



City Research Online

City St George's, University of London

Citation: Zhang, Y. (2009). Optimal Plan Design and Dynamic Asset Allocation of Defined Contribution Pension Plans: Lessons from Behavioural Finance and Non-expected Utility Theories. (Unpublished Doctoral thesis, City University London)

This is the accepted version of the paper.

This version of the publication may differ from the final published version. To cite this item please consult the publisher's version.

Permanent repository link: <https://city-ref-test.eprints-hosting.org/id/eprint/12031/>

Copyright and Reuse: Copyright and Moral Rights remain with the author(s) and/or copyright holders. Copies of full items can be used for personal research or study, educational, or not-for-profit purposes without prior permission or charge, unless otherwise indicated, provided that the authors, title and full bibliographic details are credited, a hyperlink and/or URL is given for the original metadata page and the content is not changed in any way. For full details of reuse please refer to [City Research Online policy](#).

Optimal Plan Design and Dynamic Asset Allocation of Defined Contribution Pension Plans: Lessons from Behavioural Finance and Non-expected Utility Theories

Yumeng Zhang

A thesis submitted for the award of the degree of
Doctor of Philosophy in Actuarial Science

Department of Actuarial Science and Insurance
Cass Business School
City University
London, England

June 2009

Contents

List of Tables	VI
List of Figures.....	VII
Acknowledgements	IX
Declaration.....	X
Abstract.....	XI
1 Setting the Scene	1
1.1 Statement of the Problem.....	1
1.1.1 Why defined contribution pension plans?.....	1
1.1.2 The investment problem with DC plans	2
1.2 Dynamic asset allocation	4
1.2.1 Short-term investment strategy (two-period model).....	4
1.2.2 Long-term investment strategy (multi-period model).....	6
1.2.3 Financial and human wealth	8
1.2.4 The consumption problem	9
1.2.5 Optimisation methods	10
1.3 Behavioural issues	13
1.3.1 The existing literature on DC investment	13
1.3.2 Differentiators	15
1.4 Overview of the thesis	18
2 Behavioural Features in DC Plan Arrangement.....	20
2.1 Introduction.....	20
2.1.1 Background	20
2.1.2 Literature review	22
2.2 Behavioural finance	24
2.2.1 Repeated gamble	24

2.2.2 Prospect theory and loss aversion	25
2.2.3 Mental accounting.....	27
2.2.4 Myopic loss aversion (MLA).....	28
2.2.5 Framing effect	29
2.2.6 Anchoring effect	29
2.2.7 Overconfidence and confirmation bias	30
2.2.8 Default option, Inertia and Procrastination	30
2.2.9 Hyperbolic discounting.....	31
2.2.10 Other	32
2.3 Application to DC plan arrangement	32
2.3.1 Communication.....	32
2.3.2 Decision to save	33
2.3.3 Default and other options.....	33
2.3.4 Fund selection	35
2.3.5 Fund performance data	36
2.3.6 Annuitisation decision	36
2.4 Conclusions.....	37
 3 Reexamining Behavioural Characteristics (UK Evidence)	39
3.1 Introduction.....	39
3.2 Survey	44
3.2.1 Subjects	44
3.2.2 Survey Design	46
3.2.3 Method	48
3.3 Results.....	49
3.3.1 Loss aversion parameters	49
3.3.2 Behavioural features	54
3.3.3 Other	56
3.4 Discussion	59

4 Loss Aversion Model.....	60
4.1 Introduction.....	60
4.2 Asset Allocation Problem for a DC Pension Plan	65
4.2.1 Financial assets	66
4.2.2 Labour income process	67
4.2.3 Pension fund accumulation and target levels.....	68
4.2.4 Formulation of the target driven process	71
4.2.5 Optimisation and numerical method.....	72
4.3 Empirical analysis and parameter calibration	74
4.3.1 Loss aversion parameters.....	74
4.3.2 Salary scale	75
4.3.3 Parameter calibration	76
4.4 Simulation and results.....	77
4.4.1 The use of loss aversion utility in a target driven model	77
4.4.2 Baseline case	81
4.4.3 Sensitivity analysis.....	84
4.5 Conclusions.....	98
 5 Recursive Utility Model.....	 100
5.1 Introduction.....	100
5.1.1 Allocation of consumption across the life cycle	100
5.1.2 Epstein-Zin utility	105
5.2 The model	110
5.2.1 The model structure and optimisation problem	110
5.2.2 Parameter calibration	119
5.3 Results.....	121
5.3.1 Baseline case	121
5.3.2 Sensitivity analysis.....	131
5.4 Conclusion	142

6 Conclusions.....	146
6.1 Summary	146
6.2 Future Developments	149
Bibliography	151
Appendices.....	157
A 01 – Risk attitude survey.....	157
A 02 – Numerical method of loss aversion model.....	160
A 03 – Numerical solution for the dynamic programming and integration process...	163
A 04 – Optimal asset allocation at different ages	166
A 05 – Survival probabilities table	167

List of Tables

Table 1 Survey sample (966, all adults aged 18+).....	45
Table 2 Loss aversion questions	49
Table 3 Estimated loss aversion parameters	50
Table 4 Estimated subgroup loss aversion parameters	52
Table 5 Financial knowledge: base rate.....	54
Table 6 Financial knowledge: diversification.....	55
Table 7 Framing effect.....	56
Table 8 Behavioural questions.....	58
Table 9 Levels of replacement ratios under loss aversion and quadratic utility model (over 10k simulations), without salary risk.....	80
Table 10 Levels of replacement ratios under dynamic loss aversion model and 5-year lifestyle (over 10k simulations)	83
Table 11 Reliability of different asset strategies.....	84
Table 12 Loss aversion curvatures.....	85
Table 13 Final target weight ($v_1=0.53$, $v_2=0.77$ and $\lambda=3.4$)	92
Table 14 Only final target case	93
Table 15 Loss-aversion model Vs. Power utility.....	95
Table 16 Loss-aversion model Vs. Prospect theory model.....	97
Table 17 Pension plan member types	102
Table 18 Baseline parameter values	120
Table 19 RRA and EIS values for the different types of plan member	131
Table 20 The effect of the preference parameters on the mean optimal annuitisation ratio	141

List of Figures

Figure 1: 5-year deterministic lifestyle strategy	4
Figure 2: Markowitz mean-variance theory.....	5
Figure 3: Financial and human wealth over the lifecycle up to retirement	9
Figure 4: Loss aversion value function.....	17
Figure 5: Loss aversion value function.....	26
Figure 6: Loss aversion value function.....	40
Figure 7: Prospect theory – probability weighting function	41
Figure 8: Survey sample (total respondents: 966)	44
Figure 9: Planning Retirement Age (total respondents: 269)	55
Figure 10 Labour income process.....	68
Figure 11 Expected path of target at age 20	71
Figure 12 Salary scale (scaled to 100 at age 65).....	76
Figure 13 Loss aversion Vs. quadratic utility	80
Figure 14 Histogram (LA Vs. Quadratic utility model)	81
Figure 15 Mean optimal equity asset allocation	82
Figure 16 Histogram of replacement ratio: baseline case Vs. 5-year lifestyle	83
Figure 17 Sensitivity: loss aversion coefficient (Lamda)	85
Figure 18 Case 1: $V_1 < 1$ and $V_2 < 1$ (S-shaped).....	86
Figure 19 Mean optimal equity asset allocation for case 1	87
Figure 20 Case 2: $v_1 > 1$ and $v_2 > 1$ (reverse S-shaped).....	87
Figure 21 Mean optimal equity asset allocation for case 2.....	88
Figure 22 Histogram of replacement ratios: baseline case Vs. Reverse S-shaped utility .	89
Figure 23 Case 3: $v_1 < 1$ and $v_2 > 1$ (kinked power utility).....	89
Figure 24 Mean optimal equity asset allocation for case 3.....	91
Figure 25 Histogram of replacement ratios: baseline case Vs. kinked power utility	91
Figure 26 Sensitivity: final target weight.....	92
Figure 27 Power utility case	94
Figure 28 Loss aversion utility Vs. power utility	95

Figure 29 Histogram of replacement ratio: baseline case Vs. power utility.....	95
Figure 30 baseline Vs. prospect theory setting	97
Figure 31: Decomposition of total wealth over the life cycle.....	108
Figure 32: Labour income process.....	113
Figure 33: Optimal equity asset allocation for the final time period	122
Figure 34: Mean of simulated wealth, consumption and labour income	124
Figure 35: Mean of consumption.....	124
Figure 36: Human capital.....	125
Figure 37: Optimal equity asset allocation prior to retirement	127
Figure 38: Optimal contribution rate	128
Figure 39: Mean optimal contribution rate (with lower limit of 5% per annum, dotted).....	130
Figure 40: Mean optimal equity asset allocation (with lower limit on contribution rate of 5% per annum, dotted).....	130
Figure 41: Mean optimal contribution rate (for different RRA/EIS combinations)	132
Figure 42: Mean optimal asset allocation (for different RRA/EIS combinations)	133
Figure 43: Mean optimal contribution rate (for different levels of bequest motive)	135
Figure 44: Mean optimal equity asset allocation (for different levels of bequest motive)	135
Figure 45: Mean optimal contribution rate (for different levels of personal discount factor)	137
Figure 46: Mean optimal equity asset allocation (for different levels of personal discount factor).....	137
Figure 47: Mean optimal contribution rate (for different correlation between labour income and equity returns).....	139
Figure 48: Mean optimal equity asset allocation (for different correlation between labour income and equity returns).....	139
Figure 49: Optimal equity asset allocation at age 64.....	160
Figure 50: Optimal equity asset allocation at age 20.....	162

Acknowledgements

My PhD research was funded by way of a bursary from the Department of Actuarial Science and Insurance at Cass Business School, City University.

I wish to sincerely thank my supervisor, Dr. Douglas Wright, for his constant encouragement and guidance. I was very lucky to have Dr. Wright as my supervisor. In addition to the research problem, I got much valuable professional advice from him on how to structure and present a paper with more considerations on potential audience. The more pleasurable aspect was that our discussions did not stay restricted to my PhD. We discussed travelling, books, footballs, sports gambling, etc. This has become an important part of my PhD.

I am also grateful to the following for their various contributions: Dr Russell Gerrard for his advice on the numerical methods used in this thesis; Dr. Zaki Khorasanee and Dr. Iqbal Owadally for their helpful comments on my transfer report in the first year of my PhD.

Last but not least I would like to thank my family and friends, wherever they are, particularly my Mum and Dad; and, most importantly, my wife Xue Han. The writing of a thesis can be a lonely and isolating experience. The simple thought that she is always there is really a powerful source of inspiration and energy to me.

Declaration

Powers of discretion are hereby granted to the University Librarian to allow this thesis to be copied in whole or in part without further reference to the author. This permission covers only simple copies made for study purposes, subject to the normal conditions of acknowledgement.

Abstract

The question of optimal asset allocation strategy for defined contribution (DC) pension plans is addressed. A primary motivation for this study is provided by the recent literature on behavioural finance and intertemporal life-cycle investment theory.

In this thesis two alternative utility forms are considered: loss aversion and Epstein-Zin recursive utility. We develop a dynamic-programming-based numerical model with uninsurable stochastic labour income and borrowing constraints. In the loss aversion case, members are assumed to be loss averse with a target replacement ratio at retirement and a series of suitably defined interim target prior to retirement. We also extend the intertemporal life-cycle saving and investment theory to the dynamic asset allocation problem of DC pension schemes. A new approach to model contribution and investment decisions with focus on the member's desired pattern of consumption over the lifetime (based on Epstein-Zin utility preference) is proposed.

The thesis draws on empirical evidence of salary scales and loss aversion parameters from UK households, with labour income progress estimated from the New Earnings Survey and loss aversion parameters estimated on the basis of face-to-face interviews with 966 randomly selected UK residents.

Chapter 1

Setting the Scene

1.1 Statement of the Problem

Pension planning is probably the longest and most important financial decision that an individual has to make over their lifetime. The investment and asset allocation strategy in relation to pension funds has a profound effect on the capital market. To employers, the process of choosing a pension provider is one of the most important decisions an employer can make and one which can reflect directly on business and staff retention. To employees, even a small change in saving and investment decisions can have significant influence on their retirement standard of living.

1.1.1 Why defined contribution pension plans?

Basically, there are two types of pension plans: Defined Benefit (DB) and Defined Contribution (DC). Generally, in a Defined Benefit (DB) pension plan, the amount of retirement income of employees depends on the number of years they worked for the employers and the level of their salary when they retire, i.e. their final salary. A Defined Contribution (DC) pension plan is also called a money purchase plan. The level of contributions (made by employer and employee) is preset, but the amount of retirement income is not. The investment performance and longevity risks of the pension are borne by the employee, not the employer.

Important changes are occurring in the overall occupational pensions landscape. Traditionally, DB plans have been the dominant form of occupational pension provision in the UK, but a combination of increasing regulation and external market factors have led a growing number of employers to close their plans to new or even existing members and set up DC plans in their place. Some of the factors include new accounting rules forcing schemes to value their liabilities at current market rates, the prolonged equity bear market of 2000-2003, a secular fall in interest rate yields which are used to discount pension liabilities, and above all, longer life expectancy. The government is aiming to shift the current 60:40 state/private pension sector ratio to 40:60. An increasing number of workers now have to rely on defined contribution schemes to provide their future retirement income, either through a scheme set up by their employer or a personal pension as a group or individual arrangement, which is primarily the reason of objectives of this research.

1.1.2 The investment problem with DC plans

DC pension plans are designed to provide pensions on retirement for members. In a DC pension scheme, the member contributes part of his or her labour income each year and builds up a pension fund before retirement. At retirement, the member annuitises part of the pension fund by buying a life annuity. Unlike defined benefit (DB) pension plans, there is no guarantee offered by the employer that a pension fund will pay out a set amount on retirement. The investment performance risk of the pension is borne by the employee, not the employer.

Members in a DC pension plan face the following risks:

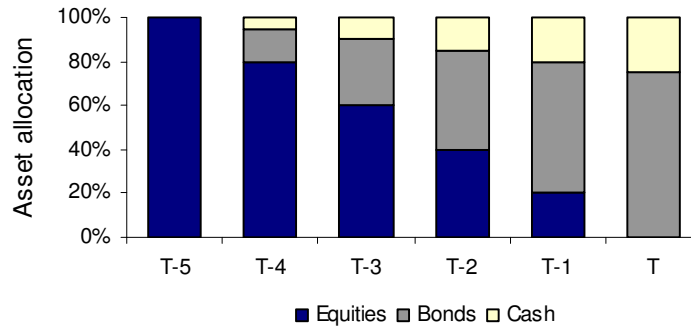
- Inflation risk. The risk that the value of assets do not keep pace with inflation thereby eroding the purchasing power of the final pension;
- Annuity risk. The risk that as the member approaches retirement the cost of purchasing an annuity fluctuates significantly;

- Investment risk. The risk that the value of the member's assets drops significantly when very close to retirement and there is little scope for recovery

Most DC plans allow members a degree of choice about how to invest their contributions. They also normally offer a default option in which the member's contribution is automatically placed if the member does not actively choose a fund. However, many members show little interest in financial matters and have little knowledge about investment. They often readily accept the default options. Choi et al. (2002) studied the tendency to accept scheme default options in the US and found that very few employees opt out of default arrangements even when they are free to do so. A similar tendency to accept the default is found in the UK. Hewitt Bacon and Woodrow, the pension consulting firm, found that around 80% of members in UK DC schemes accept the default fund choice (Bridgeland 2002). As Blake et al. (2005, p4) claimed in their studies of UK stakeholder pensions, "The vast majority of pension scheme members appear to passively accept whatever default fund the pension provider has chosen, but there is little consensus amongst providers as to what the appropriate characteristics for a default fund are, despite the importance of the choice in determining pension outcomes. In this sense, stakeholder pension schemes can be characterised as a lottery for the members".

In practice, a traditional deterministic lifestyle investment strategy (e.g. 5 year lifestyle, as shown in **Figure 1** below) is widely used by many DC pension plans as the default option. In such a strategy, the pension wealth is invested entirely in high risk assets (e.g. equities) when the member is young. Then, as the individual approaches retirement, the assets are switched gradually into lower risk assets such as bonds and cash. This provides a pre-determined switching strategy for members who do not wish to take an active role and represents a compromise between risk and reward.

Figure 1: 5-year deterministic lifestyle strategy



But, recent research suggests that a traditional deterministic lifestyle investment strategy still leaves members with considerable exposure to volatility in the years preceding that gradual switch. The range of potential results (in terms of both final fund levels and retirement income) from a traditional lifestyle strategy is huge. This makes it very difficult for a DC member to have any idea of what level of retirement income they can expect. For DC members who seek greater certainty in their retirement planning, the asset strategy adopted needs to be far more focused on achieving their retirement targets.

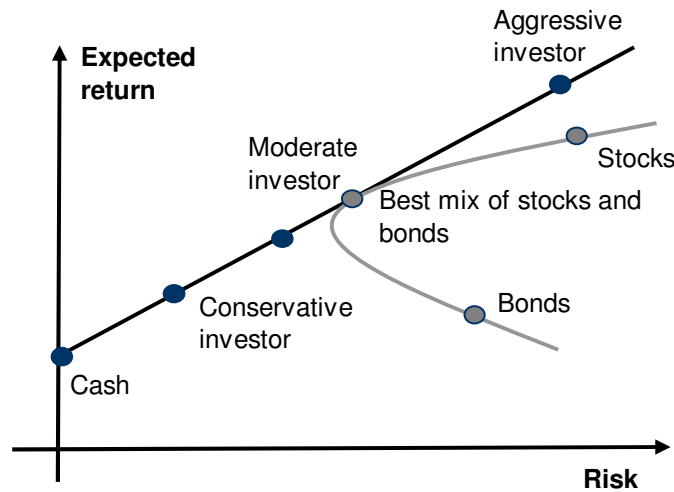
1.2 Dynamic asset allocation

1.2.1 Short-term investment strategy (two-period model)

Modern finance theory started with the mean-variance analysis of Markowitz (1952), which shows how investors should select assets if they only care about mean and variance of their end-of-period wealth. His analysis is shown in **Figure 2**.

A riskless asset (cash) corresponds to a point on the y-axis. The tangent point where the straight line touches the curved line is the market portfolio, which is the best mix of risky assets. Investors can pick a point somewhere on this upward-sloping line depending on their risk attitudes. This is also called mutual fund separation theorem.

Figure 2: Markowitz mean-variance theory



If we assume that investors have utility defined over final wealth (i.e. making investment decisions to maximise the expected utility of final wealth), in order to get a tractable solution, several assumptions about the utility function and distribution of asset returns need to be made. Some popular assumptions are the following:

1. Quadratic Utility, in which case, $U(W_{t+1}) = aW_{t+1} - bW_{t+1}^2$, absolute and relative risk aversion increases with wealth. No distributional assumptions on asset returns are needed.
2. Exponential Utility, in which case, $U(W_{t+1}) = -\exp(-\theta W_{t+1})$, absolute risk aversion is a constant θ , relative risk aversion increases with wealth. Assets returns are assumed to be normally distributed.
3. Power Utility, in which case, $U(W_{t+1}) = (W_{t+1}^{1-\gamma} - 1)/(1-\gamma)$, absolute risk aversion decline with wealth and relative risk aversion is a constant γ . Asset returns are lognormally distributed.

1.2.2 Long-term investment strategy (multi-period model)

Intuitively, long-term investors are different from short-term investors, because long-term investors have a longer investment horizon and they could change the portfolio decision when any new information arrives or simply just as time passes. However, according to the analysis of Markowitz, it seems that all investors who only care about mean and variance of final portfolio wealth should hold the same portfolio of assets, no matter whether they have a short-term or a long-term investment horizon. Is this really correct?

The answer is yes, as long as the following conditions are met:

- 1) the risky asset returns are independent identically distributed (IID);
- 2) investment opportunity set is constant (i.e. constant risk-free rate, constant expected return and constant volatility);
- 3) investors' relative risk aversion coefficient does not depend systematically on their wealth.

These points have been understood for many years. In a discrete time framework, Samuelson (1963, 1969) proposed a dynamic programming model to explain this. We will discuss this method later but the main idea is to use a series of two-period problems to solve the multi-period optimisation problem recursively. In the last time period, the long-term investor actually faces a two-period optimisation problem and will become a short-term investor. Since the asset returns are IID and investors' risk attitudes do not change over time, the results for all the two-period maximisation problems should be the same. In other words, long-term investors should behave like short-term investors in this case.

Merton (1969, 1971) was the first to apply this approach to a continuous-time optimal investment problem. The continuous-time models can be seen as the limit of the multi-period discrete-time models when the time period becomes very small. Merton's model suggests that, in the single-risky-asset and constant-investment opportunity setting, the

optimal portfolio weight in the risky asset for an investor with power utility should equal the risk premium (μ) divided by variance of the risky asset returns (σ^2) times the coefficient of relative risk aversion (γ):

$$\alpha = \frac{\mu}{\gamma\sigma^2}$$

[1]

Another interesting point to note about long-term investors is that they are more concerned with the standard of living that can be financed by wealth, rather than the final wealth level itself. Or, as explained by Campbell and Viceira (2002), “they consume out of wealth and derive utility from consumption rather than wealth” (p37). For example, let us assume long-term investors have power utility on consumption and only have two assets to invest (one risky and one riskless). They face a wealth accumulation process as follows:

$$W_{t+1} = (W_t - C_t) \left[(1 - \alpha_t)(1 + r_f) + \alpha_t(1 + r_h) \right]$$

[2]

where r_f is the risk-free interest rate; r_h is rate of investment return of risky asset; W_t and C_t are the wealth and consumption level at time t respectively.

In this case, the optimal dynamic asset allocation on risky assets (α_t) should maximise the expected present value (EPV) of total future consumption utility, i.e.,

$$\max E_t \sum_{i=0}^{\infty} \beta^i U(C_{t+i})$$

[3]

where β is the time discount factor, representing the relative weight investors put on future consumption.

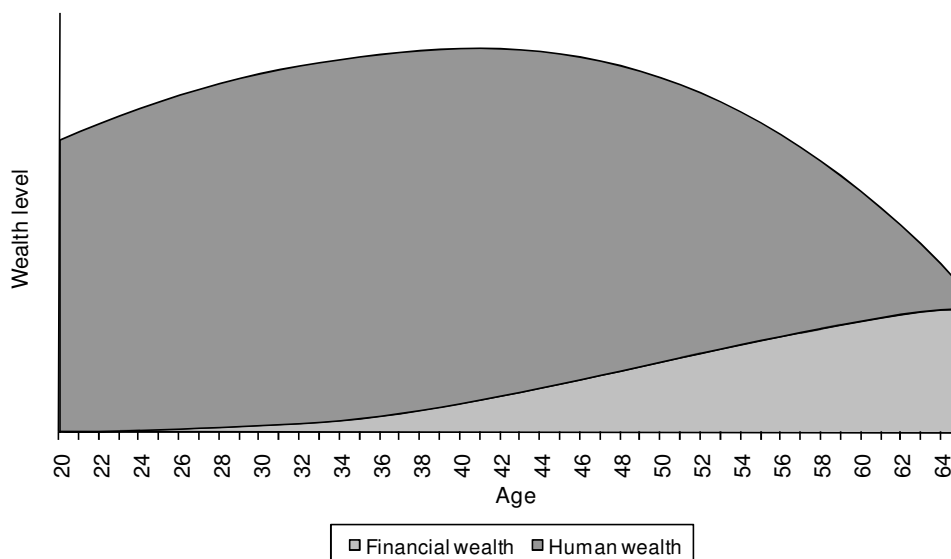
An approximate closed form solution is available for this problem. Campbell and Viceira (2002) showed that, for above long-term investors with power utility over consumption, the optimal portfolio weight should still be the same as suggested in[1], if the consumption-wealth ratio is constant over time.

1.2.3 Financial and human wealth

So far we have briefly reviewed the research history of long-term portfolio choice with financial assets only. In a realistic life-cycle saving and investment model, however, labour income is also important for long-term investors. With income risk, the optimal portfolio weight is not constant and will follow a “lifestyle” strategy. This can be explained. Human wealth can be understood as the expected net present value (NPV) of future labour income. An individual’s labour income can be seen as a dividend on the individual’s implicit holding of human wealth. The ratio of human to financial wealth is the crucial determinant of life-cycle portfolio composition.

In early life, as shown in **Figure 3** below, the ratio is large because people have little time to accumulate financial wealth and expect to receive labour earnings for many years. Given that the growth rate of labour income is close to the risk-free rate, labour income can be seen as an implicit substitute of riskless asset. Young individuals hold “too much” in this non-tradable riskless asset and therefore need to allocate most their financial wealth to risky assets to keep the overall portfolio composition constant. When they grow older, they accumulate more financial wealth and have less human wealth left (i.e. a smaller holding in this implicit non-tradable riskless asset). Thus, they need to rebalance the portfolio and increase the weight in the riskless asset.

Figure 3: Financial and human wealth over the lifecycle up to retirement



1.2.4 The consumption problem

Most economic decisions are intertemporal, as current decisions made will affect future available choices. Decisions regarding the funding and investment strategies adopted for retirement savings are a classic example of this. The retirement saving decisions made today affects not only an individual's current level of consumption but also future consumption possibilities. In other words, individuals face an intertemporal trade-off: if they save more today, they must consume less and hence their current utility declines, but they can then consume more in the future (thereby increasing future utility).

In fact, the most basic objective of a DC pension scheme (or all types of pension arrangements) is to arrange consumptions over life cycle. The member's decision of contribution rate, portfolio asset allocation and the proportion of the accumulated fund used to purchase an annuity, are all driven by his or her preference between current and future consumptions. Thus, as a result, the optimal asset strategy of DC pension plans

should depend on the pattern of consumption levels over the entire lifetime, rather than just focusing on the terminal pension wealth level at retirement.

1.2.5 Optimisation methods

The inclusion of labour income will make the life-cycle investment model more realistic. As a tradeoff, the optimisation problem becomes very difficult (if not impossible) to solve analytically. Therefore, the recent literature uses a variety of numerical methods to approximate the solution of the dynamic portfolio optimisation problem.

Dynamic programming¹

Dynamic programming was originally used in the 1940s by Richard Bellman to solve discrete-time optimisation problems. Since then, it has become one of the most fundamental building blocks of numerical methods in multi-period portfolio choice problems.

The basic idea is to turn the multi-period optimisation problem into a series of two-period optimisation problems. At the heart of dynamic programming is the value function, which represents the maximum present discounted value of the objective function onward as a function of current state variables. For each period, going backward from the next-to-last period to the beginning, the solution is found by maximising the one-step ahead expectation of the approximated value function.

To explain the idea into more detail, let us look at one simple example. At age t , an investor faces a long-term multi-period portfolio choice problem to maximise the expected present value (EPV) of total utility of consumption (U_{65}) at retirement age 65.

We simplify the problem and make the following assumptions:

- there are only two financial assets to choose: one risk-free asset (R_f) and one risky asset (R_h);

¹ Dynamic programming method is also used in our models later on to solve the optimisation problem numerically.

- The risk-less asset is assumed to yield a constant interest rate r_f p.a.
- The return on the risky asset (R_h) is assumed to be normally distributed with mean $r_f + \mu$, and volatility of σ_1 ;
- the investor has a wealth level of W_t at age t ;
- the investor has a salary level of Y_t at age t ; the growth rate of salary is given by $I_t = r_t + Z_2$ where r_t is the annual growth rate of average salary and $Z_2 \sim N(0, \sigma_2^2)$;
- the investor has time-separable power utility ($U(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}$) on consumption at age t ;
- the investor is allowed to rebalance her portfolio annually;
- the investor will consume all available wealth in the last period.

The optimisation problem is $\text{Max}_{\alpha_t, C_t} E_t \sum_{i=0}^{\infty} \beta^i U(C_{t+i})$, subject to the constraint that $W_{t+1} = (W_t + Y_t - C_t) \cdot [(1 - \alpha_t)(1 + r_f) + \alpha_t(1 + r_f + \mu + Z_1)]$, where α_t is the asset allocation in risky assets at age t , β is the time discount factor, and $Z_1 \sim N(0, \sigma_1^2)$;

We start from the next-to-last period, i.e. the period from age 64 to age 65. Specifically, at age 64, the value function is $J_{64}(W_{64}, Y_{64}) = \text{Max}_{\alpha_{64}} [U_{64} + \beta E_{64}(J_{65})]$, where $J_{65} = \frac{W_{65}^{1-\gamma}}{1-\gamma}$.

As an important step of stochastic dynamic programming, we need to discretise the state variables (wealth level and income level, in this example) first. For example, wealth and labour income can be discretised into 100 and 10 even grids, respectively, in computation, so that we can calculate the optimal control variables (α_t, C_t) and value function for each grid point on the 100 by 10 matrix.

To approximate the expectation term in the value function, by far, the most popular approach is quadrature integration. Gauss-Hermite quadrature is used to discretise shocks

(normally distributed variables for the risky asset return and salary growth rate, in this example) into several (e.g. 7) nodes, and the procedure of discretising Z_1 and Z_2 is to substitute $\sqrt{2}Z_{1,m}$ and $\sqrt{2}Z_{2,n}$ for them respectively. So,

$$E_t(J_{t+1}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J_{t+1}(W_{t+1}, Y_{t+1}) f(Z_1, Z_2) dZ_1 dZ_2 \approx \pi^{-1} \sum_{m=1}^7 \sum_{n=1}^7 w_{Z_{1,m}} w_{Z_{2,n}} \left[J_{t+1}(\sqrt{2}Z_{1,m}, \sqrt{2}Z_{2,n}; W_{t+1}, Y_{t+1}) \right]$$

, where $w_{Z_{1,m}}$, $w_{Z_{2,n}}$ and $Z_{1,m}, Z_{2,n}$ are the Gauss-Hermite quadrature weights and nodes.

As soon as we get optimal asset allocation and value function for each grid point at age 64, we can then use them to solve the optimisation problem with the same 100 by 10 matrix for previous time period, i.e. $J_{63}(W_{63}, Y_{63}) = \underset{\alpha_{63}}{\text{Max}}[U_{63} + \beta E_{63}(J_{64})]$. It is very likely that the accumulated state variable values from the previous time period are not on a grid point, in which case, some interpolation methods (e.g. bilinear, cubic spline, etc.) are employed to approximate the value function (J_{64} , in this case). The iteration process is then repeated backward until the beginning.

The above numerical method is called value function iteration. However, this approach requires knowledge of the distribution of all the shocks, so that appropriate quadrature integration (e.g. Gauss quadrature) can be used. Further, it cannot handle a large number of state variables. To overcome these limitations, Brandt et al. (2005) propose a simulation-based method based on recursive use of approximated optimal portfolio weights. The idea is to estimate asset return moments with a large number of simulated sample paths then approximate the value function using a Taylor series expansion. Also, if the returns are path-dependent, it would be necessary to regress the return variable on the simulated state variables from the previous time period, before using the Taylor expansion with conditional return moments.

In a continuous-time framework, Bellman extends earlier work with William R. Hamilton and Carl G. J. Jacobi and derives Hamilton-Jacobi-Bellman (HJB) equation, which represents the fundamental partial differential equation obeyed by the optimal value function. This then becomes the so called stochastic optimal control theory.

Since most of the models in my PhD research are based on a discrete-time setting, continuous-time models are out of the scope of this thesis. Continuous-time models appear to be a fruitful area of future research. The biggest benefit of using continuous-time models is that the mathematical calculations are easier in continuous time (and therefore it might be easier to get closed-form solutions). But if we need to use numerical methods to solve the problem, discrete-time and continuous-time models are in fact very similar in terms of methodology.

1.3 Behavioural issues

In a DC pension plan, the investment performance risk is borne by the employee, not the employer. The participants have more responsibilities in terms of deciding how much to save and how to invest the funds. DC plans have traditionally been regarded as employee-directed with employees seen as the active agents and the employer thought to play a minimal decision-making role. But, in fact, pension plan design is not a neutral vehicle within which participants make their own rational choices based on rational expectations. Instead, participants have a strong tendency to choose default options in DC schemes. They can be easily influenced by plan design, both in the saving area and in the investment decision-making as well.

1.3.1 The existing literature on DC investment

Cairns (1994) reviewed and divided the objectives of defined contribution (DC) pension research into two categories: (a) members are told the likely future amount of pension and expect the target to be attained, in which case, we need the actual pension to be as close to this level as possible; or (b) members are told the actual pension only at the date of retirement, in which case, we would need to maximise the expected utility of the net replacement ratio at retirement.

If equity returns are mean-reverting, to hold equity for a long period before retirement can be justified because the volatility of equity returns will be lower over a longer time period. But Howie and Davies (2002) finds limited evidence on mean-reversion of UK equity prices (or time diversification, which is mathematically the same thing). Also, as discussed earlier, to have a fixed asset allocation is not appropriate for long-term investors. DC pension plan members normally have a 20 to 30 year investment horizon and expect to receive many years of pension income after retirement. Thus, it is also necessary to consider non-pension assets² (e.g. labour income) and have a more dynamic investment solution.

With regard to the dynamic asset allocation problem for DC pension plans, most of the existing literature investigates the optimal dynamic asset strategy of DC pensions by assuming a fixed contribution rate (e.g. 10%, 12%) and maximising the expected utility of the terminal replacement ratio (i.e. pension as a proportion of final salary) at retirement (for example, Cairns et al. (2006)) or by minimising the expected present value of the total disutility³ until retirement (for example, Haberman and Vigna (2002)).

Haberman and Vigna (2002) derived a dynamic-programming-based formula for the optimal investment allocation in DC schemes and considered three risk measures in analyzing the final net replacement ratio: the probability of failing the target, the mean shortfall and the value at risk (VaR). They suggested that the risk profile of the individual and the trade-off between different risk measures of the downside risk are important factors to be taken into consideration. According to their research, only risk averse members should adopt a lifestyle strategy and the switching point depends on the degree of risk aversion of members and the time to retirement, i.e. the more risk averse, the sooner the switch; the longer the accumulation period, the later the switch.

² In real life, other non-pension assets such as housing also plays an important role in the member's financial planning. However, this is out of the scope of our models in this thesis.

³ The disutility is normally defined over the deviation of actual fund level from interim and final target fund levels.

Cairns et al. (2004) investigated a model incorporating asset risk, salary risk and interest-rate risk and proposed a new form of terminal utility function by using the plan members' final salary as a numeraire. They showed that the use of a stochastic asset allocation strategy (which they called stochastic lifestyle) could enhance the welfare relative to deterministic lifestyle and benchmark mixed funds strategy.

1.3.2 Differentiators

Application of behavioural features in DC plan design

As argued by Mitchell and Utkus (2004), defined contribution (DC) plans can provide real retirement security, but only if participants utilise them appropriately and make optimal investment decisions. There is growing evidence suggesting that there are key behaviour barriers preventing participants from doing so. One obvious solution to dealing with significant behavioural barriers to the effective use of DC plans for retirement provision is to offer some form of education to participants, but intelligent plan design is also required when some participants show little interest in financial matters and readily accept default options.

Representing an alternative way of looking at financial market, behavioural finance research is quite helpful for pension plan design. Behavioural finance is a combination of psychology and economics that investigates what happens in markets in which some of the agents display behavioural limitations and complications. Most behavioural studies have an empirical component in common and show a high predictive value. The fertilisation of economics with psychology and empirical evidence makes it interesting and promising in the pension plan design field. We believe behavioural studies are useful tools to improve both the design of pension schemes and the efficiency of communication.

Loss-aversion dynamic asset allocation

The concept of risk and risk aversion are cornerstones of economic modelling. Within the expected utility framework, the only explanation for risk aversion is that the utility

function for wealth is concave. People are assumed to be risk-averse expected utility maximisers and make rational choices based on rational expectations. To date, the investment problem of DC pension plans has been addressed principally for life cycle expected utility optimizers.

However, as will be discussed later in Chapter 2, observed behaviour appears to invalidate expected utility theory as a descriptive model. Instead the use of loss aversion (LA) utility seems quite promising and helpful in modelling the optimal dynamic asset allocation of DC pension funds.

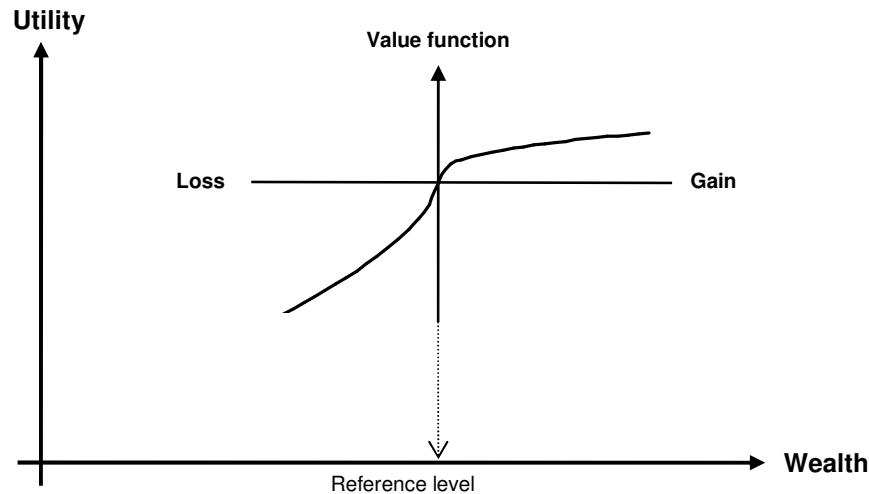
The idea of “loss aversion” was first proposed by Kahneman and Tversky (1979) within the framework of prospect theory⁴. As one of its distinguishing features, the loss aversion value function is defined on gains and losses of wealth relative to a reference point, rather than absolute levels of total wealth (as is the case with the traditional ideas of utility theory).

