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Citation: Linckelmann, M. (2006). Simple fusion systems and the Solomon 2-local groups. *Journal of Algebra*, 296(2), pp. 385-401. doi: 10.1016/j.jalgebra.2005.09.024

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Link to published version: <https://doi.org/10.1016/j.jalgebra.2005.09.024>

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SIMPLE FUSION SYSTEMS AND THE SOLOMON 2-LOCAL GROUPS

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ABSTRACT. We introduce a notion of simple fusion systems which imitates the corresponding notion for finite groups and show that the fusion system on the Sylow-2-subgroup of a 7-dimensional spinor group over a field of characteristic 3 considered by Ron Solomon [18] and by Ran Levi and Bob Oliver [11] is simple in this sense.

INTRODUCTION

The bigger picture which motivates the content of the present paper is the intuition, formulated by D. J. Benson in [3], that associated with each fusion system on a finite p -group in the sense of Puig [15] there should be a p -complete topological space which generalises the concept of a classifying space of a finite group. Broto, Levi and Oliver developed in [4] a theory describing how such a space should look, leading to the notion of a p -local finite group, and they gave in particular a cohomological criterion for the existence and uniqueness of p -local finite groups. Using this criterion, Levi and Oliver showed in [11] that there is up to homotopy equivalence a unique 2-local group associated with Solomon's fusion system and they showed further that this coincides indeed with the space constructed earlier by Benson in [3]. Put in these terms, Solomon's fusion system provides an example of a simple 2-local finite group which is not the 2-complete classifying space of any finite group by [18]. In fact, Solomon's fusion system cannot even be the fusion system of any 2-block of a finite group by [9]. Besides the obvious question - can one classify simple fusion systems? - one might wonder, whether the problem of the existence and uniqueness of a p -local finite group associated with any fusion system can be reduced to simple fusion systems.

Section 1 contains a brief account of Puig's abstract notion of a fusion system and we recall in Section 2 how fusion systems occur in block theory. The following two sections introduce our notions of normal and simple fusion systems. Sections 5, 6, 7 contain simplicity results for fusion systems on dihedral 2-groups, fusion systems related to orthogonal groups and the Solomon's fusion system, respectively. Throughout this paper, p denotes a prime.

1991 *Mathematics Subject Classification.* 20C20.

1 BACKGROUND MATERIAL ON FUSION SYSTEMS

We recall basic material on Puig's axiomatisation of the local structure of blocks [15]. If P, Q, R are subgroups of a finite group G , we denote by $\text{Hom}_P(Q, R)$ the set of group homomorphisms $\varphi : Q \rightarrow R$ for which there is $y \in P$ satisfying $\varphi(u) = yuy^{-1}$ for all $u \in Q$; we write $\text{Aut}_P(Q) = \text{Hom}_P(Q, Q)$. Thus $\text{Aut}_P(Q)$ is canonically isomorphic to $N_P(Q)/C_P(Q)$; in particular $\text{Aut}_Q(Q) \cong Q/Z(Q)$ is the group of inner automorphisms of Q .

Definition 1.1. A *category on a finite p -group P* is a category $\mathcal{F}$ whose objects are the subgroups of P and whose morphism sets $\text{Hom}_{\mathcal{F}}(Q, R)$ consist, for any two subgroups Q, R of P , of injective group homomorphisms with the following properties:

- (i) if Q is contained in R then the inclusion $Q \subseteq R$ is a morphism in $\mathcal{F}$;
- (ii) for any $\varphi \in \text{Hom}_{\mathcal{F}}(Q, R)$, the induced isomorphism $Q \cong \varphi(Q)$ and its inverse are morphisms in $\mathcal{F}$;
- (iii) composition of morphisms in $\mathcal{F}$ is the usual composition of group homomorphisms.

Definition 1.2. Let $\mathcal{F}$ be a category on a finite p -group P . A subgroup Q of P is called *fully $\mathcal{F}$ -centralised* if $|C_P(R)| \leq |C_P(Q)|$ for any subgroup R of P such that $R \cong Q$ in $\mathcal{F}$, and Q is called *fully $\mathcal{F}$ -normalised* if $|N_P(R)| \leq |N_P(Q)|$ for any subgroup R of P such that $R \cong Q$ in $\mathcal{F}$.

The following definition is due to Broto, Levi and Oliver [4].

Definition 1.3. Let $\mathcal{F}$ be a category on a finite p -group P , and let Q be a subgroup of P . For any morphism $\varphi : Q \rightarrow P$ in $\mathcal{F}$, we set $N_\varphi = \{y \in N_P(Q) \mid \text{there is } z \in N_P(\varphi(Q)) \text{ such that } \varphi(yu) = z\varphi(u) \text{ for all } u \in Q\}$.

In other words, N_φ is the inverse image in $N_P(Q)$ of the group $\text{Aut}_P(Q) \cap (\varphi^{-1} \circ \text{Aut}_P(\varphi(Q)) \circ \varphi)$. Note that in particular $QC_P(Q) \subseteq N_\varphi \subseteq N_P(Q)$. Broto, Levi and Oliver use the groups N_φ in [4] to give a definition of fusion systems (called saturated fusion systems in [4]) which is equivalent to Puig's original definition (called full Frobenius systems there), which in turn has been simplified by Stancu [20]; we present here Stancu's version:

Definition 1.4. A *fusion system on a finite p -group P* is a category $\mathcal{F}$ on P such that $\text{Hom}_P(Q, R) \subseteq \text{Hom}_{\mathcal{F}}(Q, R)$ for any two subgroups Q, R of P , and such that the following two properties hold:

- (I-S) $\text{Aut}_P(P)$ is a Sylow- p -subgroup of $\text{Aut}_{\mathcal{F}}(P)$.
- (II-S) every morphism $\varphi : Q \rightarrow P$ in $\mathcal{F}$ such that $\varphi(Q)$ is fully $\mathcal{F}$ -normalised extends to a morphism $\psi : N_\varphi \rightarrow P$ (that is, $\psi|_Q = \varphi$).

The “extension axiom” (II-S) relates the role of N_φ as object of $\mathcal{F}$ to its image N_φ/Q in $\text{Aut}_{\mathcal{F}}(Q)$. We show in the following three Propositions that definition 1.4 is equivalent to the definition given in [4, 1.2] which uses the a priori stronger axioms

(I-BLO) if Q is a fully $\mathcal{F}$ -normalised subgroup of P then Q is fully $\mathcal{F}$ -centralised and $\text{Aut}_P(Q)$ is a Sylow- p -subgroup of $\text{Aut}_{\mathcal{F}}(Q)$;

(II-BLO) given any subgroup Q of P , every morphism $\varphi : Q \rightarrow P$ such that $\varphi(Q)$ is fully $\mathcal{F}$ -centralised extends to a morphism $\psi : N_\varphi \rightarrow P$ in $\mathcal{F}$ (that is, $\psi|_Q = \varphi$).

The Propositions 1.5 and 1.6 show that the axioms in 1.4 imply the “Sylow axiom” (I-BLO).

Proposition 1.5. ([20]) *Let $\mathcal{F}$ be a fusion system on a finite p -group P and let Q be a subgroup of P . If Q is fully $\mathcal{F}$ -normalised then Q is fully $\mathcal{F}$ -centralised.*

Proof. Let $\varphi : R \rightarrow Q$ be an isomorphism in $\mathcal{F}$. Assume that Q is fully $\mathcal{F}$ -normalised and that R is fully $\mathcal{F}$ -centralised. By (II-S) in 1.4 there is a morphism $\psi : RC_P(R) \rightarrow P$ in $\mathcal{F}$ such that $\psi|_R = \varphi$. Hence ψ maps $C_P(R)$ to $C_P(Q)$, which implies that $|C_P(R)| \leq |C_P(Q)|$, hence equality since R is fully $\mathcal{F}$ -centralised. Thus Q is fully $\mathcal{F}$ -centralised. $\square$

Proposition 1.6. *Let $\mathcal{F}$ be a fusion system on a finite p -group P and let Q be a subgroup of P . Then Q is fully $\mathcal{F}$ -normalised if and only if Q is fully $\mathcal{F}$ -centralised and $\text{Aut}_P(Q)$ is a Sylow- p -subgroup of $\text{Aut}_{\mathcal{F}}(Q)$.*

