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Citation: Gavranovic, Nedim (2011). Optimal asset allocation and annuitisation in a defined contribution pension scheme. (Unpublished Doctoral thesis, City University London)

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**OPTIMAL ASSET ALLOCATION AND
ANNUITISATION IN
A DEFINED CONTRIBUTION PENSION SCHEME**

by

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March 2011

PhD Thesis

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Acknowledgements

I would like to express my sincere appreciation to Professor Steven Haberman of Cass Business School. There were periods during the research for this thesis when I wondered if I would ever finish it, but his professional support and his personality gave me the strength to keep going further with the research. I think that accomplishing this thesis has been possible with his guidance, advice and assistance only. In this thesis, we measure the pensioner's utility gains from access to annuities. My utility gains from access to Professor Steven Haberman and Cass Business School are so high that it would be hard to measure.

My brother and Professor at the Faculty of Science, University of Sarajevo, provided me with the hardware needed to perform parallel computing in this thesis. This thesis would not been possible in this form without his support.

I would like to acknowledge my deep appreciation to my colleagues in Sarajevo Osiguranje d.d. Sarajevo and ASA Osiguranje d.d. Sarajevo, the two insurance companies where I used to work during the period of research for this thesis. Their friendly and professional support was very important for me.

Special thanks go to my parents, Akif and Subhija, and again brother, Haris, for their help and encouragement during the research of this thesis. I was raised in a family where science and education was one of the most important aspects of life. I dedicate this PhD thesis to them.

And of course, there is the family where I am a father and husband. I express many thanks to my wife Meliha, daughters, Naida and Maida, and my new-born son, who has not yet been named. Their support and silent tolerance towards their father's and husband's busy days is built in this thesis. I hope the new generation in my family will love science, and hope that this thesis will be at least a small stimulus for their decisions to work hard on their education.

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Abstract

In this thesis, we investigate a pensioner's gains from access to annuities. We observe a pensioner aged 65, having constant income from social security, having certain amount of pension wealth at age 65. The pensioner optimally decides each year how much of his available assets to consume, to invest into tradable assets, and how much to convert to annuities. Annuities are irreversible investments, once bought they provide income in the later years, but it is not possible to trade annuities any more. The pensioner makes optimal decisions such that the expected discounted utility from future consumption and bequest (if the pensioner has a bequest motive) is maximised. We develop and solve two models for the member of a defined contribution pension scheme in the post-retirement period.

The first one is a two assets model with stochastic inflation. We refer to this model as the inflation risk model. The pensioner in the inflation risk model has access to risk less (cash) and risky (equity) investment and to nominal and/or real annuities. The solution of this type of problem using numerical mathematics is presented in detail. We investigate different constraints on annuitisation. The main results presented and analysed are the pensioner's gains from access to certain class/classes of annuities, and also the pensioner's optimal asset allocation and annuitisation strategies such that the maximised expected discounted utility from future consumption and bequest is attained.

The second model for the pensioner in a defined contribution pension scheme is a three assets model with a stochastic interest rate. We refer to this model as the interest rate risk model. The pensioner in the interest rate risk model has access to risk less (one year bond), low risk (rolling bond with constant duration) and risky (equity) assets, and to annuities. Again, we precisely define the problem mathematically and solve it using numerical mathematics. We present and thoroughly analyse the pensioner's optimal asset allocation and optimal annuitisation such that his expected discounted utility from consumption and bequest is maximised. Particularly, we investigate in detail the dependence of the results on the value of the interest rate during the year before retirement.

After investigating the inflation risk model and interest rate risk model separately, we investigate deeper the new results obtained by introducing a stochastic interest rate. We compare the results obtained in the inflation risk model where the value of the interest rate is constant and the results in the interest rate risk model where the value of the interest rate changes. Particularly, in the interest rate risk model, we investigate deeper the dependence of the results on the value of the interest rate during the year before retirement and on the value of the interest rate during each year before annuitisation and asset allocation during the retirement period.

Chapter 1

Introduction

Financial contracts having payments dependent on a person's survival have been known for centuries. One often thinks of the tontine that was introduced as early as 17th century, as the first form of annuities. Later in history, and particularly in the 20th century, the care and support of elderly people have been dramatically improved. Nowadays, some type of income after retirement exists in almost all economically developed countries throughout the world.

Defined benefit pension schemes, either funded or not, have prevailed in the market for a long period and still do. However, the maturing of such schemes and the changing age structure of the population in many countries have opened the question of the long-term sustainability of many defined benefit schemes. Other major factors affecting defined benefit pension schemes include the employer's scheme management costs, and the identification of the costs of guarantees that the employer has promised to the scheme members. Although employees usually favour defined benefit schemes, both employers in company run schemes and states in state run schemes wish to free themselves from the rising risks by transferring these risks to the employees and pensioners. The possible solution for employers and for states may be the partial or complete switch from defined benefit to defined contribution pension schemes.

