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# **Application of Stochastic Methods in the Valuation of Social Security Pension Schemes.**

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## **APPLICATION OF STOCHASTIC METHODS IN THE VALUATION OF SOCIAL SECURITY PENSION SCHEMES**

**Subramaniam Iyer**

### **1. INTRODUCTION**

The actuarial valuation of social security pension schemes has invariably been based on the deterministic approach. Although it is recognized that the determining parameters are subject to stochastic variation, the valuation is based on constant values of the parameters, chosen so as to represent the “average” that is expected each year over the time period covered by the valuation (Iyer, 1999). Sometimes results are also presented for alternative sets of parameters (typically, for two additional sets on either side, termed “conservative” and “optimistic”), or the sensitivity of the results to individual key parameters are illustrated, but apart from having some recourse to the simulation technique, stochastic methods have not seen much application (Plamondon et.al., 2002).

This contrasts with the case of other areas of insurance, including that of occupational pensions, where stochastic methods are increasingly being applied (Daykin et.al., 1994; Booth et.al., 1999).

This paper attempts to develop a stochastic valuation method for a newly introduced social security pension scheme. The approach is not based on the simulation technique. Rather, the purpose is to derive algebraic expressions for the variances of, and the co-variances among, the important aggregates that characterize the development of the scheme, i.e. the financial projections of insured salaries and benefit expenditures and their discounted values, taking into account stochastic variation in the following factors:

- The age-specific cohort survival
- The time-specific intake of new entrants
- The time-specific real rate of return on the Fund (i.e. the return net of salary escalation).

On this basis, a method of determining premiums is developed which recognizes explicitly and quantifies the underlying stochastic variations.

In order to keep the mathematical manipulations within reasonable limits, however, the following simplifying assumptions are made in regard to the pension model:

- The scheme provides pensions only on retirement (invalidity and survivors’ pensions are not considered). There are no pensioners at the outset.
- The ages of entry and retirement are each unique, and constant over time
- The age-specific mortality rates are constant over time
- The pension is based on the final salary and is fully indexed to salary escalation.

Another important set of assumptions concerns independence. Each individual member of the scheme is assumed to be independent of every other member – whether in a different cohort or even in the same cohort - in regard to the mortality factor. Further, the investment return factor is assumed to be independent of the mortality factor and the new entrant factor.

For the development of the theory a continuous formulation is adopted. The paper does not enter into the details of practical application. However, numerical illustrations of the main results are provided in Appendix 8.

## 2. DEMOGRAPHIC PROJECTIONS, ALLOWING FOR STOCHASTIC VARIATION IN COHORT SURVIVAL

### 2.1 Notation and assumptions

In classical demographic projections, the number of survivors to various ages from a given group of individuals is projected exactly in line with the (assumed) underlying life table function. In this paper, it is recognized that cohort survival as above represents only the average progression of the cohort. The actual outcome is subject to stochastic variation around the average. The aim is to quantify the extent of the variation.

For the present purposes, the age  $x$  and time  $t$  are treated as continuous variables. The population element aged  $x$ , at time  $t$ , is, in general, denoted by  $P(x, t)$ . That is to say, the population aged between  $x$  and  $x+dx$  at time  $t$  is  $P(x, t)dx$ . When it is necessary to distinguish between the Active Population and the Retired Population, they are indicated, respectively, by the symbols  $Ac(x, t)$  and  $Re(x, t)$ . The Total Active and Retired Populations at time  $t$  are denoted by the symbols  $A(t)$  and  $R(t)$ .

The entry age is denoted by  $b$  and the retirement age by  $r$ . The initial active population,  $Ac(x, 0)$ ,  $b = x < r$ , is assumed to be given, with  $Ac(b, 0)dt$  representing the number of new entrants at  $t = 0$  – more precisely, the number of new entrants in the interval  $(-dt, 0)$ . The number of new entrants at time  $t$  – or rather, in the interval  $(t-dt, t)$  – is given by  $Ac(b, 0)e^{rt} dt$ , where  $r$  is an assumed constant. **The “dt” is hereafter omitted in the text for brevity, but should be understood as implied.**

Stochastic variation in survival is assumed to apply to each population element from the time it enters the purview of the scheme. The applicable mortality table, representing the “average” survival pattern, is denoted by  $\{l_x\}$ ,  $b = x < ?$ , where  $?$  is the limiting age, with  $? - r < r - b$  (i.e. the potential active lifetime exceeds the potential retirement lifetime).

### 2.2 Initial Population cohorts

If  $P(x, t)$  derives from an initial population cohort, the condition  $t - x + b < 0$  would apply, and the corresponding initial population element would be  $Ac(x - t, 0)$ . The number surviving to age  $x$  out of this number has the Binomial distribution with parameter  ${}_t p_{x-t}$ . Therefore (see Appendix 1, Note 1), the expected value and variance of  $P(x, t)$  are given by

$$EP(x, t) = Ac(x - t, 0) {}_t p_{x-t} \quad (1)$$

$$VP(x, t) = Ac(x - t, 0) {}_t p_{x-t} (1 - {}_t p_{x-t}) \quad (2)$$

If  $P(x, t)$  and  $P(y, u)$ , both deriving from initial population cohorts, are from different cohorts, i.e.  $x - t \neq y - u$ , they will be independent and therefore uncorrelated. However, if they derive from the same cohort ( $x - t = y - u$ ), they will be correlated. Assuming  $x > y$ , the covariance is given by (see Appendix 1, Note 1),

$$COV(P(x, t), P(y, u)) = Ac(x - t, 0) {}_t p_{x-t} (1 - {}_u p_{y-u}) \quad (3)$$

### 2.3 New Entrant cohorts

We shall first consider the case corresponding to a unit new entrant at  $t=0$ . If  $P(x, t)$  derives from a new entrant cohort ( $t-x+b > 0$ ), the size of the cohort at entry would be  $e^{(t-x+b)r}$ , and the expected value and variance of  $P(x, t)$  would be given by

$$EP(x, t) = e^{(t-x+b)r} p_b = e^{rt} \frac{D_x^{(r)}}{D_b^{(r)}} \quad (4)$$

$$VP(x, t) = e^{(t-x+b)r} p_b (1 - p_b) = e^{rt} \left[ \frac{D_x^{(r)}}{D_b^{(r)}} - \frac{D_{xx}^{(r)}}{D_{bb}^{(r)}} \right] \quad (5)$$

where  $D_x^{(r)} = l_x e^{-rx}$ ,  $D_{xx}^{(r)} = l_x l_x e^{-rx}$ .

The above formulae correspond to a unit new entrant at  $t = 0$ . The actual number of new entrants at  $t = 0$  being  $Ac(b, 0)$ , it is necessary to adjust formulae (4) and (5). The adjustment is obtained by multiplying both the formulae by  $Ac(b, 0)$ . It might be thought that formula (5) should be multiplied by  $Ac(b, 0)^2$ , but this would be incorrect, in view of the assumed independence of individual members within a cohort (see Appendix 1, Note 2). The adjusted formulae are given by

$$EP(x, t) = Ac(b, 0) e^{rt} \frac{D_x^{(r)}}{D_b^{(r)}} \quad (6)$$

$$VP(x, t) = Ac(b, 0) e^{rt} \left[ \frac{D_x^{(r)}}{D_b^{(r)}} - \frac{D_{xx}^{(r)}}{D_{bb}^{(r)}} \right] \quad (7)$$

$P(x, t)$  and  $P(y, u)$  will be independent and uncorrelated if one is from an initial population cohort and the other is from a new entrant cohort. Similarly, if  $P(x, t)$  and  $P(y, u)$ , both deriving from new entrant cohorts, are from different cohorts, i.e.  $x-t \neq y-u$ , they will be independent and therefore uncorrelated. However, if they derive from the same cohort ( $x-t = y-u$ ), they will be correlated. Assuming  $x > y$ , the covariance is given by the following formula (see Appendix 1, Note 1), including the adjustment required to take into account the actual number of new entrants at  $t = 0$ , namely  $Ac(b, 0)$ ,

$$COV(P(x, t), P(y, u)) = Ac(b, 0) e^{(t-x+b)r} p_b (1 - p_b) = Ac(b, 0) e^{rt} \frac{D_x^{(r)}}{D_b^{(r)}} (1 - p_b) \quad (8)$$

### 2.4 Total Active and Retired Population projections: Expected values and variances

Since the different cohorts are considered to be evolving independently of each other, the variances are additive across cohorts at each point in time. When the expected values of the different cohorts surviving to time  $t$  are added to produce the expected total active and retired populations, the corresponding variances can be added to obtain the variance of the total active and retired populations. Due to the differences in the formulae, the elements from the initial population cohorts and those from new entrant cohorts should be added separately, over the appropriate age ranges,  $a_1(t), a_2(t); a_0(t), a_1(t)$ . This will give rise to the following formulae for the expected value and variance of  $A(t)$  or  $R(t)$ :

$$EA(t) \text{ or } ER(t) = Ac(b,0) \left[ \int_{a_1(t)}^{a_2(t)} \frac{Ac(x-t,0)}{Ac(b,0)} {}_t p_{x-t} dx + e^{rt} \int_{a_0(t)}^{a_1(t)} \frac{D_x^{(r)}}{D_b^{(r)}} dx \right] \quad (9)$$

$$VA(t) \text{ or } VR(t) = Ac(b,0) \left[ \int_{a_1(t)}^{a_2(t)} \frac{Ac(x-t,0)}{Ac(b,0)} {}_t p_{x-t} (1 - {}_t p_{x-t}) dx + e^{rt} \int_{a_0(t)}^{a_1(t)} \left( \frac{D_x^{(r)}}{D_b^{(r)}} - \frac{D_{xx}^{(r)}}{D_{bb}^{(r)}} \right) dx \right] \quad (10)$$

For the simplified pension model assumed in this paper, with constant entry and retirement ages  $b$  and  $r$ , constant mortality table and with no pensioners at the outset of the scheme, the limits  $a_0(t), a_1(t), a_2(t)$  will be as follows (Note:  $-r < r-b$ ):

| Active/Retired population | Time range                                                   | Age range for IP<br>( $a_1(t), a_2(t)$ ) | Age range for NE<br>( $a_0(t), a_1(t)$ ) |
|---------------------------|--------------------------------------------------------------|------------------------------------------|------------------------------------------|
| Active population $A(t)$  | $0 = t < r-b$<br>$t = r-b$                                   | $b+t, r$<br>$nil$                        | $b, b+t$<br>$b, r$                       |
| Retired population $R(t)$ | $0 = t < -r$<br>$-r = t < r-b$<br>$r-b = t < -b$<br>$t = -b$ | $r, r+t$<br>$r, ?$<br>$b+t, ?$<br>$nil$  | $nil$<br>$nil$<br>$r, b+t$<br>$r, ?$     |

## 2.5 Total Active and Retired Population projections: Co-variances

$A(t)$  will be correlated with  $A(u)$ , and  $R(t)$  with  $R(u)$ , to the extent that there are common cohorts between either pair. Similarly  $R(t)$  will be correlated with  $A(u)$ ,  $u < t$ , if there are common cohorts. However,  $R(t)$  will not be correlated with  $A(u)$  if  $u > t$ , since there cannot be common cohorts between them.

Consider, for example,  $A(t)$  and  $A(u)$ , with  $u < t$ . If  $Ac(x, t)$  and  $Ac(y, u)$  are from the same cohort, then  $x-t = y-u$ , or  $x = y+t-u$ . If common cohorts exist over the range  $(c_0(t, u), c_2(t, u))$  of  $y$ , of which  $(c_0(t, u), c_1(t, u))$  concerns new entrant cohorts and  $(c_1(t, u), c_2(t, u))$  relates to initial population cohorts, using (3) and (8) above, the co-variance between  $A(t)$  and  $A(u)$  can be expressed as

$$Ac(b,0) \left[ e^{rt} \int_{c_0(t, u)}^{c_1(t, u)} \frac{D_{y+t-u}^{(r)}}{D_b^{(r)}} (1 - {}_{y-b} p_b) dy + \int_{c_1(t, u)}^{c_2(t, u)} \frac{Ac(y-u, 0)}{Ac(b,0)} {}_t p_{y-u} (1 - {}_u p_{y-u}) dy \right] \quad (11)$$

Similar formulae apply to the covariance between  $R(t)$  and  $R(u)$  and between  $R(t)$  and  $A(u)$ ,  $u < t$ . The appropriate limits of integration are obtained according to the following rules:

If either  $(a_0(t), a_1(t))$  or  $(a_0(u), a_1(u)) = nil$ , then  $(c_0(t, u), c_1(t, u)) = nil$ .

If  $a_0(t) - t + u > a_1(u)$  or  $a_1(t) - t + u < a_0(u)$  then  $(c_0(t, u), c_1(t, u)) = nil$ .

Otherwise, if  $a_0(t) - t + u < a_0(u)$  then  $c_0(t, u) = a_0(u)$ ;

if  $a_0(t) - t + u > a_0(u)$  then  $c_0(t, u) = a_0(t) - t + u$ ;

if  $a_1(t) - t + u < a_1(u)$  then  $c_1(t, u) = a_1(t) - t + u$ ;

if  $a_1(t) - t + u > a_1(u)$  then  $c_1(t, u) = a_1(u)$ .

If either  $(a_1(t), a_2(t))$  or  $(a_1(u), a_2(u)) = nil$ , then  $(c_1(t, u), c_2(t, u)) = nil$ .

If  $a_1(t) - t + u > a_2(u)$  or  $a_2(t) - t + u < a_1(u)$  then  $(c_1(t, u), c_2(t, u)) = nil$ .

Otherwise, if  $a_1(t) - t + u < a_1(u)$  then  $c_1(t, u) = a_1(u)$  ;  
 if  $a_1(t) - t + u > a_1(u)$  then  $c_1(t, u) = a_1(t) - t + u$  ;  
 if  $a_2(t) - t + u < a_2(u)$  then  $c_2(t, u) = a_2(t) - t + u$  ;  
 if  $a_2(t) - t + u > a_2(u)$  then  $c_2(t, u) = a_2(u)$  .

The limits of integration for the simplified pension model are given in Appendix 2.

## 2.6 The long-term situation

In the long-term ( $t = \infty$ ) both the active and the retired populations would consist entirely of new entrant cohorts, and the above formulae would simplify as follows:

$$EA(t) = Ac(b, 0)e^{rt} \frac{\bar{N}_b^{(r)} - \bar{N}_r^{(r)}}{D_b^{(r)}}; ER(t) = Ac(b, 0)e^{rt} \frac{\bar{N}_r^{(r)}}{D_b^{(r)}} \quad (12)$$

$$VA(t) = Ac(b, 0)e^{rt} \left[ \frac{\bar{N}_b^{(r)} - \bar{N}_r^{(r)}}{D_b^{(r)}} - \frac{\bar{N}_{bb}^{(r)} - \bar{N}_{rr}^{(r)}}{D_{bb}^{(r)}} \right]; VR(t) = Ac(b, 0)e^{rt} \left[ \frac{\bar{N}_r^{(r)}}{D_b^{(r)}} - \frac{\bar{N}_{rr}^{(r)}}{D_{bb}^{(r)}} \right] \quad (13)$$

where,  $\bar{N}_{xx}^{(r)} = \int_x^v D_{yy}^{(r)} dy$

It will be noted that although  $EA(t)$  and  $ER(t)$  and  $VA(t)$  and  $VR(t)$  increase continuously and tend to infinity as  $t$  tends to infinity (unless  $\beta = 0$ ), the respective coefficients of variation,  $CVA(t)$  and  $CVR(t)$ , are proportional to  $Ac(b, 0)^{-\frac{1}{2}} e^{-\frac{1}{2}rt}$ . Therefore, they are inversely related to the square root of the population factor  $Ac(b, 0)$  and tend to zero as  $t$  tends to infinity, provided  $\beta > 0$ . This is in conformity with the law of large numbers.

## 2.7 The Demographic Ratio

Consider the Demographic Ratio, defined as

$$DR(t) = \frac{R(t)}{A(t)} \quad (14)$$

$R(t)$  can be regarded as independent of  $A(t)$  as they are constituted by different cohorts. Therefore  $R(t)$  and  $A(t)$  are uncorrelated, and if the third and higher central moments of  $A(t)$  as well as fourth and higher powers of  $CVA(t)$  are ignored, the following approximate expressions will be obtained for the expected value and the variance of the Demographic Ratio (see Appendix 1, Note 3):

$$EDR(t) = \frac{ER(t)}{EA(t)} [1 + CVA^2(t)] \quad (15)$$

$$VDR(t) = \frac{ER^2(t)}{EA^2(t)} [CVR^2(t) + CVA^2(t)] \quad (16)$$

Equation (15) indicates that the ratio  $ER(t)/EA(t)$  is a biased estimator of the Demographic Ratio, although the bias declines with time if  $\beta > 0$ . Equation (16) shows that the variance of this ratio also tends to decline with time.

Further, it can be shown that the coefficient of variation,  $CVDR(t)$  decreases with  $t$ , tending to 0 as  $t \rightarrow \infty$ , so that the precision of the estimated Demographic Ratio increases with time. The initial population size has also a similar effect; for a given  $t$ , the larger the value of  $Ac(b, 0)$ , the

smaller will be  $CVA(t)$  and  $CVR(t)$  and the larger will be the precision of the estimated Demographic Ratio.

### 3. DEMOGRAPHIC PROJECTIONS, ALLOWING ALSO FOR STOCHASTIC INTAKE OF NEW ENTRANTS

#### 3.1 Notation and assumptions

In classical demographic projections, the annual number of new entrants is assumed deterministically. For example, the number of new entrants in the interval  $(t-dt, t)$  may be taken as  $Ac(b,0)e^{rt}dt$ , where  $r$  is an assumed parameter (see section 2.1).

