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Propagation of bending waves along the edge of a point-loaded piezoelectric plate on elastic foundation

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Abstract

An investigation on the propagation of bending waves along the edge of a piezoelectric thin plate resting on an elastic foundation (cf. Pasternak foundation) and under the action of a point load is carried out in this paper. The displacement field of the plate is based on **the** classical plate theory. The Kelvin-Voigt type viscoelastic model is used to examine the **viscous** effect on the characteristics of the bending wave. The Moore-Gibson-Thompson (MGT) thermoelastic equation is solved for an external time variant harmonic heat source at the upper surface of the plate. The piezoelectric potential is obtained for an electrode-covered upper surface and an electrically shorted lower surface. The displacements of the propagating bending wave along the edge are examined under a time-harmonic point-load acting near the vicinity of the free edge of the plate. The well-known Green function technique and Fourier transform method are implemented to generate an analytical solution for the non-homogeneous boundary value problem. We analyze and present the displacements of the bending edge wave under external loading, both analytically and graphically, and summarize the results.

Keywords: Edge wave, Piezoelectricity, Viscosity, Green function, Dispersion

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Nomenclature

Symbols	Description	Symbols	Description
C_{ij}^* (N/m^2)	Material stiffness constant	e_{kij}^* (C/m^2)	Piezoelectric constant
ψ_{ij}^* ($N/m^2.K$)	Thermal expansion coefficient	$\varkappa_{ij}^*$ ($C^2/N.m^2$)	Dielectric constant
c_v ($J/Kg.K$)	Specific heat per unit mass	Θ (K)	Thermal deviation
τ_0 (K)	Reference temperature	T_0 (s)	Relaxation parameter
k_i ($N/K.s$)	Thermal conductivity	k_i^* ($N/K.s^2$)	Thermal conductivity rate
ρ (Kg/m^3)	Material density	D_i (C/m^2)	Electric displacement
h (m)	Thickness of the plate	β (N/m^3)	Winkler constant
k_g (N/m)	Foundation shear layer constant	η_{ij} (N/m^2)	Viscosity constant
ω ($1/s$)	Angular frequency	ψ_D ($C/m^2.K$)	Thermal expansion coefficient
α_i ($1/K$)	Constants of the thermal expansion	Φ_z (V/m)	Electric field
V ($\mathcal{V}$)	Electrode covered	f_0 (K/m)	External heat

1 Introduction

Free vibration analysis of structural elements such as rod [1], beam [2–4], plate [5–7] and shell [8–10] made of composite materials is an area of intense research activities because of their applications in civil, mechanical, and aerospace engineering. Therefore, researchers have made concerted efforts to understand the properties of composite materials under different mechanical environments. One type of advanced composite is piezoelectric material, in which an electric field is generated due to a mechanical strain and vice-versa. The plate structure particularly made of piezoelectric materials is often used to manufacture pressure sensors, piezoelectric actuators, ultrasonic transducers for energy harvesting, amongst others. Therefore, numerous investigations have been conducted to understand the mechanical behaviour of piezoelectric plates. Kulikov and Plotnikova [11] established an analytical method for the solution of a laminated piezoelectric plate in a thermal environment. The mathematical analysis of an infinite piezoelectric plate with traction-free and electrically shorted boundary conditions was examined by Tiersten [12]. Li et al. [13, 14] analysed the boundaries of the unstable regions of the frequency, parametric resonances of a viscoelastic piezoelectric nanoplate and self-powered nanoribbon under the effect of thermo-electro-mechanical forces. By contrast, Khosh-goftar et al. [15] proposed that the mechanical properties can be controlled in a functionally graded cylinder for a proper distribution of thermal and piezoelectric properties. Shakeri and Mirzaeifar [16] employed finite element formulation to examine the impact of static and dynamic loads on the vibration behaviour of a thick functionally graded multilayer plate having piezoelectric surface layer. Najd et al. [17] investigated the optimal position of a piezoelectric transducer and size of sensors for lamb wave propagation in glass-fibre/epoxy plates and concluded that the thickness and size of the sensor play an important role in changing the behavior of the amplitude of S0 mode.

The analysis of a plates and their associated foundation structure under different types of external load (cf. dynamic or static) have gained considerable attention in recent years due to their practical applications in different fields like highways, airport runways, raft foundations and concrete pavements. A review of different models of the interaction between plates and beams with

elastic foundation can be found in [18]. Using the Fourier transform and Green function technique, Sun [19] obtained the analytical solution to the problem of an impulsive line load on an infinite plate supported by an elastic foundation. On the other hand, Eringen [20] investigated the impact of the stochastic load in the vibration of a damped beam and plate. Using the finite element method and discretising a flat plate into isoparametric plate elements, Wu et al. [21] demonstrated that the eccentricity of the load varies with the rigidity of the structure. Interestingly Akhavan et al. [22] extracted the closed-form solution of a thick rectangular plate with the Mindlin plate theory under the impact of uniformly distributed in-plane load. Tkacheva [23] obtained an analytical solution of a floating plate with periodic loading using the Wiener-Hopf technique and examined the impact of the loading distribution upon the amplitude of the surface waves. Responses due to different types of load on a circular plate have been investigated by Starovoitov et al. [24], Micallef et al. [25], and research on sandwich plates has been carried out by Carrera and Ciuffreda [26] and Kao et al. [27], amongst others.

One of the major concerns in the design of a thin structure is its reliability and safety. Therefore, different wave propagation techniques have been developed to monitor the integrity of the kind of structure; see Cawley and Adams [28], Messina et al. [29], Yao et al. [30], Marin et al. [31], Wu et al. [32] and Palacz and Krawczuk [33]. One such type of wave model is the **so-called** edge wave. The development of the mathematical theory of this type of wave model started with the work of Kononkov [34]. Using the Stroh-like formulation technique, Fu [35] established that there exists a unique edge wave at the free edge of an anisotropic plate. The parabolic-elliptic type nature of the edge wave on isotropic and orthotropic plates have been established by Kaplunov et al. [36] and Althobaiti et al. [37]. Zernov and Kaplunov [38] demonstrated the propagation characteristics of a three-dimensional edge wave with traction-free and fixed face boundary conditions and they established the existence of the lower order edge mode. Wilde et al. [39] proposed a refined boundary condition **when using the work of the Goldenveizer and Kolos** considering **the plate bending problem**. Several investigations have been conducted to understand the nature of this type of waves in different media. For examples, see Thompson et al. [40], Abrahams and Norris [41], Manna and Som [42,43], Zakharov [44], Som and Manna [45] and Kumari et al. [46].

Having conducted the above **literature** reviews, it appears that the existing literature lacks an important investigation on the propagation of bending edge waves under the effect of an external point-load acting near the vicinity of **the plate**. Therefore, the primary objective of this paper is to fill this gap **in the literature by focusing** on the influence of an externally applied point-load on the displacement of the bending edge wave. A time-harmonic load acting at a certain point near the vicinity of the free edge of the plate has been taken as the base in the proposed model. The concentrated point-load acting on the plate is modelled by the Dirac delta function of the space variable. The plate is of piezoelectric material with thermoelastic properties and resting on an elastic foundation. The viscosity in the material parameters is modelled using the Kelvin-Voigt viscosity theory. The Moore-Gibson-Thompson (MGT) thermoelasticity theory generates the temperature distribution of the plate. An external time harmonic heat source is considered to obtain the temperature distribution for the thin plate. The piezoelectric potential function is obtained by considering an electrically shorted lower surface and an electrode covered on the upper surface with some specific voltage for the electrode. The equation of motion for the proposed model under the externally applied point

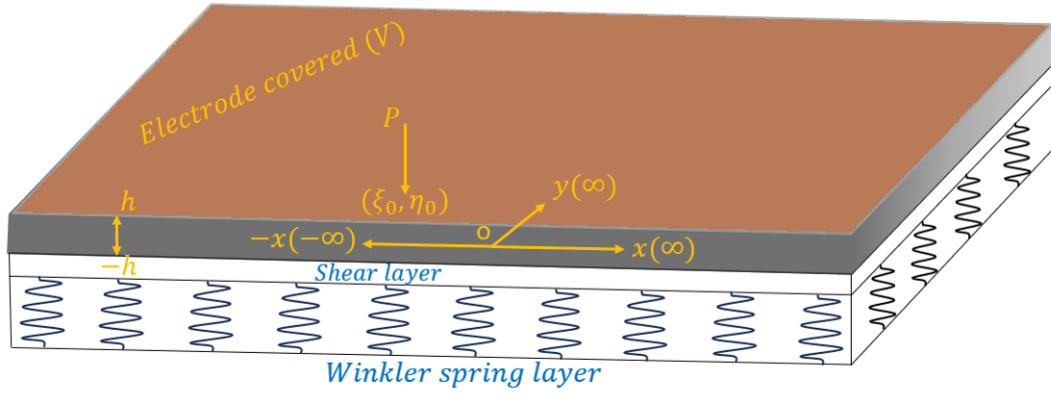


Figure 1: Geometry of the model

load is derived from the elastodynamic equations. The rotary inertia effect is taken into account in the model when formulating the equation of motion. To generate the dispersion relation and the displacement of the bending edge wave, the Fourier transform is used in the first instance to solve the equations of motion, and then, to solve the non-homogeneous boundary value problem, the Green function technique is employed. The dispersion relation is derived as implicit function of frequency, and the Fourier transforms variable. A beam-like behaviour of the thin plate is observed during the bending edge wave propagation. This was captured as a degenerate case of the plate theory. An expression of the Konenkov-type constant for the considered model is obtained, and its dependency on the frequency and wave number is observed. Graphical analysis indicates that the frequency of the edge wave increases with an increase in the voltage of the covered electrode, and the displacement amplitude decreases with an increase in the value of the foundation modulus.