1.1 Main Features of Defined Contribution Pension Scheme

The main idea behind defined contribution pension schemes (abbreviation DCPS) is the individualisation of assets as well as risks. By definition, DCPS is funded, and

member's income in retirement is the result of the available pension assets in possession.

Usually, the member of the DCPS joins the scheme in the early years of his employment, and stays involved up to the end of his life. In a pre-retirement period, the prospective pensioner contributes into his pension account and that period of the member's life is referred to as the accumulation phase. Contributions are invested into appropriate assets yielding investment returns. The member of DCPS expects the value of his assets to increase during the accumulation phase due to the contributions and due to the positive investment returns. However, the member carries the investment risks. Generally, no or little guarantee is given on asset returns.

At the end of a member's active working period of life, he has certain assets that are then used for income in retirement. The time of retirement and type of income stream can be, up to the certain limits, chosen by the member. In many countries the state provides certain income to the pensioners in the form of social security. Income from social security usually depends on the particular pre-retirement employment and is very different from state to state. There is an expectation that in the coming years, income from social security will be more or less sufficient for the basic pensioner's needs. All income above a pensioner's basic needs will probably come from his extra contribution to either defined benefit or defined contribution pension schemes. As DCPS are becoming more and more widespread throughout the world, we can expect that the pensioner in the coming years will have income from social security up to the limited level and above that he will have income directly connected to his pre-retirement contributions and investment results on the accumulated funds.

The moment of retirement is very important for the pensioner in DCPS. In many countries, the pensioner at the time of retirement chooses, with possible legal constraints, the way we will use his accumulated pension funds. We usually refer to the pension funds available to the pensioner as "pension wealth". Thus, the pensioner at the time of retirement ceases salary earnings and contributions to the pension funds and begins receiving income from social security and income from his pension wealth.

Throughout this thesis, we assume that our investigation is done on the microeconomic level. In other words, we assume that the market is exogenously given and the market is perfectly competitive. In that environment, the member is a price-taker, i.e. the member's decision cannot influence the market itself.

1.1.1 Pre-Retirement Period

The contribution into the pension fund in the pre-retirement period can be any stream of instalments, depending on legislation and on the member's preferences. The regular contribution stream, for example, can be a percentage of the salary or absolute amounts in money terms are a common type of contribution. It could be further enhanced with the regular contributions by the employer. Moreover, a single instalment, as well as any stream of irregular instalments is possible.

When the member starts with the very first pension contribution, he will usually continue contributing for the whole period while the assets are in accumulation. The collected amounts will be invested into the appropriate assets. For the whole accumulation period, there are contributions of the new amounts and investments of any new contributions, and also reinvestment of any amount earned from investments. Usually no outflow, i.e. no consumption of the pension wealth, is allowed in the accumulation period. The individual often accepts that his savings are for retirement purposes only and under that condition he is eligible to get a tax incentive (Lunnon (2002)).

The member, together with the investments advisers, will manage the assets available in the portfolio. The investment approach will balance the need for the long-term growth with a concern for risk. The way that the asset classes are managed is of particular interest and it is referred to as the strategic asset allocation. Often, the member himself will make decisions regarding the level of risk that he is willing to take, and investment advisers will create the asset allocation strategy, based on the member's preferences towards risk and returns trade off. Understanding different investment options and goals, and choosing an appropriate investment strategy are of particular importance.

There are a number of desirable requirements for the asset allocation in the accumulation period. The higher return and the lower risk are often stated as the most important requirements of the asset allocation strategy. In practice, a higher return means higher risk, and the trade-off between the two must be exercised. Many other criteria can be set up, and asset allocation can be managed and assessed in accordance with these criteria as well (Khorasanee (1995), Blake, Cairns and Dowd (2001), Haberman and Vigna (2002), Basu, Byrne and Drew (2010)).

The two main types of investment strategies are static and dynamic. If the member keeps his assets in the same proportions during the whole investment period it is referred to as the static asset allocation strategy. The dynamic strategy is the one where the member changes with time the proportions invested in available assets. The proportions in the dynamic investment strategy can be deterministic or can be stochastic processes. The latter is referred to as stochastic dynamic asset allocation.

The investment strategy usually adopted by actuaries and investment managers of DCPS in pre-retirement period is the “lifestyle strategy” (Blake, Cairns and Dowd (2001), Vigna and Haberman (2001)). The lifestyle strategy in the accumulation period means that the member switches from more to less risky assets when he is close to retirement. In practice, it means a higher proportion of stocks in earlier years and a gradual switch towards bonds and maybe cash in the years before retirement. The time when this switch begins is usually less than ten years before retirement. The switch is usually implemented gradually throughout the last five to ten years in the pre-retirement period. If the decrease of the percentage invested in the risky asset and increase of the percentage invested in less risky asset is a deterministic function of the time left to retirement, then it is referred to as a deterministic lifestyle strategy. On the other hand, if these percentages are stochastic processes, then it is referred to as a stochastic lifestyle strategy.