In this section, the number of new entrants in the interval  $(t-dt, t)$  is assumed to be  $Ac(b,0)e^{\int_0^t r(z)dz}dt$ , where  $e^{r(z)}$  for different values of  $z$  are independent identically distributed (IID) variables, with expected value and second raw moment given by

$$E(e^{r(z)}) = e^r; E(e^{2r(z)}) = e^p \quad (17)$$

so that the variance is given by

$$V(e^{r(z)}) = e^p - e^{2r} \quad (18)$$

For the variance to be positive, we should have  $p > 2r$ . When  $p = 2r$ , the variance is zero, meaning that the new entrant factor is deterministic. The symbol  $f$  is used to denote the difference  $p - 2r$ .

As regards the initial population cohorts, the formulae developed in section 2.2 above are also valid in the present case. It is therefore necessary to discuss only the new entrant cohorts.

#### 3.2 New entrant cohorts: Expected value and variance of $P(x, t)$

As in section 2, we shall first develop the formulae for a unit new entrant at time  $t = 0$ . If  $P(x, t)$  derives from a new entrant cohort, this cohort would have entered at time  $t-x+b$ , and would have

numbered  $L = e^{\int_0^{t-x+b} r(z)dz}$  at entry. In view of the IID property of  $e^{r(z)}$  we have

$$E(L) = e^{(t-x+b)r}; V(L) = e^{(t-x+b)p} - e^{2(t-x+b)r} \quad (19)$$

The conditional expectation and variance of  $P(x, t)$ , given  $L$ , are given by

$$E(P(x, t) | L) = L {}_{x-b}p_b; V(P(x, t) | L) = L {}_{x-b}p_b(1 - {}_{x-b}p_b) \quad (20)$$

The unconditional expectation of  $P(x, t)$  is therefore

$$E(P(x, t)) = EE(P(x, t) | L) = {}_{x-b}p_b E(L) = {}_{x-b}p_b e^{(t-x+b)r} = e^{rt} \frac{D_x^{(r)}}{D_b^{(r)}} \quad (21)$$

The unconditional variance of  $P(x, t)$  is obtained as  $VP(x, t) = EV(P(x, t) | L) + VE(P(x, t) | L)$

$$EV(P(x, t) | L) = {}_{x-b}p_b(1 - {}_{x-b}p_b)E(L) = e^{(t-x+b)r} {}_{x-b}p_b(1 - {}_{x-b}p_b) \quad (22)$$

$$VE(P(x, t) | L) = {}_{x-b}p_b^2 V(L) = {}_{x-b}p_b^2 (e^{(t-x+b)p} - e^{2(t-x+b)r}) \quad (23)$$

Adding (22) and (23) and simplifying, we get

$$VP(x, t) = e^{pt} \frac{D_{xx}^{(p)}}{D_{bb}^{(p)}} - e^{2rt} \left( \frac{D_x^{(r)}}{D_b^{(r)}} \right)^2 + e^{rt} \left( \frac{D_x^{(r)}}{D_b^{(r)}} - \frac{D_{xx}^{(r)}}{D_{bb}^{(r)}} \right) \quad (24)$$

The formulae (21) and (24), which correspond to a unit new entrant at  $t = 0$ , should be adjusted to correspond to the actual number of new entrants at  $t = 0$ , namely  $Ac(b, 0)$ . As in the case of

section 2 above, for the same reasons, the adjustment factor would, in either case, be  $Ac(b, 0)$ . The final expressions, therefore, are

$$EP(x, t) = Ac(b, 0)e^{rt} \frac{D_x^{(r)}}{D_b^{(r)}} \quad (25)$$

$$VP(x, t) = Ac(b, 0)[e^{pt} \frac{D_{xx}^{(p)}}{D_{bb}^{(p)}} - e^{2rt} (\frac{D_x^{(r)}}{D_b^{(r)}})^2 + e^{rt} (\frac{D_x^{(r)}}{D_b^{(r)}} - \frac{D_{xx}^{(r)}}{D_{bb}^{(r)}})] \quad (26)$$

It will be noted that (25) is identical to (6), and that (26) will reduce to (7) if  $p = 2?$

### 3.3 New entrant cohorts: Covariance between $P(x, t)$ and $P(y, u)$ , from the same cohort

Unlike the case of section 2,  $P(x, t)$  and  $P(y, u)$  will be correlated, whether or not they are from the same cohort. As before, the formulae are first developed for a unit new entrant at  $t = 0$ , and then adjusted to correspond to  $Ac(b, 0)$  new entrants at  $t = 0$ .

Since they are from the same cohort,  $x-t = y-u$ . Assume  $x > y$ .  $L$  has the same definition as in section 3.2. We use the formula (see Appendix 1, Note 4):

$$COV(P(x, t), P(y, u)) = ECOV(P(x, t), P(y, u) | L) + COV(E(P(x, t) | L), E(P(y, u) | L)) \quad (27)$$

The development of the formula proceeds as follows:

$$COV(P(x, t), P(y, u) | L) = L_{x-b} p_b (1 - {}_{y-b} p_b) \quad (28)$$

$$ECOV(P(x, t), P(y, u) | L) = {}_{x-b} p_b (1 - {}_{y-b} p_b) E(L) = e^{(t-x+b)r} {}_{x-b} p_b (1 - {}_{x-b} p_b) \quad (29)$$

$$E(P(x, t) | L) = L_{x-b} p_b; E(P(y, u) | L) = L_{y-b} p_b \quad (30)$$

$$E[E(P(x, t) | L) E(P(y, u) | L)] = {}_{x-b} p_b {}_{y-b} p_b E(L^2) \quad (31)$$

$$EE(P(x, t) | L) * EE(P(y, u) | L) = {}_{x-b} p_b {}_{y-b} p_b E^2(L) \quad (32)$$

$$COV[E(P(x, t) | L), E(P(y, u) | L)] = {}_{x-b} p_b {}_{y-b} p_b (E(L^2) - E^2(L)) \quad (33)$$

Substituting for  $E(L^2)$  and  $E(L)$  in (33), combining with (29) and simplifying, and incorporating the adjustment for correspondence with  $Ac(b, 0)$  new entrants at  $t = 0$ ,

$$COV(P(x, t), P(y, u)) = Ac(b, 0)[(e^{pt} \frac{D_x^{(p)}}{D_b^{(p)}} - e^{2rt} \frac{D_x^{(2r)}}{D_b^{(2r)}}) {}_{y-b} p_b + e^{rt} \frac{D_x^{(r)}}{D_b^{(r)}} (1 - {}_{y-b} p_b)] \quad (34)$$

It will be noted that (34) will reduce to (26) when  $x = y$ , and to (8) when  $p = 2?$ .

### 3.4 New Entrant cohorts: Covariance between $P(x, t)$ and $P(y, u)$ , from different cohorts

Since they are not from the same cohort,  $x-t \neq y-u$ . Let us assume that  $x-t < y-u$ , which means that  $P(x, t)$  is from a later cohort than  $P(y, u)$ . Let  $L_1 = e^{\int_0^{t-x+b} r(z) dz}$  and  $L_2 = e^{\int_0^{u-y+b} r(z) dz}$ . We use the following formula (see Appendix 1, Note 4)

$$ECOV(P(x, t), P(y, u) | L_1, L_2) + COV(E(P(x, t) | L_1), E(P(y, u) | L_2)) \quad (35)$$

The first term of (35) is zero, because  $P(x, t)$  and  $P(y, u)$  are uncorrelated if the respective cohorts are non-stochastic. Therefore only the second term is relevant. The derivation proceeds as follows, per unit new entrant at  $t = 0$ :

$$E(L_1) = e^{(t-x+b)r}; E(L_2) = e^{(u-y+b)r} \quad (36)$$

$$E(L_1 L_2) = E(e^{\int_0^{t-x+b} r(z) dz + \int_0^{u-y+b} r(z) dz}) = E(e^{2 \int_0^{u-y+b} r(z) dz + \int_{u-y+b}^{t-x+b} r(z) dz}) = e^{(u-y+b)r + (t-x-u+y)r} \quad (37)$$

$$EE(P(x, t) | L_1) = E(L_{1-x-b} p_b) = {}_{x-b} p_b E(L_1) \quad (38)$$

$$EE(P(y, u) | L_2) = E(L_{2-y-b} p_b) = {}_{y-b} p_b E(L_2) \quad (39)$$

$$E[E(P(x, t) | L_1) * E(P(y, u) | L_2)] = {}_{x-b} p_b {}_{y-b} p_b E(L_1 L_2) \quad (40)$$

Substituting from (36) and (37) in (38), (39) and (40), the required co-variance is obtained as (40) – (38)\*(39) which, after allowing for the actual number of new entrants at  $t = 0$ , namely  $Ac(b, 0)$  and simplification, gives the following result:

$$COV(P(x, t), P(y, u)) = Ac(b, 0) e^{rt} \frac{D_x^{(r)}}{D_b^{(r)}} [e^{fu} \frac{D_y^{(f)}}{D_b^{(f)}} - e^{ru} \frac{D_y^{(r)}}{D_b^{(r)}}] \quad (41)$$

If  $x-t > y-u$ , i.e.  $P(x, t)$  is from an earlier cohort than  $P(y, u)$ , the expression for the co-variance would become

$$Ac(b, 0) e^{ru} \frac{D_y^{(r)}}{D_b^{(r)}} [e^{ft} \frac{D_x^{(f)}}{D_b^{(f)}} - e^{rt} \frac{D_x^{(r)}}{D_b^{(r)}}]$$

It will be noted that if  $p = 2?$ , the expression within the square brackets will, in either case, disappear, making the covariance zero.

### 3.5 Total Active and Retired Population projections: Expected values, variances and co-variances

The expected values of  $A(t)$  and  $R(t)$  are not affected by the stochastic nature of new entrants. The formulae already derived in section 2 will therefore hold. However, the expressions for the variances and co-variances will be considerably more complicated. Consider the covariance between  $A(t)$  and  $A(u)$ . The general expression for this covariance would be

$$COV(A(t), A(u)) = \int_{a_0(t)}^{a_2(t)} \int_{a_0(u)}^{a_2(u)} COV(Ac(x, t), Ac(y, u)) dy dx = K + L + M \quad (42)$$

where  $K$ ,  $L$  and  $M$  are components which are discussed below.

If  $Ac(x, t)$  derives from an initial population cohort, then it will not be correlated with any other  $Ac(y, u)$  except if they are both from the same cohort. The component  $K$  refers to the integral of the co-variances among initial population cohorts, and will be the same as the second member on the right-hand-side of (11). If  $Ac(x, t)$  is from a new entrant cohort, it will be correlated with any  $Ac(y, u)$  which is from the same or another new entrant cohort, although the expression for their covariance would depend upon whether they are from the same cohort (equation (34)) or from different cohorts (equation (41)). The component  $L$  refers to the integral obtained by applying formula (41) over the whole range of integration for new entrant cohorts. The component  $M$  is a correction to  $L$  to allow for the correct formula, i.e. (34), in those cases where  $Ac(x, t)$  and  $Ac(y, u)$  are from the same cohort. The expressions for  $K$ ,  $L$  and  $M$ , after simplification, allowing for  $Ac(b, 0)$  new entrants at  $t = 0$ , are as follows (note:  $L = L_1 + L_2 - L_3$ ):

$$K = Ac(b, 0) \left[ \int_{c_1(t, u)}^{c_2(t, u)} \frac{Ac(y-u, 0)}{Ac(b, 0)} {}_t p_{y-u} (1 - {}_u p_{y-u}) dy \right] \quad (43)$$

$$L_1 = Ac(b, 0) e^{rt} e^{fu} \int_{a_0(t)}^{a_1(t)} \frac{\bar{N}_{y_0}^{(f)} - \bar{N}_{a_1(u)}^{(f)}}{D_b^{(f)}} \frac{D_x^{(r)}}{D_b^{(r)}} dx \quad (44)$$

$$L_2 = Ac(b,0)e^{ru}e^{ft} \int_{a_0(t)}^{a_1(t)} \frac{\bar{N}_{a_0(u)}^{(r)} - \bar{N}_{y_0}^{(r)}}{D_b^{(r)}} \frac{D_x^{(f)}}{D_b^{(f)}} dx \quad (45)$$

$$L_3 = Ac(b,0)e^{r(t+u)} \frac{\bar{N}_{a_0(t)}^{(r)} - \bar{N}_{a_1(t)}^{(r)}}{D_b^{(r)}} \frac{\bar{N}_{a_0(u)}^{(r)} - \bar{N}_{a_1(u)}^{(r)}}{D_b^{(r)}} \quad (46)$$

$$M = Ac(b,0)e^{rt} \int_{c_0(t,u)}^{c_1(t,u)} \frac{D_{y+t-u}^{(r)}}{D_b^{(r)}} (1 - p_{y-b}) dy \quad (47)$$

where  $\bar{N}_x^{(r)} = \int_x^w D_y^{(r)} dy$ . The values of  $a_0(t), a_1(t)$  were defined in section 2.4 and the values of  $c_0(t,u), c_1(t,u), c_2(t,u)$  were defined in section 2.5. The value of  $y_0$  is  $x-t+u$ , subject to the minimum of  $a_0(u)$  and the maximum of  $a_1(u)$ . Demographic projections are illustrated in Appendix 8, Table 1.

$K$  (43) is not affected by  $\rho$  or  $p$ , but it can be shown that  $L_1, L_2, L_3$  and  $M$  ((44)-(47)) will all increase if either  $\rho$  or  $p$  is increased. Therefore, an increase in  $\rho$  or  $p$  will lead to an increase in  $COV(A(t), A(u))$ .

It will be noted that when  $p = 2\rho$ ,  $L_1 + L_2 = L_3$ , so that the expression for the co-variance in (42) will reduce to  $K+M$ , which is identical to (11). Eventually, when the initial population has disappeared from the scene ( $t > \rho - b$ ), expression (42) for the co-variance would reduce to  $L+M$ .

The expression (42) is also valid for the variance of  $A(t)$ , which is obtained by putting  $u = t$ . Similar expressions also apply to the co-variance between  $R(t)$  and  $R(u)$  and that between  $R(t)$  and  $A(u)$ . Provided the initial relative age-distribution remains invariant, it will be evident that the coefficients of variation of  $A(t)$  and  $R(t)$  will be inversely proportional to the square root of  $Ac(b,0)$  which means that the lower the initial population, the higher the coefficients of variation.

### 3.6 Expressions for co-variances in the long-term

For  $t$  exceeding specified values, general expressions can be derived for  $COV(A(t), A(u))$ ,  $COV(R(t), R(u))$ ,  $COV(R(t), A(u))$  and  $COV(A(t), R(u))$ , where  $u = t$  (see Appendices 4, 5 and 6). In particular, the following expressions, for the variance of  $A(t)$  and  $R(t)$  and for the covariance between  $R(t)$  and  $A(t)$ , are valid for  $t > \rho - b$ :

$$VA(t) = Ac(b,0)[P_1 e^{\rho t} - P_2 e^{2\rho t} + P_3 e^{rt}] \quad (48)$$

where

$$\begin{aligned} P_1 &= \int_b^r \frac{D_x^{(f)}}{D_b^{(f)}} \frac{\bar{N}_b^{(r)} - \bar{N}_x^{(r)}}{D_b^{(r)}} dx + \int_b^r \frac{D_x^{(r)}}{D_b^{(r)}} \frac{\bar{N}_x^{(f)} - \bar{N}_r^{(f)}}{D_b^{(f)}} dx \\ P_2 &= \left[ \frac{\bar{N}_b^{(r)} - \bar{N}_r^{(r)}}{D_b^{(r)}} \right]^2 \\ P_3 &= \frac{\bar{N}_b^{(r)} - \bar{N}_r^{(r)}}{D_b^{(r)}} - \frac{\bar{N}_{bb}^{(r)} - \bar{N}_{rr}^{(r)}}{D_{bb}^{(r)}} \\ VR(t) &= Ac(b,0)[Q_1 e^{\rho t} - Q_2 e^{2\rho t} + Q_3 e^{rt}] \end{aligned} \quad (49)$$

where

$$\begin{aligned}
 Q_1 &= \int_r^v \frac{D_x^{(f)}}{D_b^{(f)}} \frac{\bar{N}_r^{(r)} - \bar{N}_x^{(r)}}{D_b^{(r)}} dx + \int_r^v \frac{D_x^{(r)}}{D_b^{(r)}} \frac{\bar{N}_x^{(f)}}{D_b^{(f)}} dx \\
 Q_2 &= \left[ \frac{\bar{N}_r^{(r)}}{D_b^{(r)}} \right]^2 \\
 Q_3 &= \frac{\bar{N}_r^{(r)}}{D_b^{(r)}} - \frac{\bar{N}_{rr}^{(r)}}{D_{bb}^{(r)}} \\
 COV(R(t), A(t)) &= Ac(b, 0) [R_1 e^{pt} - R_2 e^{2rt}] \quad (50)
 \end{aligned}$$

where

$$\begin{aligned}
 R_1 &= \frac{\bar{N}_b^{(r)} - \bar{N}_r^{(r)}}{D_b^{(r)}} \frac{N_r^{(f)}}{D_b^{(f)}} \\
 R_2 &= \frac{\bar{N}_b^{(r)} - \bar{N}_r^{(r)}}{D_b^{(r)}} \frac{\bar{N}_r^{(r)}}{D_b^{(r)}}
 \end{aligned}$$

It will be noted that if  $p = 2r$ , the first two terms of (48) will cancel out and (48) will reduce to the expression for  $VA(t)$  in (13). Similarly, (49) will reduce to the expression for  $VR(t)$  in (13), and the expression for the covariance in (50) will reduce to zero.

