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# The indecomposability of a certain bimodule given by the Brauer construction

Shigeo Koshitani and Markus Linckelmann

## Abstract.

Broué’s abelian defect conjecture [3, 6.2] predicts for a  $p$ -block of a finite group  $G$  with an abelian defect group  $P$  a derived equivalence between the block algebra and its Brauer correspondent. By a result of Rickard [11], such a derived equivalence would in particular imply a stable equivalence induced by tensoring with a suitable bimodule - and it appears that these stable equivalences in turn tend to be obtained by “gluing” together Morita equivalences at the local levels of the considered blocks; see e.g. [4, 6.3], [8, 3.1], [12, 4.1], and [13, 5.6, A.4.1]. This note provides a technical indecomposability result which is intended to verify in suitable circumstances the hypotheses that are necessary to apply gluing results as mentioned above. This is used in [7] to show that Broué’s abelian defect group conjecture holds for nonprincipal blocks of the simple Held group and the sporadic Suzuki group.

**Keywords:** Broué’s conjecture; Brauer construction; block; Brauer pair

Throughout this note,  $p$  is a prime and  $\mathcal{O}$  is a complete discrete valuation ring having an algebraically closed residue field  $k$  of characteristic  $p$ . We allow the case  $\mathcal{O} = k$ . We state our result and explain the terminology below.

**Theorem.** *Let  $G$  be a finite group, let  $b$  be a block of  $\mathcal{O}G$  and let  $(P, e)$  be a maximal  $b$ -Brauer pair. Set  $H = N_G(P, e)$ . For any subgroup  $Q$  of  $P$  denote by  $e_Q$  and  $f_Q$  the unique blocks of  $kC_G(Q)$  and  $kC_H(Q)$  satisfying  $(Q, e_Q) \subseteq (P, e)$  and  $(Q, f_Q) \subseteq (P, e)$ , respectively. Let  $f$  be a primitive idempotent in  $(\mathcal{O}Gb)^{\Delta H}$  such that  $\text{Br}_{\Delta P}(f)e = e$  and set  $X = \mathcal{O}Gf$ . Then, as  $\mathcal{O}(G \times H)$ -module  $X$  is indecomposable with vertex  $\Delta P$ , and for any subgroup  $Q$  of  $Z(P)$  the  $k(C_G(Q) \times C_H(Q))$ -module  $e_Q X(\Delta Q) f_Q$  is up to isomorphism the unique indecomposable direct summand of  $e_Q kC_G(Q) f_Q$  with vertex  $\Delta P$ .*

This Theorem is used in [7] to verify Broué’s abelian defect group conjecture for nonprincipal blocks of the simple Held group and the sporadic Suzuki group.

Given a finite group  $G$ , a *block of  $\mathcal{O}G$*  is a primitive idempotent in  $Z(\mathcal{O}G)$ . We denote by  $\Delta G$  the diagonal subgroup  $\Delta G = \{(g, g) \mid g \in G\}$  of  $G \times G$ . Unless stated otherwise, modules are left modules. If  $G$  and  $H$  are two finite groups, by an  $(\mathcal{O}G, \mathcal{O}H)$ -bimodule we mean a bimodule whose left and right  $\mathcal{O}$ -module structure coincide, so that we can view any such bimodule  $X$  as  $\mathcal{O}(G \times H)$ -module via  $(g, h)x = gxh^{-1}$  for any  $(g, h) \in G \times H$  and any  $x \in X$ . If furthermore  $Q$  is a common subgroup of  $G$  and  $H$ , we set  $X^{\Delta Q} = \{x \in X \mid (u, u)x = x, \forall u \in Q\} = \{x \in X \mid uxu^{-1} = x, \forall u \in Q\}$ . If  $Q$  is actually a  $p$ -group, the *Brauer construction* is defined to be the quotient  $X(\Delta Q) = X^{\Delta Q} / (\sum_{Q'} \text{Tr}_{Q'}^Q(X^{\Delta Q'}) + J(\mathcal{O})X^{\Delta Q})$ ,