The arrangement of the paper is organised as follows: Following this introduction section, section 2 contains the mathematical model of the piezoelectric viscous plate under the thermal environment, and stress displacement relation is obtained under the classical plate theory. The analytical solution (displacement and dispersion) of the proposed model is discussed in section 3. Section 4 consists of the graphical description of the analytical solution and analysis of the model. The conclusions of the investigation are summarized in section 5.

2 Constitutive relation and equation of motion

Let us consider a thin plate bounded by two planes, $z = -h$ and $z = h$, having the mid-plane at $z = 0$ as shown in Figure 1. An elastic foundation consists of a spring layer and a shear layer supported at $z = -h$. A rectangular Cartesian coordinate system is considered in which the xy -plane passes through the mid-plane of the plate. A hexagonal 6mm crystals class of piezoelectric material plate is considered, where the z -axis is considered as the axis of symmetry. The stress-strain relation for a thermally affected piezoelectric plate under the Kelvin-Voigt viscoelastic model can be written

as [47]

$$\sigma_{ij} = C_{ijkl}^* \gamma_{kl} - e_{kij}^* \Phi_k - \psi_{ij}^* \Theta, \quad (1)$$

$$D_i = e_{ikl}^* \gamma_{kl} + \varkappa_{ij}^* \Phi_k - \psi_D^* \Theta. \quad (2)$$

In equations (1-2), C_{ijkl} , e_{kij} , γ_{kl} , Φ_k , ψ_{ij} , $\varkappa_{ij}$, ψ_D , D_i ($i, j, k, l = 1, 2, 3$) and Θ represent material stiffness constants, piezoelectric constant, strain tensor, electric field vector, thermal expansion coefficients, dielectric constant, thermal expansion coefficients related to piezoelectricity electric displacement and [thermal deviation from reference temperature](#) (τ_0), respectively, in which $|\frac{\theta}{\tau_0}| \ll 1$.

Also, $\psi^* = C^* \alpha$, where $\psi^* = [\psi_{11}^* \ \psi_{33}^*]^T$, $C^* = \begin{pmatrix} C_{11}^* + C_{12}^* & C_{13}^* \\ 2C_{13}^* & C_{33}^* \end{pmatrix}$ and $\alpha = [\alpha_1 \ \alpha_3]^T$, α_i are constants of the thermal expansion, and $*$ denote the complex quantity and take the form as [48] $P^* = (\cdot) + i\eta_{ij}\partial_t(\cdot)$, where, η_{ij} ($i, j = 1, 2, 3$) represent the complex materials stiffness constant, which satisfy the positive definiteness of elastic strain energy and dissipation energy i.e.,

$$C_{ijkl}\gamma_{ij}\gamma_{kl} > 0 \text{ and } \eta_{ijkl}\partial_t(\gamma_{ij})\partial_t(\gamma_{kl}) > 0, \quad \forall \ \gamma_{ij}. \quad (3)$$

The strain of the piezoelectric plate is formulated by assuming that the thickness of the plate is small as compared to the wavelength of the considered wave and that bending occurs along the transverse direction of the flat plate. Therefore, under this assumption, the strain and displacement of the thin plate undergoing bending motion will be governed by the relation given by

$$\gamma_{xx} = -z\partial_{xx}w, \quad \gamma_{yy} = -z\partial_{yy}w \text{ and } \gamma_{xy} = -z\partial_{xy}w, \quad (4)$$

where, $w(x, y, t)$ is the transverse displacement of the plate. ∂_{ii} represent the partial derivative with respect to the variable i .

The elastodynamic equation for the piezoelectric materials can be written as [47],

$$\partial_j \sigma_{ij} = \rho \partial_{tt} u_i \text{ and } \partial_i D_i = 0, \quad i, j = x, y, z, \quad (5)$$

[In equation \(5\)](#), u_i ($i = 1, 2, 3$) denotes the displacement of the medium along the coordinate axes, and ρ represents the density of the medium.

Since $\sigma_{zz} = 0$ (under classical thin plate assumption), we can obtain from equation (1)

$$\gamma_{zz} = -\frac{C_{13}^*}{C_{33}^*} \gamma_{ll} + \frac{e_{33}^*}{C_{33}^*} \Phi_z + \frac{\psi_{33}^*}{C_{33}^*} \Theta, \quad l = x, y. \quad (6)$$

Now, using the relation of equation (6), equations (1) and (2) can be written as

$$\sigma = S \cdot \gamma, \quad (7a)$$

where,

$$\sigma = (\sigma_{xx} \ \sigma_{yy} \ D_z \ \sigma_{xy})^T, \gamma = (\gamma_{xx} \ \gamma_{yy} \ \Phi_z \ \Theta \ \gamma_{xy})^T \text{ and } S = \begin{pmatrix} C_1 & C_2 & -\tilde{e}_{31} & \psi_1 & 0 \\ C_2 & C_1 & -\tilde{e}_{31} & \psi_1 & 0 \\ \tilde{e}_{31} & \tilde{e}_{31} & \varkappa_{3p} & \psi_D & 0 \\ 0 & 0 & 0 & 0 & (C_1 - C_2) \end{pmatrix}, \quad (7b)$$

the expression of each term in the matrix are defined in the appendix.

Now substitute the value of D_z from equation (7a) into the equation of motion (5) by considering $\Phi_z = -\partial_z\phi$, we can obtain

$$\kappa_{3p}\partial_{zz}\phi = -\tilde{e}_{31}\nabla^2 w - \psi_D\partial_z\Theta, \quad (8)$$

where, $\nabla^2 \equiv \partial_{xx} + \partial_{yy}$.

In the considered model the thermal effect of the thin plate is governed by the MGT thermoelasticity theory which can be written as [49]

$$\rho c_v(1 + T_0\partial_t)\partial_{tt}\Theta = -\psi_{11}^*\tau_0(1 + T_0\partial_t)\partial_{tt}\gamma u + (k_1\partial_t + k_1^*)\partial_{ll}\Theta + (k_3\partial_t + k_3^*)\partial_{zz}\Theta, \quad l = x, y. \quad (9)$$

3 Analytical solution

3.1 Solution for the thermal equation

If we assume $\partial_{ll}\Theta \ll \partial_{zz}\Theta$, then the equation (9) becomes

$$\rho c_v(1 + T_0\partial_t)\partial_{tt}\Theta = -\psi_{11}^*\tau_0(1 + T_0\partial_t)\partial_{tt}\gamma u + (k_3\partial_t + k_3^*)\partial_{zz}\Theta. \quad (10)$$

Also, let us consider the solution of the above equation to be of the form

$$w = w_0(x, y)e^{-i\omega t} \text{ and } \Theta = \Theta_0(x, y, z)e^{-i\omega t}, \quad (11)$$

where, ω represent the angular or circular frequency of the bending edge wave.