The expected values  $EA(t)$  and  $ER(t)$  being the same as those in (12), it will be seen the coefficients of variation of  $A(t)$  and  $R(t)$  are given by the expressions

$$CVA^2(t) = \frac{1}{Ac(b, 0)} \left[ \frac{P_1}{P_2} e^{(p-2r)t} - 1 + \frac{P_3}{P_2} e^{-rt} \right] \quad (51)$$

$$CVR^2(t) = \frac{1}{Ac(b, 0)} \left[ \frac{Q_1}{Q_2} e^{(p-2r)t} - 1 + \frac{Q_3}{Q_1} e^{-rt} \right] \quad (52)$$

In addition, if  $Corr(R(t), A(t))$  denotes the correlation coefficient between  $A(t)$  and  $R(t)$ , the ratio of  $COV(R(t), A(t))$  to the product  $EA(t)ER(t)$  will be given by

$$Corr(R(t), A(t)) CVA(t) CVR(t) = \frac{1}{Ac(b, 0)} \left[ \frac{R_1}{R_2} e^{(p-2r)t} - 1 \right] \quad (53)$$

It will be noted that when  $p > 2r$ , all the three expressions will diverge to infinity as  $t$  tends to infinity. For a given  $t$ , a decrease in the initial population size will lead to an increase in all the three coefficients of variation. It can be shown that an increase in  $p$  will have a similar effect.

### 3.7 The Demographic Ratio

Considering the Demographic Ratio  $DR(t)$ , unlike the case in section 2.7,  $R(t)$  and  $A(t)$  will now be correlated. If, in addition to what was ignored in section 2.7, the second and higher powers of  $Corr(R(t), A(t)) CVA(t) CVR(t)$  and correlations between  $R(t)$  and  $A(t)$  of order higher than one are ignored, the following approximate expressions can be derived (see Appendix 1, Note 3):

$$EDR(t) = \frac{ER(t)}{EA(t)} [1 + CVA^2(t) - Corr(R(t), A(t)) CVA(t) CVR(t)] \quad (54)$$

$$VDR(t) = \frac{ER^2(t)}{EA^2(t)} [CVR^2(t) + CVA^2(t) - 2Corr(R(t), A(t)) CVA(t) CVR(t)] \quad (55)$$

Unlike section 2.7, these two expressions, individually, will diverge to infinity as  $t \rightarrow \infty$ . However, the coefficient of variation  $CVDR(t)$  can be shown to tend to zero as  $t \rightarrow \infty$ . For a given

$t$ ,  $CVDR(t)$  will increase with decreasing initial population size and with increasing  $p$ . These effects are illustrated numerically in Appendix 8 (Tables 6 and 7).

#### 4. FINANCIAL PROJECTIONS (STATIC SALARY BASIS)

##### 4.1 Notation and assumptions

As in classic projections, it is assumed that the growth of an individual's salary is governed by a salary escalation due to the increase in the general level of earnings, on top of the progression due to age/seniority along a salary scale. By static salary basis, it is meant that salary escalation is not taken into account directly in the projections (although the salary scale effect is). The assumption is also made that pensions are fully adjusted to salary escalation. In this circumstance, the analysis of ratios between benefit and salary projections (e.g. pay-as-you-go premiums) will be equivalent to that based on projections allowing directly for salary escalation. In fact, salary escalation is allowed for, indirectly, in the assumption concerning the investment return (see section 5, below).

The average annual salary of the  $Ac(x, t)$  persons aged  $x$ , at time  $t$ , is denoted by  $sa(x, t)$  and the annual pension amount of the  $Re(x, t)$  pensioners is denoted by  $pe(x, t)$ .  $sa(x, 0)$  at the start of the scheme is assumed to be given. The average annual salary at entry of the  $Ac(b, 0)$  persons entering at  $t = 0$  is  $sa(b, 0)$ . The total annual salary of the  $Ac(x, t)$  persons is denoted by  $Sa(x, t)$  and the total annual pensions of the  $Re(x, t)$  retired persons is denoted by  $Be(x, t)$ . Pensions are assumed to be based on the final salary at retirement and the proportionate pension rate for entry age  $x$  is denoted by  $k(x)$ .

Individual salaries progress according to the salary scale  $s(x)$ ,  $b = x = r$ , which is assumed to remain constant over time. Since salary escalation over time is not directly taken into account, the salary at entry for all new entrant cohorts would be  $sa(b, 0)$ .

##### 4.2 Expected values, variances and co-variances

The following relationships hold between the financial and demographic projection elements:

$$Sa(x, t) = Ac(x, t)sa(x, t); Be(x, t) = Re(x, t)pe(x, t) \quad (56)$$

where, for the initial population cohorts

$$sa(x, t) = sa(x-t, 0) \frac{s(x)}{s(x-t)}; pe(x, t) = sa(x-t, 0) \frac{s(r)}{s(x-t)} k(x-t) \quad (57)$$

and for the new entrant cohorts

$$sa(x, t) = sa(b, 0) \frac{s(x)}{s(b)}; pe(x, t) = sa(b, 0) \frac{s(r)}{s(b)} k(b) \quad (58)$$

The expected values of the financial projection elements are therefore related to those of the corresponding demographic projection elements as follows:

$$ESa(x, t) = sa(x, t)EAc(x, t); EBe(x, t) = pe(x, t)ERe(x, t) \quad (59)$$

Based on (9), this leads to the following expressions for  $ES(t)$  and  $EB(t)$ :

$$ES(t) = Sa(b, 0) \left[ \int_{a_1(t)}^{a_2(t)} \frac{Sa(x-t, 0)}{Sa(b, 0)} \frac{s(x)}{s(x-t)} {}_tP_{x-t} dx + e^{rt} \int_{a_0(t)}^{a_1(t)} \frac{D_x^{s(r)}}{D_b^{s(r)}} dx \right] \quad (60)$$

$$EB(t) = Sa(b, 0) \left[ \int_{a_1(t)}^{a_2(t)} \frac{Sa(x-t, 0)}{Sa(b, 0)} k(x-t) \frac{s(r)}{s(x-t)} {}_tP_{x-t} dx + k(b) e^{rt} \int_{a_0(t)}^{a_1(t)} \frac{s(r) D_x^{(r)}}{D_b^{s(r)}} dx \right] \quad (61)$$

where  $D_x^{s(r)} = s(x)D_x^{(r)}$ .

The co-variances of the financial projection elements are related to those of the corresponding demographic projection elements as follows:

$$COV(Sa(x, t), Sa(y, u)) = sa(x, t)sa(y, u)COV(Ac(x, t), Ac(y, u)) \quad (62)$$

$$COV(Be(x, t), Be(y, u)) = pe(x, t)pe(y, u)COV(Re(x, t), Re(y, u)) \quad (63)$$

$$COV(Be(x, t), Sa(y, u)) = pe(x, t)sa(y, u)COV(Re(x, t), Ac(y, u)) \quad (64)$$

These adjustments should be incorporated in (42) to (47), as appropriate, to obtain the expressions for the various co-variances of the financial projections. In order to illustrate this, the expression for  $COV(S(t), S(u))$ ,  $u = t$ , is indicated below:

$$COV(S(t), S(u)) = K + L_1 + L_2 - L_3 + M \quad (65)$$

$$K = Sa(b, 0) \left[ \int_{c_1(t, u)}^{c_2(t, u)} \frac{Sa(y - u, 0)}{Sa(b, 0)} sa(y - u, 0) \frac{s(y + t - u)s(y)}{s(y - u)^2} {}_t p_{y-u} (1 - {}_u p_{y-u}) dy \right] \quad (66)$$

$$L_1 = Sa(b, 0)sa(b, 0)e^{rt}e^{fu} \int_{a_0(t)}^{a_1(t)} \frac{\bar{N}_{y_0}^{s(f)} - \bar{N}_{a_1(u)}^{s(f)}}{D_b^{s(f)}} \frac{s(r)D_x^{(r)}}{D_b^{s(r)}} dx \quad (67)$$

$$L_2 = Sa(b, 0)sa(b, 0)e^{ru}e^{ft} \int_{a_0(t)}^{a_1(t)} \frac{\bar{N}_{a_0(u)}^{s(r)} - \bar{N}_{y_0}^{s(r)}}{D_b^{s(r)}} \frac{s(r)D_x^{(f)}}{D_b^{s(f)}} dx \quad (68)$$

$$L_3 = Sa(b, 0)sa(b, 0)e^{r(t+u)} \frac{\bar{N}_{a_0(t)}^{s(r)} - \bar{N}_{a_1(t)}^{s(r)}}{D_b^{s(r)}} \frac{\bar{N}_{a_0(u)}^{s(r)} - \bar{N}_{a_1(u)}^{s(r)}}{D_b^{s(r)}} \quad (69)$$

$$M = Sa(b, 0)sa(b, 0)e^{rt} \int_{c_0(t, u)}^{c_1(t, u)} \frac{D_{y+t-u}^{(r)}}{D_b^{(r)}} \frac{s(y + t - u)s(y)}{s(b)^2} (1 - {}_{y-b} p_b) dy \quad (70)$$

where  $\bar{N}_x^{s(r)} = \int_x^w D_y^{s(r)} dy$ . Expressions for  $COV(B(t), B(u))$  and  $COV(B(t), S(u))$  and for  $COV(S(t), B(u))$ ,  $u = t$ , can be developed on similar lines (see Appendix 3). The formula for  $COV(S(t), S(u))$  is also valid for  $VS(t)$ , with  $t = u$ , and the formula for  $COV(B(t), B(u))$  is valid for  $VB(t)$ . Financial projections are illustrated in Appendix 8, Tables 2 and 3.

Expressions (48) to (53) concerning the demographic projections in the mature situation can be adapted to the financial projections and will lead to similar expressions. The analysis of the effects of the parameters  $\beta$  and  $p$  and of the size of the initial population will be similar to that in sections 3.5 and 3.6 and will lead to the same conclusions.

### 4.3 The capitalized value of pension awards

Let  $Bc(t)$  denote the capitalized value of the pensions awarded to those retiring at time  $t$ , and  $a^*$  the capitalized value of an indexed unit pension at age  $r$ . Then,

$$Bc(t) = c(t)Ac(r, t)a^*$$

where  $c(t) = sa(r - t, 0) \frac{s(r)}{s(r - t)} k(r - t)$  if  $t < r - b$ ;  $c(t) = sa(b, 0) \frac{s(r)}{s(b)} k(b)$  if  $t = r - b$ .

$$Ea^* = \bar{a}_r^{(d)}; Va^* = 2 \frac{\bar{a}_r^{(d)} - \bar{a}_r^{(a)}}{(a - d)} - (\bar{a}_r^{(d)})^2 \quad (\text{see Appendix 1, Note 5})$$

$c(t)$  is deterministic and  $a^*$  and  $Ac(r, t)$  can be regarded as independent. Therefore,

$$EBc(t) = c(t)EAc(r, t)Ea^* \quad (71)$$

From Appendix 1, Note 6, it follows that

$$VBc(t) = c^2(t)[VAc(r, t)(Va * + E^2 a^*) + Va * E^2 Ac(r, t)] \quad (72)$$

$EAc(r, t)$  given by (1) or (4), and  $VAc(r, t)$  by (2) or (26), depending upon whether  $t < \text{or} > r - b$ , can be substituted in the above expression..

The co-variance between  $Bc(t)$  and  $S(t)$  will be zero if  $t < r - b$ . If  $t > r - b$ ,

$$COV(Bc(t), S(t)) = Ea * k(b) \int_b^r sa(b, 0)^2 \frac{s(r)s(x)}{s(b)^2} COV(Ac(r, t), Ac(x, t)) dx$$

Substituting for  $COV(Ac(r, t), Ac(x, t))$  from the expression following (41) and simplifying, for  $t > r - b$ ,

$$COV(Bc(t), S(t)) = Ac(b, 0) Ea * sa(b, 0)^2 k(b) [e^{pt} \frac{D_r^{s(f)}}{D_b^{s(f)}} - e^{2rt} \frac{D_r^{s(r)}}{D_b^{s(r)}}] \frac{\overline{N}_b^{s(r)} - \overline{N}_r^{s(r)}}{D_b^{s(r)}} \quad (73)$$

## 5. DISCOUNTED VALUES OF SALARY AND BENEFIT PROJECTIONS

Discounting of projections is relevant in the context of funded financial systems. The following development is based on the **force of “real interest”**, defined here as nominal interest less salary escalation. By using this force of interest to discount the projections on the **static salary basis**, salary escalation is effectively taken into account.

### 5.1 Notation and assumptions

In the classical approach, the force of real interest, denoted by  $\mathbf{d}$ , would be assumed to be a constant throughout the period of projections (or taking different constant values over successive sub-intervals). In this paper, however, the force of real interest  $d(t)$  is assumed to be distributed such that  $e^{-d(z)}$  for different values of  $z$  are independent, identically distributed (IID) variables with expected value and second raw moment given by

$$E(e^{-d(z)}) = e^{-d} ; E(e^{-2d(z)}) = e^{-a} \quad (74)$$

so that the variance is given by

$$V(e^{-d(z)}) = e^{-a} - e^{-2d} \quad (75)$$

For the variance to be positive, we should have  $\mathbf{a} < 2\mathbf{d}$ . If  $\mathbf{a} = 2\mathbf{d}$  the variance is zero, meaning that the interest factor is deterministic. The symbol  $\mathbf{?}$  is used to denote the difference  $\mathbf{a} - \mathbf{d}$ .

### 5.2 Case of deterministic projections

We shall first consider the case where the projections themselves are deterministic, so that only the interest element is stochastic. Let  $S(t)$  and  $B(t)$  denote the salary and benefit projections on a **static salary basis**, ignoring salary escalation. It is assumed that pensions are fully indexed to salary escalation. Let  $DS(t)$  and  $DB(t)$  denote the values of  $S(t)$  and  $B(t)$  discounted to time  $t = 0$ , and  $TDS(t)$  and  $TDB(t)$  the totals of the discounted values over  $(0, t)$ .

In view of the IID properties described in section 5.1,

$$EDS(t) = E(S(t) e^{-\int_0^t d(z) dz}) = S(t) e^{-dt} ; EDS^2(t) = E(S(t)^2 e^{-2\int_0^t d(z) dz}) = S(t)^2 e^{-at} \quad (76)$$

$$ETDS(t) = E[\int_0^t S(u) e^{-\int_0^u d(z) dz} du] = \int_0^t S(u) e^{-du} du \quad (77)$$

$$VDS(t) = EDS^2(t) - [EDS(t)]^2 = S(t)^2 [e^{-at} - e^{-2dt}] \quad (78)$$

$$COV(DS(t), DB(t)) = S(t)B(t)[e^{-at} - e^{-2dt}] \quad (79)$$

Squaring  $TDS(t)$  we have

$$TDS^2(t) = \int_0^t S(u)e^{-\int_0^u d(z)dz} du \int_0^t S(v)e^{-\int_0^v d(z)dz} dv = 2 \int_0^t S(u)e^{-\int_0^u d(z)dz} du \int_0^u S(v)e^{-\int_0^v d(z)dz} dv$$

Noting that  $u > v$ , the above expression can be simplified as

$$TDS^2(t) = 2 \int_0^t S(u) \int_0^u S(v) e^{-2 \int_0^v d(z)dz - \int_v^u d(z)dz} dv du$$

Taking expectations we have

$$ETDS^2(t) = 2 \int_0^t S(u) e^{-du} du \int_0^u S(v) e^{-(a-d)v} dv \quad (80)$$

Therefore the variance of  $TDS(t)$  is given by

$$VTDS(t) = 2 \int_0^t S(u) e^{-du} \int_0^u S(v) e^{-qv} dv du - \left[ \int_0^t S(u) e^{-du} du \right]^2 \quad (81)$$

The above formula can also be expressed as

$$VTDS(t) = 2 \int_0^t S(u) e^{-du} \int_0^u S(v) [e^{-qv} - e^{-dv}] dv du \quad (82)$$

It can similarly be shown that the covariance between  $TDS(t)$  and  $TDB(t)$  is given by

$$\int_0^t S(u) e^{-du} \int_0^u B(v) [e^{-qv} - e^{-dv}] dv du + \int_0^t B(u) e^{-du} \int_0^u S(v) [e^{-qv} - e^{-dv}] dv du \quad (83)$$

It can be shown that when  $a = 2d$ , i.e. the interest element is deterministic, the expression for  $VTDS(t)$  in (82) will vanish. Two further useful results are the following:

$$COV(TDB(t), DB(t)) = B(t) e^{-dt} \int_0^t B(u) [e^{-qu} - e^{-du}] du \quad (84)$$

$$COV(TDS(t), DB(t)) = B(t) e^{-dt} \int_0^t S(u) [e^{-qu} - e^{-du}] du \quad (85)$$

### 5.3 Case of stochastic projections

In developing the formulae for this case, the assumption is made that the interest element is independent of the mortality and new entrant elements. Consider first,  $DS(t)$ .

$$DS(t) = S(t) e^{-\int_0^t d(z)dz} = S(t)\Phi(t) \quad (86)$$

$S(t)$  depends on the mortality and new entrant elements, while  $\Phi(t)$  depends on the interest element. Therefore they are independent of each other.

$$EDS(t) = ES(t)E\Phi(t) = e^{-dt}ES(t) \quad (87)$$

where  $ES(t)$  is given by (60).  $VDS(t)$  is derived as follows:

$$\begin{aligned} VDS(t) &= ES^2(t)E\Phi^2(t) - (ES(t))^2(E\Phi(t))^2 = ES^2(t)e^{-2dt} - E(S(t))^2 e^{-2dt} \\ VDS(t) &= VS(t)e^{-2dt} + (ES(t))^2[e^{-2dt} - e^{-2dt}] \end{aligned} \quad (88)$$