Proof. Assume that Q is fully $\mathcal{F}$ -normalised. Then Q is fully CF -centralised by 1.5. Choose Q to be of maximal order such that $\text{Aut}_P(Q)$ is not a Sylow- p -subgroup of $\text{Aut}_{\mathcal{F}}(Q)$. Then Q is a proper subgroup of P by 1.4.(I-S). Choose a p -subgroup S of $\text{Aut}_{\mathcal{F}}(Q)$ such that $\text{Aut}_P(Q)$ is a proper normal subgroup of S . Let $\varphi \in S - \text{Aut}_P(Q)$. Since φ normalises $\text{Aut}_P(Q)$, for every $y \in N_P(Q)$ there is $z \in N_P(Q)$ such that $\varphi(yu) = z\varphi(u)$ for all $u \in Q$. In other words, $N_\varphi = N_P(Q)$. Since Q is fully $\mathcal{F}$ -normalised, it follows from 1.4.(II-S) that there is an automorphism ψ of $N_P(Q)$ in $\mathcal{F}$ such that $\psi|_Q = \varphi$. Since φ has p -power order, by decomposing ψ into its p -part and its p' -part we may in fact assume that ψ has p -power order. Let $\tau : N_P(Q) \rightarrow P$ be a morphism in $\mathcal{F}$ such that $\tau(N_P(Q))$ is fully $\mathcal{F}$ -normalised. Now $\tau\psi\tau^{-1}$ is a p -element in $\text{Aut}_{\mathcal{F}}(\tau(N_P(Q)))$, thus conjugate to an element in $\text{Aut}_P(\tau(N_P(Q)))$. Therefore we may choose τ in such a way that there is $y \in N_P(\tau(N_P(Q)))$ satisfying $\tau\psi\tau^{-1}(v) = yv$ for any $v \in \tau(N_P(Q))$. Since $\psi|_Q = \varphi$, the automorphism $\tau\psi\tau^{-1}$ of $\tau(N_P(Q))$ stabilises $\tau(Q)$. Thus $y \in N_P(\tau(Q))$. Since Q is fully $\mathcal{F}$ -normalised we have $N_P(\tau(Q)) \subseteq \tau(N_P(Q))$, hence $\psi(u) = \tau^{-1}(y)u$ for all $u \in N_P(Q)$. But then in particular $\varphi \in \text{Aut}_P(Q)$, contradicting our initial choice of φ . The converse is easy since $|N_P(Q)| = |\text{Aut}_P(Q)| \cdot |C_P(Q)|$. $\square$

The next Proposition shows that the axioms in 1.4 imply also the “extension axiom” (II-BLO).

Proposition 1.7. ([20]) *Let $\mathcal{F}$ be a fusion system on a finite p -group P , let Q be a subgroup of P and let $\varphi : Q \rightarrow P$ be a morphism in $\mathcal{F}$ such that $\varphi(Q)$ is fully $\mathcal{F}$ -centralised. Then there is a morphism $\psi : N_\varphi \rightarrow P$ in $\mathcal{F}$ such that $\psi|_Q = \varphi$.*

Proof. Let $\rho : \varphi(Q) \rightarrow P$ be a morphism in $\mathcal{F}$ such that $R = \rho(\varphi(Q))$ is fully $\mathcal{F}$ -normalised. Then $\rho \circ \text{Aut}_P(\varphi(Q)) \circ \rho^{-1}$ is a p -subgroup of $\text{Aut}_{\mathcal{F}}(R)$. Moreover, by 1.6, the group $\text{Aut}_P(R)$ is a Sylow- p -subgroup of $\text{Aut}_{\mathcal{F}}(R)$. Thus there is $\alpha \in \text{Aut}_{\mathcal{F}}(R)$ such that $\alpha \circ \rho \circ \text{Aut}_P(\varphi(Q)) \circ \rho^{-1} \circ \alpha^{-1} \subseteq \text{Aut}_P(R)$. This means that after replacing ρ by $\alpha \circ \rho$, we may assume that $N_\rho = N_P(\varphi(Q))$. In particular, ρ extends to a morphism $\sigma : N_P(\varphi(Q)) \rightarrow P$. But then $N_\varphi \subseteq N_{\rho \circ \varphi}$, hence $\rho \circ \varphi$ extends to a morphism $\tau : N_\varphi \rightarrow P$. Then $\tau(N_\varphi) \subseteq \sigma(N_P(\varphi(Q)))$, and hence we get a morphism $\sigma^{-1}|_{\tau(N_\varphi)} \circ \tau : N_\varphi \rightarrow P$ which extends φ as required. $\square$

Definition 1.8. Let $\mathcal{F}$ be a fusion system on a finite p -group P and let Q be a subgroup of P .

- (i) Q is $\mathcal{F}$ -centric if $C_P(R) = Z(R)$ for any subgroup R of P such that $R \cong Q$ in $\mathcal{F}$.
- (ii) Q is $\mathcal{F}$ -radical if $O_p(\text{Aut}_{\mathcal{F}}(Q)/\text{Aut}_Q(Q)) = 1$.
- (iii) Q is $\mathcal{F}$ -essential if Q is $\mathcal{F}$ -centric, $Q \neq P$, and $\text{Aut}_{\mathcal{F}}(Q)/\text{Aut}_Q(Q)$ has a strongly p -embedded proper subgroup M (that is, M contains a Sylow- p -subgroup S of $\text{Aut}_{\mathcal{F}}(Q)/\text{Aut}_Q(Q)$ such that ${}^\varphi S \cap S = \{1\}$ for every $\varphi \in \text{Aut}_{\mathcal{F}}(Q)/\text{Aut}_Q(Q) - M$).
- (iv) Q is weakly $\mathcal{F}$ -closed if for every morphism $\varphi : Q \rightarrow P$ in $\mathcal{F}$ we have $\varphi(Q) = Q$.
- (v) Q is strongly $\mathcal{F}$ -closed, if for any subgroup R of P and any morphism $\varphi : R \rightarrow P$ in $\mathcal{F}$ we have $\varphi(R \cap Q) \subseteq Q$.

If Q is $\mathcal{F}$ -centric, then Q is fully $\mathcal{F}$ -centralised, and if Q is $\mathcal{F}$ -essential, then Q is $\mathcal{F}$ -radical. If Q is strongly $\mathcal{F}$ -closed then Q is weakly $\mathcal{F}$ -closed. One easily checks that if Q is strongly $\mathcal{F}$ -closed then for any subgroup R of P and any morphism $\varphi : R \rightarrow P$ in $\mathcal{F}$ we have in fact $\varphi(R \cap Q) = \varphi(R) \cap Q$. Indeed, the left side is contained in the right side by the above definition, and the other inclusion is obtained by applying this inclusion to $\varphi(R)$ and the morphism φ^{-1} viewed as morphism from $\varphi(R)$ to P .

Definition 1.9 Let $\mathcal{F}$ be a category on a finite p -group P , and let Q be a subgroup of P . We define the category $N_{\mathcal{F}}(Q)$ on $N_P(Q)$ by $\text{Hom}_{N_{\mathcal{F}}(Q)}(R, R') = \{\varphi : R \rightarrow R' \mid \varphi \text{ extends to a morphism } \psi : QR \rightarrow QR' \text{ in } \mathcal{F} \text{ such that } \psi(Q) = Q\}$, for any two subgroups R, R' of $N_P(Q)$. Similarly, we define the category $C_{\mathcal{F}}(Q)$ on $C_P(Q)$ by $\text{Hom}_{C_{\mathcal{F}}(Q)}(R, R') = \{\varphi : R \rightarrow R' \mid \varphi \text{ extends to a morphism } \psi : QR \rightarrow QR' \text{ in } \mathcal{F} \text{ such that } \psi|_Q = \text{Id}_Q\}$.

We have clearly inclusions of categories $C_{\mathcal{F}}(Q) \subseteq N_{\mathcal{F}}(Q) \subseteq \mathcal{F}$. If $\mathcal{F} = N_{\mathcal{F}}(Q)$ for some subgroup Q of P , then clearly Q is strongly $\mathcal{F}$ -closed. The converse of this statement is not true, in general. If $\mathcal{F}$ is a fusion system on P such that $\mathcal{F} = C_{\mathcal{F}}(Z)$ for some (necessarily central) subgroup Z of P then the category on P/Z induced by $\mathcal{F}$ is a fusion system on P/Z , denoted by $\mathcal{F}/Z$. In that case, if $\mathcal{F}'$ is a fusion system on P contained in $\mathcal{F}$ we have $\mathcal{F}' = \mathcal{F}$ if and only if $\mathcal{F}'/Z = \mathcal{F}/Z$; this follows from

Alperin's fusion theorem 1.11 below together with the fact that if Q is a subgroup of P then the canonical map $\text{Aut}_{\mathcal{F}}(Q) \rightarrow \text{Aut}_{\mathcal{F}/Z}(Q/Z)$ has a p -group as kernel because any p' -automorphism of Q/Z lifts to a p' -automorphism of Q .

