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Bounds for Hochschild cohomology of block algebras

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Abstract

We show that for any block algebra B of a finite group over an algebraically closed field of prime characteristic p the dimension of $HH^n(B)$ is bounded by a function depending only on the nonnegative integer n and the defect of B . The proof uses in particular a theorem of Brauer and Feit which implies the result for $n = 0$.

Let p be a prime and k an algebraically closed field of characteristic p . Let G be a finite group and B a block algebra of kG ; that is, B is an indecomposable direct factor of kG as a k -algebra. A *defect group* of B is a minimal subgroup P of G such that B is isomorphic to a direct summand of $B \otimes_{kP} B$ as a B - B -bimodule. The defect groups of B form a G -conjugacy class of p -subgroups of G , and the *defect* of B is the integer $d(B)$ such that $p^{d(B)}$ is the order of the defect groups of B . The *weak Donovan conjecture* states that the Cartan invariants of B are bounded by a function depending only on the defect $d(B)$ of B . As a consequence of a theorem of Brauer and Feit [3], the number of isomorphism classes of simple B -modules is bounded by a function depending only on $d(B)$. Thus the weak Donovan conjecture would imply that the dimension of a basic algebra of B is bounded by a function depending on $d(B)$. This in turn would imply that the dimension of the term in any fixed degree n of the Hochschild complex of a basic algebra of B is bounded by a function depending on n and $d(B)$; since Hochschild cohomology is invariant under Morita equivalences, we would thus get that the dimension of $HH^n(B)$ is bounded by a function depending on n and $d(B)$. The purpose of this note is to show that this consequence of the weak Donovan conjecture does indeed hold.

Theorem 1. *There is a function $f : \mathbb{N}_0 \times \mathbb{N}_0 \rightarrow \mathbb{N}_0$ such that for any integer $n \geq 0$, any finite group G and any block algebra B of kG with defect d we have*

$$\dim_k(HH^n(B)) \leq f(n, d)$$

For $n = 0$ this follows from the aforementioned theorem of Brauer and Feit [3], since $HH^0(B) \cong Z(B)$. Using Tate duality, the theorem above extends to Tate cohomology for negative n . A result of Külshammer and Robinson [7, Theorem 1] implies that it suffices to show theorem 1 for finite groups with a non-trivial normal p -subgroup. We follow a slightly different strategy in the proof below, reducing the problem directly to finite groups with a non-trivial central p -subgroup.

Remark 2. We make no effort to construct a best possible bound; we define the function f in theorem 1 inductively as follows: we set $f(0, 0) = 1$, $f(n, 0) = 0$ for $n > 0$; for all $d > 0$, $f(0, d)$ is the largest integer less or equal to the bound $\frac{1}{4}p^{2d} + 1$ given in the Brauer-Feit theorem (one

could, of course, also take $f(0, 1) = p$ and $f(0, d) = p^{2d-2}$ for $d \geq 2$; cf. [6, Ch. VII, 10.14]), and for $n > 0$, $d > 0$ we set

$$f(n, d) = p \cdot c(d) \cdot \sum_{i=0}^n f(i, d-1)$$

where $c(d)$ is the maximum of the numbers of subgroups in any finite group of order p^d .

Let G be a finite group and U a kG -module. We denote as usual by U^G the subspace of G -fixed points in U . If H is a subgroup of G then $U^G \subseteq U^H$, and there is a *trace map* $\text{tr}_H^G : U^H \rightarrow U^G$ sending $u \in U^H$ to $\sum_{x \in [G/H]} xu$, where $[G/H]$ is a set of representatives of the H -cosets in G ; one checks that this map is independent of the choice of $[G/H]$ and that its image, denoted U_H^G , is contained in U^G . For Q a p -subgroup of G , we denote the Brauer construction of U with respect to Q by $U(Q) = U^Q / \sum_{R: R < Q} U_R^Q$ and by $\text{Br}_Q^U : U^Q \rightarrow U(Q)$ the canonical surjection, called *Brauer homomorphism*. A block algebra B of kG can be viewed as an indecomposable $k(G \times G)$ -module, with $(x, y) \in G \times G$ acting by left multiplication with x and right multiplication with y^{-1} . For H a subgroup of G , we denote by ΔH the ‘diagonal’ subgroup $\Delta H = \{(h, h) \mid h \in H\}$ in $G \times G$. In particular, the action of ΔG on B can be identified with the conjugation action of G on B . The Brauer construction applied to B with respect to ΔQ is canonically isomorphic to $kC_G(Q)c$, where Q is a p -subgroup of G and $c = \text{Br}_{\Delta Q}(1_B)$. A *B-Brauer pair* is a pair (Q, e) consisting of a p -subgroup Q of G and of a block idempotent e of $kC_G(Q)$ satisfying $e\text{Br}_Q(1_B) = e$. The set of B -Brauer pairs is a G -poset in which the maximal pairs are all conjugate. The maximal B -Brauer pairs are exactly the B -Brauer pairs (Q, e) for which Q is a defect group of B . See [2] and [9, §11, §40] for details. In what follows we use without further comment the canonical graded isomorphism $HH^*(B) \cong H^*(\Delta G; B)$; see [8, (3.2)]. The following result is certainly well-known but not always stated in exactly the form we need it; we therefore give a proof for the convenience of the reader.