$ETDS(t)$  and  $VTDS(t)$  are derived as follows:

$$ETDS(t) = E \int_0^t DS(u) du = \int_0^t EDS(u) du = \int_0^t ES(u) e^{-du} du \quad (89)$$

$$VTDS(t) = \int_0^t \int_0^t COV(DS(u), DS(v)) dv du = 2 \int_0^t \int_0^u COV(DS(u), DS(v)) dv du$$

$$= 2 \left[ \int_0^t \int_0^u E[DS(u)DS(v)] dv du - \int_0^t \int_0^u EDS(u)EDS(v) dv du \right] = 2[A - B]$$

$$E[DS(u)DS(v)] = E[\Phi(u)S(u)\Phi(v)S(v)] = E[\Phi(u)\Phi(v)]E[S(u)S(v)]$$

$$E[\Phi(u)\Phi(v)] = E[e^{-\int_0^u d(z)dz} e^{-\int_0^v d(z)dz}] = E[e^{-2 \int_0^v d(z)dz - \int_v^u d(z)dz}] = e^{-av - d(u-v)} = e^{-du} e^{-(a-d)v}$$

$$A = \int_0^t e^{-du} \int_0^u e^{-(a-d)v} E(S(u)S(v)) dv du$$

$$B = \int_0^t e^{-du} \int_0^u e^{-dv} ES(u)ES(v) dv du$$

$$VTDS(t) = 2 \left[ \int_0^t e^{-du} \int_0^u e^{-qv} COV(S(u), S(v)) dv du + \int_0^t e^{-du} ES(u) du \int_0^u (e^{-qv} - e^{-dv}) ES(v) dv \right] \quad (90)$$

When  $S(t)$  is deterministic,  $COV(S(u), S(v)) = 0$  and the first member of (90) will vanish so that (90) reduces to (82). When  $a = 2d$ , i.e. the interest element is deterministic, the second member of (90) will vanish and (90) will reduce to

$$VTDS(t) = 2 \int_0^t e^{-du} \int_0^u e^{-dv} COV(S(u), S(v)) dv du$$

The expression for  $VTDB(t)$  will be similar to (90). It can be shown on the same lines that the covariance between  $TDS(t)$  and  $TDB(t)$  will be given by

$$\begin{aligned} & \int_0^t e^{-du} \int_0^u e^{-qv} COV(S(u), B(v)) dv du + \int_0^t e^{-du} ES(u) du \int_0^u (e^{-qv} - e^{-dv}) EB(v) dv \\ & + \int_0^t e^{-du} \int_0^u e^{-qv} COV(B(u), S(v)) dv du + \int_0^t e^{-du} EB(u) du \int_0^u (e^{-qv} - e^{-dv}) ES(v) dv \quad (91) \end{aligned}$$

Two further useful results are:

$$COV(DB(t), TDB(t)) = e^{-dt} \int_0^t [e^{-qu} COV(B(t), B(u)) + (e^{-qu} - e^{-du}) EB(t) EB(u)] du \quad (92)$$

$$COV(DB(t), TDS(t)) = e^{-dt} \int_0^t [e^{-qu} COV(B(t), S(u)) + (e^{-qu} - e^{-du}) EB(t) ES(u)] du \quad (93)$$

Discounted financial projections are illustrated in Appendix 8, Table 4.

Let the two terms of (90) be indicated by  $VTDS1(t)$  and  $VTDS2(t)$ . An increase in either  $d$  or  $a$  will lead to a decrease in either term and therefore in  $VTDS(t)$ .  $VTDS1(t)$  involves the factor  $Ac(b,0)$  whereas  $VTDS2(t)$  involves  $Ac(b,0)^2$  and  $VTDS2$  will predominate, so that the co-variances between  $S(t)$  and  $S(u)$  will not have much effect on  $VTDS(t)$  unless the initial population is relatively small. The same conclusions apply to  $VTDB(t)$  and  $COV(TDS(t), TDB(t))$ . For a given  $t$ , the coefficients of variation  $CVTDS(t)$  and  $CVTDB(t)$  will increase with decreasing initial population size and with increasing  $p$  or decreasing  $a$ . These effects will be reflected in the stochastic premiums (see section 6).

#### 5.4 Convergence of the expressions in the long-term

Based on the expressions for  $COV(S(t), S(u))$  etc. – see section 4.2 and Appendix 3 – long-term expressions for these co-variances can be derived and further, long-term expressions for the variances of and the co-variances between totals of discounted values can be developed (see Appendices 4, 5 and 6).

The development in Appendix 4 shows that, for any  $t$  exceeding a sufficiently large value  $t_0 (> 2(r-b))$

$$VTDS(t) - VTDS(t_0) = F(t_0) - F(t) \quad (94)$$

$F(t)$  is given by an expression of the form,

$$F(t) = F_1 e^{-(d-r)t} + F_2 e^{-(a-p)t} + F_3 e^{-(a-2r)t} + F_4 e^{-(a-r)t} + F_5 e^{-2(d-r)t} \quad (95)$$

Given the conditions (a)  $p > 2r$  and (b)  $a < 2d$ , the further conditions required for the expression in (95) to tend to 0 as  $t$  tends to  $\infty$  are:

(c)  $d > r$  and (d)  $a > p$ .

For, (a) and (d) imply that  $a > 2r$ . The limiting value of  $VTDS(t)$  will then be given by,

$$VTDS(\infty) = VTDS(t_0) + F(t_0) \quad (96)$$

It turns out that  $VTDB(t) - VTDB(t_0)$  (for  $t_0 > (r-b) + 2(v-r)$ ) and  $Cov(TDB(t), TDS(t)) - Cov(TDB(t_0), TDS(t_0))$  (for  $t_0 > 2(r-b) + (v-r)$ ) too have a structure similar to (94) and (95) (see Appendices 5 and 6), and will converge as in (96) under the same conditions. Further, based on (89), it can be shown that,  $ETDS(t) - ETDS(t_0)$  (for  $t_0 > r-b$ ), and  $ETDB(t) - ETDB(t_0)$  (for  $t_0 > v-b$ ), respectively take the forms  $P(t_0) - P(t)$  and  $Q(t_0) - Q(t)$ , where  $P(t)$  and  $Q(t)$  are given by,

$$P(t) = \frac{Sa(b,0)}{d-r} \frac{\bar{N}_b^{s(r)} - \bar{N}_r^{s(r)}}{D_b^{s(r)}} e^{-t(d-r)}; Q(t) = \frac{Sa(b,0)}{d-r} \frac{\bar{N}_r^{s(r)}}{D_b^{s(r)}} k(b) e^{-t(d-r)} \quad (97)$$

It follows that  $ETDS(t)$  and  $ETDB(t)$  will converge as  $t \rightarrow \infty$  subject to condition © above ( $d > ?$ ).

## 6 APPLICATION OF STOCHASTIC METHODS IN THE FINANCING OF SOCIAL SECURITY PENSION SCHEMES

### 6.1 General principles

The classical method of determining premiums for any given financial system (see Iyer, 1999, chapter 1) is on the basis of deterministic projections, which is effectively in terms of expected values. Any margins which are then added to absorb the effect of chance variations are necessarily arbitrary. However, since expressions for the variances and co-variances are now available, an improved method can be applied incorporating the appropriate adjustment for chance variations.

Any financial system essentially seeks a level premium, which, when applied to a function  $f(S)$  of projected salaries, will balance another function  $g(B)$  of projected benefits. The classical method determines the premium  $pr$  by the formula

$$pr = \frac{Eg(B)}{Ef(S)} \quad (98)$$

Following the method outlined in Booth et. al., 1999 (p. 640) an alternative approach (based on the so-called *percentile* principle) is to consider a variable  $u$  defined by

$$u = kf(S) - g(B) \quad (99)$$

An assumption is required concerning the statistical distribution of  $u$ . The normal distribution could possibly be justified on the basis of the central limit theorem, when the population size is sufficiently large. The premium  $pr^*$  (referred to in the sequel as the stochastic premium) is determined such that the probability of failure of the financial system (i.e. that  $u$  will be negative) is less than a pre-determined  $e$ .  $pr^*$  will be given by the larger root of the equation

$$E^2(u) - y_e^2 V(u) = 0 \quad (100)$$

where,  $y_e$  is the  $(1-e)^{th}$  percentile of the appropriate distribution.

Alternatively, given any premium  $pr^{**}$ , the probability that its application will lead to the failure of the financial system can be determined.

## 6.2 The pay-as-you-go system

In the framework of the continuous formulation in this paper, the pay-as-you-go system aims at balancing the income and expenditure at a given point in time. Given projections established at  $t=0$ , the relevant functions for the pay-as-you-go premium at time  $t=n$  are

$$f(S) = S(n); g(B) = B(n)$$

The classical premium is given by

$$PAYG(n) = \frac{ES(n)}{EB(n)}$$

The stochastic premium  $PAYG^*(n)$  is given by the larger root of (100), where

$$E(u) = kES(n) - EB(n); V(u) = k^2VS(n) + VB(n) - 2kCOV(S(n), B(n)) \quad (101)$$

## 6.3 The Average Premium system

The Average Premium system aims to balance expenditure over successive intervals of several years through the application of a level premium over each interval, allowing for any accumulated fund at the start of the interval. The relevant functions corresponding to the interval  $(0, n)$ , starting with a fund  $F(0)$ , are

$$f(S) = TDS(n); g(B) = TDB(n) - F(0)$$

The classical premium is given by

$$AVP(0, n) = \frac{ETDB(n) - F(0)}{ETDS(n)}$$

The stochastic premium is the larger root of (100), where

$$E(u) = F(0) + kETDS(n) - ETDB(n) \\ V(u) = k^2VTDS(n) + VTDB(n) - 2kCOV(TDS(n), TDB(n)) \quad (102)$$

## 6.4 The Reserve Ratio system

The Reserve Ratio system applies a level premium over an interval of years such that a Fund is accumulated bearing a specified ratio to the level of the expenditure at the end of the period. For the interval  $(0, n)$ , the relevant functions, allowing for a reserve ratio  $\Lambda$  and an initial fund  $F(0)$  are

$$f(S) = TDS(n); g(B) = TDB(n) + \Lambda DB(n) - F(0)$$

The classical premium is given by

$$RRP(0, n, \Lambda) = \frac{ETDB(n) + \Lambda EDB(n) - F(0)}{ETDS(n)}$$

The stochastic premium is the larger root of (100), where

$$E(u) = F(0) + kETDS(n) - ETDB(n) - \Lambda EDB(n)$$

$$V(u) = k^2VTDS(n) + VTDB(n) + \Lambda^2VDB(n) - 2kCOV(TDS(n), TDB(n)) \\ - 2k\Lambda COV(TDS(n), DB(n)) + 2\Lambda COV(TDB(n), DB(n)) \quad (103)$$

### 6.5 The Terminal Funding system

In the framework of the continuous formulation in this paper, the Terminal Funding system fully capitalizes the pensions awarded at any time  $t$  by the corresponding contribution income. For determining the Terminal Funding premium at  $t = n$ , based on projections established at  $t = 0$ , the relevant functions are

$$f(S) = S(n); g(B) = Bc(n)$$

The classical method is based on the formula

$$TFP(n) = \frac{EBc(n)}{ES(n)} \quad (104)$$

The stochastic premium would be the larger root of (100), where

$$E(u) = kES(n) - EBc(n) \quad (105)$$

$$V(u) = k^2VS(n) + VBc(n) - 2kCOV(Bc(n), S(n)) \quad (106)$$

### 6.6 The General Average Premium system

The General Average Premium system is based on the concept of a constant premium that will ensure the financial equilibrium of a pension scheme throughout the infinite lifetime of the scheme following an actuarial valuation (at  $t = 0$ ), taking credit for the accumulated fund. The General Average Premium system can be regarded as the limiting form of the Average Premium system, when  $n \rightarrow \infty$ . Provided the necessary conditions for convergence are met (i.e.  $d > \rho$  and  $a > p$  – see section 5.4, above), the formulae and the procedure of section 6.3 above can be applied, by substituting  $ETDS(\infty)$  for  $ETDS(n)$ ,  $VTDS(\infty)$  for  $VTDS(n)$ , etc.

### 6.7 Sensitivity of the premiums to population size and parameters

In contrast to the classical premiums, the stochastic premiums are affected by the size of the initial population, being inversely related to the population size. However, the sensitivity to a change in the population size is considerably higher for the Pay-as-you-go and Terminal Funding systems, as compared to the other systems. This is to be explained by the fact that the projected values (on which the *PAYG* and *TF* systems depend directly) are much more sensitive to population size changes than the totals of discounted projected values (to which the other systems relate). In all cases, the stochastic premium tends to the classical premium as the population size tends to infinity.

As regards the effect of the parameters, the classical premiums are affected by  $\rho$  and  $d$  but are independent of the secondary parameters  $p$  and  $a$ . Stochastic *PAYG* and *TFP* premiums are affected by  $p$ , with an increase in  $p$  leading to an increase in the premiums, but are not affected by  $a$ . Stochastic *AVP* and *RRP* premiums are affected by both  $p$  and  $a$ , increasing with increasing  $p$  and decreasing with increasing  $a$ , although the effect of changing  $p$  is relatively marginal. These effects of changing population size and changing parameters on the stochastic premiums are illustrated in Appendix 8, Tables 5 to 8.

## 7. CONCLUSION

The traditional, deterministic, approach to the actuarial valuation of social security pension schemes has continued practically unchanged to this day. This paper has explored the possibility of applying stochastic methods, drawing on the techniques being applied in other areas of insurance but tailoring them to the special characteristics of social security pensions. Although it is based on certain simplifying assumptions, the paper has shown that it is feasible to apply stochastic methods in this area as well.

The paper has brought out several interesting features of the stochastic aspects of social security pension projections. It has demonstrated in particular that deterministic projections can be highly misleading when the population size is small. Fortunately most social security pension schemes, being national in scope, have high coverage in terms of numbers of individuals. Nevertheless, the paper has shown that stochastic methods can throw considerably more light on the evolution of a social security pension scheme and therefore enable more informed policy decisions to be taken.

The paper has been largely theoretical and further work is certainly required, in particular, to facilitate practical application. It is hoped that the paper will succeed in drawing attention to this potential new area of application of stochastic methods, stimulate interest in further research and eventually, lead to the adoption of stochastic methods in social security pension scheme valuations.

## REFERENCES

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## APPENDIX 1 MATHEMATICAL NOTES

### Note 1 (sections 2.2, 2.3)

Consider a cohort of size  $N$  at age  $z$ , which is subject to a given life table  $\{l_x\}$ . Let  $N(y)$  and  $N(x)$  denote the survivors to ages  $y$  and  $x$  ( $z < y < x$ ), where  $y$  and  $x$  are not necessarily integral. Then

$N(y)$  has the binomial distribution with parameters  $N$  and  $p = \frac{l_y}{l_z}$  and we have

$$E(N(y)) = Np; V(N(y)) = Np(1-p)$$

Let  $p' = \frac{l_x}{l_z}$  and  $p'' = \frac{l_y}{l_x}$ . Then  $p' = pp''$ .

$$E(N(x)) = Np'; V(N(x)) = Np'(1-p')$$

$$E(N(x) | N(y)) = N(y)p''$$

$$\begin{aligned} E(N(x) * N(y)) &= E(N(y) * E(N(x) | N(y))) = E(N^2(y)p'') \\ &= p''[V(N(y)) + E^2(N(y))] = p''[Np(1-p) + N^2p^2] = Npp''(1-p) + N^2p^2p'' \\ &= Np'(1-p) + Np * Np' = Np'(1-p) + E(N(x))E(N(y)) \end{aligned}$$

Therefore the covariance between  $N(x)$  and  $N(y)$  is given by

$$COV(N(x), N(y)) = Np'(1-p) = N \frac{l_x}{l_z} (1 - \frac{l_y}{l_x})$$

### Note 2 (section 2.3)

A parallel can be drawn with the distinction between  $u$  and  $v$ , defined by

$$u = nx; v = \sum_{i=1}^n x_i$$

where the  $x_i$ 's are independent variables having the same distribution as  $x$ . We have

$$E(u) = E(v) = nE(x)$$

But the variances are not equal. They are given by

$$V(u) = n^2V(x); V(v) = nV(x)$$

### Note 3 (sections 2.7 and 3.7)

Let  $x$  and  $y$  be two variables and let  $u = \frac{x}{y}$ . Let the main parameters of  $x$  and  $y$  be:

$$E(x) = m; V(x) = s^2; E(y) = n; V(y) = q^2; COV(x, y) = k$$

Approximate expressions for  $E(u)$  and  $V(u)$  can be derived as follows:

$$E(u) = E\left(\frac{x}{y}\right) = E\left(\frac{x-m+m}{y-n+n}\right) = \frac{m}{n} E\left(1 + \frac{x-m}{m}\right) \left(1 + \frac{y-n}{n}\right)^{-1} = \frac{m}{n} \left(1 - \frac{k}{mn} + \frac{q^2}{n^2}\right)$$

$$E(u^2) = \frac{m^2}{n^2} E\left(1 + \frac{x-m}{m}\right)^2 \left(1 + \frac{y-n}{n}\right)^{-2} = \frac{m^2}{n^2} \left(1 + \frac{s^2}{m^2} + 3\frac{q^2}{n^2} - 4\frac{k}{mn}\right)$$

$$V(u) = E(u^2) - E^2(u) = \frac{m^2}{n^2} \left(\frac{s^2}{m^2} + \frac{q^2}{n^2} - 2\frac{k}{mn}\right)$$

**Note 4 (sections 3.3 and 3.4)**

Let  $x, y$  be two variables, either being dependant on  $z$ . Then, the conditional co-variance between  $x$  and  $y$  is given by,

$$\text{Cov}(x, y | z) = E(xy | z) - E(x | z)E(y | z)$$

Taking the expectation over the range of  $z$ ,

$$ECov(x, y | z) = EE(xy | z) - E[E(x | z)E(y | z)]$$

The co-variance between the conditional expectations is given by

$$\text{Cov}(E(x | z), E(y | z)) = E[E(x | z)E(y | z)] - EE(x | z)EE(y | z)$$

By adding the two results, and noting that  $EE$  signifies unconditional expectation, it follows that

$$\text{Cov}(x, y) = ECov(x, y | z) + \text{Cov}(E(x | z), E(y | z))$$

The above result is valid even if  $z$  represents a group of variables. Therefore, if  $x$  depends on  $z$  and  $y$  depends on  $w$ ,

$$\text{Cov}(x, y) = ECov(x, y | z, w) + \text{Cov}(E(x | z), E(y | w))$$

**Note 5 (section 4.3)**

These formulae for  $Ea^*$  and  $Va^*$  correspond to formulae (24.60) and (24.62) on pp. 644-645 of Booth et. al., 1999, but relate to the continuous formulation adopted in this paper. When  $a = 2d$ , the formula for  $Va^*$  reduces to formula (24.32) on p.629 of Booth et. al.

