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On $H^*(\mathcal{C}; k^\times)$ for fusion systems

Markus Linckelmann

Abstract

We give a cohomological criterion for the existence and uniqueness of solutions of the 2-cocycle gluing problem in block theory. The existence of a solution for the 2-cocycle gluing problem is further reduced to a property of fusion systems of certain finite groups associated with the fusion system of a block.

1 Introduction

Given a block b of a finite group G over an algebraically closed field k of characteristic p with defect group P and associated fusion system $\mathcal{F}$, there is, for any $\mathcal{F}$ -centric subgroup Q of P , a canonically determined element $\alpha_Q \in H^2(\text{Aut}_{\mathcal{F}}(Q); k^\times)$, by [13, 1.12]. It is conjectured in [14, 4.2] that this family can be glued together to a class $\alpha \in H^2(\mathcal{F}^c; k^\times)$ satisfying $\alpha|_{\text{Aut}_{\mathcal{F}}(Q)} = \alpha_Q$ for any Q belonging to the full subcategory $\mathcal{F}^c$ of $\mathcal{F}$ -centric subgroups of P . We describe a cohomological criterion for the existence and uniqueness of α . This can be formulated more generally for EI -categories. Following [18, 9.2], an EI -category is a small category $\mathcal{C}$ with the property $\text{End}_{\mathcal{C}}(X) = \text{Aut}_{\mathcal{C}}(X)$ for any object X in $\mathcal{C}$. The set $[\mathcal{C}]$ of isomorphism classes of objects in $\mathcal{C}$ becomes then a partially ordered set via $[X] \leq [Y]$ if $\text{Hom}_{\mathcal{C}}(X, Y)$ is non-empty, for X, Y objects in $\mathcal{C}$ and $[X], [Y]$ their respective isomorphism classes. A morphism $\varphi : X \rightarrow Y$ in $\mathcal{C}$ need not induce a map between the automorphism groups of X, Y ; the *subdivision* of $\mathcal{C}$ is a tool to address this issue; this is the category defined as follows. The objects of $S(\mathcal{C})$ are faithful functors $\sigma : [m] \rightarrow \mathcal{C}$, where m is a nonnegative integer and the totally ordered set $[m] = \{0 < 1 < \dots < m\}$ is viewed as category in the obvious way; a morphism in $S(\mathcal{C})$ from σ to another object $\tau : [n] \rightarrow \mathcal{C}$ is a pair (α, μ) consisting of an injective order preserving map $\alpha : [m] \rightarrow [n]$ and an isomorphism of functors $\mu : \sigma \cong \tau \circ \alpha$. The composition of (α, μ) with another morphism (β, ν) from τ to $\rho : [r] \rightarrow \mathcal{C}$ is defined by $(\beta, \nu) \circ (\alpha, \mu) = (\beta \circ \alpha, (\nu \alpha) \circ \mu)$, where $\nu \alpha : \tau \circ \alpha \cong \rho \circ \beta \circ \alpha$ is induced by precomposing ν with α . Loosely speaking, $S(\mathcal{C})$ consists of chains of non-isomorphisms in $\mathcal{C}$. It is easy to see that (α, μ) induces a group homomorphism $\text{Aut}_{S(\mathcal{C})}(\tau) \rightarrow \text{Aut}_{S(\mathcal{C})}(\sigma)$ mapping $(\text{Id}_{[n]}, \gamma)$ to $(\text{Id}_{[m]}, \mu^{-1} \circ (\gamma \alpha) \circ \mu)$, for any automorphism γ of the functor τ , where $\gamma \alpha$ is the induced automorphism of $\tau \circ \alpha$. Clearly $S(\mathcal{C})$ is again an EI -category. The partially ordered set $[S(\mathcal{C})]$, viewed as topological space, is called the *orbit space* of $\mathcal{C}$. We denote by **Ab** the category of abelian groups.

Theorem 1.1. *Let $\mathcal{C}$ be a finite EI -category and k an algebraically closed field. For any positive integer i there is a canonical functor $\mathcal{A}^i : [S(\mathcal{C})] \rightarrow \mathbf{Ab}$ sending $[\sigma] \in [S(\mathcal{C})]$ to $H^i(\text{Aut}_{S(\mathcal{C})}(\sigma); k^\times)$. If $H^1([S(\mathcal{C})]; k^\times) = H^2([S(\mathcal{C})]; k^\times) = 0$ then $H^1(\mathcal{C}; k^\times) \cong \lim_{[S(\mathcal{C})]}(\mathcal{A}^1)$ and if also $H^3([S(\mathcal{C})]; k^\times) = H^4([S(\mathcal{C})]; k^\times) = 0$ then there is an exact sequence of abelian groups*

$$0 \rightarrow H^1([S(\mathcal{C})]; \mathcal{A}^1) \rightarrow H^2(\mathcal{C}; k^\times) \rightarrow \lim_{[S(\mathcal{C})]}(\mathcal{A}^2) \rightarrow H^2([S(\mathcal{C})]; \mathcal{A}^1) \rightarrow H^3(\mathcal{C}; k^\times)$$

In particular, the group $H^2(\mathcal{C}; k^\times)$ is finite, of order coprime to $\text{char}(k)$ if $\text{char}(k)$ is positive.

As mentioned above, the motivation for considering the map $H^2(\mathcal{C}; k^\times) \rightarrow \lim_{[S(\mathcal{C})]}(\mathcal{A}^2)$ comes from block theory, which is the reason for stating the above theorem for $k^\times$, but it is worth noting that the exact sequence in the theorem holds for any abelian group instead of $k^\times$. In order to be more precise, let b be a block of a finite group G over an algebraically closed field k of positive characteristic p , let P be a defect group of b and $\mathcal{F}$ an associated fusion system on P . Set $\mathcal{C} = \mathcal{F}^c$, the full subcategory $\mathcal{F}$ consisting of all $\mathcal{F}$ -centric subgroups of P . Then $\mathcal{C}$ is a right ideal in $\mathcal{F}$, and hence $[S(\mathcal{C})]$ is contractible by [16, 1.1]; in particular, $H^i([S(\mathcal{C})]; k^\times) = 0$ for $i > 0$. Thus $\mathcal{C}$ satisfies the hypotheses of Theorem 1.1. As a consequence of work of Külshammer and Puig [13, 1.8, 1.12] in conjunction with Dade's splitting theorem on fusion, the block b determines for every $\sigma \in S(\mathcal{C})$ a class $\alpha_\sigma \in H^2(\text{Aut}_{S(\mathcal{C})}(\sigma); k^\times)$, and the family (α_σ) of these classes determines an element β in $\lim_{[S(\mathcal{C})]}(\mathcal{A}^2)$. Denote by γ the image of β in $H^2([S(\mathcal{C})]; \mathcal{A}^1)$. Then, by Theorem 1.1 above, the gluing problem [14, 4.2] has a solution if $\gamma = 0$, and the solution is unique if $H^1([S(\mathcal{C})]; \mathcal{A}^1) = 0$. In particular, if $H^1([S(\mathcal{C})]; \mathcal{A}^1) = H^2([S(\mathcal{C})]; \mathcal{A}^1) = 0$ then we have an isomorphism $H^2(\mathcal{C}; k^\times) \cong \lim_{[S(\mathcal{C})]}(\mathcal{A}^2)$, and so the gluing problem [14, 4.2] would have a solution for any block b with fusion system $\mathcal{F}$. This isomorphism holds trivially if $\mathcal{F} = N_{\mathcal{F}}(P)$ (which includes the case where P is abelian) and if P is a tame 2-group, in which case the right side is well-known to be zero, and the left side is zero by a result of S. Park [21]. In general, this isomorphism is relevant for the block theoretic reformulation of Alperin's weight conjecture in terms of Bredon cohomology in [15, 4.3]. The purpose of the next result is to reduce the vanishing of $H^2([S(\mathcal{F}^c)]; \mathcal{A}^1)$ further to a statement on finite groups. If all else fails, this points at least hypothetically to a strategy to solve the existence part of the gluing problem using the classification of finite simple groups. For any finite group G , denote by $\Delta_p(G)$ the partially ordered G -set of chains $\sigma = Q_0 < Q_1 < \dots < Q_m$ of nontrivial p -subgroups Q_i of G and denote by $[\Delta_p(G)]$ the set of G -conjugacy classes of chains in $\Delta_p(G)$, viewed as partially ordered set via taking subchains. Denote by $\mathcal{N}_G : [\Delta_p(G)] \rightarrow \mathbf{Ab}$ the covariant functor sending the G -conjugacy class $[\sigma]$ of the chain $\sigma \in \Delta_p(G)$ as above to the abelian group $\mathcal{N}_G([\sigma]) = \text{Hom}(N_G(\sigma); k^\times)$, where $N_G(\sigma)$ is the intersection of the normalisers $N_G(Q_i)$, $0 \leq i \leq m$. As before, one checks that this is a well-defined functor which does not depend, up to unique isomorphism of functors, on the choice of a representative σ of $[\sigma]$.