Proposition 1.10. ([15]) *Let $\mathcal{F}$ be a fusion system on a finite p -group P , and let Q be a subgroup of P . If Q is fully $\mathcal{F}$ -centralised, then $C_{\mathcal{F}}(Q)$ is a fusion system on $C_P(Q)$; if Q is fully normalised, then $N_{\mathcal{F}}(Q)$ is a fusion system on $N_P(Q)$.*

A proof of this Proposition can be found in [4, A6] (applied to the cases where the group K occurring in the statement of [4, A6] is either trivial or equal to $\text{Aut}(Q)$). By the previous remarks, Proposition 1.10 implies that if Q is fully $\mathcal{F}$ -centralised then $C_{\mathcal{F}}(Q)/Z(Q)$ is a fusion system on $C_P(Q)/Z(Q)$. The following result is Alperin's fusion theorem [1], refined by Goldschmidt [8], and extended to arbitrary fusion systems by Puig [15].

Theorem 1.11. *Let $\mathcal{F}$ be a fusion system on a finite p -group P . Every isomorphism in $\mathcal{F}$ can be written as a composite of finitely many isomorphisms $\varphi : Q \cong R$ in $\mathcal{F}$ such that either $\varphi = \alpha|_Q$ for some $\alpha \in \text{Aut}_{\mathcal{F}}(P)$ or there is an $\mathcal{F}$ -essential subgroup E of P containing both Q , R , and an automorphism $\beta \in \text{Aut}_{\mathcal{F}}(E)$ such that $\varphi = \beta|_Q$.*

Lemma 1.12. ([15]) *Let $\mathcal{F}$ be a fusion system on a finite p -group P . Let Q, R be $\mathcal{F}$ -centric subgroups of P such that $Q \subseteq R$, and let $\varphi \in \text{Aut}_{\mathcal{F}}(R)$. We have $\varphi|_Q = \text{Id}_Q$ if and only if $\varphi \in \text{Aut}_{Z(Q)}(R)$.*

Proof. Assume that $\varphi|_Q = \text{Id}_Q$. We proceed by induction over $[R : Q]$. Consider first the case where Q is normal in R . Let $u \in Q$ and $v \in R$. Then ${}^v u \in Q$, hence ${}^v u = \varphi({}^v u) = \varphi({}^v)u$, and thus $v^{-1}\varphi(v) \in C_R(Q) = Z(Q)$, or equivalently, $\varphi(v) = vz$ for some $z \in Z(Q)$. If φ has order prime to p in $\text{Aut}(R)$ this forces $\varphi = \text{Id}_R$. Therefore we may assume that the order of φ is a power of p . Upon replacing R by a fully $\mathcal{F}$ -normalised $\mathcal{F}$ -conjugate we may assume that $\varphi \in \text{Aut}_P(R)$. Since φ restricts to Id_Q and since Q is $\mathcal{F}$ -centric this implies that $\varphi \in \text{Aut}_{Z(Q)}(R)$. This proves 1.12 if Q is normal in R . In general, if $\varphi|_Q = \text{Id}_Q$ then $\varphi(N_R(Q)) = N_R(Q)$. Thus $\varphi|_{N_R(Q)} \in \text{Aut}_{Z(Q)}(N_R(Q))$ by the previous paragraph. Hence there is $z \in Z(Q)$ such that $c_z \circ \varphi|_{N_R(Q)} = \text{Id}_{N_R(Q)}$, where c_z is the automorphism of R given by conjugation with z . By induction we get $c_z \circ \varphi \in \text{Aut}_{Z(N_R(Q))}(Q)$. As all involved groups are $\mathcal{F}$ -centric we have $Z(N_R(Q)) \subseteq Z(Q)$, and thus $\varphi \in \text{Aut}_{Z(Q)}(R)$ as claimed. The converse is trivial. $\square$

Lemma 1.13. *Let $\mathcal{F}$ be a fusion system on a finite p -group P , let Q, R be $\mathcal{F}$ -centric subgroups of P such that $Q \subseteq R$, and let $\varphi, \varphi' \in \text{Hom}_{\mathcal{F}}(R, P)$ such that $\varphi|_Q = \varphi'|_Q$. Then $\varphi(R) = \varphi'(R)$.*

Proof. Let $v \in N_R(Q)$. For every $u \in Q$ we have $\varphi({}^v u) = \varphi'({}^v u)$, hence $\varphi(v)^{-1}\varphi'(v) \in C_P(\varphi(Q)) = Z(\varphi(Q))$. It follows that $\varphi(N_R(Q)) = \varphi'(N_R(Q))$. By 1.12, $\varphi|_{N_R(Q)}$ and

$\varphi'_{N_R(Q)}$ differ by conjugation with an element in $Z(Q)$, and we may therefore assume that their restrictions to $N_R(Q)$ actually coincide. The equality $\varphi(R) = \varphi'(R)$ follows by induction. $\square$

Given a fusion system $\mathcal{F}$ on a finite p -group P , we denote by $\mathcal{F}^c$ the full subcategory of $\mathcal{F}$ -centric subgroups of P ; we denote by $\bar{\mathcal{F}}$ the *orbit category* of $\mathcal{F}$, which has the same objects as $\mathcal{F}$ but whose sets of morphisms are the quotient sets $\text{Hom}_{\bar{\mathcal{F}}}(Q, R) = \text{Aut}_R(R) \backslash \text{Hom}_{\mathcal{F}}(Q, R)$ of morphisms in $\mathcal{F}$ modulo inner automorphisms of the corresponding subgroups of P . We denote by $\bar{\mathcal{F}}^c$ the image in $\bar{\mathcal{F}}$ of $\mathcal{F}^c$. The category $\mathcal{F}$ has the property that every morphism is a monomorphism, and every endomorphism is an automorphism. The orbit category $\bar{\mathcal{F}}$ has still the property that every endomorphism is an automorphism, but not every morphism is a monomorphism, in general. As observed in [14] in the context of fusion systems of finite groups, the straightforward consequence of 1.12 is that in the opposite category $(\bar{\mathcal{F}}^c)^0$ every morphism is a monomorphism, or equivalently:

Proposition 1.14. *Let $\mathcal{F}$ be a fusion system on a finite p -group P . Every morphism in the category $\bar{\mathcal{F}}^c$ is an epimorphism.*

Proof. Let Q, R, S be $\mathcal{F}$ -centric subgroups of P , let $\varphi \in \text{Hom}_{\mathcal{F}}(Q, R)$ and let $\psi, \psi' \in \text{Hom}_{\mathcal{F}}(R, S)$. Assume that the images of $\psi \circ \varphi$ and $\psi' \circ \varphi$ in $\text{Hom}_{\bar{\mathcal{F}}^c}(Q, S)$ coincide. Up to replacing ψ' by some S -conjugate, we may assume that $\psi \circ \varphi = \psi' \circ \varphi$. Thus the restrictions to $\varphi(Q)$ of ψ, ψ' coincide. It follows from 1.13 that $\psi(R) = \psi'(R)$. Thus $\psi^{-1} \circ \psi'$ is an automorphism of R which restricts to the identity on $\varphi(Q)$, hence $\psi^{-1} \circ \psi' \in \text{Aut}_{Z(\varphi(Q))}(R)$ by 1.12. Thus the images of ψ, ψ' in the orbit category are equal. $\square$

2 FUSION SYSTEMS OF FINITE GROUPS AND p -BLOCKS

For expository purpose, we describe in this section briefly the well-known examples which motivate Puig's definition of a fusion system.

Definition 2.1 Let G be a finite group, and let P be a Sylow- p -subgroup of G . We denote by $\mathcal{F}_P(G)$ the category on P whose morphisms are the group homomorphisms $\varphi : Q \rightarrow R$ for which there is an element $x \in G$ such that $\varphi(u) = xux^{-1}$ for all $u \in Q$.

Equivalently, $\text{Hom}_{\mathcal{F}_P(G)}(Q, R) = \text{Hom}_G(Q, R)$; in particular, $\text{Aut}_{\mathcal{F}_P(G)}(Q) = \text{Aut}_G(Q) \cong N_G(Q)/C_G(Q)$. We leave the elementary proof of the following well-known statement to the reader.

Theorem 2.2. *Let G be a finite group, and let P be a Sylow- p -subgroup of G .*

- (i) *The category $\mathcal{F}_P(G)$ is a fusion system on P .*
- (ii) *A subgroup Q of P is fully $\mathcal{F}_P(G)$ -centralised if and only if $C_P(Q)$ is a Sylow- p -subgroup of $C_G(Q)$.*

(iii) A subgroup Q of P is fully $\mathcal{F}_P(G)$ -normalised if and only if $N_P(Q)$ is a Sylow- p -subgroup of $N_G(Q)$.