As shown in **Figure 4** below, the two key properties of the loss aversion value function are:

- (i) it is S-shaped (i.e. convex below the reference point and concave above it), implying that individuals are risk seeking in the domain of losses and risk averse in the domain of gains; and
- (ii) it is asymmetric (i.e. steeper below the reference point than above, because of the effect of the loss aversion ratio λ), implying that individuals are more sensitive to losses than to gains.

⁴ Kahneman and Tversky (1979) developed this theory to remedy the descriptive failures of subjective expected utility (SEU) theories of decision making.

Figure 4: Loss aversion value function



We can incorporate this behaviour research finding into the investment solution of DC pension plans. In a DC pension plan, prior to retirement, the members can be thought to have a final target fund level at retirement and a series of consistent interim targets before retirement. Members are assumed to be loss averse with respect of these targets (which define the reference points in the loss aversion framework above) and make asset allocation decisions to maximise the sum of expected present value (EPV) of loss aversion value function at each age until retirement.

Consumption problem

Most economic decisions are intertemporal, as the current decisions made will affect the future available choices. Decisions regarding the funding and investment strategies adopted for retirement savings are a classic example of this. The retirement saving decisions made today affect not only an individual's current level of consumption but also future consumption possibilities. In other words, individuals face an intertemporal

trade-off: if they save more today, they must consume less and hence their current utility declines, but they can then consume more in the future (thereby increasing future utility).

In fact, the most basic objective of a DC pension scheme (or all types of pension arrangements) is to plan consumptions over life cycle. The member's decision of contribution rate, portfolio asset allocation and the proportion of the accumulated fund used to purchase an annuity, are all driven by her preference between current and future consumptions. Thus, as a result, the optimal asset strategy of DC pension schemes should depend on the pattern of consumption levels over the entire lifetime, rather than just focusing on the terminal pension wealth level at retirement.

1.4 Overview of the thesis

The thesis consists of six chapters and is structured as follows:

Chapter 1 introduces the background of the investment problem of DC pension scheme, including a brief review on existing literature in the area of optimal asset allocation problem. Research motivations are discussed in chapter 1 as well.

In Chapter 2, we review the behavioural features that are relevant to the work of DC plan design and communication. This is necessary especially given that some approach taken in current DC plans may be counterproductive in encouraging retirement saving and helping members make appropriate investment decisions.

Chapter 3 is devoted to a survey we did with sponsorship from Distribution Technology Ltd. The survey was conducted on a face-to-face interview basis from 14th April 2005 to 19th April 2005. A total of 966 responses were received. The results help us to investigate how people's attitudes to risk vary during long-term financial decision making. This

chapter provides the empirical evidence of behavioural utility parameters on UK households.

This is followed, in Chapter 4, by incorporating the survey results into a loss-aversion-based model to investigate the optimal investment strategy for DC scheme members. Members are assumed to be loss averse with a target fund level at retirement and a series of suitably defined interim targets prior to retirement, and are assumed to make asset allocation decisions with the aim of maximising the expected present value (EPV) of their total loss aversion value function over the period until retirement.

Chapter 5 is based on a different research idea, intertemporal saving and investment. I built an intertemporal investment model for DC pension plan, in which case, the plan members' contribution and investment decisions all depend on their preferences of consumption levels over their entire lifetime. Epstein-Zin preference is used to separate risk aversion and elasticity of intertemporal substitution (EIS). The effects of risk aversion and EIS on contribution rate and asset strategy over the life cycle are also considered.

The thesis concludes with Chapter 6, which highlights the main contributions of the research, explains some of the shortcomings of the models, and looks to the future of asset allocation strategy of DC pension plans.

Chapter 2

Behavioural Features in DC Plan Arrangement

2.1 Introduction

2.1.1 Background

Given that in many countries, social security pensions are either non-existent or provided at only very low levels, and given that employers are moving away from providing defined benefit (DB) pension plans, it is increasingly becoming the responsibility of individuals to make adequate retirement provision for themselves. An increasing number of workers now have to rely on defined contribution (DC) plans to provide their future retirement income, either through a plan set up by their employer or a personal pension as a group or individual arrangement.

DC plans have traditionally been regarded as employee-directed with employees seen as the active agents and employer thought to play a minimal decision-making role. In fact, pension plan design is not a neutral vehicle within which participants make their own rational choices based on rational expectations. As argued by Mitchell (2004), “Being good at retirement savings requires accurate estimates of uncertain future processes, including lifetime earnings, asset returns, tax rates, family and health status, and longevity. In order to solve this problem, the human brains as a calculating machine

would need to have the capacity to solve many decades-long time value of money problem, with massive uncertainties as to stochastic cash flows and their timing”. Many workers do not have particularly firm convictions about their desired saving behaviour. They can be easily influenced by plan design, both in the saving area and in the investment decision-making as well.

According to the annual DC/AVC Survey of Hewitt Bacon & Woodrow’s (2004), only 3% of employers think that DC members have a good understanding of the funding levels required to build sufficient savings for retirement. In US, according to survey by John Hancock insurance company (2003), 42% of the respondents said they had little or no investment knowledge, a further 38% stated they were “somewhat knowledgeable”, only 20% of regarded themselves as knowledgeable investors. Alistair Byrne (2004b) did a similar survey in UK on the members of a mid-sized occupational pension plan and found that the results were broadly consistent with the US findings in that many employees show limited knowledge and interest in their pension arrangements. According to the survey by Office of Fair Trading (1997), half of the respondents agreed or strongly agreed that “I have found all the information I have seen, and the advice I have received, on pensions very confusing.”

All the above evidence shows that the plan design and investment option offering will have substantial implications to the DC pension plan members. If DC pension plan members have behavioural biases and are not rational agents, then we are in fact transferring the long-term investment risk to many individuals who cannot make optimal decisions. This will have a potentially costly long-term consequences to the economy. Thus, it is important that investment arrangements in a DC pension plan are carried out with the knowledge of members’ potential biases and errors in decision making. Representing an alternative way of looking at financial market, behavioural finance is a combination of psychology and economics that investigates what happens in markets in which some of the agents display human limitations and complications.

2.1.2 Literature review

There has been considerable amount of research carried out in the applications of behavioural studies in capital market, but relatively few of them deal with pension design and the implications to the work of pension actuaries. By reviewing several behavioural features, Mitchell and Utkus (2004, p30) illustrated how behavioural research of the last few years had fundamentally challenged the ways in which plan sponsors, retirement service providers and policy makers should think about retirement plan design in the future. They proposed several insights in pension plan design: behavioural research challenged the notion that workers are rational, autonomous and can exercise unbiased judgment in their retirement plans; sponsors and policymakers can affect members' saving and investment decisions by choosing different default structures; some approach taken in current DC plans may be counterproductive in encouraging retirement saving; education in DC plans has its effective limits. The plan members' tendency to choose default options in DC plans tells us that the plan design has a significant impact on scheme members. Blake et al. (2005, p4) claimed in their studies of UK stakeholder pension, "The vast majority of pension scheme members appear to passively accept whatever default fund the pension provider has chose, but there is little consensus amongst providers as to what the appropriate characteristics for a default fund are, despite the importance of the choice in determining pension outcomes. In this sense, stakeholder pension schemes can be characterised as a lottery for the members".

Some researchers tried to use the findings in behavioural study to improve the pension plan design. Taylor (2000) discussed the implications of behavioural finance on actuarial work including anchoring effect, prospect theory, framing, myopic loss aversion, overconfidence and mental accounting⁵. Sykes (2004) discussed the ideas of behavioural finance and their application to the areas in which actuaries work, also highlighted several issues by means of examples.

⁵ These concepts will be explained and discussed later on in this chapter.

Much of the other existing literature explaining saving behaviour in DC pension plan focuses on how participation and saving rates vary according to plan design. Choi et al. (2002) investigate the role of inertia in default options and found that employees' choices can be easily swayed by the default options established by their employer. Papke (2004) finds that having the ability to direct the asset allocation of contributions to an employer-sponsored saving plan leads to a large 36 percent point increase in the probability of participating. But this is not to say, having a greater number of funds available should make members in DC schemes more willing to invest. Actually, Iyengar, Huberman and Jiang (2003) show a strong negative relationship between the number of funds offered in DC schemes and average participation rates. They find that increasing the number of funds offered by 10 lead to a 1.5 to 2 percentage point decline in the participation rate. Also, behavioural studies find evidence that when an employee is offered a number of funds to choose, there is a bias towards dividing the money evenly among the funds offered. The asset allocation an investor chooses will depend on the arrays of funds offered in the retirement plan. For example, employees may evenly allocate asset when they are offered one equity fund and one bond fund. But if another equity fund were added, the allocation to equities would jump to two thirds.

As mentioned by Van Der Sar (2004), most behavioural studies have an empirical component in common and show a high predictive value. It is maybe true that most of the behavioural research findings can not provides a sound basis for a normative theory of asset allocation. However, as will be discussed in this chapter, we believe behavioural studies are definitely useful tools to design DC pension plans and improve the efficiency of communication. As claimed by Sykes (2004), financial experts should be concentrating their efforts on identifying inefficiencies and designing the necessary products to offset them, rather than trying to decide which theory is a better explanation of current behaviour.

In this chapter, I choose some distinguishing features that seem particularly relevant to DC pension arrangement and then discuss their possible applications in the different processes of DC pension arrangement. This chapter will proceed as follows: in Section

2.2, all relevant behavioural features are reviewed; applications to DC pension arrangement are elaborated in Sections 2.3 and concluded in Section 2.4.

2.2 Behavioural finance

2.2.1 Repeated gamble

The literature on repeated gambles can be traced to 1963. Samuelson asked his colleague whether he would be willing to accept the following bet: a 50 percent chance to win \$200 and a 50 percent chance to lose \$100. The colleague turned him down and offered his rationale as: “I won’t bet because I would feel the \$100 loss more than the \$200 gain.” However, the colleague announced that he was happy to accept 100 such bets. After that, many academics tried to investigate why individuals make decisions as Samuelson’s colleague does.

They argued individuals tend to perceive and evaluate changes of wealth (gains and losses) rather than final wealth positions as assumed in the expected utility framework. More importantly, they are more sensitive to losses than to gains. To explain the effect, if we assume Samuelson’s colleague has a value function as follows,

$$v(x) = \begin{cases} x, & \text{if } x \geq 0 \\ 3x, & \text{if } x < 0 \end{cases} \quad [4]$$

where is x the potential outcome.

he would reject the lottery with 50% chance of \$200 gain and 50% chance of \$100 loss. This can also helps to explain why Samuelson’s colleague is willing to accept 100 such bets. For example, when there are two consecutive such bets, he actually faces another

lottery with 25% chance to win \$400, 50% chance to win \$100 and 25% chance to lose \$200. According to the above value function, he would feel indifferent to 2 gambles and be willing to take a sequence of more than 2 gambles.

Benartzi and Thaler (1999) studied the decision making of multiple plays of a gamble or investment and showed repeated play of a positive expected value gamble is more attractive if they are shown as the explicit resulting distribution of possible outcomes. They tried to apply the finding to retirement investing and found that subjects were willing to invest up to 90% of their investment funds in stocks when they were shown distributions of long-run returns rather than one year.

However, the above value function [4] does not fully capture the empirically observed attitude towards risk. Individuals display diminishing sensitivity in both gain and loss. Kahneman and Tversky (1979, 1992) discussed the issue in detail and made it popular through their prospect theory.

2.2.2 Prospect theory and loss aversion

As one of its distinguishing features, in the prospect theory, there is a value function defined on gains and losses relative to a reference point, rather than absolute levels of total wealth.

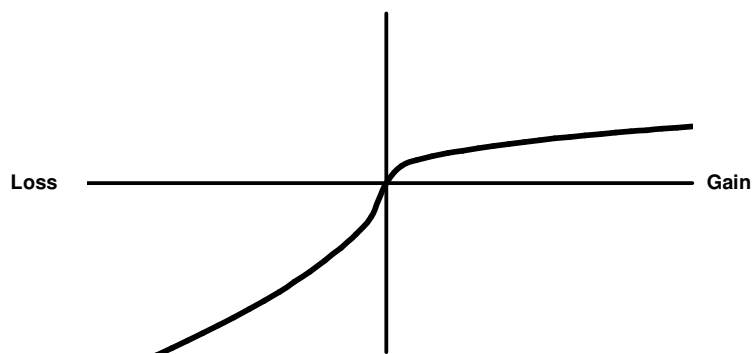
Tversky and Kahneman (1979, 1992) suggested a value function as follows:

$$V(x) = \begin{cases} x^{\nu_1} & \text{if } x \geq 0 \\ -\lambda(-x)^{\nu_2} & \text{if } x < 0 \end{cases}, \quad [5]$$

where $\lambda \geq 1$, $0 \leq \nu_1 \leq 1$ and $0 \leq \nu_2 \leq 1$. This function reflects loss aversion via parameter λ and diminishing sensitivity via parameter ν_1, ν_2 . As shown in

Figure 5 below, two important properties of the value function are that, first, it is S shaped (convex below the reference point and concave above it), i.e. people are risk seeking in the domain of losses and risk averse in the domain of gains. The smaller the ν_1, ν_2 , the more risk averse the individual is in the gain domain and the more risk seeking in the loss domain; second, it is asymmetric, steeper below the reference point than above because of loss aversion. The reference point is determined by each individual as a point of comparison. Based on an experiment conducted using a group of 25 graduate students, Tversky and Kahneman (1992) suggest that $\lambda = 2.25$ and $\nu_1 = \nu_2 = 0.88$.

Figure 5: Loss aversion value function



Prospect theory receives increasing attention in economic analysis because of the fact that it can explain many phenomena which it is hard to explain in expected utility framework (e.g. premium puzzle⁶). Rabin and Thaler (2001) argued the expected utility theory is manifestly not close to the right explanation for most risk attitudes and “we have also been surprised by economists’ reluctance to acknowledge the descriptive inadequacies of expected utility theory”. They claimed that loss aversion and the tendency to isolate each

⁶ Equity premium is defined as the difference in returns between stocks and a risk-free asset such as treasury bill. Equity premium puzzle refers to the argument that stocks have outperformed bonds over the last century by a surprisingly large margin, and that investors will be unreasonably risk averse to be willing to hold bonds at all. Mehra and Prescott (1985) estimated that investors would have to have coefficients of relative risk aversion in excess of 30 to explain the historical equity premium, whereas previous estimates and theoretical arguments suggest that the actual figure is close to 1.

risky choice should replace expected utility theory as the descriptive theory of risk attitudes.

2.2.3 Mental accounting

Meanwhile, people show a tendency to separate related events and decisions and find it difficult to aggregate events. The process by which decision-makers formulate problems for themselves is called mental accounting. People vary in their attitudes to risk between their mental accounts. Rather than netting out all gains and losses, people set up a series of “mental accounts” and view individual decisions as relating to one or another of these accounts.

Let us use an example to explain the effect of mental accounting. Kahneman and Tversky (1982) asked people the following questions: “Imagine that you have decided to see a play where admission is \$10 per ticket. As you enter the theatre you discover that you have lost a \$10 note. Would you still pay \$10 for a ticket to the play?” 88% of the people said they would still buy a ticket.

They then changed the question to the following:” Imagine that you have decided to see a play and paid the admission price of \$10 per ticket. As you enter the theatre you discover you have lost the ticket. The seat was not marked and ticket cannot be recovered. Would you pay another \$10 for another ticket?” In this case, only 46% of people said they would buy another ticket.

In the above example, we all understand that the loss of ticket and the loss of the \$10 are mathematically the same. However, people seem to set up a “theatre ticket” mental account in the second question and add the cost of buying another ticket (\$10) to the total cost of “theatre ticket”. Then they decided \$20 per ticket was too expensive. In our everyday lives, we can actually also find lots of example about mental accounting. For example, some people do not want to buy life assurance products; meanwhile, they are willing to spend lots of monies on gambling.

2.2.4 Myopic loss aversion (MLA)

In 1995, Benartzi and Thaler (1995) put forward an explanation for the equity premium puzzle by combining the above two behavioural concepts: loss aversion and mental accounting. The behavioural hypothesis of myopic loss aversion (MLA) assumes that people are myopic in evaluating outcomes over time, and are more sensitive to losses than to gains. “Myopic” is put here to describe the short-sightedness that induces a decision maker to evaluate each alternative within a sequence independently, whereas a rational decision maker would evaluate the sequence as a whole.

Gneezy and Potters (1997) tested this hypothesis in laboratory experiments. In the experiments, subjects were confronted with a sequence of 12 identical but independent rounds of a lottery. As a crucial feature of the design, there were two different treatments. In “high frequency treatment”, the subjects played rounds one by one (i.e. they were not allowed to bet on round 2 until they were informed about the realisation of the lottery in round 1, as so on). In “low frequency treatment”, however, subjects played the rounds in blocks of three. For example, at the beginning of round 1, subjects had to decide how much to bet in rounds 1, 2 and 3 (these three bets were restricted to be equal). They were then informed about the combined realisation for round 1, 2 and 3 (i.e. they only knew the aggregate result for 3 rounds rather the results for any particular round). The basic idea behind the two treatments is to manipulate the evaluation period. In “low frequency treatment”, the frequency of choice and information feedback was lower than “high frequency treatment”. The results of their experimental studies suggests that the more frequently returns are evaluated, the more risk averse investors will be, which is in line with the MLA hypothesis. Thaler et al. (1997) examined how individuals split money between two assets with different levels of risk by conducting an experiment to 80 undergraduate students at the University of California at Berkeley and made similar conclusions.

All the above evidence on MLA suggests that the willingness to invest in the risky asset is influenced by a simultaneous manipulation of feedback frequency and period of commitment. If participants receive less frequent feedback and are forced to make a binding multi-period decision, they evaluate the assets less myopically and are more willing to accept risk. Rather than using additive approach as above researchers did, Langer and Weber (2003) used multiplicative approach⁷ and found that binding decisions and providing less frequent feedback seemed to help people to be more willing to accept risk and make long-term investment decisions. But they claimed that there was no simple effect from combining commitment and feedback and “it seems that if people are committed to their decisions, more frequent feedback is helpful, because over time it becomes more salient that occasional losses are outweighed by ultimate gains”.

2.2.5 Framing effect

Many individuals deviate from standard economic theory in another important way. How the question is asked or “framed”, in particular, the wording in terms of gains and losses can have an enormous impact on the decision made. Mitchell (2004) argued that “rational economic agents would not be expected to vary their responses to a question based on how it is asked. But in practice, many people do exactly that, both in the saving area and in the investment decision-making as well”.

2.2.6 Anchoring effect

Anchoring is a psychological concept which is used to describe the common human tendency to rely too heavily on a piece of information when making decisions. Or, as defined by Taylor (2000), anchoring effect refers to the notion that people base perception on past experience or “expert” opinion, which they amend to allow emerging deviations from the current conditions. Investors tend to use initial condition to justify the

⁷ In multiplicative approach, people receive an initial endowment that is transferred from period to period and can be reinvested together with its returns, which resembles the investment process more closely.

current decision making. For example, subsequent portfolio changes tend to be made with reference to initial allocation decision, rather than on some absolute basis.

2.2.7 Overconfidence and confirmation bias

Individuals tend to overestimate their own abilities, knowledge and skills. The classical economic equilibrium theory asserts that when agents receive heterogeneous information, they tend to communicate their private signals through purchase and sell orders. Thus, assuming all agents are rational, it may be worth for an investor to revise his private opinion in the light of new information coming from the activity of other investors. However, the investors are subject to overconfidence and will only concentrate on their own signal, even if that of other investors is different. This behavioural feature leads to inadequate revision of opinions and excessive trading.

Furthermore, they would see a positive event as confirmation that their approach was correct. This is also called “hindsight bias”, reflecting the situation that events that happen will be thought of as having been predictable prior to an event and that events that do not happen will be thought of as having been unlikely to happen prior to the event.

2.2.8 Default option, Inertia and Procrastination

What this suggests, is that many individuals do not have particularly firm convictions about their desired saving behaviour. In Madrian and Shea’s (2001) analysis of automatic enrolment, they showed that after automatic enrolment was introduced, plan participation rates jumped from 37 percent to 86 percent for new hires. The impact of automatic enrolment is an illustration of a broader behavioural phenomenon, namely “default option”. When confronted with difficult decisions, individuals tend to adopt heuristics that simplify the complex problem they face. One simple heuristic is to accept the available default option, rather than making an active choice. Choi et al. (2002) studied

the tendency to accept scheme default options in US and found that very few employees opt out of default arrangements even when they are free to do so⁸. A similar tendency to accept the default is found in the UK. Hewitt Bacon and Woodrow, the pension consulting firm, found that around 80% of members in UK DC schemes accept the default fund choice (Bridgeland, 2002). Currently, deterministic lifestyle investment strategy is widely used in the default fund offered by many UK DC pension providers⁹.

Inertia is also sometimes called status quo bias, which means people would prefer to stick with their current position than move to new position. They would need a positive incentive to be persuaded to make a change. In the same analysis by Madrian and Shea's (2001), they showed that the benefit of higher plan participation rates appeared to be offset by a profound level of inertia. Most participants remained at the default saving and conservative investment choices set for them by their employer. Once enrolled, participants made few active changes to the contribution rates or investment mixes selected for them by their employers.

2.2.9 Hyperbolic discounting

Hyperbolic discounting effect means that individuals' short-term discount rate is higher than long-term discount rate. Thaler (1981) discussed this feature with an interesting example: some people prefer "one apple today" to "two apple tomorrow", but at the same time they prefer "two apples in one year plus one day" to "one apple in one year". In other words, people seem to over value the immediate utility loss and behave irrationally in short-term decision making. In contrast, hyperbolic discounting implies that people are more likely to make rational commitments long in advance that they would never make if the commitment required immediate action.

⁸ According to their studies, in the schemes they investigated, between 42% and 71% of participants accept the default contribution rate and between 48% and 81% of scheme assets are invested in the default fund.

⁹ According to "Annual Defined Contribution Industry Survey 2003" conducted by JP Morgan Fleming, in 2003, 77% of DC schemes use lifestyle default option.

2.2.10 Other

There are some other interesting behavioural findings which will be relevant to DC plan arrangement:

- Representativeness. People naturally group information and rely on these grouping when making future decisions. They consider more probable that which they find easier to imagine.
- Home bias. Many investors are most comfortable with investments they know or are most familiar with. For example, many investors prefer to invest in domestic stocks, even though they should rationally diversify their portfolios to include overseas stocks.

2.3 Application to DC plan arrangement

As discussed earlier in this chapter, some approaches taken in the DC pension plans may be counterproductive in encouraging retirement saving and helping members make appropriate investment decisions. Thus, this section looks at some possible applications of behaviour finance in the DC plan arrangement.

2.3.1 Communication

In DC pension plans, members are responsible for deciding to join a plan and how to invest their monies. An effective communication strategy is therefore crucial to ensure members make appropriate decisions in terms of both take-up and contribution rate. Attitudes to risk are fundamental to the optimal design of DC pension plans. Without a good understanding of members' attitude to risk, all the subsequent communication can be wasted. To avoid this, we can classify the members into three key categories: aggressive, balanced and conservative, by asking a few simple survey questions. The communication is then tailored for each category of members. In addition, a good communication strategy should be tailored for individual members at different ages. For

example, 25-year old members are unlikely to pay much attention to their pension plan but have a long investment horizon. In this case, the communication tools offered to them should focus on encouraging making enough contributions and investing in long-term return generating asset classes.

Meanwhile, a good communication strategy should be designed with consideration of the behavioural features of members. As discussed in the previous section, due to framing effect, members vary their response to questions depending on how they are framed, such as numbering, order and degree of difference between options. In explaining investment risk, compared with probability and fancy stochastic graphic projection, natural frequencies are more intuitive and easier for readers to understand properly. Also, members would feel more comfortable to hear that “there is 50% chance that the funding level will rise above 100%” than “there is 50% chance that the funding level will remain or drop below 100%”.

2.3.2 Decision to save

Due to hyperbolic discount rate, people like to postpone the painful decisions in the future. Pension members are easily convinced to commit a savings plan starting sometime later (e.g., one year). By using the implication of this behavioural feature, Benartzi and Thaler (2004) proposed so-called “Save More for Tomorrow” or “SMarT” pension plan. Under “SMarT” plan, members consent to allow their contribution rate to increase as part of salary increase in the future, in other words, people postpone the “painful” decision. According to their study, the employee who elected to join the “SMarT” plan contribute 3 percent more over three years than those who have not joined “SMarT”.

2.3.3 Default and other options

Many members show little interests in financial matters and have little knowledge about the investment. As explained in previous section, when confronted with difficult decisions, individuals tend to adopt heuristics that simplify the complex problem they face. The simplest option is actually the non-decision, i.e. doing nothing. To protect the plan members who have little financial knowledge when they make important investment decisions, we need give them a default option to select based on the presumption that the option is likely to be a reasonably good choice.

To those members who have interest and knowledge in planning their retirement, we need to give them enough rights to make their own decisions. Behavioural finance theory tells us that if the employers offer a large range of funds, this can have an adverse effect on the decision process by confusing the members. In addition, members are likely to divide the money evenly among the funds offered. So the asset allocation members choose will depend strongly on the arrays of funds offered. When confronted with a range of funds with different risk profiles, people tend to select from the middle of the range, regardless of whether this suits their risk preference.

Thus, to find an appropriate default fund and find a good balance between default and other options is crucial to DC pension design. Myners Report¹⁰ recommended that when DC plans offer default options, trustees should ensure that an objective is set for that option, including expected risks and return and that they have offered a sufficient range of funds to satisfy the risk and return combinations reasonable for most members. Trying to find an appropriate default fund and find a good balance between default and other options is important to DC pension design.

Set the UK stakeholder DC pension market as an example. As explained in Blake et al. (2005), current situation is that around 85% of members are not interested in investment choices and just take the default options. A small percentage, around 10% to 15% will be

¹⁰ In March 2000, the Chancellor of the Exchequer (HM Treasury) commissioned Paul Myners to conduct a review of institutional investment in the UK. The Myners Report entitled "Institutional Investment in the UK: a review" was published on 6 March 2001.

prepared to make choices within a limited range of options. Only quite few members, less than 5%, will be looking for a wide range of options, which we should not target on when we design the scheme. The current investment decision-making process is too complicated because almost every plan has its own default options. In theory, ideal situation is that there are only limited types of default funds in the DC pension market, or even only one default fund exists, which is a “long-term pension market portfolio”. In this case, the members will be more willing to save and invest according to behavioural finance theory, the investment risk placed on members in DC scheme is better managed, retirement system would be more flexible and the cost of transferring fund between employers can be reduced to a minimal level.

2.3.4 Fund selection

In practice, we can face a lot of problems related to behavioural finance when we select fund managers (especially active fund managers) for the pension plan. For example, if the active managers have yet to obtain any outperformance above their benchmarks, they might tend to be optimistic about their methods and believe it is just unproven; if they had achieved outperformance, they will believe that it demonstrates their process works. In other words, they can be over confident in either situation of the market. In addition, many members or investment consultants are affected by anchoring effect as discussed in section 2.2.6. For the selected fund managers, their past performance is an anchor. If they used to perform well, we are more willing to be optimistic and tolerant of them.

With knowledge of the impact of above behavioural features, we should be able to remove the bias in the information provided by a manager. For example, we can setup a set of objective and rigorous fund selection criteria to avoid the above behavioural bias. When the fund performance is communicated to the members, transparency and an objective outlook is quite important. In addition, the process should be made easy for members to change the asset allocation decision as their circumstances change.

2.3.5 Fund performance data

In investment, loss aversion is also believed to manifest itself in what is known as the “disposition effect”. That is people appear to realise gains too quickly in the fear that they may make a loss. As explained in section 2.2.4, myopic loss aversion theory tells us that employees are more willing to accept risk when they evaluate less often and the frequent feedback would prevent them from adopting appropriate long-term investment strategy. So, for the benefits of the members, it will actually be better off if we provide them less frequent investment performance data.

2.3.6 Annuitisation decision

The annuitisation decision (including the timing of an annuity buy out and the form of the annuity benefits), is a very important event for DC members. This decision will be heavily influenced by factors like longevity, the outlook for inflation, and other considerations like value of non-pension assets and current state of health. Longevity risk¹¹ can be eliminated by purchasing a life annuity at retirement, however, as an important problem of DC pension plans, many members do not choose to buy annuities and instead take too much cash lump sum at retirement.

One obvious reason for this is bequest motive (which we will explain and discuss in details later on in Chapter 4 and Chapter 5). Members are concerned that they might die shortly after buying the annuity and the lump sum used to buy annuity is no longer available to make bequests. In addition, there are also some possible behavioural explanations for why members choose the lump sum over the annuity:

¹¹ Because people do not know precisely how long they will live, they run the risk of exhausting their assets before dying. Such risk exposure can be reduced by consuming less per year during retirement, but of course this simply elevates the chances that a retiree might die with “too much” wealth left over. One way to offset longevity risk is to buy an annuity with all or part of one’s retirement assets. Single premium lifelong annuities are relatively appealing, since they continue to pay benefits as long as the retiree lives, irrespective of whether the retiree outlives the life tables.

- Overconfidence – members underestimate how much they need to live on after retirement
- Hyperbolic discounting – members are very “myopic” and would treat the lump sum payment at retirement as the current “one apple”!

To encourage members to choose annuity over the lump sum, we can use the following behavioural features:

- Framing effect. Choice can be framed in a way that causes people to overvalue the annuity and undervalue the lump sum.
- Hyperbolic discounting. As explained in Section 2.2.9, members are more likely to make rational commitments long in advance that they would never make if the commitment required immediate action. Thus, we can let members make annuitisation decisions several years ahead of retirement (e.g. 5 years before retirement) rather than at retirement.

2.4 Conclusions

DC pension plans can provide real retirement security, but only if participants utilise them appropriately and make the right investment decisions. It would be a challenge to provide the expertise to support good decision-making in the DC context. The research we describe in this chapter tells us that some approaches taken in current DC plans may be counterproductive in encouraging retirement saving. The investment decisions made by employees in DC plans also vary considerably depending on how their investment opportunities are described and the manner and frequency with which they receive feedback on their investment returns. With more education, more information on investment risks and expected returns and better communication, the members may begin to act more rationally. However, this will take time and has its own limit as well.

I choose some features that seem particularly relevant to DC pension arrangement and then discuss their possible applications in the different processes of DC pension arrangement. It is important that actuaries are alert to the use of these factors when they design the plan and give financial advice. It is hoped that the research will have a profound impact on the way actuaries now view varied aspects of financial life and manage retirement systems.

Chapter 3

Reexamining Behavioural Characteristics (UK Evidence)

3.1 Introduction

Representing an alternative way of looking at financial market, behavioural finance investigates what happens in markets where some of the agents display human limitations and complications. The fertilisation of economics with psychological and evidence makes it interesting and promising.

The literature of behavioural research can be traced to 1963. As discussed in previous chapter (see Section 2.2.1 and 2.2.2), to explain the behaviour of Samuelson's colleague, some academics argued that individuals tend to perceive and evaluate changes of wealth (gains and losses) rather than final wealth positions as assumed in the expected utility framework. For example, Kahneman and Tversky (1979) suggested a value function as follows:

$$v(x) = \begin{cases} x^{\alpha} & \text{if } x \geq 0 \\ -\lambda(-x)^{\beta} & \text{if } x < 0 \end{cases}$$

[6]

where $\lambda \geq 1$, $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1$. This function reflects loss aversion (representing the phenomenon that investors are more sensitive to losses than to gains) via parameter

λ and diminishing sensitivity via parameter α and β . Kahneman and Tversky (1979) investigated the issue in detail and made it popular through their prospect theory:

$$U = w(p_1)v(x_1) + w(p_2)v(x_2) + \dots$$

[7]

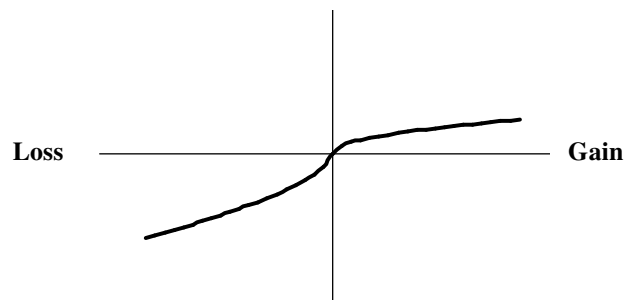
where $x_1, x_2, \dots$ are the potential outcomes, $p_1, p_2, \dots$ are their respective probabilities and $w(\cdot)$ is the probability weighting function.

Prospect theory differs from expected utility theory in two important respects:

- 1) Prospect theory has a value function $v(x)$ defined on gains and losses relative to a reference point, rather than absolute levels of total wealth. Two important properties of the value function are that, first, it is S shaped (convex below the reference point and concave above it), i.e. people are risk seeking in the domain of losses and risk averse in the domain of gains; second, it is asymmetric. The value function has a kink at the reference point, with the slope of the loss function steeper than the gain function. The ratio of these slopes at the reference point is a measure of loss aversion.

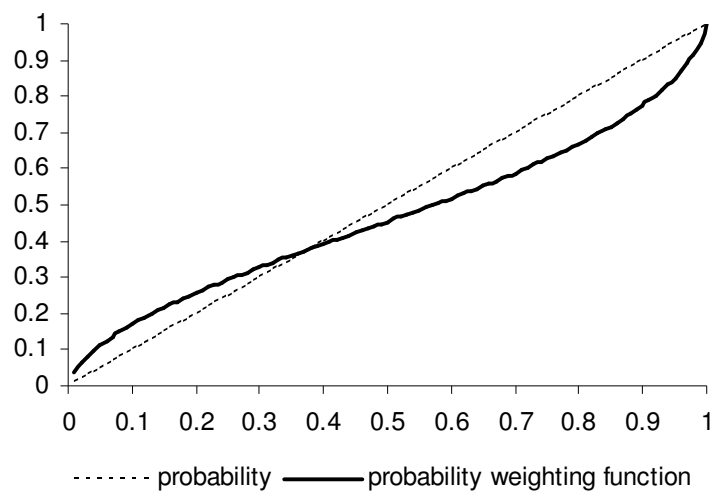
Figure 6: Loss aversion value function

$$(\lambda > 1, v_1 < 1, v_2 < 1)$$



- 2) Prospect theory also has a probability weighting function $w(p)$. It postulates that decision weights tend to overweight small probabilities and underweight moderate and high probabilities, as shown in **Figure 7**.

Figure 7: Prospect theory – probability weighting function



Loss aversion receives increasing attention in economic analysis because of the fact that it can explain many phenomena that are hard to explain in expected utility framework. As discussed in Section 2.2.2, Benartzi and Thaler (1995) tried to explain premium puzzle with loss aversion and argue that in a myopic evaluation, the volatile return of an equity investment looks particularly unattractive, i.e. Myopic Loss Aversion (MLA). “Myopic” is put here to describe the short-sightedness that induces a decision maker to evaluate each alternative within a sequence independently, whereas a rational decision maker would evaluate the sequence as a whole. Gneezy and Potters (1997) tested this hypothesis in laboratory experiments. The results of their experimental studies suggests that the more frequently returns are evaluated, the lower are the average level of investments in the equity market, i.e. the more risk averse investors will be, which is in line with the MLA hypothesis. Thaler et al. (1997) examined how individuals split money between

two assets with different levels of risk by conducting an experiment to 80 undergraduate students at the University of California at Berkeley and made similar conclusions.

Recently, loss aversion has also been applied to asset pricing by assuming that investors are loss averse over the fluctuations in the value of their financial wealth (e.g. Berkelaar and Kouwenberg (2003)). Barberis et al. (2001) showed that loss aversion helped explain the high mean, excess volatility, and the predictability of equity returns, as well as their low correlation with consumption growth. By using an example where an expected utility maximiser turn down a 50/50 gamble between losing \$100 and winning \$2.5m (due to diminishing marginal utility), Rabin and Thaler (2001) argued the expected utility theory is manifestly not close to the right explanation for most risk attitudes and “we have also been surprised by economists’ reluctance to acknowledge the descriptive inadequacies of expected utility theory” (p229). They claimed that loss aversion and the tendency to isolate each risky choice should replace expected utility theory as the descriptive theory of risk attitudes.

The growing literature of behavioural finance, in particular loss aversion utility theory raises important questions about how individual investors will behave under such behavioural utilities. In other words, to understand the values of loss aversion parameters (v_1, v_2 and λ) of individuals and how the values change with profiling characteristics (e.g. sex, age, marital status, standard region, etc.) becomes more important.

