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Citation: Pothos, E. M. & Bussemeyer, J. R. (2022). Quantum Cognition. Annual Review of Psychology, 73(1), pp. 749-778. doi: 10.1146/annurev-psych-033020-123501

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Link to published version: <https://doi.org/10.1146/annurev-psych-033020-123501>

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Quantum cognition

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Running title: quantum cognition

Running text word count: 12,995

Abstract

Uncertainty is an intrinsic part of life; most events, affairs, or questions are uncertain. A key problem in behavioral sciences is how the mind copes with uncertain information. Quantum probability theory offers a set of principles for inference, which align well with intuition about psychological processes in certain cases: cases when it appears that inference is contextual, the mental state changes as a result of previous judgments, or there is interference between different possibilities. We motivate the use of quantum theory in cognition and its key characteristics. For each of these characteristics, we review relevant quantum cognitive models and empirical support. The scope of quantum cognitive models encompasses fallacies in decision making (such as the conjunction fallacy or the disjunction effect), question order effects, conceptual combination, evidence accumulation, perception, over/ under distribution effects in memory and more. Quantum models often formalize psychological ideas previously expressed in heuristic terms, allow unified explanations of previously disparate findings, and have led to several, surprising novel predictions. We also cast a critical eye on quantum models and consider some of their shortcomings and issues regarding their further development.

Introduction

Information in our environment is mostly uncertain and much of cognition is about managing this uncertainty to generate useful conclusions. For example, we worry about things like whether it will rain tonight (a natural worry of Londoners); whether next year will be free from dangerous viruses; and about the cause of some funny spots suddenly appearing on our child's face. What are the foundations for our capacity for probabilistic inference? This is a question which encompasses research related to three Nobel prizes (Simon in 1978; Kahneman in 2002; Thaler in 2017), a philosophical debate which goes back to Antiquity, and surprising implications for norms for correct reasoning.

Probabilities are used to quantify uncertainty and make inferences from uncertain premises. A probability theory is a set of mathematical axioms for how to combine and update probabilities. This review concerns three overarching traditions regarding the relevance of probability theory to cognition. First, we have Bayesian/ Classical Probability Theory (CPT). CPT axioms embody some of our basic intuitions regarding how to deal with probabilities. In a famous quote by Laplace (1816, cited in Perfors et al., 2011), CPT is described as “nothing but common sense reduced to calculation.” The axioms of CPT are only four, yet they are the foundation of a mathematical edifice which encompasses any kind of probabilistic reasoning. CPT cognitive models have clearly attracted a great interest in the last few decades (Griffiths et al., 2010; Oaksford & Chater, 1994; Tenenbaum et al., 2011). Second, there are heuristics and biases, a toolbox of rules which offer fast and frugal accounts that describe numerous behavioral findings. Heuristics and biases have also attracted a large amount of interest (Gigerenzer & Todd, 1999; Kahneman et al., 1982). Third, there is Quantum Probability Theory (QPT), which is a newer direction, and is the focus of the present review. QPT is a general probability theory, like CPT, that is, a set of rules for how to combine and update probabilities. QPT and CPT axioms are different and so we often reach different conclusions if we employ QPT versus CPT. We can consider any of the questions above, e.g., will it rain tonight, and compute the corresponding probabilities either with CPT or QPT.

Some readers may have come across quantum mechanics, which is a theory of physics. The pioneering physicists who developed quantum mechanics soon realized that CPT was not suitable for this new physical theory – it seemed that uncertain information for microscopic particles obeyed probability rules different from the familiar ones from CPT. So, together with a new physics theory, they developed a new theory of probability as well – what we call QPT. QPT is the theory of probability from

quantum mechanics, without any of the physics. An important qualification is that the use of QPT in cognitive science makes no assumptions regarding the nature of brain neurophysiology; all current quantum cognitive models do not rely on a quantum brain hypothesis, which has been heatedly contested (Hameroff, 2007; Litt et al., 2006).

As with the physicists who developed quantum mechanics, some pioneering researchers in psychology have asked whether there are cognitive phenomena for which CPT or heuristic explanations are not sufficiently satisfactory. These researchers have initiated the quantum cognition research programme (Aerts & Aerts, 1995; Atmanspacher et al., 2002; Bordley, 1998; Khrennikov, 1999). However, the application of these ideas to empirical data started becoming more widespread after a special issue in the *Journal of Mathematical Psychology*, about 10 years ago (2009, Vol 53, 5). It is now a good point to bring together and evaluate the key ideas and advances. Previous overviews are in Ashtiani and Azgomi (2015), Bruza et al. (2015), Busemeyer and Bruza (2011), and Pothos and Busemeyer (2013). Haven and Khrennikov (2013) and Wendt (2015) describe applications beyond psychology in social sciences. These references also serve as tutorials for the QPT formalism, as well as Yearsley (2017) and Yearsley and Busemeyer (2016). Note, describing these contributions as the ‘quantum’ cognition research programme has two purposes. First, it brings together models which employ broadly similar mathematical tools and concepts, those from QPT. Second, it implies a commitment to the specific way in which probabilities are computed, from quantum mechanics, and the associated mathematical theorems (the Born rule, Gleason’s theorem, the Kochen-Specker and Bell theorems, Lüder’s postulate).

The general questions which guide this review include: how general is the applicability of probabilistic reasoning in cognitive processing? If we see a part of cognition as a probabilistic engine, when is the mind better described by CPT vs. QPT principles? What is the importance of context in cognition and how can we formalize contextual influences? How can prior decisions shape subsequent thought?

Motivation for QPT in cognition

For the average psychologist, the proposal that quantum theory might have something to do with cognition initially shades between ambitious and implausible. Early QPT models have been met with skepticism. These early models therefore had to focus on behavioral findings which have proved the most persistently challenging for classical (CPT or otherwise) formalisms. In this section we outline some of the findings which initially motivated QPT models, reserving additional empirical coverage for later.

CPT inference is considered to be rational (de Finetti et al., 1993). Additionally, employing CPT forces decisions to be consistent with the basic CPT axioms. These and other powerful arguments have been made for adopting CPT in inference (Griffiths et al., 2010). So why are there apparent discrepancies between CPT prescription and human behavior? The key problem is that full CPT inference can be intractably complex, for many real problems; even limited situations can require effortful computation. Limiting CPT to make processes manageable by realistic agents is the fundamental problem of bounded rationality (Simon, 1955) – and of course there is no single answer. Moreover, there is abundant evidence that human inference is characterized by a mix of more analytic and more intuitive inference, where the former can be sometimes associated with more accurate approximations of CPT processes (Elqayam & Evans, 2013; Fernbach & Sloman, 2009; Kahneman, 2001; Sloman, 1996).

Despite the many successes of CPT, a large body of empirical findings has accumulated, which are hard (though not impossible) to reconcile with CPT cognitive models. These findings challenge the cognitive ubiquity of some of the most basic CPT intuitions. Consider the conjunction fallacy (Tversky & Kahneman, 1983). In one example, participants were told of a hypothetical person, Linda, who was described to look like a feminist, but not a bank teller. Participants were asked to rank order the likelihood of several statements about Linda. The critical statements were that Linda is a feminist, Linda is a bank teller, and Linda is a feminist and bank teller. The results indicated that $P(\text{feminist}) > P(\text{feminist} \& \text{bank teller}) > P(\text{bank teller})$. The conjunction fallacy refers to the finding that the conjunction is judged more probable than the bank teller possibility alone.

Why is this result problematic for CPT? The formal foundations of CPT are essentially set theory. To compute CPT probabilities, we calculate fractions for the outcomes of interest relative to all possible outcomes. This is best explained with the conjunction fallacy variant from Tentori et al. (2004), which concerned Scandinavian individuals and the probabilities that they might have blue eyes and blond hair. The key finding was $P(\text{blond hair} \& \text{blue eyes}) > P(\text{blond hair})$. The $P(\text{blond hair})$ would be computed as the fraction of blond Scandinavians relative to all Scandinavians. We can imagine enumerating Scandinavian individuals, selecting out the ones with blond hair, and then from this subset selecting out the ones with blue eyes (and blond hair). It seems clearly incorrect that there will be more Scandinavian individuals in the conjunctive set than the set corresponding to the individual premise, yet this is what people do – and us, erudite readers, may also find it hard to avoid the intuition that the conjunction seems more probable than the individual statements (Gilboa, 2000). Tentori et al.'s (2004) formulation makes the set-theoretic structure of the problem obvious, but the CPT situation is identical whether we employ frequentist probabilities (as in Tentori et al., 2004) or probabilities as subjective

degrees of belief (Tversky & Kahneman, 1983). The conjunction fallacy is an extensively replicated finding and has resisted all kinds of disambiguation manipulations to ensure that participants correctly understand the conjunction and individual statements as intended (as opposed to, for example, understanding ‘bank teller’ as ‘bank teller & not feminist’; Dulany & Hilton, 1991; Moro, 2009).

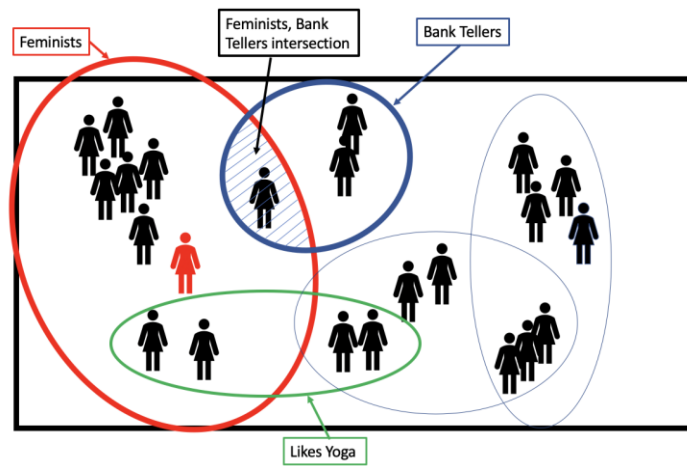


Figure 1. A CPT representation for the conjunction fallacy in Tversky and Kahneman (1983). We consider a large set of all possible realizations for Linda that we can imagine. The red, blue ellipses correspond to the subsets of possible Lindas consistent with the bank teller and feminist properties (and other subsets correspond to other properties). The conjunction is the intersection between these two subsets. The ‘real’ Linda is shown in red, representing all possible information about Linda.

