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On Morita and derived equivalences for cohomological Mackey algebras

Markus Linckelmann and Baptiste Rognerud

Abstract

By results of the second author, a source algebra equivalence between two p -blocks of finite groups induces an equivalence between the categories of cohomological Mackey functors associated with these blocks, and a splendid derived equivalence between two blocks induces a derived equivalence between the corresponding categories of cohomological Mackey functors. The main result of this paper proves a partial converse: an equivalence (resp. Rickard equivalence) between the categories of cohomological Mackey functors of two blocks of finite groups induces a permeable Morita (resp. derived) equivalence between the two block algebras.

1 Introduction

Let p be a prime and $\mathcal{O}$ a complete local principal ideal domain with residue field $k = \mathcal{O}/J(\mathcal{O})$ of characteristic p ; we allow the case $\mathcal{O} = k$. Let G be a finite group. The blocks of $\mathcal{O}G$ are the primitive idempotents in $Z(\mathcal{O}G)$. For b a block of $\mathcal{O}G$, denote by $\text{coMack}(G, b)$ the abelian category of cohomological Mackey functors of G associated with b , with coefficients in the category $\text{mod}(\mathcal{O})$ of finitely generated $\mathcal{O}$ -modules. The second author showed in [11] that if two block algebras $\mathcal{O}Gb$ and $\mathcal{O}Hc$ of finite groups G and H are splendidly Morita or derived equivalent, then $\text{coMack}(G, b)$ and $\text{coMack}(H, c)$ are equivalent or derived equivalent, respectively. We show that a Morita or Rickard equivalence between $\text{coMack}(G, b)$ and $\text{coMack}(H, c)$ induces a Morita or derived equivalence between $\mathcal{O}Gb$ and $\mathcal{O}Hc$.

Theorem 1.1. *Let G, H be finite groups, b a block of $\mathcal{O}G$ and c a block of $\mathcal{O}H$. An equivalence between the abelian categories $\text{coMack}(G, b)$ and $\text{coMack}(H, c)$ (resp. a Rickard equivalence between their chain homotopy categories) induces a permeable Morita equivalence (resp. a derived equivalence) between the block algebras $\mathcal{O}Gb$ and $\mathcal{O}Hc$.*

If $\mathcal{O}$ has characteristic zero, we obtain a converse to [11, Proposition 4.5].

Corollary 1.2. *Suppose that $\mathcal{O}$ has characteristic zero. The abelian categories $\text{coMack}(G, b)$ and $\text{coMack}(H, c)$ are equivalent if and only if b and c are splendidly Morita equivalent.*

By a result of Scott [12] and Puig [8], two blocks are splendidly Morita equivalent if and only if they have isomorphic source algebras. Theorem 1.1 will be proved in Section 4 as a consequence of the description of the category $\text{coMack}(G, b)$ in terms of a source algebra of b in [5], and the two theorems below. Corollary 1.2 follows from this and Weiss' criterion [13], implying that if $\text{char}(\mathcal{O}) = 0$, then a permeable Morita equivalence is splendid.

Theorem 1.3. *Let A and B be symmetric $\mathcal{O}$ -algebras. Let X be a finitely generated $\mathcal{O}$ -free A -module and let Y be a finitely generated $\mathcal{O}$ -free B -module. Suppose that A is isomorphic to a direct summand of X as an A -module, and that B is isomorphic to a direct summand of Y as a B -module. Set $E = \text{End}_A(X)$ and $F = \text{End}_B(Y)$. A Morita equivalence between E and F induces a Morita equivalence between A and B which restricts to an equivalence between $\text{add}(X)$ and $\text{add}(Y)$.*

A Rickard equivalence between two algebras A and B consists of a bounded complex X of A - B -bimodules and a bounded complex Y of B - A -bimodules such that the terms of X, Y are finitely generated projective as left and right modules, such that we have homotopy equivalences $X \otimes_B Y \simeq A$ in the homotopy category $K(A \otimes_{\mathcal{O}} A^{\text{op}})$ of finitely generated A - A -bimodules and $Y \otimes_A X \simeq B$ in the corresponding homotopy category $K(B \otimes_{\mathcal{O}} B^{\text{op}})$. A Rickard equivalence induces a derived equivalence between A and B .

Theorem 1.4. *Let A and B be symmetric $\mathcal{O}$ -algebras. Let X be a finitely generated $\mathcal{O}$ -free A -module and let Y be a finitely generated $\mathcal{O}$ -free B -module. Suppose that A is isomorphic to a direct summand of X as an A -module, and that B is isomorphic to a direct summand of Y as a B -module. Set $E = \text{End}_A(X)$ and $F = \text{End}_B(Y)$. A Rickard equivalence between E and F induces a derived equivalence between A and B .*

Since the categories of cohomological Mackey functors in the statement of Theorem 1.1 are equivalent to the module categories of the corresponding Mackey algebras, the notion of Rickard equivalences extends in the obvious way to categories of cohomological Mackey functors. By results of Rickard in [9] and [10], if two symmetric $\mathcal{O}$ -algebras are derived equivalent, then they are Rickard equivalent, but such a Rickard equivalence may not be related in an obvious way to a given derived equivalence. This is essentially the reason why the conclusions in the theorems above are formulated in terms of derived equivalences rather than Rickard equivalences.

Notation. For A an algebra, we denote by A^{op} the opposite algebra. An A -module is a unital left module, unless stated otherwise. We denote by $\text{mod}(A)$ the category of finitely generated A -modules, and we identify $\text{mod}(A^{\text{op}})$ with the category of finitely generated unital right A -modules. For U a finitely generated A -module, we denote by $\text{add}(U)$ the full subcategory of $\text{mod}(A)$ consisting of all modules which are isomorphic to finite direct sums of summands of U . We denote by $\text{Ch}(A)$ the category of chain complexes of finitely generated A -modules, and by $K(A)$ the corresponding homotopy category.