1.1.2 Post-Retirement Period

In the post-retirement period, the member’s contributions into the pension fund terminate, and the consumptions of the assets accumulated prior to the time of retirement commence. We differentiate income and consumption in retirement. In this thesis we assume that income in retirement comes from social security and from annuities bought earlier in retirement. Consumption is the amount that the pensioner actually consumes. The amounts used for purchasing annuities are deemed as change of the form of the pension wealth, and purchasing annuities is neither income nor consumption. During one period, for example one year, income can be smaller, equal or larger than consumption. If the income is larger than the consumption in certain periods then the difference between income and consumption is simply added to the pension wealth. Otherwise, the positive difference between consumption and income is deducted from the pension wealth.

Different types of income streams in a retirement period, i.e. pension income in DCPS are common nowadays (Collinson (1999)). Lunnon (2002) categorises income in retirement in three main groups: annuities, income drawdown and the combination of these two.

The annuity is a financial contract, usually offered by an insurance company, to provide a given income on a regular basis from the moment when an annuity is bought until the annuitant's death. By taking an annuity, the member transfers investment and longevity risks to the insurance company. In other words, he completely gives up the control of his assets in exchange for certain type of predefined income while he is alive. It usually means that at the time of death, no pension assets can be bequeathed. Different types of annuities exist. The income taken can be constant in nominal terms, constant in real terms or variable. The member can choose a single or joint annuity, with or without a guaranteed term. Bequeathing some assets on death can be specially arranged. Also, the frequency and timing of annuity payments needs to be defined (Blake (1995)).

On the other side of the spectrum of income plans in retirement is income drawdown, sometimes also referred to as self-investment in retirement or self-annuitisation. By taking income drawdown, the member keeps the control of the allocation of his pension wealth in retirement. In order to provide income in retirement, he deducts certain amounts from the pension fund from time to time. In contrast to annuitisation, self-annuitisation involves a positive probability that the member will run out of pension wealth while still alive. This is sometimes referred to as the probability of ruin, or to be more precise the probability of receiving income in retirement from social security only. If no annuity is taken, the member is exposed to the risk of "living too long" and running out of benefits from his own pension wealth. Because the assets stay in the actual possession of the member, all assets not consumed at the moment of death will be bequeathed.

Lump sum withdrawal can be deemed as a part of self-annuitisation. However, we separate these two by defining that self-annuitisation is the regular withdrawal of smaller amounts, while lump sum is withdrawal of larger amounts once or just a few times during retirement. For example, lump sum withdrawal could be withdrawing a certain percentage of the pension wealth at retirement in order to repay outstanding loans, or withdrawing larger amounts for medical expenses in old age. We prefer to separate lump sum consumption from self-annuitisation.

Income in retirement is the combination of social security income, lump sum withdrawal, income from annuities and self-annuitisation.

The member of DCPS can usually choose certain options for the type of income in retirement, asset allocation in retirement, and any guarantee and the bequest. Applying different options for income in retirement one can easily end up with the combination of social security income, lump sum withdrawal, income from annuities and self-annuitisation. The plans for income and consumption in retirement are usually influenced by the legislation. Often, this legislation significantly differs from country to country. It is usual that the government is eager not to have old age people with no assets, and often limits the freedom of choosing asset allocation options and income options in retirement. By limiting the options for management of the pension wealth, income in retirement is controlled, and consequently the consumption is limited as well. Legislation usually imposes these limitations such that the pensioner is not in a position to consume his pension wealth “too early”.

In the UK for example, there was a legislation limitation that the pensioner can defer annuitisation up to the age 75. In the period of deferment he could consume 35% to 100% of income that he would have been receiving by purchasing a single-life non-increasing annuity at the moment of retirement from a reasonably competitive insurance company. Further, at the time of retirement he was allowed to withdraw 25% of the available pension wealth as a lump sum. However, these rules have been changed recently and there is no compulsory annuitisation at age 75 any longer (for details, see The United Kingdom Government Actuary's Department (GAD) website www.gad.gov.uk).

1.2 Asset Allocation and Annuitisation in Retirement in DCPS

The analysis of DCPS is usually done separately for the accumulation period and for the decumulation period. One reason for this approach could come from the real life experience. The time of retirement is a turning point in life, the end of salary earning and accumulation, i.e. end of a saving strategy for the retirement period and the beginning of the decumulation and income from the social security and from the assets in possession, i.e. beginning of the pension consumption strategy. The other reason lies in the complexity of the models investigating both phases at the same time.

Lately, there appears to be more diversity in the choice of the asset allocation and annuitisation options in the retirement. Which asset allocation and annuitisation strategy will be adopted by the pensioner depends on many factors, and different strategies can be optimal depending on different criteria applied.