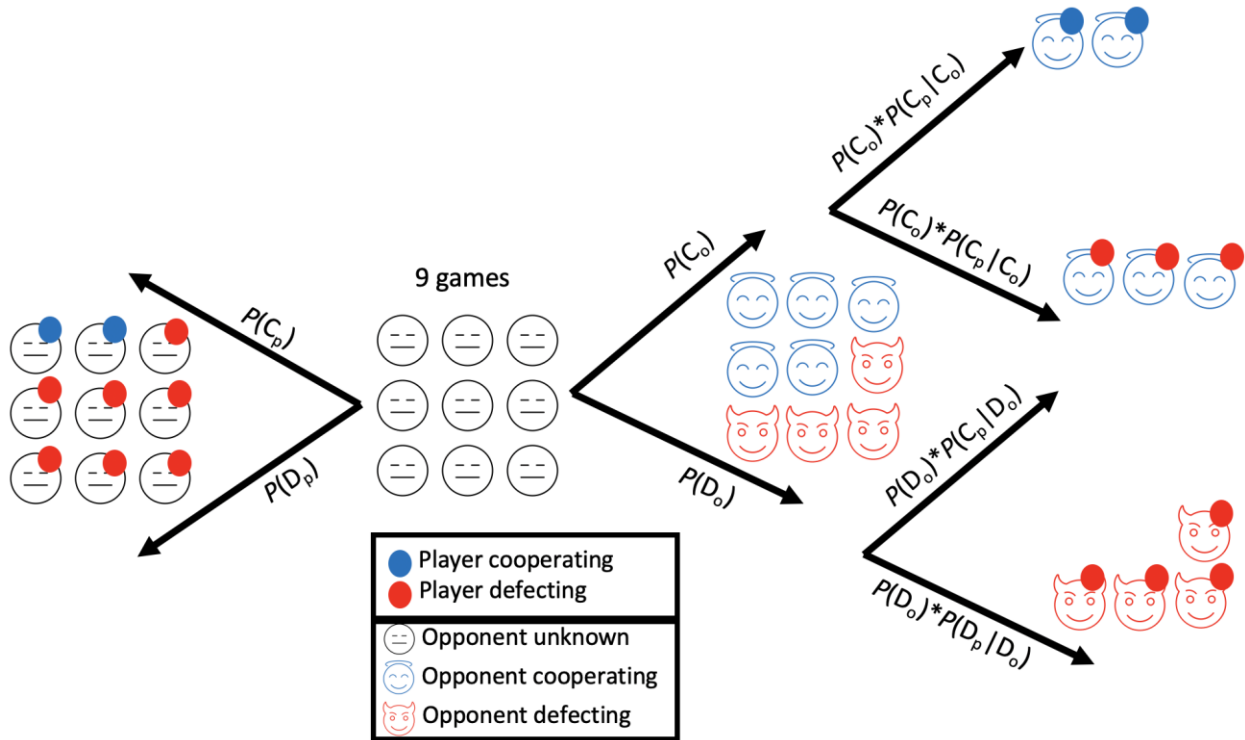


Figure 2. On the left vs. right, the player decides to defect or cooperate without and with knowledge of the opponent's action, respectively. The law of total probability means that the number of times the player cooperates is cleanly split into the number of times the player cooperates *and* the opponent cooperates vs. defects (there is the same number of blue/ red dots on the left and right of the diagram). C_p , D_p refer to the player cooperating or defecting and C_o , D_o to the opponent cooperating or defecting.

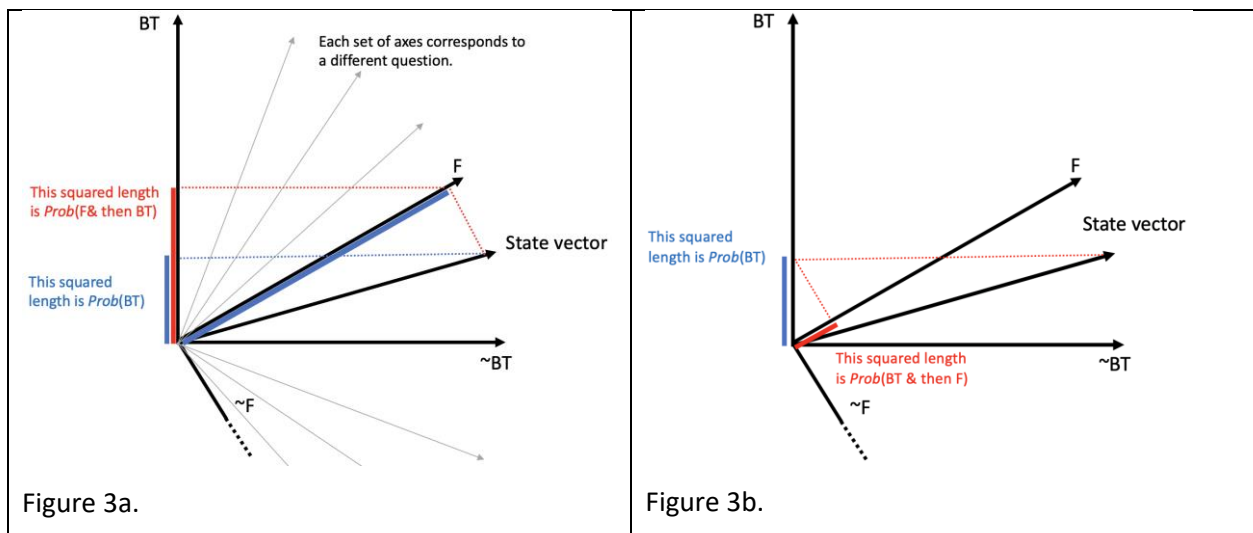


Figure 3. A QPT representation of the Linda problem in two dimensions. Particular axis (one-dimensional subspaces, which are also called rays) correspond to question outcomes, with BT=Bank Teller, F=Feminist, and ~ =Not. Probabilities are computed as the squared length of the projection of the state vector onto the corresponding axis. In Figure 3a, the red bar along the BT ray illustrates the $P(F \text{ \& then } BT)$. In Figure 3b, the red bar along the F ray illustrates the $Prob(BT \text{ \& then } F)$.

There is also a disjunction fallacy – the probability that Linda is a bank teller or a feminist is judged to be less likely than feminist alone (Bar-Hillel & Neter, 1993; Carlson & Yates, 1989) -- and other related fallacies including unpacking effects (Rottenstreich & Tversky, 1997; Sloman et al., 2004), and more complex conjunctions (Gronchi & Strambini, 2017; Winman et al., 2010). Conjunction and disjunction type of fallacies also occur in conceptual combination, called over-extensions and under-extensions respectively (Aerts, Sozzo, & Veloz, 2018; Hampton 1988a, 1988b), and in memory processes (Brainerd & Reyna, 2008; Brainerd et al., 1999, 2015).

Consider next a prisoner's dilemma game, in which two players make a choice to defect or cooperate and receive payoffs depending on their combined choices. Shafir and Tversky (1992) designed a study using this game to test a rational axiom of decision making called the sure thing principle (Savage, 1954). According to this principle, if under each possible state of the world you always prefer action A over B, then you should prefer action A over B even when the state of the world is unknown. Shafir and Tversky (1992) tested this principle by examining three conditions: one in which the player was informed that the opponent had already defected; another in which the player was informed that the opponent cooperated; and a third in which the opponent's play remained unknown. Many players chose to defect when the opponent defected and when the opponent cooperated, but then switched and decided to cooperate in the unknown case, violating the sure thing principle. This result was called the disjunction effect and shows a violation of the law of total probability (Figure 2). As for CFs, violations of the law of total probability challenge CPT intuition at a basic level. The number of times a participant cooperates splits cleanly into the number of times the participant cooperates and the opponent cooperates or defects, $P(C_{participant}) = P(C_{participant} \& C_{opponent}) + P(C_{participant} \& D_{opponent}) = P(C_{participant} | C_{opponent}) \cdot P(C_{opponent}) + P(C_{participant} | D_{opponent}) \cdot P(D_{opponent})$. The latter expression involving conditional probabilities corresponds to what is measured and shows that $P(C_{participant})$ is bounded between $P(C_{participant} | C_{opponent})$ and $P(C_{participant} | D_{opponent})$.

Contrary to this prediction, the probability to cooperate in the unknown condition was 37%, which is higher than the probability to cooperate both when the opponent was known to defect (3%) and to cooperate (16%; Shafir and Tversky, 1992; see also Busemeyer et al., 2006; Croson, 1999; Tesar, 2020, Kvam, Busemeyer, & Lambert-Mogiliansky, 2014). Similar results were observed in other tasks, notably a two-stage gambling task (Broekaert et al., 2020; Tversky & Shafir, 1992) and a categorization – decision paradigm (Townsend et al., 2000; Wang & Busemeyer, 2016).