**Note 6 (section 4.3)**

Let  $x$  and  $y$  be two independent variables, and let  $u = xy$ .

$$E(xy) = E(x)E(y)$$

$$V(u) = E(u^2) - E^2(u) = E(x^2y^2) - E^2(x)E^2(y)$$

Since  $x$  and  $y$  are independent, so are their squares. Therefore,

$$E(x^2y^2) = E(x^2)E(y^2) = (V(x) + E^2(x))(V(y) + E^2(y))$$

It follows that

$$V(u) = V(x)V(y) + V(x)E^2(y) + V(y)E^2(x)$$

**APPENDIX 2**  
**LIMITS OF INTEGRATION  $c_0(t, u)$ ,  $c_1(t, u)$  AND  $c_2(t, u)$  FOR**  
**SIMPLIFIED PENSION SCHEME**

| <u>Co-variants</u> | <u>Range of t</u>                                                             | <u>Range of u</u>                                                                                                                                                                                     | <u>Range NE</u><br>$c_0, c_1(t, u)$                                                                                                      | <u>Range IP</u><br>$c_1, c_2(t, u)$                                                                                                 |
|--------------------|-------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------|
| $A(t)$ and $A(u)$  | $t < r-b$<br>$t = r-b$<br>$t > r-b$                                           | $u < t$<br>$u < t-r+b$<br>$u > t-r+b$                                                                                                                                                                 | $b, b+u$<br>$nil$<br>$b, r-t+u$                                                                                                          | $b+u, r-t+u$<br>$nil$<br>$nil$                                                                                                      |
| $R(t)$ and $R(u)$  | $t < ?-r$<br>$?-r = t < r-b$<br>$r-b = t < ?-b$<br>$t > u = r-b$<br>$t = ?-b$ | $u < t$<br>$t > u = t-?+r$<br>$u < t-?+r$<br>$r-b > u > t-?+r$<br>$u < t-?+r$<br>$t > u = t-?+r$<br>$u < t-?+r$                                                                                       | $nil$<br>$nil$<br>$nil$<br>$r, b+u$<br>$nil$<br>$nil$<br>$r, ?-t+u$<br>$nil$                                                             | $r, r+u$<br>$r, ?-t+u$<br>$nil$<br>$b+u, ?-t+u$<br>$r, ?-t+u$<br>$nil$<br>$nil$                                                     |
| $R(t)$ and $A(u)$  | $t < ?-r$<br>$?-r = t < r-b$<br>$r-b = t < ?-b$<br>$t = ?-b$                  | $u < t$<br>$t > u = t-?+r$<br>$u < t-?+r$<br>$t > u = r-b$<br>$r-b > u > t-?+r$<br>$t-r+b = u < t-?+r$<br>$u < t-r+b$<br>$t > u = t-?+r$<br>$t-r+b = u < t-?+r$<br>$t-?+b = u < t-r+b$<br>$u < t-?+b$ | $nil$<br>$nil$<br>$nil$<br>$r-t+u, r$<br>$r-t+u, b+u$<br>$r-t+u, b+u$<br>$b, b+u$<br>$r-t+u, r$<br>$r-t+u, ?-t+u$<br>$b, ?-t+u$<br>$nil$ | $r-t+u, r$<br>$r-t+u, r$<br>$r-t+u, ?-t+u$<br>$nil$<br>$b+u, r$<br>$b+u, ?-t+u$<br>$b+u, ?-t+u$<br>$nil$<br>$nil$<br>$nil$<br>$nil$ |

**APPENDIX 3**  
**EXPRESSIONS FOR  $COV(B(t), B(u))$  AND  $COV(S(t), B(u))$**

**Expression for  $COV(B(t), B(u))$ ,  $u = t$**

$$COV(B(t), B(u)) = K + L_1 + L_2 - L_3 + M$$

$$K = Sa(b, 0) \left[ \int_{c_1(t, u)}^{c_2(t, u)} \frac{Sa(y-u, 0)}{Sa(b, 0)} sa(y-u, 0) \frac{s(r)^2}{s(y-u)^2} {}_t p_{y-u} (1 - {}_u p_{y-u}) k(y-u)^2 dy \right]$$

$$L_1 = Sa(b, 0) sa(b, 0) e^{rt} e^{fu} \int_{a_0(t)}^{a_1(t)} \frac{\bar{N}_{y_0}^{(f)} - \bar{N}_{a_1(u)}^{(f)}}{D_b^{s(f)}} \frac{s(r)^2 D_x^{(r)}}{D_b^{s(r)}} k(b)^2 dx$$

$$L_2 = Sa(b, 0) sa(b, 0) e^{ru} e^{ft} \int_{a_0(t)}^{a_1(t)} \frac{\bar{N}_{a_0(u)}^{(r)} - \bar{N}_{y_0}^{(r)}}{D_b^{s(r)}} \frac{s(r)^2 D_x^{(f)}}{D_b^{s(f)}} k(b)^2 dx$$

$$L_3 = Sa(b, 0) sa(b, 0) e^{r(t+u)} s(r)^2 \frac{\bar{N}_{a_0(t)}^{(r)} - \bar{N}_{a_1(t)}^{(r)}}{D_b^{s(r)}} \frac{\bar{N}_{a_0(u)}^{(r)} - \bar{N}_{a_1(u)}^{(r)}}{D_b^{s(r)}} k(b)^2$$

$$M = Sa(b, 0) sa(b, 0) e^{rt} \int_{c_0(t, u)}^{c_1(t, u)} \frac{D_{y+t-u}^{(r)}}{D_b^{(r)}} \frac{s(r)^2}{s(b)^2} (1 - {}_{y-b} p_b) k(b)^2 dy$$

**Expression for  $COV(S(t), B(u))$ ,  $u = t$**

$$COV(S(t), B(u)) = K + L_1 + L_2 - L_3 + M$$

$$K = Sa(b, 0) \left[ \int_{c_1(t, u)}^{c_2(t, u)} \frac{Sa(y-u, 0)}{Sa(b, 0)} sa(y-u, 0) \frac{s(y+t-u)s(r)}{s(y-u)^2} {}_t p_{y-u} (1 - {}_u p_{y-u}) k(y-u) dy \right]$$

$$L_1 = Sa(b, 0) sa(b, 0) e^{rt} e^{fu} \int_{a_0(t)}^{a_1(t)} \frac{\bar{N}_{y_0}^{(f)} - \bar{N}_{a_1(u)}^{(f)}}{D_b^{s(f)}} \frac{s(r) D_x^{s(r)}}{D_b^{s(r)}} k(b) dx$$

$$L_2 = Sa(b, 0) sa(b, 0) e^{ru} e^{ft} \int_{a_0(t)}^{a_1(t)} \frac{\bar{N}_{a_0(u)}^{(r)} - \bar{N}_{y_0}^{(r)}}{D_b^{s(r)}} \frac{s(r) D_x^{s(f)}}{D_b^{s(f)}} k(b) dx$$

$$L_3 = Sa(b, 0) sa(b, 0) e^{r(t+u)} s(r) \frac{\bar{N}_{a_0(t)}^{s(r)} - \bar{N}_{a_1(t)}^{s(r)}}{D_b^{s(r)}} \frac{\bar{N}_{a_0(u)}^{(r)} - \bar{N}_{a_1(u)}^{(r)}}{D_b^{s(r)}} k(b)$$

$$M = Sa(b, 0) sa(b, 0) e^{rt} \int_{c_0(t, u)}^{c_1(t, u)} \frac{D_{y+t-u}^{(r)}}{D_b^{(r)}} \frac{s(y+t-u)s(r)}{s(b)^2} (1 - {}_{y-b} p_b) k(b) dy$$

**Expression for  $COV(S(u), B(t))$ ,  $u = t$**

$$COV(B(t), S(u)) = K + L_1 + L_2 - L_3 + M$$

$$K = Sa(b, 0) \left[ \int_{c_1(t, u)}^{c_2(t, u)} \frac{Sa(y-u, 0)}{Sa(b, 0)} sa(y-u, 0) \frac{s(r)s(y)}{s(y-u)^2} {}_t p_{y-u} (1 - {}_u p_{y-u}) k(y-u) dy \right]$$

$$L_1 = Sa(b, 0) sa(b, 0) e^{rt} e^{fu} \int_{a_0(t)}^{a_1(t)} \frac{\bar{N}_{y_0}^{s(f)} - \bar{N}_{a_1(u)}^{s(f)}}{D_b^{s(f)}} \frac{s(r) D_x^{(r)}}{D_b^{s(r)}} k(b) dx$$

$$\begin{aligned}
L_2 &= Sa(b,0)sa(b,0)e^{ru}e^{ft}\int_{a_0(t)}^{a_1(t)}\frac{\bar{N}_{a_0(u)}^{s(r)}-\bar{N}_{y_0}^{s(r)}}{D_b^{s(r)}}\frac{s(r)D_x^{(f)}}{D_b^{s(f)}}k(b)dx \\
L_3 &= Sa(b,0)sa(b,0)e^{r(t+u)}s(r)\frac{\bar{N}_{a_0(t)}^{s(r)}-\bar{N}_{a_1(t)}^{s(r)}}{D_b^{s(r)}}\frac{\bar{N}_{a_0(u)}^{s(r)}-N_{a_1(u)}^{s(r)}}{D_b^{s(r)}}k(b) \\
M &= Sa(b,0)sa(b,0)e^{rt}\int_{c_0(t,u)}^{c_1(t,u)}\frac{D_{y+t-u}^{(r)}}{D_b^{(r)}}\frac{s(r)s(y)}{s(b)^2}(1-\frac{y-b}{p_b})k(b)dy
\end{aligned}$$

#### APPENDIX 4 LONG-TERM EXPRESSIONS FOR $COV(A(t),A(u))$ , $COV(S(t),S(u))$ and $VTDS(t)$

##### Co-variance between $A(t)$ and $A(u)$ , ( $t = 2(r-b)$ , $u = t$ )

The formulae and expressions will be identical to those for  $COV(S(t),S(u))$  below, except that the salary scale function will be eliminated throughout. Thus, all commutation functions will be replaced by the corresponding functions without the salary scale being incorporated (replace  $D_x^{s(r)}$  by  $D_x^{(r)}$ ,  $\bar{N}_x^{s(r)}$  by  $\bar{N}_x^{(r)}$  etc). Moreover, the adjustment factor should be  $Ac(b,0)$  (instead of  $Sa(b,0)sa(b,0)$ ).

**Co-variance between  $S(t)$  and  $S(u)$ , ( $t = 2(r-b)$ ,  $u = t$ ), according to range of  $u$**   
The adjustment factor  $Sa(b,0)sa(b,0)$  should apply to all the following formulae.

**$u = r - b$ :**  $COV(S(t),S(u)) = e^{rt} f(u)$ , where

$$f(u) = \frac{\bar{N}_b^{s(r)} - \bar{N}_r^{s(r)}}{D_b^{s(r)}} [e^{fu} \frac{\bar{N}_b^{s(f)} - \bar{N}_{b+u}^{s(f)}}{D_b^{s(f)}} - e^{ru} \frac{\bar{N}_b^{s(r)} - \bar{N}_{b+u}^{s(r)}}{D_b^{s(r)}}]$$

where  $? = p - ?$ .

**$r - b < u = t - (r - b)$ :**  $COV(S(t),S(u)) = k_1 e^{pt} e^{-fz} - k_2 e^{2rt} e^{-rz}$

where  $z = t - u$ ;

$$\begin{aligned}
k_1 &= \frac{\bar{N}_b^{s(r)} - \bar{N}_r^{s(r)}}{D_b^{s(r)}} \frac{\bar{N}_b^{s(f)} - \bar{N}_r^{s(f)}}{D_b^{s(f)}} \\
k_2 &= [\frac{\bar{N}_b^{s(r)} - \bar{N}_r^{s(r)}}{D_b^{s(r)}}]^2
\end{aligned}$$

**$u > t - (r - b)$ :**  $COV(S(t),S(u)) = e^{pt} f_1(z) - e^{2rt} f_2(z) + e^{rt} f_3(z)$

where  $z = t - u$ ;

$$\begin{aligned}
f_1(z) &= e^{-fz} [\frac{\bar{N}_b^{s(r)} - \bar{N}_{b+z}^{s(r)}}{D_b^{s(r)}} \frac{\bar{N}_b^{s(f)} - \bar{N}_r^{s(f)}}{D_b^{s(f)}} + \int_{b+z}^r \frac{\bar{N}_{x-z}^{s(f)} - \bar{N}_r^{s(f)}}{D_b^{s(f)}} \frac{D_x^{s(r)}}{D_b^{s(r)}} dx] + e^{-rz} \int_{b+z}^r \frac{\bar{N}_b^{s(r)} - \bar{N}_{x-z}^{s(r)}}{D_b^{s(r)}} \frac{D_x^{s(f)}}{D_b^{s(f)}} dx \\
f_2(z) &= e^{-rz} [\frac{\bar{N}_b^{s(r)} - \bar{N}_r^{s(r)}}{D_b^{s(r)}}]^2 \\
f_3(z) &= \int_b^{r-z} \frac{D_{y+z}^{s(r)}}{D_b^{s(r)}} \frac{s(y)}{s(b)} (1 - \frac{l_y}{l_b}) dy
\end{aligned}$$

### Variance of $TDS(t)$

In the following,  $f(u), k_1, k_2, f_1(z), f_2(z), f_3(z)$  denote the functions occurring in the expressions relating to  $COV(S(t), S(u))$ , after incorporating the adjustment factor.

Let the two components of (90) be denoted by  $VTDS1(t)$  and  $VTDS2(t)$ .

For  $t = 2(r - b)$ ,

$$VTDS1(t) - VTDS1(2(r - b)) = H(2(r - b)) - H(t)$$

where,

$$\begin{aligned} H(t) &= H_1 e^{-t(d-r)} + H_2 e^{-t(a-p)} + H_3 e^{-t(a-2r)} + H_4 e^{-t(a-r)} \\ H_1 &= \frac{2}{d-r} \left[ \int_0^{r-b} e^{-qu} f(u) du + k_1 \frac{e^{-(r-b)(q-f)}}{q-f} - k_2 \frac{e^{-(r-b)(q-r)}}{q-r} \right] \\ H_2 &= \frac{2}{a-p} \left[ \int_0^{r-b} e^{qz} f_1(z) dz - k_1 \frac{e^{(r-b)(q-f)}}{q-f} \right] \\ H_3 &= \frac{2}{a-2r} \left[ k_2 \frac{e^{(r-b)(q-r)}}{q-r} - \int_0^{r-b} e^{qz} f_2(z) dz \right] \\ H_4 &= \frac{2}{a-r} \int_0^{r-b} e^{qz} f_3(z) dz \end{aligned}$$

For  $t = r - b$ ,

$$\begin{aligned} VTDS2(t) - VTDS2(r - b) &= G(r - b) - G(t) \\ G(t) &= G_1 e^{-t(d-r)} + G_2 e^{-t(a-2r)} + G_3 e^{-2t(d-r)} \\ k &= Sa(b, 0) \frac{\overline{N}_b^{s(r)} - \overline{N}_r^{s(r)}}{D_b^{s(r)}} \\ G_1 &= \frac{2}{d-r} \left[ k \int_0^{r-b} (e^{-qu} - e^{-du}) ES(u) du + k^2 \left\{ \frac{e^{-(r-b)(q-r)}}{q-r} - \frac{e^{-(r-b)(d-r)}}{d-r} \right\} \right] \\ G_2 &= \frac{-2k^2}{(q-r)(a-2r)} \\ G_3 &= \frac{k^2}{(d-r)^2} \end{aligned}$$

It follows that for  $t_0 \geq 2(r - b)$ ,

$$\begin{aligned} VTDS(t) - VTDS(t_0) &= F(t_0) - F(t) \\ F(t) &= F_1 e^{-(d-r)t} + F_2 e^{-(a-p)t} + F_3 e^{-(a-2r)t} + F_4 e^{-(a-r)t} + F_5 e^{-2(d-r)t} \end{aligned}$$

where  $F_1 = H_1 + G_1; F_2 = H_2; F_3 = H_3 + G_2; F_4 = H_4; F_5 = G_3$ .

Note 1. The expressions for  $Cov(S(t), S(u))$  are obtained by simplifying the respective integrals, within the appropriate integration limits (section 2.4 and Appendix 2).

Note 2. The expression for  $VTDS1(t) - VTDS1(2(r-b))$  is obtained by expressing the relevant integral as the sum of sub-integrals, as follows (and then simplifying each):

$$\int_{2(r-b)}^t e^{-du} \int_0^u e^{-qv} \text{Cov}(S(u), S(v)) dv du = \int_{2(r-b)}^t e^{-du} \left[ \int_0^{r-b} + \int_{r-b}^{u-r+b} + \int_{u-r+b}^u \right] e^{-qv} \text{Cov}(S(u), S(v)) dv du$$

A similar procedure applies to the expression for  $VTDS2(t) - VTDS2(r-b)$ .