Remark 1.5. For the purpose of proving Theorems 1.3 and 1.4 it would be sufficient to require that every projective indecomposable A -module is isomorphic to a direct summand of X , or equivalently, that the category $\text{proj}(A)$ of finitely generated projective A -modules is contained in the category $\text{add}(X)$. Since A is symmetric, this condition is equivalent to X having a generator and a cogenerator as a direct summand. This is the condition which appears for instance in work of Auslander [1], introducing the notion of representation dimension, and subsequently in work of Iyama [4], where the finiteness of the representation dimension of Artin algebras is proved.

Remark 1.6. The theme of extending/restricting derived equivalences between two finite-dimensional algebras over a field to/from suitable idempotent condensations appears in the work of many authors, such as [2] and [3]. As pointed out by the referee, it would certainly be interesting to explore the possibility of common generalisations of the results of the present paper and some of the results

in the above mentioned references. There is some overlap with methods used in [6] and [2]; we mention this in the Remarks 2.5 and 3.3 below.

It is tempting to speculate, whether the above theorems might possibly be of some use towards Broué's abelian defect conjecture, by playing the conjecture back to a question of derived equivalences between certain endomorphism algebras with interesting structural properties. Just to give an example of what we have in mind here, one could ask whether - at least in some special cases - the derived equivalence between blocks with abelian defect groups predicted by Broué's abelian defect conjecture become more accessible for the quasi-hereditary covers (in the sense of Rouquier) of the block algebras.

2 On relatively $\mathcal{O}$ -injective modules

Let A be an $\mathcal{O}$ -algebra. Suppose that A is free of finite rank as an $\mathcal{O}$ -module. Let U be a finitely generated left A -module. The module U is called *relatively $\mathcal{O}$ -projective* if U is isomorphic to a direct summand of $A \otimes_{\mathcal{O}} V$ for some $\mathcal{O}$ -module V . Thus U is projective if and only if U is relatively $\mathcal{O}$ -projective and $\mathcal{O}$ -free. If U is indecomposable and relatively $\mathcal{O}$ -projective, then U is isomorphic to a direct summand of either A or $A/\pi^n A$ for some positive integer n , because an indecomposable $\mathcal{O}$ -module is isomorphic to either $\mathcal{O}$ or $\mathcal{O}/\pi^n \mathcal{O}$ for some positive integer n .

Dually, U is called *relatively $\mathcal{O}$ -injective* if U is isomorphic to a direct summand of $\text{Hom}_{\mathcal{O}}(A, V)$ for some $\mathcal{O}$ -module V , where the left A -module structure on $\text{Hom}_{\mathcal{O}}(A, V)$ is given by $(b \cdot \varphi)(a) = \varphi(ab)$ for all $a, b \in A$ and $\varphi \in \text{Hom}_{\mathcal{O}}(A, V)$. As before, if U is indecomposable relatively $\mathcal{O}$ -injective, then U is isomorphic to a direct summand of either $A^* = \text{Hom}_A(A, \mathcal{O})$ or of $\text{Hom}_{\mathcal{O}}(A, \mathcal{O}/\pi^n \mathcal{O})$ for some positive integer n . Note that for $n = 1$ this yields the injective $k \otimes_{\mathcal{O}} A$ -modules. It is well-known that if U is $\mathcal{O}$ -free, then U is relatively $\mathcal{O}$ -injective if and only if $k \otimes_{\mathcal{O}} U$ is injective as a $k \otimes_{\mathcal{O}} A$ -module; we include short proofs of this and related facts for the convenience of the reader.

Lemma 2.1. *Let A be an $\mathcal{O}$ -algebra which is free of finite rank as an $\mathcal{O}$ -module, and let U be an A -module which is free of finite rank as an $\mathcal{O}$ -module. Then U is projective if and only if $k \otimes_{\mathcal{O}} U$ is a projective $k \otimes_{\mathcal{O}} A$ -module.*

Proof. If U is projective, then $k \otimes_{\mathcal{O}} U$ is obviously projective as a $k \otimes_{\mathcal{O}} A$ -module. Suppose conversely that $k \otimes_{\mathcal{O}} U$ is a projective $k \otimes_{\mathcal{O}} A$ -module. Then there is a projective A -module P such that $k \otimes_{\mathcal{O}} P \cong k \otimes_{\mathcal{O}} U$ as A -modules. Since P is projective, it follows that the obvious map $P \rightarrow k \otimes_{\mathcal{O}} P \cong k \otimes_{\mathcal{O}} U$ lifts to a map $P \rightarrow U$. This map is surjective by Nakayama's lemma (we use here that U is finitely generated as an $\mathcal{O}$ -module). Since both P and U are $\mathcal{O}$ -free of the same rank (equal to the dimension of $k \otimes_{\mathcal{O}} U$), it follows that the map $P \rightarrow U$ obtained in this way is an isomorphism. $\square$

Lemma 2.2. *Let A be an $\mathcal{O}$ -algebra which is free of finite rank as an $\mathcal{O}$ -module, and let U be an A -module which is free of finite rank as an $\mathcal{O}$ -module. Then U is relatively $\mathcal{O}$ -injective if and only if the right A -module $U^* = \text{Hom}_{\mathcal{O}}(U, \mathcal{O})$ is projective.*

Proof. Since duality preserves finite direct sums and since U is projective (resp. relatively $\mathcal{O}$ -injective) if and only if all direct summands of U have the same property, we may assume that U is indecomposable. Since also U is $\mathcal{O}$ -free, it follows that U is relatively $\mathcal{O}$ -injective if and only if

U is isomorphic to a direct summand of $A^* = \text{Hom}_{\mathcal{O}}(A, \mathcal{O})$. Applying $\mathcal{O}$ -duality, this is the case if and only if U^* is isomorphic to a direct summand of $A^{**} \cong A$ as a right A -module, hence if and only if U^* is projective as a right A -module. $\square$