Then after substituting equation (11) into the equation (10) and simplifying, we obtained a second order non-homogeneous ordinary differential equation in thermal deviation, which can be written as

$$\partial_{zz}\Theta_0 + \zeta_1^2\Theta_0 = z\zeta_2\nabla^2 w_0, \quad (12)$$

where, $\zeta_1^2 = \frac{\tilde{\zeta}_1^2}{h^2}$ and $\zeta_2 = \frac{\tau_0}{h^2}\tilde{\zeta}_2$, with $\tilde{\zeta}_1^2 = \frac{(-1+i\tilde{T}_0\Omega)\Omega^2}{\tilde{\alpha}(k^*-i\Omega)}$, $\tilde{\zeta}_2 = \tilde{\psi}_{kc}\delta\psi_*\frac{(1-i\tilde{T}_0\Omega)\Omega^2}{(k^*-i\Omega)}$ and $\delta\psi_* = (1 - i\omega\delta_{11}) + \nu_{21}(1 - i\omega\delta_{12}) + \nu_{31}(1 - i\omega\delta_{13})\alpha_{31}$. The dimensionless quantities used above are defined in the appendix. Let us consider the thermal boundary condition as

$$\left. \begin{aligned} \partial_z\Theta_0 &= f_0e^{i\omega t}, & z &= h, \\ &= 0, & z &= -h, \end{aligned} \right\} \quad (13)$$

where, f_0 is external thermal heat. The solution of the equation (12) subjected to the boundary condition (13) can be obtained as

$$\Theta(x, y, z, t) = \Theta_0e^{-i\omega t} = \frac{1}{2\zeta_1^3} \left((f_0\zeta_1^2 - 2\zeta_2\nabla^2 w) \frac{\sin(z\zeta_1)}{\cos(h\zeta_1)} - f_0\zeta_1^2 \frac{\cos(z\zeta_1)}{\sin(h\zeta_1)} + 2z\zeta_1\zeta_2\nabla^2 w \right). \quad (14)$$

3.2 Solution for the piezoelectric potential

Let us consider that an electrode covers the upper surface of the plate and is electrically shorted on the lower surface. Then, the corresponding piezoelectric boundary conditions at both surfaces of the plate can be written as [47]

$$\left. \begin{aligned} \phi &= \tilde{V}, & z &= h, \\ &= 0, & z &= -h. \end{aligned} \right\} \quad (15)$$

Now, the solution of equation (8) can be obtained by substituting equation (14) together with the piezoelectric boundary condition (15). In this way, the explicit form of the piezoelectric potential can be obtained as

$$\phi(x, y, z, t) = \frac{1}{2h\zeta_1^4\kappa_{3p}}(\tilde{e}_{31}h\zeta_1^4\nabla^2 w(h^2 - z^2) - \psi_{De}(f_0\zeta_1^2(h+z) + h\zeta_2\nabla^2 w(\zeta_1^2(z^2 - h^2) - 2)) + \psi_{De}h\frac{\cos(\zeta_1 z)}{\cos(h\zeta_1)}(f_0\zeta_1^2 - 2\zeta_2\nabla^2 w) + \psi_{De}f_0h\zeta_1^2\frac{\sin(\zeta_1 z)}{\sin(h\zeta_1)} + \zeta_1^4V_0(h+z)\kappa_{3p}). \quad (16)$$

The bending moment for the considered model will be of the form

$$M_{ij} = \int_{-h}^h z\sigma_{ij}dz, \quad i, j = x, y. \quad (17)$$

Next, substitute equations (7a), (14) and (16) into the equation (17) and after simplifying we get the bending and twisting moment in dimensionless form as

$$M_{\xi\xi} = c_{11}h^2 \left(M_1\partial_{\xi\xi}U + M_2\partial_{\eta\eta}U - \frac{\tilde{e}_3fe_{3D}}{\tilde{\zeta}_1^3\kappa_D}\Psi_D(-\tilde{\zeta}_1 + \tan(\tilde{\zeta}_1)) - \frac{\psi_{1D}}{\tilde{\zeta}_1^3}\tilde{\alpha}_1f(\tan(\tilde{\zeta}_1) - \tilde{\zeta}_1) \right), \quad (18)$$

$$M_{\eta\eta} = c_{11}h^2 \left(M_2\partial_{\xi\xi}U + M_1\partial_{\eta\eta}U - \frac{\tilde{e}_3fe_{3D}}{\tilde{\zeta}_1^3\kappa_D}\Psi_D(-\tilde{\zeta}_1 + \tan(\tilde{\zeta}_1)) - \frac{\psi_{1D}}{\tilde{\zeta}_1^3}\tilde{\alpha}_1f(\tan(\tilde{\zeta}_1) - \tilde{\zeta}_1) \right), \quad (19)$$

$$M_{\xi\eta} = C_{11}h^2(M_1 - M_2)\partial_{\xi\eta}U, \quad (20)$$

where,

$$M_1 = -\frac{2}{3}\tilde{c}_1 - \frac{2}{3}\tilde{e}_3\frac{e_{3D}^2}{\kappa_D} + 2\frac{\tilde{e}_3\tilde{\tau}_0}{\tilde{\zeta}_1^5\kappa_D}\Psi_D\tilde{\zeta}_2e_{3D}\left(\frac{-\tilde{\zeta}_1(3 + \tilde{\zeta}_1^2)}{3} + \tan(\tilde{\zeta}_1)\right) + 2\left(-\frac{\tilde{\zeta}_1^3}{3} - \tilde{\zeta}_1 + \tan(\tilde{\zeta}_1)\right)\tilde{\alpha}_1\tilde{\tau}_0\psi_{1D}\frac{\tilde{\zeta}_2}{\tilde{\zeta}_1^5}, \quad (21)$$

$$M_2 = -\frac{2}{3}\mu\tilde{c}_2 - \frac{2}{3}\tilde{e}_3\frac{e_{3D}^2}{\kappa_D} + 2\frac{\tilde{e}_3\tilde{\tau}_0}{\tilde{\zeta}_1^5\kappa_D}\Psi_D\tilde{\zeta}_2e_{3D}\left(\frac{-\tilde{\zeta}_1(3 + \tilde{\zeta}_1^2)}{3} + \tan(\tilde{\zeta}_1)\right) + 2\left(-\frac{\tilde{\zeta}_1^3}{3} - \tilde{\zeta}_1 + \tan(\tilde{\zeta}_1)\right)\tilde{\alpha}_1\tilde{\tau}_0\psi_{1D}\frac{\tilde{\zeta}_2}{\tilde{\zeta}_1^5}, \quad (22)$$

in which the above dimensionless quantities are defined in the appendix.

In-plane force resultant for the thin elastic plate can be obtained by using the relation given by

$$N_{ii} = \int_{-h}^h \sigma_{ii}dz, \quad i = x, y. \quad (23)$$

By using equation (7a) the expression of resultant in-plane force of equation (23) can be obtained as

$$N_{ii} = \left(Ve_{3D} + \frac{\psi_{1D}\tilde{\alpha}_1f}{\tilde{q}_1^2} \right) C_{11}h, \quad i = \xi, \eta. \quad (24)$$

The first term of equation (5) can be written in terms of bending moment and resultant in-plane force under the external force acting at the lower surface of the plate by means of an elastic foundation, which is given by

$$\partial_{xx}M_{xx} + 2\partial_{xy}M_{xy} + \partial_{yy}M_{yy} + \partial_x(N_{xx}\partial_xw) + \partial_y(N_{yy}\partial_yw) + (\tilde{\beta}w - \tilde{k}_s\nabla^2w) = \rho h\partial_{tt}w - \frac{\rho h^3}{12}\partial_{tt}\nabla^2w, \quad (25)$$

with $\tilde{\beta}$ and $\tilde{k}_s$ are the Winkler and shear layer modulus of the foundation, respectively. Let us consider the displacement will be of the form (dimensionless) $U(\xi, \eta, \tau) = u(\xi, \eta)e^{-i\Omega\tau}$, then the dimensionless form of the equation of motion becomes

$$M_1\nabla^4u - \left(\frac{\tilde{e}_3^2}{\tilde{\zeta}_1^3\kappa_D}\Psi_D\tilde{\zeta}_2e_{3D} + \frac{\psi_{1D}\tilde{\alpha}_1}{\tilde{\zeta}_1^3} \right) (-\tilde{\zeta}_1 + \tan(\tilde{\zeta}_1))\nabla^2f + \frac{\psi_{1D}\tilde{\alpha}_1}{\tilde{\zeta}_1^2}(\partial_{\xi}f\partial_{\xi}u + \partial_{\eta}f\partial_{\eta}u) + \left(Ve_{3D} + \frac{\psi_{1D}\tilde{\alpha}_1f}{\tilde{\zeta}_1^2} \right) \nabla^2u - \frac{(\beta u - k_s\nabla^2u)}{C_{11}} = -\Omega^2u + \frac{1}{12}\Omega^2\nabla^2u, \quad (26)$$

If f is constant then the above equation can be reduced to the form

$$M_1 \nabla^4 u + a \nabla^2 u + bu = 0, \quad (27)$$

where, $a = \left(V e_{3D} + \frac{\psi_{1D} \tilde{\alpha}_1 f}{\tilde{\zeta}_1^2} - \frac{1}{12} \Omega^2 + k_s \right)$ and $b = (-\beta + \Omega^2)$.

If a time harmonic external point-load acts at the vicinity of the free edge of the plate at (x_0, y_0) in the form $P \Delta(x - x_0) \Delta(y - y_0) e^{-i\omega t}$, then the above equation can be written as

$$M_1 \nabla^4 u + a \nabla^2 u + bu = P \delta(\xi - \xi_0) \delta(\eta - \eta_0), \quad |\xi_0| < \infty \text{ and } 0 \leq \eta_0, \quad (28)$$

where, $\delta(\cdot)$ is the Dirac delta function.

