Young's modulus E , shear modulus (or modulus of rigidity) G , and density ρ , as the material properties. The area and the second moment of area of the beam cross-section are A and I respectively, so that the bending rigidity, shear rigidity, mass per unit length, and mass moment of area per unit length for the beam are EI , kAG , ρA and ρI , respectively, with k being the shear correction factor also known as shape factor. In some measure and probably to a great extent, the shear correction factor k , accounts for the non-uniform shear stress distribution through the beam cross-section as opposed to the uniform shear stress distribution necessitated in the formulation of the Timoshenko-Ehrenfest beam theory. As shown in Fig. 1, node 1 of the beam is located at the origin O and node 2 is at the other end at a distance L from the origin. A compressive axial load P as shown in the figure, is assumed to be positive, acting through the centroid of the cross-section, but P can also be negative so that tension is included in the theory.

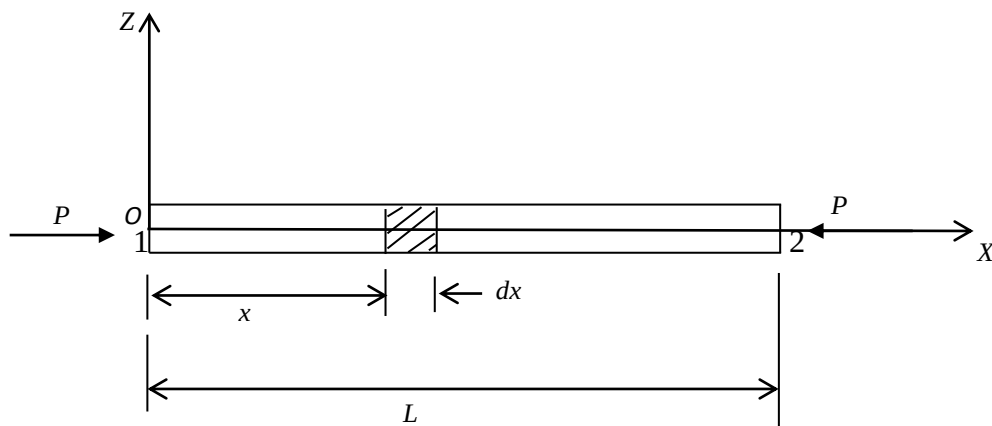


Fig. 1. Coordinate system and notation for an axially loaded Timoshenko-Ehrenfest beam.

Using linear small deflection theory, the kinetic energy (T) and potential energy (V) of an axially loaded Timoshenko-Ehrenfest beam in the usual notation can be expressed as [28, 29]

$$T = \frac{1}{2} \int_0^L (\rho A \dot{w}^2 + \rho I \dot{\theta}^2) dx; \quad V = \frac{1}{2} \int_0^L \{EI (\theta')^2 + kAG(w' - \theta)^2 - \frac{1}{2} \int_0^L P(w')^2\} dx \quad (1)$$

where $w(x, t)$ is the flexural or transverse deflection in the Z -direction and θ is the bending rotation of the beam cross-section. A prime and an over-dot here, and in the subsequent text, represent differentiation with respect to the length coordinate x and time t , respectively. Now, Hamilton's principle is applied to derive the governing differential equation.

Hamilton's principle states [14, 15]

$$\delta \int_{t_1}^{t_2} (T - V) dt = 0 \quad (2)$$

where t_1 and t_2 are the time interval in the dynamic trajectory, and δ is the usual variational operator.

The governing differential equations of motion for the axially loaded Timoshenko-Ehrenfest beam in free vibration are now derived by substituting the kinetic energy (T) and the potential energy (V) expressions given by Eq. (1) into Eq. (2), and then using the δ operator and next integrating by parts, and finally collecting terms. In an earlier publication, the entire procedure to generate the governing differential equations of motion and natural boundary conditions for bar or beam type structures using Hamilton's principle, was automated by Banerjee et al. [30] through the application of symbolic computation. In this way, the governing differential equations of the axially loaded Timoshenko-Ehrenfest beam in free vibration are obtained as [31-34]

$$EI\theta'' + kAG(w' - \theta) - \rho I\ddot{\theta} = 0 \quad (3)$$

$$kAG(w'' - \theta') - Pw'' - \rho A\ddot{w} = 0 \quad (4)$$

Assuming harmonic oscillation so that $w(x, t) = W(x)e^{i\omega t}$ and $\theta(x, t) = \Theta(x)e^{i\omega t}$ with ω being the angular or circular frequency of oscillation in rad/s, $W(x)$ and $\Theta(x)$ are the amplitudes of flexural displacement and bending rotation, respectively and $i = \sqrt{-1}$. By introducing the non-dimensional length parameter $\xi = x/L$, the above two partial differential equations can be combined into one (ordinary) differential equation to give [35]

$$\left(D^4 + \frac{b^2 c^2}{1-p^2 s^2} D^2 - \frac{b^2(1-b^2 r^2 s^2)}{1-p^2 s^2}\right) H = 0 \quad (5)$$

where D is the differential operator $d/d\xi$ and

$$H = W \text{ or } \Theta \quad (6)$$

and the rest of the parameters in Eq. (5) are as follows.

$$\left. \begin{aligned} b^2 &= \rho A \omega^2 L^4 / EI \\ p^2 &= PL^2 / EI \\ r^2 &= I / AL^2 \\ s^2 &= EI / kAGL^2 \\ c^2 &= p^2 / b^2 + r^2(1 - p^2 s^2) + s^2 \end{aligned} \right\} \quad (7)$$

The general solutions for W and Θ resulting from Eq. (5) are given by

$$W(\xi) = A_1 \cosh \alpha \xi + A_2 \sinh \alpha \xi + A_3 \cos \beta \xi + A_4 \sin \beta \xi \quad (8)$$

$$\theta(\xi) = B_1 \sinh \alpha \xi + B_2 \cosh \alpha \xi + B_3 \sin \beta \xi + B_4 \cos \beta \xi \quad (9)$$

where α and β are given by

$$\alpha = \frac{b}{\bar{p}\sqrt{2}} \left[\mp c^2 + \left\{ c^4 + \frac{4}{b^2} \bar{p}^2 (1 - b^2 r^2 s^2) \right\}^{1/2} \right]^{1/2} \quad (10)$$

with

$$\bar{p}^2 = 1 - p^2 s^2 \quad (11)$$

and A_1 - A_4 and B_1 - B_4 are two different sets of integration constants.

It can be shown with the help of Eq. (3) or Eq. (4) and that the constants of A_1 - A_4 and B_1 - B_4 of Eqs. (8) and (9) are related as follows [35]

$$B_1 = \frac{k_\alpha}{L} A_1; \quad B_2 = \frac{k_\alpha}{L} A_2; \quad B_3 = -\frac{k_\beta}{L} A_3; \quad B_4 = \frac{k_\beta}{L} A_4 \quad (12)$$

where

$$k_\alpha = \frac{\bar{p}^2 \alpha^2 + b^2 s^2}{\alpha}; \quad k_\beta = \frac{\bar{p}^2 \beta^2 - b^2 s^2}{\beta} \quad (13)$$

Thus, the bending rotation $\theta(\xi)$ of Eq. (9) can be expressed as

$$\theta(\xi) = \frac{1}{L} (A_1 k_\alpha \sinh \alpha \xi + A_2 k_\alpha \cosh \alpha \xi - A_3 k_\beta \sin \beta \xi + A_4 k_\beta \cos \beta \xi) \quad (14)$$

Referring to Fig. 2, the end conditions (or boundary conditions) for W and θ at nodes 1 and 2 can now be applied to Eqs. (8) and (14) to eliminate the constants A_1 - A_4 and formulate the shape functions.

The end or boundary conditions are:

$$\text{At node 1 } (x=0, \text{ i.e., } \xi=0), W = W_1 \text{ and } \theta = \theta_1 \quad (15)$$

$$\text{At node 2 } (x=L, \text{ i.e., } \xi=1), W = W_2 \text{ and } \theta = \theta_2 \quad (16)$$

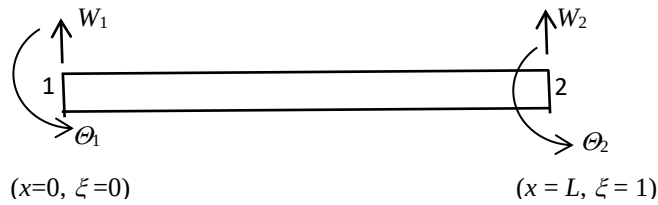


Fig. 2. Boundary conditions for displacements and rotations of an axially loaded Timoshenko-Ehrenfest beam.

Substituting Eqs. (15) and (16) into Eqs. (8) and (14) gives the following matrix relationship

$$\begin{bmatrix} W_1 \\ \theta_1 \\ W_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \alpha/L & 0 & \beta/L \\ C_{h\alpha} & S_{h\alpha} & C_\beta & S_\beta \\ k_\alpha S_{h\alpha}/L & k_\alpha C_{h\alpha}/L & -k_\beta S_\beta/L & k_\beta C_\beta/L \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} \quad (17)$$

or

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \alpha/L & 0 & \beta/L \\ C_{h\alpha} & S_{h\alpha} & C_\beta & S_\beta \\ k_\alpha S_{h\alpha}/L & k_\alpha C_{h\alpha}/L & -k_\beta S_\beta/L & k_\beta C_\beta/L \end{bmatrix}^{-1} \begin{bmatrix} W_1 \\ \theta_1 \\ W_2 \\ \theta_2 \end{bmatrix} \quad (18)$$

where

$$C_{h\alpha} = \cosh \alpha; \quad S_{h\alpha} = \sinh \alpha; \quad C_\beta = \cos \beta; \quad S_\beta = \sin \beta \quad (19)$$

Now, Eqs. (8) and (14) can be written in the following matrix form

$$\begin{bmatrix} W(\xi) \\ \theta(\xi) \end{bmatrix} = \begin{bmatrix} \cosh \alpha & \sinh \alpha & \cos \beta & \sin \beta \\ (k_\alpha/L) \sinh \alpha & (k_\alpha/L) \cosh \alpha & -(k_\beta/L) \sin \beta & (k_\beta/L) \cos \beta \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} \quad (20)$$

Substituting the constant vector comprising A_1 to A_4 from Eq. (18) into Eq. (20) gives

$$\begin{bmatrix} W(\xi) \\ \theta(\xi) \end{bmatrix} = \begin{bmatrix} \cosh \alpha & \sinh \alpha & \cos \beta & \sin \beta \\ (k_\alpha/L) \sinh \alpha & (k_\alpha/L) \cosh \alpha & -(k_\beta/L) \sin \beta & (k_\beta/L) \cos \beta \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \alpha/L & 0 & \beta/L \\ C_{h\alpha} & S_{h\alpha} & C_\beta & S_\beta \\ k_\alpha S_{h\alpha}/L & k_\alpha C_{h\alpha}/L & -k_\beta S_\beta/L & k_\beta C_\beta/L \end{bmatrix}^{-1} \begin{bmatrix} W_1 \\ \theta_1 \\ W_2 \\ \theta_2 \end{bmatrix} \quad (21)$$

or

$$\begin{bmatrix} W(\xi) \\ \theta(\xi) \end{bmatrix} = \begin{bmatrix} N_{w1} & N_{w2} & N_{w3} & N_{w4} \\ N_{\theta1} & N_{\theta2} & N_{\theta3} & N_{\theta4} \end{bmatrix} \begin{bmatrix} W_1 \\ \theta_1 \\ W_2 \\ \theta_2 \end{bmatrix} \quad (22)$$

i.e.,

$$\delta = \mathbf{N} \delta_N \quad (23)$$

where δ comprises the displacements ($W(\xi)$, $\theta(\xi)$) within the element whereas δ_N corresponds to nodal displacements (W_1 , θ_1 , W_2 , θ_2) and $\mathbf{N}$ is the shape function of the axially loaded Timoshenko-Ehrenfest beam with elements (N_{w1} , N_{w2} , N_{w3} , N_{w4}) and ($N_{\theta1}$, $N_{\theta2}$, $N_{\theta3}$, $N_{\theta4}$) Thus, the shape function $\mathbf{N}$ which relates displacements with the element to nodal displacements can be written as

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_w \\ \mathbf{N}_\theta \end{bmatrix} = \begin{bmatrix} N_{w1} & N_{w2} & N_{w3} & N_{w4} \\ N_{\theta1} & N_{\theta2} & N_{\theta3} & N_{\theta4} \end{bmatrix} \quad (24)$$