Empirical estimates of loss aversion are typically in the neighbourhood of 2.5, meaning that the disutility of giving something up is twice as great as the utility of acquiring it. Tversky and Kahneman (1992) carried out an experiment on 25 graduate students and suggested that the loss aversion coefficient (λ) should equal to 2.25 and the curvatures for gains (v_1) and losses (v_2) should be both equal to 0.88. This parameterisation then became well known and frequently used by other researchers. However, Tversky and Kahneman (1992)’s experiment and most of the existing empirical studies of loss aversion parameters are based on surveys with students in the university (e.g. Brooks and Zank (2005)), who are relatively well educated and rational individuals. Hwang and

Satchell (2005) use 20 years of monthly UK and US asset allocation data to empirically investigate admissible ranges for the loss aversion parameters in financial market. Their study proposes a long-term average value of $\lambda = 3$ for the loss aversion ratio, and suggest that this value should be adjusted upwards and downwards by 1.5 for bull and bear markets respectively. They suggest curvature parameters of $v_1 = 0.75$ and $v_2 = 0.95$, again implying that investors are risk averse with respect to gains and risk seeking with respect to losses.

The aim of this study is to provide some UK empirical evidence on individuals' risk attitude in terms on loss aversion parameters (v_1, v_2 and λ). This helps us to compare and contrast with existing US findings (e.g. Tversky and Kahneman (1992)). Also, we proposed several behavioural characteristics questions in the survey to investigate how individuals make their saving and investment decisions.

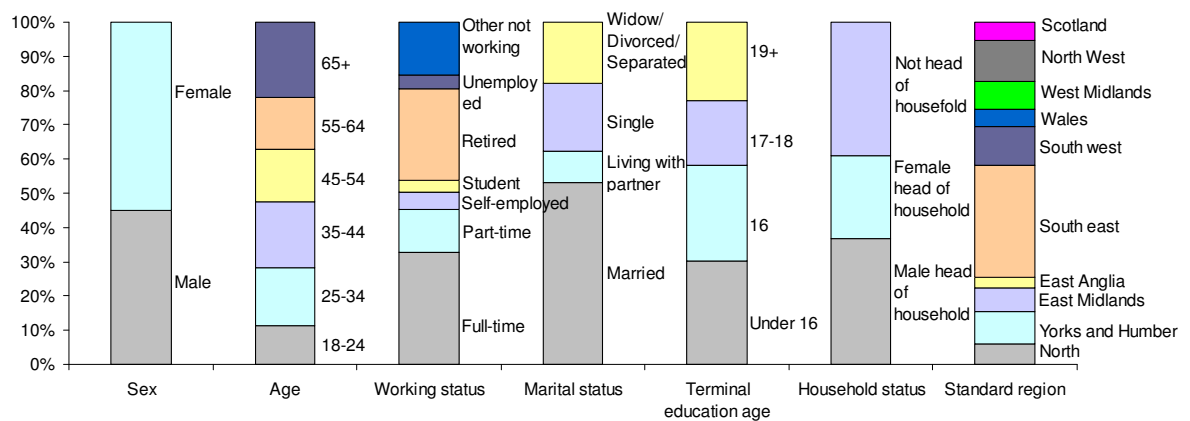
The outline of this chapter is as follows. Section 3.2 describes the survey design and estimation method, while Section 3.3 provides the survey results and analysis. Section 3.4 summarises and concludes the chapter.

3.2 Survey

3.2.1 Subjects

The survey was conducted on a face-to-face interview basis from 14th April 2005 to 19th April 2005¹². A total of 966 responses were received, who are classified with nine profiling characteristics, i.e. sex, age, marital status, terminal education age, standard region, working status, household status and household income (as shown in **Figure 8** and **Table 1**).

Figure 8: Survey sample (total respondents: 966)



¹² The survey was sponsored by Distribution Technology Ltd.

Table 1 Survey sample (966, all adults aged 18+)¹³

Sex		
Male	433	45%
Female	533	55%
Age		
18-24	110	11%
25-34	162	17%
35-44	186	19%
45-54	151	16%
55-64	144	15%
65+	213	22%
Marital Status		
Married	514	53%
Living with partner	86	9%
Single	192	20%
Widowed/Divorced/Separated	174	18%
Terminal Education Age		
Under 16	281	29%
16	260	27%
17-18	177	18%
19+	214	22%
Working Status		
Full-time	316	33%
Part-time	122	13%
Self-employed	48	5%
Student	34	4%
Retired	258	27%
Unemployed	38	4%
Other not working	150	16%
Household Status		
Male head of house hold (hoh)	354	37%
Female head of household (hoh)	234	24%
Not head of household (hoh)	378	39%
Standard Region		
North	58	6%
Yorks and Humber	90	9%
East Midlands	68	7%
East Anglia	31	3%
South East	315	33%
South West	110	11%
Wales	49	5%
West Midlands	78	8%
North West	116	12%
Scotland	51	5%

¹³ To investigate the representativeness of our sample, we compared it with UK Census 2001. In UK Census 2001 dataset, the split between male and female is 49% and 51%; the split by ages is 18-24 (11%), 25-34(18%), 35-44(19%), 45-54(17%), 55-64 (14%) and equal or above 65 (21%); the split for single, married and widowed/divorced/separated is 44%, 41% and 15%. We can see that our sample is quite representative to the UK population.

3.2.2 Survey Design

The survey consists of three components: classification questions, behavioural characteristics/financial knowledge questions and risk preference questions (see Appendix 01).

In the classification questions, in addition to the nine profiling questions, i.e. sex, age, marital status, terminal education age, standard region, working status, household status, household income and social class, we also asked about people's expected retirement age and health status. This helps us to understand how the expected retirement age when people grow older.

In the behavioural characteristics questions, we designed several straight forward questions to find out more about individuals' knowledge and interests in financial planning.

In addition, we proposed five questions as follows to get a better understanding of how individuals make their investment decisions (see Appendix 01 for details):

- *Question 6: how you allocate money for retirement needs?*
- *Question 8: how often you review savings and investments*
- *Question 10: do you act in spontaneous or unplanned way?*
- *Question 11: do you make plans and stick to them?*
- *Question 12: are you able to control impulsive feelings?*

As the crucial part of the survey, risk reference part contains six simple lottery questions (Question 13 to 16, as illustrated in Appendix 01). The estimated results of the loss aversion parameters (i.e. v_1 , v_2 and λ) will be based on individuals' answers to these six questions:

- *Question 13: In a coin tossing contest, what would the minimum prize money have to be to persuade you to take part if you stood to lose £100?*
- *Question 14: In a coin tossing contest, what would the minimum prize money have to be to persuade you to take part if you stood to lose £1000?*
- *Question 15: In a coin tossing contest, what would the minimum prize money have to be to persuade you to take part if you stood to lose £10000?*
- *Question 16: In a coin tossing contest, what is the maximum amount of money you would be prepared to lose if the prize money was £100?*
- *Question 17: In a coin tossing contest, what is the maximum amount of money you would be prepared to lose if the prize money was £1000?*
- *Question 18: In a coin tossing contest, what is the maximum amount of money you would be prepared to lose if the prize money was £10000?*

Some special considerations and efforts were made to keep the above questions unbiased:

- (i) **Choices tasks vs. matching task.** We used so called “matching task” approach to investigate people’s attitudes to risk to avoid suggestions by proving choices. There are two common experimental procedures for eliciting risk attitudes: choice tasks and matching tasks. In choice tasks, subjects are asked to choose between two gambles. Choice tasks are sometimes accused of “simply recovering the expectations of the experimenters that guided the experimental design” (Frederick, Loewenstein and O’Donoghue, 2002, p48) because results from choice tasks can be affected by the given choices. In matching tasks, respondents fill in the blank to equate two options (e.g. in a simple coin toss, individuals choose a value X such that $£0 = 50\%$ chance of winning $£X$ plus a 50% change of losing $£100$). In this case, an exact figure can be imputed from a single response because subjects give an indifference point.
- (ii) **Framing effect.** Individuals’ answers can be affected by how the question is “framed”, in particularly, by the wording in terms of gains and losses. To avoid this impact, in the six lottery questions, three of them are framed in the sense of “gains” and the other three are framed in the sense of “losses”.

(iii) **Anchoring effect.** Anchoring effect suggests that when respondents are asked to make decisions on a series of questions, the first question they face often influences their subsequent answers. Thus, we have ensured that questions 13-18 are asked in random order to avoid this effect.

(iv) **Real rewards vs. hypothetical rewards.** We use “hypothetical rewards” method in this study. The use of real rewards is generally desirable for obvious reasons, but hypothetical rewards actually have some advantages. In studies involving hypothetical rewards, respondents can be presented with a wide range of reward amounts, including losses and large gains, both of which are generally infeasible in studies involving real outcomes¹⁴.

3.2.3 Method

According to the prospect theory, the interplay of loss aversion value function and probability weighting function controls the investor’s decision making. The loss aversion parameter estimation results therefore depend on the probability value and probability weighting function form being used. To keep the analysis consistent with existing literatures, we only use 50% probability in our gambling questions and follow the probability weighting function setting¹⁵ used in Tversky and Kahneman (1992):

$$w^+(p) = \frac{p^r}{(p^r + (1-p)^r)^{\frac{1}{r}}}, \quad w^-(p) = \frac{p^\delta}{(p^\delta + (1-p)^\delta)^{\frac{1}{\delta}}}$$

[8]

¹⁴ Of course, as a trade off, we realise that the disadvantage of hypothetical choice data is the uncertainty about whether people are motivated, or capable of, accurately predicting what they would do if outcomes were real.

¹⁵ This form of function has been used by most researchers in past literatures. As explained by Tversky and Kahneman, it has several useful features: it has only one parameter; it encompasses weighting functions with both concave and convex regions; and more importantly, it provides a reasonably good approximation to probabilities in the range between 0.05 and 0.95.

where $r = 0.61$ and $\delta = 0.69$.

3.3 Results

3.3.1 Loss aversion parameters

As explained in the previous section, we ask six simple lottery questions in the survey, with the first three questions (Q13-Q15) fixing the loss amount, the last three questions (Q16-Q18) fixing the gain amount, as shown in **Table 2**. 371 individuals gave quantitative answers¹⁶ to all the six questions (i.e. there are 595 respondents chose “would not take part” or “do not know” in at least one of the six questions). Therefore we use this sample to investigate loss aversion parameters.

Table 2 Loss aversion questions

Question	50% Gain	50% Loss
13	?	100
14	?	1000
15	?	10000
16	100	?
17	1000	?
18	10000	?
Sample: 371		

To estimate the loss aversion parameters (v_1, v_2 and λ), we define A as the loss amount, B as the gain amount, then, $B^{\eta} w^+(0.5) = \lambda A^{v_2} w^-(0.5)$ ¹⁷, and,

$$v_1 \ln B + \ln(w^+(0.5)) = \ln \lambda + v_2 \ln A + \ln(w^-(0.5))$$

[9]

¹⁶ The survey results confirmed some weakness of matching task approach. More than half of the respondents did not give a quantitative answer to question 13-18.

¹⁷ The aim of this study is to estimate the parameters of loss aversion value function proposed by prospect theory (Kahneman and Tversky (1979)), we therefore follow the method of Tversky and Kahneman (1992) and assume this function form still holds.

For each individual in the above sample, we will obtain six equations in the same form as equation [9] with different gain and loss amounts. The loss aversion parameters (v_1, v_2 and λ) are then estimated by using least square method, i.e.,

$$\text{Min} \sum_{i=1}^6 \left[\text{Ln} \hat{\lambda} + \hat{v}_2 \text{Ln} A_i + \text{Ln} (w^-(0.5)) - \hat{v}_2 \text{Ln} B_i - \text{Ln} (w^+(0.5)) \right]^2 \quad [10]$$

Table 3 Estimated loss aversion parameters

	v_1	v_2	λ
Median	0.53	0.77	3.4
25%	0.36	0.47	2.0
75%	0.69	1.21	4.1
Sample: 371			

As shown in **Table 3**, the results¹⁸ suggest the median values of three parameters as $v_1 = 0.53$, $v_2 = 0.77$ and $\lambda = 3.4$ (with the 25% quartile being $v_1 = 0.36$, $v_2 = 0.47$ and $\lambda = 2$, and 75% quartile being $v_1 = 0.69$, $v_2 = 1.21$ and $\lambda = 4.1$). The above results tell us that, on average, individuals are risk averse with respect to gains ($v_1 < 1$) and risk seeking with respect to losses ($v_2 < 1$). This is consistent with the prospect theory setting as explained in Section 3.1. Meanwhile, compared with the results ($v_1 = 0.88, v_2 = 0.88$ and $\lambda = 2.25$) of prospect theory which is suggested by Tversky and Kahneman (1992), our estimation results are different in several respects:

- Loss aversion coefficient. Our results suggest average individuals are 3.4 times as sensitive to losses as to gains. In other words, individuals in our sample are more loss averse than the people suggested by the Tversky and Kahneman (1992) and Hwang and Satchell (2005)

¹⁸ The results from the process above are 371 estimated values of v_1, v_2 and λ for each individual.

- Sensitivity to marginal losses and gains. While the prospect theory implies people have the same sensitivity to marginal losses and gains ($v_1 = v_2 = 0.88$), our results suggest that they are more sensitive to marginal losses than to the equivalent marginal gains ($v_1 < v_2$).
- It is worth noticing that the individuals above the upper quartile have different risk attitudes. While they are more sensitive to losses than to gains, they are risk averse with respect to both losses and gains ($v_1 < 1$ and $v_2 > 1$).

Some possible reasons for the above difference are:

- We have a much large sample size (371). Tversky and Kahneman (1992) estimated their loss aversion parameters on only 25 graduate students from Berkeley and Stanford (12 males and 13 females). Furthermore, graduate students are relatively well educated and rational individuals.
- Our estimation results are based on UK individuals. The difference may simply reflect the different risk attitudes of UK and US individuals.
- The questions in Tversky and Kahneman (1992) can be biased due to anchoring effect. As explained in Section 3.2.2, we have made special efforts to avoid this anchoring effect in our survey results.

Given the relatively larger sample of response, we can separate individuals into several subgroups according to their gender, age, standard region, working status, terminal education age, marital status and household status. **Table 4** shows the loss aversion parameter estimations for each group.

Table 4 Estimated subgroup loss aversion parameters

	Responses	%	V1	V2	Lambda
Total sample (371)			0.53	0.77	3.42
Gender					
Male	182	49.1%	0.56	0.77	3.29
Female	189	50.9%	0.52	0.75	3.55
Age					
18-24	60	16.2%	0.63	0.96	2.37
25-34	82	22.1%	0.58	1.00	2.28
35-44	85	22.9%	0.56	0.77	3.50
45-54	54	14.6%	0.47	0.69	3.65
55-64	46	12.4%	0.49	0.66	3.75
65+	44	11.9%	0.36	0.60	4.12
Standard region					
North	26	7.0%	0.48	0.55	3.82
Yorks and Humber	36	9.7%	0.41	0.66	3.94
East Midlands	33	8.9%	0.64	1.00	2.33
East Anglia	8	2.2%	0.57	1.25	2.54
South East	129	34.8%	0.56	0.73	3.52
South West	39	10.5%	0.56	0.90	2.49
Wales	14	3.8%	0.51	0.58	3.73
West Midlands	28	7.5%	0.54	0.62	3.65
North West	37	10.0%	0.56	0.64	3.65
Scotland	21	5.7%	0.58	1.25	2.00
Working status					
Full-time	151	40.7%	0.56	0.86	3.02
Part-time	49	13.2%	0.54	0.77	3.50
Self-employed	20	5.4%	0.50	0.57	3.89
Student	16	4.3%	0.64	0.99	2.22
Retired	60	16.2%	0.41	0.64	3.94
Unemployed	20	5.4%	0.61	0.75	3.58
Other not working	55	14.8%	0.52	0.68	3.65
Terminal education age					
Under 16	83	22.4%	0.47	0.69	3.92
16	104	28.0%	0.56	0.83	3.17
17-18	96	25.9%	0.57	0.80	3.28
19+	88	23.7%	0.51	0.69	3.50
Marital status					
Married	197	53.1%	0.48	0.68	3.65
Living with partner	44	11.9%	0.60	1.00	2.19
Single	88	23.7%	0.64	1.00	2.28
Widowed/Divorced/Separated	42	11.3%	0.49	0.66	4.03
Household status					
Male hoh	133	35.8%	0.49	0.69	3.50
Female hoh	71	19.1%	0.52	0.75	3.52
Not hoh	167	45.0%	0.56	0.81	3.29

Following observations can be made from **Table 4**:

- Our estimation results for the whole sample ($v_1 = 0.53$, $v_2 = 0.77$ and $\lambda = 3.42$) are reasonably robust to different profiling characteristics. For all the subgroups, individuals are more sensitive to marginal gains than to marginal losses, i.e. $v_1 < v_2$. Within all the estimated results for the three parameters, v_1 varies from 0.41 to 0.64, v_2 varies from 0.55 to 1.25 and λ varies from 2 to 4.12.
- Gender effect. On average, females are more loss averse than males; meanwhile females are more risk averse than males in the region of gains and more risk seeking in the region of losses, although this effect is not very significant according to our survey results.
- Age effect. Individual's risk attitudes are very sensitive to age. They become more loss averse when they grow old (with $\lambda = 2.37$ for individuals aged between 18 and 24, and $\lambda = 4.12$ for individuals aged over 65). For all the age bands, people are more sensitive to marginal gains than to marginal losses, i.e. $v_1 < v_2$. Unsurprisingly, people also become more risk averse (with respect to gains) when they grow old and approach retirement. Here, it is interesting to note that our estimation results for individuals aged between 18 and 24 are close to the results suggested by Tversky and Kahneman (1992) ($v_1 = 0.88$, $v_2 = 0.88$ and $\lambda = 2.25$).
- Geographic location effect. In terms of loss aversion ratio λ , people living in Scotland are least loss averse, while people living in Yorks and Humber are most loss averse. While there can be various reasons causing the outlier with Scottish people, there is some consistency between their relatively lower λ (meaning being less loss averse) and higher v_2 (meaning being less risk seeking with respect to losses).
- Working status effect. Self-employed and retired individuals have the highest loss aversion ratios; students have the lowest loss aversion ratios ($\lambda = 2.22$). In addition, students are least risk averse with respect to gains and least risk seeking with respect to losses.

- Education effect. The effect of education received on individuals' risk attitudes is not very significant according to our survey results. However, it is interesting to note that the risk attitudes of well educated individuals (with terminal education age over 19 in our survey) move towards the least educated group (with terminal education age under 16 in our survey). Of course, this is not conclusive.
- Marital status effect. Individual's risk attitudes are very sensitive to marital status as well. Unmarried (single and living with partner) individuals are less loss averse than individuals who are married or were married (widowed/divorced/separated).
- Household effect. From the survey results, we can see status of household (i.e. whether with a male or female head of household) does not have a strong impact on individuals' risk attitudes.

3.3.2 Behavioural features

Financial knowledge

The survey results shows lots of people have little interest and knowledge about saving and investment. We can see from **Table 5** below that, 364 (38%) respondents "don't know" the "current rate of interest level of saving account", and another 38% respondents give the wrong answers. Only 22% of total sample seem to know the base rate.

Table 5 Financial knowledge: base rate

0.1-0.9%	5(1%)
1.0-1.9%	33(3%)
2.0-2.9%	48(5%)
3.0-3.9%	114(12%)
4.0-4.9%	162(17%)
5.0+ %	217(22%)
No interest	23(2%)
Don't know	364(38%)

Sample: 966

Also, as shown in **Table 6**, when we ask "if a person spreads their investments over a number of assets or assets types successfully, what do you expect to happen to the level

of risk in their portfolio” (Question 9 in our survey), 78 (8%) respondents “don’t understand the question”, 282 (29%) have “no ideas”, 251 (26%) think it will “go up”, 147 (15%) think it will “stay the same”, only 208 respondents (22%) give the right answer.

Table 6 Financial knowledge: diversification

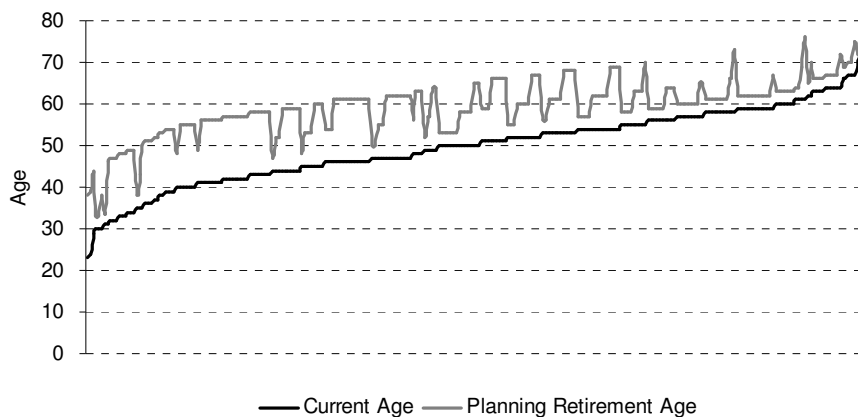
Go up	251 (26%)
Stay the same	147 (15%)
Go down	208 (22%)
No idea	282 (29%)
Do not understand	78 (8%)

Sample: 966

Overconfidence

Figure 9 shows the current age and planning retirement age of 269 respondents, who gave quantitative answers to the question 3 - “when plan to retire”. Clearly, individuals are over confident in planning retirement and they will change and postpone their expected retirement ages over time when they grow old.

Figure 9: Planning Retirement Age (total respondents: 269)



Framing effect

Three pairs of questions (i.e. Q13 and Q16, Q14 and Q17, Q15 and Q18) with the same gambling amount in the terms of loss or gain are asked to investigate the framing effect. We find framing effect plays an important role in affecting people's financial decision makings. For example, as shown in **Table 7**, among 453 risk-averse individuals according to their answers to question 16, 13% of them change to be risk-seeking in question 13 and 29% decline to take part, with another 5% saying that they do not know this time. Also, it seems that framing effect becomes stronger with the increase of gambling amount. For example, when the gambling amount increases, within the initial risk-averse sample (453 individuals), fewer people would stick to their risk attitudes (with percentages dropped from 88% to 86% in Q17 and Q18, from 53% to 30% in Q13 and Q15).

Table 7 Framing effect

	“loss” sense questions			“gain” sense questions		
	Q13	Q14	Q15	Q16	Q17	Q18
Risk-averse answers	239/453 (53%)	194/453 (43%)	138/453 (30%)	Sensible answers (453 risk-averse subjects)	398/453 (88%)	389/453 (86%)
risk-seeking answers	58/453 (13%)	47/453 (10%)	54/453 (12%)		6/453 (1%)	0/453 (0%)
Would not take part	132/453 (29%)	190/453 (42%)	238/453 (53%)		45/453 (10%)	54/453 (12%)
Don't know	24/453 (5%)	22/453 (5%)	23/453 (5%)		4/453 (1%)	10/453 (2%)

3.3.3 Other

In the survey, we proposed five questions which we think can help us understand how individuals make investment decisions. The results are shown in **Table 8** below. Some observations are:

- A majority of individuals (72%) of individuals insist that they will never or only occasionally make decisions in a spontaneous or unplanned way. However, only 50%

of the sample will always make plans and stick to them; while only 55% of the respondents think they are able to always control impulsive feelings.

- In terms of the frequency of reviewing savings and investments, only 36% of the respondents are willing to review their savings and investment frequently (weekly, monthly or quarterly).
- 42% of the respondents prefer to “have one pot of money for all different needs”.

Table 8 Behavioural questions

	(%)
Total sample (966)	
Q6. How to allocate money for retirement needs	
You allocate different pots of money to different needs	30%
You have different pots of money, but do not allocate them to different needs	15%
You have one pot of money for all your different needs	42%
Do not know	13%
Q8. How often to review savings and investments?	
Weekly	16%
Monthly	14%
Quarterly	9%
Annually	30%
Never	16%
Do not have savings/investments	10%
Do not know	4%
Refused	1%
Q10. Do you act in a spontaneous or unplanned way?	
Yes, always	26%
Occasionally	43%
No, never	29%
Do not know	2%
Q11. Do you make plans and stick to them?	
Yes, always	50%
Occasionally	40%
No, never	9%
Do not know	1%
Q12. Are you able to control impulsive feeling?	
Yes, always	55%
Occasionally	36%
No, never	8%
Do not know	1%

3.4 Discussion

Motivated by the existing behavioural finance and loss aversion theory literature, we conducted a face-to-face survey on 966 randomly selected UK residents (for more details, see Appendix 02). The main contribution of this paper is to provide UK empirical evidence on individuals' risk attitude in terms on loss aversion parameters (v_1 , v_2 and λ). This enables us to compare and contrast UK case study evidence with existing US findings (e.g. Tversky and Kahneman (1992)).

The median estimated results of the loss aversion parameters are:

- the loss aversion ratio is $\lambda = 3.4$,
- the curvature for gains is $v_1 = 0.53$, and
- the curvature for losses is $v_2 = 0.77$.

The results are broadly consistent with the well known prospect theory setting (Tversky and Kahneman (1992)) and suggest that UK investors are more loss averse than the Tversky and Kahneman's original group of US students. To further investigate the loss aversion parameters of individual investors, we also separate individuals into several subgroups according to their gender, age, standard region, working status, terminal education age, marital status and household status, and estimate the loss aversion parameters for each group. The results help us to understand the relevant sensitivities of these factors on individuals' risk attitudes.

In addition, the survey results show many individual struggle to understand and make saving and investment decision. This is consistent with the principles of behavioural finance theory, which suggests individuals' decision makings are not always in a rational and unbiased manner assumed by expected utility theory.

Chapter 4

Loss Aversion Model

4.1 Introduction

As discussed in Chapter 2 and Chapter 3, the idea of “loss aversion” was first proposed by Kahneman and Tversky (1979) within the framework of prospect theory¹⁹. As one of the distinguishing features, the loss aversion value function is defined on gains and losses of wealth relative to a reference point, rather than absolute levels of total wealth (as is the case with the more traditional ideas of utility theory). Specifically, the loss aversion utility function is defined as follows:

$$V(x) = \begin{cases} \frac{(X - F)^{v_1}}{v_1} & \text{if } X \geq F \\ -\lambda \frac{(F - X)^{v_2}}{v_2} & \text{if } X < F \end{cases} \quad [11]$$

where

- X is the wealth amount,
- F is the reference wealth level,

¹⁹ Kahneman and Tversky (1979) developed this theory to remedy the descriptive failures of subjectively expected utility (SEU) theories of decision making.

- v_1 and v_2 are the curvature parameters for gains and losses respectively, with $0 < v_1 < 1$ and $0 < v_2 < 1$, and
- $\lambda > 0$ is the loss aversion ratio (i.e. investors are assumed to be λ times more sensitive to losses than to gains).

The two key properties of the loss aversion utility function are:

- it is S-shaped (i.e. convex below the reference point and concave above it), implying that individuals are risk seeking in the domain of losses and risk averse in the domain of gains; and
- it is asymmetric (i.e. steeper below the reference point than above, because of the effect of the loss aversion ratio λ), implying that individuals are more sensitive to losses than to gains.

Loss aversion theory has become increasingly popular in recent economic studies of asset allocation issues, largely because it can better explain many observed behavioural traits in investment decision-making that are hard to rationalise in an expected utility setting. For example, as mentioned in Section 2.2.2, Benartzi and Thaler (1995) explain the “equity premium puzzle” in terms of myopic loss aversion (MLA) by investors, whereby investors are both loss averse and evaluate their portfolios too frequently. And, Rabin and Thaler (2001) argue that expected utility theory does not always explain the behaviour of investors in a risky environment (especially the hesitation over risky monetary prospects even when they involve an expected gain) and the authors comment that they “have also been surprised by economists’ reluctance to acknowledge the descriptive inadequacies of expected utility theory”.

This chapter considers the optimal age-dependent investment strategy for defined contribution (DC) pension plans when plan members experience loss aversion. We use a two-asset, dynamic-programming-based numerical model with uninsurable labour income and borrowing constraints. Members are assumed to be loss averse with respect

to a target replacement ratio²⁰ on retirement at age 65 and a series of interim targets at regular intervals prior to retirement which reflect the discounted value of the target fund level. They are also assumed to make asset allocation decisions with the aim of maximising the discounted sum of the expected present value (EPV) of the LA functions (one for each reference point to be defined later) over the period until retirement. Within this proposed framework, it will be shown that the optimal dynamic asset allocation strategy is the target-driven strategy known as “threshold” or “funded status” that was considered in Blake et al (2001). With this strategy, the weight in equities is increased if the accumulating fund is below target and is decreased if the fund is above target. Over time, as the retirement date approaches, the overall equity weight falls. Although this is similar to what happens in “lifestyle” strategies, the target-driven strategy is very different. Whilst “deterministic” lifestyle strategies typically involve switching from 100% equities only in the last 5 to 10 years before retirement and often end up holding 100% of the fund in bond-type assets at retirement, the optimal strategy under loss aversion involves a much more gradual reduction in the equity holding, beginning at about age 40, and retains a significant equity holding of around 40% at retirement. We also show that under loss aversion, the risk of failing to attain the desired replacement ratio at retirement is significantly reduced in comparison with target-driven strategies derived from maximising either a quadratic or a power utility function.

In a DC pension plan, a member contributes part of his/her salary each year to building a pension fund for retirement. The accumulated fund is then used to buy a life annuity to provide a pension income after retirement. Deterministic lifestyle investment strategies are widely used by many such pension plans as the default investment option, with the aim of achieving the desired replacement ratio at retirement. However, as will be shown below, there can be substantial uncertainty over the size of the fund at retirement when a lifestyle investment strategy is used and this makes it difficult for the plan member to be confident about the level of retirement income.²¹ Thus, for DC plan members seeking

²⁰ The ratio of pension income divided by salary level at retirement.

²¹ This is based on the assumption that members pay a constant contribution rate each year (as assumed by most researchers in the literature). In the real world, members would be able to assist in achieving the desired target replacement ratio by adjusting the annual contribution rate. However, previous behavioural research findings (e.g. Madrian and Shea (2001)) suggest that, once enrolled, plan members make few active changes to the contribution rates

greater certainty in their retirement planning, the plan's investment strategy needs to be far more focused on achieving the target pension.

We will assume that plan members evaluate the plan's investment performance on an annual basis. Members have a final target replacement ratio at retirement and a series of discounted final targets before retirement. They are assumed to be loss averse with respect to these targets (which define the reference points in the LA framework outlined above) and to make asset allocation decisions to maximise the discounted sum of the EPV of the LA functions at each age until retirement.

Having target fund levels when formulating the optimal investment strategy for a DC pension plan is not a new idea. Vigna and Haberman (2001, 2002) derive a dynamic-programming-based formula for the optimal investment allocation in DC plans. In their model, members are assumed to face a quadratic cost (or disutility) function each year based on actual and targeted fund levels and to make investment decisions that minimise the cost of deviations of the fund from these corresponding targets. Their analysis suggests that a lifestyle investment strategy remains optimal for a risk averse member and that the age at which the member begins to switch from equities to bonds depends on both the member's risk aversion and age when the plan started: the more risk averse the member or the longer the accumulation period prior to retirement, the earlier the switch to bonds. However, as the authors acknowledge, one obvious limitation of this approach is that the quadratic cost function equally penalises both under- and over-performance relative to the specified targets.

Our proposed model differs from existing literature in three significant respects:

- (i) Loss aversion utility. As discussed above, most of the existing studies (e.g. Haberman and Vigna (2002), Gerrard et al. (2004)) assume that the individual plan member has a quadratic utility function with respect to deviations in the

or investment decisions. This is known as the "inertia effect".

actual fund from the corresponding targets. Given the behavioural traits exhibited by many investors, we believe it is more appropriate to use a loss aversion utility function instead in the asset allocation model of a DC pension plan;

- (ii) Stochastic salary process. Previous studies (e.g. Haberman and Vigna (2001)) have been based on a simplistic deterministic salary model, whereas this paper uses a more realistically-calibrated stochastic salary model; and
- (iii) Choice of the investment targets. To choose an appropriate investment target is crucial in this type of target driven approach. Previous studies have been oversimplified by assuming fixed interim and final targets (based in a constant investment return). In our model, both the interim and final fund targets are path-dependent (and vary over time in accordance with the member's actual current salary and required replacement ratio at retirement).

Our baseline simulation draws on empirical evidence from UK households, with the loss aversion parameters estimated from the results of a specially commissioned survey of almost 1,000 randomly selected individual investors on attitudes to risk and loss (see Blake et al (2008a)). This survey suggests that UK investors are significantly more loss averse than was suggested by the original Kahneman and Tversky study (which was based on a much smaller survey of US graduate students only).

The rest of the chapter is organised as follows. Section 4.2 formulates the target driven asset allocation problem for a DC pension plan and outlines our model including optimisation method and Section 4.3 presents the empirical part of this paper including the parameter calibration for our baseline simulation; in Section 4.4, we provide some justifications for the use of loss aversion utility first and then analyse the simulation results of the optimal asset allocations from our model, with the main conclusions described in Section 4.5.

4.2 Asset Allocation Problem for a DC Pension Plan

In this section, we describe the two-asset discrete-time model with constant investment opportunity set²² (i.e. a constant risk-free rate, constant risk premium and constant volatility of return on the risky asset) used in the simulation process. All investment returns in the model are in real term (i.e. inflation is not considered in the model).

A number of assumptions have been made:

- (i) Members are assumed to join the pension scheme at age 20 (without bringing a transfer value) and the retirement age is fixed at 65 (i.e. at time $T = 45$);
- (ii) Contributions, expressed as a percentage of current salary, are assumed to be fixed and paid annually in advance;
- (iii) We assume that members target a chosen replacement ratio at retirement (in practice, this would often be 0.5 or, even, 0.667), which then enables us to estimate a target value for the pension fund at retirement (based on projected final salary and the expected purchase price for a whole of life annuity at the date of retirement);
- (iv) Members are assumed to evaluate the investment performance of the portfolio annually before retirement. At each age prior to retirement, the final target fund is adjusted (to reflect actual current salary and, hence, projected final salary), and members can also be considered to have a series of future interim targets in the period prior to retirement (corresponding to the discounted value of the final target at each future age, allowing for the then future contributions to be paid);
- (v) At each age, we give the same weight to all future interim targets and a higher weight to the final target (to reflect the fact that final target is more important than interim

²² This assumption is made to facilitate the numerical optimisation method used in the model. In real life, investment returns of some risky assets, e.g. equities may be mean reverting. This will mean that the optimal asset allocation in such risky asset needs to be slightly higher at each year, compared to what our model suggests.

targets). These weighting coefficients are used to balance the importance of these interim and final targets; and

- (vi) members are assumed to be loss averse and to make investment decisions with the aim of maximising the expected present value (EPV) of the loss aversion utility function (see Section 2.3 below).

4.2.1 Financial assets

There are two underlying assets in which the pension plan can invest:

- a risk-free asset (e.g. a cash fund), and
- a risky asset (e.g. an equity fund)

The risk-free asset is assumed to yield a constant return of r per annum. The annual return on the risky asset in year t , R_t , is given by²³:

$$R_t = r + \mu + \sigma \times Z_{1,t}$$

[12]

where

- μ is the annual risk premium on the risky asset,
- σ is the annual volatility of return on the risky asset, and
- $Z_{1,t}$ is a series of independent and identically distributed standard Normal random variables. While we realise that the normal distribution of risky asset return is a simplified assumption, this help facilitate the numerical optimisation method.

²³ In our current model, the investment opportunity set is assumed constant. This will greatly facilitate the numerical method of calculating optimal asset allocation. However, it will certainly be an important improvement to investigate the loss aversion model under a stochastic interest rate (e.g. Cox-Ingersoll-Ross model), which will allow us to incorporate the annuity risk (with annuity price based on stochastic interest rate) into the model. Also, the normal distribution assumption is made help facilitate the numerical optimisation method. One potential improvement is to assume that equity returns are regime switching. We will leave this to our future research.

4.2.2 Labour income process

Before retirement, the annual rate of labour income growth in year t , I_t , is given by:

$$I_t = r_I + \frac{S_t - S_{t-1}}{S_{t-1}} + \sigma_1 \times Z_{1,t} + \sigma_2 \times Z_{2,t} \quad [13]$$

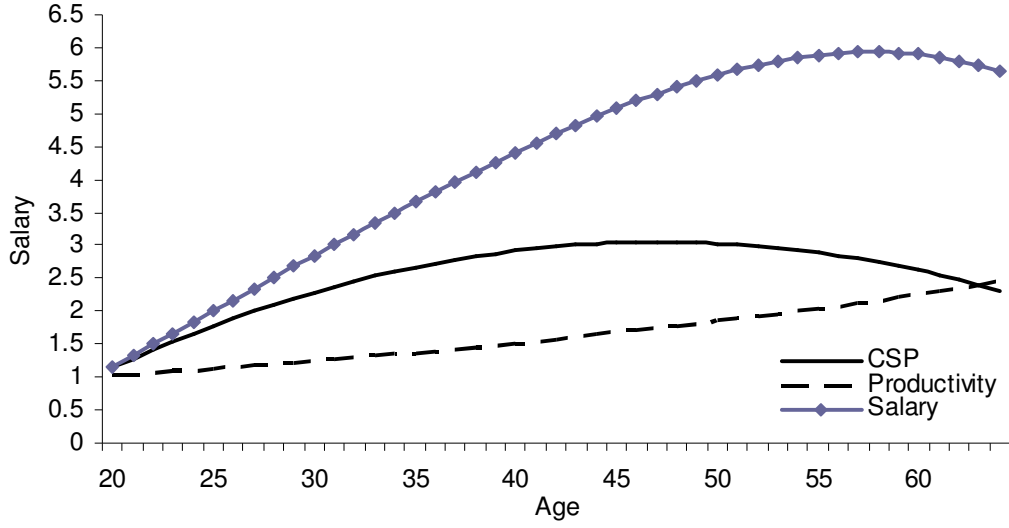
where

- r_I is the long-term annual rate of growth in national average earnings, reflecting productivity growth;
- S_t is the salary scale or career salary profile (CSP) at time t , so that $\frac{S_t - S_{t-1}}{S_{t-1}}$ reflects the promotional salary increase between time $t-1$ and t ;
- σ_1 represents the volatility of a shock from equity returns. In this way, we allow for possible correlation between labour income growth and equity returns²⁴.
- σ_2 represents the volatility of the annual rate of salary growth, and
- $Z_{2,t}$ is a series of independent and identically distributed standard Normal random variables.