The mechanical boundary conditions at the free edge for the considered model will be of the form

$$\left. \begin{aligned} M_2 \partial_{\xi\xi} u + M_1 \partial_{\eta\eta} u &= A, \\ M_1 \partial_{\eta\eta\eta} u + (2M_1 - M_2) \partial_{\xi\xi\eta} u &= 0, \end{aligned} \right\} \quad (29)$$

where, $A = \left(\frac{\tilde{e}_{3D}^2 f}{\tilde{\zeta}_1^3} \Psi_D e_{3D} + \frac{\psi_{1D} \tilde{\alpha}_1 f}{\tilde{\zeta}_1^2} \right) (\tan(\tilde{\zeta}_1) - \tilde{\zeta}_1)$.

3.3 Solution for the external load

To generate the analytical solution to the boundary value problem, the Fourier transform technique to the ξ variable is implemented. Thus, the equation of motion (28) and the boundary conditions (29) will be of the form.

$$a_1 \partial_{\eta\eta\eta} u^F - a_2 \partial_{\eta\eta} u^F - a_3 u^F = P_1 e^{i\xi_0 p} \delta(\eta - \eta_0). \quad (30)$$

and

$$\left. \begin{aligned} M_1 \partial_{\eta\eta} u^F - p^2 M_2 u^F &= A \delta(p), \\ M_1 \partial_{\eta\eta\eta} u^F - p^2 (2M_1 - M_2) \partial_{\eta} u^F &= 0, \end{aligned} \right\} \quad (31)$$

where, $a_1 = M_1$, $a_2 = 2p^2 M_1 - a$, $a_3 = ap^2 - b - M_1 p^4$ ($\mathcal{F}(\xi) = p$, is the Fourier transform variable), $\mathcal{F}(1) = \delta(p)$, $\mathcal{F}(u) = u^F$ and $P_1 = \frac{P}{\sqrt{2\pi}}$.

The Green function technique is used to solve this non-homogeneous equation (30) together with the non-homogeneous boundary conditions (31) and the radiation condition of the edge wave, i.e., $u^F \rightarrow 0$ as $\eta \rightarrow \infty$. Therefore, the Green function corresponding to the equation (30) can be written as

$$\mathcal{G}(\eta, v) = \left. \begin{aligned} &= \frac{(e^{-\lambda_1 \eta} - u_{c1} e^{-\lambda_2 \eta})(e^{-\lambda_1 v} - u_{c2} e^{-\lambda_2 v})}{a_1 (\lambda_1 - \lambda_2) e^{-(\lambda_1 + \lambda_2) \eta} (u_{c1} - u_{c2})}, & 0 \leq \eta < v, \\ &= \frac{(e^{-\lambda_1 v} - u_{c1} e^{-\lambda_2 v})(e^{-\lambda_1 \eta} - u_{c2} e^{-\lambda_2 \eta})}{a_1 (\lambda_1 - \lambda_2) e^{-(\lambda_1 + \lambda_2) \eta} (u_{c1} - u_{c2})}, & v \leq \eta, \end{aligned} \right\} \quad (32)$$

where $u_{c1} = \frac{M_1 \lambda_1^2 - p^2 M_2}{M_1 \lambda_2^2 - p^2 M_2}$, $u_{c2} = \frac{\lambda_1 (-M_1 \lambda_1^2 + p^2 (2M_1 - M_2))}{\lambda_2 (-M_1 \lambda_2^2 + p^2 (2M_1 - M_2))}$ and

$$\lambda_i = \sqrt{\frac{a_2 + (-1)^i \sqrt{a_2^2 + 4a_1 a_3}}{2a_1}}, \quad (i = 1, 2) \text{ with } Re(\lambda_i) > 0. \quad (33)$$

The general solution of the non-homogeneous boundary value problem (equations (30-31)) is given by

$$u^F = B e^{-\lambda_1 \eta} + \frac{A \delta(p) - B (M_1 \lambda_1^2 - p^2 M_2)}{(M_1 \lambda_2^2 - p^2 M_2)} e^{-\lambda_2 \eta} + \mathcal{G}(\eta, \eta_0), \quad (34)$$

B is an arbitrary constant.

3.4 Dispersion analysis

We can obtain relation between the values of λ from equation (33) which can be expressed as

$$\left. \begin{aligned} \lambda_1^2 + \lambda_2^2 &= p^2 \left(2 - \frac{a}{p^2 M_1} \right), \\ (\lambda_1 \lambda_2)^2 &= p^4 \left(1 + \frac{b - ap^2}{p^4 M_1} \right). \end{aligned} \right\} \quad (35)$$

The dispersion relation for bending edge wave can be obtained from equation (34) by equating zero to the denominator of u^F . Then, after simplifying, we can obtain

$$(\lambda_1 \lambda_2)^2 = p^4 (1 - \Gamma_+^4), \quad (36)$$

where, $\Gamma_+^4 = (1 - \chi) \left(3(\chi - 1) + 2 - \frac{a}{M_1 p^2} \frac{\chi}{\chi - 1} + 2\sqrt{2\chi^2 - 2\chi + 1 - \frac{a\chi}{M_1 p^2}} \right)$ and $\chi = \frac{M_2}{M_1}$.

Therefore, the dispersion relation takes the form

$$b + \Gamma_+^4 p^4 M_1 = ap^2. \quad (37)$$

After some algebraic manipulation, we can rewrite the dispersion relation as

$$M_1(1 - \chi)(3(\chi - 1) + 4)p^4 - (\tilde{a} - k_s)p^2 - \frac{p^2 \Omega^2}{12} - \beta + \Omega^2 = 0, \quad (38)$$

where, $\tilde{a} = a - \tilde{k}_g + \frac{\Omega^2}{12}$. This equation holds if $\Gamma_1 = \sqrt{1 + 2\chi^2 - 2\chi - \frac{a\chi}{M_1 p^2}} \sim 1$ and $\Gamma_2 = \frac{a\chi}{M_1 p^2(\chi - 1)} \sim 0$ for the higher values of p .

Therefore, from the equation (38), we can obtain a piezoelectric viscoelastic beam equation under a thermal environment, which can be written as

$$M_1(1 - \chi)(3(\chi - 1) + 4)\partial_{\xi\xi\xi\xi}u - \tilde{a}\partial_{\xi\xi}u + \frac{1}{12}\partial_{\xi\xi\tau\tau}u - \beta u + k_s\partial_{\xi\xi}u - \partial_{\tau\tau}u = 0. \quad (39)$$

The equation (39) implies that under the wave form $u(\xi, \eta, \tau) = g(\eta)e^{i(p\xi - \Omega\tau)}$, the nature of a thermally affected plate under the bending edge wave is similar to the wave propagation on a thermally affected beam supported by a Pasternak elastic foundation. The equation (39) is similar to the equation obtained by Kaplunov and Nobili [50] for an isotropic elastic plate with Pasternak elastic foundation, except for the complex coefficient of the derivative operator.

3.4.1 Reduction to the isotropic plate with Winkler foundation

The obtained dispersion relation can be reduced to the isotropic model by substituting $c_{11} = c_{33} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$, $c_{12} = C_{13} = \frac{E\nu}{(1+\nu)(1-2\nu)}$, $a = 0$, then $\chi = \nu$ and $M_1 = -\frac{2}{3}\frac{E}{(1-\nu)}$, where E and ν are the Young's modulus and Poisson's ratio respectively. Then the dispersion relation (37) will take the form

$$\Omega^2 - \beta = -M_1 p^4 \Gamma_+^4, \quad (40)$$

which is similar to the dispersion relation obtained by Kaplunov et al. [36].

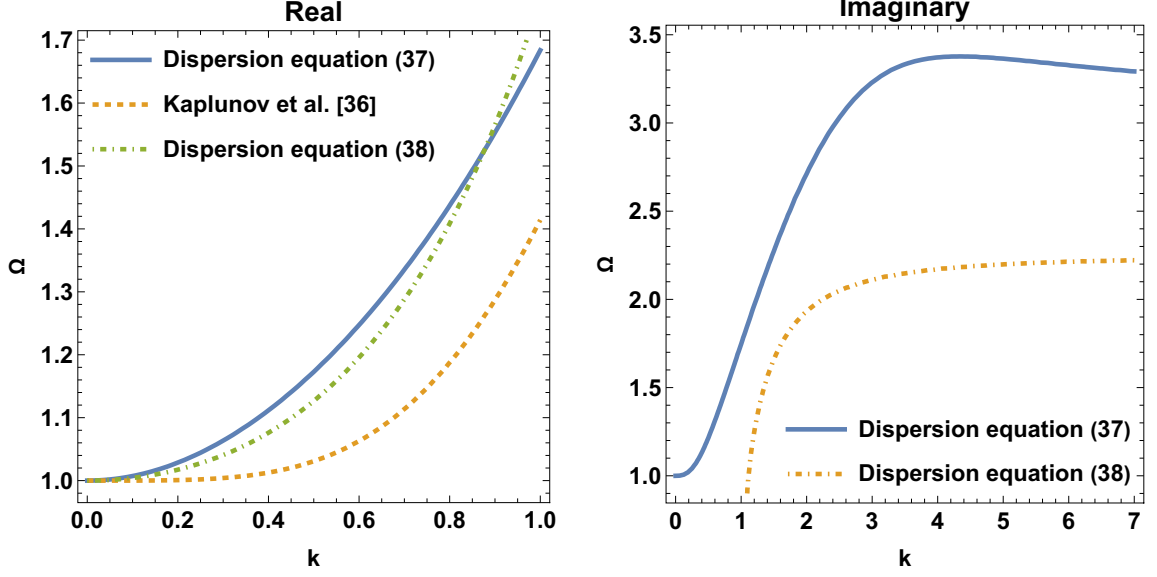


Figure 2: Dispersion portrait of edge wave (cf. equation (37))