Another example of a puzzling result lacking a natural classical explanation concerns order effects in judgment. A popular example is the following. Using a Gallup opinion poll, Moore (2002) examined the probability of ‘yes’ responses to the pair of questions ‘is Clinton honest’ and ‘is Gore honest’, presented in both possible orders. When the Clinton question was first, $P(\text{Clinton}_{yes}) = 0.50$, $P(\text{Gore}_{yes}) = 0.68$. With the reverse order, Moore (2002) observed $P(\text{Gore}_{yes}) = 0.60$, $P(\text{Clinton}_{yes}) = 0.57$. This is a surprising difference both because the putative cause, the change in ordering, appears innocuous and because of the practical importance of the result: for many public debates, Gallup polls can have a substantial influence in opinion, especially for undecided individuals. Similar order effects have been reported in assessing evidence, for example, concerning the probability of a disease given two pieces of evidence in different orders (with participants who were medical professionals! Bergus et al., 1998) and in mock-jury decision situations (McKenzie, Lee, & Chen, 2002; Trueblood & Busemeyer, 2011). Order effects also occur with similarity judgments. For example, Tversky (1977) found that judging the similarity of Korea to China produced higher ratings than judging these two countries the other way around. Explanations for question order effects invariably invoke ideas related to the constructive nature of judgment, for example, answers to earlier questions activate thoughts which affect our perspective for later ones (Hogarth & Einhorn, 1992; Schwarz, 2007). These order effects can naturally be explained by the non-commutative nature of QPT. However, they are more challenging for CPT, because conjunction in CPT is commutative, which means that order does not matter: $Prob(A\&B) = Prob(B\&A)$. Commutativity sometimes makes intuitive sense, but yet again we are confronted with empirical situations which diverge from this CPT intuition.

People are sensitive to measurements, such as asking an opinion, and taking a measurement can change their future behavior. This is difficult for CPT theories, because when asked a question (e.g., is Linda a feminist), the answer is based on existing (albeit not accessed) knowledge and the answer does not create any new information. No Bayesian updating is required, and so the answer should not affect responses to future questions. Surprisingly, human cognition sometimes does not work like this. Asking for an opinion or making a judgment can apparently change the relevant mental state, as

evidenced by effects on subsequent behavior (Ariely, 2008; Brehm, 1956; Lichtenstein & Slovic, 2006; Sharot, et al., 2010; Schwarz, 2007). For example, in a hypothetical legal case, Holyoak and Simon (1999) showed that the evaluation of arguments changed to become more consistent with the produced verdict. Such effects can be called constructive influences, because the act of e.g. making a decision helps ‘construct’ a particular mental state. These constructive influences naturally agree with QPT because it was originally formulated by physicists to account for the fact that measurement can change and disturb the state of a system (a cognitive system in our case).

The tension between such so-called fallacies and (especially) CPT is not insurmountable, and so these findings do not *disprove* the applicability of CPT. We will later review some principled attempts to reconcile fallacies, such as the conjunction fallacy, with CPT (Costello & Watts, 2014; Tentori, Crupi, & Russo, 2013; Zhu, Sanborn, & Chater, 2020). However, for many researchers, fallacies do reveal a persistent tension between fundamental CPT principles and human intuition. We will also consider psychological accounts of the fallacies outside CPT, spanning ideas across areas as diverse as cognitive and social psychology. These ideas are often expressed in the form of simple heuristics, which are principles intended to explain specific aspects of cognition (Gigerenzer & Goldstein, 1996; Hertwig et al., 2013; Tversky & Kahneman, 1983).

There is little doubt that at least some of cognition relies on probabilistic reasoning captured by CPT reasoning, while other parts of cognition rely on a toolbox of simple heuristics. However, these two opposing views cannot easily be integrated in a formal way with each other. One key idea motivating the use of quantum theory in cognition is that it lies between CPT models (which are constrained by strict axiomatic rules) and simple heuristics (which are free from any axiomatic constraints) by employing axioms that are less constraining than CPT, but more formal and systematic than simple heuristics.

The consideration of QPT in behavioral modeling is appealing for three reasons. First, it provides a coherent explanation for the wide range of puzzling findings summarized above, using a common set of principles. For example, the same QPT principles that account for probability judgment fallacies, such as the CF, can also account for the disjunction effect in decision making. Such common explanations reveal links between findings, which have not been considered together before. Second, QPT enriches psychological theory with several novel concepts, such as incompatibility, superposition, collapse, and entanglement, which are the foundation for new hypotheses and have led to the discovery of new phenomena. Finally, in a certain formal way, it represents the next step away from CPT (Sorkin, 1994), if computations are required to have a more local focus. As we will explore shortly, in QPT there are

compatible and incompatible questions. For compatible questions, everything is classical; for incompatible ones, apparent classical errors and inconsistencies arise. QPT can be seen as a more local version of CPT, where instead of having a large space of classical questions, we are classical only within smaller subsets of questions compatible with each other; across subsets, questions are incompatible (cf., Fernbach & Sloman, 2009; Lewandowsky, Kalish, & Ngang, 2002; Pothos et al., in press; Trueblood, Yearsley, & Pothos, 2017).

Basic description of QPT

The predominance of CPT intuition has been so complete, it is hard to imagine alternatives. To understand how QPT provides an alternative form of probabilistic intuition, we briefly consider the picture of the world according to each probability theory, using Linda from the conjunction fallacy example. In each case, we have to represent our uncertain information about Linda, formulate different questions about possible characteristics she might have, and then estimate the probabilities of question outcomes (see also Busemeyer & Bruza, 2012; Khrennikov, 2010).

CPT begins with a sample space, such as the rectangle of Figure 1, which contains all the various possible realizations for Linda, represented by the female characters. A possible outcome to a question, such as whether or not Linda is a feminist, is represented by a subset of the sample space, such as the red bounded ellipse for yes to feminism. The outcome from a pair of questions, such as whether Linda is a feminist and bank teller, is represented by the intersection of subsets, as in the shaded region of overlap between the red and blue ellipses. The beliefs a person has about these questions are represented by a probability function that assigns a probability to each subset. For example, the probability that Linda is a feminist is the probability assigned to the red ellipse. The probabilities assigned to the union of mutually exclusive events must add. The larger the subset for a question outcome, the more possible Lindas we can imagine consistent with this question outcome, and the more likely this question outcome is – that is, the probability of a question outcome depends on the size of the corresponding subset. Crucially, we could resolve all possible questions about Linda and identify the ‘real’ Linda in the sample space.

Note that the subset for bank teller is formed by the union of the subset for bank teller and feminist with the subset for bank teller and not feminist. Consequently, the probability of bank teller equals the sum of the probabilities assigned to these two events, and so must exceed the probability of the single bank teller and feminist event. This illustrates why the conjunction fallacy is so puzzling according to CPT.

QPT begins with what is essentially a vector space, such as the 2-dimensional space shown in Figure 3. A vector can be thought of as a line with a specific direction and length, e.g., the ‘arrows’ in Figure 3. Such a vector space contains all possible outcomes for questions about Linda. For example, for the question ‘is Linda a bank teller’, there are two question outcomes (yes or no), represented by two unit-length vectors at 90-degree angles to each other, basically forming the x-y axes. Specifically, in Figure 3, the outcome that Linda is a bank teller is represented by the vertical axis and that she is not a bank teller by the horizontal axis. The answers to a different question, like the feminist question, can be represented by a different pair of orthogonal vectors rotated by some angle. That is, the x-y axes for different questions are related to each other through simple rotations. A vector representing a question outcome spans a one-dimensional subspace, called a ray. Subspaces are really important in QPT; QPT is a way to assign probabilities to subspaces and the key difference between QPT and CPT is that the latter involves assigning probabilities to subsets.

The set of beliefs a person has about these questions is represented by a (unit length) state vector, as labeled in Figure 3. The probability of a question outcome is obtained by projecting (laying down) the state vector onto the subspace representing the answer, and then computing the squared length. In Figure 3, the projection onto feminism is the blue bar on the feminist line and the probability equals the squared length of this bar. To compute the conjunction of two question outcomes, we typically have to employ a sequential projection, which corresponds to resolving one question after the other. For example, suppose we are interested in $P(F \& BT)$. It can be seen in Figure 3 that there is no single ray corresponding to both feminism and being a bank teller. Therefore, we have to compute $P(F \& \text{then } BT)$, which involves two steps. First, project the state vector onto the F ray. Second, project this previous projection onto the BT ray (follow the two red perforated lines in Figure 3a). The $P(F \& \text{then } BT)$ is the squared length of the last projection, indicated by the red bar along the BT ray. Alternatively, suppose we want to compute $P(BT \& \text{then } F)$. We first project the state vector to the BT ray and then project the resulting projection to the F ray; again, follow the red perforated line in Figure 3b, noting that the last projection is indicated by the red bar along the F ray.

These simple computations illustrate two key properties of QPT. First, $P(F \& \text{then } BT) \neq P(BT \& \text{then } F)$, that is a sequence of projections can be *non-commutative*. Unlike CPT, order matters and such non-commutativity leads to interference effects (which we will consider in detail later). Second, every time we resolve a question, the state vector has to change in a specific way. This is the QPT property of vector collapse (which we will also consider later).

As for CPT, we might wonder what happens if we try to resolve as much uncertainty about Linda as possible, by considering all the possible questions about Linda. Suppose first that we determine that Linda is a feminist. We would then have to place the state vector along the feminist subspace, since in this way we have maximum overlap between state vector and subspace. If next we decide that Linda is not a bank teller, we would align the state vector with the subspace for not-bank-teller. But a state vector aligned with the not-bank-teller subspace has partial overlap with both the feminist and non-feminist subspaces. That is, resolving the bank teller question made us uncertain about the feminist one!

In general, in QPT, it is often impossible to resolve all questions concurrently and certainty about one question introduces uncertainty about others, that is, there are uncertainty relations. The impossibility of complete knowledge contrasts sharply with CPT intuition. Psychologically, at a broad level, these ideas resonate with a view of cognition as strongly context and perspective dependent (Fodor, 1983). Answers to questions, like bank teller and feminism in Figure 3, for which there are uncertainty relations, are called incompatible and are unique to QPT. Two questions are incompatible if the presence of one question alters our perspective for the other, that is, if the questions are contextual; or if the questions are such that answering one question disturbs the relevant 'system' for the other one. For example, we might want to ask a teenager the pair of questions 'how well is the revision of your paper going' and 'what will you do this Friday night'. We can imagine that the first question makes the teenager anxious, so that he understands the second question differently than if we had asked the second question individually. Other pairs of answers can be compatible, such as for example, whether Linda is a feminist or not and whether she has black hair or not (e.g., it seems unlikely that for these questions, answering one changes our perspective for the other). For compatible questions, the probabilities add up just like in CPT. So henceforth when contrasting QPT with CPT, we will mostly focus on incompatible questions.