## APPENDIX 5

### LONG-TERM EXPRESSIONS FOR $COV(R(t), R(u))$ , $COV(B(t), B(u))$ and $VTDBt$

#### Co-variance between $R(t)$ and $R(u)$ , ( $t = (r-b)+2(?-r)$ , $u = t$ )

The formulae and expressions will be identical to those for  $COV(B(t), B(u))$  below, except that the salary scale functions ( $s(b)$ ,  $s(r)$ ) and the pension rate function  $k(b)$  will be suppressed throughout. Moreover, the *adjustment factor should be  $Ac(b,0)$  (instead of  $Sa(b,0)sa(b,0)$ )*.

**Co-variance between  $B(t)$  and  $B(u)$ , ( $t = (r-b)+2(?-r)$ ,  $u = t$ ), according to range of  $u$**   
*The adjustment factor  $Sa(b,0)sa(b,0)$  should apply to all the following formulae.*

**$r-b = u = ? - b$ :**  $COV(B(t), B(u)) = e^{rt} f(u)$ , where

$$f(u) = \frac{\overline{N}_r^{(r)}}{D_b^{(r)}} \left[ e^{fu} \frac{\overline{N}_r^{(f)} - \overline{N}_{b+u}^{(f)}}{D_b^{(f)}} - e^{ru} \frac{\overline{N}_r^{(r)} - \overline{N}_{b+u}^{(r)}}{D_b^{(r)}} \right] \frac{s(r)^2}{s(b)^2} k(b)^2$$

where  $? = p - ?$ .

**$? - b < u = t - (? - r)$ :**  $COV(B(t), B(u)) = k_1 e^{pt} e^{-fz} - k_2 e^{2rt} e^{-rz}$

where  $z = t - u$ ;

$$k_1 = \frac{\overline{N}_r^{(r)}}{D_b^{(r)}} \frac{\overline{N}_r^{(f)}}{D_b^{(f)}} \frac{s(r)^2}{s(b)^2} k(b)^2$$

$$k_2 = \left[ \frac{\overline{N}_r^{(r)}}{D_b^{(r)}} \right]^2 \frac{s(r)^2}{s(b)^2} k(b)^2$$

**$u > t - (? - r)$ :**  $COV(B(t), B(u)) = e^{pt} f_1(z) - e^{2rt} f_2(z) + e^{rt} f_3(z)$

where  $z = t - u$ ;

$$f_1(z) = \left[ e^{-fz} \left[ \frac{\overline{N}_r^{(r)} - \overline{N}_{r+z}^{(r)}}{D_b^{(r)}} \frac{\overline{N}_r^{(f)}}{D_b^{(f)}} + \int_{r+z}^v \frac{\overline{N}_{x-z}^{(f)}}{D_b^{(f)}} \frac{D_x^{(r)}}{D_b^{(r)}} dx \right] + e^{-rz} \int_{r+z}^v \frac{\overline{N}_r^{(r)} - \overline{N}_{x-z}^{(r)}}{D_b^{(r)}} \frac{D_x^{(f)}}{D_b^{(f)}} dx \right] \frac{s(r)^2}{s(b)^2} k(b)^2$$

$$f_2(z) = e^{-rz} \left[ \frac{\overline{N}_r^{(r)}}{D_b^{(r)}} \right]^2 \frac{s(r)^2}{s(b)^2} k(b)^2$$

$$f_3(z) = \left[ \int_r^{v-z} \frac{D_{y+z}^{(r)}}{D_b^{(r)}} \left( 1 - \frac{l_y}{l_b} \right) dy \right] \frac{s(r)^2}{s(b)^2} k(b)^2$$

### Expression for the variance of $TDB(t)$

In what follows, the functions  $f(u), k_1, k_2, f_1(z), f_2(z), f_3(z)$  refer to those corresponding to  $COV(B(t), B(u))$ , after incorporating the adjustment factor.

As in the case of  $VTDS(t)$  (see (90)), let  $VTDB(t) = VTDB1(t) + VTDB2(t)$ .

For  $t = (r - b) + 2(v - r)$ ,

$$VTDB1(t) - VTDB1((r - b) + 2(v - r)) = H(2(r - b)) - H(t)$$

where,

$$\begin{aligned} H(t) &= H_1 e^{-t(d-r)} + H_2 e^{-t(a-p)} + H_3 e^{-t(a-2r)} + H_4 e^{-t(a-r)} \\ H_1 &= \frac{2}{d-r} \left[ \int_{r-b}^{v-b} e^{-qu} f(u) du + k_1 \frac{e^{-(v-b)(q-f)}}{q-f} - k_2 \frac{e^{-(v-b)(q-r)}}{q-r} \right] \\ H_2 &= \frac{2}{a-p} \left[ \int_0^{v-r} e^{qz} f_1(z) dz - k_1 \frac{e^{(v-r)(q-f)}}{q-f} \right] \\ H_3 &= \frac{2}{a-2r} \left[ k_2 \frac{e^{(v-r)(q-r)}}{q-r} - \int_0^{v-r} e^{qz} f_2(z) dz \right] \\ H_4 &= \frac{2}{a-r} \int_0^{v-r} e^{qz} f_3(z) dz \end{aligned}$$

For  $t = v - b$ ,

$$VTDB2(t) - VTDB2(v - b) = G(r - b) - G(t)$$

$$G(t) = G_1 e^{-t(d-r)} + G_2 e^{-t(a-2r)} + G_3 e^{-2t(d-r)}$$

$$k^* = Sa(b, 0) \frac{\overline{N}_r^{s(r)}}{D_b^{s(r)}}$$

$$G_1 = \frac{2}{d-r} \left[ k^* \int_0^{v-b} (e^{-qu} - e^{-du}) EB(u) du + k^{*2} \left\{ \frac{e^{-(v-b)(q-r)}}{q-r} - \frac{e^{-(r-b)(d-r)}}{d-r} \right\} \right]$$

$$G_2 = \frac{-2k^{*2}}{(q-r)(a-2r)}$$

$$G_3 = \frac{k^{*2}}{(d-r)^2}$$

It follows that for  $t_0 \geq (r - b) + 2(v - r)$ ,

$$VTDB(t) - VTDB(t_0) = F(t_0) - F(t)$$

$$F(t) = F_1 e^{-(d-r)t} + F_2 e^{-(a-p)t} + F_3 e^{-(a-2r)t} + F_4 e^{-(a-r)t} + F_5 e^{-2(d-r)t}$$

where  $F_1 = H_1 + G_1; F_2 = H_2; F_3 = H_3 + G_2; F_4 = H_4; F_5 = G_3$ .

Note. The procedures for the development and simplification of the above expressions are similar to those adopted in Appendix 4 (see Notes 1 and 2, Appendix 4).

**APPENDIX 6**  
**LONG-TERM EXPRESSIONS FOR  $COV(R(t), A(u))$ ,  $COV(B(t), S(u))$  and  $Cov(TDB(t), TDS(t))$**

**Co-variance between  $R(t)$  and  $A(u)$ , ( $t = 2(r-b) + (?-r)$ ,  $u = t$ )**

The formulae and expressions will be identical to those for  $COV(B(t), S(u))$  below, except that the pension rate function ( $k(b)$ ) and the salary scale function ( $s(x)$ ) will be suppressed throughout. Thus, all commutation functions will be replaced by the corresponding functions without the salary scale being incorporated (replace  $D_x^{s(r)}$  by  $D_x^{(r)}$ ,  $\bar{N}_x^{s(r)}$  by  $\bar{N}_x^{(r)}$  etc). Moreover, the *adjustment factor* should be  $Ac(b, 0)$  (instead of  $Sa(b, 0)sa(b, 0)$ ).

**Co-variance between  $B(t)$  and  $S(u)$ , ( $t = 2(r-b) + (?-r)$ ,  $u = t$ ), according to range of  $u$**   
*The adjustment factor  $Sa(b, 0)sa(b, 0)$  should apply to all the following formulae.*

**$u = r - b$ :**  $COV(B(t), S(u)) = e^{rt} f(u)$ , where

$$f(u) = \frac{\bar{N}_r^{(r)}}{D_b^{(r)}} [e^{fu} \frac{\bar{N}_b^{s(f)} - \bar{N}_{b+u}^{s(f)}}{D_b^{s(f)}} - e^{ru} \frac{\bar{N}_b^{s(r)} - \bar{N}_{b+u}^{s(r)}}{D_b^{s(r)}}] \frac{s(r)}{s(b)} k(b)$$

where  $? = p - ?$ .

**$r - b < u = t - (? - b)$ :**  $COV(B(t), S(u)) = k_1 e^{pt} e^{-fz} - k_2 e^{2rt} e^{-rz}$

where  $z = t - u$ ;

$$k_1 = \frac{\bar{N}_r^{(r)}}{D_b^{(r)}} \frac{\bar{N}_b^{s(f)} - \bar{N}_r^{s(f)}}{D_b^{s(f)}} \frac{s(r)}{s(b)} k(b)$$

$$k_2 = \frac{\bar{N}_r^{(r)}}{D_b^{(r)}} \frac{\bar{N}_b^{s(r)} - \bar{N}_r^{s(r)}}{D_b^{s(r)}} \frac{s(r)}{s(b)} k(b)$$

**$t - (? - b) < u = t - (r - b)$ :**  $COV(B(t), S(u)) = e^{pt} f_1(z) - e^{2rt} f_2(z) + e^{rt} f_3(z)$

where  $z = t - u$ ;  $f_1(z) =$

$$[e^{-fz} \{ \frac{\bar{N}_r^{(r)} - \bar{N}_{b+z}^{(r)}}{D_b^{(r)}} \frac{\bar{N}_b^{s(f)} - \bar{N}_r^{s(f)}}{D_b^{s(f)}} + \int_{b+z}^v \frac{\bar{N}_{x-z}^{s(f)} - \bar{N}_r^{s(f)}}{D_b^{s(f)}} \frac{D_x^{(r)}}{D_b^{(r)}} dx \} + e^{-rz} \int_{b+z}^v \frac{\bar{N}_b^{s(r)} - \bar{N}_{x-z}^{s(r)}}{D_b^{s(r)}} \frac{D_x^{(f)}}{D_b^{(f)}} dx] \frac{s(r)}{s(b)} k(b)$$

$$f_2(z) = e^{-rz} \frac{\bar{N}_r^{(r)}}{D_b^{(r)}} \frac{\bar{N}_b^{s(r)} - \bar{N}_r^{s(r)}}{D_b^{s(r)}} \frac{s(r)}{s(b)} k(b)$$

$$f_3(z) = [ \int_b^{v-z} \frac{D_{y+z}^{(r)}}{D_b^{(r)}} \frac{s(y)}{s(b)} (1 - \frac{l_y}{l_b}) dy ] \frac{s(r)}{s(b)} k(b)$$

**$t - (r - b) < u = t - (? - r)$ :**  $COV(B(t), S(u)) = e^{pt} f_4(z) - e^{2rt} f_2(z) + e^{rt} f_5(z)$

where  $z = t - u$ ;

$$f_4(z) = [e^{-fz} \int_r^v \frac{\bar{N}_{x-z}^{s(f)} - \bar{N}_r^{s(f)}}{D_b^{s(f)}} \frac{D_x^{(r)}}{D_b^{(r)}} dx + e^{-rz} \int_r^v \frac{\bar{N}_b^{s(r)} - \bar{N}_{x-z}^{s(r)}}{D_b^{s(r)}} \frac{D_x^{(f)}}{D_b^{(f)}} dx] \frac{s(r)}{s(b)} k(b)$$

$$f_5(z) = [ \int_{r-z}^{v-z} \frac{D_{y+z}^{(r)}}{D_b^{(r)}} \frac{s(y)}{s(b)} (1 - \frac{l_y}{l_b}) dy ] \frac{s(r)}{s(b)} k(b)$$

**$u > t - (? - r)$ :**  $COV(B(t), S(u)) = e^{pt} f_6(z) - e^{2rt} f_2(z) + e^{rt} f_7(z)$

where  $z = t - u$ ;

$$f_6(z) = [e^{-fz} \int_r^{r+z} \frac{\overline{N}_{x-z}^{s(f)} - \overline{N}_r^{s(f)}}{D_b^{s(f)}} \frac{D_x^{(r)}}{D_b^{(r)}} dx + e^{-rz} \{ \int_r^{r+z} \frac{\overline{N}_b^{s(r)} - \overline{N}_{x-z}^{s(r)}}{D_b^{s(r)}} \frac{D_x^{(f)}}{D_b^{(f)}} dx + \frac{\overline{N}_b^{s(r)} - \overline{N}_r^{s(r)}}{D_b^{s(r)}} \frac{D_{r+z}^{(f)}}{D_b^{(f)}} \} ] \frac{s(r)}{s(b)} k(b)$$

$$f_7(z) = [ \int_{r-z}^r \frac{D_{y+z}^{(r)}}{D_b^{(r)}} \frac{s(y)}{s(b)} (1 - \frac{l_y}{l_b}) dy ] \frac{s(r)}{s(b)} k(b)$$

### Co-variance between $A(t)$ and $R(u)$ , ( $t = ? - b$ , $u = t$ )

The formulae and expressions will be identical to those for  $COV(S(t), B(u))$  below, except that the pension rate function ( $k(b)$ ) and the salary scale function ( $s(x)$ ) will be suppressed throughout. Thus, all commutation functions will be replaced by the corresponding functions without the salary scale being incorporated (replace  $D_x^{s(r)}$  by  $D_x^{(r)}$ ,  $\overline{N}_x^{s(r)}$  by  $\overline{N}_x^{(r)}$  etc). Moreover, the adjustment factor should be  $Ac(b, 0)$  (instead of  $Sa(b, 0)sa(b, 0)$ ).

### Co-variance between $S(t)$ and $B(u)$ , ( $t = ? - b$ , $u = t$ ), according to range of $u$

The adjustment factor  $Sa(b, 0)sa(b, 0)$  should apply to all the following formulae.

**$r - b = u < ? - b$ :**  $COV(S(t), B(u)) = e^{rt} g(u)$ , where

$$g(u) = \frac{\overline{N}_b^{s(r)} - \overline{N}_r^{s(r)}}{D_b^{s(r)}} [ \frac{\overline{N}_r^{(f)} - \overline{N}_{b+u}^{(f)}}{D_b^{(f)}} - \frac{\overline{N}_r^{(r)} - \overline{N}_{b+u}^{(r)}}{D_b^{(r)}} ] \frac{s(r)}{s(b)} k(b)$$

where  $? = p - ?$ .

**$u = ? - b$ :**  $COV(S(t), B(u)) = h_1 e^{pt} e^{-fz} - h_2 e^{2rt} e^{-rz}$

where  $z = t - u$ ;

$$h_1 = \frac{\overline{N}_r^{(f)}}{D_b^{(f)}} \frac{\overline{N}_b^{s(r)} - \overline{N}_r^{s(r)}}{D_b^{s(r)}} \frac{s(r)}{s(b)} k(b)$$

$$h_2 = \frac{\overline{N}_r^{(r)}}{D_b^{(r)}} \frac{\overline{N}_b^{s(r)} - \overline{N}_r^{s(r)}}{D_b^{s(r)}} \frac{s(r)}{s(b)} k(b)$$

### Co-variance between $TDS(t)$ and $TDB(t)$

In the following,  $f(u), k_1, k_2, f_1(z), f_2(z), f_3(z), f_4(z), f_5(z), f_6(z), f_7(z)$  denote the functions occurring in the expressions relating to  $COV(B(t), S(u))$  and  $g(u), h_1, h_2$  denote the functions occurring in the expressions for  $COV(S(t), B(u))$ , after incorporating the respective adjustment factors.

Let the four components of (91) be denoted by  $CovTDBTDS1(t)$ ,  $CovTDBTDS2(t)$ ,  $CovTDSTDB1(t)$  and  $CovTDSTDB2(t)$ .