Lemma 2.3. *Let A be an $\mathcal{O}$ -algebra which is free of finite rank as an $\mathcal{O}$ -module, and let U be an A -module which is free of finite rank as an $\mathcal{O}$ -module. Then U is relatively $\mathcal{O}$ -injective if and only if $k \otimes_{\mathcal{O}} U$ is an injective $k \otimes_{\mathcal{O}} A$ -module.*

Proof. We may assume that U is indecomposable. If U is relatively $\mathcal{O}$ -injective, then U is isomorphic to a direct summand of $A^* = \text{Hom}_{\mathcal{O}}(A, \mathcal{O})$. Thus $k \otimes_{\mathcal{O}} U$ is isomorphic to a direct summand of $(k \otimes_{\mathcal{O}} A)^* = \text{Hom}_k(k \otimes_{\mathcal{O}} A, k)$, and hence $k \otimes_{\mathcal{O}} U$ is injective as a $k \otimes_{\mathcal{O}} A$ -module. Suppose conversely that $k \otimes_{\mathcal{O}} U$ is injective as a $k \otimes_{\mathcal{O}} A$ -module. Then $k \otimes_{\mathcal{O}} U$ is isomorphic to a direct summand of the k -dual $(k \otimes_{\mathcal{O}} A)^*$ of $k \otimes_{\mathcal{O}} A$. Thus the k -dual $(k \otimes_{\mathcal{O}} U)^*$ of $k \otimes_{\mathcal{O}} U$ is isomorphic to a direct summand of the regular right $k \otimes_{\mathcal{O}} A$ -module $k \otimes_{\mathcal{O}} A$, hence is projective as a right $k \otimes_{\mathcal{O}} A$ -module. By the obvious version of Lemma 2.1 for right modules, the $\mathcal{O}$ -dual U^* of U is a projective right A -module. Thus U is relatively $\mathcal{O}$ -injective, by Lemma 2.2. $\square$

Let A be a finite-dimensional k -algebra. The k -dual U^* of a right A -module U is then a left A -module, and for any idempotent e in A , we have a canonical isomorphism $(Ue)^* \cong e(U^*)$ of left eAe -modules. This isomorphism is induced by restricting k -linear maps $U \rightarrow k$ to Ue . It can be regarded as a special case of an adjunction isomorphism: since $Ue \cong U \otimes_A Ae$, we have a natural isomorphism $\text{Hom}_k(U \otimes_A Ae, k) \cong \text{Hom}_A(Ae, \text{Hom}_k(U, k))$. The left side is $(Ue)^*$, and the right side is $\text{Hom}_A(Ae, U^*) \cong e(U^*)$. Applied to $U = eA$ this yields an isomorphism of left eAe -modules $e((eA)^*) \cong (eAe)^*$. We use this in the proof of the following lemma.

Lemma 2.4. *Let A be a finite-dimensional k -algebra. Suppose that for any primitive idempotent i in A the left A -module Ai is injective if and only if the right A -module iA is injective. Let J be a set of pairwise orthogonal representatives of the conjugacy classes of primitive idempotents j with the property that Aj is injective. Set $e = \sum_{j \in J} j$. Then the k -algebra eAe is selfinjective.*

Proof. By the assumptions, the right A -module eA is both projective and injective. Thus its k -dual $(eA)^* = \text{Hom}_k(eA, k)$ is a projective and injective left A -module. It follows that any indecomposable direct summand of $(eA)^*$ is isomorphic to a direct summand of Ae , by the choice of e . The indecomposable direct summands of Ae in any decomposition of Ae are pairwise nonisomorphic. Similarly for eA . Since Ae and eA have the same number of indecomposable direct factors, it follows that $(eA)^* \cong Ae$ as left A -modules. Multiplying both modules on the left by e yields an isomorphism of left eAe -modules $e(eA)^* \cong eAe$. The left side is isomorphic to $(eAe)^*$, and hence eAe is injective as a left eAe -module as required. $\square$

Remark 2.5. Lemma 2.4 can be proved as a consequence of [6, Theorem 1.5]. To see this, it suffices to show that the class of projective-injective indecomposable A -modules in Lemma 2.4 is invariant under the Nakayama functor ν . If i is a primitive idempotent in A such that the projective indecomposable A -module Ai is injective, then the right A -module $iA \cong \text{Hom}_A(Ai, A)$ is projective and injective by the assumptions in the Lemma, and hence $\nu(Ai) = (iA)^*$ is projective and injective.

If a finite-dimensional k -algebra A has the property that for any primitive idempotent i the left A -module Ai is injective if and only if the right A -module iA is injective, then clearly any

block algebra of A and any algebra Morita equivalent to A inherit the analogous property. This property applies to the Yoshida type endomorphism algebras. This will follow from some general considerations, based on the usual translation between direct summands of a module and projective modules over its endomorphism algebra.

Lemma 2.6. *Let A be a finite-dimensional k -algebra, X a finite-dimensional left A -module, and set $E = \text{End}_A(X)$. Let M be a direct summand of X .*

- (i) *If $\text{Hom}_A(X, M)$ is injective as an E^{op} -module, then M is injective as an A -module.*
- (ii) *If $\text{Hom}_A(M, X)$ is injective as an E -module, then M is projective as an A -module.*