Theorem 1.2. *Let $\mathcal{F}$ be a fusion system on a finite p -group P . Suppose that for any finite group G isomorphic to $\text{Aut}_{\mathcal{F}}(Q)/\text{Inn}(Q)$ for some $\mathcal{F}$ -centric subgroup Q of P we have $H^1([\Delta_p(G)]; \mathcal{N}_G) = 0$. Then $H^2([S(\mathcal{F}^c)]; \mathcal{A}^1) = 0$; in particular, the canonical map $H^2(\mathcal{F}^c; k^\times) \rightarrow \lim_{[S(\mathcal{F}^c)]}(\mathcal{A}^2)$ is surjective.*

We do not know whether $H^1([\Delta_p(G)]; \mathcal{N}_G)$ vanishes for all finite groups G and do not have, at this point, enough backup from examples to make this a conjecture. Theorem 1.2 will follow from a spectral sequence using a filtration indexed by isomorphism classes of $\mathcal{F}$ -centric subgroups, by making use of the fact that the cohomology of $\mathcal{A}^1$ may be calculated using *normal* chains of subgroups of P . After reviewing briefly some basic facts on functor cohomology in section 2 we consider regular EI -categories and prove Theorem 1.1 in section 3. This is followed by a section proving Theorem 1.2 and a section 5 on regular functors between EI -categories, which in turn is used in the last section to show that in order to calculate $H^*(\mathcal{C}; k^\times)$ for a right ideal $\mathcal{C}$ of a fusion system, we may replace $\mathcal{C}$ by its image in the orbit category or its inverse image in a centric linking system.

2 Background

We collect, mostly from [5], [6], [9], [10], [11], [23], [24], background material on functor cohomology, which we will use without further comment. See also [25] for a broader exposition and further references. A *right ideal* in a category $\mathcal{C}$ is a full subcategory $\mathcal{D}$ of $\mathcal{C}$ with the property that if $\varphi : X \rightarrow Y$ is a morphism in $\mathcal{C}$ and if X belongs to $\mathcal{D}$, then also Y belongs to $\mathcal{D}$. For $\mathcal{C}$ a small category denote by $\hat{\mathcal{C}}$

and $\tilde{\mathcal{C}}$ the categories of covariant and contravariant functors from $\mathcal{C}$ to $\mathbf{Ab}$, respectively. The cohomology $H^*(\mathcal{C}; \mathcal{A})$ or $H^*(\mathcal{C}^{op}; \mathcal{A})$ of a functor $\mathcal{A}$ in $\tilde{\mathcal{C}}$ or $\tilde{\mathcal{C}}$, respectively, is the graded abelian group defined as follows: for any nonnegative integer n denote by $H^n(\mathcal{C}; \mathcal{A})$ or $H^n(\mathcal{C}^{op}; \mathcal{A})$ the n -th right derived functor of the limit functor $\lim_{\mathcal{C}}$ over $\mathcal{C}$ from $\tilde{\mathcal{C}}$ or $\tilde{\mathcal{C}}$ to $\mathbf{Ab}$, respectively. If A is an abelian group and $\mathcal{A}$ the constant covariant functor on $\mathcal{C}$ taking the value A , we write $H^n(\mathcal{C}; A)$ instead of $H^n(\mathcal{C}; \mathcal{A})$; similarly for the contravariant constant functor. It is well-known that $H^*(\mathcal{C}; A) \cong H^*(\mathcal{C}^{op}; A)$. If $\Phi : \mathcal{C} \rightarrow \mathcal{D}$ is a covariant functor, we denote by $\Phi^* : \tilde{\mathcal{D}} \rightarrow \tilde{\mathcal{C}}$ the induced restriction functor sending $\mathcal{A}$ in $\tilde{\mathcal{C}}$ to $\mathcal{A} \circ \Phi$; we use (abusively) the same notation for the restriction functor $\tilde{\mathcal{D}} \rightarrow \tilde{\mathcal{C}}$. We denote by $\Phi_*, \Phi_! : \tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{D}}$ the left and right adjoint (Kan extension functors) of Φ^* , and as before, use the same notation for the categories $\tilde{\mathcal{C}}, \tilde{\mathcal{D}}$. By [24, 1.4.(ii)] or [11, 5.3] or [5, Appendix II, Thm. 3.6] there is a cohomology spectral sequence, called *base change spectral sequence*

$$E_2^{p,q} = H^p(\mathcal{D}^{op}; R^q\Phi_!(\mathcal{A})) \Rightarrow H^{p+q}(\mathcal{C}^{op}; \mathcal{A})$$

for any $\mathcal{A}$ in $\tilde{\mathcal{C}}$, where $R^q\Phi_!$ is the q -th right derived functor of $\Phi_!$. It is well-known (see e.g. [5, Appendix II, §3] for the homology version) that $R^q\Phi_!$ can be computed explicitly by

$$R^q\Phi_!(\mathcal{A})(Y) = H^q((\Phi^Y)^{op}; \mathcal{A}^Y)$$

where $\mathcal{A}$ is in $\tilde{\mathcal{C}}$, Y is an object in $\mathcal{D}$, where Φ^Y is the category with objects pairs (X, φ) with X an object in $\mathcal{C}$ and $\varphi : \Phi(X) \rightarrow Y$ a morphism in $\mathcal{D}$; a morphism in Φ^Y from (X, φ) to (X', φ') is a morphism $\alpha : X \rightarrow X'$ in $\mathcal{C}$ satisfying $\varphi' \circ \Phi(\alpha) = \varphi$, and $\mathcal{A}^Y$ is the functor obtained by restricting $\mathcal{A}$ to Φ^Y via the forgetful functor $\Phi^Y \rightarrow \mathcal{C}$ sending (X, φ) to X . We will need the base change spectral sequence only the case where $\mathcal{C}$ is an *EI*-category and $\mathcal{D}$ a partially ordered set. In that case, for any Y in $\mathcal{D}$, the category Φ^Y can be identified with the category $\mathcal{C}_{\leq Y}$ consisting of all objects X in $\mathcal{C}$ such that $\text{Hom}_{\mathcal{D}}(\Phi(X), Y)$ is non-empty, and hence the base change spectral sequence takes the well-known form

$$E_2^{p,q} = H^p(\mathcal{D}^{op}; Y \mapsto H^q(\mathcal{C}_{\leq Y}; \mathcal{A})) \Rightarrow H^{p+q}(\mathcal{C}^{op}; \mathcal{A})$$

for any $\mathcal{A}$ in $\tilde{\mathcal{C}}$, where in the expression $H^q(\mathcal{C}_{\leq Y}; \mathcal{A})$ the restriction of $\mathcal{A}$ to $\mathcal{C}_{\leq Y}$ is abusively again denoted by $\mathcal{A}$. If $\Phi_!$ is exact the base change spectral sequence collapses. This happens in particular if Φ has a right adjoint $\Psi : \mathcal{D} \rightarrow \mathcal{C}$, for in that case the right adjoint $\Phi_!$ of Φ^* is isomorphic to the exact restriction functor Ψ^* . Thus, if Φ has a right adjoint we have $H^*(\mathcal{D}^{op}; \Phi_!(\mathcal{A})) \cong H^*(\mathcal{C}^{op}; \mathcal{A})$ for any $\mathcal{A}$ in $\tilde{\mathcal{C}}$ (cf. [9, 3.1]).

3 Regular *EI*-categories and proof of Theorem 1.1

Following [15, 2.1], an *EI*-category $\mathcal{C}$ is called *regular* if for any two objects X, Y in $\mathcal{C}$ such that $\text{Hom}_{\mathcal{C}}(X, Y) \neq \emptyset$ the group $\text{Aut}_{\mathcal{C}}(X)$ acts regularly (i.e. transitively and freely) on $\text{Hom}_{\mathcal{C}}(X, Y)$. Any morphism in a regular *EI*-category is a monomorphism. For any *EI*-category $\mathcal{C}$ the subdivision $S(\mathcal{C})$ is regular, and if $\mathcal{C}$ is regular, there is a contravariant functor from $\mathcal{C}$ to the category of groups sending an object X to its automorphism group $\text{Aut}_{\mathcal{C}}(X)$ and a morphism $\varphi : X \rightarrow Y$ in $\mathcal{C}$ to the unique map $\text{Aut}_{\mathcal{C}}(Y) \rightarrow \text{Aut}_{\mathcal{C}}(X)$ which sends $\sigma \in \text{Aut}_{\mathcal{C}}(Y)$ to the unique $\rho \in \text{Aut}_{\mathcal{C}}(X)$ satisfying $\varphi \circ \rho = \sigma \circ \varphi$; we use the regularity of $\mathcal{C}$ for the existence and uniqueness of ρ (cf [15, 2.2]). For any *EI*-category $\mathcal{C}$, the subdivision $S(\mathcal{C})$ comes with canonical functors from $S(\mathcal{C})$ to $\mathcal{C}$ and $\mathcal{C}^{op}$ sending an object $\sigma : [m] \rightarrow \mathcal{C}$ in $S(\mathcal{C})$ to $\sigma(m)$ and $\sigma(0)$, respectively. If every isomorphism class of $\mathcal{C}$ has a unique element then $S(\mathcal{C})$ is equivalent to the opposite of the category $s(\mathcal{C})$ defined in [23, §1], and hence [23, 1.5] implies that the canonical functor $S(\mathcal{C}) \rightarrow \mathcal{C}$ induces, for any abelian group A , an isomorphism $H^*(S(\mathcal{C}); A) \cong H^*(\mathcal{C}; A)$. For regular *EI*-categories the appropriate base change spectral sequence specialises to the following spectral sequence:

Theorem 3.1. *Let $\mathcal{C}$ be a regular EI-category and A an abelian group. There is a cohomology spectral sequence $E_2^{p,q} = H^p([\mathcal{C}]^{op}; [X] \mapsto H^q(\text{Aut}_{\mathcal{C}}(X); A)) \Rightarrow H^{p+q}(\mathcal{C}; A)$.*