The task of generating the shape function in short, compact, and concise algebraic form by performing the matrix inversion and multiplication steps of Eq. (21) was somehow challenging. This activity was greatly assisted by applying symbolic computation [12, 13]. Thus, the explicit expressions for the shape functions $\mathbf{N}$ containing $N_{w1}, N_{w2}, N_{w3}, N_{w4}$ and $N_{\theta1}, N_{\theta2}, N_{\theta3}, N_{\theta4}$ terms were derived by extensive use of the symbolic computation package REDUCE [13]. These are given by

$$\begin{aligned}
 N_{w1} &= (\mu_1 k_\beta \cosh \alpha \xi + \mu_2 k_\beta \sinh \alpha \xi + \mu_3 k_\alpha \cos \beta \xi - \mu_2 k_\alpha \sin \beta \xi) / \Delta \\
 N_{w2} &= (-\mu_4 \cosh \alpha \xi + \mu_3 \sinh \alpha \xi + \mu_4 \cos \beta \xi + \mu_1 \sin \beta \xi) L / \Delta \\
 N_{w3} &= (\mu_6 k_\alpha k_\beta \cosh \alpha \xi - \mu_5 k_\beta \sinh \alpha \xi - \mu_6 k_\alpha k_\beta \cos \beta \xi + \mu_5 k_\alpha \sin \beta \xi) / \Delta \\
 N_{w4} &= (\mu_7 \cosh \alpha \xi + \mu_6 k_\beta \sinh \alpha \xi - \mu_7 \cos \beta \xi - \mu_6 k_\alpha \sin \beta \xi) L / \Delta \\
 N_{\theta1} &= (\mu_2 \cosh \alpha \xi + \mu_1 \sinh \alpha \xi - \mu_2 \cos \beta \xi - \mu_3 \sin \beta \xi) k_\alpha k_\beta / (\Delta L) \\
 N_{\theta2} &= (\mu_3 k_\alpha \cosh \alpha \xi - \mu_4 k_\alpha \sinh \alpha \xi + \mu_1 k_\beta \cos \beta \xi - \mu_4 k_\beta \sin \beta \xi) / \Delta \\
 N_{\theta3} &= (-\mu_5 \cosh \alpha \xi + \mu_6 k_\alpha \sinh \alpha \xi + \mu_5 \cos \beta \xi + \mu_6 k_\beta \sin \beta \xi) k_\alpha k_\beta / (\Delta L) \\
 N_{\theta4} &= (\mu_6 k_\alpha k_\beta \cosh \alpha \xi + \mu_7 k_\alpha \sinh \alpha \xi - \mu_6 k_\alpha k_\beta \cos \beta \xi + \mu_7 k_\beta \sin \beta \xi) / \Delta
 \end{aligned} \tag{25}$$

where

$$\begin{aligned}
 \mu_1 &= k_\alpha (1 - C_{h\alpha} C_\beta) - k_\beta S_{h\alpha} S_\beta \\
 \mu_2 &= k_\alpha S_{h\alpha} C_\beta + k_\beta C_{h\alpha} S_\beta \\
 \mu_3 &= k_\alpha S_{h\alpha} S_\beta + k_\beta (1 - C_{h\alpha} C_\beta) \\
 \mu_4 &= k_\alpha C_{h\alpha} S_\beta - k_\beta S_{h\alpha} C_\beta \\
 \mu_5 &= k_\alpha S_{h\alpha} + k_\beta S_\beta \\
 \mu_6 &= C_{h\alpha} - C_\beta \\
 \mu_7 &= k_\alpha S_\beta - k_\beta S_{h\alpha} \\
 \Delta &= (k_\alpha^2 - k_\beta^2) S_{h\alpha} S_\beta + 2k_\alpha k_\beta (1 - C_{h\alpha} C_\beta)
 \end{aligned} \tag{26}$$

3. Frequency-dependent mass matrix

The shape functions derived in Section 2 can now be used to derive the frequency-dependent mass matrix $\mathbf{m}$ of an axially loaded Timoshenko-Ehrenfest beam as follows [2, 36]

$$\mathbf{m} = \int_V \rho \mathbf{N}^T \mathbf{N} dv \tag{27}$$

where T denotes a transpose.

When considering a uniform axially loaded Timoshenko-Ehrenfest beam as in the present case, the volume integral of Eq. (27) reduces to a single integral with respect to the length parameter x to arrive at the following expression.

$$\mathbf{m}(\omega) = \int_0^L \begin{bmatrix} N_{w1} & N_{\theta1} \\ N_{w2} & N_{\theta2} \\ N_{w3} & N_{\theta3} \\ N_{w4} & N_{\theta4} \end{bmatrix} \begin{bmatrix} \rho A & 0 \\ 0 & \rho I \end{bmatrix} \begin{bmatrix} N_{w1} & N_{w2} & N_{w3} & N_{w4} \\ N_{\theta1} & N_{\theta2} & N_{\theta3} & N_{\theta4} \end{bmatrix} dx \tag{28}$$

or,

$$\mathbf{m}(\omega) = L \int_0^1 \begin{bmatrix} N_{w1} & N_{\theta1} \\ N_{w2} & N_{\theta2} \\ N_{w3} & N_{\theta3} \\ N_{w4} & N_{\theta4} \end{bmatrix} \begin{bmatrix} \rho A & 0 \\ 0 & \rho I \end{bmatrix} \begin{bmatrix} N_{w1} & N_{w2} & N_{w3} & N_{w4} \\ N_{\theta1} & N_{\theta2} & N_{\theta3} & N_{\theta4} \end{bmatrix} d\xi =$$

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{12} & m_{22} & m_{23} & m_{24} \\ m_{13} & m_{23} & m_{33} & m_{34} \\ m_{14} & m_{24} & m_{34} & m_{44} \end{bmatrix} \quad (29)$$

Substituting Eqs. (24) and (25) into Eq. (29) and carrying out the matrix multiplications and integrations algebraically with the help of symbolic computation [12, 13] yielded explicit expressions for the elements of the frequency-dependent mass matrix $\mathbf{m}(\omega)$ as follows. (Generation of these explicit expressions would have been difficult, if not impossible, without the application of symbolic computation.)

$$m_{11}(\omega) = m_{33}(\omega) = \rho AL \frac{\Phi_1}{2\alpha\alpha_1\beta\Delta^2} \quad (30)$$

$$m_{12}(\omega) = -m_{34}(\omega) = \rho AL \frac{\Phi_2 L}{2\alpha\alpha_1\beta\Delta^2} \quad (31)$$

$$m_{13}(\omega) = \rho AL \frac{\Phi_3}{2\alpha\alpha_1\beta\Delta^2} \quad (32)$$

$$m_{14}(\omega) = -m_{23}(\omega) = \rho AL \frac{\Phi_4 L}{2\alpha\alpha_1\beta\Delta^2} \quad (33)$$

$$m_{22}(\omega) = m_{44}(\omega) = \rho AL \frac{\Phi_5 L^2}{2\alpha\alpha_1\beta\Delta^2} \quad (34)$$

$$m_{24}(\omega) = \rho AL \frac{\Phi_6 L^2}{2\alpha\alpha_1\beta\Delta^2} \quad (35)$$

where

$$\Phi_1 = 2\alpha\alpha_1\beta_2\mu_2\mu_3k_\alpha^2C_\beta^2 + 4\alpha\beta\mu_2k_\alpha k_\beta(\zeta_1 - \zeta_2r^2k_\alpha k_\beta)C_\beta C_{h\alpha} - \alpha\alpha_1\beta_2\delta_1k_\alpha^2C_\beta S_\beta + 4\alpha\beta k_\alpha k_\beta(\lambda_1 - \kappa_4r^2k_\alpha k_\beta)C_\beta S_{h\alpha} + 2\alpha_1\beta\beta_1\mu_1\mu_2k_\beta^2C_{h\alpha}^2 - 4\alpha\beta k_\alpha k_\beta(\lambda_1r^2k_\alpha k_\beta + \kappa_4)C_{h\alpha}S_\beta + \alpha_1\beta\beta_1\gamma_1k_\beta^2C_{h\alpha}S_{h\alpha} - 4\alpha\beta\mu_2k_\alpha k_\beta(\zeta_2 + \zeta_1r^2k_\alpha k_\beta)S_\beta S_{h\alpha} + \Gamma_1 \quad (36)$$

$$\Phi_2 = -\alpha\alpha_1\beta_2\varepsilon_3k_\alpha C_\beta^2 + 2\alpha\beta\{k_\beta(\eta_3r^2k_\alpha^2 + \kappa_3) + k_\alpha(\eta_1r^2k_\beta^2 + \kappa_2)\}C_\beta C_{h\alpha} + \alpha\alpha_1\beta_2\psi_4k_\alpha C_\beta S_\beta - 2\alpha\beta\gamma_1(\zeta_3r^2k_\alpha k_\beta + \zeta_4)C_\beta S_{h\alpha} + \alpha_1\beta\beta_1\varepsilon_3k_\beta C_{h\alpha}^2 + 2\alpha\beta\gamma_1(\zeta_4r^2k_\alpha k_\beta - \zeta_3)C_{h\alpha}S_\beta - \alpha_1\beta\beta_1\varepsilon_2k_\beta C_{h\alpha}S_{h\alpha} - 2\alpha\beta\{r^2k_\alpha k_\beta(\kappa_2k_\alpha + \kappa_3k_\beta) - (\eta_3k_\alpha + \eta_1k_\beta)\}S_\beta S_{h\alpha} + \Gamma_2 \quad (37)$$

$$\Phi_3 = -\alpha\alpha_1\beta_2\nu_3k_\alpha^2C_\beta^2 - 2\alpha\beta\{\mu_2\mu_6k_\alpha k_\beta(\alpha_1\bar{p}^2r^2k_\alpha k_\beta - \alpha_4) - k_\alpha k_\beta(\beta_3\mu_1\mu_5 - \beta_4\mu_3\mu_5)\}C_\beta C_{h\alpha} - \alpha\alpha_1\beta_2\nu_4k_\alpha^2C_\beta S_\beta + 2\alpha\beta k_\alpha k_\beta(\beta_4\xi_1 + 2\beta_3\mu_2\mu_5)C_\beta S_{h\alpha} + \alpha_1\beta\beta_1\xi_2k_\beta^2C_{h\alpha}^2 - 2\alpha\beta k_\alpha k_\beta(\beta_3\xi_1 - 2\beta_4\mu_2\mu_5)C_{h\alpha}S_\beta + \alpha_1\beta\beta_1\phi_1k_\beta^2C_{h\alpha}S_{h\alpha} - 2\alpha\beta k_\alpha k_\beta\{\mu_2\mu_6(\alpha_4r^2k_\alpha k_\beta + \alpha_1\bar{p}^2) - \mu_5(\beta_3\mu_3 + \beta_4\mu_1)\}S_\beta S_{h\alpha} + \Gamma_3 \quad (38)$$

$$\Phi_4 = \alpha\alpha_1\beta_2\rho_4k_\alpha C_\beta^2 - 2\alpha\beta\{\mu_6k_\alpha k_\beta(\beta_3\mu_1 - \beta_4\mu_3) + \mu_2\mu_7(\alpha_1\bar{p}^2r^2k_\alpha k_\beta - \alpha_4)\}C_\beta C_{h\alpha} - \alpha\alpha_1\beta_2\xi_3k_\alpha C_\beta S_\beta + 2\alpha\beta(\beta_4\xi_4 - 2\beta_3\mu_2\mu_6k_\alpha k_\beta)C_\beta S_{h\alpha} + \alpha_1\beta\beta_1\phi_2k_\beta C_{h\alpha}^2 - 2\alpha\beta(2\beta_4\mu_2\mu_6k_\alpha k_\beta + \beta_3\xi_4)C_{h\alpha}S_\beta + \alpha_1\beta\beta_1\nu_2k_\beta C_{h\alpha}S_{h\alpha} - 2\alpha\beta(\beta_3\rho_4k_\beta + \beta_4\phi_2k_\alpha)S_\beta S_{h\alpha} + \Gamma_4 \quad (39)$$

$$\begin{aligned}\Phi_5 = & -2\alpha\alpha_1\beta_2\mu_1\mu_4C_\beta^2 - \alpha\alpha_1\beta_2\delta_2C_\beta S_\beta - 4\alpha\beta\mu_4(\zeta_2r^2k_\alpha k_\beta - \zeta_1)C_\beta C_{h\alpha} + 4\alpha\beta(\eta_4r^2k_\alpha k_\beta - \kappa_1)C_\beta S_{h\alpha} - \\ & 2\alpha_1\beta\beta_1\mu_3\mu_4C_{h\alpha}^2 + 4\alpha\beta(\kappa_1r^2k_\alpha k_\beta + \eta_4)C_{h\alpha}S_\beta + \alpha_1\beta\beta_1(\mu_3^2 + \mu_4^2)C_{h\alpha}S_{h\alpha} - 4\alpha\beta(\lambda_3r^2k_\alpha k_\beta + \psi_1)S_\beta S_{h\alpha} + \Gamma_5\end{aligned}\quad (40)$$