In this thesis we want to investigate optimal asset allocation and annuitisation strategies for the pensioner retiring at age 65, with a certain pension wealth at that age, with a certain last salary received at age 65, with a certain replacement rate at age 65, with a certain income from social security during retirement period, with certain personal preferences towards risk and bequest, and with certain limitations on his asset allocation and annuitisation strategies. The pensioner in this thesis wishes to maximise utility drawn from consumptions during retirement and also from bequeathing assets to his heirs if the pensioner has a bequest motive. We want to develop optimal asset allocation and annuitisation strategies for the pensioner wishing to maximise expected discounted utility drawn from future consumption and bequest.

Besides the pensioner's "ordinary" consumption, there is usually a need for certain lump sum consumption related to health costs in retirement, loan repayment expenditure, or some other consumption needed in special cases. These expenditures can be significant in terms of amounts and can happen just once or several times in retirement. This kind of expenditure will have its influence on a pensioner's optimal asset allocation and annuitisation.

In this thesis, we particularly concentrate on adding the risk of inflation in the model where nominal and real annuities are available in the market and on adding the risk of a real interest rate in the model where only real annuities are available. We develop one model with two assets with a constant interest rate in real terms and random inflation and with the presence of nominal and real annuities, and another model with three assets with no inflation but with a random real interest rate and with bonds.

We recognise here that the analysis of both interest risk and inflation risk in a single model would be an interesting problem to investigate. If we develop a single model with both risks then the two models investigated here would be special cases of the more general model. Furthermore, it would be possible to investigate the possible correlation between these two risks. However, adding both of these risks at the same time seems to be too time-consuming for calculating results. Also, the results would be too complicated to draw the conclusions about the influence of each of the inflation and interest risk individually. We chose to investigate two models separately in this

thesis and leave the derivation and investigation of the model addressing both risks in the same time for further research.

Inflation risk for the pensioner purchasing nominal annuities and the risk of stochastic interest rate have not been deeply investigated so far in the post-retirement models with constraints. Our work in this thesis can be deemed as an extension of known models and results of optimal asset allocation and annuitisation in retirement in the directions of adding one more source of risk, inflation risk in combination with nominal annuities, and stochastic interest rate risk in combination with real annuities. We will develop the criteria for comparing the results in terms of the pensioner's welfare and give numerical valuation related to these risks.

1.2.1 Asset Allocation

In this thesis we investigate in detail two models, an inflation risk model and an interest rate risk model. In the inflation risk model, the pensioner can invest in equities with a random return and cash with a constant return in real terms. In the interest rate risk model, the pensioner can invest in equities as a high-risk asset with a random return, in long-term bonds as a low-risk asset with a random return and a one-year bond as risk free asset. In each case, we develop the optimal asset allocation strategy with maximising expected discounted utility as a criterion. We will have no borrowing constraints in our models and results. No borrowing constraints become sometimes very hard to apply in continuous time models. In discrete time models, such as the ones developed and investigated here, no borrowing constraints are easily handled and many other constraints can be applied as well.

The pensioner can choose the asset allocation for the pension wealth only. At age 65, he possesses certain pension wealth and if annuities are available he can purchase a certain amount of annuities using his pension wealth and the rest is available for investment. Annuities can be deemed as an irreversible risk free investment, and income from social security can also be deemed as already bought annuities. From that point of view, we can expect that if more annuities are bought and if income from social security is higher then more available pension wealth will be invested into risky assets. However, some part of the assets available for investment will still be invested into low-risk and risk free assets. A precisely developed model and calculation of the numerical results from the model will give us an idea about the pensioner's optimal asset allocation.

We develop optimal asset allocation as function of the state variables, where the state variables are known values of the variables which influence future developments. Once knowing those functions, we can also make a sample of random realisations and investigate behaviours of optimal asset allocation paths for the pensioner.

As we will see in this thesis, there is a whole range of different pensioner's optimal asset allocation strategies depending on the different assumptions.

1.2.2 Annuitisation

In this thesis we investigate three main environments regarding the annuitisation possibility for the pensioner. The first one is where no annuitisation is possible at any age, the second one is where annuitisation is available at age 65 only, and the third one is where no constraints on annuitisation are imposed where annuitisation is possible at any age.

In the inflation risk model, the pensioner can purchase nominal or real annuities, and in the interest rate risk model he can purchase real annuities only. Actually, in the interest rate risk model no inflation is present and nominal annuities are the same as real ones.

Again, we calculate optimal annuitisation as function of the state variables and investigate the characteristics of these functions. If no annuitisation is allowed for the pensioner then the results under this assumption are used as the benchmark for investigating how much benefit the pensioner has from annuitisation. If annuitisation is allowed at age 65 only, then we get one single number as the optimal annuitisation strategy and this number depends on the assumptions. If annuities are available at any age, then we get results which depend on the state variables during retirement. If annuities are available at any age, we make a sample of random realisations of random variables and get random paths of optimal annuitisation for the pensioner. Once we have a random sample of the paths of optimal annuitisation as well as optimal asset allocation throughout the retirement we are in a position to investigate these random paths in statistical terms determining mean values, quantiles and so on.