Proposition 3. *Let G be a finite group, B be a block algebra of kG and Q a p -subgroup of G . Set $b = 1_B$ and $c = \text{Br}_Q(b)$. Suppose that $c \neq 0$ and set $B_Q = kC_G(Q)cb$. Then we have a direct sum decomposition of $kN_{G \times G}(\Delta Q)$ -modules*

$$\text{Res}_{N_{G \times G}(\Delta Q)}^{G \times G}(B) = B_Q \oplus C_Q$$

such that multiplication by b is an isomorphism of $kN_{G \times G}(\Delta Q)$ -modules $kC_G(Q)c \cong B_Q$ and such that $C_Q(\Delta Q) = \{0\}$.

The proof we present here uses the following well-known lemma, which is a special case of expressing relative projectivity in terms of the splitting of adjunction maps (the general theme behind this is developed in [4], [5], for instance).

Lemma 4. *Let $\alpha : B \rightarrow A$ be a homomorphism of k -algebras. Suppose that B is isomorphic to a direct summand of A as a B - B -bimodule. Then α is injective and $\text{Im}(\alpha)$ is a direct summand of A as a B - B -bimodule.*

Proof. The left or right action of an element $b \in B$ on A is given by left or right multiplication with $\alpha(b)$. Let $\iota : B \rightarrow A$ and $\pi : A \rightarrow B$ be B - B -bimodule homomorphisms satisfying $\pi \circ \iota = \text{Id}_B$. Then $\iota(1_B)$ commutes with $\text{Im}(\alpha)$, the map β sending $a \in A$ to $a\iota(1_B)$ is an A - B -bimodule

endomorphism of A , and we have $\beta(\alpha(b)) = \alpha(b)\iota(1_A) = \iota(b)$, hence $\beta \circ \alpha = \iota$. Thus $\pi \circ \beta \circ \alpha = \text{Id}_B$, which shows that as a B - B -bimodule homomorphism, α is split injective with $\pi \circ \beta$ as a retraction. $\square$

Proof of Proposition 3. For any block of $kN_G(Q)$ which appears in $kN_G(Q)c$, the block B of kG is the corresponding ‘induced’ block. By [1, §14, Lemma 1], $kN_G(Q)c$ is isomorphic to a direct summand of B as a $kN_G(Q)$ - $kN_G(Q)$ -bimodule, and thus of cBc , as a $kN_G(Q)c$ - $kN_G(Q)c$ -bimodule. By lemma 4, multiplication by b induces an algebra homomorphism $kN_G(Q)c \rightarrow cBc$ which is split injective as a homomorphism of $kN_G(Q)c$ - $kN_G(Q)c$ -bimodules. Since $kC_G(Q)c$ is a direct summand of $kN_G(Q)c$ as an $N_{G \times G}(\Delta Q)$ -module we get that $kC_G(Q)c \cong B_Q$ and that B_Q is a direct summand of B as an $N_{G \times G}(\Delta Q)$ -module. Moreover, $B(\Delta Q) \cong B_Q$, and hence any complement C_Q of B_Q in B , as an $N_{G \times G}(\Delta Q)$ -module, satisfies $C_Q(\Delta Q) = \{0\}$. $\square$

We will make use of the following well-known fact on transfer in cohomology (we include a short proof for the convenience of the reader).

Lemma 5. *Let G be a finite group, H a subgroup of G and V a kH -module. Let U be a direct summand of $\text{Ind}_H^G(V)$. Then $H^*(G; U) = \text{tr}_H^G(H^*(H; \text{Res}_H^G(U)))$.*

Proof. By Higman’s criterion there is a kH -endomorphism φ of U such that $\text{Id}_U = \text{tr}_H^G(\varphi)$. Let $n \geq 0$ and let $\zeta : \Omega^n(k) \rightarrow U$ be a kG -homomorphism, representing an element in $H^n(G; U)$. Then $\zeta = \text{Id}_U \circ \zeta = \text{tr}_H^G(\varphi \circ \zeta)$, whence the result. $\square$

This is applied in the following situation:

Lemma 6. *Let G be a finite group, B a block algebra of kG and P a defect group of B . We have $H^*(\Delta G; B) = \text{tr}_{\Delta P}^{\Delta G}(H^*(\Delta P; B))$.*