$$\begin{aligned}\Phi_6 = & \alpha\alpha_1\beta_2(\mu_4\mu_6k_\alpha + \mu_1\mu_7)C_\beta^2 + 2\alpha\beta\{\mu_4\mu_6(\alpha_1\bar{p}^2r^2k_\alpha k_\beta - \alpha_4) + \mu_7(\beta_3\mu_1 - \beta_4\mu_3)\}C_\beta C_{h\alpha} + \\ & \alpha\alpha_1\beta_2\phi_3C_\beta S_\beta - 2\alpha\beta(\beta_3\xi_1 - 2\beta_4\mu_4\mu_7)C_\beta S_{h\alpha} - \alpha_1\beta\beta_1\lambda_2C_{h\alpha}^2 - 2\alpha\beta(\beta_4\xi_1 + 2\beta_3\mu_4\mu_7)C_{h\alpha}S_\beta + \\ & \alpha_1\beta\beta_1\phi_4C_{h\alpha}S_{h\alpha} + 2\alpha\beta\{\mu_4\mu_6(\alpha_4r^2k_\alpha k_\beta + \alpha_1\bar{p}^2) + \mu_7(\beta_3\mu_3 + \beta_4\mu_1)\}S_\beta S_{h\alpha} + \Gamma_6\end{aligned}\quad (41)$$

with

$$\alpha_1 = \alpha^2 + \beta^2; \quad \alpha_2 = \alpha^2 - \beta^2; \quad \alpha_3 = -\alpha k_\alpha + \beta k_\beta; \quad \alpha_4 = \beta k_\alpha - \alpha k_\beta \quad (42)$$

$$\beta_1 = 1 + k_\alpha^2 r^2; \quad \beta_2 = 1 - k_\beta^2 r^2; \quad \beta_3 = \alpha k_\alpha k_\beta r^2 - \beta; \quad \beta_4 = \beta k_\alpha k_\beta r^2 + \alpha \quad (43)$$

$$\theta_1 = 1 - k_\alpha^2 r^2; \quad \theta_2 = 1 + k_\beta^2 r^2; \quad \theta_3 = k_\alpha^2 - k_\beta^2; \quad \theta_4 = \mu_3^2 + \mu_4^2 \quad (44)$$

$$\gamma_1 = \mu_1^2 + \mu_2^2; \quad \gamma_2 = \mu_2^2 + \mu_3^2; \quad \gamma_3 = \mu_1^2 - \mu_2^2; \quad \gamma_4 = \mu_1^2 - \mu_3^2 \quad (45)$$

$$\delta_1 = \mu_2^2 - \mu_3^2; \quad \delta_2 = \mu_1^2 - \mu_4^2; \quad \delta_3 = \mu_3^2 - \mu_4^2; \quad \delta_4 = \mu_1^2 + \mu_4^2 \quad (46)$$

$$\varepsilon_1 = \mu_1\mu_4 + \mu_2\mu_3; \quad \varepsilon_2 = \mu_1\mu_4 - \mu_2\mu_3; \quad \varepsilon_3 = \mu_1\mu_3 - \mu_2\mu_4; \quad \varepsilon_4 = \mu_1\mu_2 - \mu_3\mu_4 \quad (47)$$

$$\zeta_1 = \alpha\mu_3 + \beta\mu_1; \quad \zeta_2 = \alpha\mu_1 - \beta\mu_3; \quad \zeta_3 = \alpha\mu_2 + \beta\mu_4; \quad \zeta_4 = \alpha\mu_4 - \beta\mu_2 \quad (48)$$

$$\eta_1 = \alpha\mu_1^2 + \beta\mu_2\mu_4; \quad \eta_2 = \alpha\mu_2^2 - \beta\mu_2\mu_3; \quad \eta_3 = \beta\mu_3^2 + \alpha\mu_2\mu_4; \quad \eta_4 = -\beta\mu_4^2 + \alpha\mu_1\mu_3 \quad (49)$$

$$\kappa_1 = \alpha\mu_4^2 + \beta\mu_1\mu_3; \quad \kappa_2 = \alpha\mu_3^2 - \beta\mu_2\mu_4; \quad \kappa_3 = -\beta\mu_1^2 + \alpha\mu_2\mu_4; \quad \kappa_4 = \alpha\mu_2^2 - \beta\mu_1\mu_3 \quad (50)$$

$$\lambda_1 = \alpha\mu_1\mu_3 + \beta\mu_2^2; \quad \lambda_2 = k_\beta\mu_4\mu_6 - \mu_3\mu_7; \quad \lambda_3 = \alpha\mu_3\mu_4 + \beta\mu_1\mu_4; \quad \lambda_4 = \alpha\mu_1\mu_7 - \beta\mu_3\mu_7 \quad (51)$$

$$\psi_1 = \alpha\mu_1\mu_4 - \beta\mu_3\mu_4; \quad \psi_2 = \alpha\mu_3\mu_5 + \beta\mu_1\mu_5; \quad \psi_3 = \alpha\mu_3\mu_6 + \beta\mu_1\mu_6; \quad \psi_4 = \mu_1\mu_2 + \mu_3\mu_4 \quad (52)$$

$$\nu_1 = k_\alpha\mu_3 - k_\beta\mu_1; \quad \nu_2 = k_\beta\mu_2\mu_6 + \mu_1\mu_7; \quad \nu_3 = k_\beta\mu_2\mu_6 + \mu_3\mu_5; \quad \nu_4 = k_\beta\mu_3\mu_6 - \mu_2\mu_5 \quad (53)$$

$$\xi_1 = k_\alpha\mu_3\mu_6 - k_\beta\mu_1\mu_6; \quad \xi_2 = k_\alpha\mu_2\mu_6 - \mu_1\mu_5; \quad \xi_3 = k_\alpha\mu_2\mu_6 + \mu_3\mu_7; \quad \xi_4 = k_\alpha\mu_3\mu_7 - k_\beta\mu_1\mu_7 \quad (54)$$

$$\rho_1 = \alpha k_\alpha \mu_3^2 - \beta k_\beta \mu_1^2; \quad \rho_2 = \beta k_\alpha \mu_3^2 + \alpha k_\beta \mu_1^2; \quad \rho_3 = \beta \mu_2^2 r^2 k_\beta^2 - \mu_2 \mu_3; \quad \rho_4 = k_\alpha \mu_3 \mu_6 - \mu_2 \mu_7 \quad (55)$$

$$\phi_1 = k_\alpha \mu_1 \mu_6 - \mu_2 \mu_5; \quad \phi_2 = k_\beta \mu_1 \mu_6 + \mu_2 \mu_7; \quad \phi_3 = k_\alpha \mu_1 \mu_6 - \mu_4 \mu_7; \quad \phi_4 = k_\beta \mu_3 \mu_6 - \mu_4 \mu_7 \quad (56)$$

and

$$\begin{aligned}\Gamma_1 = & -\alpha\alpha_1\beta\gamma_4r^2k_\alpha^2k_\beta^2 + 2\alpha\alpha_1\rho_3k_\alpha^2 + 2\alpha_2\mu_2\zeta_1r^2k_\alpha^2k_\beta^2 + \alpha\alpha_1\beta\gamma_2k_\alpha^2 - 4\alpha\beta\mu_2\zeta_1k_\alpha k_\beta + \alpha\alpha_1\beta\gamma_3k_\beta^2 - \\ & 2\alpha_1\beta\mu_1\mu_2k_\beta^2\end{aligned}\quad (57)$$

$$\begin{aligned}\Gamma_2 = & -\alpha\alpha_1\beta(\varepsilon_1\theta_1k_\beta + \varepsilon_4\theta_2k_\alpha) + \alpha_1\mu_1\mu_3(\alpha\beta_2k_\alpha - \beta\beta_1k_\beta) - \alpha_2\mu_2\mu_4(\alpha\beta_2k_\alpha + \beta\beta_1k_\beta) - 2\alpha\beta(\rho_2r^2k_\alpha k_\beta + \\ & \rho_1)\end{aligned}\quad (58)$$

$$\Gamma_3 = \alpha\alpha_1\beta\mu_6k_\alpha k_\beta(\mu_1\theta_1k_\beta - \mu_3\theta_2k_\alpha) + \alpha_2\mu_2\mu_6k_\alpha k_\beta(\beta\beta_1k_\beta + \alpha\beta_2k_\alpha) - \alpha\mu_3\mu_5k_\alpha^2(\alpha_2r^2k_\beta^2 - \alpha_1) - \beta\mu_1\mu_5k_\beta^2(\alpha_2r^2k_\alpha^2 - \alpha_1) - \alpha\alpha_1\beta\mu_2\mu_5\theta_3 - 2\alpha\beta k_\alpha k_\beta(\alpha_1\mu_2\mu_5r^2k_\alpha k_\beta - \psi_2) \quad (59)$$

$$\Gamma_4 = \alpha\mu_3\mu_6k_\alpha^2(\alpha_2r^2k_\beta^2 - \alpha_1) + \beta\mu_1\mu_6k_\beta^2(\alpha_2r^2k_\alpha^2 - \alpha_1) + \alpha\alpha_1\beta(\mu_1\mu_7\theta_1k_\beta + \mu_2\mu_6\theta_3) + \alpha_2\mu_2\mu_7(\beta\beta_1k_\beta + \alpha\beta_2k_\alpha) - \alpha\alpha_1\beta\mu_3\mu_7\theta_2k_\alpha + 2\alpha\beta k_\alpha k_\beta(\alpha_1\mu_2\mu_6r^2k_\alpha k_\beta - \psi_3) \quad (60)$$

$$\Gamma_5 = \alpha\alpha_1\beta\delta_3r^2k_\alpha^2 + 2\beta\mu_3\mu_4(\alpha_1r^2k_\alpha^2 - \alpha_2) + \alpha\alpha_1\beta(\delta_4r^2k_\beta^2 + \gamma_4) - 2\alpha\mu_1\mu_4(\alpha_1r^2k_\beta^2 - \alpha_2) + 2\alpha\beta\mu_4(2\zeta_2r^2k_\alpha k_\beta + \alpha_1\mu_4) \quad (61)$$

$$\Gamma_6 = -\alpha\alpha_1\beta\mu_6(\mu_3\theta_1k_\beta + \mu_1\theta_2k_\alpha) - \alpha_2\mu_4\mu_6(\beta\beta_1k_\beta + \alpha\beta_2k_\alpha) + \alpha\alpha_1\beta\mu_4\mu_7\theta_3r^2 - \beta\mu_3\mu_7(\alpha_1r^2k_\alpha^2 - \alpha_2) + \alpha\mu_1\mu_7(\alpha_1r^2k_\beta^2 - \alpha_2) - 2\alpha\beta(\lambda_4r^2k_\alpha k_\beta + \alpha_1\mu_4\mu_7) \quad (62)$$

4. Frequency-dependent elastic stiffness matrix

The shape functions derived in Section 2 can likewise be used to derive the frequency-dependent elastic stiffness matrix of the axially loaded Timoshenko-Ehrenfest beam by using the following relationship [14, 37]

$$\mathbf{k}(\omega) = L \int_0^1 [(\mathbf{N}'_\theta)^T \quad (\mathbf{N}'_w - \mathbf{N}_\theta)^T] \begin{bmatrix} EI & 0 \\ 0 & kAG \end{bmatrix} \begin{bmatrix} \mathbf{N}'_\theta \\ \mathbf{N}'_w - \mathbf{N}_\theta \end{bmatrix} d\xi \quad (63)$$

Substituting $\mathbf{N}_w$ and $\mathbf{N}_\theta$ from Eq. (24) into Eq. (63) gives

$$\begin{aligned} \mathbf{k}(\omega) &= L \int_0^1 \begin{bmatrix} N'_{\theta_1} & N'_{w_1} - N_{\theta_1} \\ N'_{\theta_2} & N'_{w_2} - N_{\theta_2} \\ N'_{\theta_3} & N'_{w_3} - N_{\theta_3} \\ N'_{\theta_4} & N'_{w_4} - N_{\theta_4} \end{bmatrix} \begin{bmatrix} EI & 0 \\ 0 & kAG \end{bmatrix} \begin{bmatrix} N'_{\theta_1} & N'_{\theta_2} & N'_{\theta_3} & N'_{\theta_4} \\ N'_{w_1} - N_{\theta_1} & N'_{w_2} - N_{\theta_2} & N'_{w_3} - N_{\theta_3} & N'_{w_4} - N_{\theta_4} \end{bmatrix} d\xi \\ &= \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{12} & k_{22} & k_{23} & k_{24} \\ k_{13} & k_{23} & k_{33} & k_{34} \\ k_{14} & k_{24} & k_{34} & k_{44} \end{bmatrix} \end{aligned} \quad (64)$$