Following Alperin-Broué [2], there is a fusion system on a defect group of a p -block of a finite group which generalises the definition of $\mathcal{F}_P(G)$ above in the sense, that it coincides with $\mathcal{F}_P(G)$ if the considered block is the principal p -block of G . In order to describe this briefly, let k be a field of characteristic p , let G be a finite group, and let b be a block of kG ; that is, b is a primitive idempotent in $Z(kG)$. A b -Brauer pair is a pair (Q, f) consisting of a p -subgroup Q of G and a block f of $kC_G(Q)$ such that $\text{Br}_Q(b)f = f$. Here $\text{Br}_Q : (kG)^Q \rightarrow kC_G(Q)$ is the Brauer homomorphism mapping any element of $C_G(Q)$ to itself and any non trivial Q -conjugacy class sum of elements in G to zero. By [2], the set of b -Brauer pairs admits a partial order “ $\subseteq$ ” which is compatible with the action of G by conjugation on this set, such that the maximal b -Brauer pairs form a single G -conjugacy class. Given a maximal b -Brauer pair (P, e) , for every subgroup Q of P there is a unique block e_Q of $kC_G(Q)$ such that $(Q, e_Q) \subseteq (P, e)$, and the group P is called a *defect group of the block b* . The choice of a maximal b -Brauer pair gives rise to a category on P (we follow the notation of [10]):

Definition 2.3. Let G be a finite group, let b be a block of kG , and let (P, e) be a maximal b -Brauer pair. For any subgroup Q of P , denote by e_Q the unique block of $kC_G(Q)$ such that $(Q, e_Q) \subseteq (P, e)$. We denote by $\mathcal{F}_{(P,e)}(G, b)$ the category on P whose morphisms are the group homomorphisms $\varphi : Q \rightarrow R$ for which there is an element $x \in G$ such that $\varphi(u) = xux^{-1}$ for all $u \in Q$ such that $xe_Qx^{-1} = e_{xQx^{-1}}$, or equivalently, such that ${}^x(Q, e_Q) \subseteq (R, e_R)$, where Q, R are subgroups of P .

If S is a Sylow- p -subgroup of G containing the defect group P of b , then clearly $\mathcal{F}_{(P,e)}(G, b)$ is a subcategory of $\mathcal{F}_S(G)$, but it is not in general a full subcategory, because the elements x in G used to define the morphisms in $\mathcal{F}_{(P,e)}(G, b)$ have to fulfill the additional compatibility property ${}^x(Q, e_Q) \subseteq (R, e_R)$. If b is the *principal block of kG* (that is, b is the unique block of kG not contained in the augmentation ideal of kG), then P is a Sylow- p -subgroup of G and e_Q is the principal block of $kC_G(Q)$ for any subgroup Q of P , and hence $\mathcal{F}_{(P,e)}(G, b) = \mathcal{F}_P(G)$ in this case. The following statement, which generalises 2.2, is essentially a reformulation of results in [2]; we sketch a proof for the convenience of the reader:

Theorem 2.4. Let G be a finite group, let b be a block of kG , and let (P, e) be a maximal b -Brauer pair. For every subgroup Q of P , denote by e_Q the unique block of $kC_G(Q)$ such that $(Q, e_Q) \subseteq (P, e)$.

(i) The category $\mathcal{F}_{(P,e)}(G, b)$ is a fusion system on P .

(ii) A subgroup Q of P is fully $\mathcal{F}_{(P,e)}(G, b)$ -centralised if and only if $C_P(Q)$ is a defect group of $kC_G(Q)e_Q$.

(iii) A subgroup Q of P is fully $\mathcal{F}_{(P,e)}(G, b)$ -normalised if and only if $N_P(Q)$ is a defect group of $kN_G(Q, e_Q)e_Q$.

Note that the last statement makes sense, as e_Q remains a block for the group $N_G(Q, e_Q)$ by [2, (2.9)]. The automorphism group in $\mathcal{F}_{(P,e)}(G, b)$ of a subgroup Q of P is isomorphic to $N_G(Q, e_Q)/C_G(Q)$. Thanks to the preceding Theorem, we can apply Alperin's fusion theorem to the fusion system $\mathcal{F}_{(P,e)}(G, b)$, which implies in particular, that $\mathcal{F}_{(P,e)}(G, b)$ is completely determined by the automorphism groups $N_G(Q, e_Q)/C_G(Q)$ for the $\mathcal{F}_{(P,e)}(G, b)$ -essential subgroups Q of P . Specialising Theorem 2.4 to the case where b is the principal block of kG yields Theorem 2.2.

Proof of Theorem 2.4. We prove first (ii) and (iii). By [12, 7.6], for every subgroup Q of P the group $C_P(Q)$ is contained in a defect group of e_Q as block of $kC_G(Q)$, and there is $x \in G$ such that ${}^x(Q, e_Q) \subseteq (P, e)$ and such that $C_P({}^xQ)$ is a defect group of xe_Q as block of $kC_G({}^xQ)$. From this follows (ii). By [2, (2.9)], e_Q remains a block of $kN_G(Q, e_Q)$. As before, $N_P(Q)$ is contained in a defect group of e_Q as block of $kN_G(Q, e_Q)$, and there is $x \in G$ such that ${}^x(Q, e_Q) \subseteq (P, e)$ and such that $N_P({}^xQ)$ is a defect group of xe_Q as block of $kN_G({}^x(Q, e_Q))$. This proves (iii).

In order to see (i), observe first that $\mathcal{F}_{(P,e)}(G, b)$ is clearly a category on P in the sense of 1.1. By Brauer's First Main Theorem [23, (40.14)], the group $N_G(P, e)/PC_G(P)$ is a p' -group (called inertial quotient of b), and hence the group $\text{Aut}_{\mathcal{F}_{(P,e)}(G, b)}(P) \cong N_G(P, e)/C_G(P)$ has $\text{Aut}_P(P)$ as Sylow- p -subgroup. In particular, the Sylow axiom (I-S) holds. It remains to verify that $\mathcal{F}_{(P,e)}(G, b)$ has also the property (II-S). Let Q, R be subgroups of P such that $N_P(R)$ is a defect group of e_R as block of $kN_G(R, e_R)$, and let $x \in G$ such that ${}^x(Q, e_Q) = (R, e_R)$. Denote by $\varphi : Q \rightarrow P$ the morphism in $\mathcal{F}_{(P,e)}(G, b)$ defined by $\varphi(u) = {}^xu$ for all $u \in Q$. Then $N_\varphi = \{y \in N_P(Q) \mid \text{there is } z \in N_P(R) \text{ such that } {}^{xy}u = {}^{zx}u \text{ for all } u \in Q\}$. Thus ${}^xN_\varphi \subseteq N_P(R)C_G(R)$. Since R is fully $\mathcal{F}_{(P,e)}(G, b)$ -normalised, $N_P(R)$ is a defect group of e_R viewed as block of $kN_G(R, e_R)$ by (ii), and hence $N_P(R)$ is still a defect group of e_R viewed as block of $N_P(R)C_G(R)$. Therefore $(N_P(R), e_{N_P(R)})$ is a maximal $(N_P(R)C_G(R), e_R)$ -Brauer pair (cf. [23, (40.15)]) and contains hence a $C_G(R)$ -conjugate of every other $(N_P(R)C_G(R), e_R)$ -Brauer pair (cf. [2, 3.10]). Thus there is $c \in C_G(R)$ such that ${}^{cx}(N_\varphi, e_{N_\varphi}) \subseteq (N_P(R), e_{N_P(R)})$. Hence $\psi : N_\varphi \rightarrow P$ defined by $\psi(n) = {}^{cx}n$ for all $n \in N_\varphi$ is a morphism in $\mathcal{F}_{(P,e)}(G, b)$ which extends φ . $\square$

For future reference we include another obvious reformulation of some results in [2].

Proposition 2.5. *Let G be a finite group, let b be a block of kG , and let (P, e) be a maximal b -Brauer pair. For every subgroup Q of P , denote by e_Q the unique block of $kC_G(Q)$ such that $(Q, e_Q) \subseteq (P, e)$. Set $\mathcal{F} = \mathcal{F}_{(P,e)}(G, b)$.*

- (i) *If Q is a fully $\mathcal{F}$ -centralised subgroup of P then $(C_P(Q), e_{QC_P(Q)})$ is a maximal $(C_G(Q), e_Q)$ -Brauer pair and we have $\mathcal{F}_{(C_P(Q), e_{QC_P(Q)})}(C_G(Q), e_Q) = C_{\mathcal{F}}(Q)$.*
- (ii) *If Q is a fully $\mathcal{F}$ -normalised subgroup of P then $(N_P(Q), e_{N_P(Q)})$ is a maximal $(N_G(Q, e_Q), e_Q)$ -Brauer pair and we have $\mathcal{F}_{(N_P(Q), e_{N_P(Q)})}(N_G(Q, e_Q), e_Q) = N_{\mathcal{F}}(Q)$.*

Proof. (i) Suppose that Q is fully $\mathcal{F}$ -centralised. By 2.4.(ii), $C_P(Q)$ is a defect group of e_Q as block of $C_G(Q)$. We have $C_{C_G(Q)}(C_P(Q)) = C_G(QC_P(Q))$, hence $(C_P(Q), e_{QC_P(Q)})$ is a maximal $(C_G(Q), e_Q)$ -Brauer pair. Similarly, for any subgroup R of $C_P(Q)$, the pair (R, e_{QR}) is a $(C_G(Q), e_Q)$ -Brauer pair contained in $(C_P(Q), e_{QC_P(Q)})$. If R, S are subgroups of $C_P(Q)$ and $x \in C_G(Q)$ such that ${}^x(R, e_{QR}) \subseteq (S, e_{QS})$ then the group homomorphism from R to S induced by conjugation with x extends to a group homomorphism from QR to QS which is the identity on Q . Statement (i) follows.