1.3 Structure of the Thesis

After the introduction in Chapter 1, we present the review of literature relevant for the investigation done in this thesis in Chapter 2. We present a wide range of the literature relevant for optimal asset allocation in both pre- and post-retirement periods and in both discrete and continuous time framework. Other authors investigate optimal annuitisation in both a continuous and discrete time framework, and sometimes they investigate the post-retirement period only and sometimes it is the whole lifecycle with annuities after retirement. Some of the literature is not directly relevant for our investigation but is relevant in terms of the way that authors approach the problem of optimal asset allocation and annuitisation and the way they approach the problem of maximising expected discounted utility.

In Chapter 3, we develop the inflation risk model, where one asset is a riskless investment in cash and the second one is a risky investment in equities. The pensioner has access to annuities with or without constraints. We assume that the pensioner retirement period starts at age 65 with a given amount of pension wealth and with a given last salary as well as income from social security during retirement. We assume that the pensioner receives the last salary income at age 65 and then from age 66 to the end of life he receives income from social security and from annuities if the pensioner converts part of his pension wealth into annuities. We develop the model which allow for nominal and real annuities in retirement. The model is quite general in terms of possible application for investigation of different constraints on investment and annuitisation strategies. If the pensioner purchases nominal annuities in the inflation risk model, then income from nominal annuities is subject to the yearly correction in real terms due to the influence of inflation. We investigate constant and stochastic inflation. The results are presented for different cases where the pensioner faces different constraints on his access to annuities. We investigate the case of no access to annuities, the case of access to nominal annuities at age 65 only, access to real annuities at age 65 only, access to nominal annuities at any age, access to real annuities at any age, and the case of access to nominal and real annuities at any age. Using numerical mathematics, we find and investigate the pensioner's optimal consumption, optimal asset allocation and optimal annuitisation strategies in order to maximise his expected discounted utility from the consumption and from a bequest if he has a bequest motive.

In Chapter 4, we develop the model with three assets, the first one being a riskless investment in a one year bond, the second one being a low risk investment in a rolling bond with constant duration, and the third one being a risky investment in equities.

All values are in real terms in this model. The model itself allows for any duration of the rolling bonds and even for different durations for different ages. We develop a discrete time and space model for interest rate and define the formula for the prices of bonds of any duration. Although we set no borrowing constraints, the simple changes in the model allow it to be used for different assumptions on the constraints. The pensioner has access to annuities with or without constraints. The pensioner aims to maximise his expected discounted utility from the consumption and from a bequest if he has a bequest motive. Again, the pensioner starts his retirement period at age 65 with a certain amount of pension wealth at retirement, he receives the very last salary at age 65, and from age 66 to the end of his life he receives income from social security and from annuities if some annuities are bought during retirement. Although the model allows for any constraints on annuitisation, we investigate the results related to the three main cases of optimal annuitisation policies. In the first case, the pensioner has no access to annuities, in the second one he has access to annuities at age 65 only, and in the third case we investigate the pensioner having access to annuities during the whole retirement period. Again, we find and investigate the numerical results of the pensioner's optimal consumption, optimal asset allocation and optimal annuitisation such that his expected discounted utility from the consumption and from a bequest if he has a bequest motive is maximised. In Chapter 4, we are more focused on comparing the results between the cases, where we investigate the gains in expected discounted utility due to access of annuities.

In Chapter 5, we investigate further the results from Chapter 4 but now we are focused, within a given case, on the results depending on the value of the interest rate during the year prior to retirement. We also make some comparison with the chosen comparable results in Chapter 3.

The most important findings of the developed models, and the conclusions drawn from the numerical results based on the inflation risk model and interest rate risk model are presented in Chapter 6. We also provide a discussion on possible future research based on the results obtained in this thesis.

In the final section of the thesis, we give the entire list of references related to this thesis and three Appendices. In the first Appendix, we present the way to decrease the number of state variables from four to three state variables in the inflation risk model. In the second Appendix, we present the technique used for excluding income as a state variable in the interest rate risk model. These two Appendices are very important because excluding income as a state variable significantly increases the speed needed

for obtaining numerical solutions. In the third Appendix, we present the numerical values of the bond prices, obtained for the values of the parameters used in the numerical results in Chapters 4 and 5.