Performing all matrix operations and subsequent integrations using REDUCE [13], explicit algebraic expressions of the elements of the frequency-dependent stiffness matrix $\mathbf{k}(\omega)$ are generated, as follows:

$$k_{11}(\omega) = k_{33}(\omega) = \left(\frac{EI}{L^3}\right) \frac{\Psi_1}{2\alpha\alpha_1\beta s^2\Delta^2} \quad (65)$$

$$k_{12}(\omega) = -k_{34}(\omega) = \left(\frac{EI}{L^3}\right) \frac{\Psi_2L}{2\alpha\alpha_1\beta s^2\Delta^2} \quad (66)$$

$$k_{13}(\omega) = \left(\frac{EI}{L^3}\right) \frac{\Psi_3}{2\alpha\alpha_1\beta s^2\Delta^2} \quad (67)$$

$$k_{14}(\omega) = -k_{23}(\omega) = \left(\frac{EI}{L^3}\right) \frac{\Psi_4L}{2\alpha\alpha_1\beta s^2\Delta^2} \quad (68)$$

$$k_{22}(\omega) = k_{44}(\omega) = \left(\frac{EI}{L^3}\right) \frac{\Psi_5L^2}{2\alpha\alpha_1\beta s^2\Delta^2} \quad (69)$$

$$k_{24}(\omega) = \left(\frac{EI}{L^3}\right) \frac{\Psi_6L^2}{2\alpha\alpha_1\beta s^2\Delta^2} \quad (70)$$

where

$$\begin{aligned}\Psi_1 = & 2\alpha\alpha_1\bar{\alpha}_4\mu_2\mu_3k_\alpha^2C_\beta^2 - 4\alpha\beta\mu_2k_\alpha k_\beta(\alpha\beta\zeta_1s^2k_\alpha k_\beta + \zeta_2g_\alpha g_\beta)C_\beta C_{h\alpha} - \alpha\alpha_1\bar{\alpha}_4\delta_1k_\alpha^2C_\beta S_\beta - \\ & 4\alpha\beta k_\alpha k_\beta(\alpha\beta\lambda_1s^2k_\alpha k_\beta + \kappa_4g_\alpha g_\beta)C_\beta S_{h\alpha} + 2\alpha_1\bar{\alpha}_1\beta\mu_1\mu_2k_\beta^2C_{h\alpha}^2 + 4\alpha\beta k_\alpha k_\beta(\alpha\beta\kappa_4s^2k_\alpha k_\beta - \lambda_1g_\alpha g_\beta)C_{h\alpha}S_\beta + \\ & \alpha_1\bar{\alpha}_1\beta\gamma_1k_\beta^2C_{h\alpha}S_{h\alpha} + 4\alpha\beta\mu_2k_\alpha k_\beta(\alpha\beta\zeta_2s^2k_\alpha k_\beta - \zeta_1g_\alpha g_\beta)S_\beta S_{h\alpha} + \bar{\Gamma}_1\end{aligned}\quad (71)$$

$$\begin{aligned}\Psi_2 = & -\alpha\alpha_1\bar{\alpha}_4\varepsilon_3k_\alpha C_\beta^2 + 2\alpha\beta(-\alpha\beta\kappa_2s^2k_\alpha^2k_\beta - \alpha\beta\kappa_3s^2k_\alpha k_\beta^2 + \eta_3g_\alpha g_\beta k_\alpha + \eta_1g_\alpha g_\beta k_\beta)C_\beta C_{h\alpha} + \\ & \alpha\alpha_1\bar{\alpha}_4\psi_4k_\alpha C_\beta S_\beta + 2\alpha\beta\nu_1(\alpha\beta\zeta_4s^2k_\alpha k_\beta - \zeta_3g_\alpha g_\beta)C_\beta S_{h\alpha} + \alpha_1\bar{\alpha}_1\beta\varepsilon_3k_\beta C_{h\alpha}^2 + 2\alpha\beta\nu_1(\alpha\beta\zeta_3s^2k_\alpha k_\beta + \\ & \zeta_4g_\alpha g_\beta)C_{h\alpha}S_\beta - \alpha_1\bar{\alpha}_1\beta\varepsilon_2k_\beta C_{h\alpha}S_{h\alpha} - 2\alpha\beta\{\alpha\beta s^2k_\alpha k_\beta(\eta_3k_\alpha + \eta_1k_\beta) + g_\alpha g_\beta(\kappa_2k_\alpha + \kappa_3k_\beta)\}S_\beta S_{h\alpha} + \bar{\Gamma}_2\end{aligned}\quad (72)$$

$$\begin{aligned}\Psi_3 = & -\alpha\alpha_1\bar{\alpha}_4\nu_3k_\alpha^2C_\beta^2 - 2\alpha\beta k_\alpha k_\beta\{\alpha\beta s^2k_\alpha k_\beta(\alpha_4\mu_2\mu_6 - \mu_5\zeta_1) + g_\alpha g_\beta(\alpha_1\bar{p}^2\mu_2\mu_6 - \zeta_2\mu_5)\}C_\beta C_{h\alpha} - \\ & \alpha\alpha_1\bar{\alpha}_4\nu_4k_\alpha^2C_\beta S_\beta - 2\alpha\beta k_\alpha k_\beta\{\beta\mu_6\nu_1(\alpha^2s^2k_\alpha k_\beta - g_\alpha g_\beta) - 2\alpha\bar{\beta}_3\mu_2\mu_5\}C_\beta S_{h\alpha} + \alpha_1\bar{\alpha}_1\beta\xi_2k_\beta^2C_{h\alpha}^2 - \\ & 2\alpha\beta k_\alpha k_\beta\{\alpha\mu_6\nu_1(\beta^2s^2k_\alpha k_\beta + g_\alpha g_\beta) + 2\beta\bar{\beta}_2\mu_2\mu_5\}C_{h\alpha}S_\beta + \alpha_1\bar{\alpha}_1\beta\phi_1k_\beta^2C_{h\alpha}S_{h\alpha} + \\ & 2\alpha\beta k_\alpha k_\beta\{\alpha\beta s^2k_\alpha k_\beta(\alpha_1\bar{p}^2\mu_2\mu_6 - \mu_5\zeta_2) - g_\alpha g_\beta(\alpha_4\mu_2\mu_6 - \mu_5\zeta_1)\}S_\beta S_{h\alpha} + \bar{\Gamma}_3\end{aligned}\quad (73)$$

$$\begin{aligned}\Psi_4 = & \alpha\alpha_1\bar{\alpha}_4\rho_4k_\alpha C_\beta^2 - 2\alpha\beta\{\mu_6k_\alpha k_\beta(\beta\bar{\beta}_2\mu_3 + \alpha\bar{\beta}_3\mu_1) + \alpha\alpha_4\beta\mu_2\mu_7s^2k_\alpha k_\beta + \alpha_1\bar{p}^2\mu_2\mu_7g_\alpha g_\beta\}C_\beta C_{h\alpha} - \\ & \alpha\alpha_1\bar{\alpha}_4\xi_3k_\alpha C_\beta S_\beta - 2\alpha\beta\{2\alpha\bar{\beta}_3\mu_2\mu_6k_\alpha k_\beta + \beta\bar{\beta}_2\mu_7(\mu_3k_\alpha - \mu_1k_\beta)\}C_\beta S_{h\alpha} + \alpha_1\bar{\alpha}_1\beta\phi_2k_\beta C_{h\alpha}^2 + \\ & 2\alpha\beta\{2\beta\bar{\beta}_2\mu_2\mu_6k_\alpha k_\beta - \alpha\bar{\beta}_3\mu_7(\mu_3k_\alpha - \mu_1k_\beta)\}C_{h\alpha}S_\beta + \alpha_1\bar{\alpha}_1\beta\nu_2k_\beta C_{h\alpha}S_{h\alpha} + 2\alpha\beta\{\mu_6k_\alpha k_\beta(\beta\bar{\beta}_2\mu_1 - \alpha\bar{\beta}_3\mu_3) + \\ & \mu_2\mu_7(\alpha\alpha_1\bar{p}^2s^2k_\alpha k_\beta - \alpha_4g_\alpha g_\beta)\}S_\beta S_{h\alpha} + \bar{\Gamma}_4\end{aligned}\quad (74)$$

$$\begin{aligned}\Psi_5 = & -2\alpha\alpha_1\bar{\alpha}_4\mu_1\mu_4C_\beta^2 - 4\alpha\beta\mu_4(\beta\bar{\beta}_2\mu_3 + \alpha\bar{\beta}_3\mu_1)C_\beta C_{h\alpha} - \alpha\alpha_1\bar{\alpha}_4\delta_2C_\beta S_\beta + 4\alpha\beta(\beta\bar{\beta}_2\mu_4^2 + \alpha\bar{\beta}_3\mu_1\mu_3)C_\beta S_{h\alpha} - \\ & 2\alpha_1\bar{\alpha}_1\beta\mu_3\mu_4C_{h\alpha}^2 + 4\alpha\beta(\alpha\bar{\beta}_3\mu_4^2 - \beta\bar{\beta}_2\mu_1\mu_3)C_{h\alpha}S_\beta + \alpha_1\bar{\alpha}_1\beta\theta_4C_{h\alpha}S_{h\alpha} + 4\alpha\beta\mu_4(\beta\bar{\beta}_2\mu_1 - \alpha\bar{\beta}_3\mu_3)S_\beta S_{h\alpha} + \bar{\Gamma}_5\end{aligned}\quad (75)$$

$$\begin{aligned}\Psi_6 = & \alpha\alpha_1\bar{\alpha}_4(\mu_4\mu_6k_\alpha + \mu_1\mu_7)C_\beta^2 + 2\alpha\beta\{\mu_4\mu_6(\alpha\bar{\beta}_3k_\alpha - \beta\bar{\beta}_2k_\beta) + \mu_7(\alpha\beta\zeta_1s^2k_\alpha k_\beta + \zeta_2g_\alpha g_\beta)\}C_\beta C_{h\alpha} + \\ & \alpha\alpha_1\bar{\alpha}_4\phi_3C_\beta S_\beta - 2\alpha\beta\{\alpha\bar{\beta}_3\mu_6(\mu_3k_\alpha - \mu_1k_\beta) + 2\beta\bar{\beta}_2\mu_4\mu_7\}C_\beta S_{h\alpha} - \alpha_1\bar{\alpha}_1\beta\lambda_2C_{h\alpha}^2 + 2\alpha\beta\{\beta\bar{\beta}_2\mu_6(\mu_3k_\alpha - \\ & \mu_1k_\beta) - 2\alpha\bar{\beta}_3\mu_4\mu_7\}C_{h\alpha}S_\beta + \alpha_1\bar{\alpha}_1\beta\phi_4C_{h\alpha}S_{h\alpha} - 2\alpha\beta\{\mu_4\mu_6(\beta\bar{\beta}_2k_\alpha + \alpha\bar{\beta}_3k_\beta) + \mu_7(\beta\bar{\beta}_2\mu_1 - \alpha\bar{\beta}_3\mu_3)\}S_\beta S_{h\alpha} + \\ & \bar{\Gamma}_6\end{aligned}\quad (76)$$

with

$$g_\alpha = \alpha - k_\alpha; \quad g_\beta = \beta - k_\beta \quad (77)$$

$$\bar{\alpha}_1 = \alpha^2s^2k_\alpha^2 + g_\alpha^2; \quad \bar{\alpha}_2 = \alpha^2s^2k_\alpha^2 - g_\alpha^2; \quad \bar{\alpha}_3 = \beta^2s^2k_\beta^2 + g_\beta^2; \quad \bar{\alpha}_4 = \beta^2s^2k_\beta^2 - g_\beta^2 \quad (78)$$

$$\bar{\beta}_1 = \alpha k_\alpha g_\beta^2 + \beta k_\beta g_\alpha^2; \quad \bar{\beta}_2 = \alpha^2s^2k_\alpha k_\beta - g_\alpha g_\beta; \quad \bar{\beta}_3 = \beta^2s^2k_\alpha k_\beta + g_\alpha g_\beta; \quad \bar{\beta}_4 = k_\alpha^2g_\beta^2 + k_\beta^2g_\alpha^2 \quad (79)$$

and

$$\bar{\Gamma}_1 = \alpha\alpha_1\bar{\alpha}_2\beta\mu_1^2k_\beta^2 - \alpha^3\beta s^2k_\alpha^2k_\beta^2(\alpha^2\mu_2^2 - \beta^2\mu_3^2) - 2\alpha\alpha_2\beta s^2k_\alpha^2k_\beta^2(\alpha\mu_1\mu_2 - \beta\mu_2\mu_3) + \alpha\bar{\alpha}_3\beta^3\gamma_2k_\alpha^2 + \alpha^3\beta\gamma_2g_\beta^2k_\alpha^2 + 2\alpha\alpha_1\mu_2\mu_3g_\beta^2k_\alpha^2 + 4\alpha\beta g_\alpha g_\beta k_\alpha k_\beta(\alpha\mu_1\mu_2 - \beta\mu_2\mu_3) + \alpha\alpha_1\beta\mu_2^2g_\alpha^2k_\beta^2 - 2\alpha_1\beta\mu_1\mu_2g_\alpha^2k_\beta^2 \quad (80)$$