Proof. Suppose that $\text{Hom}_A(X, M)$ is injective as an E^{op} -module. Let $\iota : M \rightarrow N$ be an injective A -homomorphism. We need to show that ι is split. The map $\eta : \text{Hom}_A(X, M) \rightarrow \text{Hom}_A(X, N)$ sending $\varphi \in \text{Hom}_A(X, M)$ to $\iota \circ \varphi$ is an injective homomorphism of E^{op} -modules. Thus η is split injective. Therefore there is an E^{op} -homomorphism $\epsilon : \text{Hom}_A(X, N) \rightarrow \text{Hom}_A(X, M)$ satisfying $\epsilon \circ \eta = \text{Id}$, the identity on $\text{Hom}_A(X, M)$. By the usual general abstract nonsense, ϵ is induced by an A -homomorphism $\pi : N \rightarrow M$; that is, we have $\epsilon(\psi) = \pi \circ \psi$. Since $\varphi = \epsilon(\eta(\varphi)) = \pi \circ \iota \circ \varphi$ for all $\varphi \in \text{Hom}_A(X, M)$. Applying this with φ a projection of X onto M implies that $\pi \circ \iota = \text{Id}_M$, and hence ι is split. Thus M is an injective A -module. This proves (i). The proof of (ii) is dual; we sketch the steps. Let $\pi : N \rightarrow M$ be a surjective A -homomorphism. Precomposing with π induces an injective homomorphism $\beta : \text{Hom}_A(M, X) \rightarrow \text{Hom}_A(N, X)$, which by the assumptions, is split injective. Any splitting of β is induced by precomposing with an A -homomorphism $M \rightarrow N$, which is then shown to be a section of π . This proves (ii). $\square$

We will use the following elementary fact on tensor products of finitely generated projective modules.

Lemma 2.7. *Let A be an $\mathcal{O}$ -algebra which is finitely generated as an $\mathcal{O}$ -module. Let e be an idempotent in A . Suppose that U is a projective left A -module which is a finite direct sum of direct summands of Ae , or that V is a projective right A -module which is a finite direct sum of direct summands of eA . The inclusions $eU \subseteq U$ and $Ve \subseteq V$ induce an isomorphism $Ve \otimes_{eAe} eU \cong V \otimes_A U$.*

Proof. The maps $Ve \otimes_{eAe} eU \rightarrow V \otimes_A U$ induced by the inclusions $eU \subseteq U$ and $Ve \subseteq V$ are a natural transformation from the bifunctor $(U, V) \mapsto Ve \otimes_{eAe} eU$ to the bifunctor $(U, V) \mapsto V \otimes_A U$. This natural transformation is $\mathcal{O}$ -linear in both arguments, so it suffices to show that it yields an isomorphism if $U = Ae$ or if $V = eA$. If $U = Ae$, then $Ve \otimes_{eAe} eU = Ve \otimes_{eAe} eAe \cong Ve \cong V \otimes_A Ae = V \otimes_A U$. A similar argument shows that if $V = eA$, then we have an isomorphism $Ve \otimes_{eAe} eU \cong V \otimes_A U$, proving the statement. $\square$

Lemma 2.8. *Let A be an $\mathcal{O}$ -algebra which is finitely generated as an $\mathcal{O}$ -module. Let U be a bounded complex of finitely generated projective A -modules. Suppose that $k \otimes_{\mathcal{O}} U$ has a contractible direct summand W as a complex of $k \otimes_{\mathcal{O}} A$ -modules. Then U has a contractible direct summand V such that $k \otimes_{\mathcal{O}} V = W$.*

Proof. Set $\bar{A} = k \otimes_{\mathcal{O}} A$ and $\bar{U} = k \otimes_{\mathcal{O}} U$. Since U is bounded, the algebra $\text{End}_{\text{Ch}(A)}(U)$ is finitely generated as an $\mathcal{O}$ -module, and $\text{End}_{\text{Ch}(\bar{A})}(\bar{U})$ is finite-dimensional. Denote by C the ideal in $\text{End}_{\text{Ch}(A)}(U)$ of chain maps $\psi : U \rightarrow U$ which satisfy $\psi \sim 0$; that is, C is the kernel of the

canonical algebra homomorphism $\text{End}_{\text{Ch}(A)}(U) \rightarrow \text{End}_{K(A)}(U)$. Similarly, denote by D the kernel of the canonical algebra homomorphism $\text{End}_{\text{Ch}(\bar{A})}(\bar{U}) \rightarrow \text{End}_{K(\bar{A})}(\bar{U})$. Since the components of U are projective, any homotopy $\bar{U} \rightarrow \bar{U}[-1]$ lifts to a homotopy $U \rightarrow U[-1]$. It follows that the canonical map $C \rightarrow D$ is surjective. The summand W of $\bar{U}$ corresponds to an idempotent η in $\text{End}_{\text{Ch}(\bar{A})}(\bar{U})$. Since W is contractible, this idempotent is contained in D . Thus, by standard lifting theorems, there is an idempotent $\hat{\eta}$ in C which lifts η . Thus $V = \hat{\eta}(U)$ is a contractible direct summand of U which lifts W . $\square$

The following result identifies bounded complexes of finitely generated modules which are both projective and injective.

Lemma 2.9. *Let A be a finite-dimensional k -algebra. Let U be a bounded chain complex of finitely generated projective A -modules. Suppose that U has no nonzero contractible direct summand as a chain complex, and that for any bounded above acyclic chain complex C in $\text{Ch}(A)$ we have $\text{Hom}_{K(A)}(C, U) = \{0\}$. Then the components of U are injective.*

Proof. Let $\alpha : U \rightarrow I_U$ be an injective resolution of U ; that is, I_U is a bounded above chain complex of finitely generated injective A -modules and α is a quasi-isomorphism. Denote by $\underline{\alpha}$ the image of α in $\text{Hom}_{K(A)}(U, I_U)$. Consider the associated exact triangle in $K(A)$,

$$U \xrightarrow{\underline{\alpha}} I_U \longrightarrow C(\alpha) \longrightarrow U[1]$$

Since α is a quasi-isomorphism, it follows its cone $C(\alpha)$ is acyclic. By the assumptions on U , the morphism $C(\alpha) \rightarrow U[1]$ in this triangle is zero. Therefore, by a standard property of triangulated categories (see e. g. [15, Lemma 3.4.9]), the morphism $\underline{\alpha}$ is a split monomorphism in $K(A)$. That is, there exists a chain map $\delta : I_U \rightarrow U$ such that $\delta \circ \alpha \sim \text{Id}_U$.