In Figure 3, the sequential probability $P(F \text{ \& then } BT)$ is greater than the direct probability, $P(BT)$, that is, even with this simple representation, we have an example where $P(F \text{ \& then } BT) > P(BT)$. Such an approach, suitably generalized, could therefore be a cognitive model for the conjunction fallacy (Busemeyer et al., 2011). An interesting question is, why would someone ever think to use this approach to model the conjunction fallacy? There are three parts to the answer. First, CPT does not appear to offer a natural approach to understanding the conjunction fallacy (but note that there are more elaborate CPT models, which we will consider later). Second, heuristic explanations developed for the original conjunction fallacy, based on e.g. the similarity between Linda and her various possible

characteristics (Tversky & Kahneman, 1983) are intuitive, but vague and with limited predictive value. Third, since the conjunction fallacy problem is about probabilistic judgment, we might be biased to develop a model for the conjunction fallacy based on some kind of probability theory. What is a probability theory, not constrained in the same way as CPT, which is consistent with the intuitions expressed in the heuristic, similarity-based models for the conjunction fallacy? The most immediate answer is QPT.

One of the important points in the present review is that QPT provides a notion of probabilistic correctness that is an alternative to CPT. Classically, the conjunction fallacy appears blatantly incorrect, in the same way that $1+1=3$ appears incorrect (Figure 1). Yet if we accept the QPT axioms, a conjunction fallacy can seem reasonable. Moreover, QPT inference can be said to be rational on exactly the same basis as CPT. A hallmark of rationality in probabilistic inference is the Dutch book theorem. Suppose you have to assign probabilities to a combination of bets. The Dutch book theorem is that if probabilities obey a small set of requirements, then you are guaranteed to be protected from a certain loss (otherwise, there may be combinations of bets such as, whatever happens, you will suffer a net loss). CPT is consistent with the Dutch book theorem requirements (Oaksford & Chater, 2007), but, interestingly, so is QPT (Pothos et al., 2017). How can a decision, like the conjunction fallacy, be both correct and incorrect, both rational and irrational? This boils down to whether the corresponding questions can be assumed to be compatible or incompatible. If the latter, then QPT is the appropriate theory of inference; if the former then CPT is (in which case QPT agrees with CPT). So, QPT will be the correct and rational way to approach certain questions.

In the case of an imaginary Linda, a person's understanding of what is meant by bank teller in the context of questions about feminism might be different from the understanding emerging without that context, in which case the conjunction fallacy is not unreasonable. However, sometimes naïve observers represent questions as incompatible, when it seems clear that they are not. In the case of a Scandinavian person, we can easily check whether or not a person has blue eyes and blond hair, and so these questions should be treated as compatible. We may conclude that the conjunction fallacy in this case is a mental error of representing as incompatible questions which are not so. Why might this occur? There is evidence that lack of familiarity or reduced effort increases the likelihood of incompatible representations (Nilsson, Rieskamp, & Jenny, 2013; Trueblood et al., 2017). So, we can provisionally assume that QPT representations are less demanding than CPT ones and offer a more default option under reflexive versus reflective modes of reasoning (Kahneman, 2001). These issues are

not without practical importance. For example, in medical or legal decision making, the rational status of an inference can be of paramount importance (Wojciechowski & Pothos, 2018).

In subsequent sections, we will consider in more detail key QPT features, which distinguish QPT from CPT and offer novel directions for behavioral research. In doing so, we will describe some of the main QPT cognitive models and how they account for the puzzling findings reviewed earlier.

Quantum Interference

Quantum interference is a unique property which concerns a breakdown of the law of total probability. Suppose we give the decision maker some preliminary information, and then we compare two conditions: in the first condition, we simply measure event B alone; in the second condition, we first measure event A and then measure event B. What we expect from the CPT law of total probability is the following

$$P(B) = P(A \text{ and } B) + P(\sim A \text{ and } B). \quad (1)$$

Now let us see how this works with QPT. After being given the preliminary information, the decision maker is in a state represented by state vector ψ . A positive outcome from the measurement of B is represented by a projector P_B (a matrix). It projects the state vector onto the subspace representing B by the matrix product $P_B \cdot \psi$ (above we considered one-dimensional subspaces, but a subspace can have any dimensionality). The probability of a positive outcome to question B then equals the squared length $\|P_B \cdot \psi\|^2$ (the double vertical bars indicate length). Consider another projector P_A , which is used to project onto positive outcome for question A, and define $I - P_A$ as the orthogonal projector that projects on the negative outcome for A. Then we can re-write the projector for B as the sum of two products as $P_B = P_B P_A + P_B (I - P_A)$ so that the probability of B can be broken down as follows

$$\|P_B \cdot \psi\|^2 = \|P_B P_A \cdot \psi + P_B (I - P_A) \cdot \psi\|^2 = \|P_B P_A \cdot \psi\|^2 + \|P_B (I - P_A) \cdot \psi\|^2 + \Delta \quad (2)$$

where Δ is the sum of cross products produced by squaring the sum. The first term $\|P_B P_A \cdot \psi\|^2$ on the right of Equation 2 is $P(A \text{ and } B)$, and the second term $\|P_B (I - P_A) \cdot \psi\|^2$ is $P(\sim A \text{ and } B)$. The last term Δ is called the interference term, which can be positive, negative, or zero. If the measures are compatible, then Δ is zero, and QPT satisfies the CPT law of total probability; but if the measures are incompatible, then Δ can be negative or positive, and so the CPT law of total probability is violated. It is so as to emphasize the importance of the order, that we usually write $P(A \text{ \& then } B)$ for $P(A \text{ and } B)$,

when A is measure first. Now we will show how this simple property of interference accounts for a variety of puzzling findings mentioned earlier.

Probability judgments.

CPT assumes that we can resolve any combination of questions concurrently, so that we can assign a probability for any combination of question outcomes. In fact, this is required because of a property called closure – this is why CPT calculations are intractable for bounded rational agents (CPT theorists are aware of this problem and incorporate it in their modelling, e.g., Lake, Salakhutdinov, & Tenenbaum, 2015; Tenenbaum et al., 2011). By contrast, in QPT, uncertainty relations preclude the concurrent resolution of incompatible questions. Pairs of questions have to be evaluated in a sequential way, one question outcome at a time. For example, for the conjunction fallacy, we cannot evaluate $P(\text{feminist} \& \text{bank teller})$, rather we have to commit to an order and evaluate e.g. $P(\text{feminist} \& \text{then bank teller})$.

Bussemeyer et al. (2011, 2015; Franco, 2009; see also Aerts, 2009; Gronchi & Strambini, 2017; Miyadera & Phillips, 2012) showed how QPT can account for not only the conjunction fallacy, but also the disjunction fallacy, unpacking effects, and more complex conjunctions. We illustrated a “toy” version of their model with Figure 3, however, Bussemeyer et al. (2011) used a more general model, based on subspaces of arbitrary dimensionality. In Equation 2, we assume that vector ψ is determined by the Linda story; we use a projector P_B to represent a positive answer to the bank teller question and a projector P_F to represent a positive answer to the feminism question. We assume that the bank teller, feminist questions are incompatible so that the interference term $\Delta \neq 0$. A conjunction fallacy in the observed direction is produced by having $\Delta < -\|P_B(I - P_F) \cdot \psi\|^2$, a requirement which can be justified from the way different question outcomes about Linda are expected to correlate. A similar argument can be used for the disjunction fallacy. This is a simple illustration of how, in QPT, it can be entirely correct to have a (sequential) conjunction as more probable than an individual statement. Psychologically, participants may find it hard to imagine Linda as a bank teller from the initial perspective of the story. But once they accept that Linda is a feminist, they might think that bank teller may not be such an unlikely profession for feminists. QPT sequential processes are like successive abstractions; with each question evaluated, some of the original information is lost, but new insights and perspectives may be acquired.

The same principles, and interference effects, have been used to account for over and under extensions of membership judgments that occur with conceptual combinations (e.g., Hampton 1988a, Osherson & Smith, 1981; Storms et al., 1999). An over-extension effect occurs when the membership of

an item is stronger for a conjunction of two concepts as compared to either concept individually. A classic example is a goldfish, which is rated as a better example of the combined concept pet-fish than either the concepts of pet or fish individually. An under-extension effect occurs when the strength of membership of an item is weaker for a disjunction of two concepts as compared to the individual concepts. For example, an ashtray is considered a better example of home furnishings as compared to home furnishings or furniture. These and related findings were modelled with QPT models by Aerts and colleagues (2009; Aerts & Gabora, 2005, Aerts, Sozzo, and Veloz, 2015). Such models have been intended to cover overextension and underextension effects in conceptual combination (Hampton, 1988a, 1988b), as well as non-compositionality.

Question order effects.

According to QPT, both the conjunction fallacy and question order effects arise from the same principle: interference, which results from incompatibility. The co-occurrence of the two effects is an important a priori prediction from QPT models, which was confirmed by Yearsley and Trueblood (2018; see also Stolarz-Fantino et al., 2003, Experiment 2; Gavanski & Roskos-Ewoldsen, 1991). Yearsley and Trueblood additionally showed that the extent of conjunction fallacies was within the bounds predicted by QPT.