$$\bar{\Gamma}_2 = -\alpha\alpha_1\bar{\alpha}_2\beta\varepsilon_1k_\beta + \alpha\alpha_1\alpha_3\beta\mu_1\mu_3s^2k_\alpha k_\beta + \alpha\alpha_1\bar{p}^2\alpha_2\beta\mu_2\mu_4s^2k_\alpha k_\beta + 2\alpha^2\beta^2\rho_1s^2k_\alpha k_\beta - \alpha\alpha_1\bar{\alpha}_3\beta\varepsilon_4k_\alpha - \alpha_1\bar{\beta}_1\varepsilon_3 - 2\alpha\beta\eta_3g_\alpha g_\beta k_\alpha - 2\alpha\beta\eta_1g_\alpha g_\beta k_\beta \quad (81)$$

$$\bar{\Gamma}_3 = \alpha\alpha_1\bar{\alpha}_2\beta\mu_1\mu_6k_\alpha k_\beta^2 - \alpha\alpha_1\bar{p}^2\alpha_2\beta\mu_2\mu_6s^2k_\alpha^2k_\beta^2 - \alpha\alpha_1\beta^3\mu_3\mu_6s^2k_\alpha^2k_\beta^3 + \alpha\alpha_1\alpha_2\beta\mu_2\mu_5s^2k_\alpha^2k_\beta^2 + \alpha^2\alpha_2\beta\mu_1\mu_5s^2k_\alpha^2k_\beta^2 - \alpha\alpha_2\beta^2\mu_3\mu_5s^2k_\alpha^2k_\beta^2 - \alpha\alpha_1\mu_6g_\beta^2k_\alpha^2k_\beta(\mu_2 + \beta\mu_3) + 2\alpha\beta g_\alpha g_\beta k_\alpha k_\beta(\alpha_1\bar{p}^2\mu_2\mu_6 - \zeta_2\mu_5) - \alpha\alpha_1\beta\bar{\beta}_4\mu_2\mu_5 - \alpha\alpha_1\mu_3\mu_5g_\beta^2k_\alpha^2 - \alpha_1\beta\xi_2g_\alpha^2k_\beta^2 \quad (82)$$

$$\bar{\Gamma}_4 = -\alpha\alpha_1\alpha_2\beta\mu_2\mu_6s^2k_\alpha^2k_\beta^2 - \alpha\alpha_2\beta\mu_6\zeta_2s^2k_\alpha^2k_\beta^2 + \alpha\alpha_1\bar{\alpha}_2\beta\mu_1\mu_7k_\beta - \alpha\alpha_1\alpha_2\beta\mu_2\mu_7\bar{p}^2s^2k_\alpha k_\beta + \alpha\alpha_1\beta\bar{\beta}_4\mu_2\mu_6 + \alpha\alpha_1\mu_3\mu_6g_\beta^2k_\alpha^2 - \alpha\alpha_1\beta^3\mu_3\mu_7s^2k_\alpha k_\beta^2 + 2\alpha\beta g_\alpha g_\beta(\mu_6\zeta_2k_\alpha k_\beta + \alpha_1\bar{p}^2\mu_2\mu_7) - \alpha\alpha_1\mu_7g_\beta^2k_\alpha(\mu_2 + \beta\mu_3) - \alpha_1\beta\phi_2g_\alpha^2k_\beta \quad (83)$$

$$\bar{\Gamma}_5 = -\alpha^3\alpha_1\beta\delta_3s^2k_\alpha^2 + 2\alpha_1\bar{\alpha}_1\beta\mu_3\mu_4 + 4\alpha\beta\mu_4(\alpha\beta\zeta_1s^2k_\alpha k_\beta + \zeta_2g_\alpha g_\beta) + \alpha\alpha_1\beta^3\delta_4s^2k_\beta^2 + 2\alpha\alpha_1\bar{\alpha}_4\mu_1\mu_4 + \alpha\alpha_1\beta\delta_4g_\beta^2 + \alpha\alpha_1\beta\delta_3g_\alpha^2 \quad (84)$$

$$\bar{\Gamma}_6 = -\alpha\alpha_1\bar{\alpha}_2\beta\mu_3\mu_6k_\beta + \alpha\alpha_1\alpha_2\beta\mu_4\mu_6\bar{p}^2s^2k_\alpha k_\beta - \alpha\alpha_1\bar{\alpha}_2\beta\mu_4\mu_7 - \alpha_1\bar{\alpha}_1\beta\mu_3\mu_7 - \alpha\alpha_1\bar{\alpha}_3\beta\mu_1\mu_6k_\alpha - 2\alpha\beta\mu_7(\alpha\beta\zeta_1s^2k_\alpha k_\beta + \zeta_2g_\alpha g_\beta) + \alpha_1\bar{\beta}_1\mu_4\mu_6 - 2\alpha\alpha_1\beta\mu_4\mu_6\bar{p}^2g_\alpha g_\beta - \alpha\alpha_1\bar{\alpha}_4\mu_1\mu_7 - \alpha\alpha_1\bar{\alpha}_3\beta\mu_4\mu_7 \quad (85)$$

and all the rest of the terms have already been defined before.

5. Frequency-dependent geometric stiffness matrix

The geometric stiffness matrix of a beam of length L under the action of a compressive load P in terms of the shape functions in the usual notation, is given by [15]

$$\mathbf{k}_G = -\frac{P}{L} \int_0^1 \begin{bmatrix} N'_{w1} \\ N'_{w2} \\ N'_{w3} \\ N'_{w4} \end{bmatrix} \begin{bmatrix} N'_{w1} & N'_{w2} & N'_{w3} & N'_{w4} \end{bmatrix} d\xi = \begin{bmatrix} k_{11}^G & k_{12}^G & k_{13}^G & k_{14}^G \\ k_{12}^G & k_{22}^G & k_{23}^G & k_{24}^G \\ k_{13}^G & k_{23}^G & k_{33}^G & k_{34}^G \\ k_{14}^G & k_{24}^G & k_{34}^G & k_{44}^G \end{bmatrix} \quad (86)$$

Note that the negative sign in front of Eq. (86) arises because compressive force has been taken to be positive.

Substituting the expressions for the shape functions from Eqs. (24)-(25) into Eq. (86) and performing the differentiation and integration algebraically through the application of symbolic computation [12, 13], yielded the terms of the geometric stiffness matrix explicitly and concisely, as follows.

$$\begin{aligned}
k_{11}^G &= k_{33}^G = -\frac{P}{L} \left(\frac{\bar{\Theta}_1}{2\alpha_1\Delta} \right) \\
k_{12}^G(\omega) &= -k_{34}^G(\omega) = -\frac{P}{L} \left(\frac{\bar{\Theta}_2 L}{2\alpha_1\Delta} \right) \\
k_{13}^G(\omega) &= -\left(\frac{P}{L} \right) \left(\frac{\bar{\Theta}_3}{2\alpha_1\Delta} \right) \\
k_{14}^G(\omega) &= -k_{23}^G(\omega) = -\left(\frac{P}{L} \right) \left(\frac{\bar{\Theta}_4 L}{2\alpha_1\Delta} \right) \\
k_{22}^G(\omega) &= k_{44}^G(\omega) = -\left(\frac{P}{L} \right) \left(\frac{\bar{\Theta}_5 L^2}{2\alpha_1\Delta} \right) \\
k_{24}^G(\omega) &= -\left(\frac{P}{L} \right) \left(\frac{\bar{\Theta}_6 L^2}{2\alpha_1\Delta} \right)
\end{aligned} \tag{87}$$

where

$$\begin{aligned}
\bar{\Theta}_1 &= -2\alpha_1\beta\mu_2\mu_3k_\alpha^2C_\beta^2 - 4\alpha\beta\mu_2\zeta_2k_\alpha k_\beta C_\beta C_{h\alpha} + \alpha_1\beta\delta_1k_\alpha^2C_\beta S_\beta - 4\alpha\beta\kappa_4k_\alpha k_\beta C_\beta S_{h\alpha} + 2\alpha\alpha_1\mu_1\mu_2k_\beta^2C_{h\alpha}^2 - \\
&4\alpha\beta\lambda_1k_\alpha k_\beta C_{h\alpha}S_\beta + \alpha\alpha_1\gamma_1k_\beta^2C_{h\alpha}S_{h\alpha} - 4\alpha\beta\mu_2\zeta_1k_\alpha k_\beta S_{h\alpha}S_\beta + \Sigma_1
\end{aligned} \tag{88}$$

$$\begin{aligned}
\bar{\Theta}_2 &= \alpha_1\beta\varepsilon_3k_\alpha C_\beta^2 + 2\alpha\beta(\alpha_1\mu_2\mu_4\bar{p}^2 + \rho_2)C_\beta C_{h\alpha} - \alpha_1\beta\psi_4k_\alpha C_\beta S_\beta - 2\alpha\beta\nu_1\zeta_3C_\beta S_{h\alpha} + \alpha\alpha_1\varepsilon_3k_\beta C_{h\alpha}^2 + \\
&2\alpha\beta\nu_1\zeta_4C_{h\alpha}S_\beta - \alpha\alpha_1\varepsilon_2k_\beta C_{h\alpha}S_{h\alpha} - 2\alpha\beta(\kappa_2k_\alpha + \kappa_3k_\beta)S_{h\alpha}S_\beta + \Sigma_2
\end{aligned} \tag{89}$$

$$\begin{aligned}
\bar{\Theta}_3 &= \alpha_1\beta\nu_3k_\alpha^2C_\beta^2 - 2\alpha\beta k_\alpha k_\beta(\alpha_1\mu_2\mu_6\bar{p}^2 - \mu_5\zeta_2)C_\beta C_{h\alpha} + \alpha_1\beta\nu_4k_\alpha^2C_\beta S_\beta + 2\alpha\beta k_\alpha k_\beta(\beta\mu_6\nu_1 + 2\alpha\mu_2\mu_5)C_\beta S_{h\alpha} + \\
&\alpha\alpha_1\xi_2k_\beta^2C_{h\alpha}^2 - 2\alpha\beta k_\alpha k_\beta(\alpha\nu_1\mu_6 - 2\beta\mu_2\mu_5)C_{h\alpha}S_\beta + \alpha\alpha_1\phi_1k_\beta^2C_{h\alpha}S_{h\alpha} - 2\alpha\beta(\alpha_4\mu_2\mu_6 - \zeta_1\mu_5) + \Sigma_3
\end{aligned} \tag{90}$$

$$\begin{aligned}
\bar{\Theta}_4 &= -\alpha_1\beta\rho_4k_\alpha C_\beta^2 - 2\alpha\beta(\zeta_2\mu_6k_\alpha k_\beta + \alpha_1\mu_2\mu_7\bar{p}^2)C_\beta C_{h\alpha} + \alpha_1\beta\xi_3k_\alpha C_\beta S_\beta - 2\alpha\beta(2\alpha\mu_2\mu_6k_\alpha k_\beta - \\
&\beta\nu_1\mu_7)C_\beta S_{h\alpha} + \alpha\alpha_1\phi_2k_\beta C_{h\alpha}^2 - 2\alpha\beta(2\beta\mu_2\mu_6k_\alpha k_\beta + \alpha\mu_7\nu_1)C_{h\alpha}S_\beta + \alpha\alpha_1\nu_2k_\beta C_{h\alpha}S_{h\alpha} - 2\alpha\beta(\zeta_1\mu_6k_\alpha k_\beta + \\
&\alpha_4\mu_2\mu_7)S_{h\alpha}S_\beta + \Sigma_4
\end{aligned} \tag{91}$$

$$\begin{aligned}
\bar{\Theta}_5 &= 2\alpha_1\beta\mu_1\mu_4C_\beta^2 - 4\alpha\beta\zeta_2\mu_4C_\beta C_{h\alpha} + \alpha_1\beta\delta_2C_\beta S_\beta + 4\alpha\beta\eta_4C_\beta S_{h\alpha} - 2\alpha\alpha_1\mu_3\mu_4C_{h\alpha}^2 + 4\alpha\beta\kappa_1C_{h\alpha}S_\beta + \\
&\alpha\alpha_1\theta_4C_{h\alpha}S_{h\alpha} - 4\alpha\beta\zeta_1\mu_4S_{h\alpha}S_\beta + \Sigma_5
\end{aligned} \tag{92}$$