Since U is bounded, the algebra $\text{End}_{\text{Ch}(A)}(U)$ is finite-dimensional. We use as before the fact that idempotents in this algebra correspond to direct summands of U as a chain complex, and that contractible direct summand correspond to those idempotents which are in the kernel of the canonical algebra homomorphism $\text{End}_{\text{Ch}(A)}(U) \rightarrow \text{End}_{K(A)}(U)$.

By the assumptions on U , this kernel contains no idempotents, and hence is contained in the radical $J(\text{End}_{\text{Ch}(A)}(U))$. Since $\delta \circ \alpha$ maps to the identity in $\text{End}_{K(A)}(U)$, it follows that $\delta \circ \alpha$ is invertible in $\text{End}_{\text{Ch}(A)}(U)$. Thus the chain map $\beta = (\delta \circ \alpha)^{-1} \circ \delta$ satisfies $\beta \circ \alpha = \text{Id}_U$. This shows that U is isomorphic to a direct summand of I_U . In particular, the components of U are injective. $\square$

3 Proof of Theorems 1.3 and 1.4

An $\mathcal{O}$ -algebra A is *symmetric* if A is free of finite rank as an $\mathcal{O}$ -module, and if A is isomorphic to its $\mathcal{O}$ -dual A^* as an A - A -bimodule. One of the special features of a symmetric $\mathcal{O}$ -algebra A is that the two duality functors with respect to A and $\mathcal{O}$ are isomorphic; that is, for any left A -module U , there is an isomorphism $\text{Hom}_A(U, A) \cong U^*$ of right A -modules which is natural in U . More precisely, any choice of a bimodule isomorphism $A \cong A^*$ induces such an isomorphism of duality functors as follows: if $s \in A^*$ is the image of 1 under a bimodule isomorphism $A \cong A^*$, then the map sending $\varphi \in \text{Hom}_A(U, A)$ to $s \circ \varphi \in U^*$ is an isomorphism, for any A -module U . The naturality implies in particular that this isomorphism is an isomorphism as right $\text{End}_A(U)$ -modules.

Lemma 3.1. *Let A be a finite-dimensional k -algebra and let X be a finite-dimensional A -module. Set $E = \text{End}_A(X)$. If X has a direct summand isomorphic to A as an A -module, then X is projective as an E -module. If in addition A is symmetric, then X^* is projective as an E^{op} -module.*

Proof. If A is isomorphic to a direct summand of X , then $\text{Hom}_A(A, X)$ is a projective E -module, and clearly $\text{Hom}_A(A, X) \cong X$. If in addition A is symmetric, then we have a natural isomorphism $X^* \cong \text{Hom}_A(X, A)$, and this is a projective E^{op} -module. $\square$

Proposition 3.2. *Let A be a symmetric k -algebra and X a finite-dimensional A -module. Suppose that A is isomorphic to a direct summand of X . Set $E = \text{End}_A(X)$. Let U be a direct summand of X . The following are equivalent.*

- (i) *The A -module U is projective.*
- (ii) *The E^{op} -module $\text{Hom}_A(X, U)$ is injective.*
- (iii) *The E -module $\text{Hom}_A(U, X)$ is injective.*

Proof. If $\text{Hom}_A(X, U)$ is an injective E^{op} -module, then by Lemma 2.6 (i) the A -module U is injective, hence also projective, since we assume that A is symmetric. Thus (ii) implies (i). Similarly, if $\text{Hom}_A(U, X)$ is injective, then by Lemma 2.6 (ii) the A -module U is projective. Thus (iii) implies (i). In order to show that (i) implies (ii) and (iii), it suffices to show that $\text{Hom}_A(X, A)$ is an injective E^{op} -module and that $\text{Hom}_A(A, X)$ is an injective E -module. Thus it suffices to show that their duals $\text{Hom}_A(X, A)^*$ and $\text{Hom}_A(A, X)^*$ are projective as modules over E and E^{op} , respectively. Since A is symmetric, we have a natural isomorphism $\text{Hom}_A(X, A) \cong \text{Hom}_k(X, k) = X^*$. The naturality implies in particular, that these isomorphisms are isomorphisms as E^{op} -modules. Thus we have an isomorphism of E -modules $\text{Hom}_A(X, A)^* \cong X$, and this is a projective E -module by Lemma 3.1. Similarly, note that $\text{Hom}_A(A, X) \cong X$, and hence that $\text{Hom}_A(A, X)^* \cong X^*$, which is indeed a projective E^{op} -module, again by Lemma 3.1. $\square$

Proof of Theorem 1.3. We use the notation of Theorem 1.3. By Proposition 3.2 and Lemma 2.3, projective indecomposable modules over E and F which are also relatively $\mathcal{O}$ -injective, correspond to the indecomposable summands of A and B , respectively. Denote by e an idempotent in E such that $e(X) \cong A$, and denote by f an idempotent in F such that $f(Y) \cong B$. Then Ee is a direct sum of indecomposable E -modules which are projective and relatively $\mathcal{O}$ -injective, and any indecomposable E -module which is projective and relatively $\mathcal{O}$ -injective is isomorphic to a direct summand of Ee . The right E -module eE , the left F -module Ff and the right F -module fF have the analogous properties.