(ii) Suppose that Q is fully $\mathcal{F}$ -normalised. By 2.4.(iii), $N_P(Q)$ is a defect group of e_Q as block of $N_G(Q, e_Q)$. We have $C_{N_G(Q)}(C_P(Q)) = C_G(N_P(Q))$, hence $(N_P(Q), e_{N_P(Q)})$ is a maximal $(N_G(Q, e_Q), e_Q)$ -Brauer pair. Similarly, for any subgroup R of $N_P(Q)$, the pair (R, e_{QR}) is a $(N_G(Q, e_Q), e_Q)$ -Brauer pair contained in $(N_P(Q), e_{N_P(Q)})$. If R, S are subgroups of $N_P(Q)$ and $x \in N_G(Q, e_Q)$ such that ${}^x(R, e_{QR}) \subseteq (S, e_{QS})$ then the group homomorphism from R to S induced by conjugation with x extends to a group homomorphism from QR to QS which restricts to an automorphism of Q in $\text{Aut}_{\mathcal{F}}(Q)$. The result follows. $\square$

3 NORMAL FUSION SYSTEMS

Definition 3.1 Let $\mathcal{F}$ be a category on a finite p -group P , and let $\mathcal{F}'$ be a category on a subgroup P' of P . We say that $\mathcal{F}$ *normalises* $\mathcal{F}'$ if P' is strongly $\mathcal{F}$ -closed and if for every isomorphism $\varphi : Q \rightarrow Q'$ in $\mathcal{F}$ and any two subgroups R, R' of $Q \cap P'$ we have

$$\varphi \circ \text{Hom}_{\mathcal{F}'}(R, R') \circ \varphi^{-1}|_{\varphi(R)} \subseteq \text{Hom}_{\mathcal{F}'}(\varphi(R), \varphi(R')) .$$

We say that $\mathcal{F}'$ is *normal in* $\mathcal{F}$ and write $\mathcal{F}' \trianglelefteq \mathcal{F}$ if $\mathcal{F}'$ is contained in $\mathcal{F}$ and $\mathcal{F}$ normalises $\mathcal{F}'$.

In other words, $\mathcal{F}$ normalises $\mathcal{F}'$ if for any isomorphism $\varphi : Q \rightarrow Q'$ in $\mathcal{F}$ and any morphism $\psi : R \rightarrow R'$ in $\mathcal{F}'$ such that $\langle R, R' \rangle \subseteq Q$, we have $\langle \varphi(R), \varphi(R') \rangle \subseteq P'$ and the induced morphism $\varphi \circ \psi \circ \varphi^{-1} : \varphi(R) \rightarrow \varphi(R')$ is a morphism in $\mathcal{F}'$. Note that this implies that we have in fact an equality

$$\varphi \circ \text{Hom}_{\mathcal{F}'}(R, R') \circ \varphi^{-1}|_{\varphi(R)} = \text{Hom}_{\mathcal{F}'}(\varphi(R), \varphi(R')) .$$

Indeed, the left side is contained in the right side by the definition, and the other inclusion follows from applying this inclusion to $\varphi^{-1}, \varphi(R), \varphi(R')$ instead of φ, R, R' , respectively. Applied to $R = R'$ and $S = \varphi(R)$ and making use of Alperin's fusion theorem this implies in particular that if R, S are subgroups of P' which are isomorphic in $\mathcal{F}$ then $\text{Aut}_{\mathcal{F}'}(R) \cong \text{Aut}_{\mathcal{F}'}(S)$.

The unique category on the trivial subgroup $\{1\}$ of P is a fusion system which is normal in any fusion system $\mathcal{F}$ on P . The obvious motivating example for the definition of normal fusion systems is this:

Proposition 3.2. *Let G be a finite group, let P be a Sylow- p -subgroup of G , and let N be a normal subgroup of G . We have $\mathcal{F}_{P \cap N}(N) \trianglelefteq \mathcal{F}_P(G)$.*

Proof. Trivial. $\square$

Proposition 3.3. *Let $\mathcal{F}$ be a fusion system on a finite p -group P . Then $\mathcal{F}_P(P)$ is normal in $\mathcal{F}$ if and only if $\mathcal{F} = N_{\mathcal{F}}(P)$.*

Proof. Suppose that $\mathcal{F}_P(P) \trianglelefteq \mathcal{F}$. Then in particular for any morphism $\varphi : R \rightarrow P$ in $\mathcal{F}$ and any $u \in N_P(R)$ there is $v \in N_P(\varphi(R))$ such that $\varphi(ur) = v\varphi(r)$ for all $r \in R$. Whenever $\varphi(R)$ is fully $\mathcal{F}$ -centralised, φ extends to a morphism $\psi : N_P(R) \rightarrow P$ in $\mathcal{F}$. In particular, this holds if R , and hence $\varphi(R)$, are $\mathcal{F}$ -centric. But then also $N_P(R)$ and $\psi(N_P(R))$ are $\mathcal{F}$ -centric. Inductively, it follows that φ can be extended to an automorphism of P belonging to $\mathcal{F}$. Thus, by Alperin's fusion theorem, we get $\mathcal{F} = N_{\mathcal{F}}(P)$. The converse is easy. $\square$

In fact, Proposition 3.3 remains true with P replaced by any subgroup of P (cf. [21, 6.2] or [13, Corollary 2]).

Proposition 3.4. *Let $\mathcal{F}$ be a fusion system on a finite p -group P . If Q is a strongly $\mathcal{F}$ -closed abelian subgroup of P then $\mathcal{F}_Q(Q)$ is normal in $\mathcal{F}$.*

Proof. Since Q is abelian, the only morphisms in $\mathcal{F}_Q(Q)$ are inclusions $R \subseteq R'$ of subgroups R, R' of Q . Since Q is strongly $\mathcal{F}$ -closed, the result follows. $\square$

Proposition 3.5. *Let $\mathcal{F}, \mathcal{F}'$ be fusion systems on a finite p -group P such that $\mathcal{F}'$ is normal in $\mathcal{F}$. Then for every subgroup Q of P the index $[\text{Aut}_{\mathcal{F}}(Q) : \text{Aut}_{\mathcal{F}'}(Q)]$ is prime to p .*

Proof. Let Q be a subgroup of P , and let $\varphi : Q \rightarrow R$ be an isomorphism in $\mathcal{F}$ such that the subgroup R of P is fully $\mathcal{F}$ -normalised. Then $\text{Aut}_P(R)$ is a Sylow- p -subgroup of $\text{Aut}_{\mathcal{F}}(R)$ by 1.5, and $\text{Aut}_P(Q) \subseteq \text{Aut}_{\mathcal{F}'}(R)$. Since $\mathcal{F}'$ is normal in $\mathcal{F}$, it follows that the Sylow- p -subgroup $\varphi^{-1} \circ \text{Aut}_P(R) \circ \varphi$ of $\text{Aut}_{\mathcal{F}}(Q)$ is contained in $\text{Aut}_{\mathcal{F}'}(Q)$. Thus the index of $\text{Aut}_{\mathcal{F}'}(Q)$ in $\text{Aut}_{\mathcal{F}}(Q)$ is prime to p . $\square$

Remark 3.6. Proposition 3.5 is not true, in general, without the assumption that $\mathcal{F}'$ is normal in $\mathcal{F}$. Consider the case of a fusion system $\mathcal{F}$ on P such that there is a subgroup Q of P which is fully $\mathcal{F}$ -centralised but not fully $\mathcal{F}$ -normalised, and set $\mathcal{F}' = \mathcal{F}_P(P)$. Then $\text{Aut}_P(Q) = \text{Aut}_{\mathcal{F}'}(Q)$ is not a Sylow- p -subgroup of $\text{Aut}_{\mathcal{F}}(Q)$. The following is an example for this situation.