Figure 10 shows the expected salary process over time up to retirement, allowing for both productivity growth (at a rate, r_I , of 2% p.a.) and promotional salary increases (i.e. career salary profile, CSP).

²⁴ Same approach has been used by other researchers, e.g. Cairns, Blake and Dowd (2006). Here, it is worth noting that this correlation might not be instantaneous in real life, given it is normal to a lagged effect between the stock market and labour income shocks. But this is out of the scope of our model.

Figure 10 Labour income process



4.2.3 Pension fund accumulation and target levels

The level of the pension fund at time $(t+1)$, denoted by f_{t+1} , is given by:

$$f_{t+1} = (f_t + \pi \times Y_t) \times [1 + r + \theta_t \times (\mu + \sigma_1 \times Z_{1,t+1})]$$

[14]

for $t = 0, 1, 2, \dots, T-1$, where

- $T = 45$ is the time of retirement;
- f_t is the fund level at time t , with $f_0 = 0$;
- Y_t is the labour income received at time t
 - where $Y_t = Y_{t-1} \times \exp(I_t)$, and $Y_0 = 1$;
- π is the fixed contribution rate, payable annually in advance; and
- θ_t is the proportion of fund invested in risky asset during year $(t, t+1)$

- a non-borrowing constraint is imposed, so that $0 \leq \theta_t \leq 1$.

Setting the final target

We will assume that member has a target replacement ratio at retirement of $\frac{2}{3} \times$ salary at retirement. Given a risk-free interest rate (in real term) of $r = 2\%$ per annum and using mortality in accordance with the projected PMA92 table²⁵, the price of a whole life annuity (payable annually in arrears) on retirement at exact age 65 is $a_{65}^{2\%} = 14.87$.

Assuming an initial salary at age 20 of $Y_0 = 1$ unit, the expected salary at retirement, denoted by $E_0(Y_{T=45})$, is 5.7526 units. Then, at time 0, the target final fund level at retirement is $F_0(T) = \frac{2}{3} \times 5.75 \times 14.87 = 57$ units.

After age 20, for each year before retirement, the final target fund level will depend on the current actual salary level and expected growth rate up to retirement. Thus, given a salary of Y_t units at time t , the expected salary at retirement is $E_t(Y_T) = Y_t \times \frac{E_0(Y_{T=45})}{E_0(Y_t)}$

(allowing for actual salary growth prior to time t and expected salary growth after time t). Then, the final target fund level at retirement is

$$F_t(T) = 0.667 \times Y_t \times \frac{5.75}{E_0(Y_t)} \times 14.87 = 57 \times \frac{Y_t}{E_0(Y_t)} \quad (\text{i.e. the final target fund level is}$$

adjusted to reflect actual salary growth up to time t).

²⁵ PMA92 is a mortality table for male pension annuitants in the UK based on experience over the period 1990-92. In this case, we use the projected mortality rates for 2010, denoted PMA92 (C2010), published by the Continuous Mortality Investigation (CMI) Bureau in February 2004.

²⁶ Based on real labour income growth of 2% per annum and using the salary scale described later in Section 3.2.

Setting the discounted interim targets

By assuming a suitable discount rate, denoted by $r_{discount}$, and an annual contribution rate of π , the interim targets in the period before retirement can then be derived recursively from the final target. Thus, at time t , the path-dependent interim target levels can be derived recursively as follows:

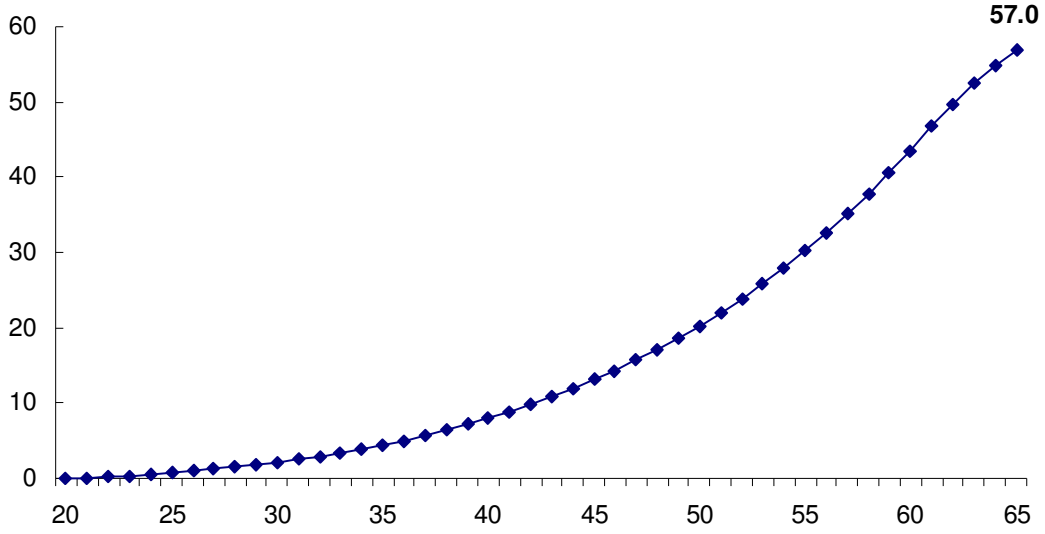
$$F_t(s) + \pi \times E_t(Y_s) = F_t(s+1) \times (1 + r_{discount}) \quad [15]$$

for $t = 0, 1, \dots, T-1$ and $s = t, \dots, T-1$, with $F_0(0) = 0$ and $F_t(T) = 57 \times Y_t / E_0(Y_t)$.

In this paper, we use the yield on AA-grade corporate bonds with a term in excess of 15 years as the discount rate to calculate discounted final target. This is consistent with the method for valuing the liabilities of defined benefit pension plans applied by global pension accounting standards (e.g. FAS 158, FRS 17 or IAS 19). As of 30 September 2008, the credit spread on Iboxx Sterling-denominated Corporate AA Over 15 years was 2.3%, so we use $r_{discount} = r + 0.023$ in our model.

Figure 11 shows the discounted interim targets and final target at age 20, using this approach together with an annual contribution rate of $\pi = 10\%$.

Figure 11 Expected path of target at age 20



4.2.4 Formulation of the target driven process

Based on the current actual fund level at time t , f_t , and the current target fund level (reflecting past and expected future salary growth), $F_t(t)$, we assume that a plan member faces a loss aversion utility function each year defined as:

$$U(t) = \begin{cases} \frac{(f_t - F_t(t))^{v_1}}{v_1} & \text{if } f_t \geq F_t(t) \\ -\lambda \frac{(F_t(t) - f_t)^{v_2}}{v_2} & \text{if } f_t < F_t(t) \end{cases}$$

[16]

However, it is reasonable to assume that, at any time t prior to retirement, the final target is significantly more important to members than the interim target. So, we apply a weighting coefficient, ω_1 , to the final target and lower weighting coefficient, ω_0 , to each

of the interim targets. Then, at time t , the present value of the total loss aversion utility function up to retirement is given by:

$$G_t = \sum_{s=t}^{T-1} \beta^{s-t} \times \omega_0 \times U(s) + \beta^{T-t} \times \omega_1 \times U(T) \quad [17]$$

for $t = 0, 1, 2, \dots, T-1$, where β is the intertemporal discount factor. Thus, the model proposed here is to maximise the expected present value of the total utility prior to retirement by focusing on both the interim and final targets.

It is important to note that the formulation of equation [17] can be interpreted as a generalisation of the optimal asset allocation problem of DC pension plans considered in much of the existing literature. In particular, two different methodologies have been used to investigate the optimal dynamic asset strategy of DC plans:

- Cairns et al. (2006) maximised the expected utility of the terminal replacement ratio at retirement, which can be represented a special case in this model with $F_t(t) = 0$ (for all $t < T$) and $\omega_0 = 0$; or
- Vigna and Haberman(2001) minimised the expected present value of the total disutility at each age prior retirement, which can be represented in the above framework with utility function $U(t) = (f_t - F_t)^2$

4.2.5 Optimisation and numerical method

Then, the set of optimal equity allocation proportions at each time t , $\{\theta_t : t = 0, 1, 2, \dots, T-1\}$, can be determined as follows:

$$\max_{\theta_t : 0 \leq \theta_t \leq 1} E \left(\sum_{s=t}^{T-1} \beta^{s-t} \times \omega_0 \times U(s) + \beta^{T-t} \times \omega_1 \times U(T) \right)$$

[18]

subject to the constraint that $f_{t+1} = (f_t + \pi \times I_t) \times [1 + r + \theta_t(\mu + \sigma_1 \times Z_{1,t})]$.

An analytical solution to this problem does not exist because we do not have an explicit solution for the expectation in above expression. Thus, we must use numerical dynamic programming methods to maximise the value function and derive the optimal control parameters for the asset allocation proportions.

The idea is to use the terminal utility function on retirement at age 65, $U(65)$, to compute (from equation [18]) the corresponding value function at age 64) and then to iterate this procedure backwards.

A crucial first step in the stochastic dynamic programming approach requires us to discretise the state space and shocks in the stochastic processes (i.e. equity return and labour income growth). Thus, wealth and labour income level are discretised into 100 and 10 evenly-spaced grid points respectively in computation. Also, the normally-distributed shocks in both the equity return and labour income growth are discretised into 9 nodes. The expected utility level at time t , $U(t)$, is then computed using these nodes and the relevant weights attached to each²⁷. Clearly, the choice of the number of nodes is subjective, but it is felt that this choice represents an appropriate trade-off between accuracy and speed of computation. After determining the optimal value of the control variable θ_t at each grid point, we then substitute these values in to equation [18] and solve the maximisation problem for the previous age. This process is then iterated backwards until age 20. Details of the dynamic programming and integration process are illustrated in Appendix 02.

²⁷ This method is known as Gaussian quadrature numerical integration. For more details, see Judd (1998), page 257-266.

4.3 Empirical analysis and parameter calibration

The baseline simulation of the optimal asset allocation for a typical member of our model defined contribution pension plan draws on empirical evidence of loss aversion parameters and salary scales from UK households.

4.3.1 Loss aversion parameters

As discussed in previous section, in the existing literature on loss aversion, the curvature parameter for gains, v_1 , and for losses, v_2 , are commonly assumed between 0 and 1, which implies that individuals are risk averse with respect to gains and risk seeking with respect to losses.

Based on an experiment conducted on a group of 25 graduate students, Tversky and Kahneman (1992) suggest that US individuals are 2.25 times more sensitive to losses than to gains (i.e. $\lambda = 2.25$) and have gain and loss curvature parameters both equal to 0.88 (i.e. $v_1 = v_2 = 0.88$). And, Hwang and Satchell (2005) use 20 years of monthly UK and US asset allocation data to empirically investigate admissible ranges for the loss aversion parameters. Their study proposes a long-term average value of $\lambda = 3$ for the loss aversion ratio, and suggest that this value should be adjusted upwards and downwards by 1.5 for bull and bear markets respectively. They suggest curvature parameters of $v_1 = 0.75$ and $v_2 = 0.95$, again implying that investors are risk averse with respect to gains and risk seeking with respect to losses. However, Levy and Levy (2002), amongst others, suggest that both v_1 and v_2 should be greater than one (implying that investors are risk seeking with respect to gains and risk averse with respect to losses), so favouring the reversed 'S'-shaped utility function proposed by Markowitz (1952).

To estimate the loss aversion parameters for UK investors, we conducted a face-to-face survey on 966 randomly selected UK residents in 2005 (see Appendix 2 for more details). The result suggests that:

- the loss aversion ratio is $\lambda = 3.4$,
- the curvature for gains is $v_1 = 0.53$, and
- the curvature for losses is $v_2 = 0.77$ ²⁸

This means that, as suggested previously, members are risk seeking with respect of the wealth level below a specified reference point and risk averse above the reference point. Also, given a value of $\lambda = 3.4$, these findings suggest that UK investors are significantly more loss averse²⁹ than would be implied by the Tversky and Kahneman framework (based, as mentioned above, on a much smaller sample of US graduate students).

4.3.2 Salary scale

Real salary growth for a pension plan member is assumed to consist of a stochastic element reflecting general growth in national average earnings and a deterministic promotional *increase* element (in line with a specified age-dependent salary scale). Following the work of Blake et al. (2007), we use the following quadratic function to model the salary scale:

$$S_{20+t} = 1 + k_1 \times \left(-1 + \frac{t}{45} \right) + k_2 \times \left[-1 + 4 \times \frac{t}{45} + 3 \times \left(\frac{t}{45} \right)^2 \right] \quad [19]$$

The parameters k_1 and k_2 are estimated by least square method using average male salary data (across all occupations) reported in the 2005 Annual Survey of Hours and

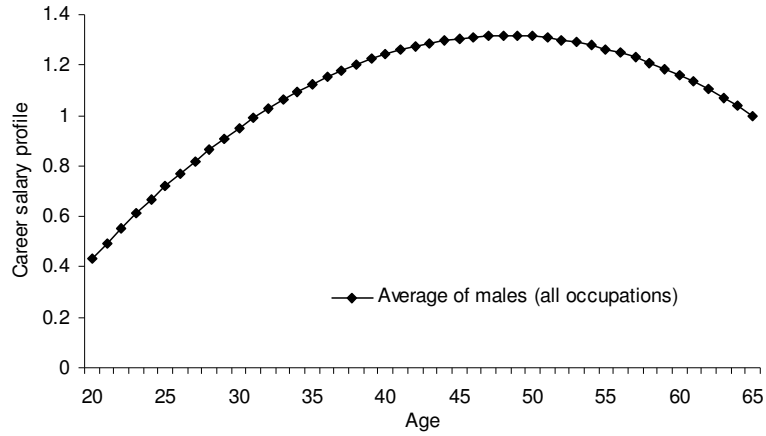
²⁸ with the 25% quartile being $v_1 = 0.36$, $v_2 = 0.47$ and $\lambda = 2$, and 75% quartile being $v_1 = 0.69$,

$v_2 = 1.21$ and $\lambda = 4.12$). More details can be found in a forth coming paper by Blake, Wright and Zhang.

²⁹ They are 3.4 times as sensitive to losses as to gains. In other words, when failing to achieve a target fund level, members in the loss-aversion framework feel 3.4 times as much pain as members in an equivalent power utility model.

Earnings. The estimated parameter values are $k_1 = -0.1865$ and $k_2 = 0.7537$ for all male workers, which leads to the age-dependent promotional scale illustrated in Figure 12 below. The results suggest that individuals can expect to achieve maximum earnings (in real terms) in their early 50s, with income reducing thereafter (when many choose to work part-time as retirement approaches).

Figure 12 Salary scale (scaled to 100 at age 65)



4.3.3 Parameter calibration

In the baseline case, we begin with a set of parameter values as follows:

Asset returns		Labour income process	
Risk-free return, r	0.02	r_I	0.02
Equity risk premium, μ	0.04	hedgeable volatility, σ_1	0.05
Volatility of equity return, σ	0.2	non-hedgeable volatility, σ_2	0.02
Discount factor, β	0.97	k_1	-0.1865
		k_2	0.7537
Loss aversion parameters		Other	
Loss aversion ratio, λ	3.4	Contribution rate, π	9%
Curvature for gains, ν_1	0.53	Weight of final target, ω_1	2.0
Curvature for losses, ν_1	0.77	Weight of interim targets, ω_0	1.0

It is assumed members join the plan with no transfer value (i.e. $f_0 = 0$) and the annual contribution rate is fixed at 9%. Using an equity premium of 4% per annum³⁰ (as opposed to the historical long-term average of closer to 6% per annum) is a fairly common choice in recent literature (e.g. Gomes and Michaelides (2005)).

As mentioned above, it is reasonable to assume that fund members will consider the final target to be significantly more important than each of the interim targets. Thus, following the work of Vigna and Haberman (2001), we assume initially that final target is twice as important as interim target in our baseline case (i.e. $\omega_0 = 1.0$ and $\omega_1 = 2.0$). However, in Section 4.4, we will analyse the sensitivity of the results obtained to using different final target weights.

As for the loss aversion utility function parameters (v_1, v_2 and λ), we use our survey estimate ($v_1 = 0.53, v_2 = 0.77$ and $\lambda = 3.4$) to run the baseline simulations. Sensitivity of the results to the chosen loss aversion utility parameters will also be examined later.

4.4 Simulation and results

4.4.1 The use of loss aversion utility in a target driven model

The use of loss aversion utility can be justified by the empirical behavioural research evidence which shows it is a more realistic representation of investors' financial decision

³⁰ Recent research in the area of lifecycle asset allocation tends to use a conservative equity premium setting mainly because the presence of labour income substantially increases the demand for stocks.

making, as we discussed earlier. However, it is also of interest to see if there is any tangible added value to use loss aversion utility in an asset allocation model of DC pension plans in terms of absolute final fund level or the certainty of achieving final target level.

As explained in section 4.1, previous researchers (e.g. Vigna and Haberman (2001, 2002)) propose a target-based asset allocation model for DC plans based on a quadratic disutility (cost) function. Several important assumptions made in their models include:

- (i) No labour income risk is considered (i.e. salary process is deterministic);
- (ii) Target of annual investment return is fixed;
- (iii) The expected present value (EPV) of total cost function until retirement is minimised to derive optimal asset allocation.

Using the same target driven approach (with deterministic salary process) and baseline parameter settings (as shown in section 4.3.3 before), we have compared our loss aversion utility based model with the quadratic model as used in Vigna and Haberman (2001). Specially, to keep the two models comparable, the annual salary growth rate (equation [13]) is simplified to $I_t = r_t + \frac{S_t - S_{t-1}}{S_{t-1}}$, to remove the stochastic component;

Also, the target annual investment return is fixed at 4.4% p.a. (representing an investment strategy consisting 60% in equities and 40% in cash). The final target level is still 57.0 in this case.

The optimisation problem is to maximise $G_t = \sum_{s=t}^{T-1} \beta^{s-t} \times \omega_0 \times U(s) + \beta^{T-t} \times \omega_1 \times U(T)$,

while different utility functions are to be used:

- (i) For our loss aversion utility based model, as defined in equation [16] before, we have;

$$U(t) = \begin{cases} \frac{(f_t - F_t(t))^{v_1}}{v_1} & \text{if } f_t \geq F_t(t) \\ -\lambda \frac{(F_t(t) - f_t)^{v_2}}{v_2} & \text{if } f_t < F_t(t) \end{cases}$$

(ii) For quadratic utility model, we have $U(t) = -(f_t - F_t)^2$.

The result of the optimisation process is a set of optimal control variables (i.e. equity allocation proportions at each time, θ_t , for $t = 0, 1, 2, \dots, T-1$) on each grid point for each time period. We now generate 10,000 simulations of the future economic experience over the period up to retirement based on the shocks on risky asset return ($Z_{1,t}$) and derive the optimal asset allocation for each path.

Figure 13 and **Table 9** below show the results of optimal asset allocation and final fund level over 10000 simulations. From **Figure 14** and **Table 9**, we can see that the use of loss aversion increases both mean and median level of final fund level at retirement. The discontinuity in the frequency distribution in **Figure 14** occurs because of the discontinuity in the loss aversion utility function. This is as expected because under loss aversion utility based model, the outperformance over target is not penalised and plan members are also encouraged to take more risk when the fund underperforms targets. As a result, the optimal investment strategy adopted is more aggressive.

Figure 13 Loss aversion Vs. quadratic utility

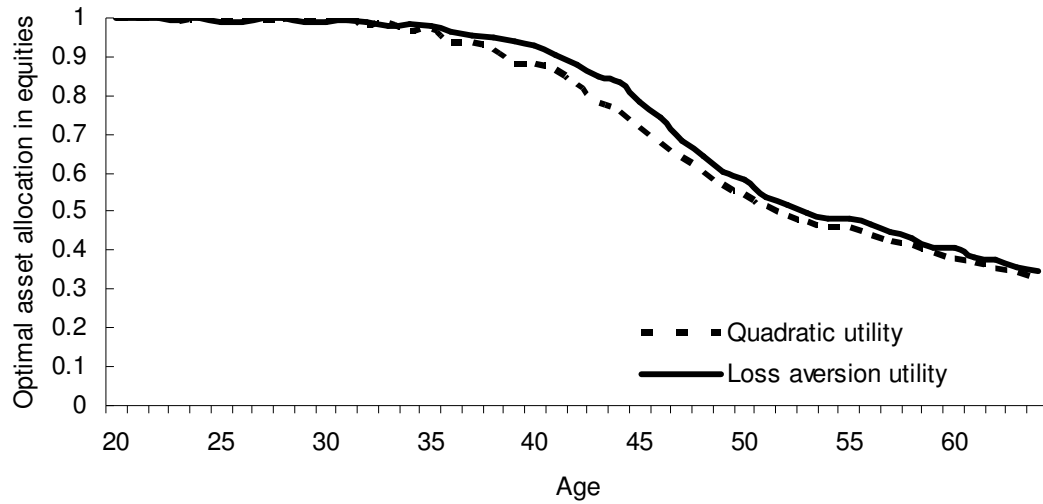
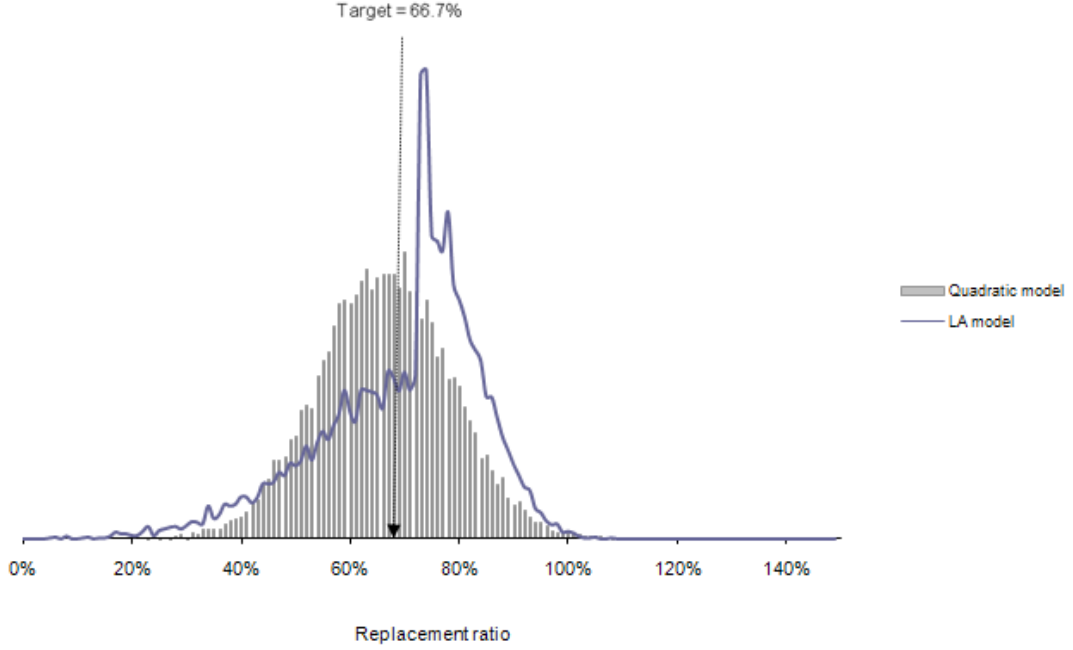


Table 9 Levels of replacement ratios under loss aversion and quadratic utility model (over 10k simulations), without salary risk

	Loss aversion utility	Quadratic utility
Target	66.7%	66.7%
Mean	70.0%	66.7%
25%	61.2%	58.6%
Median	73.7%	66.7%
75%	80.0%	74.6%
Probability of achieving target	65.9%	50.5%

Figure 14 Histogram (LA Vs. Quadratic utility model)

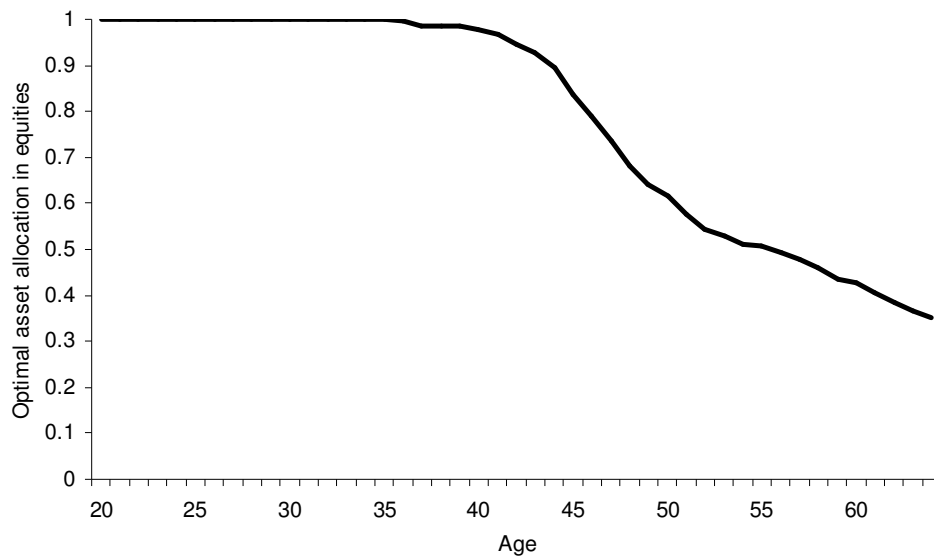


4.4.2 Baseline case

Having justified the use of loss aversion utility in a target driven framework, let us look at the simulation results of the model. Now, stochastic salary process (i.e. equation [3]) is used in the model. Again, we generate 10,000 simulations of the future economic experience over the period up to retirement based on the shocks on risky asset return ($Z_{1,t}$) and uninsurable labour income growth ($Z_{2,t}$). In each time period, as we can only tell the optimal asset allocation on each grid point in the 100 by 10 state variable matrix, bilinear interpolation method is used to derive the optimal value for the control variables (θ_t) for scenarios which lie in the state space but on the grid points.

The mean of the optimal equity allocation over 10000 simulations at each age is shown in **Figure 15**.

Figure 15 Mean optimal equity asset allocation



These results confirm the suitability of a lifestyle asset allocation strategy, which is widely adopted by many investment managers in defined contribution pension plans worldwide. However, as shown in **Figure 16**, the range of potential levels of replacement ratio at retirement from a traditional 5-year lifestyle strategy³¹ is huge. This makes it very difficult for a DC member to have any idea of what level of retirement income they can expect. The final fund levels from our loss aversion based dynamic asset allocation are also shown in the histogram in **Figure 16**. We can see that the dynamic target driven model significantly improve the certainty of traditional lifestyle strategy in terms of probability of achieving replacement ratio target at retirement.

³¹ In this case, under the 5-year deterministic lifestyle strategy, member's investment in equity fund will be gradually switched into cash fund as follows (100% before age 61, 80% at age 61, 60% at age 62, 40% at age 63, 20% at age 64 and 0% at age 65).

Figure 16 Histogram of replacement ratio: baseline case Vs. 5-year lifestyle

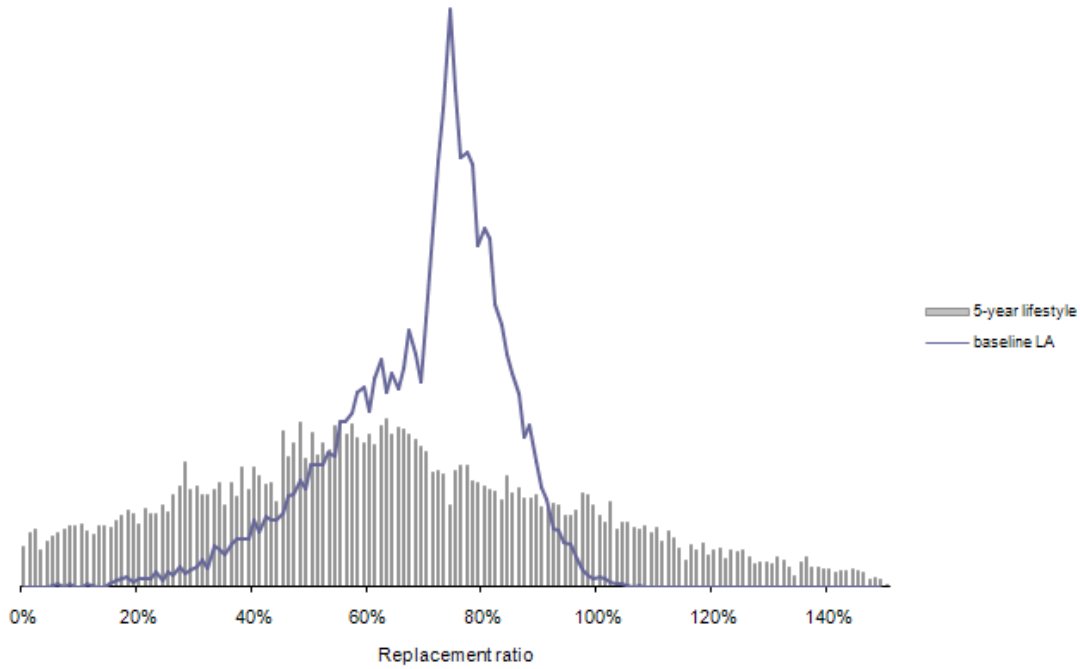


Table 10 Levels of replacement ratios under dynamic loss aversion model and 5-year lifestyle (over 10k simulations)

	Loss aversion utility	5-year lifestyle
Target	66.7%	66.7%
Mean	68.9%	67.2%
25%	60.4%	42.9%
Median	72.3%	65.0%
75%	78.8%	89.9%
Probability of achieving target	63.8%	47.2%

We can also compare this dynamic investment strategy with other investment strategies with fixed asset allocation. As shown in **Table 11** below, the optimal dynamic strategy has the largest probability of achieving final replacement ratio target (63.8%).

Table 11 Reliability of different asset strategies

	Optimal	Fixed asset decision			
Equity weight	dynamic	100%	90%	75%	50%
Target replacement ratio	66.7%	66.7%	66.7%	66.7%	66.7%
Mean	68.9%	70.9%	61.2%	60.0%	52.7%
Median	72.3%	69.73%	58.5%	56.9%	49.6%
Probability of achieving target	63.8%	50.7%	41.3%	38.4%	29.6%

If members invest 100% of portfolio in equity throughout the period, they have a 50.7% chance of achieving final target. If they follow a binding 50% equity-50% cash strategy, they only have a 29.6% chance of achieving the final target. The dynamic investment strategy we proposed is much more focused on achieving targets. The plan members in our model surrender some chances to get a very high replacement ratio and in return achieve a more certain level of living standard at retirement (as measured by replacement ratio).

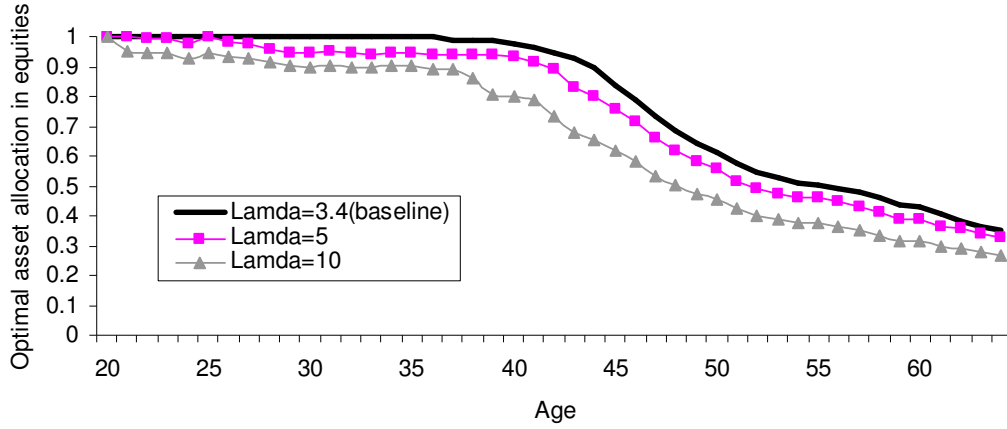
4.4.3 Sensitivity analysis

In the following paragraphs, we will investigate more about sensitivity and importance of the loss aversion ratio, curvatures, interim and final targets.

Loss aversion ratio

Loss aversion function [16] is S-shaped. When λ increases, the member becomes more loss averse (i.e. more sensitive to losses than to gains). As a result, as illustrated in **Figure 17**, the greater the value of the LA ratio, λ , the lower the optimal proportion of the fund invested in the risky asset at each age.

Figure 17 Sensitivity: loss aversion coefficient (Lamda)



Loss aversion curvatures

In addition, the loss aversion value function does not need to be S-shaped (i.e. $v_1 < 1$ and $v_2 < 1$). In this section, we investigate the change of asset strategy under different curvature settings. We have 3 different cases: case 1 ($v_1 < 1$ and $v_2 < 1$), case 2 ($v_1 > 1$ and $v_2 > 1$, i.e. a reversed S-shaped curve) and case 3 ($v_1 < 1$ and $v_2 > 1$, i.e. a kinked power utility curve). A larger curvature on losses than gains means investors are more sensitive to the changes in losses than to the equivalent changes in gains (and vice versa).

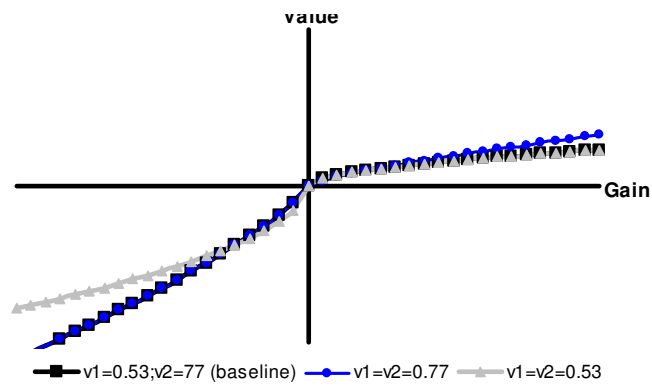
Table 12 Loss aversion curvatures

	Case1			Case2		Case3
Curvatures (v_1, v_2)	$v_1=0.53,$ $v_2=0.77$	$v_1=0.77,$ $v_2=0.77$	$v_1=0.53,$ $v_2=0.53$	$v_1=1.4,$ $v_2=1.4$	$v_1=1.4,$ $v_2=1.6$	$v_1=0.53,$ $v_2=1.4$
Target level of replacement ratio	66.7%	66.7%	66.7%	66.7%	66.7%	66.7%
Mean actual final level	68.9%	69.9%	70.2%	71.7%	72.0%	69.1%
Median actual final level	72.3%	73.3%	73.4%	67.7%	68.2%	67.9%
Probability of achieving target	63.8%	65.7%	65.8%	55.1%	56.1%	54.9%

Case 1: S-shaped ($v_1 < 1$ and $v_2 < 1$)

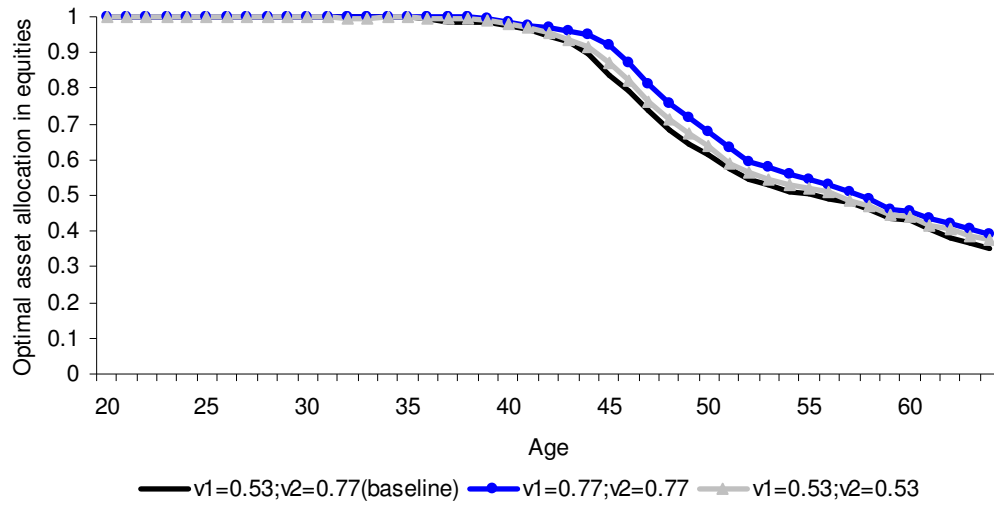
In case one, we have $v_1 < 1$ and $v_2 < 1$. Investors are risk averse with respect to the fund level above the target point and risk seeking with respect to the level below the target point (see **Figure 18**).