For order effects, QPT models go a step further. QPT makes an a priori, quantitative prediction regarding order effects called the QQ equality (Wang et al. ,2014). Consider a typical question order experiment in which two binary valued questions are asked, back-to-back, but in different orders. The basic model in this case is very simple: the probability for a pair of answers to questions A and then B is computed from the product of projectors $\|P_B P_A \psi\|^2$, and the probability for the opposite order of questions is based on the reverse product $\|P_A P_B \psi\|^2$. Despite its simplicity, this model makes the following prediction for any dimensionality and any pair of projectors:

$$\begin{aligned}
 QQ = & \left[P(A_{yes} \text{ \&then } B_{no}) + P(A_{no} \text{ \&then } B_{yes}) \right] \\
 & - \left[P(B_{yes} \text{ \&then } A_{no}) + P(B_{no} \text{ \&then } A_{yes}) \right] = 0
 \end{aligned}
 \quad \text{Equation (3)}$$

The QQ equality has been considered one of the most important a priori predictions from QPT. Wang et al. (2014) examined question order effects across 70 national surveys in the USA, with participants varying between 651 and 3,006, and reported good consistency with the QQ equality.

A related experimental direction concerns the ABA paradigm, whereby participants see one question, followed by a different one, followed finally by the original one. With such a paradigm, a QPT approach predicts that if the A, B questions are incompatible, the second instance of A (measure A then

B then A again) might produce responses different from the first instance, because of interference; but not (or less so) if the questions are compatible (Khrennikov et al., 2014). Examining corresponding predictions is complicated by response biases from just remembering the first question. Nevertheless, Busemeyer and Wang (2017) obtained results supporting the QPT predictions.

Decision making.

We next turn to the disjunction effect, as observed in e.g. a prisoner's dilemma game (Shafir and Tversky, 1992; see also Busemeyer et al., 2006; Croson, 1999; Kvam, Busemeyer, & Lambert-Mogiliansky, 2014; Tesar, 2020). Recall that the main finding is that the probability to cooperate in the unknown condition is much higher than in both known conditions, violating the CPT law of total probability (Equation 1). Pothos and Busemeyer (2009) also applied Equation 2 to account for the disjunction effect, by using a projector P_{PD} for projecting the state vector on the subspace for the player deciding to defect; and using another incompatible projector P_{OD} for the opponent's decision to defect. Pothos and Busemeyer (2009) went a step further by using the payoffs from the game and a cognitive dissonance principle from social psychology (Festinger, 1957) to build the projectors, P_{PD} , P_{OD} ; they could then quantitatively predict the sign and magnitude of the interference term Δ required to account for the disjunction effect. More recent work by Broekaert et al. (2020) used a model very similar to the one from Pothos and Busemeyer (2009) to account for the disjunction effect obtained with a two-stage gambling task (Tversky & Shafir, 1992). Other QPT accounts of the disjunction effect for the prisoners' dilemma game are presented in Denolf et al. (2017), Asano et al. (2011), and Martínez-Martínez and Sánchez-Burillo (2016); and other QPT accounts of the disjunction effect for the two-stage gambling task include Khrennikov and Haven (2009) and Yukalov and Sornette (2011).

QPT interference was also the basis for Wang and Busemeyer's (2016, see also Busemeyer, et al., 2009) model for the disjunction effect in the categorization – decision paradigm (Townsend et al., 2000). Wang and Busemeyer (2016) built the corresponding projectors, employing payoffs in the task and the same cognitive dissonance principles, as above, to predict the sign and magnitude of the interference term Δ . An interesting alternative approach for the computation of interference terms in this paradigm is Moreira and Wichert's (2016), who used a quantum-like network (Tucci, 1995) to associate interference terms with image similarities.

Memory Recognition.

Brainerd, Reyna, and colleagues (Brainerd & Reyna, 2008; Brainerd et al., 2010) asked, in one paradigm, participants to encode a set of memory targets, for example, a word list. In test participants were presented with the targets, related distractors that were semantically related to the targets, and

unrelated distractors. The test probes included “is it a Target” (T), “is it a related distractor” (R), “is it a target or a related distractor” (T or R), which produce probabilities $Prob(T)$, $Prob(R)$, $Prob(T \text{ or } R)$ respectively. Classically, because the categories of T and R were mutually exclusive, we expect that $Prob(T) + Prob(R) = Prob(T \text{ or } R)$. The major finding is overdistribution such that $Prob(T) + Prob(R) > Prob(T \text{ or } R)$. It appears that some items are being remembered as both presented and not presented (Brainerd & Reyna, 2008; Brainerd et al., 2010). An analogous subadditivity effect was also observed, $Prob(T) + Prob(R) + Prob(NR) > 1$, where NR is a non-related distractor, and T, R, NR comprise the set of mutually exclusive and exhaustive cues. Brainerd et al. (2013, 2015) developed a quantum recognition memory model that was viewed as a formalization of fuzzy trace theory (Reyna, 2008; Reyna & Brainerd, 1995). However, it relied on compatible measurements rather than using interference produced by incompatible measurements. Later, Denolf and Lambert-Mogiliansky (2016) proposed a quantum memory model in which the verbatim information and gist information were represented by incompatible measurements and used interference to account for the results. The latter approach was further extended and more rigorously tested by Trueblood and Hemmer (2017; see also Broekaert & Bussemeyer, 2017).

Comparison with alternative models.

Overall, QPT models have provided fairly straightforward accounts of a wide range of puzzling findings, which have resisted compelling classical descriptions for several decades. They have provided a unified account of these various findings, such as the conjunction fallacy for probability judgments and the disjunction effect in decision making, as arising from the common explanatory idea of interference produced by incompatible projectors. Moreover, they have led to surprising predictions, such as the QQ equality, which have pushed our understanding of constraints in behavior. However, some of these findings have been intensely studied before, at least separately. How do QPT models compare to these alternative explanations? We evaluate some of these alternative explanations, proceeding from more heuristic to more formal accounts.

First let's examine some heuristic alternative accounts. Tversky and Kahneman's (1983) explanation of the conjunction fallacy was that it was generated by a representativeness heuristic, according to which probabilistic judgments depend on the similarity between an instance (e.g., Linda) and a category (e.g., feminists). Such an approach is very intuitive but lacks precision. QPT can be viewed as a way to formalize representativeness, because probabilities are computed as the overlap between the state vector and the subspace representing a question outcome (Sloman, 1993). The advantage is

that QPT can then encompass related findings beyond the conjunction fallacy and reveal commonalities between seemingly disparate phenomena. Shafir and Tversky's (1992) explanation for the disjunction effect was based on the idea of failure of consequential reasoning (for the prisoner's dilemma finding): when the opponent's play is known, the player can readily evaluate the consequences of each action; but when the opponent's play is unknown, these consequences become unclear. Once again, this explanation is intuitive but imprecise, and QPT provides a way to make these ideas quantitatively rigorous: the effect of the two lines of reasoning that are clear when the opponent's play is known are cancelled out by interference, Δ , when the opponent's play is unknown. The advantage of the quantum model is that it can lead to new predictions, notably order effects that have been shown to moderate the disjunction effect (Broekaert et al., 2020). Concerning question order effects, a well-known idea from social psychology is that these effects arise because earlier questions create a unique context or perspective for evaluating subsequent ones (Schwarz, 2007). For example, an earlier judgment can activate thoughts or perspectives that alter perception of subsequent ones. Or it is possible that a choice biases a re-interpretation of preferences to avoid cognitive dissonance (Festinger, 1957). Again, the QPT model is completely consistent with these intuitive ideas from social psychology, but the advantage of QPT is the capability of making precise a priori predictions, such as the QQ equality.

Now let us turn to some more formal alternative accounts. A fairly natural hypothesis regarding the conjunction fallacy is that it might arise from an erroneous cognitive rule for combining probabilities for a conjunction. Averaging models assume that both conjunctions and disjunctions are based on averages of the probabilities of the conjuncts (Abelson, Leddo, & Gross, 1987; Fantino, Kulik, & Stolarz-Fantino, 1997; Nilsson et al., 2009), but with different weights for conjunction and disjunction. One problem with these models is that they provide no account of conditional probabilities. Additionally, unless weights change in a post hoc way, such models predict conjunction fallacies regardless of causal links between the conjuncts (high rate of conjunction fallacy) or not (lower rate). However, we expect a conjunction fallacy in cases such as "Mr. F has had one or more heart attacks and Mr. F is older than 55" but not in "Mr. F has had one or more heart attacks and Mr. G is older than 55" (Tversky and Kahneman, 1983). Causal strength between the individual premises is captured particularly well by models such as inductive confirmation, which purports that probabilistic assessment follows the extent to which particular pieces of information strengthen or weaken our beliefs in an initial hypothesis (Tentori et al., 2013). A weakness of inductive confirmation is that it does not provide any explanation for disjunction errors.

Another influential approach for the conjunction fallacy is that probabilistic computation is classical, but in an error prone way. Costello and Watts (2014; Costello, Watts, & Fisher, 2018; Zhu, Sanborn, & Chater, 2020; see also earlier work by Dougherty et al., 1999) proposed that the mind computes probabilities via a mental sampling process, either retrieving relevant instances from one's experience (e.g., how many women like Linda have I come across?) or generating relevant instances through a mental simulation. Importantly, the mental sampling processes for different questions are independent from each other, so that it is perfectly possible to have an estimate for a conjunction higher than an estimate for an individual premise. With an additional assumption that more complex probability terms, such as conjunctions and disjunctions, are more error prone than simpler terms, it is possible to produce conjunction and disjunction fallacies, as well as a series of probabilistic identities, which appear to be satisfied by human judgments (Costello & Watts, 2018). The main concern with this work is that it has been difficult to provide a consistent approach to conditional probabilities, with differences across publications, and so there is limited coverage of question order effects (Zhu et al., 2020). Also, some of the empirical results appear to go against predictions from CPT noise models (Tentori et al., 2013; Yearsley & Trueblood, 2017). However, there is a key idea consistent with QPT models, namely the requirement of independent sampling for the conjunctions and the individual probabilities. QPT essentially formalizes this idea, in the way that incompatibility breaks up a sample space into smaller ones. For the Linda problem, instead of having a single sample space as in Figure 1, we have a separate sample space each time a question is assessed that is incompatible with the previous one (Hughes, 1989). Note, QPT models can also embody a distinction between errorless and error-prone measurement, but in Busemeyer et al.'s (2011) model this was not needed to cover the conjunction fallacy.