Proof. Denote by $\Phi : S(\mathcal{C}) \rightarrow [\mathcal{C}]^{op}$ the canonical functor sending an object $\sigma : [m] \rightarrow \mathcal{C}$ in $S(\mathcal{C})$ to the isomorphism class $[\sigma(0)]$ in $[\mathcal{C}]$. One checks that for any X in $\mathcal{C}^{op}$ we have $S(\mathcal{C})_{\leq [X]} = S(\mathcal{C}_{\geq X})$. Thus the base change spectral sequence associated with Φ takes the form

$$E_2^{p,q} = H^p([\mathcal{C}]^{op}; [X] \mapsto H^q(S(\mathcal{C}_{\geq X}); \mathcal{A})) \Rightarrow H^{p+q}(S(\mathcal{C}); \mathcal{A})$$

As mentioned above, we have $H^{p+q}(S(\mathcal{C}); \mathcal{A}) \cong H^{p+q}(\mathcal{C}; \mathcal{A})$ and $H^q(S(\mathcal{C}_{\geq X}); \mathcal{A}) \cong H^q(\mathcal{C}_{\geq X}; \mathcal{A})$. By 3.3 below, this is isomorphic to $H^q(\text{Aut}_{\mathcal{C}}(X); A)$, whence the result. $\square$

Lemma 3.2. *Let $\mathcal{C}$ be a small category, X an object in $\mathcal{C}$ and denote by $\mathcal{D}$ the full subcategory of $\mathcal{C}$ having X as unique object. Suppose that for any object Y in $\mathcal{C}$ there is a morphism $\iota_Y : X \rightarrow Y$ in $\mathcal{C}$ such that the map $\text{Aut}_{\mathcal{C}}(X) \rightarrow \text{Hom}_{\mathcal{C}}(X, Y)$ sending $\alpha \in \text{Aut}_{\mathcal{C}}(X)$ to $\iota_Y \circ \alpha$ is a bijection. Then the inclusion functor $\Phi : \mathcal{D} \rightarrow \mathcal{C}$ has a right adjoint $\Psi : \mathcal{C} \rightarrow \mathcal{D}$.*

Proof. Define Ψ on objects by $\Psi(Y) = X$ for all Y in $\mathcal{C}$. For a morphism $\varphi : Y \rightarrow Z$ in $\mathcal{C}$, define the morphism $\Psi(\varphi) \in \text{Aut}_{\mathcal{C}}(X) = \text{Aut}_{\mathcal{D}}(X)$ as follows. By the assumptions there is a unique $\beta \in \text{Aut}_{\mathcal{C}}(X)$ such that $\varphi \circ \iota_Y = \iota_Z \circ \beta$. We set $\Psi(\varphi) = \beta$. A trivial verification shows that this construction is functorial and yields a right adjoint for Φ . $\square$

Lemma 3.3. *Let $\mathcal{C}$ be a regular EI-category, X an object in $\mathcal{C}$ and A an abelian group. Restriction induces an isomorphism $H^*((\mathcal{C}_{\geq X})^{op}; A) \cong H^*(\text{Aut}_{\mathcal{C}}(X); A)$.*

Proof. For any object Y in $\mathcal{C}_{\geq X}$ choose a morphism $\iota_Y : X \rightarrow Y$. Since $\mathcal{C}$ is regular, composition with ι_Y induces a bijection $\text{Aut}_{\mathcal{C}}(X) \cong \text{Hom}_{\mathcal{C}}(X, Y)$. Thus 3.2 applies, showing that the inclusion functor $\{X\} \rightarrow \mathcal{C}_{\geq X}$ has a right adjoint Ψ . But then the base change spectral sequence associated with this inclusion functor collapses and yields the isomorphism as stated. $\square$

Proof of Theorem 1.1. We use the notation of 1.1. The spectral sequence 3.1 applied to the regular EI-category $S(\mathcal{C})$ and the abelian group $k^\times$ takes the form

$$E_2^{p,q} = H^p([S(\mathcal{C})]; \mathcal{A}^q) \Rightarrow H^{p+q}(S(\mathcal{C}); k^\times)$$

That is, the E_2 -page has the form

$$\begin{array}{cccccc} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ H^0([S(\mathcal{C})]; \mathcal{A}^2) & H^1([S(\mathcal{C})]; \mathcal{A}^2) & H^2([S(\mathcal{C})]; \mathcal{A}^2) & H^3([S(\mathcal{C})]; \mathcal{A}^2) & \cdots & \cdots \\ H^0([S(\mathcal{C})]; \mathcal{A}^1) & H^1([S(\mathcal{C})]; \mathcal{A}^1) & H^2([S(\mathcal{C})]; \mathcal{A}^1) & H^3([S(\mathcal{C})]; \mathcal{A}^1) & \cdots & \cdots \\ k^\times & H^1([S(\mathcal{C})]; k^\times) & H^2([S(\mathcal{C})]; k^\times) & H^3([S(\mathcal{C})]; k^\times) & \cdots & \cdots \end{array}$$

This spectral sequence approximates $H^*(S(\mathcal{C}); k^\times) \cong H^*(\mathcal{C}; k^\times)$. Thus if the groups $H^1([S(\mathcal{C})]; k^\times)$ and $H^2([S(\mathcal{C})]; k^\times)$ are zero there is no differential starting or ending at the coordinates $(0, 1)$, and hence we get the isomorphism $H^1(\mathcal{C}; k^\times) \cong H^0([S(\mathcal{C})]; \mathcal{A}^1) = \lim_{[S(\mathcal{C})]}(\mathcal{A}^1)$ as stated in 1.1. Suppose now in addition that also $H^3([S(\mathcal{C})]; k^\times) = H^4([S(\mathcal{C})]; k^\times) = 0$. There is no nonzero differential starting or ending at $E_2^{1,1} = H^1([S(\mathcal{C})]; \mathcal{A}^1)$, and hence, again since this spectral sequence approximates $H^*(\mathcal{C}; k^\times)$, we get an injective map $H^1([S(\mathcal{C})]; \mathcal{A}^1) \rightarrow H^2(\mathcal{C}; k^\times)$. The cokernel of this map is the kernel of the differential

$$E_2^{0,2} = \lim_{[S(\mathcal{C})]}(\mathcal{A}^2) \longrightarrow E_2^{2,1} = H^2([S(\mathcal{C})]; \mathcal{A}^1)$$

because all differentials starting at $(0, 2)$ from page 3 onwards are zero by the assumptions. Since from page 3 onwards there is no nonzero differential ending or starting at $(2, 1)$ it follows that the cokernel of the last map is a subgroup of $H^3(\mathcal{C}; k^\times)$. This proves the exactness of the sequence stated in 1.1, and as pointed out earlier, this part of the argument is valid for any abelian group instead of $k^\times$. The groups $\mathcal{A}^1([\sigma]) \cong \text{Hom}(\text{Aut}_{S(\mathcal{C})}(\sigma), k^\times)$ and $\mathcal{A}^2([\sigma]) = H^2(\text{Aut}_{S(\mathcal{C})}(\sigma); k^\times)$ are finite, of order coprime to $\text{char}(k)$ if $\text{char}(k)$ is positive, hence $H^1([S(\mathcal{C})]; \mathcal{A}^1)$ and $\lim_{[S(\mathcal{C})]}(\mathcal{A}^2)$ are finite, proving the finiteness of $H^2(\mathcal{C}; k^\times)$ with the properties as stated. $\square$

4 Proof of Theorem 1.2

The terminology on fusion systems we use follows [17]; in particular, a fusion system means a saturated fusion system in the sense of [3]. See [22, §2] or [3, Appendix] or [17, §3] for details regarding normalisers, centralisers and quotients in fusion systems. For $\mathcal{C}$ a right ideal in a fusion system $\mathcal{F}$ on a finite p -group P denote by $S_{<}(\mathcal{C})$ the full subcategory of all $\sigma : [m] \rightarrow \mathcal{C}$ in $S(\mathcal{C})$ such that for $0 \leq i < j \leq m$ we have $\sigma(i) < \sigma(j)$ and the morphism $\sigma(i < j)$ from $\sigma(i)$ to $\sigma(j)$ is the inclusion map. By [16, 4.2], the subcategory $S_{<}(\mathcal{C})$ is equivalent to $S(\mathcal{C})$. We denote by $S_{\triangleleft}(\mathcal{C})$ the full subcategory of all $\sigma : [m] \rightarrow \mathcal{C}$ in $S_{<}(\mathcal{C})$ such that for $0 \leq i \leq j \leq m$ the subgroup $\sigma(j)$ is normal in $\sigma(i)$, or equivalently, such that $\sigma(i)$ is normal in the maximal subgroup $\sigma(m)$ of the chain of subgroups σ . The proof of 1.2 is based on the following spectral sequence.

Theorem 4.1. *Let $\mathcal{F}$ be a fusion system on a finite p -group P and $\mathcal{C}$ a right ideal in $\mathcal{F}$. Let $\{R_q \mid 0 \leq q \leq r\}$ be a set of representatives of the $\mathcal{F}$ -isomorphism classes of subgroups of P belonging to $\mathcal{C}$ such that R_q is fully $\mathcal{F}$ -normalised for $0 \leq q \leq r$ and such that $|R_q| \geq |R_{q+1}|$ for $1 \leq q \leq r$. Denote by $\mathcal{C}_q$ the right ideal in $N_{\mathcal{F}}(R_q)/R_q$ consisting of all non trivial subgroups of $N_P(R_q)/R_q$, where $1 \leq q \leq r$. Let $\mathcal{A} : [S_{\triangleleft}(\mathcal{C})] \rightarrow \mathbf{Ab}$ be a covariant functor. Then, for $1 \leq q \leq r$, the functor $\mathcal{A}$ induces a covariant functor $\mathcal{N}^q : [S_{\triangleleft}(\mathcal{C}_q)] \rightarrow \mathbf{Ab}$ and there is a spectral sequence*

$$E_1^{p,q} \Rightarrow H^{p+q}([S_{\triangleleft}(\mathcal{C})]; \mathcal{A})$$

with the following properties.