Figure 18 Case 1: $V_1 < 1$ and $V_2 < 1$ (S-shaped)



We present two different cases on v_1 and v_2 here. Keeping the other curvature parameter unchanged, we increase the value of v_1 or reduce the value of v_2 . As illustrated in **Figure 19**, when the curvature for gains (v_1) increase, investors become less risk averse with respect of gains, so the asset strategy becomes more aggressive. When the curvature for losses (v_2) decreases, investors become more risk seeking with respect of losses. Because the loss aversion ratio λ remains the same (which means individuals' sensitivities to loss are still the same), this change will not affect the shape of dynamic asset allocation much. However, it is interesting to notice from **Table 12** that this change does significantly increase the probability of achieving final target.

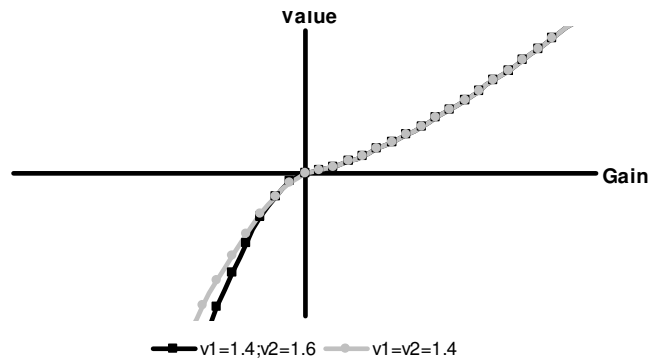
Figure 19 Mean optimal equity asset allocation for case 1



Case 2: reverse S-shaped ($v_1 > 1$ and $v_2 > 1$)

When $v_1 > 1$ and $v_2 > 1$, investors are instead risk loving for gains and risk averse for losses (as suggested by Levy and Levy (2002)), favouring a reverse S-shaped curve as shown in the **Figure 20** below.

Figure 20 Case 2: $v_1 > 1$ and $v_2 > 1$ (reverse S-shaped)



In this case, the member's dynamic asset allocation decision strategy becomes more volatile. Because when the actual fund level goes above the target, the member will be more willing to increase equity investment and achieve a very good level of pension income; however, when the actual fund level falls below the target, the member will be more risk averse and less willing to increase the investment in equities and this makes it difficult to get fund level back to the target. Therefore, it is harder to achieve the final target. As a result, the frequency distribution will be more volatile and more positively skewed. **Figure 22** and **Table 12** confirm this effect.

Figure 21 Mean optimal equity asset allocation for case 2

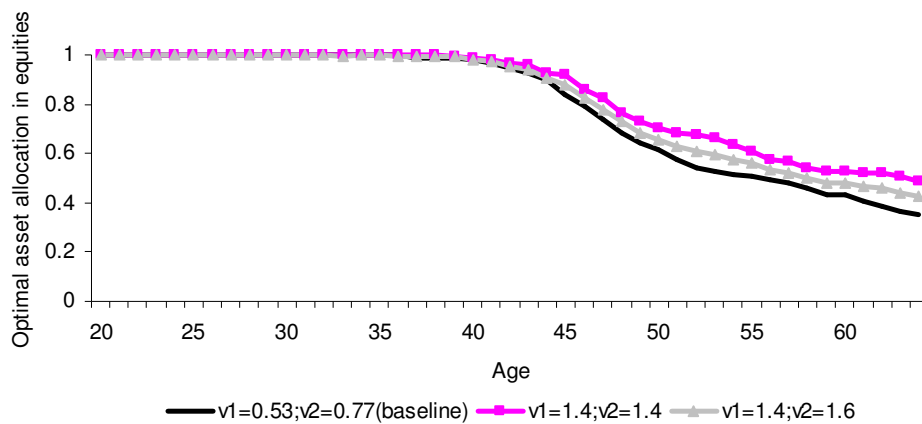
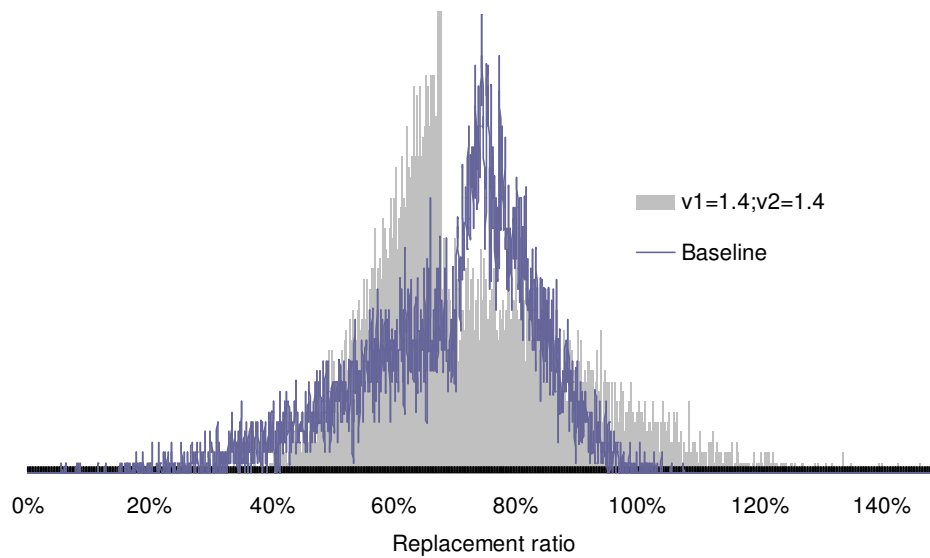


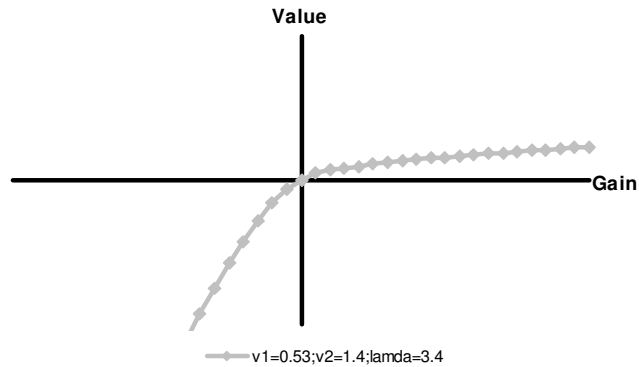
Figure 22 Histogram of replacement ratios: baseline case Vs. Reverse S-shaped utility



Case 3: kinked power utility ($v_1 < 1$ and $v_2 > 1$)

When $v_1 < 1$ and $v_2 > 1$, the members are risk averse for both gains and losses (see **Figure 23**).

Figure 23 Case 3: $v_1 < 1$ and $v_2 > 1$ (kinked power utility)



In this case, same as loss investors in our baseline simulations, investors are risk averse with respect of the fund level below the reference point. Meanwhile, when the actual fund falls below the target point, they will still be risk averse and less willing to increase equity investment. As shown in **Figure 24**, the asset strategy will be more conservative compared with baseline setting (in which members become risk lovers with respect to loss).

And we can tell from **Figure 25** that it significantly reduces the probability of achieving final target (with a chance of 54.9% in our simulations).

Figure 24 Mean optimal equity asset allocation for case 3

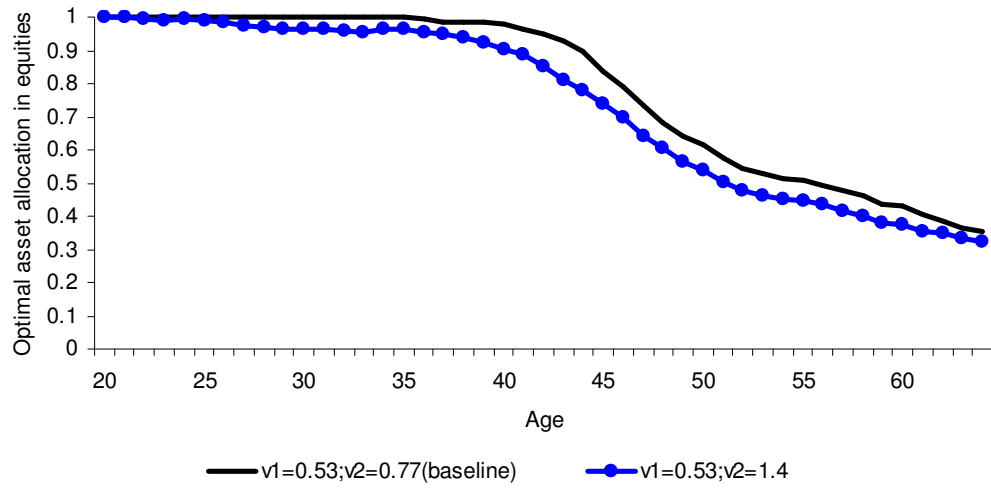
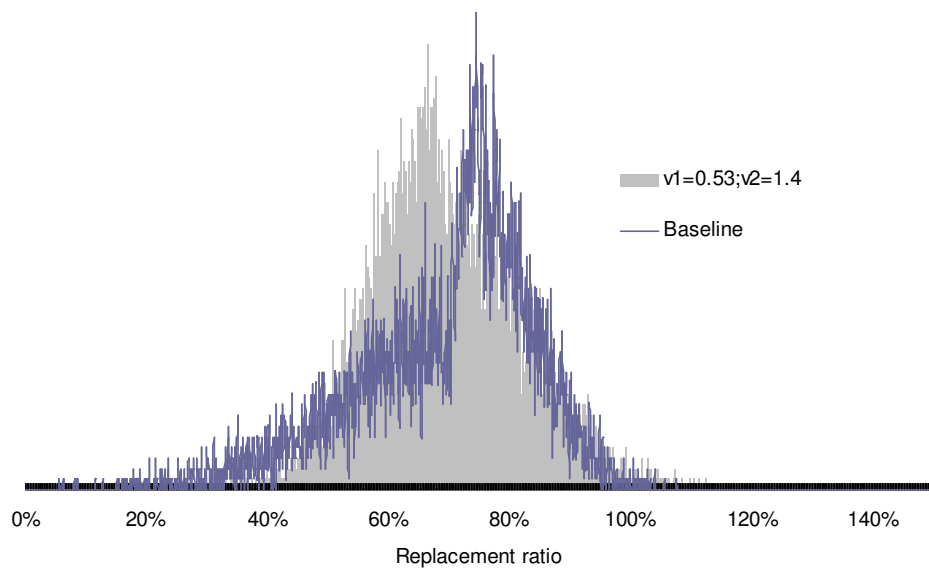


Figure 25 Histogram of replacement ratios: baseline case Vs. kinked power utility



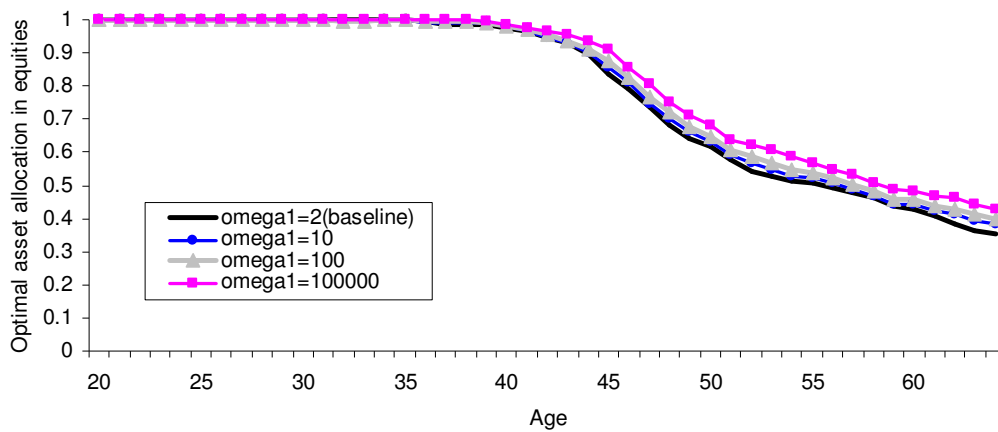
Final target weight

When the final target weight becomes larger, it becomes more important to the member to achieve the final target and investors are less committed to meeting interim targets. In this case, as shown in the **Figure 26** below, we need a more aggressive strategy to increase the probability of achieving the final target. We can see from **Table 13**, in terms of the probability of achieving final target, the benefits of applying a weighting factor of more than 10 on final target (and adopting a more aggressive investment strategy) is marginally diminished. The results seem suggest our baseline setting ($\omega_1 = 2$) is sensible.

Table 13 Final target weight ($v_1=0.53$, $v_2=0.77$ and $\lambda=3.4$)

Final target weight	1	2(baseline)	10	100	100k
Target replacement ratio	66.7%	66.7%	66.7%	66.7%	66.7%
Mean	68.8%	68.9%	70.2%	70.5%	70.9%
Median	71.9%	72.3%	73.3%	73.3%	73.4%
Probability of achieving target	62.4%	63.8%	65.8%	66.3%	66.6%

Figure 26 Sensitivity: final target weight



Final target only case

An extreme case is when only the final target is considered important, i.e. there is no feedback information before retirement (and thus no interim targets). To investigate the optimal asset allocation in this case, we can assume all the interim target equal to zero and set the weight on final target to infinity. We generate the same random shocks as in the baseline case, and find the optimal asset allocation for each path. We have the results in **Table 14**, compared with the baseline case (i.e. the dynamic strategy which considers both final and interim targets). The results suggest the importance of considering interim targets when we model the optimal investment strategies for a DC pension plan. We can see that the dynamic strategy considering both interim and final targets significantly enhances the security of the schemes (with higher probability of achieving final target).

Table 14 Only final target case

Optimal equity weight	Considering interim targets	Final target only
Target level of replacement ratio	66.7%	66.7%
Mean actual final level	70.9%	68.2%
Median actual final level	73.4%	70.5%
Probability of achieving target	66.6%	60.7%

Comparison with power utility model

It may be not very surprising to know that a dynamic investment strategy considering interim targets can bring more security (in terms of the likelihood of achieving final target). Thus, it is of interest to see how our model compares with existing models where members are assumed to only care about final target and do not have any utility on achieve interim ones before retirement.

Further to the discussion in above section and the assumption of no interim targets, we can also compare our loss aversion model with a power utility model, where members have a power utility on final fund level. This power utility framework has been used by many other authors, such as Boulrier et al. (2001), Cairns et al. (2006).

Specifically, this is done by setting the reference wealth level, F , in the loss aversion function shown in equation [16] to zero (leading to a concave utility function, as shown in **Figure 27**) and setting the final target weight to infinity (i.e. $\omega_l = 100k$). In this case, the member is always risk averse: $U(f_t) = \frac{(f_t)^{v_1}}{v_1}$, where $v_1 = 0.53$.

Figure 27 Power utility case

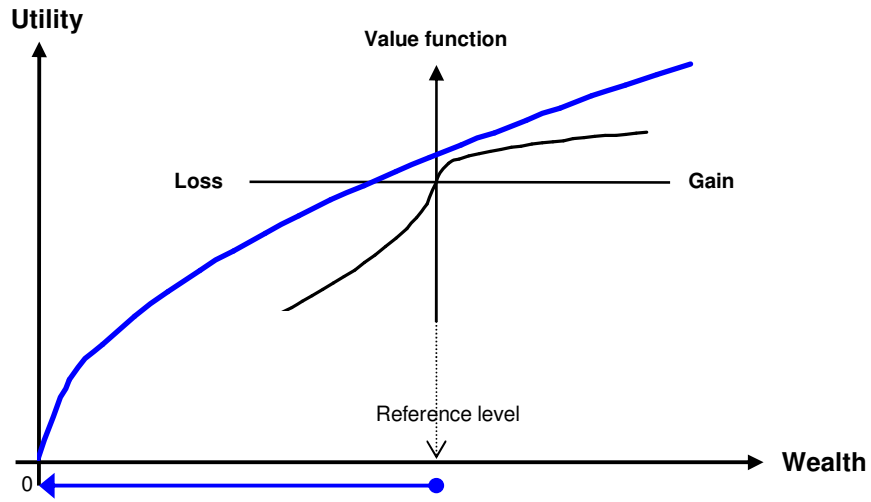


Figure 28 shows the mean optimal allocation in the risky asset at each age for both the loss-aversion and the power utility models, based on 10,000 simulations (and using the same random shocks in both the asset return and income growth processes). We can see that, compared with power utility settings, loss averse investors are more committed to meeting final target and therefore adopt a more conservative dynamic asset allocation strategy.

Figure 28 Loss aversion utility Vs. power utility

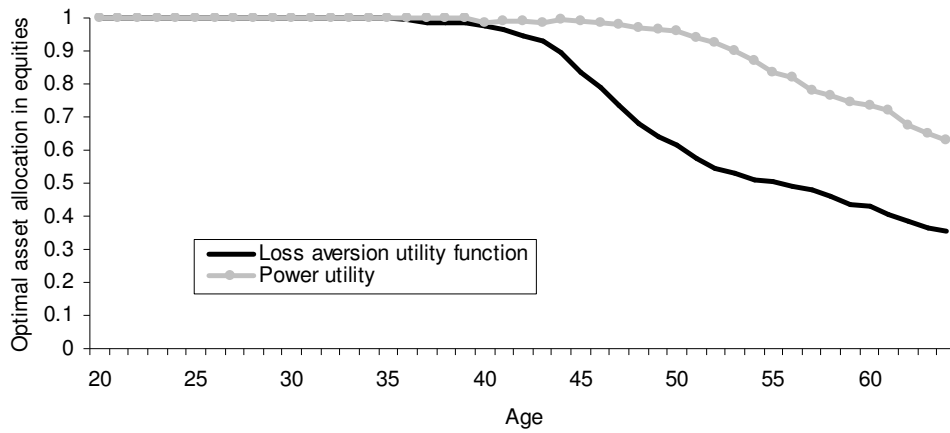


Figure 29 Histogram of replacement ratio: baseline case Vs. power utility

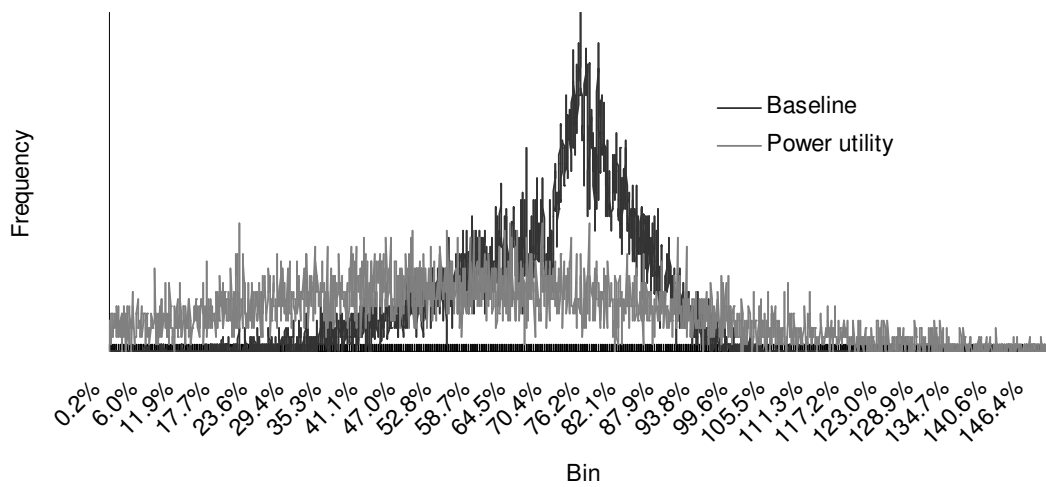


Table 15 Loss-aversion model Vs. Power utility

	Loss aversion model	Power utility model
Target level of replacement ratio	66.7%	66.7%
Mean actual final level	70.9%	60.0%
Median actual final level	73.4%	57.1%
Probability of achieving target	66.6%	39.0%

As illustrated in **Figure 29** and **Table 9**, the dynamic asset allocation strategy under the loss aversion model increases the probability of achieving final target (from 39.0% to 66.6%, in our simulations). This is because power utility maximisers only make investment decisions with the aim to maximise expected final fund level at retirement; on the contrary, loss averse investors adjust their investment decisions dynamically with the aim of achieving the target at retirement. In other words, when the expected final replacement ratio is better than the target, loss averse members will reduce the equity investment to avoid potential “loss”; when the expected final replacement ratio falls below the target, they will increase the equity investment dramatically in order to get back to the target quickly. In **Table 9**, the higher replacement ratio and probability under loss aversion model demonstrates the value of using loss aversion utility function in asset allocation strategy models of DC pension schemes.

Tversky-Kahneman setting

Given the prospect theory setting is well recognised and widely used, we have run 10000 simulations for members with the loss aversion parameters suggested by Tversky and Kahneman (1992), i.e. $v_1 = v_2 = 0.88$ and $\lambda = 2.25$. The mean of the optimal equity weight at each age, for both this case and the baseline case above (i.e. $\lambda = 3.4$, $v_1 = 0.53$ and $v_2 = 0.77$), are shown below. As illustrated in **Figure 30**, compared to our baseline case, the Tversky-Kahneman case has a smaller loss aversion ratio ($\lambda = 2.25 < 3.4$), which implies that the member is less sensitive to losses, and has a lower relative risk aversion in both the gain and loss regions ($1 - v_1 < 1 - 0.53$, $1 - v_2 < 1 - 0.77$), implying the member is willing to take more risk in both gain and loss regions. Thus, we have a more aggressive asset allocation strategy. **Table 15** shows that the investment strategy suggested by the prospect theory calibration leads to a lower probability of achieving target at retirement.

Figure 30 baseline Vs. prospect theory setting

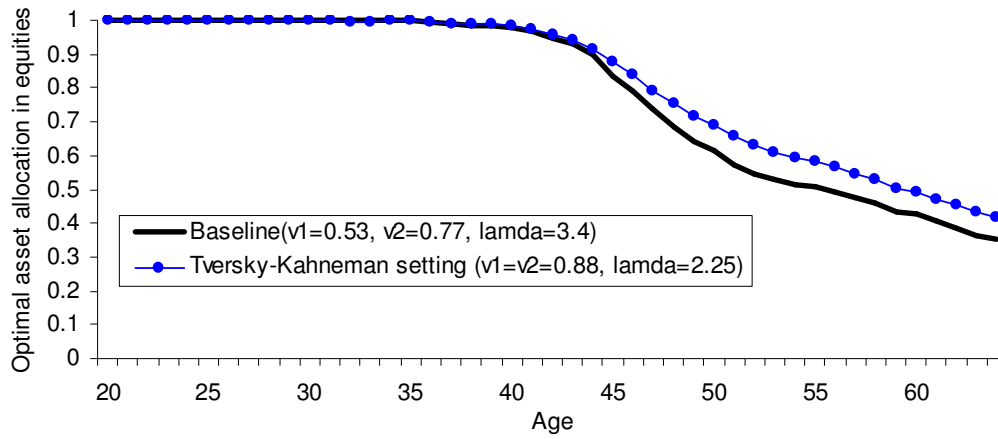


Table 16 Loss-aversion model Vs. Prospect theory model

	Loss aversion model	Prospect theory model
Target level of replacement ratio	66.7%	66.7%
Mean actual final level	70.9%	70.6%
Median actual final level	73.4%	74.0%
Probability of achieving target	66.6%	65.1%

4.5 Conclusions

In this chapter, we propose a loss aversion based numerical model to solve the optimal asset allocation problem of defined contribution pension plans. The risks from a traditional deterministic lifestyle strategy are much higher than generally understood. For DC plan members who seek greater certainty in their retirement planning, the investment strategies need to be far more focused on achieving their retirement targets.

A new type of target driven approach is used to derive the dynamic optimal asset allocation. The key findings of this paper include:

- Based on the results of a face-to-face survey on 966 randomly selected UK residents conducted in 2005, we suggest that UK households are more loss averse than Tversky and Kahneman's original group of students. Specially, our survey results suggest $v_1=0.53$, $v_2=0.77$ and $\lambda=3.4$, which means investors are 3.4 times as sensitive to losses as to gains and they are more sensitive to marginal losses than the equivalent marginal gains (i.e. $v_1 < v_2$).
- The new dynamic asset allocation strategy is much more focused on achieving their retirement targets. Compared with traditional deterministic lifestyle investment strategies, the target driven investment strategy used in this chapter significantly improve the certainty of members' retirement planning in a DC plan. As a trade off, under this dynamic target driven strategy, members need to surrender some potential good results of pension income at retirement.
- Compared with an equivalent power utility model which also has been used by other researchers, loss averse investors are more committed to achieving interim and final targets fund levels and therefore adopt a more conservative asset allocation strategy. And for loss averse investors, the more important the final target is, the more aggressive the dynamic asset allocation strategy is required.

How often the members are given fund performance information is likely to influence the relative importance of interim targets.

- Interim targets are essential in this loss-aversion-based model. To incorporate interim targets can significantly enhance the security of the schemes. In our simulations, the probability of achieving final target is 60.6% when interim targets are considered compared with a chance of 51.6% if only final fund level at retirement is targeted.
- By comparing with quadratic function as used in previous studies, we demonstrate the value of using loss aversion utility in a target driven asset allocation model. Loss aversion model helps to increase both mean and median of final fund level.

Recently, a number of fund managers have launched dynamic de-risk investment solutions for DC pension plans with targets defined on investment return or annuity amount. A standard feature of these solutions is that when the fund outperforms the pre-determined target, additional assets above the target level are “banked” by switching to risk-free investment (e.g. cash). So while they have some features in common with our framework, they do not explicitly consider issues such as path-dependent targets or members’ utility. Our framework is therefore much more general than the solutions currently being implemented in practice. In addition, the model is reasonably easy to implement as well. The outputs of our dynamic asset allocation model are a set of optimal control variable (portfolio weight to equities) on each grid points for each age. With information of the member’s actual salary and pension fund level, an optimal investment strategy can be derived immediately.

In this chapter, we justified the added value of using loss aversion utility function. However, we notice that this is based on the assumption of fixed contribution rates. In real life, if the expected utility theory still holds, power utility maximisers are to increase equity investment as their wealth increases; but meanwhile they can reduce the amount of contribution paid into their pension plans so that they get more utility from consumption before retirement. In the next chapter, we try to address this issue and look at the optimal investment strategy when members are allowed to change contribution decisions.

Chapter 5

Recursive Utility Model

5.1 Introduction

5.1.1 Allocation of consumption across the life cycle

A typical individual's life cycle consists of a period of work followed by a period of retirement. Individuals therefore need to reallocate consumption from their working life – when the lifetime's income is earned – to retirement – when there might be no other resources available, except possibly a subsistence level of support from the state. A defined contribution (DC) pension plan can achieve this reallocation in a way that is consistent with the preferences of the individual plan member. There are three key preferences to take into account.

The first relates to the desire to smooth consumption across different states of nature in any given time period. The second relates to the desire to smooth consumption across different time periods. Saving for retirement involves the sacrifice of certain consumption today in exchange for, generally, uncertain consumption in the future. This uncertainty arises because both future labour income and the returns on the assets in which retirement savings are invested are uncertain. The plan member therefore needs to form a view on both the trade-off between consumption in different states of nature in the same time period and the trade-off between consumption in different time periods. Attitudes to these

trade-offs will influence the optimal funding and investment strategies of the pension plan.

In a DC pension plan, the member allocates part of the labour income earned each year to the pension plan in the form of a contribution and, thus, builds up a pension fund prior to retirement. Then, on retirement, the member uses a proportion of the accumulated pension fund to purchase a life annuity. The decisions regarding the contribution rate each year before retirement (i.e., the funding strategy) and the annuitisation ratio (i.e., the proportion of the fund at retirement that is used to purchase a life annuity) are both driven by the member's preference between current and future consumption, as well as the desire to leave a bequest, the third key preference that we need to take into account. Should the member die before retirement, the entire accumulated pension fund will be available to bequest; after retirement, only that part of the residual pension fund that has not been either annuitised or spent can be bequested.

The investment strategy (i.e., the decision about how to invest the accumulated fund across the major asset categories, such as equities and bonds) will influence the volatility of the pension fund and, hence, consumption in different time periods, and so will be influenced by the member's attitude to that volatility.

In this chapter, we investigate the optimal funding and investment strategies in a DC pension plan³². To do this, we use a model that differs radically from existing studies in this field in three key respects.

The first key feature of the model is the assumption of Epstein-Zin recursive preferences by the plan member. This enables us to separate relative risk aversion (RRA) and the elasticity of intertemporal substitution (EIS). Risk aversion is related to the desire to stabilise consumption across different states of nature in a given time period (e.g., an

³² This research focuses on the investment and funding strategies for a DC plan during the accumulation stage and the only form of saving we allow is pension saving. Non-pension saving, housing-related investments and post-retirement investment strategies are beyond the scope of this study.

individual with a high degree of risk aversion wishes to avoid consumption uncertainty in that period, and, in particular, a reduction in consumption in an unfavourable state of nature) and the EIS measures the desire to smooth consumption over time (e.g., an individual with a low EIS wishes to avoid consumption volatility over time, and, in particular, a reduction in consumption relative to the previous time period)³³. Thus, risk aversion and EIS are conceptually distinct and, ideally, should be parameterised separately. In this paper, we consider four different types of member according to different RRA and EIS combinations, as shown in **Table 17**.

Table 17 Pension plan member types		
	High RRA (risk averse)	Low RRA (risk tolerant)
Low EIS (likes consumption smoothing)	• risk-averse member who dislikes consumption volatility over time • low equity allocation, particularly as retirement approaches • e.g., low income member with dependants	• risk-tolerant member who dislikes consumption volatility over time • high equity allocation at all ages • e.g., low income member without dependants
High EIS (accepts consumption volatility)	• risk-averse member who does not mind consumption volatility over time • low equity allocation, particularly as retirement	• risk-tolerant member who does not mind consumption volatility over time • high equity allocation at all ages

³³ The EIS is defined as

$$\varphi = - \frac{\frac{d(c_{t1}/c_{t2})}{c_{t1}/c_{t2}}}{\frac{d[U'(c_{t1})/U'(c_{t2})]}{U'(c_{t1})/U'(c_{t2})}} = - \frac{d[\ln(c_{t1}/c_{t2})]}{d[\ln(U'(c_{t1})/U'(c_{t2}))]},$$

where c_{t1} is consumption in period $t1$ and $U'(c_{t1})$ is the marginal utility of c_{t1} , etc. The sign and size of the EIS reflects the relationship between the substitution effect and income effect of a shock to a state variable, such as an increase in the risk-free interest rate. The substitution effect is always negative, since current consumption decreases when the risk-free rate increases because future consumption becomes relatively cheap and this encourages an increase in savings. The income effect will be positive if an increase in the risk-free rate (which induces an increase in wealth) leads to an increase in current consumption; it will be negative otherwise. If the income effect dominates, the EIS will be negative and an increase in the risk-free rate leads to an increase in current consumption. If the substitution effect dominates (which is the usual assumption), the EIS will be positive and an increase in the risk-free rate leads to a decrease in current consumption. If the income and substitution effects are of equal and opposite sign, the EIS will be zero and current consumption will not change in response to an increase in the risk-free rate: in other words, consumption will be smooth over time in response to interest rate volatility.

	approaches • e.g., high income member with dependants	• e.g., high income member without dependants
--	---	---

Within the commonly used power utility framework, the coefficient of relative risk aversion (RRA) is the reciprocal of EIS (see, for example, Campbell and Viceira (2002)). This restriction has been criticised because it does not reflect empirical observations. For example, based on the consumption capital asset pricing model³⁴, Schwartz and Torous (1999) disentangle these two concepts using the term structure of asset returns. Using US data, their best estimate for RRA is 5.65 (with a standard error of 0.22) and their best estimate of the EIS is 0.226 (with a standard error of 0.008). Thus, a high RRA is associated with a low level of EIS, but the estimated parameter values do not have the reciprocal relationship assumed by power utility. Blackburn (2006) also rejects the reciprocal relationship on the basis of a time series of RRA and EIS parameters estimated from observed S&P 500 option prices for a range of different expiry dates between 1996 and 2003³⁵.

The second key feature of the model is the recognition that the optimal investment strategy will depend not just on the properties of the available financial assets, but also on the plan member's human capital. A commonly used investment strategy in DC pension plans is "deterministic lifestyling". With this strategy, the pension fund is invested entirely in high risk assets, such as equities, when the member is young. Then, at some arbitrary date prior to retirement (e.g., 10 years), the assets are switched gradually (and usually linearly) into lower risk assets such as bonds and cash. However, there has been no strong empirical evidence to date demonstrating that this is an optimal strategy.

If equity returns are assumed to be mean reverting over time, then the lifestyle strategy of holding the entire fund in equities for an extended period prior to retirement may be justified, as the volatility of equity returns can be expected to decay over time (as a result

³⁴ Breeden's 1979 extension of the traditional CAPM which estimates future asset prices based on aggregate consumption rather than the return on the market portfolio.

³⁵ In particular, Blackburn (2006) found that, over the period 1996 to 2003, the level of risk aversion changed dramatically whilst the level of elasticity of intertemporal substitution stayed reasonably constant.

of the effect of “time diversification”). However, there is mixed empirical evidence about whether equity returns are genuinely mean reverting: Blake (1996), Lo and Mackinley (1988) and Poterba and Summers (1988) find supporting evidence for the UK and US, while Howie and Davies (2002) and Kim et al (1991) find little support for the proposition in the same countries. We would therefore not wish an optimal investment strategy to rely on the assumption of mean reversion holding true in practice.

A more appropriate justification for a lifestyle investment strategy comes from recognising the importance of human capital in individual financial planning. Human capital (i.e., the net present value of an individual’s future labour income) can be interpreted as a bond-like asset in which future labour income is the “dividend” on the individual’s implicit holding of human capital. Young pension plan members therefore implicitly have a significant holding of bond-like assets and, thus, should weight the financial element of their overall portfolio towards equity-type assets³⁶. But to date, there has been no quantitative research exploring the human capital dimension in a DC pension framework.

This chapter presents an intertemporal model to solve the life-cycle asset allocation problem for a DC pension plan member. The model assumes two assets (a risky equity fund and a risk-free cash fund), a constant investment opportunity set (i.e., a constant return on the risk-free asset, and a constant expected return and volatility on the risky asset) and stochastic labour income. We consider two aspects of labour income risk: the volatility of labour income and the correlation between labour income and equity returns which determines the extent to which labour income affects portfolio choice (e.g., a positive correlation reduces the optimal asset allocation to equities).

The third key feature of the model concerns the annuitisation decision at retirement. A member with a strong “bequest” savings motive will not wish to annuitise all the accumulated pension wealth. In our model, the member chooses to annuitise a proportion

³⁶ Note this argument might not be appropriate for more entrepreneurial individuals whose pattern of future labour income growth corresponds more to equity than to bonds.

of the accumulated pension fund at retirement by buying a life annuity which will generate a return linked to bonds. We denote this proportion the “annuitisation ratio”. This ratio is chosen to maximise the expected utility level at retirement when annuity income replaces labour income. The member invests the residual wealth that is not annuitised in higher returning assets in line with the RRA. The member can draw an income from the residual wealth to enhance consumption in retirement, but, unlike the life annuity, the residual wealth can be bequeathed when the individual dies.

Before considering the model in more detail, we will review Epstein-Zin utility.

5.1.2 Epstein-Zin utility

The classical dynamic asset allocation optimisation model was introduced by Merton (1969, 1971), and shows how to construct and analyse optimal dynamic models under uncertainty. Ignoring labour income, in a single risky asset and constant investment opportunity setting, the optimal portfolio weight in the risky asset for an investor with a power utility function $U(W) = (W)^{1-\gamma} / (1-\gamma)$ (where W is wealth and γ is the coefficient of relative risk aversion) is given by:

$$\alpha = \frac{\mu}{\gamma\sigma^2}$$

[20]

where μ and σ^2 are the excess return on the risky asset and the variance of the return on the risky asset, respectively. The investment opportunity set is assumed to be constant.

Equation [20] is appropriate for a single-period myopic investor, rather than a long-term investor such as a pension plan member. Instead of focusing on the level of wealth itself, long-term investors focus on the consumption stream that can be financed by a given level of wealth. As described by Campbell and Viceira (2002, p37), “they consume out of wealth and derive utility from consumption rather than wealth”. Consequently, current

saving and investment decisions are driven by preferences between current and future consumption.

To account for this, Epstein and Zin (1989) proposed a discrete-time recursive utility function³⁷. Recursive utility preferences focus on the trade-off between current-period utility and the utility to be derived from all future periods. The following Epstein-Zin recursive utility has become a standard tool in intertemporal investment models, but has not hitherto been applied to pension plans:

$$V_t = \left\{ (1-\beta) C_t^{\frac{1-\frac{1}{\varphi}}{\varphi}} + \beta \left[E_t \left(V_{t+1}^{1-\gamma} \right) \right]^{\frac{1-\frac{1}{\varphi}}{1-\gamma}} \right\}^{\frac{1}{1-\frac{1}{\varphi}}} \quad [21]$$

where

- V_t is the utility level at time t ,
- β is the individual's personal discount factor for each year,
- C_t is the consumption level at time t ,
- γ is the coefficient of relative risk aversion (RRA), and
- φ is the elasticity of intertemporal substitution (EIS).

The recursive preference structure in [21] is helpful in two ways: first, it allows a multi-period decision problem to be reduced to a series of one-period problems (from time t to time $t+1$); and second, as mentioned previously, it enables us to separate RRA and EIS.