Chapter 2

Literature Review

Lifecycle models follow an individual throughout his lifetime and investigate income and consumption patterns. A member's DCPS starts at very early ages and in the pre-retirement period he earns a salary, consumes and saves for the DCPS fund. In the post-retirement period, the member receives income from social security, from annuities if any, and consumes. In both periods, if pension wealth exists, he also invests these assets. In this thesis we investigate the post-retirement period with a particular emphasis on the advantages coming from access to annuities. However, we will give some literature related to the pre-retirement period in order to see the ideas about optimal asset allocation that is also relevant to investment strategies in the post-retirement period, and also some literature related to the lifecycle as post-retirement period is one part of a pensioner's lifecycle.

The basic idea of lifecycle consumption can be given as follows. People generate income applying their labour and have desires and needs to consume. However, income and consumption do not match each other throughout the whole of life. In their early working ages, people usually spend more than they earn and generally not much is saved. The salary growth is the fastest for this age group. The early working-age period is followed by ages 40 to 50 years, when earning is higher than needs for consumption and the worker is aware of his lifecycle. This age group saves the most. Then, near the end of the working age, salary growth slows down and or even a salary decrease is experienced. However, the worker is fully aware of the approaching retirement period of life and tends to save more for old age. A retiree does not earn any more, but still has needs and desires to spend. This is financed from the assets accumulated throughout the working period of life and from social security income, or in other words from consumption given up in the working period of life. Figure 2.1 graphically shows this process.

Consumption, Earning and Saving Patterns in the Lifecycle Model

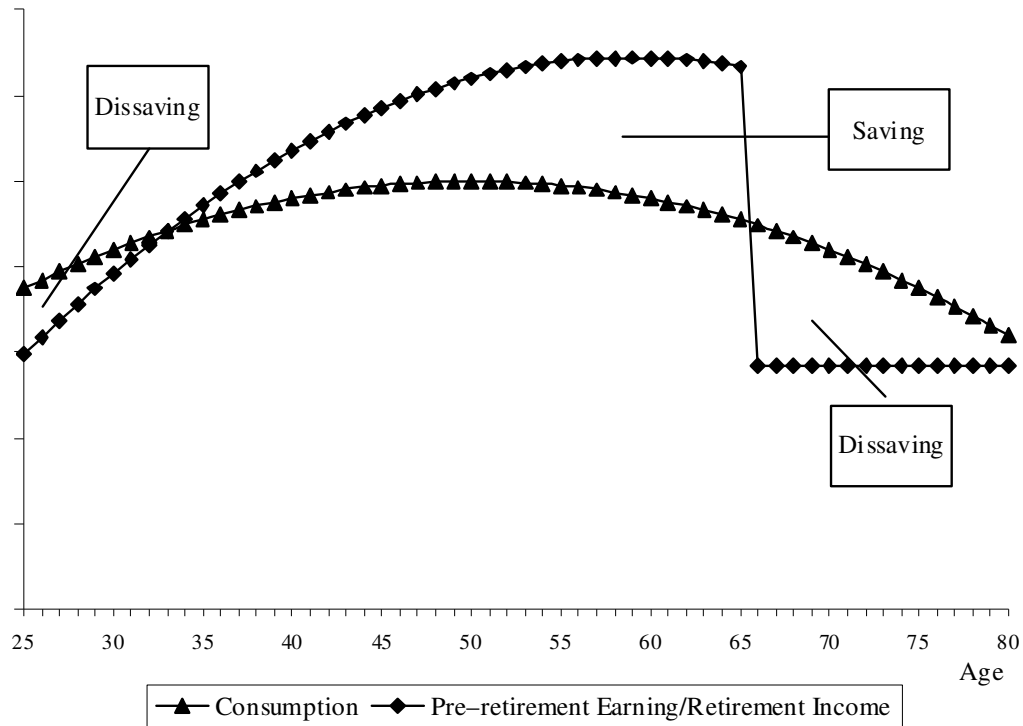


Figure 2.1 Lifecycle patterns

DCPS models for the pre-retirement period usually have characteristics of long term investment models with periodic contributions and no or little initial wealth. The duration of the pre-retirement models can be fixed assuming a fixed date of retirement, but also can have uncertain duration assuming that the member can decide to work a few years more or less. On the other hand, the models for the post-retirement period will have characteristics of asset allocation and annuitisation with a single contribution at the beginning and the stream of consumption afterwards. In addition, the typical feature of the post-retirement model is the assumption of an uncertain horizon, as income is needed as long as the member is alive and the time of death is not certain.

Ando and Modigliani (1963) investigate implications and real world empirical evidence of lifecycle approaches to income and consumption.

In the context of the whole spectrum of possible types of incomes in retirement, the results of Yaari (1965) are particularly interesting. In his seminal paper, he argues that the lifecycle consumer will always annuitise all his available assets assuming an actuarially fair annuity market. Yet, Yagi and Nishigaki (1993) show that in the case

of an imperfect capital market and level annuities, a pensioner without a bequest motive will still keep some marketable wealth.