3.4.2 Reduction of the Kononkov-type constant for transversely isotropic plate

If we remove the piezoelectric and thermoelastic properties from the model then, the Kononkov-type constant for edge wave propagation on a transversely isotropic viscoelastic plate will be of the form given by

$$\Gamma_{\pm t}^4 = (1 - \chi) \left(3(\chi - 1) + 2 - \frac{a}{M_1 p^2} \frac{\chi}{\chi - 1} \pm 2\sqrt{2\chi^2 - 2\chi + 1 - \frac{a\chi}{M_1 p^2}} \right), \quad \chi = \frac{M_2}{M_1}, \quad (41)$$

where, $a = k_s - \frac{1}{12}\Omega^2$, $M_1 = -\frac{2}{3}\tilde{c}_1$ and $M_2 = -\frac{2}{3}\tilde{c}_2\mu$.

4 Graphical descriptions

The analytical solutions are evaluated numerically, and represented by graphs in this section, where a particular material is considered for illustrative purpose, to investigate the influence of the mechanical conditions used in the model. Here, a complex form of wavenumber in the form $p = k(1 + i\vartheta)$ is considered. Graphical representation is implemented using the Mathematica software by considering some particular values of the dimensionless mechanical parameters of the medium, and they are listed below [51]:

$c_{11} = 74.1 \times 10^9 \text{ N/m}^2$, $c_{12} = 30.8 \times 10^9 \text{ N/m}^2$, $c_{13} = 39.3 \times 10^9 \text{ N/m}^2$, $c_{33} = 83.6 \times 10^9 \text{ N/m}^2$, $e_{31} = -0.16 \text{ C/m}^2$, $e_{33} = 0.347 \text{ C/m}^2$, $\rho = 5504 \text{ Kg/m}^3$, $h = 0.01 \text{ m}$, $\frac{\rho c_v}{k_3} = 10^{-6} \text{ m}^2/\text{s}$, $T_0 = 0.1 \text{ s}$, $\alpha_1 = 6.21 \times 10^5 \text{ 1/K}$, $\alpha_{33} = 5.15 \times 10^5 \text{ 1/K}$, $\psi_{De} = 170 \text{ C/m}^2 \cdot \text{K}$, $\psi_{De} = 170 \text{ C/m}^2 \cdot \text{K}$, $f = 0.8$, $V = 0.8$, $\beta = 1$, $k_s = 0.2$, $\tilde{\psi}_{kc} = 0.5$, $\delta_{11} = 0.02$, $\delta_{12} = 0.5$, $\delta_{13} = 0.01$, $\delta_{33} = 0.4$, $\delta_{e31} = 0.01$, $\delta_{e33} = 0.09$, $\tilde{k}^* = 0.5$, $\tau_0 = 0.2$, $\xi_0 = 1$, $\eta_0 = 0$, $\eta = 0$, $P = 1$, $\delta_{\kappa 33} = 0.2$, $\delta_{\psi D} = 0.2$, $\vartheta = 0.01$.

Figure [2] represents the behaviour of frequency for the thermally affected thin plate. An opposite behaviour of the frequency is observed for real and imaginary parts of the dispersion relation, where the imaginary part represents damping frequency of the bending edge wave on the plate. Viscosity in

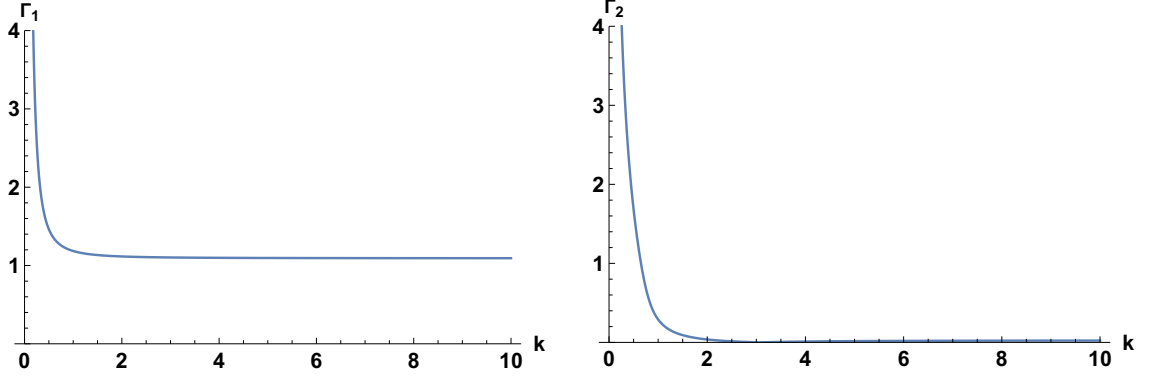


Figure 3: Approximating the beam like behaviour for $\Gamma_1 \sim 1$ and $\Gamma_2 \sim 0$

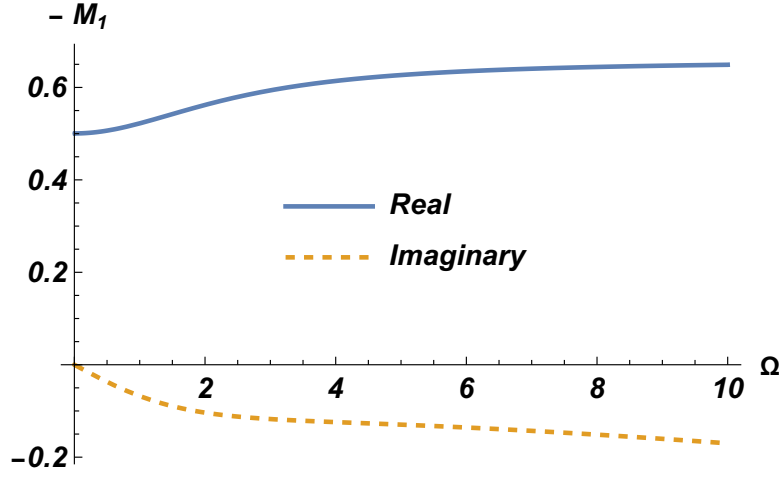


Figure 4: M_1 vs angular frequency (Ω)

the model leads to a decreasing dispersion curve for large wavenumber values in the imaginary case. The cut-off frequency generated due to the presence of the Winkler spring layer is the same as that of the isotropic case [36], which only depends upon the spring modulus. This implies that the cut-off frequency of the bending edge wave in the case of a thermally affected piezoelectric plate is not affected by the mechanical conditions. Also, when compared with an isotropic plate, the frequency of bending edge waves increases particularly with increasing wave number.

Figure [3] describes the behaviour of the functions Γ_1 and Γ_2 which have been considered to generate the beam dispersion relation. It can be observed from the figure that bounds of the functions Γ_1 and Γ_2 are obtained for the change in the values of the wavenumber, i.e., these two functions converge to 1 and 0 respectively for lower values of the wavenumber.

Figure [4] illustrates the behaviour of the rigidity of the thermally affected piezoelectric plate over the different frequencies. The dependency of the rigidity of this model explicitly upon the wave frequency is due to the consideration of the viscosity and thermoelasticity in the model. It can be ascertained from the figure that the frequency increases with decrease in rigidity in real cases but increases in imaginary cases.