Example 3.7. Let $G = S_8$ be the symmetric group on eight letters, set $E_1 = \langle (15)(26)(37)(48) \rangle$, $E_2 = \langle (13)(24), (57)(68) \rangle$, $E_4 = \langle (12), (34), (56), (78) \rangle$. Then $P = (E_4 \rtimes E_2) \rtimes E_1$ is a Sylow-2-subgroup of G . Set $\mathcal{F} = \mathcal{F}_P(G)$. The subgroup E_4 of P is $\mathcal{F}$ -centric, hence $Q = E_4 \rtimes \langle (13)(24)(57)(68) \rangle$ and $R = E_4 \rtimes E_1$ are $\mathcal{F}$ -centric as well. Conjugating Q by $(35)(46)$ yields R , hence $Q \cong R$ in $\mathcal{F}$. Clearly Q is

normal in P ; in particular, Q is fully $\mathcal{F}$ -normalised. Conjugating $(15)(26)(37)(48) \in R$ by $(13)(24) \in E_2$ yields $(17)(28)(35)(46)$. This is not an element in R since 7 does not belong to the R -orbit of 1 (which is equal to $\{1, 2, 5, 6\}$). Thus R is not normal in P , and hence R is not fully $\mathcal{F}$ -normalised.

4 SIMPLE FUSION SYSTEMS

Definition 4.1 A fusion system $\mathcal{F}$ on a non trivial finite p -group P is called *simple* if $\mathcal{F}$ has no proper non trivial normal fusion subsystem.

In view of work of Broto, Levi, Oliver [4] - introducing p -local finite groups as a generalisation of classifying spaces associated with fusion systems - we extend this terminology in the obvious way: a p -local finite group is called *simple* if its underlying fusion system is simple. In order to avoid confusion we point out that this definition is different from previous similar definitions such as fusion-simple groups (in a group theoretic context) or the notion of simple fusion systems introduced in [15].

Certainly the fusion system $\mathcal{F}_P(G)$ of a finite simple group G (with Sylow- p -subgroup P) does not have to be simple, but conversely, if a simple fusion system $\mathcal{F}$ on a finite p -group P is equal to $\mathcal{F}_P(G)$ for some finite group G containing P as Sylow- p -subgroup, then G can be chosen to be simple:

Proposition 4.2. *Let $\mathcal{F}$ be a simple fusion system on some finite p -group P . Suppose that $\mathcal{F} = \mathcal{F}_P(G)$ for some finite group G having P as Sylow- p -subgroup. If $O_{p'}(G) = 1$ and if $\mathcal{F}_P(H) \neq \mathcal{F}_P(G)$ for any proper subgroup H of G containing P , then G is simple. In particular, if G has minimal order such that P is a Sylow- p -subgroup of G and such that $\mathcal{F} = \mathcal{F}_P(G)$, then G is simple.*

Proof. Suppose that $O_{p'}(G) = 1$ and that $\mathcal{F}_P(H) \neq \mathcal{F}_P(G)$ for any proper subgroup H of G containing P . Let N be a non-trivial normal subgroup of G . Then $N \cap P$ is a Sylow- p -subgroup of N , and $\mathcal{F}_{N \cap P}(N)$ is a normal fusion system in $\mathcal{F}_P(G)$. As $O_{p'}(G) = 1$, we have $N \cap P \neq 1$. As $\mathcal{F}_P(G)$ is simple, this forces $P \subseteq N$ and $\mathcal{F}_P(N) = \mathcal{F}_P(G)$, hence $N = G$ by the assumptions. Let now G be a finite group of minimal order such that P is a Sylow- p -subgroup of G and such that $\mathcal{F} = \mathcal{F}_P(G)$. Then $O_{p'}(G) = 1$, because the canonical map $G \rightarrow G/O_{p'}(G)$ induces an isomorphism of fusion systems. By the minimality of G , we have $\mathcal{F}_P(H) \neq \mathcal{F}_P(G)$ for any proper subgroup H of G containing P . Thus the second statement follows from the first. $\square$

Proposition 4.3. *Let P be a finite p -group. Then $\mathcal{F}_P(P)$ is simple if and only if P is cyclic of order p .*

Proof. By 3.4, for every subgroup Z of $Z(P)$ we have $\mathcal{F}_Z(Z) \leq \mathcal{F}_P(P)$, from which the statement follows. $\square$

Proposition 4.4. *Let P be a finite abelian p -group and let $\mathcal{F}$ be a fusion system on P . Then $\mathcal{F}$ is simple if and only if P has order p and $\mathcal{F} = \mathcal{F}_P(P)$.*

Proof. If $\mathcal{F}$ is simple, then $\mathcal{F} = \mathcal{F}_P(P)$ by 3.4, and hence $|P| = p$ by 4.3. The converse is clear. $\square$

The following Proposition is due to the referee and has greatly simplified the original version of this paper.

Proposition 4.5. *Let $\mathcal{F}, \mathcal{F}'$ be fusion systems on a finite p -group P such that $\mathcal{F}' \trianglelefteq \mathcal{F}$ and such that $\text{Aut}_{\mathcal{F}}(P) = \text{Aut}_{\mathcal{F}'}(P)$. Then $\mathcal{F}' = \mathcal{F}$.*

Proof. Suppose that $\mathcal{F}' \neq \mathcal{F}$. Let Q be a subgroup of maximal order such that $\text{Aut}_{\mathcal{F}'}(Q) \neq \text{Aut}_{\mathcal{F}}(Q)$. By the assumptions, Q is a proper subgroup of P . Since $\mathcal{F}'$ is normal in $\mathcal{F}$ we may assume that Q is fully $\mathcal{F}$ -normalised. Then $\text{Aut}_P(Q)$ is a Sylow- p -subgroup of $\text{Aut}_{\mathcal{F}}(Q)$. Moreover, $\text{Aut}_{\mathcal{F}'}(Q)$ is a normal subgroup of $\text{Aut}_{\mathcal{F}}(Q)$ containing $\text{Aut}_P(Q)$, and hence, by the Frattini argument, we have $\text{Aut}_{\mathcal{F}}(Q) = N_{\text{Aut}_{\mathcal{F}}(Q)}(\text{Aut}_P(Q))\text{Aut}_{\mathcal{F}'}(Q)$. By the extension axiom (II-S) in 1.4 every automorphism of Q in $N_{\text{Aut}_{\mathcal{F}}(Q)}(\text{Aut}_P(Q))$ extends to an automorphism of $N_P(Q)$ in $\mathcal{F}$, hence in $\mathcal{F}'$ by the maximality assumption on Q . This in turn implies that $N_{\text{Aut}_{\mathcal{F}}(Q)}(\text{Aut}_P(Q)) \subseteq \text{Aut}_{\mathcal{F}'}(Q)$, leading to the contradiction $\text{Aut}_{\mathcal{F}}(Q) = \text{Aut}_{\mathcal{F}'}(Q)$. $\square$

Corollary 4.6. *Let $\mathcal{F}$ be a fusion system on a finite p -group P . Assume that $\text{Aut}_{\mathcal{F}}(P) = \text{Aut}_P(P)$ and that P has no proper non trivial strongly $\mathcal{F}$ -closed subgroup. Then $\mathcal{F}$ is simple.*

Proof. Let $\mathcal{F}'$ be a fusion system on a non trivial subgroup P' of P such that $\mathcal{F}' \trianglelefteq \mathcal{F}$. Then P' is strongly $\mathcal{F}$ -closed, hence $P' = P$ by the assumptions. Since $\text{Aut}_P(P) \subseteq \text{Aut}_{\mathcal{F}'}(P) \subseteq \text{Aut}_{\mathcal{F}}(P)$, the assumptions imply further that $\text{Aut}_{\mathcal{F}'}(P) = \text{Aut}_{\mathcal{F}}(P)$. Thus $\mathcal{F}' = \mathcal{F}$ by 4.5. $\square$

Corollary 4.7. *Let $\mathcal{F}$ be a fusion system on a finite p -group P . Suppose that P is generated by the set of its subgroups of order p , that all subgroups of order p in P are $\mathcal{F}$ -conjugate and that $\text{Aut}_{\mathcal{F}}(P) = \text{Aut}_P(P)$. Then $\mathcal{F}$ is simple.*

Proof. Let Q be a non-trivial strongly $\mathcal{F}$ -closed subgroup of P . Since all subgroups of order p of P are $\mathcal{F}$ -conjugate it follows that Q contains all subgroups of order p of P . But then $Q = P$ by the assumptions on P , and hence $\mathcal{F}$ is simple by 4.6. $\square$

5 DIHEDRAL 2-LOCAL GROUPS

In order to illustrate the terminology from previous sections, we determine for any fusion system on a dihedral 2-group all normal subsystems. In this section we set $P = \langle x \rangle \rtimes \langle t \rangle$, such that $x^{2^n} = 1 = t^2$ for some integer $n \geq 2$ and $txt = x^{-1}$; that is, P is a dihedral 2-group of order $2^{n+1} \geq 8$.