In the rest of this chapter, we will firstly focus our attention on the assumptions needed for developing DCPS models. Then, we will refer to the relevant models and the results derived from these models. We will group these models into five categories. The categories are: pre-retirement discrete time, pre-retirement continuous time, post-retirement discrete time, post-retirement continuous time, and the combination of pre- and post-retirement (lifecycle). The models for DCPS fit either into one of these categories, or into a mixture of them. Although we investigate discrete time models in this thesis, continuous time models are interesting for drawing ideas from them and because discrete time models are actually discrete time versions of continuous time models. Generally, continuous time models better represent the real world. The advantage of a discrete time model over continuous time one is the possibility to solve the problem on computers, and sometimes to obtain the results numerically while the analytical solution is currently not known.

2.1 Assumptions for Asset Allocation and Annuitisation Modelling

Investment models for DCPS are usually long term investment models. The long term character and periodic or continuous random fluctuations impose the need for introducing probabilistic and stochastic models. However, introducing “too many variables” representing the real world better and using the stochastic models for all of them leaves us with models that are too complex and mathematically intractable.

The asset allocation model should be a good enough representation of the real world to lead us to relevant and useful conclusions. At the same time it should be simple enough for handling mathematically or numerically such that the results are obtainable in a reasonable time. Only then can conclusions be drawn.

The starting point for any actuarial modelling is setting certain assumptions. The assumptions are very important and should be deemed as a theoretical environment where investigation is valid. Let us start with the usual market assumptions.

A frequently met assumption for modelling DCPS is that the financial market is frictionless. A theoretical trading environment where all costs and restraints associated with transactions are non existent is referred as a frictionless market.

Market developments and decreasing trading costs in recent years make this assumption sound.

The models usually ignore taxation. There are different taxation systems for DCPS throughout the world and taxes can significantly influence the relevant amounts and processes (Davis 1995). Sometimes, we can also assume that the taxation is implicitly included in the rates used in the model.

An important assumption for modelling DCPS is the type of stochastic process in the model. If we allow that trading and flow of money into or out of a fund are done only at distinct time-points with time intervals between them, then we choose a discrete time model. Otherwise, if we assume that changes happen as momentarily change at each point of time in a specified time interval then we are in a continuous time framework. We deal with these types of models separately in the sections 2.2, 2.3 and 2.4 of this chapter.

2.1.1 Utility Function

Wishing to make a theoretical model for asset allocation and annuitisation, and to find the superior asset allocation and annuitisation strategy, we have to define in which sense a certain strategy is superior to the others. Before defining criteria, we need to define what we want to compare. In the post-retirement period, it seems appropriate to compare utilities drawn from consumption and, if a bequest motive exists, from bequeathing assets to heirs as a result of a certain strategy. Using a utility function, we can distinguish the risk preferences of different investors. In the pre-retirement period, it can be the utility from the accumulated pension wealth at the time of retirement, or the utility from pension ratio. If one investigates the lifecycle, then the utility drawn from consumption throughout life time and possibly from the bequest in retirement can be the appropriate criterion for more or less successful asset allocation and annuitisation strategy. Pratt (1969) discussed preference ordering and introduced Pratt–Arrow’s measures of risk aversion that are defined as:

$$A(W) = -\frac{U''(W)}{U'(W)}, \quad \text{Absolute risk aversion (ARA) function,}$$

$$R(W) = -\frac{U''(W)W}{U'(W)}, \quad \text{Relative risk aversion (RRA) function,}$$

where U is the utility function. It is common to order preferences towards asset allocation and annuitisation in DCPS models using one of these two measures. The cases when $A(W)$ and $R(W)$ are constant functions of wealth W are referred to as constant absolute risk aversion (CARA) and constant relative risk aversion (CRRA), respectively.

2.1.2 Interest Rate and Assets

The usual assumption for the interest rate in the model is one of the following:

- a constant interest rate,
- identically independently (iid) random variables for each time period,
- discrete time stochastic process in a discrete time framework,
- continuous time stochastic process in the continuous time framework.

If the interest rate depends on time, we will usually denote the time dependence of the rate by the subscript only, i.e., $r(t) = r_t$. For random variables, we will use superscript $\sim$ above the variable.

A fixed interest rate does not seem to be the best assumption for long-term models. However, the model is usually significantly simpler with a constant interest rate and the results obtained using the fixed interest rate can give us an indication about results in the cases that are more sophisticated. When the interest rate is assumed to be modelled by independent identically distributed random values for each consecutive time intervals then it is often assumed that random values are taken from normal or log-normal distribution. Its mathematical formulation in a case of iid normal distribution from interval to interval is as follows

$$\tilde{r}_t \sim N(\mu, \sigma),$$

while in a case of log-normal iid it is

$$\tilde{r}_t = e^{\tilde{x}_t},$$

where

$$\tilde{x}_t \sim N(\mu, \sigma).$$

Among continuous time models for interest rate, we will confine ourselves to the models having stochastic differential equation of the type

$$d\tilde{r}_t = \mu(t, r_t)dt + \sigma(t, r_t)d\tilde{\varepsilon}(t) \quad (2.1)$$

where $\tilde{\varepsilon}(t)$ is the standard Brownian motion, $\mu(t, r_t)$ is the drift coefficient and $\sigma(t, r_t)$ is the diffusion coefficient and the initial condition r_0 is given constant. $\mu(t, r_t)$ and $\sigma(t, r_t)$ are deterministic functions.