From equation (34) we can obtain two values of Γ such as

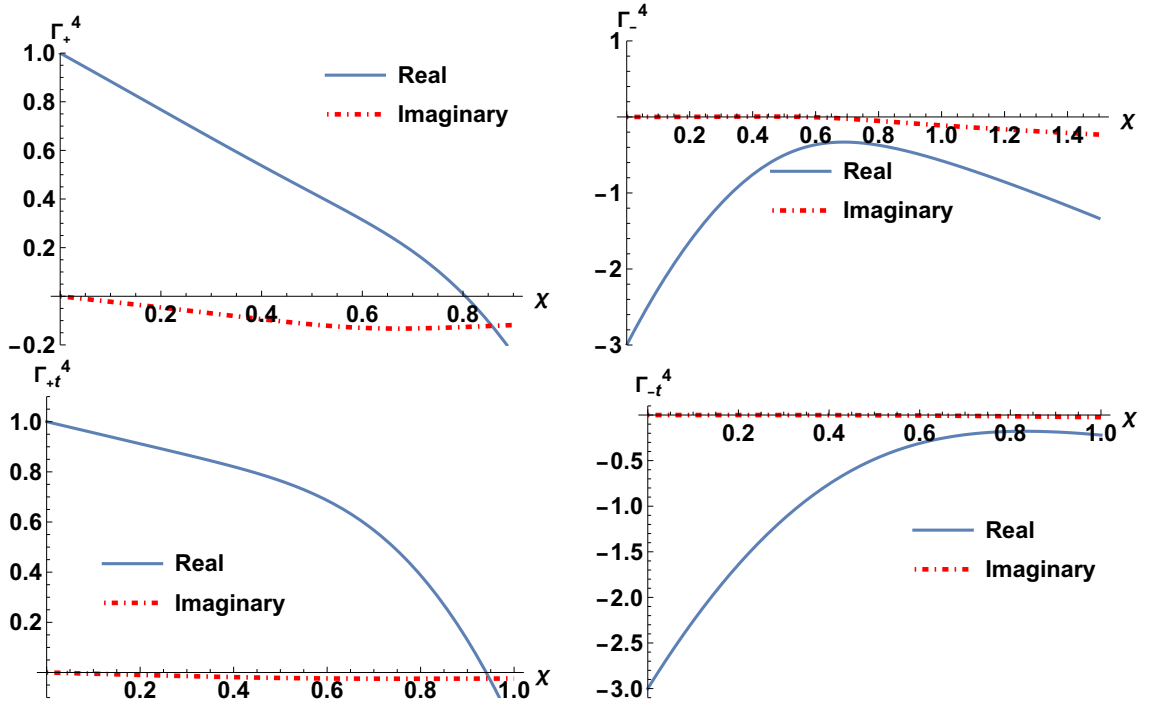


Figure 5: Behaviour of Konenkov-type constant for the considered model ($\Gamma_{\pm}^4$ and $\Gamma_{\pm t}^4$)

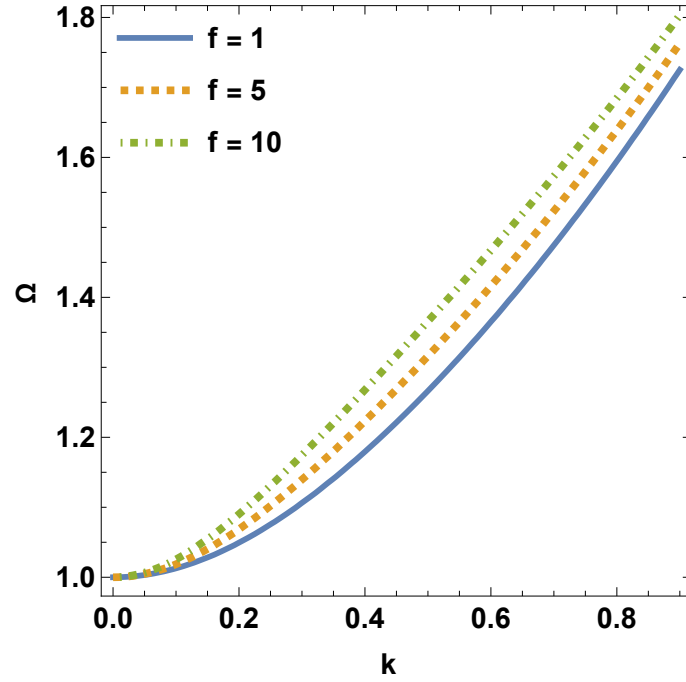


Figure 6: Effect of the external heat (f) in the dispersion of edge wave for $\frac{\rho c_v}{k_3} = 10^{-2} \text{ m}^2/\text{s}$, $T_0 = 10^{-4} \text{ s}$, $\tilde{k}^* = 10$ and $\tau_0 = 10$

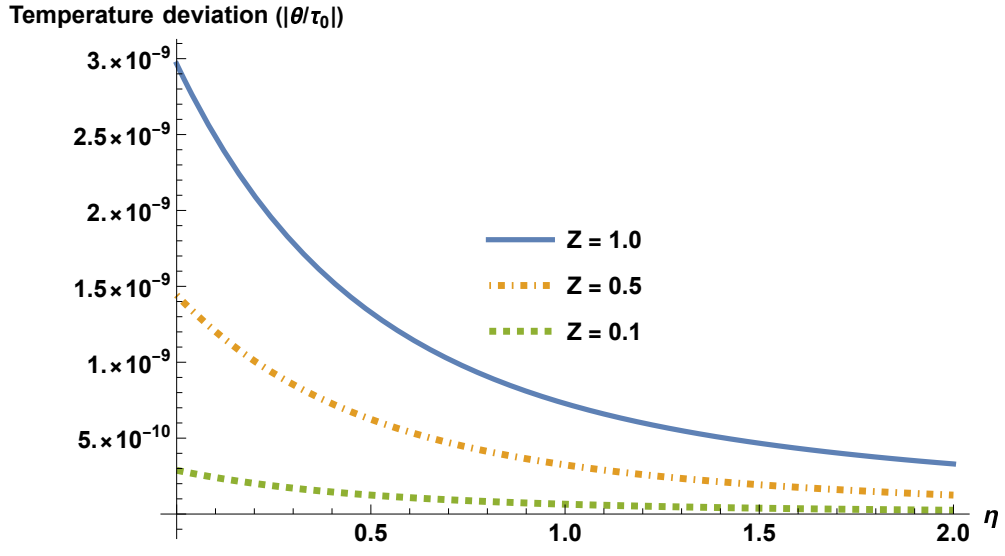


Figure 7: Temperature deviation along η direction for bending edge wave for $T_0 = 0.01$ s, $\Omega = 2$, $\tilde{k}^* = 0.001$ and $\tau_0 = 2$, $\xi_0 = 0$, $\eta_0 = 0$, and $k = 1.3$

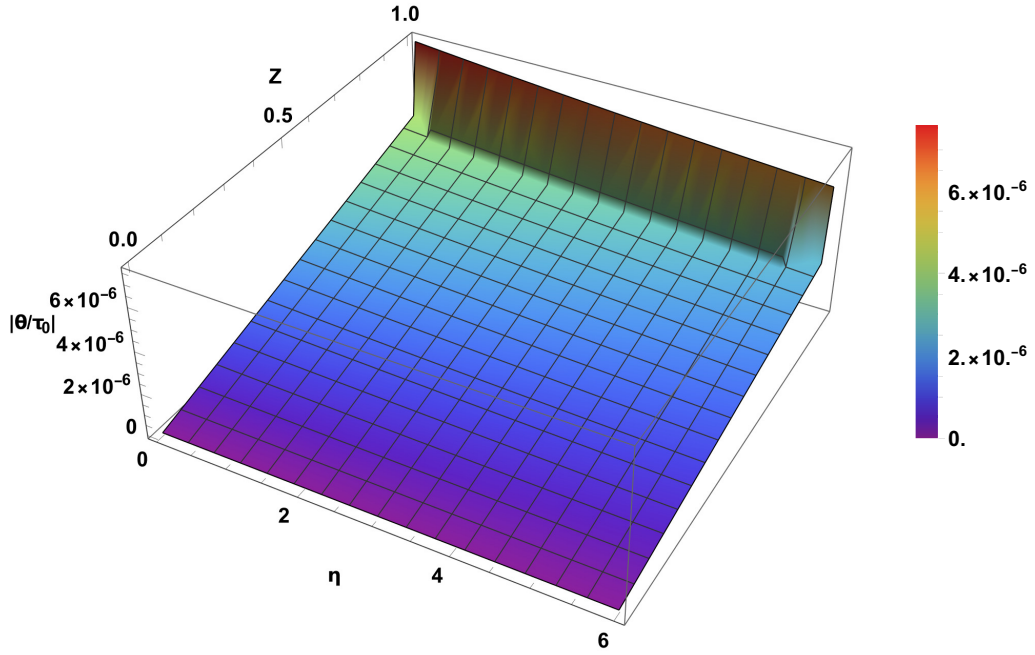


Figure 8: Temperature distribution along transverse and lateral direction for $T_0 = 10^{-4}$, $h = 0.001$ m, $\frac{\rho c_v}{k_3} = 0.3 \times 10^{-4}$ m²/s, $\Omega = 1.8$, $f = 0.5$, $V = 1$, $\beta = 0.1$, $k = 0.9$, $\xi_0 = 0$, $\eta_0 = 0$, $\tilde{k}^* = 1$ and $\tau_0 = 10$