- (i) We have $E_1^{p,q} = H^{p+q-1}([S_{\triangleleft}(\mathcal{C}_q)]; \mathcal{N}^q)$ for $p+q \geq 2$ and $1 \leq q \leq r$.
- (ii) We have $E_1^{p,q} = \{0\}$ if $q < 0$ or $q > r$ or $p+q < 0$, we have $E_1^{p,0} = \{0\}$ for $p \neq 0$, and we have $E_1^{0,0} = \mathcal{A}([P])$.
- (iii) If there is an integer $m \geq 1$ such that $H^m([S_{\triangleleft}(\mathcal{C}_q)]; \mathcal{N}^q) = \{0\}$ for $1 \leq q \leq r$ then $H^{m+1}([S_{\triangleleft}(\mathcal{C})]; \mathcal{A}) = \{0\}$.
- (iv) If $\mathcal{A}$ vanishes on all chains of length zero in $[S_{\triangleleft}(\mathcal{C})]$ then also $E_1^{p,q} = H^0([S_{\triangleleft}(\mathcal{C}_q)]; \mathcal{N}^q)$ for $p+q = 1$ and $1 \leq q \leq r$, and $E_1^{p,q} = \{0\}$ if $p+q = 0$ or $q = 0$.

We break up the proof in several steps. Let $\mathcal{F}$ be a fusion system on a finite p -group P , where p is a prime, and let $\mathcal{C}$ be a right ideal in $\mathcal{F}$. Following the notation in [16, 4.1] the category $S_{\triangleleft}(\mathcal{C})$ is the full subcategory of $S(\mathcal{C})$ whose elements can be denoted as chains

$$\sigma = Q_0 < Q_1 < \cdots < Q_m$$

of subgroups Q_i of P belonging to $\mathcal{C}$ such that Q_i is normal in Q_m for $0 \leq i \leq m$. Chains of this type were introduced in [12]. Given such a chain $\sigma = Q_0 < Q_1 < \cdots < Q_m$ we denote, for any integer i such that $0 \leq i \leq m$, by $\sigma \setminus i$ the chain

$$\sigma \setminus i = Q_0 < \cdots < Q_{i-1} < Q_{i+1} < \cdots < Q_m$$

obtained from removing Q_i . Inclusion of chains yields a canonical morphism $\sigma \setminus i \rightarrow \sigma$ in $S_{\triangleleft}(\mathcal{C})$. For any fully $\mathcal{F}$ -normalised subgroup R of P in $\mathcal{C}$ we denote by $S_{R\triangleleft}(\mathcal{C})$ the full subcategory of $S_{\triangleleft}(\mathcal{C})$ consisting of all chains $\sigma = Q_0 < Q_1 < \dots < Q_m$ with $Q_0 \cong R$ in $\mathcal{F}$ and $m \geq 1$. We denote by $\mathcal{C}_R$ the right ideal in $N_{\mathcal{F}}(R)/R$ consisting of all non-trivial subgroups of $N_P(R)/R$. We keep this notation throughout this section.

Lemma 4.2. *Let $\sigma = (Q_0 < Q_1 < \dots < Q_m)$ be a chain in $S_{\triangleleft}(\mathcal{F})$ and let R be a fully $\mathcal{F}$ -normalised subgroup of P such that $R \cong Q_0$ in $\mathcal{F}$. Then there is a chain $\tau = (R_0 < R_1 < \dots < R_m)$ in $S_{\triangleleft}(\mathcal{F})$ such that $\tau \cong \sigma$ and $R_0 = R$.*

Proof. By standard properties of fusion systems [17, 2.6] there is a morphism $\varphi : N_P(Q_0) \rightarrow P$ such that $\varphi(Q_0) = R$, because R is fully $\mathcal{F}$ -normalised. Taking $\tau = \varphi(\sigma) = (\varphi(Q_0) < \varphi(Q_1) < \dots < \varphi(Q_m))$ proves the lemma. $\square$

Lemma 4.3. *Let $\sigma = (Q_0 < Q_1 < \dots < Q_m)$ and $\tau = (R_0 < R_1 < \dots < R_m)$ be chains in $S_{\triangleleft}(\mathcal{F})$ such that $Q = Q_0 = R_0$ is fully $\mathcal{F}$ -normalised. Suppose that $m > 0$. Then $\sigma \cong \tau$ in $S_{\triangleleft}(\mathcal{F})$ if and only if $\sigma \setminus 0 \cong \tau \setminus 0$ in $S_{\triangleleft}(N_{\mathcal{F}}(Q))$.*

Proof. Any isomorphism $\sigma \cong \tau$ induces an automorphism on $Q = Q_0 = R_0$, and hence an isomorphism $\sigma \setminus 0 \cong \tau \setminus 0$ in $S_{\triangleleft}(N_{\mathcal{F}}(Q))$. Conversely, any isomorphism $\sigma \setminus 0 \cong \tau \setminus 0$ in $S_{\triangleleft}(N_{\mathcal{F}}(Q))$ induces an automorphism on Q , hence an isomorphism $\sigma \cong \tau$. $\square$

Lemma 4.4. *Suppose that $\mathcal{F} = N_{\mathcal{F}}(Q)$ for some normal subgroup Q of P . Let $\sigma = (Q_0 < Q_1 < \dots < Q_m)$ and $\tau = (R_0 < R_1 < \dots < R_m)$ be two chains in $S_{\triangleleft}(\mathcal{F})$ such that $Q \subseteq Q_0$ and $Q \subseteq R_0$. Then the chains $\bar{\sigma} = (Q_0/Q < Q_1/Q < \dots < Q_m/Q)$ and $\bar{\tau} = (R_0/Q < R_1/Q < \dots < R_m/Q)$ belong to the category $S_{\triangleleft}(\mathcal{F}/Q)$, and we have $\sigma \cong \tau$ in $S_{\triangleleft}(\mathcal{F})$ if and only if $\bar{\sigma} \cong \bar{\tau}$ in $S_{\triangleleft}(\mathcal{F}/Q)$.*

Proof. Clearly the chains $\bar{\sigma}, \bar{\tau}$ are in $S_{\triangleleft}(\mathcal{F}/Q)$, and if $\sigma \cong \tau$ then $\bar{\sigma} \cong \bar{\tau}$. Suppose conversely that we have an isomorphism $\bar{\mu} = (\bar{\mu}_i)_{0 \leq i \leq m} : \bar{\sigma} \cong \bar{\tau}$ given by a family of isomorphisms $\bar{\mu}_i : Q_i/Q \cong R_i/Q$ in $\mathcal{F}/Q$. Then any representative μ_m of $\bar{\mu}_m$ is an isomorphism $Q_m \cong R_m$ which sends Q_i to R_i , for $0 \leq i \leq m$, and hence setting $\mu_i = \mu_m|_{Q_i}$ yields an isomorphism $\sigma \cong \tau$. $\square$

Proposition 4.5. *Let R be a fully $\mathcal{F}$ -normalised subgroup of P . The map sending a chain $\sigma = (Q_0 < Q_1 < \dots < Q_m)$ of positive length m in $S_{R\triangleleft}(\mathcal{C})$ to the chain $\bar{\sigma} \setminus 0 = (Q_1/R < Q_2/R < \dots < Q_m/R)$ in $S_{\triangleleft}(\mathcal{C}_R)$ induces an isomorphism of posets $[S_{R\triangleleft}(\mathcal{C})] \cong [S_{\triangleleft}(\mathcal{C}_R)]$.*

Proof. This follows from combining the three previous Lemmas. $\square$

Proof of Theorem 4.1. We use the notation from the statement of 4.1; in particular, $\{R_q \mid 0 \leq q \leq r\}$ is a system of representatives of the isomorphism classes in $\mathcal{C}$ with all R_q fully $\mathcal{F}$ -normalised and ordered in such a way that $|R_q| \geq |R_{q+1}|$ for $0 \leq q < r$. Note that since $\mathcal{C}$ is a right ideal in $\mathcal{F}$ this implies in particular that $R_0 = P$. By [16, 3.7] applied to the left ideal $[S_{\triangleleft}(\mathcal{C})]$ in $[S(\mathcal{C})]$, the cohomology of $\mathcal{A}$ is that of a cochain complex of abelian groups $C_{\triangleleft} = C_{\triangleleft}(\mathcal{A})$ which is in degree $n \geq 0$ equal to

$$C_{\triangleleft}^n = \oplus_{[\sigma] \in [S_{\triangleleft}(\mathcal{C})], |\sigma|=n} \mathcal{A}([\sigma]) .$$

The differential of this complex is an alternating sum of maps $\mathcal{A}([\sigma] < [\tau]) : \mathcal{A}([\sigma]) \rightarrow \mathcal{A}([\tau])$, where $[\sigma], [\tau]$ are isomorphism classes of chains σ, τ in $S_{\triangleleft}(\mathcal{C})$ such that $[\sigma] < [\tau]$ and such that $|\sigma| + 1 = |\tau|$. For any integer q such that $0 \leq q \leq r$ we define a full subcategory $S_{\triangleleft}^{(q)}(\mathcal{C})$ of $S_{\triangleleft}(\mathcal{C})$ as follows: a chain $\sigma = (Q_0 < Q_1 < \dots < Q_m)$ in $S_{\triangleleft}(\mathcal{C})$ belongs to $S_{\triangleleft}^{(q)}(\mathcal{C})$ if and only if $Q_0 \cong R_j$ for some $j \geq q$. The subcategory $S_{\triangleleft}^{(q)}(\mathcal{C})$ is in fact a right ideal in $S_{\triangleleft}(\mathcal{C})$: if there is a morphism from a chain