$$\begin{aligned}
\bar{\Theta}_6 &= -\alpha_1\beta(\mu_4\mu_6k_\alpha + \mu_1\mu_7)C_\beta^2 + 2\alpha\beta(\alpha_1\mu_4\mu_6\bar{p}^2 + \zeta_2\mu_7)C_\beta C_{h\alpha} - \alpha_1\beta\phi_3C_\beta S_\beta - 2\alpha\beta(\alpha\mu_6\nu_1 - \\
&2\beta\mu_4\mu_7)C_\beta S_{h\alpha} - \alpha\alpha_1(\mu_4\mu_6k_\beta - \mu_3\mu_7)C_{h\alpha}^2 - 2\alpha\beta(\beta\mu_6\nu_1 + 2\alpha\mu_4\mu_7)C_{h\alpha}S_\beta + \alpha\alpha_1\phi_4C_{h\alpha}S_{h\alpha} + 2\alpha\beta(\alpha_4\mu_4\mu_6 + \\
&\zeta_1\mu_7)S_{h\alpha}S_\beta + \Sigma_6
\end{aligned} \tag{93}$$

with

$$\Sigma_1 = \alpha_1\beta^2\gamma_2k_\alpha^2 + 2\alpha_1\beta\mu_2\mu_3k_\alpha^2 + 4\alpha\beta\zeta_2\mu_2k_\alpha k_\beta - \alpha^2\alpha_1\gamma_3k_\beta^2 - 2\alpha\alpha_1\mu_1\mu_2k_\beta^2 \tag{94}$$

$$\Sigma_2 = -\alpha_1\beta^2\varepsilon_4k_\alpha - \alpha_1\mu_1\mu_3(\alpha k_\beta + \beta k_\alpha) - \alpha_2\alpha_4\mu_2\mu_4 - 2\alpha\beta\rho_2 + \alpha^2\alpha_1\varepsilon_1k_\beta \tag{95}$$

$$\Sigma_3 = \alpha_2\alpha_4\mu_2\mu_6k_\alpha k_\beta - \alpha_1\beta^2k_\alpha^2(\mu_3\mu_6k_\beta + \mu_2\mu_5) + \alpha_1\mu_5(\alpha\mu_1k_\beta^2 - \beta\mu_3k_\alpha^2) - \alpha^2\alpha_1k_\beta^2(\mu_1\mu_6k_\alpha + \mu_2\mu_5) - \\
2\alpha\beta\zeta_2\mu_5k_\alpha k_\beta \tag{96}$$

$$\Sigma_4 = \alpha_1\mu_2\mu_6(\alpha^2k_\beta^2 + \beta^2k_\alpha^2) - \alpha_1\beta\mu_3k_\alpha(\beta\mu_7 - \mu_6k_\alpha) + 2\alpha\beta\zeta_2\mu_6k_\alpha k_\beta + \alpha_2\alpha_4\mu_2\mu_7 - \alpha\alpha_1\mu_1k_\beta(\alpha\mu_7 + \\
\mu_6k_\beta) \tag{97}$$

$$\Sigma_5 = \alpha^2\alpha_1\mu_3^2 - \alpha_1\alpha_2\mu_4^2 + 2\alpha_2\zeta_1\mu_4 + \alpha_1\beta^2\mu_1^2 \tag{98}$$

$$\Sigma_6 = \alpha_1\mu_6(\alpha^2\mu_3k_\beta - \beta^2\mu_1k_\alpha) - \alpha_2\alpha_4\mu_4\mu_6 - \alpha_1\alpha_2\mu_4\mu_7 - \alpha_2\zeta_1\mu_7 \tag{99}$$

6. Application of the frequency-dependent mass, elastic and geometric stiffness matrices

The above theory can now be applied either to an individual axially loaded Timoshenko-Ehrenfest beam or to a framework containing such beams to investigate the free vibration characteristics. This can be achieved by utilizing the Wittrick-Williams algorithm [27] which is well-suited and customarily used as a solution technique in exact free vibration analysis. However, when analysing a framework, the frequency-dependent mass and stiffness matrices in axial motion will also be needed and these can be found in [2, 11, 14, 15]. Therefore, by using standard coordinate transformations, the frequency dependent mass, elastic and geometric stiffness matrices of each member in a frame can be assembled, prior to the application of the Wittrick-Williams algorithm [27], and the subsequent determination of the natural frequencies.

It should be noted that in the limiting case when the frequency ω tends to zero, the frequency-dependent mass, elastic and geometric stiffness matrices given in Sections 3, 4 and 5, converge to the corresponding frequency-independent mass, elastic and geometric stiffness matrices of the conventional FEM. However, in the expressions for the mass, elastic and geometric stiffness elements given in Eqs. (30)-(62), Eqs. (65)-(85), and Eqs. (87)-(99), the frequency ω must not be set to exactly zero, but a small value e.g., $\omega = 10^{-5}$ rad/s can be used for most practical problems so that numerical overflow or ill-conditioning can be avoided and yet the degenerate case of the frequency-independent mass, elastic and geometric stiffness matrices as used in FEM can be obtained. These matrices resulting from the degenerate case of the present theory by using a negligibly small frequency (ω) were checked against the frequency-independent mass, elastic and geometric stiffness matrices which are generally utilised in FEM and readily available in the literature [2, 36].

The frequency-dependent mass, elastic and geometric stiffness matrices for an axially loaded Timoshenko-Ehrenfest beam $\mathbf{m}(\omega)$, $\mathbf{k}_E(\omega)$ and $\mathbf{k}_G(\omega)$ derived in Sections 3, 4 and 5 can be related to its dynamic stiffness matrix $\mathbf{k}_D(\omega)$, by using the following relationship [15].

$$\mathbf{k}_D(\omega) = \mathbf{k}_E(\omega) + \mathbf{k}_G(\omega) - \omega^2 \mathbf{m}(\omega) \quad (100)$$

Once the dynamic stiffness matrix for the axially loaded Timoshenko-Ehrenfest beam element is established using Eq. (100), the Wittrick-Williams algorithm can then be applied as a solution technique to carry out the free vibration analysis. For the general case of a frame, the frequency-dependent mass, elastic and geometric stiffness matrices for all individual elements in the frame, should be assembled using conventional transformation technique based on the orientation of the elements, as commonly employed in FEM. Thus, the overall frequency-dependent mass, elastic and geometric stiffness matrices $\mathbf{M}(\omega)$, $\mathbf{K}_E(\omega)$ and $\mathbf{K}_G(\omega)$ and hence the overall master dynamic stiffness matrix $\mathbf{K}_D(\omega)$ of the final structure can be obtained, as follows.

$$\mathbf{K}_D(\omega) = \mathbf{K}_E(\omega) + \mathbf{K}_G(\omega) - \omega^2 \mathbf{M}(\omega) \quad (101)$$

As mentioned earlier, the best way to solve the eigenvalue problem represented by $\mathbf{K}_D(\omega)\Delta_D = 0$ is to apply the well-established algorithm of Wittrick and Williams [27] to determine the natural frequencies and the subsequent mode shapes of the structure. The Wittrick-Williams algorithm has widespread coverage in the literature with literally hundreds of papers published on the subject, see for example [38-41]. In many ways, the dynamic stiffness method and the Wittrick-Williams algorithm are permanently entangled with each other, as demonstrated in the literature [38-41]. The working principle of the algorithm is essentially based on two factors

which govern the solution technique. These factors are known as (i) the sign-count $s\{\mathbf{K}_D\}$ and (ii) the j_0 count. The procedure based on these two factors, is briefly explained below.

If ω denotes the circular (or angular) frequency of a vibrating structure, then according to the Wittrick-Williams algorithm, j , the number of natural frequencies passed, as ω is increased from zero to ω^* , is given by

$$j = j_0 + s\{\mathbf{K}_D\} \quad (102)$$

where $\mathbf{K}_D$, the overall dynamic stiffness matrix of the final structure whose elements all depend on ω , is evaluated at $\omega = \omega^*$; $s\{\mathbf{K}_D\}$, the sign count, is the number of negative elements on the leading diagonal of $\mathbf{K}_D^\Delta$, $\mathbf{K}_D^\Delta$ is the upper triangular matrix obtained by applying the usual form of Gauss elimination to $\mathbf{K}_D$, and j_0 is the number of natural frequencies of the structure still lying between $\omega = 0$ and $\omega = \omega^*$ when the displacement components to which $\mathbf{K}_D$ corresponds are all zeros. (Note that the structure can still have natural frequencies when all its nodes are clamped, because exact member equations allow each individual member to displace between nodes with an infinite number of degrees of freedom, and hence an infinite number of natural frequencies between nodes.) Thus

$$j_0 = \sum j_m \quad (103)$$

where j_m is the number of natural frequencies between $\omega = 0$ and $\omega = \omega^*$ for a component member with its ends fully clamped, while the summation extends over all members of the structure. Thus, with the knowledge of Eqs. (102) and (103), it is possible to ascertain how many natural frequencies of a structure lie below an arbitrarily chosen trial frequency. This simple feature of the algorithm (coupled with the fact that successive trial frequencies can be chosen to bracket a natural frequency) can be used to converge upon any required natural frequency to any desired accuracy.

7. Numerical results and discussion

The above theory was applied to a wide range of problems concerning axially loaded Timoshenko-Ehrenfest beams, and it was established that the frequency-dependent mass, elastic, and geometric stiffness matrices give the same results that can be obtained by the conventional dynamic stiffness method which uses a single frequency-dependent matrix containing the mass, elastic and geometric stiffness properties, rather than separate mass, elastic and geometric stiffness matrices that are frequency-dependent. The literature showing the effects of shear deformation and rotatory inertia on the free vibration characteristics of (unloaded) Timoshenko-Ehrenfest beams is extensive [42-46]. The purpose of this paper is not to duplicate these results. However, the author of this paper recently published results highlighting the effects of axial load on the free vibration behaviour of beams and frameworks, but without considering the effects of rotatory inertia and shear deformation [15]. Therefore, when obtaining results from the current theory, emphasis is given on the combined effects of the shear deformation, rotatory inertia, and axial load on the free vibration characteristics of beams and frameworks. Three selective illustrative examples are presented. The first example focuses on the free vibration analysis of an axially loaded Timoshenko-Ehrenfest beam with clamped-free (C-F), simple support-simple support (S-S) and clamped-clamped (C-C) boundary conditions with the effects of axial force, but with and without the inclusion of shear deformation and rotatory inertia. The effects of these parameters are uniquely described by the terms s^2 , r^2 and p^2 of Eq. (7). The data for this example was taken from Chen et al. [47] with beam material properties, Young's modulus $E =$

210 GPa, density $\rho = 7850 \text{ kg/m}^3$ and Poisson's ratio $\nu = 1/3$. The shear modulus G was calculated by relating it to E through the Poisson's ratio ν to give $G = 3E/8$ [35]. The beam has a length $L = 0.4\text{m}$ and is of rectangular cross-section with width $b = 0.02\text{m}$, depth or height $h = 0.08\text{m}$. The shear correction factor (also known as shape factor) was set to $k = 2/3$ as used in [47]. Using these data, the beam parameters were worked out as bending rigidity $EI = 1.792 \times 10^5 \text{ Nm}^2$, shear rigidity $kAG = 8.4 \times 10^7 \text{ N}$, and mass per unit length $\rho A = 12.56 \text{ kg/m}$. The non-dimensional shear deformation and rotatory inertia parameters of the beam were worked out as $s^2 = 1/75$, $r^2 = 1/300$. The first five flexural natural frequencies of the beam in the absence of axial load, computed by the present theory are shown in Table 1, with and without the inclusion of the effects of rotatory inertia and shear deformation, i.e. using the Timoshenko-Ehrenfest and Bernoulli-Euler beam theories, respectively. The results using the Timoshenko-Ehrenfest theory agreed completely with the exact results quoted by Chen et al [47]. Table 1 results were further checked by using the frequency equation given by Huang [48] for a Timoshenko-Ehrenfest beam for different boundary conditions. Note that in Refs. [47, 48] the effect of an axial load on the natural frequencies of the beam was not considered. Also, neither Chen et al [47] nor Huang [48] presented comparable numerical results when using the Bernoulli-Euler theory, i.e., when the effects of rotatory inertia and shear deformation are ignored, which of course, can be found in standard texts [49, 50], but these are important for comparison purposes. Table 1 includes both sets of results indicating that the effects of rotatory inertia and shear deformation for the problem are, indeed, very significant, particularly for the C-C boundary condition. For instance, the differences in the fourth natural frequency for the C-F, S-S and C-C cases are 78%, 82% and 121%, respectively. Such large differences are mainly due to the low slenderness ratio of the beam considered, which is around 17.

Table 1. Natural frequencies of a Timoshenko-Ehrenfest beam for different boundary conditions in the absence of axial load.