Let M be an E - F -module and N an F - E -bimodule inducing a Morita equivalence; that is, M, N are finitely generated projective as left and right modules, and we have bimodule isomorphisms $M \otimes_F N \cong E$ and $N \otimes_E M \cong F$. A Morita equivalence between E and F preserves projective indecomposables which are also relatively $\mathcal{O}$ -injective. Thus $N \otimes_E Ee \cong Ne$ is a direct sum of summands of Ff . In particular, fNe is projective as a fFf -module. Similarly, eM , as a right F -module, is a direct sum of summands of fF , and hence eMf is projective as a right fFf -module. Using Lemma 2.7, applied to E and F instead of A and B , we have isomorphisms

$$eEe \cong eM \otimes_F Ne \cong eMf \otimes_{fFf} fNe$$

Exchanging the roles of E and F shows similarly that $fFf \cong fNe \otimes_{eEe} eMf$, and that eMf and fNe are both projective as left and as right modules. Thus the bimodules eMf and fNe induce

a Morita equivalence between eEe and fFf . Since $eEe \cong \text{End}_A(e(X)) \cong \text{End}_A(A) \cong A^{\text{op}}$ and similarly $fFf \cong B^{\text{op}}$, passing to opposite algebras yields a Morita equivalence between A and B .

We need to show that this Morita equivalence restricts to an equivalence between $\text{add}(X)$ and $\text{add}(Y)$. It suffices to show that the equivalence $\text{mod}(A) \cong \text{mod}(B)$ sends $\text{add}(X)$ to $\text{add}(Y)$. Since A and B are symmetric, it suffices to show that $fNe \otimes_{eEe} -$ sends $\text{add}(X^*)$ to $\text{add}(Y^*)$. Let V be an A -module in $\text{add}(X)$. Then $\text{Hom}_A(V, X)$ is a projective E -module. Since $N \otimes_E -$ is an equivalence of categories, it follows that $N \otimes_E \text{Hom}_A(V, X)$ is a projective F -module. Thus there is a B -module W in $\text{add}(Y)$ such that $N \otimes_E \text{Hom}_A(V, X) \cong \text{Hom}_B(W, Y)$ as F -modules. Multiplying by f yields an isomorphism of fFf -modules

$$fN \otimes_E \text{Hom}_A(V, X) \cong f\text{Hom}_B(W, Y) \cong \text{Hom}_B(W, B)$$

Since fN is a direct sum of summands of eE , it follows from Lemma 2.7 that the left side is isomorphic to

$$fNe \otimes_{eEe} e\text{Hom}_A(V, X) \cong fMe \otimes_{eEe} \text{Hom}_A(V, A)$$

Since A and B are symmetric, we have $\text{Hom}_A(V, A) \cong V^*$ and $\text{Hom}_B(W, B) \cong W^*$. This shows that $fNe \otimes_{eEe} V^* \cong W^*$, and hence the functor $fNe \otimes_{eEe} -$ sends $\text{add}(X^*)$ to $\text{add}(Y^*)$ as claimed. $\square$

Proof of Theorem 1.4. We use the notation of Theorem 1.4. Let M be a bounded complex of E - F -bimodules and N a bounded complex of F - E -bimodules inducing a Rickard equivalence; that is, the components of M, N are finitely generated projective as left and right modules, and we have homotopy equivalences of chain complexes of bimodules E - E -bimodules $M \otimes_F N \simeq E$ and F - F -bimodules $N \otimes_E M \simeq F$. Denote by e an idempotent in E such that $e(X) \cong A$, and denote by f an idempotent in F such that $f(Y) \cong B$. Multiplying the above homotopy equivalences by e and f on both sides yields homotopy equivalences of chain complexes of eEe - eEe -bimodules $eM \otimes_F Ne \simeq eEe$ and of fFf - fFf -bimodules $fN \otimes_E Mf \simeq fFf$.

We will show that there are quasi-isomorphisms

$$eMf \otimes_{fFf} fNe \rightarrow eM \otimes_F Ne$$

$$fNe \otimes_{eEe} eMf \rightarrow fN \otimes_E Mf$$

whose restriction to the left and to the right are homotopy equivalences, and we will then see that this implies that the functors $eMf \otimes_{fFf} -$ and $fNe \otimes_{eEe} -$ induce inverse derived equivalences.

The complex $Ne \cong N \otimes_E Ee$ is a bounded complex of finitely generated projective F -modules. We will show that

$$Ne \cong N_0 \oplus N_1$$

for some contractible complex N_1 and a complex N_0 whose components consist of finite direct sums of summands of Ff . In order to show this, by Lemma 2.8, we may assume that $\mathcal{O} = k$. Since Ee is injective, we have $\text{Hom}_{K(E)}(C, Ee) = \{0\}$ for any acyclic bounded above complex C of finitely generated E -modules. Since $N \otimes_E -$ induces an equivalence of homotopy categories of chain complexes $K(E) \cong K(F)$, it follows that we have $\text{Hom}_{K(F)}(D, Ne) = \{0\}$ for any acyclic bounded above complex D of finitely generated F -modules. It follows from Lemma 2.9, that the indecomposable direct summands of Ne which are not contractible consist of injective F -modules,

hence of sums of summands of Ff . Reverting to general $\mathcal{O}$, multiplying the previous isomorphism by f on the left yields an isomorphism of chain complexes of fFf -modules

$$fNe \cong fN_0 \oplus fN_1$$

such that fN_0 is a bounded complex of finitely generated projective fFf -modules, and fN_1 is a bounded contractible complex. The same argument shows that we have an isomorphism of chain complexes of right F -modules

$$eM \cong M_0 \oplus M_1$$

where M_0 is a complex of right F -modules which are finite direct sums of summands of fF , and M_1 is a contractible complex of right F -modules. Thus, as before, multiplying this isomorphism on the right by f yields an isomorphism of chain complexes of right fFf -modules

$$eMf \cong M_0f \oplus M_1f$$

such that M_0f is a bounded complex of finitely generated projective right fFf -modules and M_1f is a bounded contractible complex. Thus we have a decomposition as complexes of right eEe -modules

$$\begin{aligned} eM \otimes_F Ne &= M_0 \otimes_F Ne \oplus M_1 \otimes_F Ne \cong \\ &M_0f \otimes_{fFf} fNe \oplus M_1 \otimes_F Ne \end{aligned}$$