$\sigma = (Q_0 < Q_1 < \cdots < Q_m)$ in $S_{\triangleleft}^{(q)}(\mathcal{C})$ to a chain $\sigma' = (Q'_0 < Q'_1 < \cdots < Q'_{m'})$ in $S_{\triangleleft}(\mathcal{C})$ then in particular either $Q_0 \cong Q'_0$ or $|Q'_0| < |Q_0|$. Since $Q_0 \cong R_j$ for some $j \geq q$ this implies $Q'_0 \cong R_{j'}$ for some $j' \geq j$. Thus σ' belongs to the category $S_{\triangleleft}^{(q)}(\mathcal{C})$ as well. It follows that for $0 \leq q \leq r$ there is a subcomplex $C_{\triangleleft}^{(q)}$ of $C_{\triangleleft}$ defined by

$$(C_{\triangleleft}^{(q)})^n = \bigoplus_{\sigma \in [S_{\triangleleft}^{(q)}(\mathcal{C})], |\sigma|=n} \mathcal{A}([\sigma])$$

for $n \geq 0$. Since the category $S_{\triangleleft}(\mathcal{C})$ is filtered by the subcategories

$$S_{\triangleleft}^{(r)}(\mathcal{C}) \subseteq S_{\triangleleft}^{(r-1)}(\mathcal{C}) \subseteq \cdots \subseteq S_{\triangleleft}^{(0)}(\mathcal{C}) = S_{\triangleleft}(\mathcal{C})$$

it follows that the cochain complex $C_{\triangleleft}$ has a filtration of the form

$$\{0\} \subseteq C_{\triangleleft}^{(r)} \subseteq C_{\triangleleft}^{(r-1)} \subseteq \cdots \subseteq C_{\triangleleft}^{(0)} = C_{\triangleleft}.$$

For notational convenience we set $C_{\triangleleft}^{(q)} = 0$ for $q > r$. By [19, 2.6], the spectral sequence associated by this filtration takes the form

$$E_1^{p,q} = H^{p+q}(C_{\triangleleft}^{(q)}/C_{\triangleleft}^{(q+1)}) \Rightarrow H^{p+q}(C_{\triangleleft}).$$

The quotient complexes of this filtration are zero unless $0 \leq q \leq r$, and they vanish in negative degrees; thus we get $E_1^{p,q} = \{0\}$ for $q < 0$ or $q > r$ or $p+q < 0$ as claimed in (ii). The right side in this spectral sequence is $H^{p+q}(C_{\triangleleft}) = H^{p+q}([S_{\triangleleft}(\mathcal{C})]; \mathcal{A})$ by [16, 3.7] applied to the left ideal $[S_{\triangleleft}(\mathcal{C})]$ in $[S(\mathcal{C})]$. We need to identify $E_1^{p,q}$ in the remaining cases. For $1 \leq q \leq r$ we set $\mathcal{D}_q = S_{R_q \triangleleft}(\mathcal{C})$, the full subcategory of $S_{\triangleleft}(\mathcal{C})$ consisting of all chains $\sigma = (Q_0 < Q_1 < \cdots < Q_m)$ of positive length m such that $Q_0 \cong R_q$. By convention, $\mathcal{D}_0$ is the empty category. That is, on object sets, we have a disjoint union of full subcategories

$$S_{\triangleleft}^{(q)}(\mathcal{C}) = S_{\triangleleft}^{(q+1)}(\mathcal{C}) \cup \mathcal{D}_q \cup [R_q].$$

Thus, for $0 \leq q \leq r$, the quotient complex

$$C_{\triangleleft}^{(q)}/C_{\triangleleft}^{(q+1)}$$

is equal to

$$(C_{\triangleleft}^{(q)}/C_{\triangleleft}^{(q+1)})^n \cong \bigoplus_{[\sigma] \in [\mathcal{D}_q], |\sigma|=n} \mathcal{A}([\sigma])$$

for $n > 0$, and

$$(C_{\triangleleft}^{(q)}/C_{\triangleleft}^{(q+1)})^0 \cong \mathcal{A}([R_q])$$

with $[R_q]$ viewed as isomorphism class of chains of length zero. Note that by our conventions, for $q = 0$ this quotient complex is concentrated in degree zero, where it is equal to $\mathcal{A}([P])$. Thus in particular $E_1^{p,0} = \{0\}$ for $p \neq 0$ and $E_1^{0,0} = \mathcal{A}([P])$ which completes the proof of (ii). Set $\mathcal{C}_q = \mathcal{C}_{R_q}$ for $1 \leq q \leq r$; that is, $\mathcal{C}_q$ is the full subcategory of $N_{\mathcal{F}}(R_q)/R_q$ consisting of all non trivial subgroups of $N_P(R_q)/R_q$; by convention, $\mathcal{C}_0$ is empty. By Proposition 4.5 we have an isomorphism of posets $[\mathcal{D}_q] \cong [S_{\triangleleft}(\mathcal{C}_{R_q})]$, where $1 \leq q \leq r$. This isomorphism induces a functor

$$\mathcal{N}^q : [S_{\triangleleft}(\mathcal{C}_q)] \longrightarrow \mathbf{Ab}$$

such that

$$\mathcal{N}^q([Q_1/R_q < Q_2/R_q < \cdots < Q_m/R_q]) = \mathcal{A}([R_q < Q_1 < Q_2 < \cdots < Q_m]),$$

where $Q_1 < Q_2 < \cdots < Q_m$ is a chain in $S_{\triangleleft}(N_{\mathcal{F}}(R_q))$ with $R_q < Q_1$ and $m \geq 1$. Note that the isomorphism of posets $[\mathcal{D}_q] \cong [S_{\triangleleft}(\mathcal{C}_q)]$ sends a chain of positive length m to a chain of length $m-1$;

this accounts for the degree shift in the statement of 4.1. Thus we can reformulate our description of the quotient complex $C_{\triangleleft}^{(q)}/C_{\triangleleft}^{(q+1)}$ as follows: in degree $n > 0$ we have

$$(C_{\triangleleft}^{(q)}/C_{\triangleleft}^{(q+1)})^n \cong \bigoplus_{[\tau] \in [S_{\triangleleft}(\mathcal{C}_q)], |\tau|=n-1} \mathcal{N}^q([\tau]),$$

(which by our conventions is zero if $q = 0$ and $n > 0$) and degree zero term equal to $\mathcal{A}([R_q])$, as pointed out before. It follows again from the description of the cohomology of posets associated with subdivisions in [16, §3] that for $n > 0$ we have

$$(C_{\triangleleft}^{(q)}/C_{\triangleleft}^{(q+1)})^n \cong C_{\triangleleft}(\mathcal{N}^q)^{n-1},$$

where $C_{\triangleleft}(\mathcal{N}^q)$ is the complex which computes the cohomology of the functor $\mathcal{N}^q$ as in [16, 3.1], with $\mathcal{C}_q$ instead of $\mathcal{C}$. In other words, $(C_{\triangleleft}^{(q)}/C_{\triangleleft}^{(q+1)})$ is the mapping cone of a chain map of the form $\mathcal{A}([R_q]) \rightarrow C_{\triangleleft}(\mathcal{N}^q)$, with $\mathcal{A}([R_q])$ viewed as complex concentrated in degree zero. In particular, we have

$$H^n(C_{\triangleleft}^{(q)}/C_{\triangleleft}^{(q+1)}) \cong H^{n-1}([S_{\triangleleft}(\mathcal{C}_q)]; \mathcal{N}^q)$$

for $n \geq 2$ and $1 \leq q \leq r$. This completes the proof of (i). Furthermore, if all functors $\mathcal{N}^q$, for $1 \leq q \leq r$, have vanishing cohomology in a fixed positive degree m , then the cohomology of $C_{\triangleleft}$ vanishes in degree $m + 1$. Statement (iii) follows. Finally, if $\mathcal{A}$ vanishes on chains of length zero, then

$$C_{\triangleleft}^{(q)}/C_{\triangleleft}^{(q+1)} = C_{\triangleleft}(\mathcal{N}^q)[-1]$$

where $0 \leq q \leq r$, and so in this case we get that

$$H^n(C_{\triangleleft}^{(q)}/C_{\triangleleft}^{(q+1)}) \cong H^{n-1}([S_{\triangleleft}(\mathcal{C}_q)]; \mathcal{N}^q)$$

for all integers n . This completes the proof of (iv). $\square$

Proof of Theorem 1.2. Let $\mathcal{F}$ be a fusion system on a finite p -group P , set $\mathcal{C} = \mathcal{F}^c$ and denote by $\{R_q \mid 0 \leq q \leq r\}$ a system of representatives of the isomorphism classes in $\mathcal{C}$ with all R_q fully $\mathcal{F}$ -normalised and ordered in such a way that $|R_q| \geq |R_{q+1}|$ for $0 \leq q < r$. The plan is to apply 4.1 to the functor $\mathcal{A}^1$ restricted to $S_{\triangleleft}(\mathcal{C})$. Note that by [16, 4.7, 4.11] we have an isomorphism

$$H^*([S(\mathcal{C})]; \mathcal{A}^1) \cong H^*([S_{\triangleleft}(\mathcal{C})]; \mathcal{A}^1)$$