where in the sum  $Q'$  runs over the set of proper subgroups of  $Q$ , and where  $\text{Tr}_{Q'}^Q$  is the usual relative trace map. This construction is functorial in  $X$ . Moreover, since  $C_{G \times H}(\Delta Q) = C_G(Q) \times C_H(Q) \subseteq N_{G \times H}(\Delta Q)$ , we can regard  $X(\Delta Q)$  as a  $(kC_G(Q), kC_H(Q))$ -bimodule. When applied to  $X = \mathcal{O}G$ , there is a canonical isomorphism  $(\mathcal{O}G)(\Delta Q) \cong kC_G(Q)$ , and the map  $\text{Br}_{\Delta Q} : (\mathcal{O}G)^{\Delta Q} \rightarrow kC_G(Q)$  obtained from composing the canonical epimorphism  $(\mathcal{O}G)^{\Delta Q} \rightarrow (\mathcal{O}G)(\Delta Q)$  with this isomorphism is in fact an algebra homomorphism, called the *Brauer homomorphism*. More explicitly, every element in  $(\mathcal{O}G)^{\Delta Q}$  is an  $\mathcal{O}$ -linear combination of  $Q$ -conjugacy class sums of elements in  $G$ , and  $\text{Br}_{\Delta Q}$  maps the  $Q$ -conjugacy class sum of an element  $x \in G$  to zero unless  $x \in C_G(Q)$ , in which case  $x$  is mapped to its canonical image in  $kC_G(Q)$ .

Given a finite group  $G$  and a block  $b$  of  $\mathcal{O}G$ , a *b-Brauer pair* is a pair  $(Q, f)$  consisting of a  $p$ -subgroup  $Q$  of  $G$  and a block  $f$  of  $kC_G(Q)$  satisfying  $\text{Br}_{\Delta Q}(b)f = f$ . By results of Alperin and Broué [1], the set of  $b$ -Brauer pairs is a  $G$ -poset with a single  $G$ -conjugacy class of maximal  $b$ -Brauer pairs. If  $(P, e)$  is such a maximal  $b$ -Brauer pair then  $P$  is a defect group of  $b$ . A primitive idempotent  $i \in (\mathcal{O}Gb)^{\Delta P}$  satisfying  $\text{Br}_{\Delta P}(i) \neq 0$  is then called a *source idempotent* of the block  $b$ . Since  $\text{Br}_{\Delta P}$  is a surjective algebra homomorphism,  $\text{Br}_{\Delta P}(i)$  is a primitive idempotent in  $kC_G(P)$ , and we may thus always choose  $i$  such that  $\text{Br}_{\Delta P}(i)e \neq 0$ . By [5, 1.8], for any subgroup  $Q$  of  $P$  there is a unique block  $e_Q$  of  $kC_G(Q)$  such that  $\text{Br}_{\Delta Q}(i)e_Q \neq 0$ , and then  $e_Q$  is the unique block of  $kC_G(Q)$  such that  $(Q, e_Q) \subseteq (P, e)$ ; in particular,  $e_P = e$ . See [9] and [14] for more details and background information. For the proof of the above Theorem we need the following Lemma, the first part of which is well-known.

**Lemma.** *Let  $G$  be a finite group, let  $b$  be a block of  $\mathcal{O}G$ , let  $(P, e)$  be a maximal  $b$ -Brauer pair, and let  $i \in (\mathcal{O}Gb)^{\Delta P}$  be a source idempotent of  $b$  such that  $\text{Br}_{\Delta P}(i)e \neq 0$ . Let  $Q$  be a subgroup of  $Z(P)$  and let  $e_Q$  be the unique block of  $kC_G(Q)$  satisfying  $(Q, e_Q) \subseteq (P, e)$ . Then  $P$  is a defect group of  $e_Q$  and  $\text{Br}_{\Delta Q}(i)$  is a source idempotent of the block  $e_Q$  in  $(kC_G(Q)e_Q)^{\Delta P}$ .*