For  $t = 2(r - b) + (? - r)$ ,

$$CovTDBTDS1(t) - CovTDBTDS1(2(r - b) + (v - r)) = H(2(r - b) + (v - r)) - H(t)$$

where,

$$H(t) = H_1 e^{-t(d-r)} + H_2 e^{-t(a-p)} + H_3 e^{-t(a-2r)} + H_4 e^{-t(a-r)}$$

$$\begin{aligned}
H_1 &= \frac{1}{d-r} \left[ \int_0^{r-b} e^{-qu} f(u) du + k_1 \frac{e^{-(r-b)(q-f)}}{q-f} - k_2 \frac{e^{-(r-b)(q-r)}}{q-r} \right] \\
H_2 &= \frac{1}{a-p} \left[ \int_0^{v-r} e^{qz} f_6(z) dz + \int_{v-r}^{r-b} e^{qz} f_4(z) dz + \int_{r-b}^{v-b} e^{qz} f_1(z) dz - k_1 \frac{e^{(v-b)(q-f)}}{q-f} \right] \\
H_3 &= \frac{1}{a-2r} \left[ k_2 \frac{e^{(v-b)(q-r)}}{q-r} - \int_0^{v-b} e^{qz} f_2(z) dz \right] \\
H_4 &= \frac{1}{a-r} \left[ \int_0^{v-r} e^{qz} f_7(z) dz + \int_{v-r}^{r-b} e^{qz} f_5(z) dz + \int_{r-b}^{v-b} e^{qz} f_3(z) dz \right]
\end{aligned}$$

For  $t = ? - b$ ,

$$\begin{aligned}
\text{CovTDBTDS2}(t) - \text{CovTDBTDS2}(v-b) &= G(v-b) - G(t) \\
G(t) &= G_1 e^{-t(d-r)} + G_2 e^{-t(a-2r)} + G_3 e^{-2t(d-r)} \\
k &= Sa(b,0) \frac{\overline{N}_b^{s(r)} - \overline{N}_r^{s(r)}}{D_b^{s(r)}}; k^* = Sa(b,0) \frac{\overline{N}_r^{s(r)}}{D_b^{s(r)}} \\
G_1 &= \frac{1}{d-r} \left[ k^* \int_0^{r-b} (e^{-qu} - e^{-du}) ES(u) du + k k^* \left\{ \frac{e^{-(r-b)(q-r)}}{q-r} - \frac{e^{-(r-b)(d-r)}}{d-r} \right\} \right] \\
G_2 &= \frac{-k k^*}{(q-r)(a-2r)} \\
G_3 &= \frac{k k^*}{2(d-r)^2}
\end{aligned}$$

For  $t = ? - b$ ,

$$\begin{aligned}
\text{CovTDSTDB1}(t) - \text{CovTDSTDB1}(v-b) &= L(v-b) - L(t) \\
L(t) &= L_1 e^{-t(d-r)} + L_2 e^{-t(a-p)} + L_3 e^{-t(a-2r)} \\
L_1 &= \frac{1}{d-r} \left[ \int_{r-b}^{v-b} e^{-qu} g(u) du + h_1 \frac{e^{-(v-b)(q-f)}}{q-f} - h_2 \frac{e^{-(v-b)(q-r)}}{q-r} \right] \\
L_2 &= \frac{-h_1}{(a-p)(q-f)} \\
L_3 &= \frac{h_2}{(a-2r)(q-r)}
\end{aligned}$$

For  $t = ? - b$ ,

$$\begin{aligned}
\text{CovTDSTDB2}(t) - \text{CovTDSTDB2}(v-b) &= G(v-b) - G(t) \\
J(t) &= J_1 e^{-t(d-r)} + J_2 e^{-t(a-2r)} + J_3 e^{-2t(d-r)} \\
J_1 &= \frac{1}{d-r} \left[ k \int_0^{v-b} (e^{-qu} - e^{-du}) EB(u) du + k k^* \left\{ \frac{e^{-(v-b)(q-r)}}{q-r} - \frac{e^{-(v-b)(d-r)}}{d-r} \right\} \right] \\
J_2 &= \frac{-k k^*}{(q-r)(a-2r)} \\
J_3 &= \frac{k k^*}{2(d-r)^2}
\end{aligned}$$

It follows that for  $t_0 \geq 2(r - b) + (? - r)$ ,

$$\text{Cov}(TDS(t), TDB(t)) - \text{Cov}(TDS(t_0), TDB(t_0)) = F(t_0) - F(t)$$

$$F(t) = F_1 e^{-(d-r)t} + F_2 e^{-(a-p)t} + F_3 e^{-(a-2r)t} + F_4 e^{-(a-r)t} + F_5 e^{-2(d-r)t}$$

where  $F_1 = H_1 + G_1 + L_1 + J_1$ ;  $F_2 = H_2 + L_2$ ;  $F_3 = H_3 + G_2 + L_3 + J_2$ ;  $F_4 = H_4$ ;  $F_5 = G_3 + J_3$ .

Note. The procedures for the development and simplification of the above expressions are similar to those adopted in Appendix 4 (see Notes 1 and 2, Appendix 4).

## APPENDIX 7 LIST OF SYMBOLS

(in the approximate order in which they first appear in the text)

|                                    |                                                                                                |
|------------------------------------|------------------------------------------------------------------------------------------------|
| $P(x, t)$                          | Population aged $x$ at time $t$ (active or retired)                                            |
| $?$                                | New entrant growth parameter (new entrant intake deterministic)                                |
| $b$                                | Entry age of new entrants                                                                      |
| $r$                                | Retirement age                                                                                 |
| $?$                                | Limiting age of the mortality table                                                            |
| $Ac(x, t)$                         | Active Population, aged $x$ at time $t$                                                        |
| $Re(x, t)$                         | Retired Population, aged $x$ at time $t$                                                       |
| $EF$                               | Expected value of (any stochastic function) $F$                                                |
| $VF$                               | Variance of $F$                                                                                |
| $COV(F, G)$                        | Co-variance between (stochastic functions) $F$ and $G$ (also $CovFG$ )                         |
| $D_x^{(r)}, \bar{N}_x^{(r)}$       | Commutation functions based on force of interest $?$                                           |
| $D_{xx}^{(r)}, \bar{N}_{xx}^{(r)}$ | Special commutation functions ( $D_{xx}^{(r)} = l_x l_x e^{-rx}$ )                             |
| $(a_0(t), a_1(t))$                 | Age-limits of the surviving new entrant population at time $t$                                 |
| $(a_1(t), a_2(t))$                 | Age limits of the surviving initial population at time $t$                                     |
| $P(t)$                             | Total population at time $t$ (active or retired)                                               |
| $A(t)$                             | Total Active Population at time $t$                                                            |
| $R(t)$                             | Total Retired Population at time $t$                                                           |
| $c_0(t, u), c_1(t, u)$             | Age-limits (at $u$ ) for common new entrant cohorts between $P(t)$ and $P(u)$                  |
| $c_1(t, u), c_2(t, u)$             | Age-limits (at $u$ ) for common initial population cohorts between $P(t), P(u)$                |
| $DR(t)$                            | Demographic ratio                                                                              |
| $CVF$                              | Coefficient of variation of (a stochastic function) $F$                                        |
| $Corr(F, G)$                       | Correlation coefficient between stochastic functions $F$ and $G$ (also $CorrFG$ )              |
| $?$                                | Primary new entrant growth parameter (new entrant intake stochastic)                           |
| $p$                                | Secondary new entrant growth parameter (new entrant intake stochastic)                         |
| $f$                                | $p - ?$                                                                                        |
| $s(x)$                             | Salary scale function                                                                          |
| $D_x^{s(r)}, \bar{N}_x^{s(r)}$     | Commutation functions based on force of interest $?$ , incorporating the salary scale function |
| $sa(x, t)$                         | Average annual salary of the active population aged $x$ at time $t$                            |
| $pe(x, t)$                         | Average annual pension of the retired population aged $x$ at time $t$                          |
| $Sa(x, t)$                         | Annual salary bill of the active population aged $x$ at time $t$                               |
| $Be(x, t)$                         | Annual pension bill of the retired population aged $x$ at time $t$                             |

|                       |                                                                                                          |
|-----------------------|----------------------------------------------------------------------------------------------------------|
| $S(t)$                | Total salary bill at time $t$                                                                            |
| $B(t)$                | Total pension bill at time $t$                                                                           |
| $Bc(t)$               | Capitalized value of the pensions of those retiring at time $t$                                          |
| $a^*$                 | Capitalized value of an indexed unit pension at age $r$                                                  |
| $k(x)$                | Proportionate rate of pension for entry age $x$                                                          |
| $d$                   | Force of real interest (deterministic) or primary real interest parameter (force of interest stochastic) |
| $a$                   | Secondary real interest parameter (force of interest stochastic)                                         |
| $?$                   | $a - d$                                                                                                  |
| $DS(t)$               | Discounted value of $S(t)$                                                                               |
| $DB(t)$               | Discounted value of $B(t)$                                                                               |
| $TDS(t)$              | Total over $(0, t)$ of discounted salaries                                                               |
| $TDB(t)$              | Total over $(0, t)$ of discounted pension expenditures                                                   |
| $pr$                  | Classical (deterministic) premium                                                                        |
| $pr^*$                | Stochastic premium                                                                                       |
| $PAYG(n)$             | Pay-as-you-go premium at $t = n$                                                                         |
| $AVP(0, n)$           | Average premium over $(0, n)$                                                                            |
| $RRP(0, n : \Lambda)$ | Reserve Ratio system premium over $(0, n)$ , with ratio = ?                                              |
| $TFP(n)$              | Terminal Funding premium at $t = n$                                                                      |
| $GAP$                 | General Average Premium                                                                                  |

## APPENDIX 8 NUMERICAL ILLUSTRATIONS

### A hypothetical retirement pension scheme

The illustrations relate to a newly introduced pension scheme that provides retirement pensions, for life, from age 65. The pension rate is 1 per cent of the final salary per year of service. Pre-scheme service is not taken into account. The pension is fully adjusted, in line with the salary escalation of active persons.

The age-distribution of the initial insured active population (per 10,000), as well as of the corresponding initial salary bill, is shown in the table below. There are no pensioners at the outset. New entrants enter at age 20. The average cohort survival pattern, assumed to remain invariant, is according to the service and life tables summarized below. The progression of individual salaries, ignoring salary escalation, is according to the salary scale indicated below.

In order to illustrate the effect of changes in the population size and the parameters, the calculations have been made for the following six variants. The implied coefficients of variation of  $\exp(?t)$  (CV1) and of  $\exp(-d(t))$  (CV2) are indicated. The value of  $Ac(20,0)$  is proportional to the initial population; for the population of 10,000, its value is 291.

| <b>Variant</b> | <b>Initial population</b> | <b><math>\beta</math></b> | <b><math>\mu</math></b> | <b><math>d</math></b> | <b><math>a</math></b> | <b>CV1</b> | <b>CV2</b> |
|----------------|---------------------------|---------------------------|-------------------------|-----------------------|-----------------------|------------|------------|
| A              | 10,000                    | 0.01                      | 0.025                   | 0.03                  | 0.055                 | 7.08%      | 7.08%      |
| A1             | 1,000                     | 0.01                      | 0.025                   | 0.03                  | 0.055                 | 7.08%      | 7.08%      |
| A2             | 100                       | 0.01                      | 0.025                   | 0.03                  | 0.055                 | 7.08%      | 7.08%      |
| B              | 10,000                    | 0.01                      | 0.0205                  | 0.03                  | 0.0595                | 2.24%      | 2.24%      |
| C              | 10,000                    | 0.01                      | 0.0205                  | 0.03                  | 0.0305                | 2.24%      | 17.30%     |
| D              | 10,000                    | 0.01                      | 0.0495                  | 0.03                  | 0.0595                | 17.30%     | 2.24%      |

Tables 1 to 5 give detailed results for A, the main variant. Table 6 illustrates the sensitivity of the results to population size and Table 7 the effect of the parameters on the results. Table 8 shows the General Average Premiums for the different variants.

#### Initial population data, mortality assumptions and salary scale

| Age Group | Initial Popn | Initial Sal Bill<br>[\$'000] | Age | Salary scale | Service Table | Age | Life Table |
|-----------|--------------|------------------------------|-----|--------------|---------------|-----|------------|
| 20-25     | 1415         | 1875                         | 20  | 100          | 1000          | 65  | 1000       |
| 25-30     | 1339         | 2587                         | 25  | 165          | 995           | 70  | 861        |
| 30-35     | 1265         | 3092                         | 30  | 221          | 989           | 75  | 677        |
| 35-40     | 1193         | 3402                         | 35  | 267          | 982           | 80  | 463        |
| 40-45     | 1121         | 3539                         | 40  | 302          | 972           | 85  | 254        |
| 45-50     | 1047         | 3525                         | 45  | 328          | 958           | 90  | 101        |
| 50-55     | 967          | 3365                         | 50  | 344          | 936           | 95  | 25         |
| 55-60     | 878          | 3082                         | 55  | 350          | 903           | 100 | 0          |
| 60-65     | 776          | 2721                         | 60  | 350          | 851           |     |            |
| Totals    | 10000        | 27188                        | 65  | 350          | 775           |     |            |

Table 1. Demographic projections, Variant A

| (t) | EA    | VA      | ER   | VR    | CovRA  | CVA   | CVR   | corr(RA) | ER/EA  | EDR    | VDR      | CVDR  |
|-----|-------|---------|------|-------|--------|-------|-------|----------|--------|--------|----------|-------|
| 0   | 10000 | 0       | 0    | 0     | 0      | 0.00% | 0.00% | 0        | 0.00%  | 0.00%  | 0.000000 |       |
| 10  | 11051 | 879     | 1293 | 276   | 0      | 0.27% | 1.28% | 0        | 11.70% | 11.70% | 0.000002 | 1.31% |
| 20  | 12214 | 5542    | 2137 | 795   | 0      | 0.61% | 1.32% | 0        | 17.50% | 17.50% | 0.000006 | 1.45% |
| 30  | 13498 | 20009   | 2537 | 1099  | 0      | 1.05% | 1.31% | 0        | 18.80% | 18.80% | 0.000010 | 1.67% |
| 40  | 14918 | 52903   | 2811 | 1257  | 0      | 1.54% | 1.26% | 0        | 18.84% | 18.85% | 0.000014 | 1.99% |
| 50  | 16487 | 113940  | 3106 | 1384  | 799    | 2.05% | 1.20% | 0.0636   | 18.84% | 18.85% | 0.000019 | 2.30% |
| 60  | 18221 | 204711  | 3433 | 2345  | 7716   | 2.48% | 1.41% | 0.3522   | 18.84% | 18.85% | 0.000020 | 2.38% |
| 70  | 20137 | 334211  | 3794 | 4916  | 22276  | 2.87% | 1.85% | 0.5496   | 18.84% | 18.85% | 0.000021 | 2.41% |
| 80  | 22255 | 516308  | 4193 | 9095  | 44983  | 3.23% | 2.27% | 0.6564   | 18.84% | 18.85% | 0.000021 | 2.44% |
| 90  | 24596 | 769444  | 4634 | 15131 | 77860  | 3.57% | 2.65% | 0.7216   | 18.84% | 18.85% | 0.000022 | 2.47% |
| 100 | 27182 | 1118077 | 5122 | 23685 | 124526 | 3.89% | 3.00% | 0.7652   | 18.84% | 18.86% | 0.000022 | 2.50% |
| 110 | 30041 | 1594556 | 5660 | 35653 | 189881 | 4.20% | 3.34% | 0.7964   | 18.84% | 18.85% | 0.000023 | 2.54% |
| 120 | 33201 | 2241578 | 6256 | 52229 | 280439 | 4.51% | 3.65% | 0.8196   | 18.84% | 18.86% | 0.000024 | 2.58% |

**Table 2. Financial projections (static salary basis), Variant A**

| (t) | ES    | EB   | VS       | VB     | CovBS   | CVS   | CVB   | E(Bc) | VBc    | CovBcS  | CVBc  |
|-----|-------|------|----------|--------|---------|-------|-------|-------|--------|---------|-------|
| 0   | 27188 | 0    | 0        | 0      | 0       |       |       | 0     | 0      | 0       |       |
| 10  | 30047 | 245  | 4828     | 10     | 0       | 0.23% | 1.32% | 617   | 691    | 0       | 4.26% |
| 20  | 33208 | 919  | 23158    | 131    | 0       | 0.46% | 1.24% | 1364  | 3833   | 0       | 4.54% |
| 30  | 36700 | 1886 | 97953    | 523    | 0       | 0.85% | 1.21% | 2261  | 10523  | 0       | 4.54% |
| 40  | 40560 | 3070 | 304327   | 1321   | 0       | 1.36% | 1.18% | 3332  | 22082  | 0       | 4.46% |
| 50  | 44825 | 4386 | 726647   | 2500   | 3434    | 1.90% | 1.14% | 4143  | 23008  | 16168   | 3.66% |
| 60  | 49540 | 5319 | 1364261  | 5596   | 33134   | 2.36% | 1.41% | 4578  | 29272  | 60755   | 3.74% |
| 70  | 54750 | 5986 | 2278626  | 12244  | 95651   | 2.76% | 1.85% | 5060  | 37464  | 126862  | 3.83% |
| 80  | 60508 | 6623 | 3569542  | 22685  | 193148  | 3.12% | 2.27% | 5592  | 48187  | 222559  | 3.93% |
| 90  | 66872 | 7319 | 5369850  | 37739  | 334315  | 3.47% | 2.65% | 6180  | 62228  | 358647  | 4.04% |
| 100 | 73905 | 8089 | 7855861  | 59074  | 534686  | 3.79% | 3.00% | 6830  | 80619  | 549523  | 4.16% |
| 110 | 81677 | 8940 | 11260960 | 88926  | 815308  | 4.11% | 3.34% | 7549  | 104704 | 814319  | 4.29% |
| 120 | 90267 | 9880 | 15893360 | 130270 | 1204140 | 4.42% | 3.65% | 8342  | 136238 | 1178395 | 4.42% |

**Note: Expected values in \$1,000; variances and co-variances in \$1,000x1,000**

**Table 3a. COV(S(t), S(u)), Variant A (\$1,000x1,000)**

| (u) |       |        |        |         |         |         |         |         |         |          |          |          |
|-----|-------|--------|--------|---------|---------|---------|---------|---------|---------|----------|----------|----------|
| (t) | 10    | 20     | 30     | 40      | 50      | 60      | 70      | 80      | 90      | 100      | 110      | 120      |
| 10  | 4828  | 5695   | 8809   | 13092   | 17106   | 19020   | 21020   | 23231   | 25674   | 28374    | 31358    | 34656    |
| 20  | 5695  | 23158  | 40717  | 64446   | 88651   | 102796  | 113758  | 125722  | 138944  | 153557   | 169707   | 187555   |
| 30  | 8809  | 40717  | 97953  | 162890  | 233761  | 284327  | 320467  | 354367  | 391637  | 432825   | 478346   | 528654   |
| 40  | 13092 | 64446  | 162890 | 304327  | 454477  | 577698  | 671988  | 750722  | 829932  | 917217   | 1013682  | 1120292  |
| 50  | 17106 | 88651  | 233761 | 454477  | 726647  | 964816  | 1163587 | 1329207 | 1479408 | 1635331  | 1807319  | 1997396  |
| 60  | 19020 | 102796 | 284327 | 577698  | 964816  | 1364261 | 1716440 | 2022400 | 2290833 | 2545188  | 2813297  | 3109173  |
| 70  | 21020 | 113758 | 320467 | 671988  | 1163587 | 1716440 | 2278626 | 2787364 | 3242137 | 3654913  | 4056610  | 4483804  |
| 80  | 23231 | 125722 | 354367 | 750722  | 1329207 | 2022400 | 2787364 | 3569542 | 4291712 | 4951218  | 5564388  | 6171897  |
| 90  | 25674 | 138944 | 391637 | 829932  | 1479408 | 2290833 | 3242137 | 4291712 | 5369850 | 6381217  | 7320275  | 8209137  |
| 100 | 28374 | 153557 | 432825 | 917217  | 1635331 | 2545188 | 3654913 | 4951218 | 6381217 | 7855861  | 9257041  | 10575400 |
| 110 | 31358 | 169707 | 478346 | 1013682 | 1807319 | 2813297 | 4056610 | 5564388 | 7320275 | 9257041  | 11260960 | 13185230 |
| 120 | 34656 | 187555 | 528654 | 1120292 | 1997396 | 3109173 | 4483804 | 6171897 | 8209137 | 10575400 | 13185230 | 15893360 |