where we use the same notation $\mathcal{A}^1$ for the restriction of $\mathcal{A}^1$ to $S_{\triangleleft}(\mathcal{C})$. Note further that $\mathcal{A}^1([\sigma]) = H^1(\text{Aut}_{S(\mathcal{C})}(\sigma); k^\times) = \text{Hom}(\text{Aut}_{S(\mathcal{C})}; k^\times)$ for any σ in $S(\mathcal{C})$. Similarly, for any finite group G the cohomology of the functor $\mathcal{N}_G$ on $[\Delta_p(G)]$ is invariant under restriction to the subset of normal chains in $\Delta_p(G)$. Moreover, $[\Delta_p(G)]$ is isomorphic, as partially ordered set, to $[S(\mathcal{F}_S(G))]$, where S is a Sylow- p -subgroup of G . By [1, Prop. C], for $0 \leq q \leq r$ there is a finite group L_q such that L_q has $N_P(R_q)$ as Sylow- p -subgroup, $R_q = O_p(L_q)$, $C_{L_q}(R_q) = Z(R_q)$ and $N_{\mathcal{F}}(R_q) = \mathcal{F}_{R_q}(L_q)$. Thus $N_{\mathcal{F}}(R_q)/R_q$ is the fusion system of the finite group $L_q/R_q \cong \text{Aut}_{\mathcal{F}}(R_q)/\text{Inn}(R_q)$ on $N_P(R_q)/R_q$. It follows that if $\mathcal{C}_q$ is the right ideal in $N_{\mathcal{F}}(R_q)/R_q$ as defined in the statement of 4.1, then $S_{\triangleleft}(\mathcal{C}_q)$ can be identified with the partially ordered subset of normal chains in $\Delta_p(L_q/R_q)$, and the cohomology of the functor $\mathcal{N}_{L_q/R_q}$ remains invariant under restriction to $[S_{\triangleleft}(\mathcal{C}_q)]$, and this restriction coincides with the functor $\mathcal{N}^q$ in the statement of 4.1. Thus statement (iii) in 4.1 implies Theorem 1.2. $\square$

5 Regular functors

Definition 5.1. Let $\mathcal{C}, \mathcal{D}$ be *EI*-categories. A covariant functor $\Phi : \mathcal{C} \rightarrow \mathcal{D}$ is called *regular* if Φ induces an isomorphism $[\mathcal{C}] \cong [\mathcal{D}]$, for any two objects X, Y in $\mathcal{C}$ the map from $\text{Hom}_{\mathcal{C}}(X, Y)$ to $\text{Hom}_{\mathcal{D}}(\Phi(X), \Phi(Y))$ induced by Φ is surjective, and for any two objects X, Y in $\mathcal{C}$ such that $\text{Hom}_{\mathcal{C}}(X, Y)$ is non-empty, the group $K(X) = \ker(\text{Aut}_{\mathcal{C}}(X) \rightarrow \text{Aut}_{\mathcal{D}}(\Phi(X)))$ acts freely on $\text{Hom}_{\mathcal{C}}(X, Y)$ through composition of morphisms and induces a bijection

$$\text{Hom}_{\mathcal{C}}(X, Y)/K(X) \cong \text{Hom}_{\mathcal{D}}(\Phi(X), \Phi(Y)) .$$

If $\mathcal{C} \rightarrow \mathcal{D}$ is an extension of $\mathcal{D}$ by a functor $\mathcal{Z} : \mathcal{D}^{op} \rightarrow \mathbf{Ab}$ then the structural functor $\mathcal{C} \rightarrow \mathcal{D}$ is regular. Clearly an *EI*-category is regular if and only if the canonical functor $\mathcal{C} \rightarrow [\mathcal{C}]$ is regular. The following lemma on lifting commutative diagrams through regular functors is used below to show that regular functors induce regular functors on subdivisions.

Lemma 5.2. *Let $\mathcal{C}, \mathcal{D}$ be *EI*-categories and $\Phi : \mathcal{C} \rightarrow \mathcal{D}$ a regular functor. If*

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \mu \downarrow & & \downarrow \nu \\ V & \xrightarrow{\psi} & W \end{array}$$

is a diagram in $\mathcal{C}$ such that the diagram in $\mathcal{D}$

$$\begin{array}{ccc} \Phi(X) & \xrightarrow{\Phi(\varphi)} & \Phi(Y) \\ \Phi(\mu) \downarrow & & \downarrow \Phi(\nu) \\ \Phi(V) & \xrightarrow{\Phi(\psi)} & \Phi(W) \end{array}$$

is commutative, then there is a unique automorphism $\rho \in \text{Aut}_{\mathcal{C}}(X)$ such that $\Phi(\rho) = \text{Id}_{\Phi(X)}$ and such that the diagram in $\mathcal{C}$

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \mu \circ \rho \downarrow & & \downarrow \nu \\ V & \xrightarrow{\psi} & W \end{array}$$

is commutative.

Proof. The morphisms $\nu \circ \varphi$ and $\psi \circ \mu$ are two morphisms from X to W in $\mathcal{C}$ whose images in the morphism set $\text{Hom}_{\mathcal{D}}(\Phi(X), \Phi(W))$ are equal. Since the kernel $K(X)$ of the map $\text{Aut}_{\mathcal{C}}(X) \rightarrow \text{Aut}_{\mathcal{D}}(\Phi(X))$ acts freely on $\text{Hom}_{\mathcal{C}}(X, Y)$, inducing a bijection $\text{Hom}_{\mathcal{C}}(X, W)/K(X) \cong \text{Hom}_{\mathcal{D}}(\Phi(X), \Phi(W))$ there is a unique $\rho \in K(X)$ with the required property. $\square$

Proposition 5.3. *Let $\mathcal{C}, \mathcal{D}$ be *EI*-categories and $\Phi : \mathcal{C} \rightarrow \mathcal{D}$ a regular functor. Then Φ induces a regular functor $S(\mathcal{C}) \rightarrow S(\mathcal{D})$.*

Proof. There is an obvious functor $S(\mathcal{C}) \rightarrow S(\mathcal{D})$ sending an object $\sigma : [m] \rightarrow \mathcal{C}$ in $S(\mathcal{C})$ to the object $\Phi \circ \sigma : [m] \rightarrow \mathcal{D}$ in $S(\mathcal{D})$. Set $\bar{\sigma} = \Phi \circ \sigma$ and $|\sigma| = m$. Let $\tau : n \rightarrow \mathcal{C}$ be another object in $S(\mathcal{C})$. We

show that the map $\text{Hom}_{S(\mathcal{C})}(\sigma, \tau) \rightarrow \text{Hom}_{S(\mathcal{D})}(\bar{\sigma}, \bar{\tau})$ induced by Φ is surjective; we proceed by induction over $|\sigma|$. For $|\sigma| = 0$ this follows from the regularity of the functor $\Phi : \mathcal{C} \rightarrow \mathcal{D}$. Suppose that $|\sigma| = m$ is positive; denote by σ' the object in $S(\mathcal{C})$ obtained by deleting $\sigma(0)$. Then $|\sigma'| = m-1$, hence by induction, the map from $\text{Hom}_{S(\mathcal{C})}(\sigma', \tau)$ to $\text{Hom}_{S(\mathcal{D})}(\bar{\sigma}', \bar{\tau})$ is surjective. Using the previous lemma one sees that the map from $\text{Hom}_{S(\mathcal{C})}(\sigma, \tau)$ to $\text{Hom}_{S(\mathcal{D})}(\bar{\sigma}, \bar{\tau})$ is surjective as well. The rest is an easy verification. $\square$

Lemma 5.4. *Let $\mathcal{C}, \mathcal{D}$ be EI-categories and $\Phi : \mathcal{C} \rightarrow \mathcal{D}$ a regular functor. Let X be an object in $\mathcal{C}$ and set $Y = \Phi(X)$.*

- (i) *The partially ordered set $[\Phi^Y]$ has $[(X, \text{Id}_Y)]$ as unique maximal element.*
- (ii) *We have $\text{Aut}_{\Phi^Y}(X, \text{Id}_Y) = \ker(\text{Aut}_{\mathcal{C}}(X) \rightarrow \text{Aut}_{\mathcal{D}}(Y))$.*

Proof. For (i), let (Z, ψ) be an object in Φ^Y ; that is, Z is an object in $\mathcal{C}$ and $\psi : \Phi(Z) \rightarrow Y$ is a morphism in $\mathcal{D}$. Since Φ is regular, there is a morphism $\beta : Z \rightarrow X$ such that $\Phi(\beta) = \psi = \text{Id}_Y \circ \psi$. Thus β is a morphism in Φ^Y from (Z, ψ) to (X, Id_Y) . In other words, $[(Z, \psi)] \leq [(X, \text{Id}_Y)]$ for any object (Z, ψ) in Φ^Y . This proves (i), and (ii) is trivial. $\square$

Lemma 5.5. *Let $\mathcal{C}, \mathcal{D}$ be EI-categories and $\Phi : \mathcal{C} \rightarrow \mathcal{D}$ a regular functor. Let X be an object in $\mathcal{C}$ and set $Y = \Phi(X)$. Suppose that every morphism in $\mathcal{C}$ is a monomorphism.*

- (i) *The category Φ^Y is regular.*
- (ii) *For any object (Z, ψ) in Φ^Y we have $\text{Aut}_{\Phi^Y}(Z, \psi) = \ker(\text{Aut}_{\mathcal{C}}(Z) \rightarrow \text{Aut}_{\mathcal{D}}(\Phi(Z)))$.*