where we have made use of Lemma 2.7 for the isomorphism. We also have

$$eMf \otimes_{fFf} fNe = M_0f \otimes_{fFf} fNe \oplus M_1f \otimes_{fFf} fNe$$

In both of these isomorphisms, the right most terms are contractible as chain complexes of right eEe -modules, because M_1f is contractible as a chain complex of right fFf -modules. Thus we have a chain map of complexes of eEe - eEe -bimodules

$$eMf \otimes_{fFf} fNe \rightarrow eM \otimes_F Ne \simeq eEe$$

which restricts to a homotopy equivalence as a chain map of complexes of right eEe -modules, and, by the analogous argument, restricts to a homotopy equivalence as a chain map of complexes of left eEe -modules. In particular, this bimodule chain map is a quasi-isomorphism. Similarly, we have a bimodule quasi-isomorphism

$$fNe \otimes_{eEe} eMf \rightarrow fN \otimes_E Mf \simeq fFf$$

which restricts to homotopy equivalences on the left and on the right.

We show next that the functor $eMf \otimes_{fFf} -$ from $\text{Ch}(fFf)$ to $\text{Ch}(eEe)$ preserves quasi-isomorphisms. We use the right fFf -chain complex decomposition $eMf = M_0f \oplus M_1f$ above. If $\beta : V \rightarrow V'$ is a quasi-isomorphism in $\text{Ch}(fFf)$, then $\text{Id}_{eMf} \otimes \beta : eMf \otimes_{fFf} V \rightarrow eMf \otimes_{fFf} V'$ decomposes as a direct sum of chain maps of complexes of $\mathcal{O}$ -modules $M_0f \otimes_{fFf} V \rightarrow M_0f \otimes_{fFf} V'$ and $M_1f \otimes_{fFf} V \rightarrow M_1f \otimes_{fFf} V'$. The first of these is a quasi-isomorphism because M_0f is a bounded complex of projective right fFf -modules. The second is trivially a quasi-isomorphism, since both $M_1f \otimes_{fFf} V$ and $M_1f \otimes_{fFf} V'$ are acyclic, as M_1f is contractible as a complex of right fFf -modules. Thus the functor $eMf \otimes_{fFf} -$ induces a functor on derived categories; similarly for

$fNe \otimes_{eEe} -$. These two functors are inverse to each other as functors on the derived categories. Indeed, since the above bimodule chain map $eMf \otimes_{fFf} fNe \rightarrow eEe$ is a homotopy equivalence as chain map of complexes of right eEe -modules, it follows that for any complex U in $\text{Ch}(eEe)$, the induced chain map $eMf \otimes_{fFf} fNe \otimes_{eEe} U \rightarrow U$ is a homotopy equivalence as a chain map of complexes of $\mathcal{O}$ -modules, hence a quasi-isomorphism as a chain map of complexes of eEe -modules. The result follows. $\square$

Remark 3.3. The above proof does not show that the quasi-isomorphisms $eMf \otimes_{fFf} fNe \rightarrow eM \otimes_F Ne$ and $fNe \otimes_{eEe} eMf \rightarrow fN \otimes_E Mf$ are homotopy equivalences as bimodule chain maps. It also does not show that eMf and fNe are projective as complexes of left and right modules. In particular, this proof does not show that eMf and fNe are Rickard complexes, and it seems unclear whether the induced derived equivalence preserves the subcategories of chain complexes over $\text{add}(X)$ and $\text{add}(Y)$. The proof does show that the derived equivalence induced by M and N restricts to an equivalence

$$K^b(\text{add}(Ee)) \cong K^b(\text{add}(Ff)) ,$$

thanks to the isomorphism $Ne \cong N_0 \oplus N_1$ for some contractible complex N_1 and a complex N_0 with components in $\text{add}(Ff)$, and the analogous isomorphism for Mf , where the notation is as in the proof above. In the case where the base ring $\mathcal{O}$ is a field, this can also be deduced from [2, Theorem 4.3], making use of Remark 2.5.

4 Proof of Theorem 1.1 and Corollary 1.2

The proof of Theorem 1.1 is played back to the theorems 1.3 and 1.4, together with description of cohomological Mackey functors in terms of source algebras of blocks in [5], extending ideas going back to Yoshida [14].

Proposition 4.1. *Let A be a source algebra of a block of a finite group with defect group P . Set $X = \oplus_Q A \otimes_{\mathcal{O}Q} \mathcal{O}$ and $E = \text{End}_A(X)$, where in the direct sum Q runs over the subgroups of P . For ι a primitive idempotent in E , the following are equivalent.*

- (i) $E \circ \iota$ is a relatively $\mathcal{O}$ -injective left E -module.
- (ii) $\iota \circ E$ is a relatively $\mathcal{O}$ -injective right E -module.
- (iii) $\iota(X)$ is isomorphic to a projective indecomposable left A -module.

Proof. Note that A is a symmetric $\mathcal{O}$ -algebra and that A is isomorphic to the summand indexed by the trivial group 1 in the direct sum $X = \oplus_Q A \otimes_{\mathcal{O}Q} \mathcal{O}$. Thus the result follows from Proposition 3.2, combined with Lemma 2.3. $\square$

Proof of Theorem 1.1. Denote by P a defect group and by A a source algebra of the block b of $\mathcal{O}G$. Similarly, denote by Q a defect group and by B a source algebra of the block c of $\mathcal{O}H$. Set $X = \oplus_R A \otimes_{\mathcal{O}R} \mathcal{O}$, where R runs over the subgroups of P , and set $E = \text{End}_A(X)$. Similarly, $Y = \oplus_T B \otimes_{\mathcal{O}T} \mathcal{O}$, where T runs over the subgroups of Q , and set $F = \text{End}_B(Y)$. It follows from Theorem 1.3 (resp. Theorem 1.4) that if E and F are Morita equivalent (resp. Rickard equivalent) then A and B are permeable Morita equivalent (resp. derived equivalent). By [5, Theorem 1.1] we have $\text{coMack}(G, b) \cong \text{mod}(E^{\text{op}})$ and $\text{coMack}(H, c) \cong \text{mod}(F^{\text{op}})$, whence the result. $\square$

A permeable Morita equivalence between block algebras over k need not be splendid; see [11, Remark 4.7]. In characteristic zero, however, permeable Morita equivalences are splendid.