The conjunction fallacy and question order effects are the areas for which QPT models have been scrutinized the most. The probabilistic identities produced by Costello, Watts, and Fisher (2018; Zhu et al., 2020) are not obeyed by the QPT model, in contrast to empirical results. This is a challenge which has yet to be addressed by QPT. Boyer-Kassem, Duchene, and Guerri (2016) examined a QPT property called the law of reciprocity and reported violations of this property. However, this property is restricted to single dimensional subspaces (like those shown Figure 3), and it does not apply to higher dimensional subspaces such as those used in Busemeyer et al. (2011). Kellen, Singmann, and Batchelder (2018) and Costello and Watts (2018) both questioned the necessity of QPT to account for the QQ equality, and they constructed alternative models to account for this finding. Clearly it is always easier to make up post hoc heuristics to reproduce a known result, than predict a result a priori. QPT models

have also been compared to other models of order effects on inference, such as the anchoring and adjustment model (Hogarth & Einhorn, 1992; McKenzie et al., 2002). However, the QPT models have been shown to provide a better account for these order effects than anchoring and adjustment models (Trueblood and Busemeyer, 2011).

Superposition and state vector collapse

Recall from the QPT description of the Linda questions (Figure 3) that the beliefs of a person judging Linda were represented by a state vector lying in the vector space. This state vector was not aligned with either the vector spanning the definitely feminist ray or the vector spanning the definitely not feminist ray. Instead, the state vector is a *superposition* state with respect to feminism, because it is formed by a linear combination of these two feminism vectors. At the same time, this state vector can also be considered a superposition with respect to bank teller formed by a linear combination of the bank teller vector and not bank teller vector. More generally, the same state vector can represent beliefs for all kinds of questions about Linda. Finally, a superposition state implies that beliefs about Linda cannot be described by a probability mixture of some conjunction of answers to the feminism and bank teller questions, because this conjunction does not even exist in this picture. In short, superposition states are not the same as probability mixtures. A superposition state contains ontic uncertainty (lack of existence of a feature) about Linda, whereas a probability mixture represents epistemic uncertainty (lack of knowledge) about an existing feature of Linda. In other words, ontic uncertainty reflects the nature of the system itself, not our lack of knowledge about it (Atmanspacher & Primas, 2003; Griffiths, 2013; Spekkens, 2007). This way of interpreting uncertainty as superpositions is a mathematical requirement from a fundamental theorem in QPT (the Kochen-Specker theorem).

Reaching a decision about Linda means that the person must resolve the ontic uncertainty and (probabilistically) create an answer from the superposition state. The resolution of ontic uncertainty is called the “collapse” of the state vector and the state vector changes in a precise way, so that it aligns with the question outcome (this relates to another fundamental principle in QPT, Lüder’s projection postulate). As noted earlier, for example, if the person decides Linda is a feminist, then the new state will be aligned with the feminist ray. That is, resolving a question changes the mental state

There have been several examinations of QPT predictions regarding constructive influences and the collapse principle. White et al. (2014, 2020) employed pairs of stimuli of strong positive or negative valence, such that the second would always be rated, but the first would be rated half the time. With a

simple QPT model, they predicted that the impact of rating the first stimulus would be to make more intense the judgment for the second one (e.g., more positive if the stimulus was already positive). This prediction was called an evaluation bias and has been multiply confirmed. Yearsley and Pothos (2016) examined a prediction from the collapse principle called the quantum Zeno effect. In their experiment, participants were asked to judge the likelihood that a defendant was guilty of a crime after being provided with a sequence of evidence; the defendant was initially assumed to be innocent. In different conditions, participants were asked to make judgments after each of n pieces of information, with the period n changing across conditions. According to the collapse principle, the more the intermediate judgments (lower n), the more likely it would be that the judgments would keep collapsing the state back to the initial one, thereby resetting the person's view that Smith is innocent. This is the quantum Zeno effect, which in a decision setting is the idea that the frequency of intermediate judgments slows down opinion change (colloquially, a watched pot never boils). Zeno effects have also been investigated using quantum models with bistable perception tasks (Atmanspacher & Filk, 2010).

Another interesting application of the collapse principle concerns the so-called zero priors paradox. A requirement from Bayes's rule in CPT is that if a person assigns a 0 or 1 probability to any hypothesis, then no new evidence can change the probability from these extremes. This rule turns out to conflict with human inferences. Basieva et al. (2017) conducted two experiments in which participants judged the likelihood of guilt for various possible candidates of a crime. The initial evidence caused participants to place zero (or very low) probability on one candidate, but later evidence moved these zero priors to a high probability of guilt. In Basieva et al.'s (2017) quantum model, it was possible to move from zero priors to non-zero probability of guilt, by taking advantage of the way an earlier decision changes the mental state.

Comparison with alternative models.

As in the case for question order effects, constructive influences on judgment have long been explained by social psychologists by assuming that an earlier judgment can increase memory or attention to some information, which impacts later judgments (e.g., Schwarz, 2007). QPT is a way to formalize such ideas. Yearsley and Pothos (2016) compared their quantum model of the Zeno effect to a Bayesian updating model, but only the quantum was able to reproduce the Zeno effect. Basieva et al. (2017) compared their quantum model with a Bayesian model which assumed a very small prior instead of an exactly zero prior, but the latter was still unable to account for the large change in new evidence.

Representing similarity structure in vector spaces

The dimensionality of a vector space is determined by the number of orthogonal basis vectors (informally, the number of axes) required to span the space, that is, to reach every point by a linear combination. In Figure 3, the vector space is two-dimensional, because only two vectors are needed to span the plane, and all subspaces are rays in the plane. However, it is not necessary, and often not even psychologically reasonable, to assume that each answer to a question is represented by a ray. Instead, subspaces can be of arbitrary dimensionality, and this dimensionality can be associated with the complexity of question outcomes. This offers a way to capture differences in extent of knowledge in similarity and categorization. For example, suppose we want to represent the concepts of (Red) China versus (North) Korea, when there would be more knowledge about the former than the latter (Tversky, 1977). The higher dimensionality of the China subspace would correspond to the more available information about China, for example, about language, culture, industry, political system etc. Note, each basis vector in that subspace is not necessarily a unique feature, but rather better understood as a summary of related features, in a way analogous to the principal components in factor analysis. Differences in subspace dimensionality can be employed to model the internal structure of concepts and have allowed coverage of empirical results in the similarity literature.

The predominant approach to similarity and representation has been that concepts are points in some vector space and similarity is a simple function of the distance between points (Nosofsky, 1992; Shepard, 1987). However, if similarities are modeled in this way, then they must be consistent with the metric axioms, three simple properties that all distances must obey: minimality (the distance between a point to itself is zero), symmetry (the distance between two points is the same regardless of starting point), and the triangle inequality (the side of a triangle will always be less than the sum of the other two sides). Tversky (1977) reported similarity findings which sharply contrasted with all three metric axioms. For example, regarding symmetry, he reported that the similarity of China to Korea was judged to be lower than the similarity of Korea to China (Aguilar & Medin, 1999; Tversky & Gati, 1982). There are simple parametric ways to cover such findings within distance-based models. For example, for symmetry, distances can be multiplied by a directionality parameter which allows asymmetries (Nosofsky, 1992). However, such approaches build in the asymmetry by adding parameters rather than having it follow from more basic principles.

Pothos, Busemeyer, and Trueblood (2013; Yearsley et al., 2017) proposed that similarity can be modeled using QPT by representing the two compared objects using incompatible subspaces, so that

the order of evaluation is important. For example, the similarity of China to Korea involves thinking about China and then about Korea, with asymmetries arising from incompatibility of subspaces, and the direction of the effect resulting from the differences in subspace dimensionality (see Figure 4). This calculation is identical to that used to compute the probability of a pair of answers, so QPT assumes the same fundamental processes are used to make similarity and probability judgments (see Shafir et al., 1990, for supporting evidence). Similarity structure in the QPT model arises from the way subspaces relate to each other – Busemeyer and Wang (2017) offered a proposal for how such representations can be computed. The QPT similarity model covers Tversky's (1977) and other main findings in the similarity literature, including structural constraints in similarity judgment. For example, comparing two individuals, the hair color of the first person would be compared to the hair color of the second person, not the color of her shoes (Gentner, 1983; Goldstone, 1994). The QPT similarity model can be extended to capture structural constraints, by employing a different space for each relevant object part and combining these individual representations into an overall one (using an operation called tensor product; Pothos and Trueblood, 2015).

Comparison with alternative models.

Similarity has been intensely studied. Tversky's (1977) own proposal was that similarity depends on common and distinctive features, but with weights determined by which concept is the target and which the referent in a similarity comparison. Ashby and Perrin (1988) considered representations as distributions of perceptual effects from perceiving the same object. Then, the overlap between these distributions would determine similarity; this is their influential general recognition theory. Krumhansl (1978) adopted a modified distance metric, such that the local density of a point would compress or expand distance. Hahn, Chater, and Richardson (2003) equated similarity with algorithmic transformational complexity. Pothos et al. (2013) pointed out some difficult assumptions in all these models. For example, regarding the triangle inequality, Ashby and Perrin (1988) assumed unequal perceptual distributions and Krumhansl (1978) postulated similarity computations on subspaces of the overall space. Criticizing the QPT model, Kintsch (2014) argued that differences in extent of knowledge can be modeled without the (QPT) subspaces, in a standard representation, by employing vectors differing in length. Even though Kintsch's (2014) proposal captures differences in the extent of knowledge, the internal structure of his representations is uninformative about what this knowledge is. In the QPT model, higher dimensionality subspaces would be associated with more basis vectors, with the latter being interpretable as features (or feature bundles).