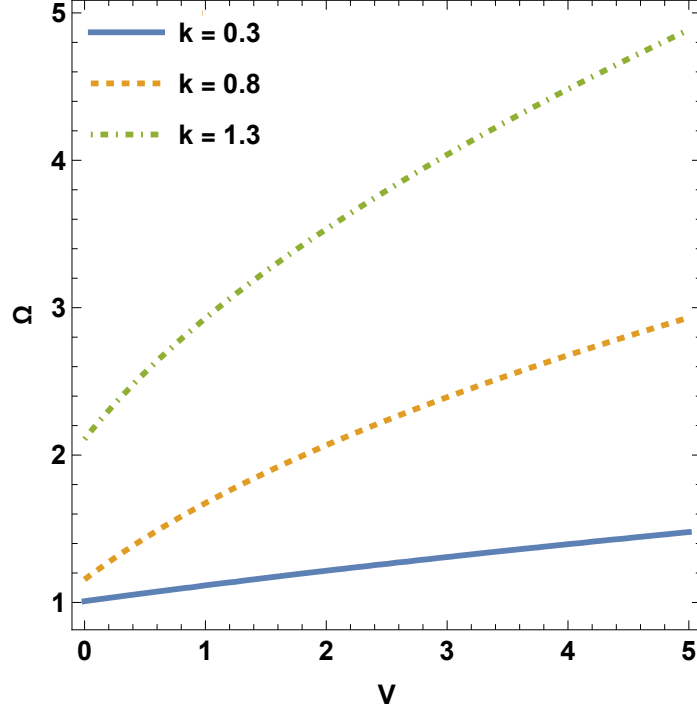


Figure 9: Frequency behaviour due to the presence of electrode-covered (V) upper surface

$$\Gamma_{\pm}^4 = (1 - \chi) \left(3(\chi - 1) + 2 - \frac{a}{M_1 p^2} \frac{\chi}{\chi - 1} \pm 2\sqrt{2\chi^2 - 2\chi + 1 - \frac{a\chi}{M_1 p^2}} \right).$$

The explicit form of this $\Gamma_{\pm}^4$ is investigated and shown in Figure [5]. It is well known that for isotropic plate, the Konenkov constant depends upon the Poisson's ratio only, as described by Norris et al. [52]. However, in the considered model, that constant depends on the piezoelectric, and thermoelastic constants. Here, two cases are considered: (i) where the piezoelectric and thermal effects are considered, and (ii) only a transversely isotropic viscoelastic plate is considered. Since two different values of Γ^4 are obtained from the equation (34), two different plots have been drawn to investigate the applicability of the values of Γ^4 for the considered model. It is clear from the examination that the values of Γ_{-}^4 and Γ_{-t}^4 are not applicable for both cases, as the values of Γ_{-}^4 and Γ_{-t}^4 lie in the negative region.

The effect of the external heat on the dispersion curve of the bending edge wave is presented in Figure [6]. It is observed that a high-frequency edge wave is generated with the increasing value of f , i.e., an external heat contributes to generating the high-frequency bending edge wave.

The temperature deviation along the lateral and transverse directions is analysed in Figure [7,8]. Figure [7] depicts that the temperature deviation along the lateral direction decays exponentially due to the displacement field of the bending edge wave. On the other hand, the temperature variation along the transverse direction is more than the lateral direction.

Since the present study is focused on the piezoelectric material plate, therefore an electrode-covered upper surface of the plate is considered, and the effects of these mechanical properties on the dispersion are investigated and shown in Figure [9]. Since an electrode is added to the upper surface of the materials, then the stiffness of the materials is increased. In the case of the surface wave, an

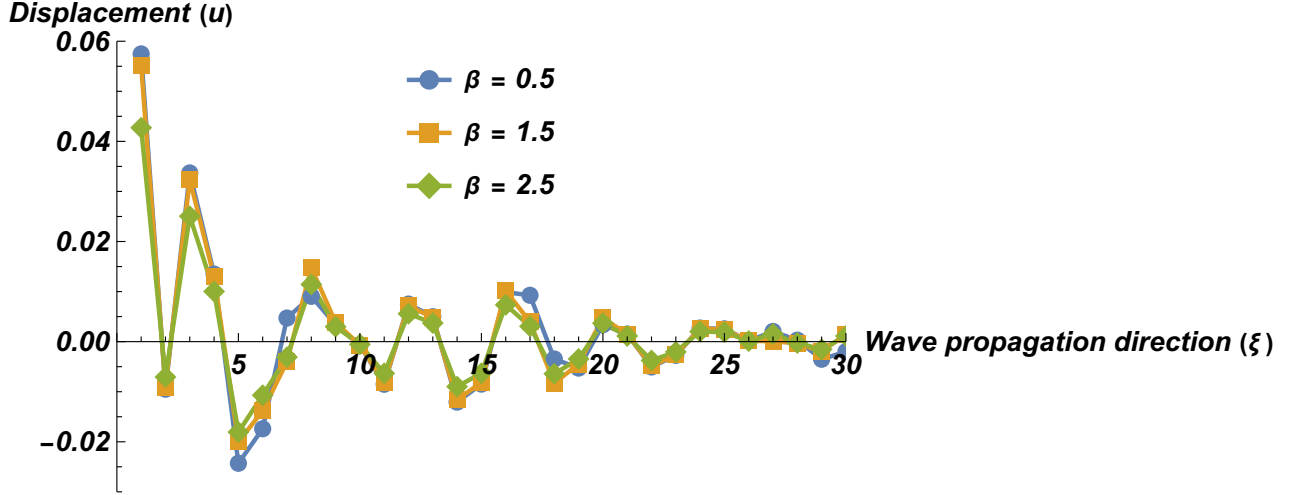


Figure 10: Displacement field with the effect of the Winkler foundation modulus for $\Omega = 1.5$, $\eta = 0$ and $\eta_0 = 0$

increase in the stiffness of the medium increases the frequency of the propagating wave. Thus, if a piezoelectric plate has a high voltage electrode covered, the frequency will increase for different wavenumbers.

The investigation of the amplitude of the displacement of the bending edge wave under the external point load is now carried out. Here, Figure [10], [11] and [12] represent the displacement due to the viscosity, foundation modulus and different load amplitude respectively. A decreasing behaviour of the displacement amplitude of the edge wave is observed for the real part of the displacement component, due to the presence of viscosity in the plate, the amplitude decreases with the increase in the distance measured along the direction of wave propagation. Figure [10,11] shows that the amplitude of the edge wave decreases as the foundation modulus (β and k_s) increases. This implies that if the foundation has a high stiffness, then the amplitude of the edge wave will be decreased. The amplitude of the edge wave will increase if the external point load magnitude is increased. The shear layer of the elastic foundation has less effect than the Winkler spring layer in the case of bending wave propagation on a thermally affected piezoelectric thin plate.

5 Summary

An in-depth mathematical investigation of bending edge waves in a thin piezoelectric plate under a thermal environment has been carried out. A non-homogeneous boundary condition corresponding to the thermal displacement and electric potential is considered in the model. The non-homogeneous boundary value problem obtained from the equation of motion of the thin plate and corresponding free edge boundary conditions is solved analytically with well-known mathematical techniques. The radiation coefficient of the bending edge wave is obtained by solving the complementary solution of the governing differential equation. The following conclusions are drawn after analysing the analytical solution followed by numerical studies and graphical representation:

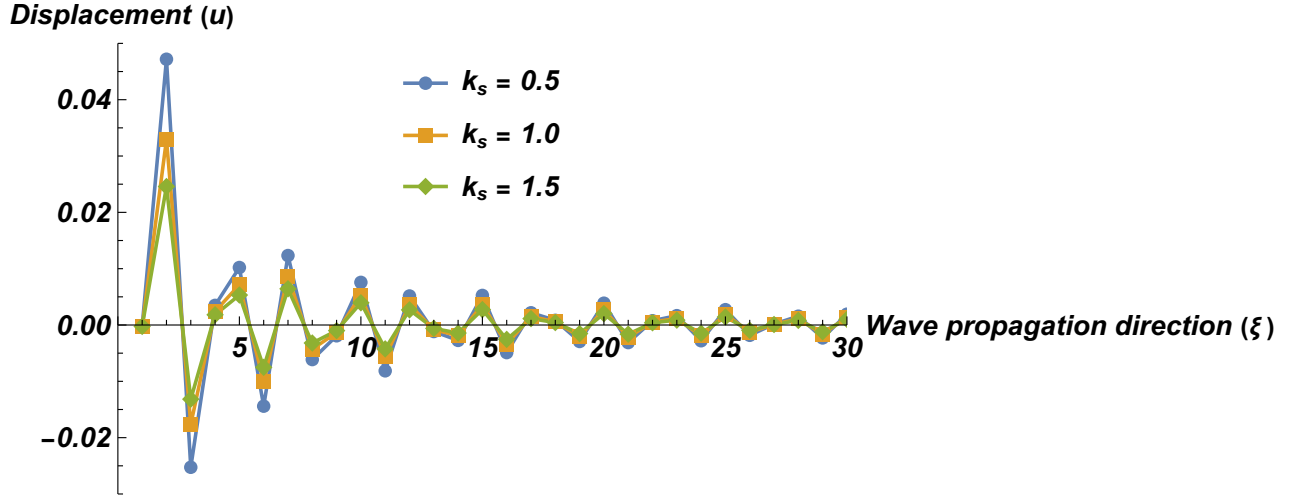


Figure 11: Effect of the displacement field due to the presence of shear layer (k_s) in the foundation for $\Omega = 1.5$, $\eta = 0$ and $\eta_0 = 0$