Natural Frequency No (i)	Natural Frequencies ω_i (rad/s)					
	Timoshenko-Ehrenfest Theory ($r^2 \neq 0, s^2 \neq 0$)			Bernoulli-Euler Theory ($r^2 = 0, s^2 = 0$)		
	Boundary Conditions			Boundary Conditions		
	C-F	S-S	C-C	C-F	S-S	C-C
1	2529.493	6838.834	12845.52	2624.853	7368.074	16702.59
2	13279.91	23190.82	28790.87	16449.67	29472.29	46041.35
3	31044.79	43443.49	47488.00	46059.56	66312.66	90259.45
4	50825.83	64939.19	67386.61	90258.35	117889.2	149203.4
5	71565.05	86710.90	88037.73	149203.5	184201.8	222884.2

Clearly, the results shown in Table 1 indicate that the effects of shear deformation and rotatory inertia for the Timoshenko-Ehrenfest beam [47] cannot be ignored, particularly for the C-C boundary conditions, and for the higher order natural frequencies. The next step in this investigation, was to investigate the effect of the axial load together with and without the effects of shear deformation and rotatory inertia. Both compressive and tensile loads were considered, and the load level when computing the natural frequencies, was based on the critical buckling load of the beam computed with and without the effects of shear deformation and rotatory inertia for each of the C-F, S-S and C-C boundary conditions.

The critical buckling load (P_{cr}) of the Timoshenko-Ehrenfest beam of [47] for C-F, S-S and C-C boundary conditions with the inclusion of shear deformation was worked out by using the theory published by Banerjee and Williams [51] and by others [52-54]. These are given by the following general formula given in [51].

$$(P_{cr})_{C-F} = \frac{\pi^2 EI}{4L^2(1+\pi^2 s^2/4)} \quad (104)$$

$$(P_{cr})_{S-S} = \frac{\pi^2 EI}{L^2(1+\pi^2 s^2)} \quad (105)$$

$$(P_{cr})_{C-C} = \frac{4\pi^2 EI}{L^2(1+4\pi^2 s^2)} \quad (106)$$

where the shear deformation parameter s^2 has already been defined in Eq. (7).

The corresponding results for the critical buckling load without the inclusion of the effects of shear deformation (P_{cr0}) was obtained by substituting $s^2 = 0$ in Eqs. (104)-(106), which can also be found in standard texts [55, 56]. Both sets of results, i.e., with $s^2 \neq 0$ and $s^2 = 0$, are shown in Table 2. The effect of shear deformation diminished the critical buckling load of the beam by 3.8%, 4.1% and 32% for the C-F, S-S and C-C boundary conditions, respectively. (Buckling being a static phenomenon, the rotatory inertia will not have any effect on the critical buckling load, as expected.) However when computing the natural frequencies for the above three boundary conditions of the beam, the axial load was taken to be $\pm 50\%$ of the corresponding critical buckling loads, respectively, i.e., $P = \pm 0.5P_{cr}$ when the effect of shear deformation is included, and $P = \pm 0.5P_{cr0}$ when the effect is excluded, noting that the positive sign for P corresponds to a compressive load whereas the negative sign corresponds to a tensile load. Based on these axial loads, the non-dimensional axial load parameter p^2 for each of C-F, S-S and C-C was worked out and subsequently used in the free vibration analysis.

Table 2. Critical buckling load of an axially loaded Timoshenko-Ehrenfest beam.

Boundary Conditions	Critical Buckling Load (kN)	
	$P_{cr} (s^2 \neq 0)$	$P_{cr0} (s^2 = 0)$
C-F	2675.47	2763.49
S-S	9768.48	11054.0
C-C	28967.8	44215.8

The first five natural frequencies of the beam with C-F, S-S and C-C boundary conditions computed using the present theory are shown in Tables 3-5, respectively, with and without the inclusion of the shear deformation, rotatory inertia, and axial load. (To be consistent with published literature, the axial natural frequencies are not included in the analysis for the results shown in Tables 3-5.)

Since symbolic algebra has been used in this paper, results in Tables 3-5 are presented with seven figure accuracy. Such a high degree of accuracy is provided so that interested readers can check their own theory and/or computer program in their research.

Table 3. Natural frequencies of an axially loaded Timoshenko-Ehrenfest beam with cantilever (C-F) boundary condition. (Results using [15] in which $r^2=0$, $s^2=0$ and $p^2 \neq 0$, are shown in the parentheses.)

Natural Frequency No (<i>i</i>)	ω_i (rad/s)			
	$r^2 = 1/300$	$r^2 = 1/300$	$r^2 = 0$	$r^2 = 1/300$
	$s^2 = 1/75$	$s^2 = 1/75$	$s^2 = 1/75$	$s^2 = 0$
	$P = 0.5P_{cr}$	$P = -0.5P_{cr}$	$P = 0$	$P = 0$
1	1833.343 (1892.152)	3032.636 (3160.457)	2546.806	2604.73
2	12595.23 (15755.89)	13926.62 (17111.891)	13734.74	15624.87
3	30398.30 (45478.92)	31677.93 (46632.88)	32516.09	41080.62
4	50101.24 (89712.38)	51539.63 (90800.10)	53609.85	74352.68
5	70744.05 (148676.9)	72357.25 (149728.2)	75556.00	11272.09

Table 4. Natural frequencies of an axially loaded Timoshenko-Ehrenfest beam with simply-supported (S-S) boundary condition. (Results using [15] where $r^2=0$, $s^2=0$ and $p^2 \neq 0$, are shown in the parentheses.)

Natural Frequency No (<i>i</i>)	ω_i (rad/s)			
	$r^2 = 1/300$	$r^2 = 1/300$	$r^2=0$	$r^2=1/300$
	$s^2 = 1/75$	$s^2 = 1/75$	$s^2=1/75$	$s^2=0$
	$P = 0.5P_{cr}$	$P=-0.5P_{cr}$	$p^2=0$	$p^2=0$
1	4835.886 (5210.015)	8375.652 (9024.011)	6926.415	7249.784
2	21148.36 (27568.81)	25066.14 (31260.09)	23855.18	27705.66
3	41058.06 (64444.33)	45702.27 (68129.79)	44867.83	58247.75
4	62110.48 (116030.5)	67646.18 (119717.0)	66897.06	95420.73
5	83383.78 (182350.5)	89910.61 (186034.7)	88934.86	136447.0

Table 5. Natural frequencies of an axially loaded Timoshenko-Ehrenfest beam with clamped-clamped (C-C) boundary condition. (Results using [15] where $r^2=0$, $s^2=0$ and $p^2 \neq 0$, are shown in the parentheses.)

Natural Frequency No (<i>i</i>)	ω_i (rad/s)			
	$r^2=1/300$	$r^2=1/300$	$r^2=0$	$r^2=1/300$
	$s^2=1/75$	$s^2=1/75$	$s^2=1/75$	$s^2=0$
	$P^2=0.5P_{cr}$	$P^2=-0.5P_{cr}$	$p^2=0$	$p^2=0$
1	9283.319 (11908.30)	15513.26 (20318.32)	12940.28	16369.99
2	23385.89 (40136.07)	33285.78 (51228.36)	29354.92	42861.56
3	40643.09 (84011.51)	53453.22 (96093.16)	48701.89	78263.59
4	58975.94 (142735.9)	74821.10 (155398.6)	69303.42	118994.8
5	78035.28 (216270.1)	96943.22 (229306.0)	90452.18	162574.0

The last two columns of the results shown in Tables 3-5 when compared with the results of Table 1, clearly indicate that the effect of shear deformation on the natural frequencies is much more pronounced than that of the rotatory inertia. Banerjee et al. [57] have recently demonstrated that the shear deformation effect will be dominant over the rotatory inertia effect for all boundary conditions of the beam. Columns 2 and 3 of Tables 3-5 show respectively, the effects of the compressive and tensile loads which are both taken to be 50% of the critical buckling load. As expected, the effect of the compressive load resulted in the reduction of the natural frequencies whereas that of the tensile load increased them. The results shown in the parentheses of columns 2 and 3 of Tables 3-5 were computed using the earlier theory of Banerjee [15] and that of Amba-Rao [58] which included the effect of axial load but excluded effects of shear deformation and rotatory inertia. A striking feature of these results when compared with the results obtained from the present theory is that there are significant differences in the natural frequencies, particularly the higher order ones, clearly suggesting that the effects of shear deformation and rotatory inertia must be included alongside the effect of the axial load when determining the natural frequencies of the type of beam that has been investigated. The beam with the C-C boundary condition is the most severely affected one, which accords with the observation made by other investigators [42-48]

The variation of the fundamental natural frequency of the beam with C-C boundary condition against the axial load is shown in Fig. 3 using the current theory by the solid line, which includes the effects of shear deformation and rotatory inertia, and by the broken (dashed) line when using the earlier theory of Banerjee [15], which excluded them. Noticeably, there are significant differences. Therefore, for this type of Timoshenko-Ehrenfest beam investigated in the literature [47], the earlier theory of Banerjee [15] will not be very accurate and thus for the statistical energy analysis method which requires modal density in the high frequency range, Banerjee's earlier theory [15] may be deficient or inadequate. As expected, the natural frequency increases with tensile load whereas it decreases with compressive load and the curves reach the horizontal axis when the natural frequency becomes zero when $P/P_{cr}=1$, implying that buckling occurs as a degenerate case of free vibration analysis at zero frequency.

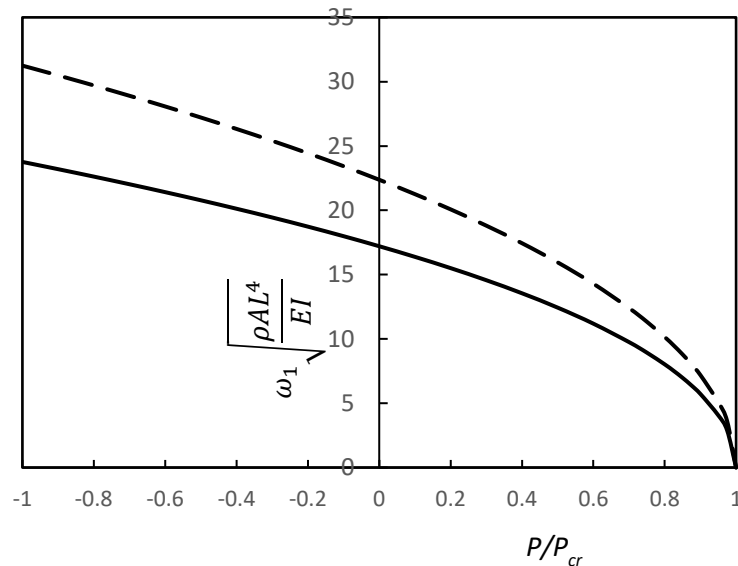


Fig. 3 Effect of axial load on the fundamental natural frequency of a clamped-clamped Timoshenko-Ehrenfest beam.

- Axially loaded Timoshenko-Ehrenfest beam
- Axially loaded Bernoulli-Euler beam

The second example is the axially loaded cantilever stepped beam shown in Fig. 4 which was recently investigated by Banerjee [15] for its free vibration characteristics, but by using Bernoulli-Euler theory, i.e., without the effects of shear deformation and rotatory inertia. The stepped beam is comprised of two parts AB and BC, as shown in Fig. 4. The first part AB has an axial rigidity E_1A_1 , flexural rigidity E_1I_1 , shear rigidity $k_1A_1G_1$, mass per unit length ρ_1A_1 and length L_1 whereas the second part BC has an axial rigidity E_2A_2 , flexural rigidity E_2I_2 , shear rigidity $k_2A_2G_2$, mass per unit length ρ_2A_2 and length L_2 . The critical buckling load (P_{cr}) of the stepped beam is established using the theory developed in Appendix A. For $E_1A_1=E_2A_2$, $E_1I_1=E_2I_2$, $k_1A_1G_1=k_2A_2G_2$ and $L_1=L_2$, the critical buckling load P_{cr} of the stepped column given by the theory in Appendix A, reduces to the degenerate case of a uniform cantilever column given by Eq. (104) where $EI=E_1I_1=E_2I_2$, $L=L_1+L_2$ and $s^2 = EIL^2/kAG$ with $kAG=k_1A_1G_1=k_2A_2G_2$. Of course, when shear deformation is ignored P_{cr} is given by $\frac{\pi^2 EI}{4L^2}$ by substituting $s^2=0$ in Eq. (104) which is a well-known result for a cantilever column.

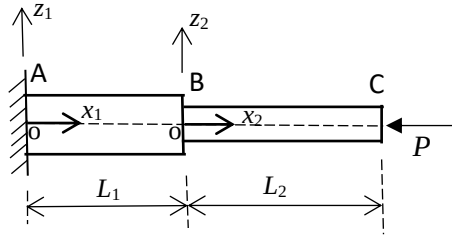


Fig. 4 An axially loaded cantilever stepped Timoshenko-Ehrenfest beam.