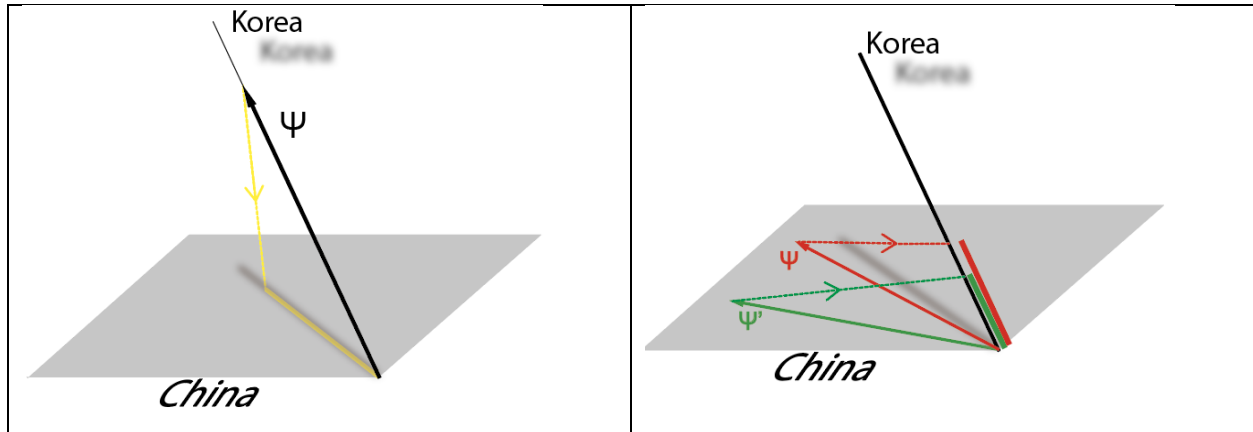


Figure 4. In the left figure, we are initially thinking about Korea (ψ is a normalized vector along the Korea ray) and the similarity to China is a projection from Korea to China. Since the Korea subspace is a ray, there is only one way to have a projection of a state from Korea to China. In the right figure, we are initially thinking about China, but in this case there are infinite possibilities for a normalized state within the China subspace (such as ψ or ψ'). So, the projection from China to Korea can likewise vary, with the maximum attained only for a specific state.

Entanglement

Although we have focused our attention on incompatible measurements, QPT also incorporates compatible questions. With compatible questions, QPT and CPT generally converge – with an important exception. With compatible questions, QPT, like CPT, can form events that are conjunctions of outcomes from a pair of measurements. The state vector in this case is a superposition over all possible conjunctions. A special type of superposition state for a pair of compatible questions is an *entangled state*, which is a state that cannot be decomposed into the product of states for each question separately. This is analogous to a dependent joint probability distribution in CPT. However, entangled quantum states can produce dependencies that are called contextual, which cannot be produced by dependent joint distributions in CPT. ‘Contextual’ refers to the idea that the meaning of one question changes depending on the other questions occurring at the time. This leads us to the notion of supercorrelation.

Any psychology student learns that the highest possible correlation between two variables is when the variables perfectly align with each other (or are perfectly opposite to each other). How can it be otherwise? Suppose we have four different binary questions denoted $a1$, $a2$, $b1$, and $b2$. We can form four 2×2 joint frequency tables produced by all question combinations, $(a1 \times b1)$, $(a1 \times b2)$, $(a2 \times b1)$, and $(a2 \times b2)$. From each of these tables we can compute a correlation, denoted as $C(a1, b2)$ for the table for $a1 \times b2$ etc. (strictly speaking these are expectation values). Then we define the quantity $CHSH = C(a1, b1) + C(a1, b2) + C(a2, b1) - C(a2, b2)$ ($CHSH$ stands for the scientists Clauser, Horn, Shimony, and Holt, who invented this quantity to test CPT versus QPT theories). Suppose that $a1, b1, a2$ all maximally correlate with each other, so that we have $CHSH = 3 - C(a2, b2)$. Classically, if $a1, b1, a2$ correlate maximally, then we must also have that $C(a2, b2) = 1$ and so $CHSH = 2$. This value of $CHSH$ is called Bell's bound (Bell, 2004).

One of the most subtle and powerful aspects of QPT is that Bell's bound can be exceeded, in which case we say the variables are *supercorrelated*. For example, we could have $C(a2, b2) = -1$, producing $CHSH = 4$ (but note that QPT allows S values only up to about 2.8, which is Tsirelson's bound). If $C(a1, b2) = 1$ and $C(a2, b2) = -1$, then it is as if question $b2$ is treated differently, depending on whether we consider it with question $a1$ or $a2$. We can say that question $b2$ depends contextually on which other question we have. Therefore, supercorrelation means perfect coordination between the a and the b questions, in a way that the b questions contextually depend on which a question we consider. Supercorrelations can arise in QPT only when the state vector for the a and b questions is entangled.

Supercorrelations indicate that questions $a1$, $a2$ cannot be understood independently from $b1$, $b2$. This lack of independence is relevant for theories of conceptual combination. For the example of a pet fish (which, as we saw, relates to a conjunction fallacy), the meaning of the combination cannot be determined by independently (i.e., compositionally; Fodor, 1994) combining the meaning from the individual concepts. Bruza et al. (2015) demonstrated supercorrelations for novel, ambiguous conceptual combinations composed of two words, such that each word had two (fairly) distinct meanings; e.g., in 'boxer bat', boxer can refer to a person or a dog and bat to a sporting equipment or an animal (for a similar proposal in memory see Bruza et al., 2009). Other researchers have focused on decisions across two pairs of questions. For example, Aerts et al. (2018) employed questions about pairs of wind directions, Cervantes and Dzhamfarov (2018) posed questions about the snow queen fairy tale, and Basieva et al. (2019) asked questions about meal choices. The latter two studies incorporated a correction in the computation of the $CHSH$ quantity, which rules out simpler accounts for the contextual result (Dzhamfarov, Zhang, & Kujala, 2016). Some researchers have also explored the possibility of

supercorrelations in time, though direct empirical confirmations have been limited (Atmanspacher & Filk, 2010; Yearsley & Pothos, 2014). Aerts et al. (2018) and Busemeyer and Wang (2018a) proposed models to account for measurement context effects that use both concepts of incompatible measurements and entangled states.

Comparison with alternative models.

Busemeyer and Wang (2018a) conducted two large experiments designed to investigate measurement context effects. In one study, participants were presented with public health service announcements and made binary decisions to six pairs of questions concerning the effectiveness of the announcements. In another study, participants viewed computer generated avatars and made binary decisions to six pairs of questions formed by four measurements concerning attitudes towards the avatars. For both studies, Busemeyer and Wang showed that QPT models were superior to Bayesian causal network models for predicting the joint frequencies in the 6 two-way tables measures. QPT models for conceptual combinations have also employed entanglement to and supercorrelations (Aerts et al., 2018).

Quantum dynamics

In psychology, probabilistic dynamic systems are most frequently modeled using Markov models, such as random walk/ diffusion models of decision making (Ratcliff et al. 2016). Quantum theory provides another general system for modeling probabilistic dynamic systems. Markov systems describe the evolution of probability distributions across time using the Kolmogorov equation. Quantum systems describe the evolution of the state vector across time using the Schrödinger equation.

One important application of both of these approaches has been to signal detection tasks, that require evidence accumulation across time. Participants are presented with information across time, so that the information is sampled from one of two possible states of nature. The information is noisy, which makes the decision difficult and take time. The decision maker's task is to determine which of the two states of nature is generating the information across time. At various points, the decision maker can report his/ her decision.

In such tasks, the Markov and QPT models are similar, but there is a critical difference when successive measurements are taken across time. The Markov model obeys a type of law of total probability across time, called the Chapman – Kolmogorov equation, but the QPT model allows violations of this law. Kvam, Pleskac, and Busemeyer (2015) tested the two types of dynamic models

using a perceptual decision paradigm, in which participants had to judge the predominant motion direction in a dynamic dot display. In a choice – confidence condition, participants studied the dots, made a decision regarding motion, studied the dots again, and finally provided a confidence rating. In a confidence only condition, participants just studied the dots and then provided the confidence rating. According to the Chapman – Kolmogorov equation, the distribution of confidence ratings should be identical across the two conditions, but this was not the case. The difference between the conditions revealed an interference effect predicted by the quantum model. Additionally, better fits were provided by Kvam et al.'s (2015) QPT model as compared to a matching Markov one.

A second critical difference between Markov and QPT models concerns the way dynamics evolve across time. Traditional Markov models of evidence accumulation predict a monotonic growth in the mean of the accumulation across time, making multiple preference reversals plausible only in multiattribute choice, and even in such cases dependent on the particular way the available options are compared (Diederich, 2003; Johnson & Busemeyer, 2005; Usher & McClelland, 2001). By contrast, the typical dynamics in QPT predict systematic oscillation effects across time. Thus, QPT dynamical models are suitable when we expect ambivalence and rumination in the decision process.

Kvam et al. (2020) tested these predictions using a preference task. Participants were given a choice between two coupons for restaurants that differed with respect to quality, cost, and distance. Participants were asked for preference ratings at several time points, with and without a prior choice. As for Kvam et al. (2015), the prior choice produced interference effects. Moreover, it was found that preferences systematically oscillated across time, as predicted by a QPT model (Figure 6).