Proof. As a $k(G \times G)$ -module, B has vertex ΔP and trivial source, thus is isomorphic to a direct summand of $\text{Ind}_{\Delta P}^{G \times G}(k)$. Mackey’s formula shows that $\text{Res}_{\Delta G}^{G \times G}(B)$ is still relatively ΔP -projective, hence lemma 5 implies the result. Alternatively, this follows from the fact that $b = 1_B$ can be written as a relative trace of the form $b = \text{Tr}_{\Delta P}^{\Delta G}(y)$ for some $y \in B^{\Delta P}$. $\square$

Proposition 7. *Let G be a finite group and B be a block algebra of kG . Set $b = 1_B$ and for every B -Brauer pair (Q, e) set $B_{(Q, e)} = kC_G(Q)eb$. Then $B_{(Q, e)}$ is a direct summand of B as a $k(C_G(Q) \times C_G(Q))\Delta Q$ -module, isomorphic to $kC_G(Q)e$. In particular, $H^*(\Delta Q; B_{(Q, e)})$ is a direct summand, as a graded vector space, of $H^*(\Delta Q; B)$, and we have*

$$H^*(\Delta G; B) = \sum_{(Q, e)} \text{tr}_{\Delta Q}^{\Delta G}(H^*(\Delta Q; B_{(Q, e)}))$$

where in the sum (Q, e) runs over a set of representatives of the G -conjugacy classes of B -Brauer pairs.

Proof. The proof adapts techniques that have been used in the proof of a result of Watanabe [10, Lemma 1]. Clearly $H^*(\Delta G; B)$ contains the right side in the displayed equation. We need to show that $H^*(\Delta G; B)$ is contained in the right side. Since any summand of the right side of the

form $\text{tr}_{\Delta Q}^{\Delta G}(H^*(\Delta Q; B_{(Q,e)}))$ depends only on the G -conjugacy class of (Q, e) it suffices to prove the inclusion

$$H^*(\Delta G; B) \subseteq \sum_Q \text{tr}_{\Delta Q}^{\Delta G}(H^*(\Delta Q; B_Q))$$

where Q runs over the p -subgroups of G for which $\text{Br}_Q(b) \neq 0$. Note that this makes sense since B_Q is a direct summand of B as a $k\Delta Q$ -module, hence $H^*(\Delta Q; B_Q)$ is a subspace of $H^*(\Delta Q; B)$, to which we then apply the transfer map $\text{tr}_{\Delta Q}^{\Delta G}$. Since $H^*(\Delta G; B) = \text{tr}_{\Delta P}^{\Delta G}(H^*(\Delta P; B))$ by lemma 6 it suffices to show that the right side contains $\text{tr}_{\Delta R}^{\Delta G}(H^*(\Delta R; B))$ for any p -subgroup R of G . This will be shown by induction. For $R = \{1\}$ this holds trivially because $B_{\{1\}} = B$ and $C_{\{1\}} = \{0\}$. For $R \neq \{1\}$ we have a direct sum decomposition $B = B_R \oplus C_R$ of $kN_{G \times G}(\Delta R)$ -modules as in proposition 3, and hence

$$H^*(\Delta R; B) = H^*(\Delta R; B_R) + H^*(\Delta R; C_R)$$

Since $C_R(\Delta R) = \{0\}$ we have

$$H^*(\Delta R; C_R) \subseteq \sum_{S; S < R} \text{tr}_{\Delta S}^{\Delta R}(H^*(\Delta S; B))$$

by lemma 5. Applying the transfer map $\text{tr}_{\Delta R}^{\Delta G}$ yields

$$\text{tr}_{\Delta R}^{\Delta G}(H^*(\Delta R; C_R)) \subseteq \sum_{S; S < R} \text{tr}_{\Delta S}^{\Delta G}(H^*(\Delta S; B))$$

hence

$$\text{tr}_{\Delta R}^{\Delta G}(H^*(\Delta R; B)) \subseteq \text{tr}_{\Delta R}^{\Delta G}(H^*(\Delta R; B_R)) + \sum_{S; S < R} \text{tr}_{\Delta S}^{\Delta G}(H^*(\Delta S; B))$$

The result follows by induction. $\square$

Lemma 8. *Let G be a finite group and B be a block algebra of kG . Set $b = 1_B$ and for every B -Brauer pair (Q, e) set $B_{(Q,e)} = kC_G(Q)eb$. For any integer $n \geq 0$ we have*

$$\dim_k(\text{tr}_{\Delta Q}^{\Delta G}(H^n(\Delta Q; B_{(Q,e)}))) \leq \dim_k(H^n(\Delta QC_G(Q); kQC_G(Q)e))$$