Proof. Let $\alpha, \beta : (Z, \psi) \rightarrow (Z', \psi')$ be two morphisms in Φ^Y . That is, Z, Z' are objects in $\mathcal{C}$ and $\psi \in \text{Hom}_{\mathcal{D}}(\Phi(Z), Y)$, $\psi' \in \text{Hom}_{\mathcal{D}}(\Phi(Z'), Y)$ such that $\psi' \circ \Phi(\alpha) = \psi = \psi' \circ \Phi(\beta)$. Since Φ is regular there are morphisms $\varphi \in \text{Hom}_{\mathcal{C}}(Z, X)$, $\varphi' \in \text{Hom}_{\mathcal{C}}(Z', X)$ such that $\Phi(\varphi) = \psi$ and $\Phi(\varphi') = \psi'$. Then $\Phi(\varphi' \circ \alpha) = \Phi(\varphi) = \Phi(\varphi' \circ \beta)$. The regularity of Φ implies that there is a unique automorphism $\gamma \in \text{Aut}_{\mathcal{C}}(X)$ such that $\varphi' \circ \alpha \circ \gamma = \varphi' \circ \beta$ and such that $\Phi(\gamma) = \text{Id}_Y$. Since φ' is a monomorphism we get that $\alpha \circ \gamma = \beta$. It follows that γ is the unique automorphism of (Z, ψ) in Φ^Y satisfying $\alpha \circ \gamma = \beta$. This proves (i). When applied to the case $(Z, \psi) = (Z', \psi')$ and $\beta = \text{Id}_Z$ this argument proves (ii). $\square$

Theorem 5.6. *Let $\mathcal{C}, \mathcal{D}$ be EI-categories and $\Phi : \mathcal{C} \rightarrow \mathcal{D}$ a regular functor. Suppose that every morphism in $\mathcal{C}$ is a monomorphism. Let A be an abelian group and for any object X in $\mathcal{C}$ denote by $K(X)$ the kernel of the group homomorphism $\text{Aut}_{\mathcal{C}}(X) \rightarrow \text{Aut}_{\mathcal{D}}(\Phi(X))$ induced by Φ . Suppose that $H^q(K(X); A) = \{0\}$ for $q \geq 0$ and any object X in $\mathcal{C}$, with respect to the trivial action of $K(X)$ on A . Then Φ induces an isomorphism $H^*(\mathcal{C}; A) \cong H^*(\mathcal{D}; A)$.*

Proof. Let Y be an object in $\mathcal{D}$. By 5.5, the category Φ^Y is regular, and by 5.4, the partially ordered set $[\Phi^Y]$ has a unique maximal element. By 3.1 applied to Φ^Y there is a spectral sequence of the form

$$H^p([\Phi^Y]; [(Z, \psi)]) \mapsto H^q(\text{Aut}_{\Phi^Y}(Z, \psi); A) \Rightarrow H^{p+q}(\Phi^Y; A)$$

Now $\text{Aut}_{\Phi^Y}(Z, \psi) = K(Z)$ by 5.5, hence the assumptions imply that this spectral sequence collapses to an isomorphism $H^p([\Phi^Y]; A) \cong H^p(\Phi^Y; A)$. But this group is zero for q positive as $[\Phi^Y]$ has unique maximal element. This shows that $R^q\Phi_!(A) = 0$ for q positive, and hence the base change spectral sequence of Φ collapses to an isomorphism $H^*(\mathcal{C}; A) \cong H^*(\mathcal{D}; A)$ as stated. $\square$

6 Further invariance properties of $H^*(\mathcal{C}; k^\times)$

Let p be a prime. A fusion system $\mathcal{F}$ of a block with defect group P of a finite p -solvable group G is, by a result of Puig, always the fusion system of a finite group L having P as Sylow- p -subgroup such that $Q = O_p(L)$ is an $\mathcal{F}$ -centric subgroup of P . The results of this section, besides providing some reduction techniques for calculating $H^*(\mathcal{F}^c; k^\times)$, imply that $H^2(\mathcal{F}^c; k^\times)$ is the p' -part of the Schur multiplier of L in that case (see 6.6 below). It is well-known that any element α in the p' -part of the Schur multiplier of L arises as Külshammer-Puig 2-cocycle of the fusion system $\mathcal{F}$ at Q in a suitable block of a finite central p' -extension $\hat{L}$ of L determined by α . We first show the invariance of $H^*(\mathcal{F}^c; k^\times)$ with respect to taking quotients by central p -subgroups. If $\mathcal{F}$ is a fusion system on a finite p -group P and Z is a subgroup of $Z(P)$ then the equality $\mathcal{F} = C_{\mathcal{F}}(Z)$ means that every morphism $\varphi : Q \rightarrow R$ in $\mathcal{F}$ can be extended to a morphism $\psi : QZ \rightarrow RZ$ in $\mathcal{F}$ such that $\psi|_Z = \text{Id}_Z$. If $\mathcal{F} = C_{\mathcal{F}}(Z)$ then $\mathcal{F}$ induces a fusion system on P/Z , denoted by $\mathcal{F}/Z$. Given a fusion system $\mathcal{F}$ on a finite p -group P , we denote as before by $\mathcal{F}^c$ the full subcategory in $\mathcal{F}$ of all $\mathcal{F}$ -centric subgroups of P ; this is a right ideal in $\mathcal{F}$. We denote by $\bar{\mathcal{F}}$ the quotient category of $\mathcal{F}$ having as objects the subgroups of P and as morphism sets the orbits $\text{Aut}_R(R) \backslash \text{Hom}_{\mathcal{F}}(Q, R)$ of the morphism set $\text{Hom}_{\mathcal{F}}(Q, R)$ with respect to the action of the group of inner automorphisms $\text{Aut}_R(R)$ of R via composition of group homomorphisms, called the *orbit category* of $\mathcal{F}$. Denote by $\bar{\mathcal{F}}^c$ the image of $\mathcal{F}^c$ in $\bar{\mathcal{F}}$. While all morphisms in $\mathcal{F}$ are monomorphisms, this is no longer true in $\bar{\mathcal{F}}$; however, all morphisms in $\bar{\mathcal{F}}^c$ are epimorphisms. For the centric linking system $\mathcal{L}$ of $\mathcal{F}$ we refer to [3, 1.7]. The following invariance properties are essentially consequences of 5.6.

Proposition 6.1. *Let $\mathcal{F}$ be a fusion system on a finite p -group P such that $\mathcal{F} = C_{\mathcal{F}}(Z)$ for some subgroup Z of $Z(P)$ and let $\mathcal{C}$ be a right ideal in $\mathcal{F}$ consisting of subgroups of P containing Z such that the image $\mathcal{C}/Z$ of $\mathcal{C}$ in $\mathcal{F}/Z$ is contained in $(\mathcal{F}/Z)^c$. Let k be an algebraically closed field of characteristic p . The canonical functor $\mathcal{C} \rightarrow \mathcal{C}/Z$ is regular and induces an isomorphism $H^*(\mathcal{C}; k^\times) \cong H^*(\mathcal{C}/Z; k^\times)$.*

Proof. Let Q, R be subgroups of P belonging to $\mathcal{C}$. Then Z is contained in Q, R by the assumptions. The morphism set $\text{Hom}_{\mathcal{F}/Z}(Q/Z, R/Z)$ is the canonical image of the morphism set $\text{Hom}_{\mathcal{F}}(Q, R)$; in particular, the canonical functor $\mathcal{C} \rightarrow \mathcal{C}/Z$ is surjective on morphisms. This implies also that if $Q/Z \cong R/Z$ in $\mathcal{C}/Z$ then $Q \cong R$ in $\mathcal{C}$, and hence we have the isomorphism of posets $[\mathcal{C}] \cong [\mathcal{C}/Z]$. If $\varphi, \psi : Q \rightarrow R$ are two morphisms in $\mathcal{C}$ whose images in $\mathcal{C}/Z$ are equal, then in particular, $\varphi(Q) = \psi(Q) \subseteq R$, and with the obvious abuse of notation we get an automorphism $\kappa = \psi^{-1} \circ \varphi$ of Q whose image in $\mathcal{C}/Z$ is the identity on Q/Z . Thus κ is the unique element of the group $K(Q) = \ker(\text{Aut}_{\mathcal{F}}(Q) \rightarrow \text{Aut}_{\mathcal{F}/Z}(Q/Z))$ satisfying $\varphi = \psi \circ \kappa$. This shows that the canonical functor $\mathcal{C} \rightarrow \mathcal{C}/Z$ is regular. Moreover, the group $K(Q)$ is an abelian p -group and since k is algebraically closed of characteristic p this implies that $H^q(K(Q); k^\times) = \{0\}$ for all $q > 0$. Thus 5.6 applies, proving the Theorem. $\square$

Proposition 6.2. *Let $\mathcal{F}$ be a fusion system on a finite group P , let $\mathcal{C}$ be a right ideal in $\mathcal{F}^c$ and let $\bar{\mathcal{C}}$ be the canonical image of $\mathcal{C}$ in the orbit category $\bar{\mathcal{F}}$. Let k be an algebraically closed field of characteristic p . The canonical functor $\mathcal{C} \rightarrow \bar{\mathcal{C}}$ induces a regular functor $S(\mathcal{C}) \rightarrow S(\bar{\mathcal{C}})$ and an isomorphism $H^*(\mathcal{C}; k^\times) \cong H^*(\bar{\mathcal{C}}; k^\times)$.*