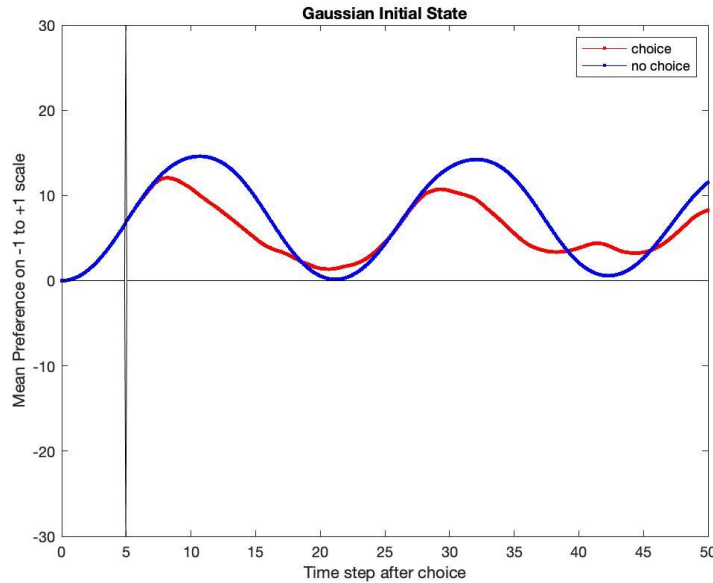


Figure 6. Oscillation and interference predicted by quantum dynamics. The vertical line indicates the time point when a choice was made for the condition with choice, followed by preference ratings across time, and the red curve represents the predictions for this condition. The blue curve represents the predictions when no choice was made at the vertical line. The choice attenuates the oscillation.

Comparisons with alternative models.

Kvam et al. (2015) compared the predictions for choice and confidence of a quantum walk model to a Markov random walk model using a Bayes factor, and the Bayes factor favored the quantum model for the majority of participants. Busemeyer, et al. (2019) compared the two models using a generalization test: the model parameters were estimated from two timing intervals for measuring confidence, and then these same parameters were used to predict the confidence ratings for a third time interval. The quantum walk model produced more accurate generalization predictions. Markov and quantum walk models have also been compared using choice and response time distributions. Navarro and Fuss (2014) found a small advantage for a quantum dynamic model over a Markov random walk model for predicting choice and response time in a signal detection type of task.

The oscillatory feature of QPT dynamics offers a precise expression of ideas in social psychology that, sometimes, we go through several cycles of back and forth before making up our mind (Brehm & Wicklund, 1970; Walster, 1964).

Integrating classical and non-classical probabilities

The final feature of QPT we discuss concerns the way QPT models can bring together both QPT and CPT probabilistic inference. The results so far indicate that QPT thinking emerges when there is less familiarity with a problem and participants are responding in a more reflexive way, and vice versa concerning CPT thinking (Trueblood et al., 2017). QPT thinking can be considered as less complex than CPT thinking. In the former case, because of incompatibility, we can embed the representations for several questions in the same low dimensionality space, but for the latter the amount of probabilistic information increases rapidly (exponentially) with increasing questions (cf. Atmanspacher & Romer, 2012; Pothos et al., in press). Therefore, a proposal is that QPT might serve as a default thinking mode, gradually replaced by CPT, as experience with a task accumulates, and depending on the importance of the task in the first place.

Trueblood et al. (2017) studied the relations between CPT and QPT in the case of causal reasoning, using Rehder's (2014) paradigm. Causal reasoning concerns how information about causes impacts on conclusions about possible effects. For example, a fallacy in causal reasoning is to confuse effects and causes, by considering the probability of an effect given a cause as the same as the probability of a cause given an effect. It appears that there is a CPT component to human causal reasoning, as well as influences from several biases. Trueblood et al. (2017; Mistry et al., 2018) proposed a model which would reflect varying degrees of quantumness: in one extreme case, all representations would be quantum (incompatible), in another classical (compatible), and intermediate cases were included too, such that some representations would be quantum and some classical. With this framework, they were able to offer a unified description of results in causal reasoning, for which Rehder (2014) had employed four separate components (a CPT one and three heuristic ones).

Summary evaluation of QPT models

QPT models have worked best when it appears that there are interference effects, the mental state is altered by earlier choices or measurements, and questions are contextual, that is, the meaning of any question depends on which other questions are present. There have been several, compelling heuristic ideas and models designed individually for each separate result, across very different theoretical traditions (e.g., Hogarth & Einhorn, 1992; Brainerd et al., 1999; Schwarz, 2007; Tversky & Kahneman, 1983). We suggest that one main advantage of quantum cognitive models is that they provide a single and common set of coherent, formal principles, to accommodate the role of measurement and

contextuality in cognition, across disparate cognitive areas. Several findings which have been considered separately, such as conjunction and disjunction fallacies, question order effects, and disjunction effects in decision making, have been shown using QPT to have the same theoretical origin – interference in this case – and explained with similar models. In other cases, separate models can be subsumed into a single QPT one (e.g., as by Trueblood et al., 2017, for Rehder, 2014).

We have reviewed several examples of successful QPT cognitive models, but not all (e.g., Al Nowaihi & Dhami, 2017; Basieva et al., 2017; Blutner, Pothos, & Bruza, 2013; Favre et al., 2016; LaMura, 2009, to mention but a few). We expect that there will be cases where QPT models fail, for example, when the circumstances are appropriate for CPT reasoning (as Trueblood et al., 2017, considered) or when neither QPT nor CPT can offer satisfactory descriptions. Indeed, there are examples of QPT papers reporting problems with QPT models (Bussemeyer, Wang, & Lambert-Mogiliansky, 2009). However, existing critical work for QPT models appears to rely on concepts already embodied in QPT (notably, the independent sampling assumption of Costello & Watts, 2014) or offer redescriptions of results in heuristic terms, which can always happen (Kellen et al., 2017) or focus on limited QPT models (Boyer-Kassem et al., 2016). Overall, most (but not all) comparisons between QPT and matched classical models (CPT or otherwise) have favored the QPT models. There is a simple explanation for this: most of QPT modeling has focused on empirical situations which appear to embody key properties of QPT, such as interference and/or collapse. QPT models suit such situations particularly well, so it is unsurprising they offer good descriptions of empirical results.

If QPT models embody both incompatible and compatible representations, but CPT ones only the latter, is it not the case that QPT models will always be more complex than CPT ones? This is not necessarily true. QPT models typically employ incompatible questions so that a complexity consideration involves a QPT model with incompatible questions and a broadly matched CPT one. A number of model comparisons have incorporated complexity and still favored QPT models over matched CPT ones (Broekaert et al., 2020; Bussemeyer, Wang, & Shiffrin, 2015; Mistry et al., 2018; Trueblood & Hemmer, 2017; Trueblood et al., 2017). Another issue concerns whether QPT models have implications for the neural substrate supporting cognition. The explanatory objective of QPT models is best stated using Griffiths et al.'s (2010; Marr, 1982) approach, as top-down or function-first models focusing on the mathematical principles that characterize behavior. As such, implications for neuronal implementation are limited and, in particular, QPT cognitive models do *not* require quantum processes at the neurophysiological level (cf. Litt et al., 2006). In any case, there have been proposals of how the

interference terms required for QPT models can naturally arise from neuronal processes (Khrennikov et al., 2018; Suppes, de Barros, & Oas, 2012).

Can the novel theoretical concepts offered by QPT models translate to generative value? We have discussed several surprising predictions, some entirely unexpected based on predominant theory. As a few main examples, the QQ equality shows that response probabilities for several different question pairs combine in a very precise way, as predicted by QPT theory (Wang et al., 2014). The co-occurrence of conjunction fallacy and question order effects is another unique prediction from QPT and that connection was never made by the earlier heuristic accounts (Yearsley & Trueblood, 2018). Even though it has been well established that decisions can alter a mental state, inertia in opinion change from multiple decisions (quantum Zeno effect) offers new support for the specific account of constructive changes in QPT (Yearsley & Pothos, 2016). Oscillations during evidence accumulation and interference effects of choice on later confidence are also surprising predictions generated by QPT (Kvam et al., 2020).

The key weakness of QPT models is that we are still unsure about what conditions are necessary or sufficient to produce incompatible representations. It is straightforward to conclude that, in some cases, CPT complexity will be intractably complex for bounded agents. But why should a drive to simplify from CPT translate to QPT, as opposed to, for example, heuristics or bounded rational approximations to CPT? There are a few possible answers. It is possible that QPT is the 'next best thing' to CPT, in that, as noted, representations are 'locally' classical, that is, there are subsets of compatible questions, but across subsets we have incompatibility and so non-classical effects (Fernbach & Sloman, 2009; Lewandowsky et al., 2002; Pothos et al., in press). Beim Graben and Atmanspacher (2006) considered how incompatibility between classical questions can sometimes emerge, if the description of the answers to some question are coarse, so that questions have some irreducible fuzziness regarding possible outcomes. Finally, Aerts and Sassoli de Bianchi (2015) proposed that the quantum probability rule in cognition arises from averaging across experimental conditions. Consider a typical decision experiment, in which participants are asked to assign probabilities to possibilities. Suppose that different participants employ different probability rules. According to Aerts and Sassoli de Bianchi (2015), averaging across participants can be shown to reduce to the QPT rule for probabilistic assignment, and this is why quantum theory appears successful in cognitive applications.

These ideas have yet to be fleshed out in a satisfactory way. However, the situation is analogous to the application of QPT in physics – successful empirical applications by far predated the developmental of foundational arguments (Hardy, 2001). Moreover, incompatibility can always be

empirically determined, e.g., with question order effects, constraining any subsequent predictions. We finish with three key points: first, if one thinks that probabilistic inference is an important part of cognition, then he or she should not be restricted to just CPT; second, QPT is the most established alternative to CPT; third, the progress of the QPT cognitive programme has so far been very promising.

Concluding comments

QPT offers to psychology several new concepts with potential explanatory value, a sophisticated framework for modeling, and promise to formalize powerful intuitions which have so far been heuristically explained. QPT cognitive models appear to work particularly well in certain empirical cases, such as when it appears that there is interference, constructive influences, or contextuality, and it is there that researchers in this community have focused their efforts. At the same time, we believe that we have only scratched the surface of QPT models, in terms of their potential to revolutionize psychological theory. This is a challenge for the next 10 years.

Acknowledgments

EMP was supported by ONRG grant N62909-19-1-2000; JRB was supported by AFOSR FA9550-15-1-0343 and U.S. National Science Foundation SES-1560554. We would like to thank Joyce Wang, Jennifer Trueblood, and James Yearsley, as well as John Wixted for his many helpful comments.

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