**Proof.** Since  $Q \subseteq Z(P)$  we have  $P \subseteq C_G(Q)$ , and hence  $P$  is a defect group of  $e_Q$  by [8, 7.6]. Now  $\text{Br}_{\Delta Q}$  maps  $(\mathcal{O}G)^{\Delta Q}$  onto  $kC_G(Q)$ ; since  $P$  normalises  $C_G(Q)$ , any  $P$ -conjugacy class of elements in  $G$  is either contained in  $C_G(Q)$  or in  $G - C_G(Q)$ . Hence  $\text{Br}_{\Delta Q}$  maps  $(\mathcal{O}G)^{\Delta P}$  onto  $(kC_G(Q))^{\Delta P}$ . This implies that  $\text{Br}_{\Delta Q}(i)$  is a primitive idempotent in  $(kC_G(Q))^{\Delta P}$ . Moreover, by [5, 1.8] we have  $\text{Br}_{\Delta Q}(i) \in kC_G(Q)e_Q$  and clearly  $\text{Br}_{\Delta P}(\text{Br}_{\Delta Q}(i)) = \text{Br}_{\Delta P}(i) \neq 0$ , which proves the second statement of the Lemma.  $\square$

**Proof of the Theorem.** Let  $\hat{e}$  be the block of  $\mathcal{O}C_G(P)$  which corresponds to the block  $e$  of  $kC_G(P)$ . Note first that  $\hat{e}$  is still a block of  $\mathcal{O}H$  with  $(P, e)$  as unique maximal Brauer pair. Let  $j \in (\mathcal{O}H\hat{e})^{\Delta P}$  be a source idempotent of  $\hat{e}$  as block of  $\mathcal{O}H$ . Then, by [6, 4.10] (or also [2, Theorem 5(ii) and p.265, line 3]) the idempotent  $i = jf$  is a source idempotent of the block  $b$  in  $(\mathcal{O}Gb)^{\Delta P}$ , and since  $f$  was chosen such that  $\text{Br}_{\Delta P}(f)e = e$  we have  $\text{Br}_{\Delta P}(i)e \neq 0$ . Let  $Q$  be a subgroup of  $Z(P)$ . By the above Lemma,  $i_Q = \text{Br}_{\Delta Q}(i)$  is a source idempotent of the block  $e_Q$ , and  $j_Q = \text{Br}_{\Delta Q}(j)$  is a source idempotent of the block  $f_Q$ . Since  $i = jf = fj$  we have  $i_Q = \text{Br}_{\Delta Q}(f)j_Q$ , and this is therefore in particular a primitive idempotent in  $(kC_G(Q)e_Q)^{\Delta P}$ .

Since  $X = \mathcal{O}Gf$  we have  $X(\Delta Q) = kC_G(Q)\text{Br}_{\Delta Q}(f)$ , and therefore

$$e_Q X(\Delta Q) j_Q = e_Q kC_G(Q) \text{Br}_{\Delta Q}(f) j_Q = e_Q kC_G(Q) i_Q .$$

As  $i_Q \in kC_G(Q)e_Q$  this implies in particular that  $e_Q X(\Delta Q) j_Q$  is non zero. The point now is that since  $i_Q$  is primitive in  $(kC_G(Q)e_Q)^{\Delta P}$ , the  $(kC_G(Q)e_Q, kP)$ -bimodule  $e_Q kC_G(Q) i_Q$  is indecomposable. Since  $kP$  is isomorphic to a subalgebra of the source algebra  $j_Q kC_H(Q) j_Q$  via multiplication by  $j_Q$ , it follows that  $e_Q X(\Delta Q) j_Q$  is indecomposable as  $(kC_G(Q)e_Q, j_Q kC_H(Q) j_Q)$ -bimodule. By [10, 3.4], the block algebra  $kC_H(Q)f_Q$  and its source algebra  $j_Q kC_H(Q) j_Q$  are Morita equivalent, which implies that indeed  $e_Q X(\Delta Q) f_Q$  is indecomposable as  $k(C_G(Q) \times C_H(Q))$ -module.