Then P has three conjugacy classes of involutions, namely the classes of the elements $z = x^{2^{n-1}}$, t and xt . Besides the trivial fusion system $\mathcal{F}_P = \mathcal{F}_P(P)$, there are two other systems, up to isomorphism. We denote by $\mathcal{F}_P^I$ the fusion system on P generated by $\mathcal{F}_P$ and an automorphism of order 3 of the Klein four group $\langle z \rangle \times \langle t \rangle$. Thus z and t are $\mathcal{F}_P^I$ -conjugate, while z and xt are not; hence there are now two $\mathcal{F}_P^I$ -conjugacy classes of involutions in P . We denote by $\mathcal{F}_P^{II}$ the fusion system on P generated by $\mathcal{F}_P$ and an automorphism of order 3 on each of the Klein four groups $\langle z \rangle \times \langle t \rangle$ and $\langle z \rangle \times \langle xt \rangle$. Thus all involutions in P are $\mathcal{F}_P^{II}$ -conjugate. Any fusion system on P is isomorphic to one of $\mathcal{F}_P$, $\mathcal{F}_P^I$, $\mathcal{F}_P^{II}$ and any of these systems appear as fusion systems $\mathcal{F}_P(G)$ of some finite group G having P as Sylow-2-subgroup (this follows easily from Erdmann's list of examples in [7]). Any 2-block of a finite group having P as defect group has 1 or 2 or 3 isomorphism classes of simple modules, and then its fusion system is isomorphic to $\mathcal{F}_P$ or $\mathcal{F}_P^I$ or $\mathcal{F}_P^{II}$, respectively. The fusion systems $\mathcal{F}_P$, $\mathcal{F}_P^I$, $\mathcal{F}_P^{II}$ correspond to the cases (bb) , (ab) , (aa) , respectively, in [6].

For notational convenience, if Q is a Klein four group, we denote by $\mathcal{F}_Q^I$ and by $\mathcal{F}_Q^{II}$ the unique fusion system on Q generated by some automorphism of order 3 of Q .

Theorem 5.1. *Let $\mathcal{F}$ be a fusion system on the dihedral 2-group P of order at least 8. Then $\mathcal{F}$ is simple if and only if $\mathcal{F} = \mathcal{F}_P^{II}$.*

One implication in 5.1 is a consequence of the following.

Lemma 5.2. *Let Q be the subgroup of index 2 of P generated by x^2 and t . Then $\mathcal{F}_Q^{II} \trianglelefteq \mathcal{F}_P^I$; in particular, Q is strongly $\mathcal{F}_P^I$ -closed and $\mathcal{F}_P^I$ is not simple.*

Proof. Observe first that $\mathcal{F}_Q^{II}$ is contained in $\mathcal{F}_P^I$, because the three classes of involutions in Q represented by z , t , x^2t are all conjugate in $\mathcal{F}_P^I$. Indeed, this is clear for z and t by the definition of $\mathcal{F}_P^I$, and moreover, $x^2t = txt^{-1}$. As $\mathcal{F}_Q^{II}$ is the unique maximal fusion system on Q , it suffices to show that Q is strongly $\mathcal{F}_P^I$ -closed, which is easy. $\square$

Proof of Theorem 5.1. All fusion systems $\mathcal{F}$ on P have the property $\text{Aut}_{\mathcal{F}}(P) = \text{Aut}_P(P)$. Let Q be a strongly $\mathcal{F}_P^{II}$ -closed subgroup of P . Then Q contains all involutions of P , as all involutions of P are $\mathcal{F}_P^{II}$ -conjugate. Hence $Q = P$, and 4.6 implies that $\mathcal{F}_P^{II}$ is simple. Conversely, $\mathcal{F}_P$ is not simple by 4.3 and $\mathcal{F}_P^I$ is not simple by 5.2. $\square$

Remark 5.3. Let q be an odd prime power. If $q \equiv \pm 1 \pmod{8}$, then the group $PSL(2, q)$ has a dihedral Sylow-2-subgroup P , and $\mathcal{F}_P(PSL(2, q)) = \mathcal{F}_P^{II}$. In particular, $\mathcal{F}_P(PSL(2, q))$ is simple in that case. If $q \equiv \pm 3 \pmod{8}$ then $PSL(2, q)$ has a Klein four group Q as Sylow-2-subgroup, and hence $\mathcal{F}_P(PSL(2, q))$ cannot be simple. As pointed out by the referee, in this case the inclusion $\mathcal{F}_Q^{II} \leq \mathcal{F}_P^I$ is realised by the inclusion $PSL_2(q) \leq PGL_2(q)$. This yields an alternative proof of 5.2.

6 THE 2-FUSION SYSTEM OF $\Omega_7(q)$, $q \equiv \pm 3 \pmod{8}$, IS SIMPLE

The group theoretic background material needed in this and the next Section can be found in [5], [16], [17], [18], [22], [24].

Theorem 6.1. *Let q be an odd prime power such that $q \equiv \pm 3 \pmod{8}$ and let P be a Sylow-2-subgroup of $\Omega_7(q)$. We have $\text{Aut}_{\Omega_7(q)}(P) = \text{Aut}_P(P)$ and P has no non-trivial proper strongly $\mathcal{F}_{\Omega_7(q)}$ -closed subgroup. In particular, the fusion system $\mathcal{F}_S(\Omega_7(q))$ is simple.*

Proof. Since $Q \equiv \pm 3 \pmod{8}$ the Sylow-2-subgroup P of $\Omega_7(q)$ is isomorphic to a Sylow-2-subgroup of the alternating group A_{12} , whose structure is as follows (cf. [16, §2]): the Thompson subgroup $A = J(P)$ is elementary abelian of order 2^6 and we have $P = A \rtimes D$ for D a dihedral group of order 8. In particular, P is generated by its set of involutions. Moreover, $Z(P)$ is a Klein four group contained in A . The statement $\text{Aut}_{\Omega_7(q)}(P) = \text{Aut}_P(P)$ is a particular case of [16, 2.1]. Let Q be a non-trivial strongly $\mathcal{F}_P(\Omega_7(q))$ -closed subgroup of P . Then in particular Q is normal in P , hence $Q \cap Z(P) \neq 1$, and so $Q \cap A \neq 1$. By the remark preceding [16, 6.3], the cases [16, 4.7.(iii)], [16, 4.8.(iii)] and [16, 6.2.(iii)] correspond to the fusion system of $\Omega_7(q)$. It follows from [16, 4.7.(iii)] that the group $\text{Aut}_{\Omega_7(q)}(A) \cong A_7$ acts irreducibly on A , and hence $A \subseteq Q$. By [16, 6.2.(iii)] every involution of P is $\Omega_7(q)$ -conjugate to an involution in A . Thus Q contains all involutions in P , and hence $Q = P$ as P is generated by its set of involutions. The simplicity of the fusion system $\mathcal{F}_{\Omega_7(q)}$ follows from 4.6. $\square$

7 THE SOLOMON 2-LOCAL FINITE GROUP $\text{Sol}(3)$ IS SIMPLE

Let q be an odd prime power such that $q \equiv \pm 3 \pmod{8}$ and let P be a Sylow-2-subgroup of the 7-dimensional spinor group $\text{Spin}_7(q)$ over $\mathbb{F}_q$. Then $\text{Spin}_7(q)$ has a central involution z such that $\text{Spin}_7(q)/\langle z \rangle \cong \Omega_7(q)$, and hence $P/\langle z \rangle$ is isomorphic to a Sylow-2-subgroup of $\Omega_7(q)$. R. Solomon showed in [18] that if $q \equiv \pm 3 \pmod{8}$, no finite group having P as Sylow-2-subgroup can have a fusion system which properly contains $\mathcal{F}_P(\text{Spin}_7(q))$, in which all involutions of P are conjugate and which has the property that $C_{\mathcal{F}}(z)/\langle z \rangle \cong \mathcal{F}_S(\Omega(7, q))$. Levi and Oliver proved in [11, 2.1], that there is actually for any odd prime power q a fusion system $\mathcal{F}_{\text{Sol}(q)}$ on P with the above properties, and that this fusion system is the underlying fusion system

of a unique 2-local finite group; we are going to call this the *Solomon 2-local finite group* $\text{Sol}(q)$. Kessar showed in [9] that the fusion system $\text{Sol}(3)$ cannot even occur as fusion system of a 2-block of a finite group with P as defect group.