Then, ignoring labour income, for an investor with Epstein-Zin utility, there is an analytical solution³⁸ for the optimal portfolio weight in the risky asset given by:

³⁷ Kreps and Porteus (1978) first developed a generalised iso-elastic utility function which distinguishes attitudes to risk from behaviour toward intertemporal substitution. Following the KP utility function, Epstein and Zin (1989, 1991) proposed a discrete-time recursive utility function that allows the separation of the risk aversion parameter from the EIS parameter. Duffie and Epstein (1992) then extended the Epstein-Zin discrete recursive utility in a continuous-time form called a stochastic differential utility (SDU) function.

³⁸ For more details, see Merton (1973) and Campbell and Viceira (2002).

$$\alpha_t = \frac{\mu_t}{\gamma \sigma_t^2} + \left(1 - \frac{1}{\gamma}\right) \times \frac{\text{cov}_t(R_{t+1}, -V_{t+1})}{\sigma_t^2}$$

[22]

This shows that the demand for the risky asset is based on the weighted average of two components. The first component is the short-term demand for the risky asset (or myopic demand, in the sense that the investor is focused on wealth in the next period). The second component is the intertemporal hedging demand, which is determined by the covariance of the risky asset return with the investor's utility per unit wealth over time. Thus, ignoring labour income, the optimal portfolio weights are constant over time, provided that the investment opportunity set remains constant over time (i.e., $\mu_t = \mu$ and $\sigma_t^2 = \sigma^2$ in [22]).

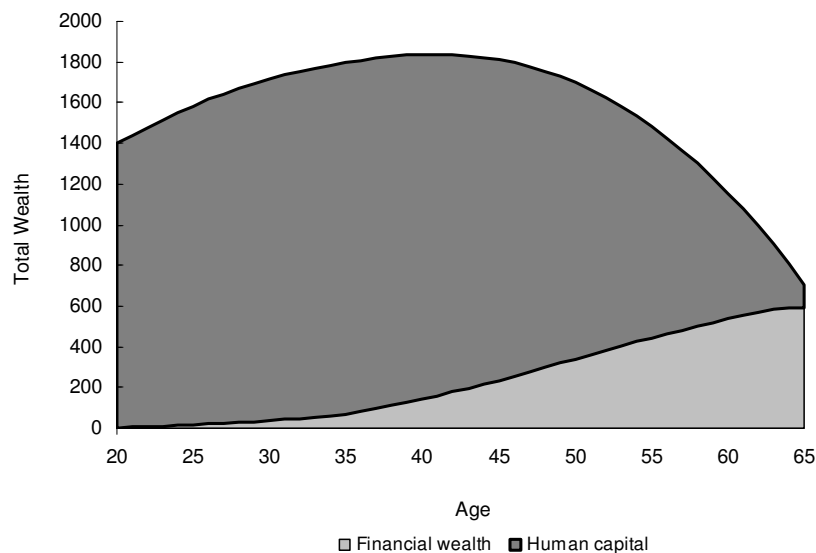
In a realistic life-cycle saving and investment model, however, labour income cannot be ignored. It is risky and cannot be capitalised and traded. But, allowing for labour income volatility in the optimisation process means that an analytical solution for the optimal asset allocation cannot be obtained. To address this, the recent literature has employed a number of numerical methods³⁹ to approximate the solution of the dynamic portfolio optimisation problem.

In the presence of income risk, the optimal portfolio weight in the risky asset is not constant, but instead follows a lifestyle strategy, as shown by Coco et al. (2005). This can be explained as follows: human capital or wealth can be thought of as the expected net

³⁹ By far, the most popular approach is value function iteration. Specifically, this involves the discretisation of the state variables by setting up a standard equally-spaced grid and solves the optimisation for each grid point at the next-to-last time period. The expectation term in the resulting Bellman equation is approximated by using quadrature integration and then the dynamic optimisation problem can be solved by backward recursion. In this case, it is possible that the accumulated state variable values from the previous time period are not represented by a grid point, in which case, an interpolation method (e.g., bilinear, cubic spline, etc.) must be employed to approximate the value function. However, this approach requires knowledge of the distribution of each of the shocks to the process, so that appropriate quadrature integration (e.g., Gauss quadrature) can be used. Furthermore, this approach cannot handle a large number of state variables. To overcome these limitations, Brandt et al. (2005) proposed a simulation method based on the recursive use of approximated optimal portfolio weights. The idea is to estimate asset return moments using a large number of simulated sample paths, and then to approximate the value function using a Taylor series expansion. If the return is path-dependent, it is necessary to regress the return variable on the simulated state variables from previous time period, before using the Taylor expansion with conditional return moments.

present value (NPV) of future labour income. Thus, an individual's labour income can be seen as the dividend on the individual's implicit holding of human capital. Hence, the ratio of human to financial wealth is a crucial determinant of the life-cycle portfolio composition. In early life, as shown in **Figure 31**, this ratio is large since the individual has had little time to accumulate financial wealth and expects to receive labour income for many years. Given that long-term average labour income growth is of a similar order of magnitude as average long-run interest rates in the UK over the last century, as explained in Cairns et al. (2006), labour income can be thought of as an implicit substitute to investing in the risk-free asset. Thus, younger individuals have a significant holding in this non-tradable risk-free asset and, therefore, should allocate most of their financial wealth to the risky asset to keep the overall portfolio composition constant, as suggested by Equation [22] above. As they grow older, individuals accumulate more financial wealth and draw down human capital. They should therefore rebalance their financial portfolio towards risk-free assets as age increases.

Figure 31: Decomposition of total wealth over the life cycle



More recently, life-cycle asset allocation models with a stochastic labour income process have been extended to include the use of a recursive utility function to allow a separation

of RRA and EIS by, for instance, Weil (1989) and Campbell and Viceira (2002). By including a fixed first-time risky-asset entry cost and adopting Epstein-Zin utility, Gomes and Michaelides (2005) present a life-cycle asset allocation model to explain the empirical observations of low stock market participation and moderate equity holdings for participants⁴⁰.

Turning to DC pension plans, most of the existing literature investigates their optimal dynamic asset allocation strategy by assuming a fixed contribution rate (e.g., 10% of salary per annum prior to retirement) and maximising the utility of the replacement ratio (i.e., pension as a proportion of final salary) at retirement (for example, Cairns et al. (2006)) or by minimising the expected present value of total disutility⁴¹ prior to retirement (for example, Haberman and Vigna (2002)). The EIS is implicitly assumed to be zero and there is no facility for adjusting the contribution rate in response to changes in salary level or in asset performance. However, in practice, most DC plans allow members to make additional voluntary contributions, and often set upper and lower limits on the contribution rate per annum.

Our aim in this study is to investigate the optimal asset allocation strategy for a DC plan member with Epstein-Zin utility, so that an individual member's investment strategy depends on the pattern of preferred consumption levels over the member's entire lifetime. We also derive the optimal profile of contribution rates over the accumulation stage of a DC plan. The rest of the chapter is structured as follows. Section 5.2 outlines the discrete-time model with Epstein-Zin utility including the parameter calibration process and optimisation method used. In Section 5.3, we generate simulations of the two key state variables (i.e., wealth and labour income) and derive the optimal funding and investment strategies for the DC pension plan; we also conduct a sensitivity analysis of the results. Section 5.4 concludes.

⁴⁰ Gomes and Michaelides (2005, page 871) argue that the less risk-averse investors have a weaker incentive to pay the fixed entry cost of equity investment, and therefore stock market participants in aggregate tend to be more risk averse.

⁴¹ The disutility is normally defined using the deviation of actual fund level from interim and final target fund levels.

5.2 The model

5.2.1 The model structure and optimisation problem

We propose a two-asset discrete-time model with a constant investment opportunity set. To ensure conformity with a DC pension plan, a number of constraints need to be specified:

- (i) pension wealth can never be negative,
- (ii) in any year prior to retirement, consumption must be lower than labour income,
- (iii) short selling of assets is not allowed,
- (iv) members are not allowed to borrow from future contributions⁴².

Members are assumed to join the pension plan at age 20 (denoted time $t=0$ below) without bringing in any transfer value from a previous plan and the retirement age is fixed at 65. Neither income tax nor inheritance tax is considered. We work in time units of one year and members are assumed to live to a maximum age of $\omega = 120$. In this model, members are allowed to make decisions on contribution rates, which is the main difference compared with the previous model setting in chapter 4.

Preferences

We assume the plan member possesses the discrete-time recursive utility function proposed by Epstein and Zin (1989):

⁴² Some studies have assumed that the member can borrow from future contributions (i.e., to incorporate a loan in the portfolio, which amounts to the present value of future contributions). In this way, Boulier et al. (2001) and Cairns et al. (2006) investigate the optimal asset allocation of DC pension plan with guaranteed benefit protection. However, there are arguments against this assumption. In most cases, this would not be allowed in practice. Also, the loan amount depends on assumptions about the level of future contributions and, in practice, there can be a lot of uncertainty about future contributions.

$$V_{20+t} = \left\{ (1 - \beta p_{20+t}) C_{20+t}^{\frac{1-\frac{1}{\varphi}}{\varphi}} + \beta \left[E_{20+t} \left(p_{20+t} \times V_{20+t+1}^{1-\gamma} + (1 - p_{20+t}) \times b \times \frac{\left(\frac{W_{20+t+1}}{b} \right)^{1-\gamma}}{1-\gamma} \right) \right]^{\frac{1-\frac{1}{\varphi}}{1-\gamma}} \right\}^{\frac{1}{1-\frac{1}{\varphi}}} \quad [23]$$

where

- V_{20+t} is the utility level at time t (or age $20+t$),
- W_{20+t} is the wealth level at time t ,
- C_{20+t} is the consumption level at time t ,
- γ is the coefficient of relative risk aversion (RRA),
- φ is the elasticity of intertemporal substitution (EIS),
- β is the discount factor for each year, and
- p_{20+t} is the one-year survival probability at time t (i.e., the probability that a member who is alive age $20+t$ survives to age $20+t+1$).

The parameter b is the “bequest intensity” and determines the strength of the bequest motive. If a member dies during the year of age $20+t$ to $20+t+1$, the deceased member will give the remaining wealth at the end of the year, W_{20+t+1} , a utility measure of $b \times (W_{20+t+1}/b)^{1-\gamma} / (1-\gamma)$. Thus, a higher value of b implies that the member has a stronger desire to bequest wealth on death.

In the final year of age $(\omega-1, \omega)$, where we have $p_{\omega-1} = 0$, equation [23] reduces to:

$$V_{\omega-1} = \left\{ C_{\omega-1}^{1-\frac{1}{\phi}} + \beta \left[E_{\omega-1} \left(b \times \frac{\left(\frac{W_{\omega}}{b} \right)^{1-\gamma}}{1-\gamma} \right)^{\frac{1-\frac{1}{\phi}}{1-\gamma}} \right]^{\frac{1}{1-\frac{1}{\phi}}} \right\}$$

[24]

which provides the terminal condition for the utility function.

Financial assets

We assume that there are two underlying assets in which the pension plan can invest⁴³:

- (i) a risk-free asset (i.e., a cash fund), and
- (ii) a risky asset (i.e., an equity fund).

The risk-free asset yields a constant rate of interest r , and the return on the risky asset in year t is given by:

$$R_{20+t} = r + \mu + \varepsilon_{20+t}$$

[25]

where

- μ is the (constant) risk premium on the risky asset, and
- $\varepsilon_{20+t} = \sigma \times Z_{1,20+t}$, where σ is the (constant) volatility of the risky asset and $Z_{1,20+t}$ is an independent and identically distributed (iid) standard Normal random variable

Whilst not necessarily corresponding with the real world, the simplified assumption of iid returns on the risky asset considerably facilitates the numerical method used. In addition, other non-pension assets such as housing is not considered in the model although they play an important role in the members' financial planning over lifetime as well.

⁴³ Asset returns are both in real term because inflation is not considered in our model.

Labour and pension income

Before retirement, the member receives an annual salary at the start of each year and contributes a proportion π_t of this into the pension plan at time t . We adopt the stochastic labour income process used in Cairns et al. (2006) which is illustrated in **Figure 32**. The growth rate in labour income prior to retirement is given by:

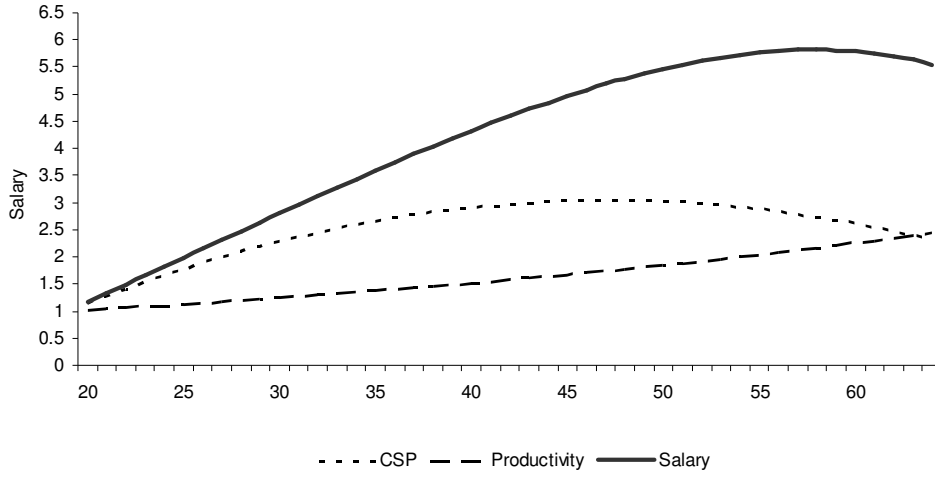
$$I_{20+t} = r_l + \frac{S_{20+t+1} - S_{20+t}}{S_{20+t}} + \sigma_1 \times Z_{1,20+t} + \sigma_2 \times Z_{2,20+t} \quad [26]$$

where

- r_l is the long-term average annual real rate of salary growth (reflecting productivity growth in the economy as a whole),
- S_{20+t} is the “career salary profile” (CSP), or salary scale, at time t , so that the term $(S_{20+t+1} - S_{20+t})/S_{20+t}$ reflects the promotional salary increase between time t and time $t+1$,
- σ_1 represents the volatility of a shock that is correlated with equity returns,
- σ_2 represents the volatility of the annual rate of salary growth, and
- $Z_{2,20+t}$ is an iid standard Normal random variable.

Equations [25] and [26] are subject to a common stochastic shock, $Z_{1,20+t}$, implying that the correlation between the growth rate in labour income and equity returns is given by $\sigma_1 / \sqrt{(\sigma_1^2 + \sigma_2^2)}$.

Figure 32: Labour income process



Following the work of Blake et al. (2007), we use a quadratic function to model the CSP:

$$S_{20+t} = 1 + h_1 \times \left(-1 + \frac{t}{45} \right) + h_2 \times \left[-1 + \frac{4t}{45} + 3 \times \left(\frac{t}{45} \right)^2 \right] \quad [27]$$

On retirement at age 65, the member is assumed to annuitise a proportion k of the accumulated pension fund by buying a life annuity, where k , the annuitisation ratio, is chosen to maximise the member's utility level at retirement. The amount of annuity income received depends on the accumulated wealth level at retirement, the annuitisation ratio and the price of a life annuity. In this model, the price of a life annuity is calculated using the risk-free return on the cash fund, so it is fixed over time and no annuity risk is considered. After retirement, the member invests the residual wealth that is not annuitised. Retirement income therefore comes from two sources: the annuity and possible withdrawals from the residual fund until death.

Wealth accumulation

Before retirement, the growth in the member's pension wealth will depend on the investment strategy adopted, the investment returns on both the risk-free asset and the risky asset, and the chosen contribution rate.

The contribution rate at time t is given by $\pi_{20+t} = (Y_{20+t} - C_{20+t})/Y_{20+t}$ (for $0 \leq t \leq 44$), where Y_{20+t} is the labour income level at time t . We require the contribution rate to be non-negative, so that $Y_{20+t} \geq C_{20+t}$ before retirement. The contribution rate is allowed to vary over time, so that consumption in any period can adjust to changes in income level and investment performance.

We also need to impose the restriction $W_{20+t} \geq 0$ (for $0 \leq t \leq 100$), to ensure that the wealth level is always non-negative at each age over the life cycle.

A proportion, α_{20+t} , of the member's pension account is assumed to be invested in the risky asset at time t . Then, for $0 \leq t \leq 43$ (i.e., up to and including the year prior to retirement), we have the following recursive relationship for the wealth process:

$$W_{20+t+1} = (W_{20+t} + \pi_{20+t} Y_{20+t}) \times [1 + r + \alpha_{20+t} (\mu + \varepsilon_{20+t})] \quad [28]$$

As mentioned above, we assume that short selling of assets is not allowed and therefore impose the restriction that $0 \leq \alpha_{20+t} \leq 1$.

At the start of the year in which the member is aged between 64 and 65, the member receives the final salary payment and makes the final contribution to the pension fund. So, we have:

$$W_{65}^- = (W_{64} + \pi_{64} Y_{64}) \times [1 + r + \alpha_{64} (\mu + \varepsilon_{64})] \quad [29]$$

At the end of this year, the member retires and chooses the annuitisation ratio k , giving a residual wealth on retirement at exact age 65 of $W_{65} = (1 - k) \times W_{65}^-$. The annuitisation

ratio k is chosen to maximise the utility level at retirement. This control variable does not appear in the utility function, but rather in the wealth constraint in the retirement year.

After retirement, the member invests a proportion α_{20+t} (for $t \geq 45$) of the residual wealth (i.e., that which was not annuitised) in the risky asset and receives annuity payments rather than labour income at the start of each year, provided that the member is still alive, so that the recursive relationship for the wealth accumulation process at this stage of the member's life cycle is given by:

$$W_{20+t+1} = \left(W_{20+t} + \frac{k \times W_{65}^-}{\ddot{a}_{65}} - C_{20+t} \right) \times [1 + r + \alpha_{20+t} (\mu + \varepsilon_{20+t})] \quad [30]$$

where $\ddot{a}_{65}$ is the price of a life annuity at age 65 (and, hence, $k \times W_{65}^- / \ddot{a}_{65}$ represents the annual annuity income after retirement).

Finally, we must constrain consumption after retirement such that

$$C_{20+t} \leq (W_{20+t} + k \times W_{65}^- / \ddot{a}_{65}).$$

The optimisation problem and solution method

The model has three control variables: the asset allocation at time t , α_{20+t} , the consumption level at time t , C_{20+t} , and the annuitisation ratio at retirement, k .

The optimisation problem is then:

$$\max_{\alpha_{20+t}, C_{20+t}, k} E(V_{20+t}) \quad [31]$$

subject to the following constraints:

- (i) for $0 \leq t \leq 43$, we have:

a) a wealth accumulation process satisfying:

$$W_{20+t+1} = (W_{20+t} + \pi_{20+t} Y_{20+t}) \times [1 + r + \alpha_{20+t} (\mu + \varepsilon_{20+t})] \geq 0,$$

b) an allocation to the risky asset satisfying $0 \leq \alpha_{20+t} \leq 1$, and

c) a contribution rate satisfying $\pi_{20+t} \geq 0$;

(ii) for $t = 44$, we have:

a) a wealth accumulation process satisfying:

$$W_{65} = (1 - k) \times ((W_{64} + \pi_{64} Y_{64}) \times [1 + r + \alpha_{64} (\mu + \varepsilon_{64})]) \geq 0,$$

b) an allocation to the risky asset satisfying $0 \leq \alpha_{64} \leq 1$,

c) a contribution rate satisfying $\pi_{64} \geq 0$, and

d) an annuitisation ratio at age 65 satisfying $0 \leq k \leq 1$;

(iii) and, for $t \geq 45$, we have:

a) a wealth accumulation process satisfying:

$$W_{20+t+1} = \left(W_{20+t} + \frac{k \times W_{65}^-}{\ddot{a}_{65}} - C_{20+t} \right) \times [1 + r + \alpha_{20+t} (\mu + \varepsilon_{20+t})] \geq 0,$$

b) an allocation to the risky asset satisfying $0 \leq \alpha_{20+t} \leq 1$, and

c) consumption satisfying $C_{20+t} \leq \left(W_{20+t} + \frac{k \times W_{65}^-}{\ddot{a}_{65}} \right)$.

The Bellman equation at time t is:

$$J_{20+t}(W_{20+t}, Y_{20+t}) = \max_{\alpha_{20+t}, \pi_{20+t}, k} \left\{ (1 - \beta p_{20+t}) C_{20+t}^{\frac{1-\frac{1}{\varphi}}{\varphi}} + \beta \left[E_{20+t} \left(p_{20+t} \times J_{20+t+1}^{1-\gamma} + (1 - p_{20+t}) \times b \times \frac{\left(\frac{W_{20+t+1}}{b} \right)^{1-\gamma}}{1-\gamma} \right)^{\frac{1-\frac{1}{\varphi}}{1-\gamma}} \right]^{\frac{1-\frac{1}{\varphi}}{1-\frac{1}{\varphi}}} \right\} \quad [32]$$

An analytical solution to this problem does not exist, because there is no explicit solution for the expectation term in the above expression. Instead, we must use a numerical solution method to maximise the value function and derive the optimal control parameters. We use the terminal utility function at age 120 to compute the corresponding value function for the previous period and iterate this procedure backwards, following a standard dynamic programming strategy.

To avoid choosing a local maximum, we discretise the control variables (i.e., asset allocation, consumption and annuitisation ratio) into equally spaced grids and optimise them using a standard grid search. As an important step in implementing the stochastic dynamic programming strategy, we need to discretise both the state space and shocks in the stochastic processes (i.e., equity return and labour income growth) first. Wealth and labour income level are discretised into 30 and 10 evenly-spaced grid points, respectively, in the computation⁴⁴. Also, the shocks in both the equity return and labour income growth processes are discretised into 9 nodes⁴⁵.

The expected utility level at time t is then computed using these nodes and the relevant weights attached to each (i.e., Gauss quadrature weights and interpolation nodes)⁴⁶. The advantage of using this set of nodes is that the state variables can be computed more quickly and precisely; however, because we have a fine grid on the control variables and a much coarser grid on the shocks, we may have some state variable values outside of the grid points in the next time period. In this case, cubic spline interpolation is employed to approximate the value function. While this approach does not significantly reduce the accuracy of the results obtained, use of a much finer grid for the shocks in the equity return and labour income growth processes would significantly increase computing time, as mentioned previously.

⁴⁴ Clearly, the choice concerning the number of nodes is subjective, but we felt that this choice represents an appropriate trade-off between accuracy and speed of computation.

⁴⁵ Again, nine nodes represents a balance between accuracy and computing time, and is a standard setting in the existing literature.

⁴⁶ For more details, see Judd (1998, page 257-266).

For each age $20+t$ prior to the terminal age of 120, we compute the maximum value function and the optimal values for the control variables at each grid point. Substituting these values in the Bellman equation, we obtain the value function of this period, which is then used to solve the maximisation problem for the previous time period. Details of the dynamic programming and integration process are given in Appendix 03. The optimal asset allocations at different ages are shown in Appendix 04. The computations were performed in MATLAB⁴⁷.

5.2.2 Parameter calibration

We begin with a standard set of baseline parameter values (all expressed in real terms) presented in Table 18.

⁴⁷ <http://www.mathworks.com/products/matlab/>. The code is available on request from the authors.

Table 18 Baseline parameter values			
Asset returns		Preference parameters	
risk-free rate, r	0.02	RRA, γ	5
equity premium, μ	0.04	EIS, ϕ	0.2
volatility, σ	0.2	bequest intensity, b	1
		discount factor, β	0.96
Mortality		Labour income process	
mortality table	PMA92 (see Appendix 2)	starting salary, Y_{20}	1
		average salary growth, r_l	0.02
		volatility of shock correlated with equity returns, σ_1	0.05
Annuity		volatility of annual rate of salary growth, σ_2	0.02
annuity price, $\ddot{a}_{65}$	15.87 (based on PMA92 and a risk-free rate of 0.02)		
		h_1	-0.276
		h_2	0.75835

The constant net real interest rate, r , is set at 2% p.a., while, for the equity return process, we consider a mean equity premium, μ , of 4% p.a. and a standard deviation, σ , of 20% p.a.. Using an equity risk premium of 4% p.a. (as opposed to the historical average of around 6%) is a common choice in recent literature (e.g., Fama and French (2002), Gomes and Michaelides (2005)). This more cautious assumption reflects the fact that the historical equity risk premium might be higher than can reasonably be expected in future, and thus will reduce the weight given to equities in the optimal portfolios obtained. We use the projected PMA92 table⁴⁸ (see Appendix 05) as the standard male mortality table, and hence, using a (real) interest rate of 2% p.a., the price of a whole life annuity from age 65 is 15.87.

⁴⁸ PMA92 is a mortality table for male pension annuitants in the UK based on experience between 1991 and 1994; here, we use the projected rates for the calendar year 2010, i.e., the table PMA92(C2010), published by the Continuous Mortality Investigation (CMI) Bureau in February 2004.

We start by presenting results for what might be considered as a relatively standard plan member, with RRA of $\gamma=5$, EIS of $\phi=0.2$ and discount factor $\beta=0.96$ ⁴⁹. As mentioned above, the bequest intensity, b , plays an important role in life-cycle saving and investment. We set b equal to unity in the baseline case (which represents a moderate level of bequest saving motive). We later conduct a sensitivity analysis on these parameter values.

The starting salary is normalised on unity. All absolute wealth and income levels are measured in units of the starting salary. In line with post-war UK experience, the annualised real growth rate of national average earnings is assumed to be 2% p.a. with a standard deviation of 2% p.a. (i.e., $r_1 = 0.02$ and $\sigma_2 = 0.02$). Following the work of Blake et al. (2007), we estimate the CSP parameters h_1 and h_2 using average male salary data (across all occupations) reported in the 2005 *Annual Survey of Hours and Earnings*. The estimated values are $h_1 = -0.276$ and $h_2 = 0.75835$ (see [27]).

5.3 Results

5.3.1 Baseline case

Optimal asset allocation assuming no bequest motive or labour income risk

As suggested by equation [22] above, the optimal portfolio composition should be constant when there is no bequest motive and labour income risk is ignored. **Figure 33** shows the optimal equity weight for the final time period (i.e., age 119 to 120) with no bequest motive for different fund and (pension) income level. As expected, we can see that when the accumulated wealth level is large (and labour income is small in comparison), the optimal asset allocation is close to the result suggested in equation [22], so that we have:

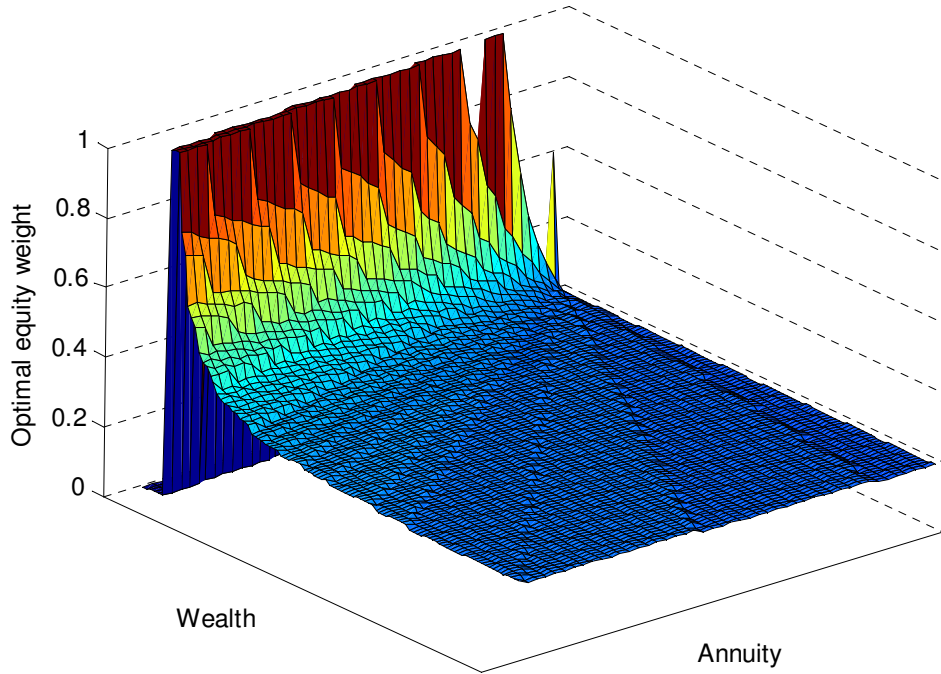
⁴⁹ This parameter constellation is common in the literature (e.g., Gomes and Michaelides (2004)). The values of RRA and EIS are also consistent with power utility for the baseline case.

$$\alpha = \frac{\mu}{\gamma\sigma^2} + \left(1 - \frac{1}{\gamma}\right) \times \frac{\text{cov}(R_{120}, -V_{120})}{\sigma^2} \approx 0.2$$

[33]

This result shows that we can approximate an analytical solution numerically reasonably accurately, thereby justifying the use of our grid search numerical method.

Figure 33: Optimal equity asset allocation for the final time period



Simulation output

The output from the optimisation exercise is a set of optimal control variables (i.e., asset allocation, α_{20+t} , and consumption level, C_{20+t}) for each time period and the optimal annuitisation ratio, k , at retirement age 65. We generate a series of random variables for

both the equity return and labour income shocks, and then generate 10,000 independent simulations of wealth and labour income levels.

Figure 34 shows the simulation means of labour income and optimal wealth and consumption levels for ages 20, 21, ..., 120, and **Figure 35** shows the consumption profile on a larger scale. We have bequest and retirement saving motives in this model. In the early years of the life cycle (i.e., up to age 45 or so), wealth accumulation is driven by the bequest intensity (i.e., the extent of the desire to protect dependants if the member dies) and by the attitude to risk (i.e., the degree of aversion to cutting consumption in unfavourable states)⁵⁰. Consumption increases smoothly during this period. Then, as the member gets older, the retirement motive becomes more important as the member recognises the need to build up the pension fund in order to support consumption after retirement. From age 45 to the retirement age of 65, the retirement savings motive dominates and the pension fund grows significantly. As a result, consumption remains almost constant during this period. After retirement, there is a large fall in consumption compared with the period immediately prior to retirement⁵¹ and thereafter consumption remains stable for the remainder of the member's lifetime.

⁵⁰ This will become clearer in section 5.3.2.

⁵¹ We are investigating this issue in our further research. In the wealth accumulation process of this model, we assume members annuitise the accumulated pension wealth at retirement and the "cash-in-hand" after retirement will be unannuitised wealth and pension income every year. If members choose to annuitise a large proportion of accumulated wealth at retirement, their "cash-in-hand" level will drop significantly after buying annuity. In the further research, we are trying to consider the present value of future pension income in the wealth accumulation process after retirement. However, to split the wealth into disposable and non-disposable wealth will involve a different numerical optimisation method which is out of the scope of this thesis.

Figure 34: Mean of simulated wealth, consumption and labour income



Figure 35: Mean of consumption

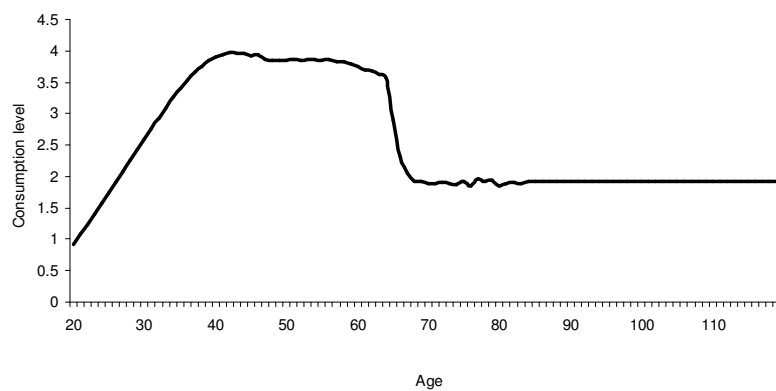


Figure 36 shows the expected NPV of total future labour income (i.e., human capital). We can see that human capital increases until about age 35. This is because of the very high rate of salary growth in the early years (relative to the discount rate applied to future labour income).

Figure 36: Human capital

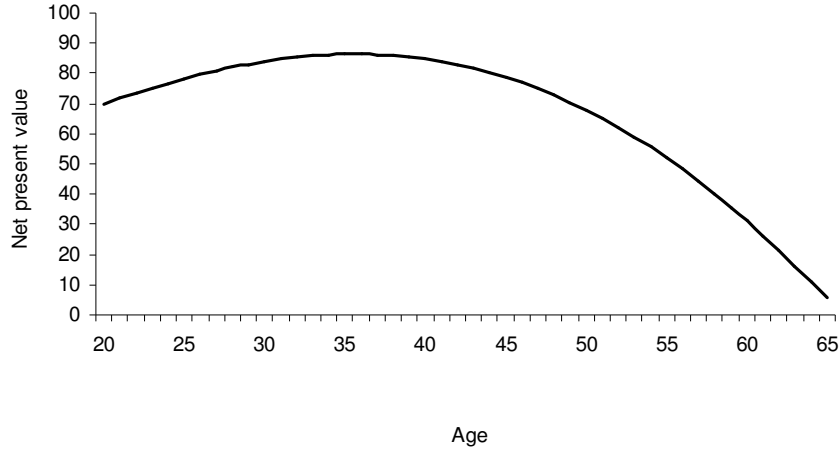


Figure 37 shows six possible optimal asset allocation profiles for equities at each age before retirement, α_{20+t} . Each profile coincides with a particular quantile from the distribution of outcomes from 10,000 simulations. These profiles are consistent with the investment strategy called “stochastic lifestyling”, first outlined in Cairns et al (2006)⁵², with a high equity weighting at younger ages and a gradual switch from equities to the risk-free financial asset as the retirement age approaches. Prior to around age 35, the member should invest all financial wealth in the risky asset, because the implicit holding in the non-tradable riskless asset (i.e., human capital) is increasing, as illustrated in **Figure 36**. However, after age 35, human capital starts to decline. The member should then begin to rebalance the financial wealth portfolio towards the risk-free financial asset to compensate for the decline in human capital. This is because the risk-free financial asset and human capital are substitutes, with the degree of substitutability inversely related to the correlation between labour income growth and equity returns, $\sigma_1 / \sqrt{(\sigma_1^2 + \sigma_2^2)}$ ⁵³. Specifically, when this correlation coefficient is high, the member’s salary growth will tend to move in line with the equity investment performance. As a result, over the life cycle, the change of the ratio of human to financial wealth becomes

⁵² See details in section 4.2.4.

⁵³ This will be discussed in detail in section 5.3.2.

more gradual. As discussed earlier, this ratio is a crucial determinant of the portfolio composition. Therefore, the optimal portfolio weight will switch from equities to the risk-free financial asset in a more gradual way.

The investment strategy is known as “stochastic lifestyling” because the optimal equity weighting over the life cycle depends on the realised outcomes for the stochastic processes driving the state variables, namely labour income and the risky financial asset. The profiles have a similar shape which can be characterised as three connecting (and approximately) linear segments. The first is a horizontal segment involving a 100% equity weighting (approximately) from age 20 to an age somewhere in the range of 40-47. The second is a steep downward segment that involves a reduction in equities to somewhere between 10-40% over a seven year period. The third is a more gentle downward sloping segment that reduces the equity weighting to somewhere between 0-10% by the retirement age. It is important to note that the profiles in **Figure 37** are not, however, consistent with the more traditional “deterministic lifestyling” strategy, which involves an initial high weighting in equities with a predetermined linear switch from equities to cash in the period leading up to retirement (typically the preceding 5 or 10 years).