Since  $X$  is a direct summand of  $\mathcal{O}Gb$  as  $\mathcal{O}(G \times H)$ -module,  $X(\Delta Q)$  is a direct summand of  $kC_G(Q)\text{Br}_{\Delta Q}(b)$  as  $k(C_G(Q) \times C_H(Q))$ -module, and hence  $e_Q X(\Delta Q) f_Q$  is a direct summand of  $e_Q kC_G(Q) f_Q$ .

Since  $f$  is primitive in  $(\mathcal{O}G)^{\Delta H}$ , the  $\mathcal{O}(G \times H)$ -module  $X$  is indecomposable. As  $\mathcal{O}(G \times G)$ -module,  $\mathcal{O}Gb$  has  $\Delta P$  as vertex. Thus  $X$  has a vertex contained in a  $(G \times G)$ -conjugate of  $\Delta P$ . Since  $\text{Br}_{\Delta P}(f)e = e \neq 0$ , we have  $X(\Delta P) \neq 0$  and thus  $X$  has  $\Delta P$  as a vertex by [14, 27.7]. Similarly, we have  $e = e \text{Br}_{\Delta P}(e_Q) = e \text{Br}_{\Delta P}(f_Q) = e \text{Br}_{\Delta P}(f)$  by [5, 1.8(3)] and the assumption. Thus, if we denote by  $\bar{f}$  the canonical image of  $f$  in  $(kG)^{\Delta H}$ , we get  $e \text{Br}_{\Delta P}(e_Q \bar{f} f_Q) = e \neq 0$ , so that  $\text{Br}_{\Delta P}(e_Q \bar{f} f_Q) \neq 0$ , hence  $(e_Q X(\Delta Q) f_Q)(\Delta P) \neq 0$ , and so  $\Delta P$  is a vertex of  $e_Q X(\Delta Q) f_Q$ .

For the last part we observe that the  $k(C_G(Q) \times C_H(Q))$ -module  $e_Q kC_G(Q) f_Q$  is a direct summand of  $kC_G(Q) f_Q = \text{Ind}_{C_H(Q) \times C_H(Q)}^{C_G(Q) \times C_H(Q)}(kC_H(Q) f_Q)$ . Moreover, the  $k(C_H(Q) \times C_H(Q))$ -module  $kC_H(Q) f_Q$  is indecomposable with  $\Delta P$  as vertex, and the normaliser of  $\Delta P$  in  $C_G(Q) \times C_H(Q)$  is contained in  $C_H(Q) \times C_H(Q)$ . Thus the Green correspondence implies that the  $k(C_G(Q) \times C_H(Q))$ -module  $kC_G(Q) f_Q$  has exactly one indecomposable direct summand with  $\Delta P$  as vertex, up to isomorphism. The result follows.  $\square$

**Remark.** With the notation of the Theorem, if  $Q$  is a subgroup of  $Z(P)$  then  $f_Q = e$ . Indeed,  $P$  is normal in  $H$ , hence in  $C_H(Q)$ , and thus every block of  $kC_H(Q)$  is contained in  $kC_H(P) = kC_G(P)$ . The last argument in the proof of the Theorem shows the seemingly stronger statement that  $e_Q X(\Delta Q) f_Q$  is the unique direct summand with vertex  $\Delta P$  of the  $k(C_G(Q) \times C_H(Q))$ -module  $kC_G(Q) f_Q$ , but since  $\text{Br}_{\Delta P}(e_Q)e = e = f_Q$ , every direct summand of  $kC_G(Q) f_Q$  with vertex  $\Delta P$  is already a direct summand of  $e_Q kC_G(Q) f_Q$ .

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