Theorem 7.1. *The Solomon 2-local finite group $\text{Sol}(3)$ is simple.*

Proof. Let $\mathcal{F}$ be the underlying fusion system of $\text{Sol}(3)$ on a Sylow-2-subgroup P of $\text{Spin}_7(3)$ as constructed in [11, §2]. The normaliser of P in $\text{Spin}_7(3)$ is the inverse image of the normaliser of a Sylow-2-subgroup of $\Omega_7(3)$, and hence $\text{Aut}_{\mathcal{F}}(P) = \text{Aut}_{\text{Spin}_7(3)}(P) = \text{Aut}_P(P)$, where the first equality uses [11, 2.1].

Let Q be a non trivial strongly $\mathcal{F}$ -closed subgroup of P . In particular, Q is strongly $\mathcal{F}_P(\text{Spin}_7(3))$ -closed. Since all involutions in P are $\mathcal{F}$ -conjugate, they are all contained in Q . Thus Q strictly contains $\langle z \rangle$. Its image $\bar{Q} = Q / \langle z \rangle$ in $\bar{P} = P / \langle z \rangle$ is strongly $\mathcal{F}_{\bar{P}}(\Omega_7(3))$ -closed. By 6.1 this forces $\bar{Q} = \bar{P}$, hence $Q = P$. Thus $\mathcal{F}$ is simple by 4.6. $\square$

8 CHARACTERISATIONS OF FUSION SYSTEMS

Proposition 4.5 would be false without the assumption on $\mathcal{F}'$ being normal in $\mathcal{F}$. For the sake of completeness, we include some statements regarding the situation of not necessarily normal subsystems.

The first result shows that a fusion system $\mathcal{F}$ on a finite p -group P is determined by its fusion on elements of order p in P and their centralisers in $\mathcal{F}$. If Q is a subgroup of P , we denote by $C_{\mathcal{F}}(Q)/Z(Q)$ the category on $C_P(Q)/Z(Q)$ whose morphisms are induced by morphisms in $C_{\mathcal{F}}(Q)$ via the canonical map $C_P(Q) \rightarrow C_P(Q)/Z(Q)$. By the remarks following 1.8, if Q is fully $\mathcal{F}$ -centralised, then $C_{\mathcal{F}}(Q)/Z(Q)$ is a fusion system on $C_P(Q)/Z(Q)$.

Proposition 8.1. *Let P be a finite p -group, and let $\mathcal{F}, \mathcal{F}'$ be fusion systems on P such that $\mathcal{F}' \subseteq \mathcal{F}$. The following are equivalent.*

- (i) $\mathcal{F} = \mathcal{F}'$.
- (ii) *For any fully $\mathcal{F}'$ -centralised subgroup Z of order p of P we have $\text{Hom}_{\mathcal{F}}(Z, P) = \text{Hom}_{\mathcal{F}'}(Z, P)$ and $C_{\mathcal{F}}(Z) = C_{\mathcal{F}'}(Z)$.*

Proof. Suppose that (ii) holds. Let Q be a non trivial subgroup of P and let $\varphi \in \text{Aut}_{\mathcal{F}}(Q)$. Let Z be a subgroup of order p of $Z(Q)$. Let $\psi : Z \rightarrow P$ be a morphism in $\mathcal{F}'$ such that $\psi(Z)$ is fully $\mathcal{F}'$ -centralised. Since $Q \subseteq C_P(Z)$, the morphism ψ extends to a morphism $\tau : Q \rightarrow P$ in $\mathcal{F}'$. In order to show that φ is a morphism in $\mathcal{F}'$, it suffices to show that $\tau \circ \varphi \circ \tau^{-1}|_{\tau(Q)} \in \text{Aut}_{\mathcal{F}'}(\tau(Q))$. Thus, after replacing Q by $\tau(Q)$, we may assume that Z is fully $\mathcal{F}'$ -centralised. By the assumptions, the morphism $\varphi^{-1}|_{\varphi(Z)} : \varphi(Z) \rightarrow Z$ belongs to $\mathcal{F}'$, and hence extends to a morphism $\kappa : Q \rightarrow P$ in $\mathcal{F}'$ (since $Q = \varphi(Q) \subseteq C_P(\varphi(Z))$). Then $\kappa \circ \varphi : Q \rightarrow P$ restricts to the identity on Z , hence $\kappa \circ \varphi$ is a morphism in $C_{\mathcal{F}}(Z) = C_{\mathcal{F}'}(Z)$. In particular, $\kappa \circ \varphi$ is a morphism

in $\mathcal{F}'$. But then so is φ , because κ is in $\mathcal{F}'$. Alperin's fusion theorem implies now (i). The converse is trivial. $\square$

Corollary 8.2. *Let $\mathcal{F}, \mathcal{F}'$ be fusion systems on a finite p -group P such that $\mathcal{F}' \trianglelefteq \mathcal{F}$. If $\text{Hom}_{\mathcal{F}}(Z, P) = \text{Hom}_{\mathcal{F}'}(Z, P)$ and $C_{\mathcal{F}}(Z)/Z$ is a simple fusion system on $C_P(Z)/Z$ for any fully $\mathcal{F}'$ -centralised subgroup Z of order p of P , then $\mathcal{F}' = \mathcal{F}$.*

Proof. We have $C_{\mathcal{F}'}(Z) \trianglelefteq C_{\mathcal{F}}(Z)$ and hence $C_{\mathcal{F}'}(Z)/Z \trianglelefteq C_{\mathcal{F}}(Z)/Z$. Thus, if $C_{\mathcal{F}}(Z)/Z$ is simple for any fully $\mathcal{F}'$ -centralised subgroup Z of order p of P , then $C_{\mathcal{F}'}(Z)/Z = C_{\mathcal{F}}(Z)/Z$. Since p' -automorphisms lift uniquely through central p -extensions this implies $C_{\mathcal{F}'}(Z) = C_{\mathcal{F}}(Z)$, hence $\mathcal{F}' = \mathcal{F}$ by 8.1. $\square$

Lemma 8.3. *Let $\mathcal{F}, \mathcal{F}'$ be fusion systems on a finite p -group P such that $\mathcal{F}' \subseteq \mathcal{F}$. Let $Q \xrightarrow{\varphi} R \xrightarrow{\psi} S$ be a sequence of two composable morphisms in $\mathcal{F}$ such that Q, R, S are $\mathcal{F}$ -centric. If any two of the three morphisms $\varphi, \psi, \psi \circ \varphi$ are in $\mathcal{F}'$, so is the third.*

Proof. If φ, ψ are in $\mathcal{F}'$, so is $\psi \circ \varphi$. If $\psi, \psi \circ \varphi$ are in $\mathcal{F}'$, then so is $\varphi = \psi^{-1}|_{\text{Im}(\psi \circ \varphi)} \circ \psi \circ \varphi$. Assume now that φ and $\psi \circ \varphi$ are morphisms in $\mathcal{F}'$. Up to replacing Q by $\varphi(Q)$, we may assume that φ is the inclusion $Q \subseteq R$. Let $v \in N_R(Q)$. Then, for any $u \in Q$, we have $\psi(vu) = \psi(v)\psi(u)$. Thus the morphism $\psi|_Q$ extends to a morphism $\tau : N_R(Q) \rightarrow P$ in $\mathcal{F}'$. By 1.11, we have $\tau(N_R(Q)) = \psi(N_R(Q))$ and hence $\psi^{-1} \circ \tau \in \text{Aut}_{Z(Q)}(N_R(Q))$ by 1.10. Thus $\psi|_{N_R(Q)}$ is a morphism in $\mathcal{F}'$. It follows inductively, that ψ is a morphism in $\mathcal{F}'$. $\square$

Proposition 8.4. *Let $\mathcal{F}, \mathcal{F}'$ be fusion systems on a finite p -group P such that $\mathcal{F}' \subseteq \mathcal{F}$. The following are equivalent.*

- (i) $\mathcal{F} = \mathcal{F}'$.
- (ii) $\text{Hom}_{\mathcal{F}}(Q, P) = \text{Hom}_{\mathcal{F}'}(Q, P)$ for every minimal $\mathcal{F}$ -centric subgroup Q of P .

Proof. Assume that (ii) holds. Let R be an $\mathcal{F}$ -centric subgroup of P , and let Q be a minimal $\mathcal{F}$ -centric subgroup of P contained in R . Let $\varphi \in \text{Hom}_{\mathcal{F}}(R, P)$. Then $\varphi|_Q \in \text{Hom}_{\mathcal{F}}(Q, P) = \text{Hom}_{\mathcal{F}'}(Q, P)$. But then $\varphi \in \text{Hom}_{\mathcal{F}'}(R, P)$ by 8.3. Alperin's fusion theorem implies (i). The converse is trivial. $\square$

Acknowledgements. The author would like to thank Ron Solomon for many helpful comments in [19] as well as the referee for his comments and Proposition 4.5.

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