Figure 37: Optimal equity asset allocation prior to retirement

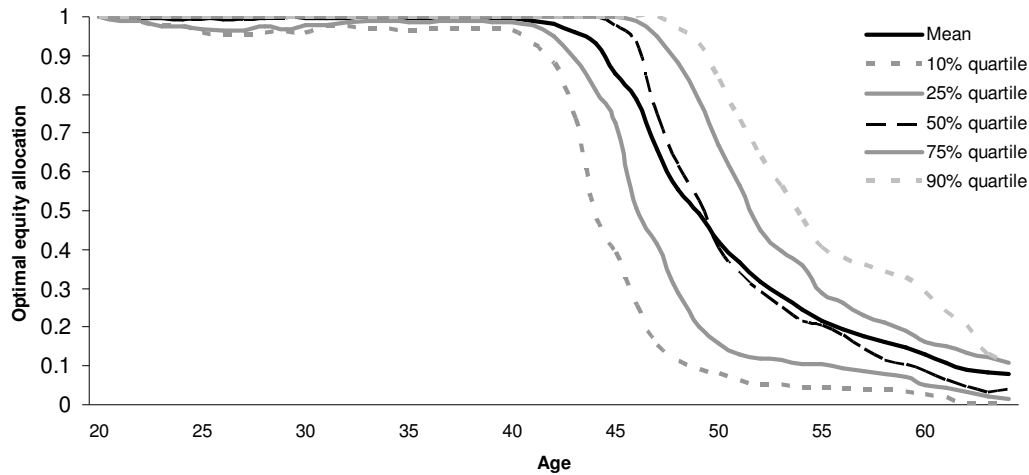
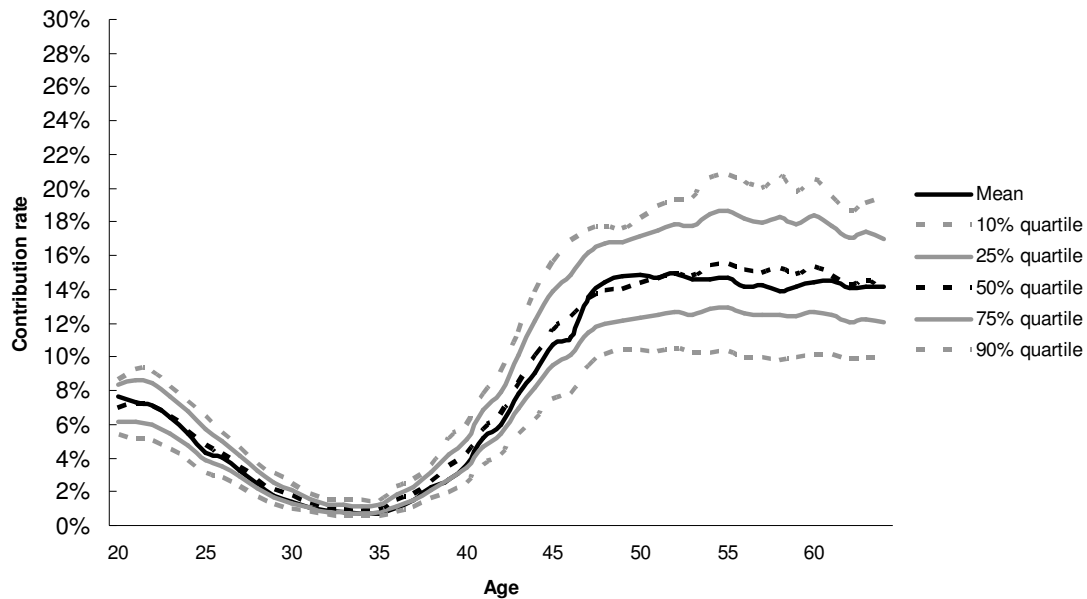


Figure 38 plots six quantiles from the distribution of optimal contribution rates, corresponding to the optimal asset allocation strategies shown in **Figure 37**. Considering the profile corresponding to the mean, the initial annual contribution rate at age 20 is just under 8% p.a.. It then decreases steadily to below 1% by age 35. This fall reflects a trade-off between the bequest motive and risk aversion, on the one hand, and the increase in human capital, on the other. Prior to age 35, when labour income is growing very rapidly and human capital is increasing, the member wishes to increase consumption and does so by reducing the contribution rate into the pension plan, despite being both risk averse and having a bequest motive⁵⁴. After age 35, however, labour income growth slows down and human capital begins to decline, and the retirement savings motive starts to become important. The contribution rate then increases steadily to almost 15% p.a. by age 48. Labour income flattens out after age 48 (see **Figure 32**) and the contribution rate then remains roughly constant until retirement.

⁵⁴ Note that pension contributions will fall by less than the contribution rate since income is growing.

Figure 38: Optimal contribution rate



The most interesting finding from **Figure 38** is that the optimal contribution rate in a DC pension plan is age-related, rather than constant. It exhibits a U-shaped pattern between age 20 and age 48 and it is only (approximately) constant after age 48. As a consequence, the contribution rate is very variable, ranging from, in the case of the mean profile, below 1% (at age 35) to almost 15% (at ages 48 and above). The other quantiles have a very similar shape: they are relatively close to the mean during the U-shaped phase, but have a wider range of post-age-48 constant rates (ranging between 10 and 20%).

An age-related pattern of contribution rates is not common in real world DC plans: for example, in the UK, there is typically a fixed standard (combined employer and employee) contribution rate varying between 8 and 10% (GAD (2006, Table 8.2)). Although age-related contribution rates are not common, minimum contributions are more so. **Figure 39** illustrates the mean optimal contribution rate over the life cycle when a lower limit of 5% p.a. is imposed on the contribution rate (the original unconstrained mean contribution rate profile is shown for comparison). In this case, the member accumulates greater pension wealth when young and, therefore, can afford a lower

contribution rate as retirement approaches. Further, because of the higher accumulated pension wealth, the member switches to the risk-free asset earlier, as shown in

Figure 40 (the original unconstrained mean optimal asset allocation is shown for comparison). Nevertheless, there is a cost from imposing this constraint: expected utility at age 20 drops by around 1% (from 2.349 to 2.329).

In our model, neither income tax nor inheritance tax is considered. However, it is worth noting that this is a simplified assumption because, in real life, the tax factors (especially the tax relief on pension contribution and inheritance tax) will affect members' consumption and contribution decisions. If the income tax and tax relief on pension contribution is considered, members (especially the higher-rate tax payers) will tend to contribute a higher percentage of their gross income before retirement, which will lead to a more conservative investment strategy. If members need to pay inheritance tax on their bequeathed wealth, they will have weaker bequest motive – the impact of bequest motive on the model outputs will be discussed later on in the sensitivity analysis section 5.3.2.

Figure 39: Mean optimal contribution rate (with lower limit of 5% per annum, dotted)

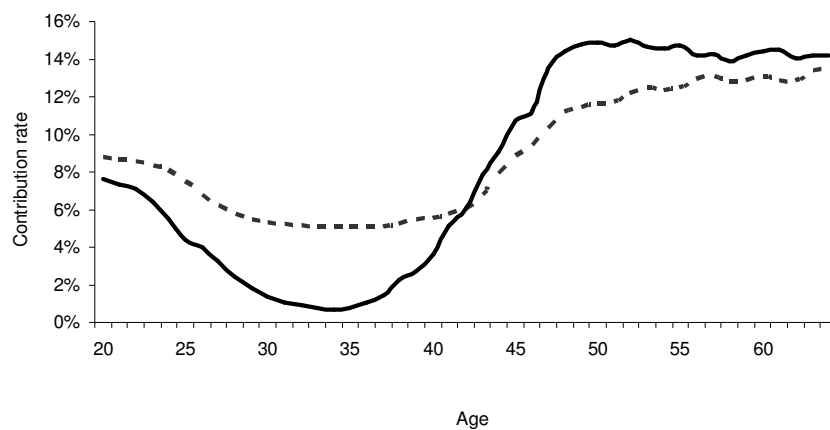
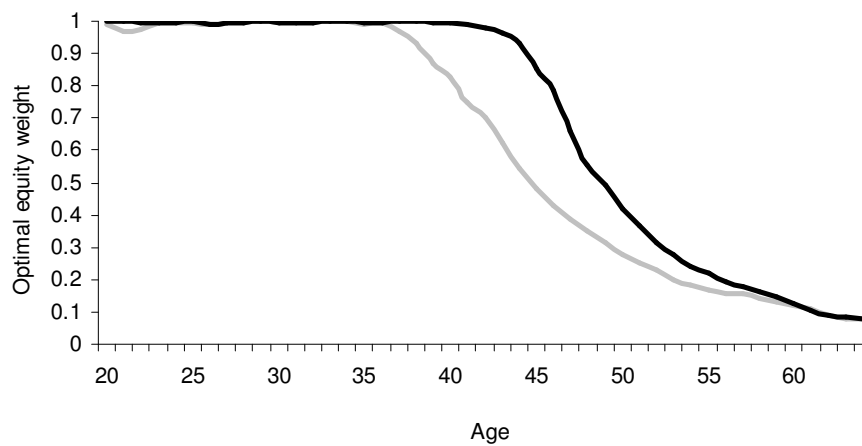


Figure 40: Mean optimal equity asset allocation (with lower limit on contribution rate of 5% per annum, dotted)



5.3.2 Sensitivity analysis

RRA and EIS

We now conduct a sensitivity analysis, by considering the simulation results for plan members with different RRA and EIS. As shown in **Table 19** (see also **Table 17**), we have four types:

- low RRA and low EIS (Type 1):
 - risk-tolerant member who dislikes consumption volatility over time
 - e.g., low income member without dependants
- low RRA and high EIS (Type 2):
 - risk-tolerant member who does not mind consumption volatility over time
 - e.g., high income member without dependants
- high RRA and low EIS (Type 3):
 - risk-averse member who dislikes consumption volatility over time
 - e.g., low income member with dependants
- high RRA and high EIS (Type 4):
 - risk-averse member who does not mind consumption volatility over time
 - e.g., high income member with dependants

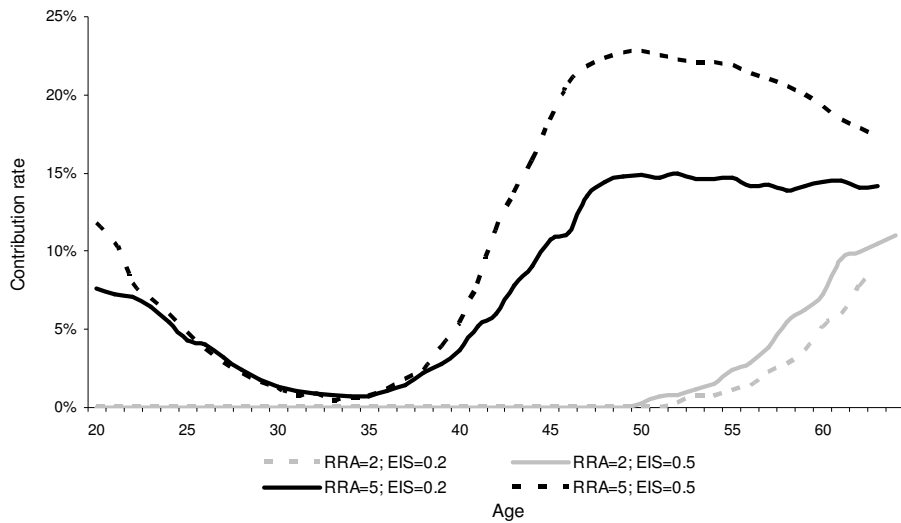
The baseline case in Section 5.3.1 dealt with Type 3 (highlighted in **Table 19**), a member with a high RRA and a low EIS (i.e., $\gamma = 5$ and $\varphi = 0.2$).

Table 19 RRA and EIS values for the different types of plan member

	RRA, γ	EIS, φ
Type 1	2	0.2
Type 2	2	0.5
Type 3	5	0.2
Type 4	5	0.5

Figure 41 shows the different patterns of optimal contribution rates corresponding to these four types. For risk-tolerant members with low RRA (i.e., Types 1 and 2), contributions prior to age 50 are negligible. From age 50 or so, the retirement savings motive kicks in and contributions into the pension plan begin. The contribution rate is lower for Type 1 (low EIS) than for Type 2 (high EIS) members (by approximately 3-4% p.a.). This lower retirement savings intensity is the result of a stronger aversion to cutting consumption: due to the lower EIS level, cuts in consumption needed to fund the pension plan are more heavily penalised in the utility function of Type 1 members than of Type 2 members.

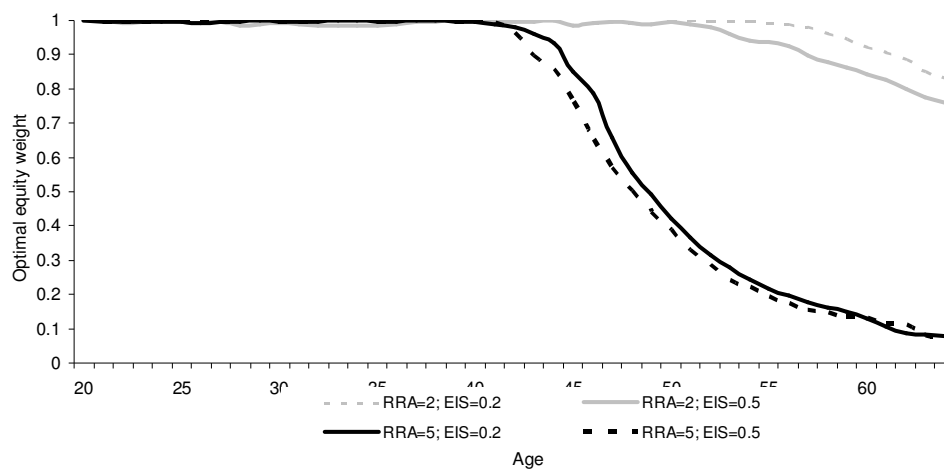
Figure 41: Mean optimal contribution rate (for different RRA/EIS combinations)



As a result of the lower mean contribution rates (particularly at younger ages), risk-tolerant members, *ceteris paribus*, accumulate a lower level of pension wealth. They therefore need (and are, of course, willing to accept) a much higher average equity weighting (and the corresponding equity premium) in the financial portfolio in an attempt to produce the desired level of retirement savings. As shown in **Figure 42**, the mean

equity allocation decreases only gradually and still exceeds 70% at retirement. The risk-tolerant member with a low EIS level (i.e., Type 1) chooses a higher equity weighting at retirement (by approximately 10%) than the risk-tolerant member with a high EIS level (i.e., Type 2). This is because, as discussed above, the annual contribution rate is lower and the reliance on the equity premium is correspondingly greater.

Figure 42: Mean optimal asset allocation (for different RRA/EIS combinations)



For risk-averse members with high RRA (i.e., Types 3 and 4), **Figure 41** shows that the bequest motive leads to much larger initial contributions (in excess of 7% p.a. at age 20 on average) than for members with low RRA, and the retirement savings motive leads to a significant rise in contributions after age 40, again compared with low RRA members. For those risk-averse members with low EIS (i.e., Type 3), the variability of the contribution rate over the working life is much less pronounced than is the case for those risk-averse members with high EIS (i.e., Type 4). The contribution rate is also lower particularly at very young and very old ages (e.g., by around 5% p.a. at both age 20 and 60 on average). The explanation is the same as given above (i.e., the reluctance of members with a lower EIS level to tolerate cuts in current consumption to fund future consumption in retirement). The fall in the contribution rate to an average of around 1%

between age 20 and age 35 for risk averse members is explained by the increase in human capital over this period which increases the desire for current consumption at the expense of saving for retirement.

As a result of higher contribution rates, risk-averse members accumulate a much higher level of pension wealth and, as can be seen in **Figure 42**, switch away from equities much earlier (from about age 40) and hold less than 10% of the pension fund in equities at retirement on average. Again, we find that a lower EIS leads to a higher equity weighting during the accumulation phase of the pension plan.

Bequest motive

Investors with a desire to bequeath the wealth to their dependants on death would be expected to save more than those who do not. **Figure 43** shows the mean optimal pattern of contribution rates for the baseline case of $\gamma=5$ and $\varphi=0.2$ and different bequest intensities. When the bequest motive is absent ($b = 0$), members consume almost all of their earnings in the early years of their working lives, and have very negligible contributions into their pension plans. By contrast, members with a high bequest intensity ($b = 2.5$) make very high contributions in the early years (14% at age 20). They also make higher contributions throughout their working lives⁵⁵, but the differences after age 35 are much less.

Figure 44 shows that the optimal equity weight in the portfolio is lower, the higher the bequest intensity. However, the effect is small because: first, the mortality rate does not vary much before retirement, and second, and more importantly, the annuitisation ratio (k) is a control variable in our model and, thus, a member with a high bequest intensity could choose to annuitise a smaller ratio of the pension fund at retirement, instead of assuming significant equity risk in an attempt to accumulate more wealth prior to retirement⁵⁶.

⁵⁵ Although it is still pulled down to below 1% between ages 20 and 35 by the effect of the increase in human capital.

⁵⁶ This is confirmed in Table 3.2 below.

Figure 43: Mean optimal contribution rate (for different levels of bequest motive)

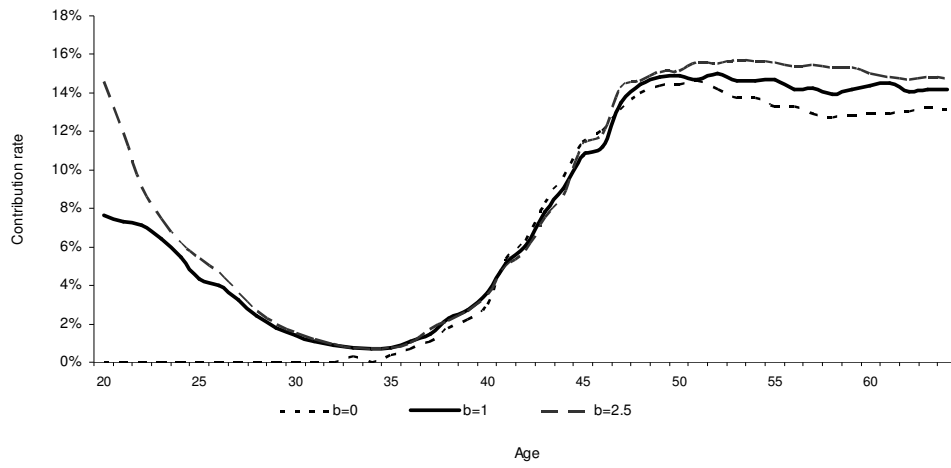
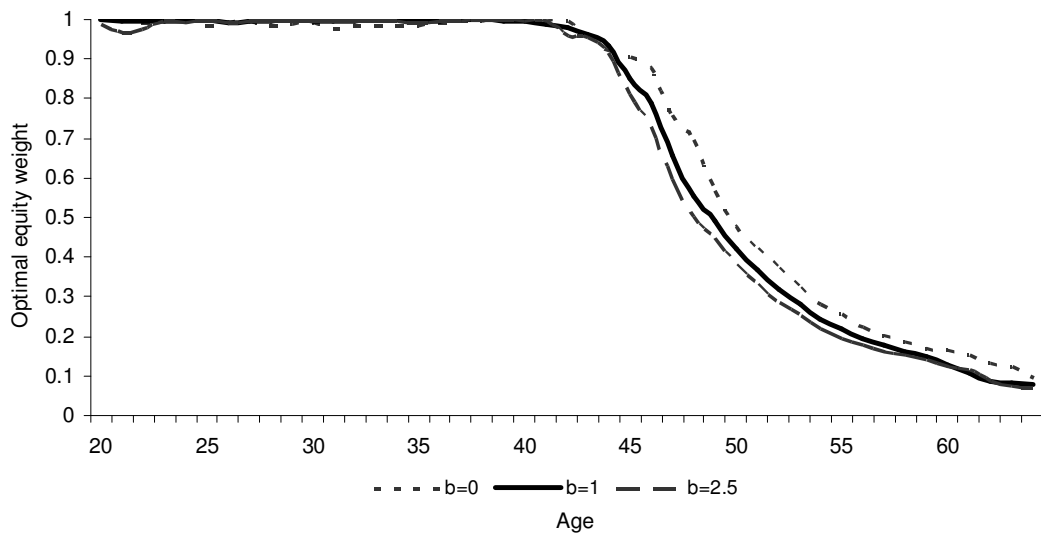


Figure 44: Mean optimal equity asset allocation (for different levels of bequest motive)



Personal discount factor

Figure 45 and **Figure 46** show the outcome from conducting a sensitivity analysis on β , the individual's personal discount factor, on the mean optimal contribution rate and asset allocation.

Individuals with a low personal discount factor (or high personal discount rate) value current consumption more highly than future consumption in comparison with individuals with a high personal discount factor. This will lead, *ceteris paribus*, to a lower contribution rate into the pension plan as shown in **Figure 45**. There will be a correspondingly slower accumulation of financial wealth and therefore a higher ratio of human to financial wealth throughout the working life. This, in turn, leads to an optimal lifestyle strategy with a higher portfolio allocation to the risky asset throughout the working life, together with a shorter switching period, as shown in **Figure 46**.

Figure 45: Mean optimal contribution rate (for different levels of personal discount factor)

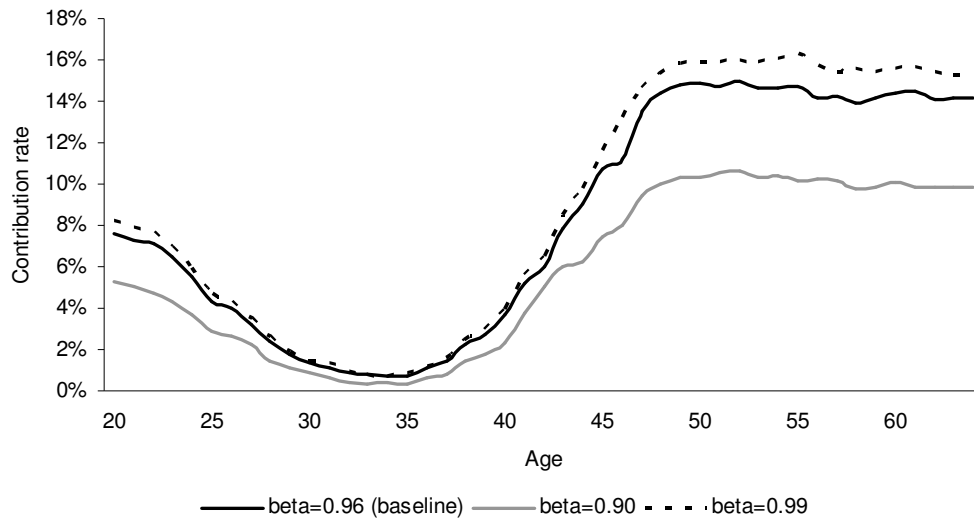
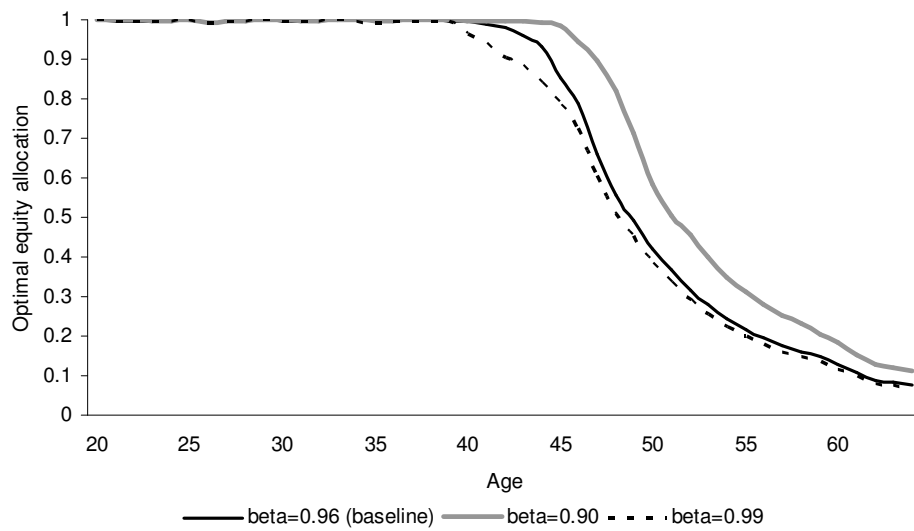


Figure 46: Mean optimal equity asset allocation (for different levels of personal discount factor)



Correlation between labour income growth and equity returns

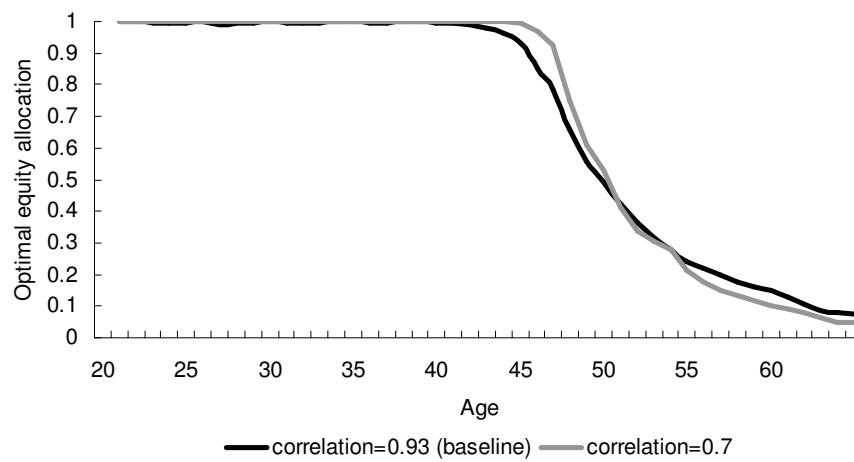
In our model, the degree of correlation between labour income growth and equity returns is controlled by $\sigma_1 / \sqrt{(\sigma_1^2 + \sigma_2^2)}$. As explained in Section 5.3.1 when this correlation coefficient is high, the member's salary growth will tend to move in line with the equity investment performance. As a result, the optimal portfolio weight will switch from equities to the risk-free financial asset in a more gradual way because the change of the ratio of human to financial wealth becomes more gradual over the life cycle.

We assume a high correlation of 0.93 in our baseline case. **Figure 47** and **Figure 48** show outcomes from conducting a sensitivity analysis on this correlation coefficient. As expected, with a lower correlation between labour income growth and equity returns (0.7), the downward switching segments become steeper.

Figure 47: Mean optimal contribution rate (for different correlation between labour income and equity returns)



Figure 48: Mean optimal equity asset allocation (for different correlation between labour income and equity returns)



Annuitisation ratio

Although the annuitisation ratio (k) is part of the budget constraint not the utility function, it is still a control variable in our model, and is chosen to maximise $E(V_{64})$, conditional on the values for RAA, EIS and the personal discount factor (see Section 5.2.1 above). **Table 20** shows the mean optimal values of k corresponding to different values for RAA, EIS and the personal discount factors.

The Table shows that there are positive relationships between RAA and k and between EIS and k , although the relationship in each case is not very strong. There is, unsurprisingly, a much stronger relationship between the bequest intensity (b) and k . When there is no bequest motive, the member annuitises the largest ratio (99.35%) of his pension wealth⁵⁷. According to Yaari (1965), when there is no bequest motive, the member should annuitise the entire accumulated pension fund at retirement. Davidoff et al (2005) extend Yaari's framework to include imperfect credit markets and habit formation (low IES) and show it is optimal to annuitise less than 100% of wealth, but the optimal proportion is still very high. We find that this is the case even when we impose a significant reduction in the discount factor (β). A discount factor of 0.9, implying a huge personal discount rate of 10% (and massive preference for current over future consumption) only reduces the optimal annuitisation ratio to 88%.

⁵⁷ The fact that 99.35% is slightly less than 100% is due to the numerical methods used to solve the problem and the estimation error caused by interpolation approximations.

Table 20 The effect of the preference parameters on the mean optimal annuitisation ratio

Preference parameter	Optimal annuitisation ratio (mean %)
RRA, γ	
2	93.17
5	94.72
EIS, ϕ	
0.2	94.72
0.5	95.88
Bequest intensity, b	
0	99.35
1	94.72
2.5	90.03
Discount factor, β	
0.90	87.66
0.96	94.72
0.99	96.43
Note: All parameters are at their baseline values (see Table 18) unless otherwise stated	

5.4 Conclusion

In this chapter, we have examined optimal funding and investment strategies in a DC pension plan using a life-cycle model that has been extended in three significant ways:

- (i) the assumption of Epstein-Zin recursive preferences by the plan member which enables a separation between the RRA and EIS,
- (ii) the introduction of human capital as an asset class along with financial assets, such as equities and cash, and
- (iii) endogenising the decision about how much to annuitise at retirement.

We also considered two important motives for saving: bequest and retirement. In addition, a plan member's personal discount rate influenced the optimal strategies.

We found that the optimal funding strategy typically exhibits a U-shaped pattern for the contribution rate in early working life and then stabilises in the period leading up to retirement. The initial high level is explained by a high bequest intensity combined with a high degree of risk aversion. The falling part of the U is explained by the increase in human capital inducing an increase in consumption at the expense of savings, while the rising part of the U is explained by the retirement savings motive. A higher personal discount rate lowers contributions at each age, while preserving the general shape of the optimal contribution profile. A lower bequest intensity reduces contributions to the pension plan in early career and in the extreme case of a zero bequest intensity completely eliminates them until mid-career when the retirement savings motive kicks in. The effect of lower risk aversion is to delay the start of contributions into the pension plan: contributions do not begin until late in the working life, but then increase steadily until the retirement age. The effect of lower EIS is to reduce contributions into the plan at all ages, since members with low EIS are reluctant to accept cuts in consumption to “pay for” the pension contributions.

We found that the optimal investment strategy is stochastic lifestyling rather than the more conventional deterministic lifestyling. While the optimal weighting in equities is initially very high and subsequently declines as the retirement date approaches, it does not do so in a predetermined manner as in the case of deterministic lifestyling. Instead the optimal equity weighting over the life cycle depends on the realisations of the stochastic processes determining labour income and equity returns. Stochastic lifestyling is justified by recognising the importance of human capital and interpreting it as a bond-like asset which deteriorates over the working life. An initial high weighting in equities is intended to counterbalance human capital in the combined “portfolio” of human capital and pension wealth. In time, the weighting in equities falls, while that in bonds rises as human capital deteriorates over the working life. When the correlation between labour income and equity returns is high, the human capital and pension wealth tend to move in line with each other; the optimal portfolio weight will therefore switch from equities to the risk-free financial asset in a more gradual way. The size of the pension fund is a crucial determinant of the optimal asset allocation. The greater the pension wealth accumulated, the more conservative the optimal asset allocation strategy will be, for a given RAA, EIS and discount rate. Also, the higher the contribution rate, the earlier the switch out of equities can be made. In our model, risk-tolerant members who value both current consumption over future consumption as smooth consumption profiles over time (i.e., have a low RRA, a high discount rate and a low EIS) accumulate the lowest pension wealth levels and therefore need to adopt the most aggressive investment strategy.

Finally, we found that the optimal annuitisation ratio was typically very high, suggesting that longevity protection is a hugely valuable feature of a well-defined DC pension plan to a rational consumer. It was negatively related to the bequest intensity and the discount rate, but not very sensitive to RRA or EIS.

The results in this chapter have some important implications for the optimal design of DC pension plans:

- An investment strategy involving a switch from equities to bonds as members approach retirement is appropriate for DC pension plans, even when equity returns are not mean reverting. However, the switch out from equities is not predetermined, but depends on what happens to equity returns. Nevertheless, the switch should typically be made earlier than in traditional lifestyle strategies (i.e., from age 45 or so rather than age 55, which is more common in practice). Also, the optimal equity weight in the portfolio typically never reduces to zero (even immediately prior to retirement), as is common in traditional lifestyle strategies.
- It is very important to incorporate the salary process in the optimal design of a DC pension plan⁵⁸. The nature of this implicit bond-like asset (i.e., human capital) has a direct impact on the optimal contribution rate and asset allocation decisions. However, for senior plan members whose salary levels (including bonus and dividends from their own stock holdings) may have a strong link with corporate profitability, the labour income growth rate may be much higher than a risk-free rate of return. In this case, this “human capital” asset may instead be thought of as more equity-like in nature (and, hence, the optimal investment strategies are likely be more weighted towards the risk-free asset).
- The results provide some justification for age-related contribution rates in DC plans. Because members tend to prefer relatively smoothed consumption growth, a plan design involving a lower contribution rate (e.g., 5% p.a. or less) when members are young, and a gradually increasing contribution rate as members get older (reaching on average 15% p.a. in the period prior to retirement) offers higher expected utility than fixed age-independent contribution rates. Greater contribution rate flexibility will be especially welcomed by members with high EIS. Such members are more sensitive to changes in financial incentives and thus are more desirous of flexible contribution rates. If they were allowed to make additional voluntary contributions (AVCs), the optimal asset allocation strategy would actually become more conservative as they approach retirement.

⁵⁸ This was first pointed out by Blake et al (2007).

- An annuity is an important component of a well-designed plan. The optimal annuitisation ratio is very high, even when there is a strong bequest motive and the plan member values current consumption highly. This is true despite the well-known aversion to annuitisation documented in Friedman and Warshawsky (1990) and Mitchell et al (1999).

Chapter 6

Conclusions

6.1 Summary

The aim of the thesis is to investigate the dynamic asset allocation of defined contribution pension funds. The main results obtained and conclusions drawn from the thesis are summarised below.

In a DC pension plan, many members do not have particularly firm convictions about their desired saving and investment behaviour. Chapter 2 tells us that the plan design and investment option offering will have substantial implications to the DC pension plan members. Important relevant behavioural features of plan members are reviewed and discussed. It is important that the education, communication and investment option offerings of DC pension plans are carried out with the knowledge of these members' potential biases in decision making.

On the basis of realising the importance of plan design of a DC pension plan, we put more attention on investigating the optimal dynamic investment strategies for DC plan members. Deterministic lifestyle strategy (e.g. with a 5-year switching period) is widely used as the default option by many DC plans. However, there has been no strong empirical evidence to date demonstrating that this is an optimal strategy. More innovative investment solutions are needed in this area. In this thesis, motivated by the recent research findings in behavioural utility and non expected utility theories, we investigate two possible alternative solutions: loss aversion utility and Epstein-Zin utility preference.

Chapter 3 provides the empirical support of the loss aversion model. A face-to-face based survey was conducted on 966 random selected UK households. The results are broadly consistent with the well known prospects theory setting and suggest that UK investors are more loss averse than the Tversky and Kahneman's original group of US students.

Chapter 4 is devoted to a loss aversion based model, where members are assumed to be loss averse with a target fund level at retirement and a series of suitably defined interim targets prior to retirement. The baseline simulation draws on empirical evidence we got from Chapter 3. We argue that, for DC plan members who seek greater certainty in their retirement planning, the asset strategies need to be far more focused on achieving their retirement targets. While this type of target-driven modelling approach has been used by other researchers, e.g. Vigna and Haberman (2001,2002), our study differs from existing studies in two respects:

- (i) Under existing research, members are assumed to have a quadratic utility function over the deviations of the fund from the corresponding targets. However, this will equally penalise both outperformance and underperformance relative to the target. We believe a loss aversion utility based model is a better reflection of members' behaviour in real life;
- (ii) In our model, the salary process of the member is realistically calibrated in a stochastic (rather than deterministic as assumed in other existing studies) way.
- (iii) Previous studies have been oversimplified by assuming fixed interim and final targets (based in a constant investment return). In our model, both the interim and final fund targets are path-dependent (and vary over time in accordance with the member's actual current salary and required replacement ratio at retirement).

By comparing with the equivalent quadratic utility, we demonstrate that use of a loss aversion framework can significantly increase the probability of achieving final targets. Relevant sensitivities on loss aversion utility parameters are also studied.

Recently, a number of fund managers have launched dynamic de-risk investment solutions for DC pension plans with targets defined on investment return or annuity amount. A standard feature of these solutions is that when the fund outperforms the pre-determined target, additional assets above the target level are “banked” by switching to risk-free investment (e.g. cash). So while they have some features in common with our framework, they do not explicitly consider issues such as path-dependent targets or members’ utility. Our framework is therefore much more general than the solutions currently being implemented in practice. In addition, the model is reasonably easy to implement as well. The outputs of our dynamic asset allocation model are a set of optimal control variable (portfolio weight to equities) on each grid points for each age. With information of the member’s actually salary and pension fund level, an optimal investment strategy can be derived immediately.

In Chapter 5, we use a model that differs radically from existing studies. We assume the optimal funding and investment strategies depend on the members’ desired pattern of consumption over the lifetime. We investigate these strategies under the assumption that the member has an Epstein-Zin utility preference, which allows a separation between risk aversion and the elasticity of intertemporal substitution, and we also take into account the member’s human capital.

We show that a stochastic lifestyle approach, with an initial high weight in equity-type investments and a gradual switch into bond-type investments as the retirement date approaches is an optimal investment strategy. In addition, the optimal contribution rate each year is not constant over the life of the plan but reflects trade-offs between the desire for current consumption, bequest and retirement savings motives at different stages in the life cycle, changes in human capital over the life cycle, and attitude to risk.

6.2 Future Developments

This thesis attempts to shed new light on the innovative dynamic asset allocation issue within a DC pension scheme framework. It is hoped that this work will lead to some further developments in this increasingly important field. There are various areas in which the analysis undertaken in this thesis may be extended and improved.

- **Stochastic interest rate**

In Chapter 4 and 5, the investment opportunity set is assumed constant. This will greatly facilitate the numerical method of calculating optimal asset allocation. However, it will certainly be an important improvement to investigate optimal investment strategies under a stochastic interest rate model (e.g. Cox-Ingersoll-Ross model), which will allow us to incorporate the annuity risk (with annuity price based on stochastic interest rate) into the model as well.

- **Non-pension saving**

In Chapter 5, for simplicity, we do not consider any non-pension saving in this paper. In practice, the pension scheme members will have other savings to act as a buffer to smooth consumption over time. However, in this case, it will be difficult to adopt this intertemporal model, unless we make some assumptions about the investment returns on these other savings too. Also, as the pension wealth will be reduced accordingly, the asset allocation strategy should be more aggressive (i.e. with a higher proportion invested in risky equity-type assets).

- **Disposable and non-disposable wealth**

In Chapter 5, under our current model setting, when plan members decide to annuitise a large proportion of the accumulated fund at retirement, the wealth level reduces significantly. This is based on the assumption that capital value of annuity investment can not be bequeathed. One possible adjustment to the model is to split the wealth into

disposable wealth (i.e. the residual wealth after annuitisation) and non-disposable wealth (i.e. the present value of future pension income) after retirement. This change will require some further adjustments in the Epstein-Zin utility function and a new state variable in the numerical optimisation process. We are investigating this issue in our further research. Full details and results will be concluded in a forthcoming paper.

- **Housing-related and other consumption.**

In Chapter 5, the model could be further improved by including other consumptions, such as mortgage repayments. But, it is out of the scope of this paper.