Proposition 4.2. *If $\mathcal{O}$ has characteristic zero, then a permeable Morita equivalence between two blocks of finite group algebras over $\mathcal{O}$ is splendid.*

Proof. Let B, B' be block algebras of some finite group algebras over $\mathcal{O}$, with defect groups P, P' , respectively. Let M be a B - B' -bimodule inducing a permeable Morita equivalence. Then $M \otimes_{B'} -$ sends the p -permutation B' -module $B' \otimes_{\mathcal{O}P'} \mathcal{O}$ to the p -permutation B -module $M \otimes_{\mathcal{O}P'} \mathcal{O}$. In particular, M is free as a left $\mathcal{O}P$ -module, as a right $\mathcal{O}P'$ -module, and $M \otimes_{\mathcal{O}P'} \mathcal{O}$ is a permutation $\mathcal{O}P$ -module. Note that since M is free as a right $\mathcal{O}P'$ -module, the quotient $M \otimes_{\mathcal{O}P'} \mathcal{O}$ is isomorphic to the fixpoints $M^{P'}$ in M with respect to the right action of P' on M . Weiss' criterion [13, Theorem 2] (adapted to the more general coefficient ring $\mathcal{O}$ in [8, Appendix 1]), applied to $P \times P'$ and the normal subgroup $1 \times P'$ instead of G and N , respectively, implies that M is a permutation $\mathcal{O}(P \times P')$ -module. $\square$

Proof of Corollary 1.2. If B and B' are splendidly Morita equivalent, then their associated categories of cohomological Mackey functors are equivalent by [11, Proposition 4.5]. Conversely, if $\text{char}(\mathcal{O}) = 0$ and if the categories of cohomological Mackey functors of B, B' are equivalent, then B, B' are permeable Morita equivalent by Theorem 1.1, hence splendidly Morita equivalent by Proposition 4.2. $\square$

5 A remark on nilpotent blocks

By results of Puig in [7] and [8], if a block b of a finite group algebra $\mathcal{O}G$ is nilpotent, then $\mathcal{O}Gb$ is Morita equivalent to $\mathcal{O}P$, and if $\mathcal{O}$ has characteristic zero, then the converse holds as well. Thus, if $\text{char}(\mathcal{O}) = 0$, then Corollary 1.2 implies that b is nilpotent with a source algebra isomorphic to $\mathcal{O}P$ if and only if $\text{coMack}(G, B) \cong \text{coMack}(P)$. For derived equivalences, Theorem 1.1 this yields the following.

Theorem 5.1. *Let G be a finite group and b a block of $\mathcal{O}G$. Suppose that $\mathcal{O}$ has characteristic zero. If the categories $\text{coMack}(G, b)$ and $\text{coMack}(P)$ are Rickard equivalent, then b is nilpotent, with defect groups isomorphic to P .*

Proof. Suppose that the categories $\text{coMack}(G, b)$ and $\text{coMack}(P)$ are Rickard equivalent. Then, by Theorem 1.1, the algebras $\mathcal{O}Gb$ and $\mathcal{O}P$ are derived equivalent. Since $\mathcal{O}P$ is split local, it follows from [15, 6.7.5] that $\mathcal{O}Gb$ and $\mathcal{O}P$ are Morita equivalent. Thus, by [8, Theorem 8.2], b is nilpotent with defect groups isomorphic to P . $\square$

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References

- [1] M. Auslander, *Representation dimension of Artin algebras*. Lecture notes, Queen Mary College, London (1971).
- [2] M. Fang, W. Hu, and S. König, *Derived equivalences, restriction to self-injective subalgebras and invariance of homological dimensions*. (2016) arXiv: 1607.03513.
- [3] W. Hu and Ch. Xi, *Derived equivalences for Φ -Auslander-Yoneda algebras*. Trans. Amer. Math. Soc. **365** (2013), 5681–5711.
- [4] O. Iyama, *Finiteness of representation dimension*, Proc. Amer. Math. Soc. **131** (2002), 1011–1014.
- [5] M. Linckelmann, *On equivalences for cohomological Mackey functors*, Represent. Theory **20** (2016), 162–171.
- [6] R. Martínez-Villa, *The stable equivalence for algebras of finite representation type*. Comm. Algebra **13** (1985), 991–1018.
- [7] L. Puig, *Nilpotent blocks and their source algebras*, Invent. Math. **93** (1988), 77–116.
- [8] L. Puig, *On the local structure of Morita and Rickard equivalences between Brauer blocks*, Progress in Math. **178**, Birkhäuser Verlag, Basel (1999)
- [9] J. Rickard, *Derived equivalences as derived functors*, J. London Math. Soc. **43** (1991), 37–48.
- [10] J. Rickard, *Splendid equivalence: derived categories and permutation modules*, Proc. London Math. Soc. **72** (1996), 331–358.
- [11] B. Rognerud, *Equivalences between blocks of cohomological Mackey algebras*, Math. Z. **280** (2015), 421–449.
- [12] L. L. Scott, unpublished notes (1990).
- [13] A. Weiss, *Rigidity of p -adic p -torsion*, Ann. of Math. **127** (1988), 317–332.
- [14] T. Yoshida, *On G -functors (II): Hecke operators and G -functors*, J. Math. Soc. Japan **35** (1983), 179–190.
- [15] A. Zimmermann, *Representation theory: A Homological Algebra Point of View*. Algebra and Applications, Springer International Publishing (2014).