Proof. Two subgroups Q, P of P are isomorphic in $\mathcal{F}$ if and only if they are isomorphic in $\bar{\mathcal{F}}$. Thus the canonical functor $\mathcal{C} \rightarrow \bar{\mathcal{C}}$ sends non-isomorphisms to non-isomorphisms, hence induces a functor $S(\mathcal{C}) \rightarrow S(\bar{\mathcal{C}})$, which in turn induces an isomorphism of partially ordered sets $[S(\mathcal{C})] \cong [S(\bar{\mathcal{C}})]$. Let $\sigma : [m] \rightarrow \mathcal{C}$ and $\tau : [n] \rightarrow \mathcal{C}$ be objects in $S(\mathcal{C})$ and denote by $\bar{\sigma}, \bar{\tau}$ their images in $S(\bar{\mathcal{C}})$. Let $(\alpha, \bar{\mu}) : \bar{\sigma} \rightarrow \bar{\tau}$ be a morphism in $S(\bar{\mathcal{C}})$; that is, $\alpha : [m] \rightarrow [n]$ is an order preserving map and $\bar{\mu} : \bar{\sigma} \cong \bar{\tau} \circ \alpha$ is a natural isomorphism. Explicitly, $\bar{\mu}$ consists of a compatible family of group isomorphisms $\bar{\mu}_i : \bar{\sigma}(i) \cong \bar{\tau}(\alpha(i))$; that is, for $0 \leq i < m$ we have $\bar{\mu}_{i+1} \circ \bar{\sigma}(i < i+1) = \bar{\tau}(\alpha(i) < \alpha(i+1)) \circ \bar{\mu}_i$. Since the functor $\mathcal{C} \rightarrow \bar{\mathcal{C}}$

is surjective on morphisms, there are group isomorphisms $\mu_i : \sigma(i) \rightarrow \tau(\alpha(i))$ in $\mathcal{C}$, but this family need not be a natural transformation. More precisely, for $0 \leq i < m$ there is a group element $v_i \in \tau(\alpha(i))$ such that if we denote by c_i the inner automorphism of $\tau(\alpha(i))$ given by conjugation with v_i , we have $c_{i+1} \circ \mu_{i+1} \circ \sigma(i < i+1) = \tau(\alpha(i) < \alpha(i+1)) \circ \mu_i$. An easy inductive argument shows that after replacing μ_{i+1} by $c_{i+1} \circ \mu_{i+1}$ we get a natural isomorphism $\mu : \sigma \cong \tau \circ \alpha$ lifting $\bar{\mu}$, which shows that the functor $S(\mathcal{C}) \rightarrow S(\bar{\mathcal{C}})$ is surjective on morphisms. Finally, if μ, μ' are two morphisms in $S(\mathcal{C})$ from σ to τ then by the regularity of $S(\mathcal{C})$ there is a unique automorphism β of σ such that $\mu' = \mu \circ \beta$. Thus, if in addition the images $\bar{\mu}, \bar{\mu}'$ of μ, μ' in $S(\bar{\mathcal{C}})$ are equal then the image $\bar{\beta}$ in $S(\bar{\mathcal{C}})$ satisfies $\mu = \mu \circ \beta$. Since every morphism in $S(\bar{\mathcal{C}})$ is a monomorphism this implies that $\bar{\beta} = \text{Id}_{\bar{\sigma}}$, and hence β belongs to the kernel $K(\sigma)$ of the canonical map from $\text{Aut}_{S(\mathcal{C})}(\sigma)$ to $\text{Aut}_{S(\bar{\mathcal{C}})}(\bar{\sigma})$, which shows that indeed the functor $S(\mathcal{C}) \rightarrow S(\bar{\mathcal{C}})$ is regular. Moreover, by standard properties of central group extensions, the group $K(\sigma)$ is a p -group. It follows that $H^q(K(\sigma); k^\times) = 0$ for positive q . Thus 5.6 implies $H^*(S(\mathcal{C}); k^\times) \cong H^*(S(\bar{\mathcal{C}}); k^\times)$, whence the stated isomorphism $H^*(\mathcal{C}; k^\times) \cong H^*(\bar{\mathcal{C}}; k^\times)$. $\square$

Proposition 6.3. *Let $\mathcal{F}$ be a fusion system on a finite p -group P having a centric linking system $\mathcal{L}$ with structural functor $\pi : \mathcal{L} \rightarrow \mathcal{F}^c$. Let $\mathcal{C}$ be a right ideal in $\mathcal{L}$ and let $\mathcal{D}$ be its image in $\mathcal{F}$ under π . Let k be an algebraically closed field of characteristic p . The functor π induces an isomorphism $H^*(\mathcal{C}; k^\times) \cong H^*(\mathcal{D}; k^\times)$.*

Proof. The centric linking system $\mathcal{L}$ is an extension of $\mathcal{F}^c$ by the centre functor $Z : \mathcal{F}^c \rightarrow \mathbb{Z}_{(p)}$ sending an $\mathcal{F}$ -centric subgroup Q of P to its centre $Z(Q)$. Thus π induces a regular functor $\mathcal{C} \rightarrow \mathcal{D}$ and for any $\mathcal{F}$ -centric subgroup of P we have $K(Q) = \ker(\text{Aut}_{\mathcal{L}}(Q) \rightarrow \text{Aut}_{\mathcal{F}}(Q)) \cong Z(Q)$, hence $H^q(K(Q); k^\times) = \{0\}$ for $q > 0$. Thus 5.6 applies and yields the isomorphism as stated. $\square$

Proposition 6.4. *Let $\mathcal{F}$ be a fusion system on a finite p -group P , let Q be a normal subgroup of P such that $\mathcal{F} = N_{\mathcal{F}}(Q)$ and let $\mathcal{C} = \mathcal{F}_{\geq Q}$ be the right ideal in $\mathcal{F}$ consisting of all subgroups of P containing Q . Let A be an abelian group. Restriction induces an isomorphism $H^*(\mathcal{C}; A) \cong H^*(\text{Aut}_{\mathcal{F}}(Q); A)$.*

Proof. Let $\mathcal{D}$ be the full subcategory of $\mathcal{C}$ having Q as unique object and let $\Phi : \mathcal{D} \rightarrow \mathcal{C}$ be the inclusion functor. For any subgroup R belonging to $\mathcal{C}$ denote by $\iota_R : Q \subseteq R$ the inclusion homomorphism. For any morphism $\varphi : Q \rightarrow R$ we have $\varphi(Q) = Q$ because $\mathcal{F} = N_{\mathcal{F}}(Q)$. Thus composition with the inclusion morphism ι_R induces a bijection $\text{Aut}_{\mathcal{F}}(Q) \cong \text{Hom}_{\mathcal{F}}(Q, R)$. It follows from 3.2 that Φ has a right adjoint $\Psi : \mathcal{C} \rightarrow \mathcal{D}$. Then Ψ^* is right adjoint to Φ^* ; in other words, $\Phi_! = \Psi^*$ is exact, and hence we have $H^*(\mathcal{D}; A) = H^*(\mathcal{C}; \Psi^*(A)) = H^*(\mathcal{C}; A)$ by [9, 3.1]. $\square$

Proposition 6.5. *Let $\mathcal{F}$ be a fusion system on a finite p -group P , let Q be an $\mathcal{F}$ -centric normal subgroup of P such that $\mathcal{F} = N_{\mathcal{F}}(Q)$ and let $\mathcal{C} = \mathcal{F}_{\geq Q}$ be the right ideal in $\mathcal{F}$ consisting of all subgroups of P containing Q . Let $\mathcal{A} : \bar{\mathcal{F}}^c \rightarrow \mathbf{Ab}$ be a contravariant functor. Restriction induces an isomorphism $H^*((\bar{\mathcal{F}}^c)^{op}; \mathcal{A}) \cong H^*(\bar{\mathcal{C}}^{op}; \mathcal{A})$.*

Proof. The inclusion functor $\Psi : \bar{\mathcal{C}} \rightarrow \bar{\mathcal{F}}^c$ has a left adjoint Φ sending R in $\bar{\mathcal{F}}^c$ to QR . Indeed, for any morphism $\varphi : R \rightarrow S$ in $\bar{\mathcal{F}}^c$ there is a morphism $\psi : QR \rightarrow QS$ in $\mathcal{C}$ extending φ , and the image of ψ in $\bar{\mathcal{C}}$ is unique since every morphism in $\bar{\mathcal{F}}^c$ is an epimorphism, which shows that there is a canonical functor Φ sending R to QR . Moreover, for R in $\bar{\mathcal{F}}^c$ and S in $\mathcal{C}$ we have $\text{Hom}_{\bar{\mathcal{C}}}(QR, S) \cong \text{Hom}_{\bar{\mathcal{F}}^c}(R, S)$, which shows that Φ is left adjoint to Ψ . Thus restriction along Ψ induces the stated isomorphism by [9, 3.1]. $\square$

Corollary 6.6. *Let $\mathcal{F}$ be a fusion system on a finite p -group P such that $\mathcal{F} = N_{\mathcal{F}}(Q)$ for some $\mathcal{F}$ -centric normal subgroup Q of P . Let L be a finite group with P as Sylow- p -subgroup such that $Q = O_p(L)$, $C_L(Q) = Z(Q)$ and such that $\mathcal{F}_P(L) = \mathcal{F}$. Then $H^*(\mathcal{F}^c; k^\times) \cong H^*(L; k^\times)$.*

Proof. We have $L/Z(Q) \cong \text{Aut}_{\mathcal{F}}(Q)$, and hence combining the above results yields $H^*(\mathcal{F}^c; k^\times) \cong H^*(\mathcal{C}; k^\times) \cong H^*(\text{Aut}_{\mathcal{F}}(Q); k^\times) \cong H^*(L; k^\times)$. $\square$

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