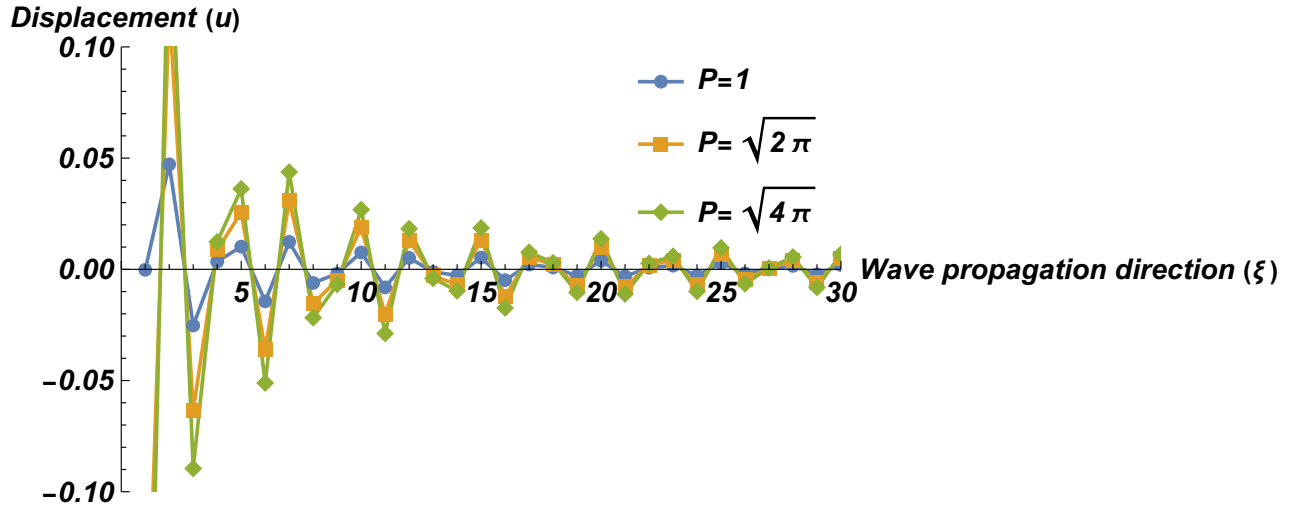


Figure 12: Displacement field under the different magnitude of the external load for $\Omega = 1.5$, $\eta = 0$ and $\eta_0 = 0$

- An increase in the temperature causes the frequency of bending edge waves increase and damping frequencies decrease in thin plates. Also, external heat contributes to the generation of high-frequency edge waves.
- The behaviour of Konenkov-type constant is similar to that of the isotropic plate except for some specific ranges for the values of χ .
- The characteristics of the bending edge wave at the edge of a thin plate is similar to the wave propagation of an elastic beam under the same mechanical conditions.
- The nature of the temperature variation along the lateral direction of the plate is similar to the displacement of the bending edge wave.
- The propagation frequency will increase when the voltage applied to the covered electrode increases.
- Wave energy decreases slowly in the direction of wave propagation.
- Foundation rigidity reduces amplitude of the bending edge wave.
- The magnitude of the load directly impacts the energy of the wave, leading to a rise in the amplitude of the wave.

The investigation carried out is significant, particularly when studying the scattering of the bending edge wave to detect any damage that occurred in the kinetic ramp or road pavement [53, 54] and any plate foundation structure made of piezoelectric materials.

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Appendix

Material constants:

$$\begin{aligned}
C_1 &= c_{11} \left[(1 - i\Omega\delta_{11}) - \frac{(1 - i\Omega\delta_{13})^2}{(1 - i\Omega\delta_{33})} S_{11} \right], \quad C_2 = c_{12} \left[(1 - i\Omega\delta_{12}) - \frac{(1 - i\Omega\delta_{13})^2}{(1 - i\Omega\delta_{33})} S_{12} \right], \\
\tilde{e}_{31} &= e_{31} \left[(1 - i\Omega\delta_{e31}) - \frac{(1 - i\Omega\delta_{e33})(1 - i\Omega\delta_{13})}{(1 - i\Omega\delta_{33})} S_{e31} \right], \quad \varkappa_{3p} = \varkappa_{33} \left[(1 - i\Omega\delta_{\varkappa33}) - \frac{(1 - i\Omega\delta_{e33})^2}{(1 - i\Omega\delta_{33})} S_{ek} \right], \\
\psi_1 &= \alpha_1 C_{11} \left[(1 - i\Omega\delta_{11}) + \nu_{21}(1 - i\Omega\delta_{12}) + \nu_{31}(1 - i\Omega\delta_{13})\alpha_{31} - S_{11} \frac{(1 - i\Omega\delta_{13})}{(1 - i\Omega\delta_{33})} (2(1 - i\Omega\delta_{13}) + \nu_{33}(1 - i\Omega\delta_{33})\alpha_{31}) \right], \\
\psi_D &= \psi_{De} \left[(1 - i\Omega\delta_{\psi D}) - S_{\psi D} \frac{(1 - i\Omega\delta_{e33})}{(1 - i\Omega\delta_{33})} (2(1 - i\Omega\delta_{13}) + \nu_{33}(1 - i\Omega\delta_{33})\alpha_{31}) \right], \\
&\text{where, } C_1 = c_{11}\tilde{c}_1, \quad C_2 = c_{12}\tilde{c}_2, \quad \tilde{e}_{31} = e_{31}\tilde{e}_{3D}, \quad \varkappa_{3p} = \varkappa_{33}\varkappa_D, \quad \psi_1 = C_{11}\alpha_1\psi_{1D}, \quad \psi_D = \psi_{De}\Psi_D.
\end{aligned}$$

Dimensionless quantities:

$$\begin{aligned}\tilde{\alpha} &= \frac{\rho c_v \sqrt{\rho}}{k_3 h \sqrt{c_{11}}}, \tilde{\psi}_{kc} = \frac{c_{11} \alpha_1 h}{\kappa_3} \sqrt{\frac{c_{11}}{\rho}}, \tilde{k}^* = \frac{k_3^* h \sqrt{\rho}}{k_3 \sqrt{c_{11}}}, \Omega^2 = \frac{h^2 \rho \omega^2}{c_{11}}, \tau = \frac{t}{h} \sqrt{\frac{c_{11}}{\rho}}, \tilde{T}_0 = \frac{T_0}{h} \sqrt{\frac{c_{11}}{\rho}}, \tilde{\tau}_0 = \frac{\psi_{De} \tau_0}{e_{31}}, \\ f &= \frac{\psi_{De} f_0 h}{e_{31}}, V = \frac{\tilde{e}_{31} \tilde{V}}{c_{11} h}, \tilde{\alpha}_1 = \frac{e_{31} \alpha_1}{\psi_{De}}, \beta = \frac{\tilde{\beta} h}{c_{11}}, k_s = \frac{\tilde{k}_s}{h c_{11}}, \xi = \frac{x}{h}, \eta = \frac{y}{h}, Z = \frac{z}{h}, \mu = \frac{c_{12}}{c_{11}}, U = \frac{w}{h}, \\ \tilde{e}_3 &= \frac{e_{31}^2}{\kappa_3 c_{11}}, \delta_{ij} = \frac{\eta_{ij}}{h \sqrt{\rho c_{ij}}}, \delta_{e31} = \frac{\eta_{e31}}{e_{31} h \sqrt{\rho c_{11}}}, \delta_{e33} = \frac{\eta_{e33}}{e_{33} h \sqrt{\rho c_{11}}}, \delta_{\kappa_{33}} = \frac{\eta_{\kappa_{33}}}{\kappa_{33} h \sqrt{\rho c_{11}}}, S_{11} = \frac{C_{13}^2}{C_{11} C_{33}}, \\ S_{12} &= \frac{C_{13}^2}{C_{12} C_{33}}, S_{e31} = \frac{C_{13} e_{33}}{e_{31} C_{33}}, S_{ek} = \frac{e_{33}^2}{k_{33} C_{33}}, \nu_{21} = \frac{C_{12}}{C_{11}}, \nu_{31} = \frac{C_{13}}{C_{11}}, \nu_{33} = \frac{C_{33}}{C_{13}}, \alpha_{31} = \frac{\alpha_3}{\alpha_1}, \delta_{\psi D} = \frac{\eta_{\psi D} e}{\psi_{De}}, \\ S_{\psi D} &= \frac{\alpha_1 e_{33} C_{13}}{C_{33} \psi_{De}}.\end{aligned}$$

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