Bibliography

Banks, James and Susann Rohwedder, 2001. "Life-cycle saving patterns and pension arrangements in the UK", *Research in Economics*, 55, 83-107.

Barberis, N., M. Huang and T. Santos, 2001. "Prospect theory and asset prices", *Quarterly Journal of Economics*, 116, 1-53.

Battocchio, Paolo and Francesco Menoncin, 2002. "Optimal Pension Management under Stochastic Interest Rate, Wages, and Inflation", *Discussion paper of IRES (Institut de Recherches Economiques et Sociales)*, http://www.ires.ucl.ac.be/DP/IRES_DP/2002-21.pdf.

Benartzi, Shlomo and Richard Thaler, 1995. "Myopic Loss Aversion and the Equity Premium Puzzle", *Quarterly Journal of Economics*, 73-92.

Benartzi, Shlomo and Richard H. Thaler, 1999. "Risk Aversion or Myopia? Choices in Repeated Gambles and Retirement Investments" *Management Science*, 45, 3: 364-381.

Berkelaar, Arjan and Roy Kouwenberg, 2003. "Optimal Portfolio Choice Under Loss Aversion", *Working Paper*, Erasmus University Rotterdam.

Bhamra, Harjoat S. and Raman Uppal, 2005. "The role of risk aversion and intertemporal substitution in dynamic consumption-portfolio choice with recursive utility", *Journal of Economic Dynamics and Control*, to appear.

Binsbergen Jules H. van and Michael W. Brandt, 2006. "Solving Dynamic Portfolio Choice Problems by Recursing on Optimized Portfolio Weights or on the Value Function", Fuqua School of Business, Duke University.

Blackburn, Douglas W., 2006. "Option implied risk aversion and elasticity of intertemporal substitution", Indiana University working paper.

Blake, David, 1996. "Efficiency, risk aversion and portfolio insurance: An analysis of financial asset portfolios held by investors in the United Kingdom, *Economic Journal*, 106, 1175-1192.

Blake, D., Andrew J.G. Cairns and Kevin Dowd, 2001. "PensionMetrics I: Stochastic pension plan design and value at risk during accumulation phase", *Insurance: Mathematics and Economics*, 29, 187-215.

Blake, David, Andrew J.G. Cairns and Kevin Dowd, 2007. "The impact of occupation and gender on pensions from defined contribution plans", *Geneva Papers on Risk & Insurance*, 32, 458-82

Boulier, Jean-Francois, Etienne Trussant and Daniele Florens, 1995. "A dynamic model for pension funds management", *Proceedings of the 5th AFIR International Colloquium*, 1, 361-384.

Boulier, Jean-Francois, ShaoJuan Huang and Gregory Taillard, 2001. "Optimal management under stochastic interest rates: the case of a protected defined contribution pension fund", *Insurance: Mathematics and Economics*, 28, 173-189.

Brandt, M. W., A. Goyal, P. Santa-Clara and J.R. Stroud, 2005. "A Simulation Approach to Dynamic Portfolio Choice with an Application to Learning About Return Predictability", *Review of Financial Studies*, 18, 3, 831-874.

Cagetti, Marco, 2003. "Wealth Accumulation over the life cycle and Precautionary Savings", *Journal of Business and Economic Statistics*, 21, 3, 339-353.

Cairns, Andrew J.G., 1994. "An Introduction to Stochastic Pension Plan Modelling", *presentation paper of workshop on Interest Rate Risk*, Vancouver, 19-20 August, 1994, <http://www.ma.hw.ac.uk/~andrewc/papers/>.

Cairns, Andrew J.G., 2000. "Some notes on the dynamics and optimal control of stochastic pension fund models in continuous time", *Astin Bulletin*, 30, 1, 19-55.

Cairns, Andrew J.G, David Blake and Kevin Dowd, 2004. "Stochastic Lifestyling: Optimal Dynamic Asset Allocation for Defined Contribution Pension Plans", *Journal of Economic Dynamics and Control*, to appear.

Campbell, John Y., João F. Cocco, Francisco J. Gomes and Pascal J. Maenhout, 1999. "Investing retirement wealth: a life-cycle model".

Campbell, John Y. and Luis M. Viceira, 2002. *Strategic Asset Allocation: Portfolio Choice for Long-Term Investors*, Oxford University Press.

Cardinale, M., G. Hatz, J. Kumar and J.M. Orszag, 2005. "Background Risk and Pensions", *Sessional meeting papers, Institute of Actuaries*, 28 November 2005, <http://www.actuaries.org.uk/files/pdf/sessional/sm20051128.pdf>.

Cocco, João F., Francisco J. Gomes and Pascal J. Maenhout, 2002. "Consumption and portfolio choice over the life-cycle", *The Review of Financial Studies*, to appear, <http://faculty.london.edu/fgomes/research.html>.

Davidoff, Thomas, Brown, Jeffrey R. and Peter A. Diamond (2005) "Annuities and Individual Welfare", *American Economic Review*, 95, 1573-1590.

Deelstra, G., Martino Grasselli and P-F Koehl, 2000. "Optimal investment strategies in a CIR framework", *Journal of Applied Probability*, 37, 936-946.

- Duffie, Darrell and Larry G. Epstein, 1992. "Stochastic Differential Utility", *Econometrica*, 60, 353-394.
- Duman, Bernard, Raman Uppal and Tan Wang, 2000. "Efficient Intertemporal Allocations with Recursive Utility", *Journal of Economic Theory*, 93, 240-259.
- Dynan, Karen, Jonathan Skinner and Stephen Zeldes, 2002. "The Importance of Bequest and Life-Cycle Saving in Capital Accumulation: A New Answer", *The American Economic Review*, 92, 2, 274-278.
- El Karoui, N., S. Peng and M.C. Quenez, 1997. "Backward stochastic differential equations in finance", *Mathematical Finance*, 7, 1-71.
- El Karoui, N., S. Peng and M.C. Quenez, 2001. "A dynamic maximum principle for the optimization of recursive utilities under constraints", *The Annals of Applied Probability*, 11, 664-693.
- Epstein, L.G. and Stanley Zin, 1989. "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework", *Econometrica* 57, 937-969.
- Epstein, L.G. and Stanley Zin, 1991. "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: An Empirical Investigation", *Journal of Political Economics*, 99, 263-286.
- Fama, Eugene and Kenneth French (2002) "The equity premium", *Journal of Finance*, 57, 637-659.
- Farah, Nathalie and Stephen E. Satchell, 2003. "A Loss Aversion Performance Measure", Faculty of Economics and Politics, Cambridge University, unpublished.
- Friedman, Benjamin and Mark Warshawsky, 1990. "The cost of annuities: Implications for saving behaviour and bequests", *Quarterly Journal of Economics*, 105, 135-154.
- Garcia, Philip, Stephen Bridges, David Hargreaves, Hui In Ng and Hemal Popat, 2004. "Demand for DC Products", *Finance and Investment Board Working Party*.
- Gerrard, Russell, Steven Haberman, Elena Vigna, 2004. "Investment Choices Post Retirement in a Defined Contribution Pension Scheme", *Insurance: Mathematics and Economics*, 35, 321-342.
- Gneezy, Uri and Jan Potters, "An Experiment on Risk Taking and Evaluation Periods", *Quarterly Journal of Economics*, 1997, 632-45.

Gomes, Francisco and Alexander Michaelides, 2002. "Life-cycle asset allocation: a model with borrowing constraints, uninsurable labor income risk and stock-market participation costs".

Gomes, Francisco and Alexander Michaelides, 2005. "Optimal Life-Cycle Asset Allocation: Understanding the Empirical Evidence", *Journal of Finance*, 60, 869-904..

Gupta, Aparna and Walter Murray. "A Framework Algorithm to Compute Optimal Asset Allocation for Retirement with Behavioral Utilities", Management Science and Engineering, Stanford University, unpublished.

Haberman, Steven and Elena Vigna, 2002. "Optimal investment strategies and risk measures in defined contribution pension schemes", *Insurance: Mathematics and Economics*, 31, 35-69.

Hoek, John van der and Michael Sherris, 2001. "A class of non-expected utility risk measures and implication for asset allocations", *Insurance: Mathematics and Economics*, 28, 69-82.

Howie, Robert, Helen Davies, 2002. "Setting investment strategy for the long term: A closer look at defined contribution investment strategy", Faculty and Institute of Actuaries, Finance and Investment Conference 2002,
http://www.actuaries.org.uk/files/pdf/library/proceedings/fin_inv/2002/Howie.pdf.

Hwang, Soosung and Stephen E. Satchell, 2005. "The Magnitude of Loss Aversion Parameters in Financial Markets", unpublished.

Josa-Fombellida, Ricardo and J.P. Rincon-Zapatero, 2001. "Minimization of risks in pension funding by means of contributions and portfolio selection", *Insurance: Mathematics and Economics*, 29, 35-45.

Kahneman, Daniel and Amos Tversky, 1979. "Prospect Theory: An Analysis of Decision under Risk", *Econometrica*, 263-91.

Kahneman, Daniel, P. Slovic and Amos Tversky, 1982. "Judgment under uncertainty: Heuristics and biases", *Cambridge University Press*..

Kim, Myung Jig , Charles R. Nelson, and Richard Startz, 1991. "Mean reversion in stock prices? A reappraisal of the empirical evidence", *The Review of Economic Studies*, 58, Special Issue: The Econometrics of Financial Markets, 515-528.

Kreps, David M. and Evan L. Porteus, 1978. "Temporal Resolution of Uncertainty and Dynamic Choice Theory", *Econometrica*, 46, 185-200.

- Langer, Thomas and Martin Weber, 2003. "Does Binding of Feedback Influence Myopic Loss Aversion? An Experimental Analysis", *Centre for Economic Policy Research, Discussion Paper*, 4084.
- Levy, M. and H. Levy, 2002. "Prospect Theory: Much Ado About Nothing?", *Management Science*, 48, 1334-1349.
- Lo, Andrew W., and A.Craig Mackinley, 1988. "Stock prices do not follow random walks: Evidence from a simple specification test", *Review of Financial Studies*, 1, 41-66.
- Markowitz, H., 1952. "Portfolio Selection", *Journal of Finance*, 7, 77-91.
- Madrian, Brigitte and Dennis Shea, 2001. "The power of suggestion: inertia in 401(k) participation and savings behaviour", *The Quarterly Journal of Economic*, 116, 1149-1187.
- Merton, Robert C., 1969. "Lifetime portfolio selection under uncertainty: The continuous-time case", *Review of Economics and Statistics*, 51, 247-257.
- Merton, Robert C., 1971. "Optimum consumption and portfolio rules in a continuous-time model", *Journal of Economic Theory*, 3, 373-413.
- Mitchell, Olivia S., James Poterba, Mark Warshawsky, and J. R. Brown, 1999. "New evidence on the money's worth of individual annuities", *American Economic Review*, 89, 1299-1318.
- Øksendal, Bernt, 1998. *Stochastic Differential Equations*, 5th Edition, Springer.
- Poterba, James M., and Lawrence H. Summers, 1988. "Mean reversion in stock prices: Evidence and Implications", *Journal of Financial Economics*, 22, 27-59.
- Rabin, Matthew and Richard H. Thaler, 2001. "Anomalies: Risk Aversion", *Journal of Economic Perspectives*, 219-232.
- Samuelson, Paul A., 1969. "Lifetime portfolio selection by dynamic stochastic programming", *Review of Economics and Statistics*, 51, 239-246.
- Schwartz, Eduardo and Walter N. Torous, 1999. "Can We Disentangle Risk Aversion from Intertemporal Substitution in Consumption", Anderson School of Management, University of California, <http://repositories.cdlib.org/anderson/fin/25-99>.
- Taylor, Nigel, 2000. "Making actuaries less human: lessons from behavioural finance", *The Staple Inn Actuarial Society (SIAC) Working Paper*.
- Thaler, Richard H., 1981. "Some Empirical Evidence on Dynamic Inconsistency", *Economic Letters*, 8, 201-207.

- Thaler, Richard H., Amos Tversky, Daniel Kahneman and Alan Schwartz, 1997. "The Effect of Myopia and Loss Aversion On Risk Taking: An Experimental Test", *The Quarterly Journal of Economics*, 647-661.
- Tversky, Amos and Daniel Kahneman, 1992. "Advances in Prospect Theory: Cumulative Representation of Uncertainty", *Journal of Risk and Uncertainty*, 297-323.
- Van der Sar, Nico L., 2004. "Behavioural Finance: How Matters Stand", *Journal of Economic Psychology*, 25, 3, 425-444.
- Vasiček, O.E., 1977. "An equilibrium characterisation of the term structure", *Journal of Financial Economics*, 5, 177-188.
- Viceira, L., 2001. "Optimal portfolio choice for long-horizon investors with nontradable labor income", *The Journal of Finance*, 56, 2, 433-470.
- Vigna, Elena and Steven Haberman, 2001. "Optimal investment strategy for defined contribution pension schemes", *Insurance: Mathematics and Economics*, 28, 233-262.
- Von Neumann, J. and O. Morgenstern, 1944. *Theory of Games and Economic Behavior*, Princeton University Press,
- Weil, Philippe, 1989. "The equity-premium puzzle and the risk-free rate puzzle", *Journal of Monetary Economics*, 24, 401-421.
- Weil, Philippe, 1990. "Non-expected Utility in Macroeconomics", *The Quarterly Journal of Economics*, 105, 1, 29-42.
- Yaari, M., 1965. "Uncertain lifetime, life insurance and the theory of the consumer", *Review of Economic Studies*, 32, 137-150.
- Yen, Simon H. and Yuan-kung Hsu Ku, 2003. "Dynamic Asset Allocation Strategy for Intertemporal Pension Fund Management with Time-Varying Volatility", *Academia Economic Papers*, 31, 229-261.

Appendices

A 01 – Risk attitude survey

Section A- Classification

Sex

1=Male

2=Female

Age

Ageband: 1="18-24"; 2="25-34"; 3="35-44"; 4="45-54"; 5="55-64"; 6="65+ "

Class

1=AB

2=C1

3=C2

4=DE

Standard region

1=North

2=Yorks and Humber

3=East Midlands

4=East Anglia

5=South East

6=South West

7=Wales

8=West Midlands

9=North West

10=Scotland

Working Status

1=Full-time

2=Part-time

3=Self-employed

4=Student

5=Retired

6=Unemployed

7=Other not working

Terminal education age

1=Under 16

2=16-17

3=18

4=19+

Marital Status

1=Married

2=Living with partner

3=Single

4=Widowed/Divorced/Separated

Household status

1=Male hoh

2=Female hoh

3=Not hoh

Household income

1=Under £6500

2=£6500-£9499

3=£9500-£15499

4=£15500-£19499

5=£20000-£24999

6=£25000-£34999

7=£35000-£49999

8=£50000+

Q.1 Total value all savings

1=£2000 or less

2=£2,001-£10,000

3=£10,001-£20,000

4=£20,001-£50,000

- 5=£50,001-£100,000
- 6=£100,001-£500,000
- 7=£500,001-£1,000,000
- 8=£1,000,000+
- 9=Don't know
- 10=Refused

Q.2 Value all borrowing

- 1=None
- 2=£2000 or less
- 3=£2,001-£10,000
- 4=£10,001-£20,000
- 5=£20,001-£50,000
- 6=£50,001-£100,000
- 7=£100,001-£500,000
- 8=£500,001-£1,000,000
- 9=£1,000,000+
- 10=Don't know
- 11=Refused

Q.3 When plan to retire

- 1=5 years or less
- 2=6-10 years
- 3=10-20 years
- 4=20+ years
- 5=Never
- 6=Don't know

Q.4 In good health

- 1=Yes
- 2=No
- 3=Refused

Section B-Behavioural Characteristics and Financial Knowledge

Q.5 Savings account rate of interest

- Type in answer
- No interests
- Don't know

Q.6 How allocate money for retirement needs

- 1=You allocate different pots of money to different needs
- 2=You have different pots of money, but don't allocate them to
- 3=You have one pot of money for all your different needs
- 4=Don't know

Q.7 Level of risk with each pot of money

- 1=Take different levels of risk with each
- 2=Take the same level of risk with each
- 3=Don't know

Q.8 How often review savings and investments

- 1=Weekly
- 2=Monthly
- 3=Quarterly
- 4=Annually
- 5=Never
- 6=Do not have savings / investments
- 7=Don't know
- 8=Refused

Q.9 Level of risk with spread assets

- 1=Go up
- 2=Stay the same
- 3=Go down
- 4=No idea
- 5=Do not understand the question

Q.10 Act in spontaneous or unplanned way

- 1=Yes - always
- 2=Occasionally
- 3=No - never
- 4=Don't know

Q.11 Make plans and stick to them

- 1=Yes - always
- 2=Occasionally
- 3=No - never
- 4=Don't know

Q.12 Ability to control impulsive feelings

1=Yes - always
2=Occasionally
3=No - never
4=Don't know

Section C- Attitudes To Risk

Q.13 In a coin tossing contest, what would the minimum prize money have to be to persuade you to take part if you stood to lose £100?

Type in answer (0 to 1,000,000)

Would not take part

Don't know

Q.14 In a coin tossing contest, what would the minimum prize money have to be to persuade you to take part if you stood to lose £1000?

Type in answer (0 to 1,000,000)

Would not take part

Don't know

Q.15 In a coin tossing contest, what would the minimum prize money have to be to persuade you to take part if you stood to lose £10000?

Type in answer (0 to 1,000,000,000)

Would not take part

Don't know

Q.16 In a coin tossing contest, what is the maximum amount of money you would be prepared to lose if the prize money was £100?

Type in answer (0 to 10,000)

Would not take part

Don't know

Q.17 In a coin tossing contest, what is the maximum amount of money you would be prepared to lose if the prize money was £1000?

Type in answer (0 to 10,000)

Would not take part

Don't know

Q.18 In a coin tossing contest, what is the maximum amount of money you would be prepared to lose if the prize money was £10000?

Type in answer (0 to 10,000)

Would not take part

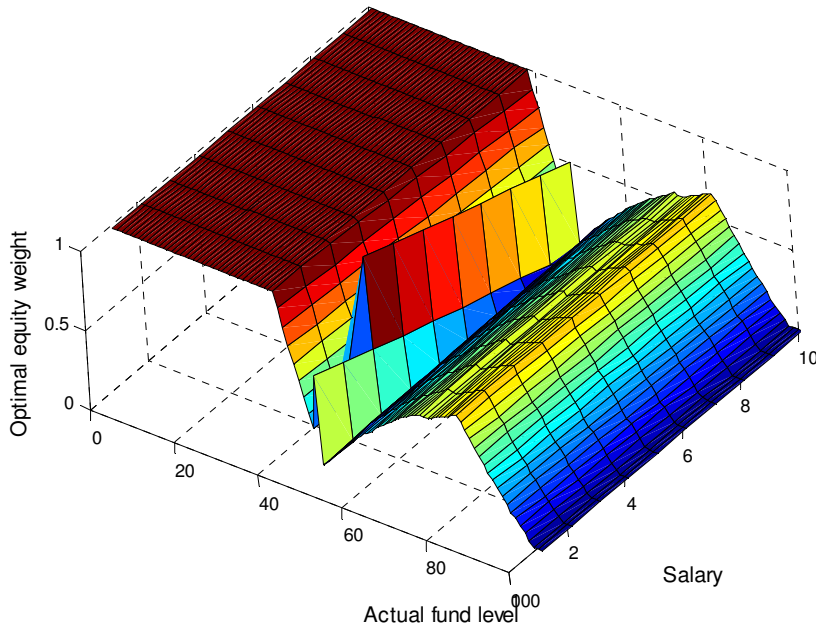
Don't know

A 02 – Numerical method of loss aversion model

The optimisation problem is $J_t(f_t, I_t) = \underset{\theta_t}{\text{Max}} E_t \left(\sum_{s=t}^{T-1} \beta^{s-t} \omega_0 V_s + \beta^{T-t} \omega_1 V_T \right)$, subject to the constraints that, $f_{t+1} = (f_t + \pi I_t) [1 + r + \theta_t (\mu + \sigma_1 Z_{1,t})]$.

At age 64, the value function is $J_{64}(f_{64}, I_{64}) = \underset{\theta_{64}}{\text{Max}} [\omega_0 V_{64} + \beta \omega_1 E_{64}(V_{65})]$. To avoid choosing local optima, we discretise the control variable (i.e. asset allocation in risky asset) into equally spaced grids and optimise them using a standard grid search. As an important step of stochastic dynamic programming, we need to discretise the state space and shocks first. Wealth and labour income level are discretised into 100 and 10 even grids respectively in computation, so that we can calculate optimal control variable and value function for each grid point, as shown in **Figure 49** below.

Figure 49: Optimal equity asset allocation at age 64



To solve the non-linear expectation part in above Bellman equation, i.e. $E_{64}(V_{65})$, Gauss-Hermite quadrature method is used, given the assumptions of normally distributed equity returns and the income growth rate. The idea is to approximate the value function by using several significant nodes of the distribution and their relevant weights. Equity return shock Z_1 and income shock Z_2 are both discretised into 9 nodes, and the procedure of discretising Z_1 and Z_2 is to substitute $\sqrt{2}Z_{1,m}$ and $\sqrt{2}Z_{2,n}$ for them respectively. So,

$$E_{64}(V_{65}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V_{65}(f_{65}, I_{65}) f(Z_1, Z_2) dZ_1 dZ_2 \approx \pi^{-1} \sum_{m=1}^9 \sum_{n=1}^9 w_{Z_{1,m}} w_{Z_{2,n}} \left[V_{65}(\sqrt{2}Z_{1,m}, \sqrt{2}Z_{2,n}; f_{65}, I_{65}) \right]$$

, where $w_{Z_{1,m}}$, $w_{Z_{2,n}}$ and $Z_{1,m}, Z_{2,n}$ are the Gauss-Hermite quadrature weights and nodes.

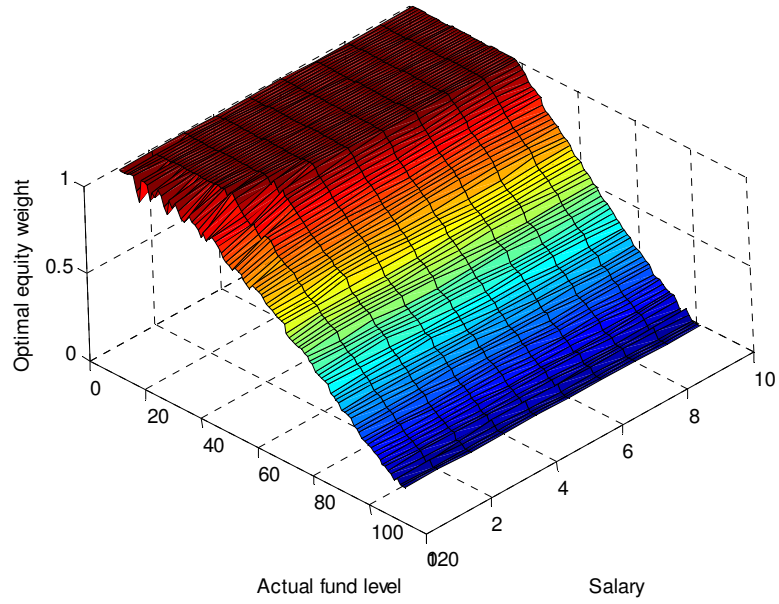
The advantage of this definition of nodes is that state variables can be computed more precisely and it costs less CPU time; however, because we have quite fine grids on control variable and much looser grid on shocks, we may have some state variable values not in grid points of next time period. Bilinear interpolation method is employed to approximate value function in this case.

Then, for every age t prior to T , we compute the value function and the optimal variables at each grid point. Substituting these values in the Bellman equation, we obtain the value function for this period, which is then used to solve the maximisation problem of the previous period. For example, at age $t < 64$, the Bellman equation for this problem is

$$\text{given by, } J_t(f_t, I_t) = \underset{\theta_t}{\text{Max}} E_t \left(\sum_{s=t}^{T-1} \beta^{s-t} \omega_0 V(s) + \beta^{T-t} \omega_1 V(T) \right) = \underset{\theta_t}{\text{Max}} \left[\omega_0 V(t) + \beta E_t(J_{t+1}) \right].$$

We set up the same 100 by 10 grids for state variables. Bilinear interpolation method is employed to approximate value function. After we get the optimal control variable at each grid point, we then substitute them in the Bellman equation and used it to solve the maximisation problem of previous time period. This process is iterated backwards until age 20. **Figure 50** illustrates the optimal equity weight at age 20.

Figure 50: Optimal equity asset allocation at age 20



A 03 – Numerical solution for the dynamic programming and integration process

The Epstein-Zin utility function at time t is as follows:

$$V_{20+t} = \left\{ (1 - \beta p_{20+t}) C_{20+t}^{\frac{1-\frac{1}{\varphi}}{\varphi}} + \beta \left[E_{20+t} \left(p_{20+t} \times V_{20+t+1}^{1-\gamma} + (1 - p_{20+t}) \times b \times \frac{\left(\frac{W_{20+t+1}}{b} \right)^{1-\gamma}}{1-\gamma} \right) \right]^{\frac{1-\frac{1}{\varphi}}{\varphi}} \right\}^{\frac{1}{1-\frac{1}{\varphi}}}$$

In the last period $(\omega-1, \omega)$, where $p_{\omega-1} = 0$, the terminal value function is given by:

$$J_{\omega-1}(W_{\omega-1}) = \max_{0 \leq C_{\omega-1} \leq W_{\omega-1}, 0 \leq \alpha_{\omega-1} \leq 1} \left\{ C_{\omega-1}^{\frac{1-\frac{1}{\varphi}}{\varphi}} + \beta \times E_{20+T} \left[b \times \frac{\left(\frac{W_{\omega}}{b} \right)^{1-\gamma}}{1-\gamma} \right]^{\frac{1-\frac{1}{\varphi}}{\varphi}} \right\}^{\frac{1}{1-\frac{1}{\varphi}}}$$

The optimisation problem is then:

$$\max_{\alpha_{20+t}, C_{20+t}, k} E(V_{20+t})$$

subject to the constraints given by:

- $W_{20+t+1} = (W_{20+t} + \pi_{20+t} Y_{20+t}) \times [1 + r + \alpha_{20+t} (\mu + \varepsilon_{20+t})]$, when $0 \leq t \leq 43$;
- $W_{65} = (1 - k) \times ((W_{64} + \pi_{64} Y_{64}) \times [1 + r + \alpha_{64} (\mu + \varepsilon_{64})])$, when $t = 44$;
- $W_{20+t+1} = \left(W_{20+t} + \frac{k \times W_{65}^-}{\ddot{a}_{65}} - C_{20+t} \right) \times [1 + r + \alpha_{20+t} (\mu + \varepsilon_{20+t})]$, when $t \geq 45$;
- $0 \leq \alpha_{20+t} \leq 1$; and
- $0 \leq k \leq 1$.

The Bellman equation is given by:

(a) for $0 \leq t \leq 43$, we have:

$$J_{20+t}(W_{20+t}, Y_{20+t}) = \max_{\alpha_{20+t}, C_{20+t}} \left\{ (1 - \beta p_{20+t}) C_{20+t}^{\frac{1}{1-\varphi}} + \beta \left[E_{20+t} \left(p_{20+t} \times J_{20+t+1}^{1-\gamma} + (1 - p_{20+t}) \times b \times \frac{\left(\frac{W_{20+t+1}}{b} \right)^{1-\gamma}}{1-\gamma} \right) \right]^{\frac{1-\frac{1}{\varphi}}{1-\gamma}} \right\}^{\frac{1}{1-\frac{1}{\varphi}}}$$

(b) for $t = 44$, we have:

$$J_{64}(W_{64}, Y_{64}) = \max_{\alpha_{64}, C_{64}, k} \left\{ (1 - \beta p_{64}) C_{64}^{\frac{1}{1-\varphi}} + \beta \left[E_{64} \left(p_{64} \times J_{65}^{1-\gamma} + (1 - p_{64}) \times b \times \frac{\left(\frac{W_{65}}{b} \right)^{1-\gamma}}{1-\gamma} \right) \right]^{\frac{1-\frac{1}{\varphi}}{1-\gamma}} \right\}^{\frac{1}{1-\frac{1}{\varphi}}}$$

(c) and, for $t \geq 45$, we have:

$$J_{20+t}(W_{20+t}, Y_{20+t}) = \max_{\alpha_{20+t}, C_{20+t}} \left\{ (1 - \beta p_{20+t}) C_{20+t}^{\frac{1}{1-\varphi}} + \beta \left[E_{20+t} \left(p_{20+t} \times J_{20+t+1}^{1-\gamma} + (1 - p_{20+t}) \times b \times \frac{\left(\frac{W_{20+t+1}}{b} \right)^{1-\gamma}}{1-\gamma} \right) \right]^{\frac{1-\frac{1}{\varphi}}{1-\gamma}} \right\}^{\frac{1}{1-\frac{1}{\varphi}}}$$

To solve the non-linear expectation part in the Bellman equation above, i.e., $E_{20+t} \left(J_{20+t+1}^{1-\gamma} \right)$, Gauss-Hermite quadrature is used to discretise Z_1 and Z_2 into 9 nodes, and the procedure of discretising Z_1 and Z_2 is to substitute $\sqrt{2}Z_{1,m}$ and $\sqrt{2}Z_{2,n}$ for them respectively.

So, we have:

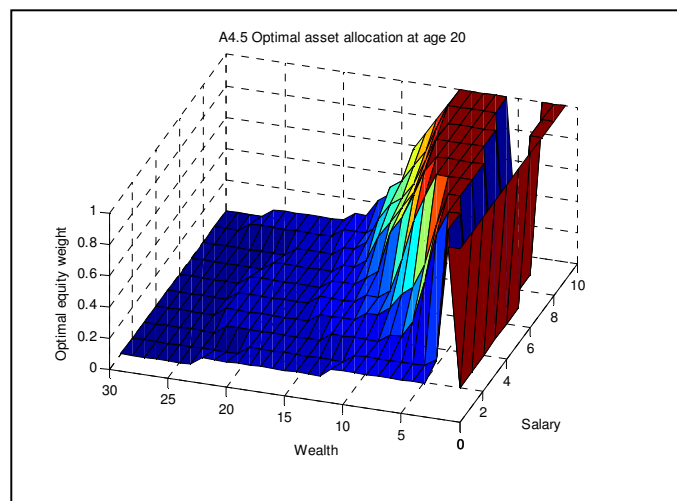
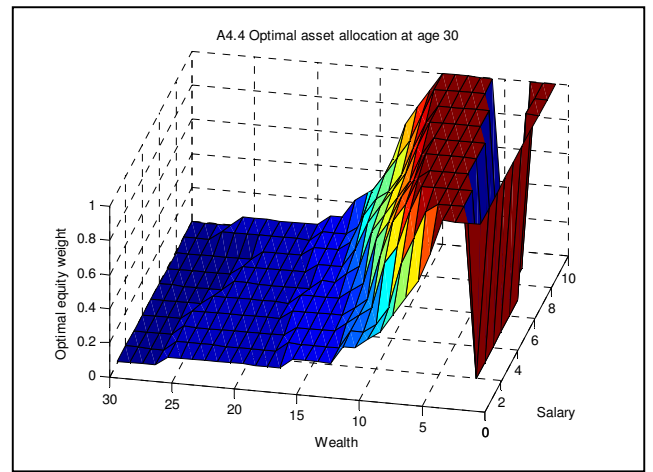
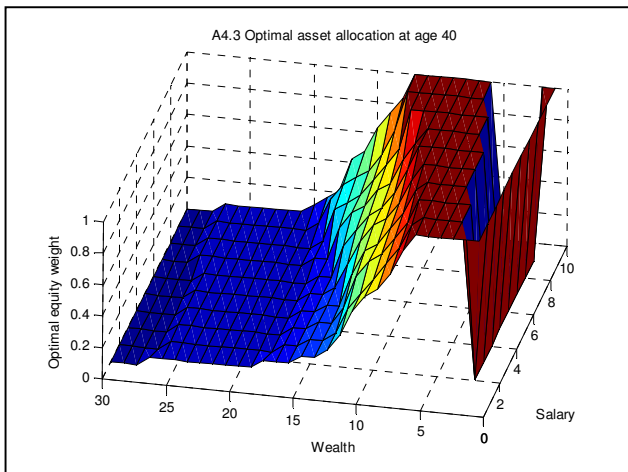
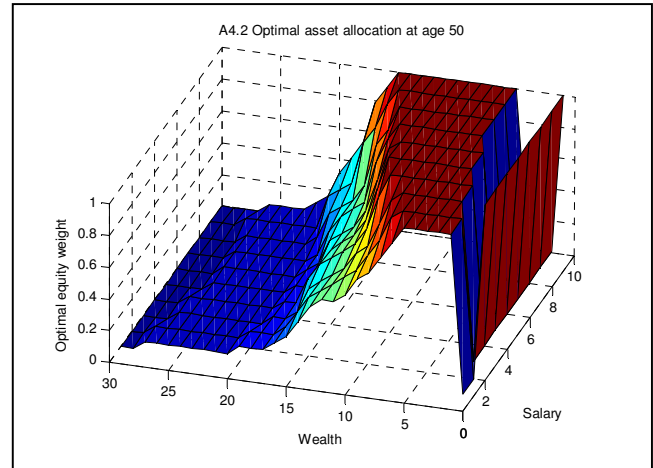
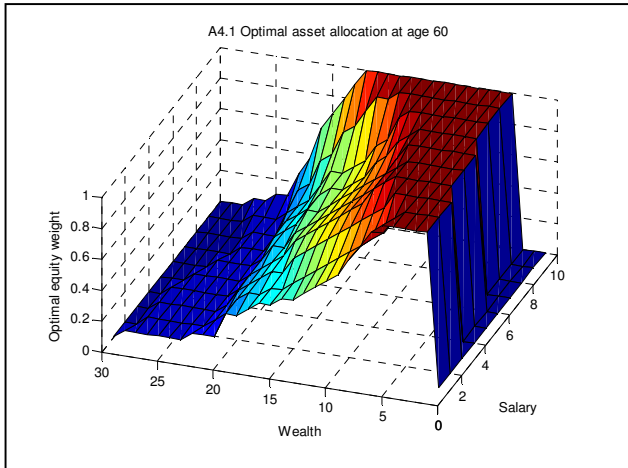
$$E_{20+t} \left(J_{20+t+1}^{1-\gamma} \right) \approx \pi^{-1} \sum_{m=1}^9 \sum_{n=1}^9 w_{Z_{1,m}} w_{Z_{2,n}} \left[J_{20+t+1} \left(\sqrt{2}Z_{1,m}, \sqrt{2}Z_{2,n}; W_{20+t+1}, Y_{20+t+1} \right) \right]^{1-\gamma}$$

where $w_{Z_{1,m}}$, $w_{Z_{2,n}}$ and $Z_{1,m}$, $Z_{2,n}$ are the Gauss-Hermite quadrature weights and nodes, and π is the mathematical constant.

To avoid choosing a local maximum, we discretise both control variables (i.e., consumption and asset allocation) into 20 equally-spaced grids and optimise using a standard grid search. Wealth and labour income level are discretised in 30 and 10 equally-spaced grids, respectively.

Substituting the expectation with the Gauss-Hermite approximated function in the Bellman equation at time t , we compute the maximum value function and optimal variables at each grid point. We then iterate the procedure back to age 20. For a point in the state space other than grid points, cubic spline interpolation is employed to approximate the optimal results.

A 04 – Optimal asset allocation at different ages



A 05 – Survival probabilities table

Age	Prob.	Age	Prob.	Age	Prob.
20	0.999812	54	0.998682	88	0.867459
21	0.999813	55	0.998465	89	0.855528
22	0.999813	56	0.998212	90	0.843024
23	0.999814	57	0.997917	91	0.829980
24	0.999815	58	0.997576	92	0.816432
25	0.999816	59	0.997180	93	0.802424
26	0.999816	60	0.996723	94	0.788008
27	0.999817	61	0.996108	95	0.773236
28	0.999817	62	0.995388	96	0.758172
29	0.999816	63	0.994549	97	0.742880
30	0.999816	64	0.993575	98	0.727427
31	0.999814	65	0.992448	99	0.711888
32	0.999813	66	0.991149	100	0.696334
33	0.999810	67	0.989657	101	0.680840
34	0.999807	68	0.987949	102	0.665482
35	0.999802	69	0.986001	103	0.650334
36	0.999797	70	0.983787	104	0.635468
37	0.999789	71	0.981282	105	0.620956
38	0.999781	72	0.978456	106	0.606867
39	0.999769	73	0.975280	107	0.593266
40	0.999755	74	0.971726	108	0.580214
41	0.999737	75	0.967762	109	0.567769
42	0.999715	76	0.963360	110	0.555986
43	0.999689	77	0.958487	111	0.546967
44	0.999657	78	0.953118	112	0.538703
45	0.999618	79	0.947223	113	0.531220
46	0.999571	80	0.940777	114	0.524541
47	0.999515	81	0.933759	115	0.518687
48	0.999448	82	0.926146	116	0.513674
49	0.999367	83	0.917925	117	0.509516
50	0.999271	84	0.909084	118	0.506224
51	0.999158	85	0.899615	119	0.503806
52	0.999024	86	0.889516	120	0.000000
53	0.998867	87	0.878794	120+	0.000000

Source: CMI mortality table PMA92(C2010).