Proof. Clearly $\dim_k(\text{tr}_{\Delta Q}^{\Delta G}(H^n(\Delta Q; B_{(Q,e)}))) \leq \dim_k(\text{tr}_{\Delta Q}^{\Delta QC_G(Q)}(H^n(\Delta QC_G(Q); B_{(Q,e)})))$. Moreover, since $B_{(Q,e)} \cong kC_Q(Q)e$ is isomorphic to a direct summand of $kQC_G(Q)e$, the lemma follows. $\square$

Lemma 9. *Let G be a finite group, B a block of kG and Z a subgroup of order p of $Z(G)$. Set $\bar{G} = G/Z$ and denote by $\bar{B}$ the image of B in $k\bar{G}$ under the canonical algebra homomorphism $kG \rightarrow k\bar{G}$. For any integer $n \geq 0$ we have*

$$\dim_k(H^n(\Delta G; B)) \leq p \cdot \sum_{i=0}^n \dim_k(H^i(\Delta \bar{G}; \bar{B}))$$

Proof. The Lyndon-Hochschild-Serre spectral sequence associated with G , Z , $\bar{G}$ and B endowed with the conjugation action of G reads

$$H^i(\Delta\bar{G}; H^j(\Delta Z; B)) \Rightarrow H^{i+j}(\Delta G; B)$$

Since ΔZ acts trivially on kG , hence on B , we have $H^j(\Delta Z; B) \cong H^j(\Delta Z; k) \otimes_k B \cong B$, where the last isomorphism uses that we have $H^j(\Delta Z; k) \cong k$ because Z is cyclic. Thus $H^n(\Delta G; B)$ is filtered by subquotients of $H^i(\Delta\bar{G}; B)$, with $0 \leq i \leq n$; in particular, $\dim_k(H^n(\Delta G; B)) \leq \sum_{i=0}^n \dim_k(H^i(\Delta\bar{G}; B))$. Let z be a generator of Z . As a $k\Delta\bar{G}$ -module, B has a filtration of the form

$$B \supseteq B(1-z) \supseteq B(1-z)^2 \supseteq \cdots \supseteq B(1-z)^{p-1} \supseteq \{0\}$$

and since B is projective as a right kZ -module, the quotient of any two consecutive terms in this filtration is isomorphic to $\bar{B}$. Thus the appropriate long exact sequences in cohomology imply that $\dim_k(H^i(\Delta\bar{G}; B)) \leq p \cdot \dim_k(H^i(\Delta\bar{G}; \bar{B}))$, whence the result. $\square$

Proof of Theorem 1. Let f be the function defined in remark 2. Note that $f(n, d) \geq f(n, d-1)$ for all $n \geq 0$ and all $d > 0$. Denote by $c(d)$ the maximum of the numbers of subgroups in finite groups of order p^d . As mentioned before, theorem 1 holds for $n = 0$. Clearly theorem 1 holds for $d = 0$ because a defect zero block is a matrix algebra. Let n and d be a positive integers. Then $\text{tr}_{\Delta I}^{\Delta G}(H^n(1; B)) = \{0\}$. Thus, by proposition 7 and lemma 8 we have $\dim_k(HH^n(B)) \leq \sum_{(Q,e)} \dim_k(HH^n(kQC_G(Q)e))$ where in the sum (Q, e) runs over a set of representatives of the G -conjugacy classes of non-trivial B -Brauer pairs. Any such pair (Q, e) has a conjugate with Q contained in a fixed defect group P , and hence the number of summands in this sum is at most $c(d)$. Moreover, $Z(QC_G(Q))$ contains $Z(Q)$, and hence $QC_G(Q)$ has a non-trivial central subgroup Z_Q of order p . After replacing (Q, e) by a suitable G -conjugate, we may assume that $QC_P(Q)$ is a defect group of e viewed as a block of $kQC_G(Q)$; in particular the defect groups of e have order at most $|P| = p^d$. Thus the defect groups of the image $\bar{e}$ of e in $kQC_G(Q)/Z_Q\bar{e}$ have order at most $|P|/p = p^{d-1}$, hence $\dim_k(HH^n(kQC_G(Q)/Z_Q\bar{e})) \leq f(n, d-1)$. It follows from lemma 9 that $\dim_k(HH^n(kQC_G(Q)e)) \leq p \cdot \sum_{i=0}^n f(i, d-1)$. Together with the above remarks we get the inequality $\dim_k(HH^n(B)) \leq p \cdot c(d) \cdot \sum_{i=0}^n f(i, d-1) = f(n, d)$, as required. $\square$

Remark 10. The strong version of Donovan's conjecture states that for a fixed integer $d \geq 0$ there should be only finitely many Morita equivalence classes of blocks with defect at most d . If true, this would imply that there are only finitely many isomorphism classes of Hochschild cohomology algebras of blocks with defect at most d ; this remains an open problem.

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