For numerical results, the data used for the axially loaded stepped beam of Fig. 4 are as follows. The stepped beam is made of steel with Young's modulus $E = 200$ GPa, density $\rho = 7850$ kg/m³ and Poisson's ratio $\nu = 1/3$. Both parts AB and BC of the beam are of solid circular cross-sections with diameters 0.03m and 0.02m, respectively. The lengths of segment AB and BC are $L_1=0.625$ m and $L_2=0.625$ m so that for the full beam $L=1.25$ m. The shear correction factor was set to 2/3. Using the buckling theory of Appendix A, the critical buckling load of the stepped beam was established at $P_{cr}=6702.8$ N with the help of Eq. (A19). The first five natural frequencies of the beam with axial load $P= \pm 4000$ N (which is around 60% of the critical buckling load) were computed with and without the effects of shear deformation and rotatory inertia and they are shown in Table 6. The effects of shear deformation and rotatory inertia on the first five natural frequencies are not so pronounced for this stepped beam, as can be seen in Table 6. The reason for this can be attributed to the fact that the stepped beam is considerably more slender (the slenderness ratio which is the ratio between the length and the radius of gyration of the cross-section is around 100) for which the shear deformation and rotatory inertia are not expected to have any major effects. This example was chosen to demonstrate the circumstances when the earlier theory of Banerjee [15] developed for axially loaded Bernoulli-Euler beams can be applied, particularly in the low frequency range. It should be noted that for higher order natural frequencies the differences in the natural frequencies with and without the inclusion of shear deformation and rotatory inertia are expected to be more pronounced than the ones shown in Table 6. Using the present theory, the variation of the fundamental natural frequency of the stepped beam with the axial load ratio P/P_{cr} is shown in Fig. 5, showing a similar trend to a uniform beam (see Fig. 3).

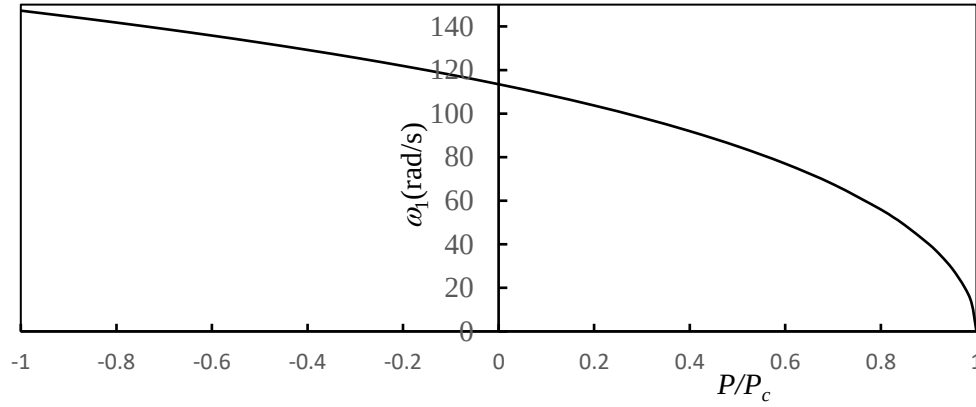


Fig. 5 Effect of axial load on the fundamental natural frequency of a cantilever stepped Timoshenko-Ehrenfest beam.

Table 6. Natural frequencies of axially loaded Timoshenko-Ehrenfest stepped beam.

Natural Frequency No (i)	ω_i (rad/s)			
	$s^2 \neq 0, r^2 \neq 0$		$s^2 = 0, r^2 = 0$ [15]	
	$P= 4000\text{N}$	$P= -4000\text{N}$	$P= 4000\text{N}$	$P= -4000\text{N}$
1	77.236	135.70	77.284	135.76
2	386.47	462.08	387.19	462.87
3	1216.6	1282.0	1223.1	1288.6
4	2249.0	2305.4	2269.1	2325.6
5	3888.5	3947.4	3942.7	4001.7

The final illustrative example is that of a plane frame shown in Fig. 7, which is taken from [59]. The node at A of the frame is fully fixed, i.e., built-in, whereas the node at B has a roller support which prevents horizontal motion but allows vertical motion and rotation. The rest of the nodes are free. Each of the nine members of the frame has the same bending rigidity $EI = 5 \times 10^6 \text{ Nm}^2$, axial rigidity $EA = 9 \times 10^8 \text{ N}$ and mass per unit length $\rho A = 35 \text{ kg/m}$. The Poisson's ratio and shear correction factor are taken as $\nu = 1/3$ and $k = 2/3$, respectively. The horizontal members AC, BD, CE and DF carry a compressive axial force of $4EI/l^2$, the diagonal members BC and CF carry a tensile axial force of $5EI/l^2$, where l is the length of the member concerned, and the vertical members AB, CD and EF are unloaded. Using the present theory, nine natural frequencies of the frame which are 1st, 2nd, 3rd, 4th, 5th, 50th, 70th, 90th and 150th natural frequencies, were computed. The results are shown in Table 7. For comparison purposes the corresponding natural frequencies using the Bernoulli-Euler theory [15] were also computed and shown in Table 7. The Bernoulli-Euler theory gave reasonably good results for the first five natural frequencies, the discrepancy with the proposed theory being less than 2.5%, but for the 50th, 70th, 90th and 150th natural frequencies, the errors incurred are 20%, 25%, 30% and 40%, respectively. For applications in statical energy analysis methods, such errors are unacceptable.

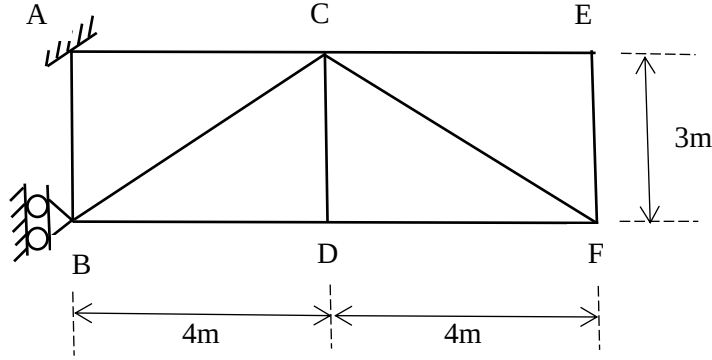


Fig. 6. A plane frame comprising axially loaded Timoshenko-Ehrenfest beams.

Table 7. Natural frequencies of a plane frame comprising axially loaded Timoshenko-Ehrenfest beams.

Natural Frequency Number (i)	Natural Frequency ω_i (rad/s)	
	Timoshenko-Ehrenfest theory	Bernoulli-Euler theory [15]
1	158.531	159.028
2	255.550	259.265
3	292.131	297.594
4	318.821	324.488
5	395.837	406.475
50	5130.79	6093.76
70	7913.96	9553.64
90	10883.9	13415.1
150	19947.5	27987.6

8. Conclusions

There was a gap in the published literature in that axially loaded Timoshenko-Ehrenfest beams and frameworks were not investigated for their free vibration characteristics by using the frequency-dependent mass, elastic and geometric stiffness matrices in an exact sense and relating them to the dynamic stiffness method. This paper has filled this gap by deriving explicit algebraic expressions for the elements of the frequency-dependent mass, elastic and geometric stiffness matrices of an axially loaded Timoshenko-Ehrenfest beam through extensive application of symbolic computation. The equivalency of the proposed method with the dynamic stiffness method has been established, and the shortcomings of earlier theories have been overcome. The theory is demonstrated by illustrative examples for free vibration analysis of axially loaded beams and frameworks through the application of the Wittrick-Williams algorithm as a solution technique. The investigation carried out is particularly useful for free vibration analysis in the high frequency range which is required in the statistical energy analysis method. The proposed theory opens the possibility of including damping in exact free vibration analysis of structures. This is a new and deeply significant development in beam and framework dynamics.

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Appendix A. Buckling of a cantilever stepped column

Referring to Fig. 4, the governing differential equations for elastic buckling of the two components AB and BC of the stepped column can be respectively, written as

$$D^2(D^2 + \alpha_1^2)z_1 = 0 \quad (A1)$$

$$D^2(D^2 + \alpha_2^2)z_2 = 0 \quad (A2)$$

where

$$\alpha_1^2 = \frac{p_1^2}{1-p_1^2 s_1^2}; \quad D = \frac{d}{d\xi_1} \quad (A3)$$

with

$$\xi_1 = \frac{x_1}{L_1}; \quad p_1^2 = \frac{PL_1^2}{E_1 I_1}; \quad s_1^2 = \frac{E_1 I_1}{k_1 A_1 G_1 L_1^2} \quad (A4)$$

and

$$\alpha_2^2 = \frac{p_2^2}{1-p_2^2 s_2^2}; \quad D = \frac{d}{d\xi_2} \quad (A5)$$

with

$$\xi_2 = \frac{x_2}{L_2}; \quad p_2^2 = \frac{PL_2^2}{E_2 I_2}; \quad s_2^2 = \frac{E_2 I_2}{k_2 A_2 G_2 L_2^2} \quad (A6)$$

The solution of Eq. (A1) for the part AB of the stepped beam of Fig. 3 leads to (see [39] for details)

$$\text{Deflection: } z_1(\xi_1) = A_1 + A_2 \xi_1 + A_3 \sin \alpha_1 \xi_1 + A_4 \cos \alpha_1 \xi_1 \quad (A7)$$

$$\text{Slope (Bending rotation): } \theta_1(\xi_1) = \frac{1}{L_1} \left(A_2 + A_3 \frac{p_1^2}{\alpha_1} \cos \alpha_1 \xi_1 + A_4 \frac{p_1^2}{\alpha_1} \sin \alpha_1 \xi_1 \right) \quad (A8)$$

$$\text{Bending moment: } M_1(\xi_1) = \frac{E_1 I_1}{L_1^2} p_1^2 (A_3 \sin \alpha_1 \xi_1 + A_4 \cos \alpha_1 \xi_1) \quad (A9)$$

$$\text{Shear force: } S_1(\xi_1) = -\frac{E_1 I_1}{L_1^3} p_1^2 A_2 \quad (A10)$$

where A_1 - A_4 are integration constants, resulting from the solution of the governing differential equation.

Similarly, the solution of Eq. (A2) for the part BC of the stepped beam of Fig. 3 leads to (see [39] for details)

$$\text{Deflection: } z_2(\xi_2) = B_1 + B_2 \xi_2 + B_3 \sin \alpha_2 \xi_2 + B_4 \cos \alpha_2 \xi_2 \quad (\text{A11})$$

$$\text{Slope (Bending rotation): } \theta_2(\xi_2) = \frac{1}{L_2} \left(B_2 + B_3 \frac{p_2^2}{\alpha_2} \cos \alpha_2 \xi_2 + B_4 \frac{p_2^2}{\alpha_2} \sin \alpha_2 \xi_2 \right) \quad (\text{A12})$$

$$\text{Bending moment: } M_2(\xi_2) = \frac{E_2 I_2}{L_2^2} p_2^2 (B_3 \sin \alpha_2 \xi_2 + B_4 \cos \alpha_2 \xi_2) \quad (\text{A13})$$

$$\text{Shear force: } S(\xi_2) = -\frac{E_2 I_2}{L_2^3} p_2^2 B_2 \quad (\text{A14})$$

where B_1 - B_4 are integration constants, resulting from the solution of the governing differential equation.

Now, the following boundary conditions can be applied to eliminate the constants A_1 - A_4 and B_1 - B_4 and arrive at the resulting buckling equation.

For the part AB (see Fig. 3):

$$\text{At } \xi_1=0, z_1=0 \text{ and } \theta_1=0 \quad (\text{A15})$$

For the part BC (see Fig. 3):

$$\text{At } \xi_2=1, M_2=0 \text{ and } S_2=0 \quad (\text{A16})$$

At the interface i.e., at the connecting point B which joins the segments AB and BC, the continuity of displacement, rotation, bending moment and shear force must be satisfied which means (i) z_1 at $\xi_1=1$ must be equal to z_2 at $\xi_2=0$, (ii) θ_1 at $\xi_1=1$ must be equal to θ_2 at $\xi_2=0$, (iii) M_1 at $\xi_1=1$ must be equal to M_2 at $\xi_2=0$ and (iv) S_1 at $\xi_1=1$ must be equal to S_2 at $\xi_2=0$, i.e.,

$$z_1(1) = z_2(0) \text{ and } \theta_1(1) = \theta_2(0) \quad (\text{A17})$$

$$M_1(1) = M_2(0) \text{ and } S_1(1) = S_2(0) \quad (\text{A18})$$

Substituting Eqs. (A15)-(A18) into Eqs.(A11)-(A14), the constants A_1 - A_4 and B_1 - B_4 can be eliminated to give the following buckling equation

$$\cos \alpha_1 \cos \alpha_2 - \left(\frac{\alpha_1 E_1 I_1 L_1}{\alpha_2 E_2 I_2 L_2} \right) \sin \alpha_1 \sin \alpha_2 = 0 \quad (\text{A19})$$

Equation (A19) can now be used to calculate the critical buckling load of a stepped column of the type shown in Fig. 3. Note that α_1 and α_2 in Eq. (A19) are functions of the axial load P which can be determined through an iterative process by using successive values of P to arrive at P_{cr} when satisfying the condition of Eq. (A19).

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