**Table 3b. COV(B(t), B(u)), Variant A (\$1,000x1,000)**

| Table 55: CCF, $\hat{C}_X(\tau)$ , $\hat{C}_Y(\tau)$ , Variant 11 (512,500,500) |    |     |     |      |      |      |       |       |       |       |       |        |
|---------------------------------------------------------------------------------|----|-----|-----|------|------|------|-------|-------|-------|-------|-------|--------|
| (t)                                                                             | 10 | 20  | 30  | 40   | 50   | 60   | 70    | 80    | 90    | 100   | 110   | 120    |
| 10                                                                              | 10 | 6   | 2   | 0    | 0    | 0    | 0     | 0     | 0     | 0     | 0     | 0      |
| 20                                                                              | 6  | 131 | 69  | 17   | 1    | 0    | 0     | 0     | 0     | 0     | 0     | 0      |
| 30                                                                              | 2  | 69  | 523 | 242  | 53   | 2    | 0     | 0     | 0     | 0     | 0     | 0      |
| 40                                                                              | 0  | 17  | 242 | 1321 | 566  | 116  | 4     | 0     | 0     | 0     | 0     | 0      |
| 50                                                                              | 0  | 1   | 53  | 566  | 2500 | 1250 | 614   | 512   | 561   | 620   | 685   | 757    |
| 60                                                                              | 0  | 0   | 2   | 116  | 1250 | 5596 | 4867  | 4604  | 4901  | 5410  | 5979  | 6608   |
| 70                                                                              | 0  | 0   | 0   | 4    | 614  | 4867 | 12244 | 12459 | 12974 | 14138 | 15619 | 17261  |
| 80                                                                              | 0  | 0   | 0   | 0    | 512  | 4604 | 12459 | 22685 | 24254 | 26020 | 28544 | 31539  |
| 90                                                                              | 0  | 0   | 0   | 0    | 561  | 4901 | 12974 | 24254 | 37739 | 41306 | 44904 | 49402  |
| 100                                                                             | 0  | 0   | 0   | 0    | 620  | 5410 | 14138 | 26020 | 41306 | 59074 | 65483 | 71700  |
| 110                                                                             | 0  | 0   | 0   | 0    | 685  | 5979 | 15619 | 28544 | 44904 | 65483 | 88926 | 99318  |
| 120                                                                             | 0  | 0   | 0   | 0    | 757  | 6608 | 17261 | 31539 | 49402 | 71700 | 99318 | 130269 |

**Table 3c. COV(B(t), S(u)), Variant A (\$1,000x1,000)**  
(u)

| (t) | 10   | 20    | 30    | 40     | 50     | 60     | 70     | 80     | 90     | 100    | 110     | 120     |
|-----|------|-------|-------|--------|--------|--------|--------|--------|--------|--------|---------|---------|
| 10  | 0    | 0     | 0     | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0       | 0       |
| 20  | 233  | 0     | 0     | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0       | 0       |
| 30  | 301  | 596   | 0     | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0       | 0       |
| 40  | 235  | 713   | 1039  | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0       | 0       |
| 50  | 437  | 1422  | 2894  | 4122   | 3434   | 3795   | 4194   | 4635   | 5123   | 5661   | 6257    | 6915    |
| 60  | 1493 | 6288  | 13573 | 22523  | 30604  | 33134  | 36619  | 40470  | 44727  | 49431  | 54629   | 60375   |
| 70  | 2191 | 11017 | 27139 | 48344  | 71160  | 87053  | 95651  | 105711 | 116829 | 129116 | 142695  | 157702  |
| 80  | 2539 | 13581 | 36843 | 71771  | 112748 | 148649 | 175089 | 193148 | 213461 | 235911 | 260722  | 288142  |
| 90  | 2810 | 15203 | 42634 | 88329  | 148513 | 208536 | 261198 | 302554 | 334315 | 369475 | 408333  | 451278  |
| 100 | 3106 | 16808 | 47368 | 100093 | 175754 | 261289 | 345668 | 421323 | 483475 | 534686 | 590920  | 653067  |
| 110 | 3432 | 18575 | 52358 | 110944 | 197430 | 303749 | 421733 | 539019 | 645998 | 736859 | 815308  | 901055  |
| 120 | 3793 | 20529 | 57864 | 122622 | 218614 | 339811 | 485386 | 646849 | 808383 | 957782 | 1087958 | 1204140 |

**Table 4. Discounted financial projections, Variant A**

| (t) | EDS  | EDB | ETDS   | ETDB | VTDS       | VTDB     | CovTDBTDS | CovTDBDS | CovTDBDB | CovTDSDS | CovTDSDB |
|-----|------|-----|--------|------|------------|----------|-----------|----------|----------|----------|----------|
| 0   | 2718 | 0   | 2718   | 0    | 0          | 0        | 0         | 0        | 0        | 0        | 0        |
| 10  | 2225 | 18  | 27114  | 75   | 9769199    | 145      | 35890     | 5579     | 45       | 1346944  | 11001    |
| 20  | 1822 | 50  | 47088  | 433  | 61922520   | 10577    | 781196    | 55947    | 1549     | 3937569  | 108919   |
| 30  | 1492 | 76  | 63441  | 1091 | 166833700  | 102562   | 4001629   | 175764   | 9037     | 6486460  | 333219   |
| 40  | 1221 | 92  | 76830  | 1952 | 317439400  | 436999   | 11398900  | 344469   | 26086    | 8459370  | 640006   |
| 50  | 1000 | 97  | 87792  | 2919 | 500367900  | 1206683  | 23775510  | 525857   | 51458    | 9715577  | 949653   |
| 60  | 818  | 87  | 96767  | 3851 | 701564700  | 2460270  | 40228420  | 673150   | 72278    | 10302890 | 1105143  |
| 70  | 670  | 73  | 104115 | 4651 | 908839500  | 4027816  | 58694000  | 758407   | 82929    | 10346790 | 1130524  |
| 80  | 548  | 60  | 110131 | 5309 | 1112756000 | 5727774  | 77612530  | 786821   | 86124    | 9990323  | 1092889  |
| 90  | 449  | 49  | 115057 | 5848 | 1306658000 | 7442042  | 96050780  | 773741   | 84691    | 9365280  | 1024626  |
| 100 | 367  | 40  | 119089 | 6289 | 1486305000 | 9095385  | 113431800 | 733242   | 80257    | 8580768  | 938873   |
| 110 | 301  | 32  | 122391 | 6651 | 1649388000 | 10640350 | 129412100 | 676302   | 74024    | 7720736  | 844820   |
| 120 | 246  | 26  | 125094 | 6947 | 1795041000 | 12050340 | 143822500 | 611058   | 66883    | 6846101  | 749157   |

**Note:** Expected values in \$10,000; variances and co-variances in \$10,000x10,000

**Table 5. Premiums, Variant A**

| (t) | PAYG system |            | TFP system |            | Period  | AVP system |            | RRP system |            |
|-----|-------------|------------|------------|------------|---------|------------|------------|------------|------------|
|     | Classical   | Stochastic | Classical  | Stochastic |         | Classical  | Stochastic | Classical  | Stochastic |
| 10  | 0.82%       | 0.83%      | 2.05%      | 2.20%      | (0,10)  | 0.28%      | 0.30%      | 0.61%      | 0.71%      |
| 20  | 2.77%       | 2.83%      | 4.11%      | 4.42%      | (0,20)  | 0.92%      | 1.03%      | 1.46%      | 1.72%      |
| 30  | 5.14%       | 5.26%      | 6.16%      | 6.63%      | (0,30)  | 1.72%      | 1.97%      | 2.32%      | 2.78%      |
| 40  | 7.57%       | 7.80%      | 8.22%      | 8.85%      | (0,40)  | 2.54%      | 2.97%      | 3.14%      | 3.80%      |
| 50  | 9.78%       | 10.14%     | 9.24%      | 9.84%      | (0,50)  | 3.33%      | 3.95%      | 3.88%      | 4.74%      |
| 60  | 10.74%      | 11.14%     | 9.24%      | 9.82%      | (0,60)  | 3.98%      | 4.77%      | 4.44%      | 5.44%      |
| 70  | 10.93%      | 11.36%     | 9.24%      | 9.80%      | (0,70)  | 4.47%      | 5.39%      | 4.82%      | 5.92%      |
| 80  | 10.95%      | 11.37%     | 9.24%      | 9.78%      | (0,80)  | 4.82%      | 5.83%      | 5.09%      | 6.25%      |
| 90  | 10.95%      | 11.38%     | 9.24%      | 9.75%      | (0,90)  | 5.08%      | 6.16%      | 5.30%      | 6.49%      |
| 100 | 10.95%      | 11.39%     | 9.24%      | 9.73%      | (0,100) | 5.28%      | 6.41%      | 5.45%      | 6.67%      |
| 110 | 10.95%      | 11.39%     | 9.24%      | 9.70%      | (0,110) | 5.43%      | 6.59%      | 5.57%      | 6.81%      |
| 120 | 10.95%      | 11.40%     | 9.24%      | 9.68%      | (0,120) | 5.55%      | 6.74%      | 5.66%      | 6.92%      |

**Table 6. Comparison of Variants A A1 A2 (CVDR and stochastic premi ums)**

| (t) | CVDR |      |       | PAYG  |       |       | TFP  |       |       |         | AVP  |      |      | RRP  |      |      |
|-----|------|------|-------|-------|-------|-------|------|-------|-------|---------|------|------|------|------|------|------|
|     | A    | A1   | A2    | A     | A1    | A2    | A    | A1    | A2    |         | A    | A1   | A2   | A    | A1   | A2   |
| 10  | 1.3% | 4.2% | 13.1% | 0.8%  | 0.9%  | 1.0%  | 2.2% | 2.5%  | 3.5%  | (0,10)  | 0.3% | 0.3% | 0.3% | 0.7% | 0.7% | 0.8% |
| 20  | 1.5% | 4.6% | 14.5% | 2.8%  | 3.0%  | 3.4%  | 4.4% | 5.1%  | 7.2%  | (0,20)  | 1.0% | 1.0% | 1.1% | 1.7% | 1.7% | 1.8% |
| 30  | 1.7% | 5.3% | 16.6% | 5.3%  | 5.5%  | 6.5%  | 6.6% | 7.7%  | 11.0% | (0,30)  | 2.0% | 2.0% | 2.1% | 2.8% | 2.8% | 2.9% |
| 40  | 2.0% | 6.3% | 19.5% | 7.8%  | 8.3%  | 10.3% | 8.9% | 10.3% | 15.1% | (0,40)  | 3.0% | 3.0% | 3.1% | 3.8% | 3.8% | 3.9% |
| 50  | 2.3% | 7.3% | 22.2% | 10.1% | 11.0% | 14.6% | 9.8% | 11.2% | 16.3% | (0,50)  | 3.9% | 4.0% | 4.1% | 4.7% | 4.8% | 4.9% |
| 60  | 2.4% | 7.5% | 22.7% | 11.1% | 12.1% | 16.7% | 9.8% | 11.1% | 16.4% | (0,60)  | 4.8% | 4.8% | 5.0% | 5.4% | 5.5% | 5.6% |
| 70  | 2.4% | 7.6% | 22.9% | 11.4% | 12.4% | 17.6% | 9.8% | 11.1% | 16.4% | (0,70)  | 5.4% | 5.4% | 5.6% | 5.9% | 5.9% | 6.1% |
| 80  | 2.4% | 7.7% | 23.1% | 11.4% | 12.4% | 18.1% | 9.8% | 11.0% | 16.6% | (0,80)  | 5.8% | 5.9% | 6.0% | 6.2% | 6.3% | 6.4% |
| 90  | 2.5% | 7.8% | 23.3% | 11.4% | 12.5% | 18.8% | 9.8% | 11.0% | 16.7% | (0,90)  | 6.2% | 6.2% | 6.3% | 6.5% | 6.5% | 6.7% |
| 100 | 2.5% | 7.9% | 23.6% | 11.4% | 12.5% | 19.6% | 9.7% | 10.9% | 17.0% | (0,100) | 6.4% | 6.4% | 6.6% | 6.7% | 6.7% | 6.8% |
| 110 | 2.5% | 8.0% | 23.9% | 11.4% | 12.5% | 20.7% | 9.7% | 10.8% | 17.3% | (0,110) | 6.6% | 6.6% | 6.8% | 6.8% | 6.8% | 7.0% |
| 120 | 2.6% | 8.1% | 24.2% | 11.4% | 12.6% | 22.1% | 9.7% | 10.7% | 17.8% | (0,120) | 6.7% | 6.8% | 6.9% | 6.9% | 6.9% | 7.1% |

**Table 7. Comparison of Variants B C D (CVDR and stochastic premiums)**

| (t) | CVDR |      |       | PAYG  |       |       | TFP  |      |       |         | AVP  |      |      | RRP  |      |      |
|-----|------|------|-------|-------|-------|-------|------|------|-------|---------|------|------|------|------|------|------|
|     | B    | C    | D     | B     | C     | D     | B    | C    | D     |         | B    | C    | D    | B    | C    | D    |
| 10  | 1.3% | 1.3% | 1.4%  | 0.8%  | 0.8%  | 0.8%  | 2.2% | 2.2% | 2.2%  | (0,10)  | 0.3% | 0.3% | 0.3% | 0.6% | 0.8% | 0.6% |
| 20  | 1.3% | 1.3% | 2.0%  | 2.8%  | 2.8%  | 2.8%  | 4.4% | 4.5% | 4.4%  | (0,20)  | 1.0% | 1.2% | 1.0% | 1.5% | 2.1% | 1.5% |
| 30  | 1.4% | 1.4% | 3.1%  | 5.2%  | 5.2%  | 5.4%  | 6.6% | 6.7% | 6.7%  | (0,30)  | 1.8% | 2.3% | 1.8% | 2.5% | 3.5% | 2.5% |
| 40  | 1.4% | 1.4% | 4.5%  | 7.7%  | 7.7%  | 8.1%  | 8.8% | 8.9% | 9.0%  | (0,40)  | 2.7% | 3.6% | 2.7% | 3.4% | 5.0% | 3.4% |
| 50  | 1.3% | 1.3% | 6.0%  | 10.0% | 10.0% | 10.7% | 9.8% | 9.9% | 10.2% | (0,50)  | 3.5% | 5.0% | 3.6% | 4.2% | 6.4% | 4.2% |
| 60  | 1.3% | 1.3% | 7.0%  | 11.0% | 11.0% | 12.0% | 9.8% | 9.9% | 10.2% | (0,60)  | 4.3% | 6.1% | 4.3% | 4.8% | 7.5% | 4.8% |
| 70  | 1.3% | 1.3% | 8.1%  | 11.2% | 11.2% | 12.5% | 9.7% | 9.9% | 10.3% | (0,70)  | 4.8% | 7.0% | 4.8% | 5.2% | 8.2% | 5.2% |
| 80  | 1.2% | 1.2% | 9.3%  | 11.2% | 11.2% | 12.8% | 9.7% | 9.9% | 10.4% | (0,80)  | 5.2% | 7.7% | 5.2% | 5.5% | 8.7% | 5.5% |
| 90  | 1.2% | 1.2% | 10.8% | 11.2% | 11.2% | 13.1% | 9.7% | 9.9% | 10.6% | (0,90)  | 5.5% | 8.1% | 5.5% | 5.7% | 9.0% | 5.7% |
| 100 | 1.1% | 1.1% | 12.4% | 11.1% | 11.1% | 13.6% | 9.7% | 9.9% | 10.8% | (0,100) | 5.7% | 8.4% | 5.7% | 5.9% | 9.2% | 5.9% |
| 110 | 1.1% | 1.1% | 14.3% | 11.1% | 11.1% | 14.1% | 9.7% | 9.9% | 11.0% | (0,110) | 5.9% | 8.7% | 5.9% | 6.0% | 9.4% | 6.0% |
| 120 | 1.1% | 1.1% | 16.4% | 11.1% | 11.1% | 14.9% | 9.7% | 9.9% | 11.4% | (0,120) | 6.0% | 8.9% | 6.0% | 6.1% | 9.5% | 6.1% |

**Note to Tables 5, 6 and 7: The RRP system premium relates to the reserve ratio of 5.**

**Table 8. Limiting values of discounted financial projections and GAP**

| Variant | ETDS   | ETDB | VTDS        | VTDB      | CovTDBTDS  | GAP         |              |
|---------|--------|------|-------------|-----------|------------|-------------|--------------|
|         |        |      |             |           |            | (classical) | (stochastic) |
| A       | 137415 | 8296 | 2645254877  | 20719273  | 229947655  | 6.04%       | 7.35%        |
| A1      | 13742  | 830  | 26950756    | 209307    | 2330706    | 6.04%       | 7.36%        |
| A2      | 1374   | 83   | 319328      | 2304      | 26430      | 6.04%       | 7.45%        |
| B       | 137415 | 8296 | 234251067   | 1766318   | 19973878   | 6.04%       | 6.51%        |
| C       | 137415 | 8296 | 51928158764 | 533555426 | 5210354509 | 6.04%       | 9.97%        |
| D       | 137415 | 8296 | 313130969   | 2040346   | 24619168   | 6.04%       | 6.52%        |

**Note: Expected values in \$10,000; variances and co-variance in \$10,000x10,000**

**Note to tables 5 to 8: Stochastic premiums are based on the Normal distribution and correspond to the 95<sup>th</sup> percentile (e = 0.05)**

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