In this case, the interest rate r_t is referred to as the diffusion process whose dynamics are defined above by Ito's stochastic integral. Four models (Vasicek 1977, Cox, Ingersoll and Ross 1985, Hull and White 1990) given in Table 2.1 are often used when modelling interest rates as a stochastic process

Model name	Formula	Parameters
Vasicek	$d\tilde{r}_t = (b - ar_t)dt + \sigma_r d\tilde{\varepsilon}$	$a, \sigma_r > 0, b \geq 0$
Cox–Ingersoll–Ross	$d\tilde{r}_t = a(b - r_t)dt + \sigma_r \sqrt{r_t} d\tilde{\varepsilon}$	$a, b, \sigma_r > 0$
Hull–White (extended Vasicek)	$d\tilde{r}_t = (\Theta(t) - a(t)r_t)dt + \sigma(t)d\tilde{\varepsilon}$	$a(t), \Theta(t), \sigma(t) > 0$
Hull–White (extended Cox–Ingersoll–Ross)	$d\tilde{r}_t = (\Theta(t) - a(t)r_t)dt + \sigma(t)\sqrt{r_t}d\tilde{\varepsilon}$	$a(t), \Theta(t), \sigma(t) > 0$

Table 2.1 Some interest rate models

In our notation, the name of the function and the name of the constant will always be different. Although we will always have a unique name for each drift and diffusion function or coefficient, often the names will differ in subscript only. In addition, we will sometimes omit writing the arguments of the functions. For example, $\sigma(t)$ is the function, and even if sometimes written without arguments, i.e. σ , the meaning is the same, while σ_r is the constant because it is defined as such.

When talking about the interest rate, we should state if it is a nominal or real interest rate. Due to the long term nature of the models and possible diminishing value of money amounts in real terms, the effect of inflation can have a significant influence on DCPS models (Meredith et al 2003). Inflation has its own dynamics and can be modelled similarly as the interest rate. However, the explicit inclusion of inflation makes models significantly more complex and harder to handle.

Asset prices are the building blocks of each DCPS model. When modelling asset prices in a continuous time framework, the usual assumption is that the asset prices follow a geometric Brownian motion (GBM). Mathematically, GBM is defined by

$$d\tilde{S}(t) = \alpha S(t)dt + \sigma S(t)d\tilde{\epsilon}(t),$$

with initial condition $S(0) = s_0$ where $\tilde{\epsilon}(t)$ is a standard Brownian motion, α is a drift coefficient and σ is a diffusion coefficient. s_0 , α and σ are known constants.

The type and number of assets included in the model depend largely on the type of the model itself and the objectives of the investigation. It is argued that different rates of return of the portfolios are a consequence of the allocation of assets to particular classes and not of the particular chosen assets in the class. From that point of view, modelling each particular stock and bond will probably not significantly improve the results. Many authors model one risk free asset representing cash in the portfolio, and one or two risky assets, representing bond (low risk and less variable asset) and equity (high risk and more variable asset) portfolios. A low risk low return asset can be deemed to be the bond portfolio and high risk high return asset can be deemed as a representative index from the stock market. In other words, proportions held in these three assets represent the proportions held in appropriate asset classes. If we assume a constant interest rate, we can analyse only two assets, one risk free and another risky.

All of the models in Table 2.1 above possess an affine term structure (Duffie and Kan 1996), i.e. $B(t, T, r_t)$ – price at time t of the zero-coupon bond paying 1 at time T can be written as

$$B(t, T, r_t) = e^{A(t, T) - C(t, T)r_t}, \quad \text{for } 0 \leq t \leq T,$$

for all $T \in (0, T^*]$. In an arbitrage free bond market, (see definition 2.2 below), functions $A(t, T)$ and $C(t, T)$ are determined from the term structure partial differential equation

$$\begin{cases} B_t + (\mu - \lambda\sigma)B_r + \frac{1}{2}\sigma^2 B_{rr} - rB = 0 \\ B(T, T, r_T) = 1 \end{cases}$$

where $\lambda(t)$ is the market price of risk of the zero-coupon bond $B(t, T, r_t)$. If we assume that $t = 0$, then we will often omit writing the first variable in $B(t, T, r_t)$ and write $B(T, r_0)$

If the nominal interest rate is assumed to follow the Vasicek model defined above, then the nominal interest rate can have negative values. In that case, the zero-coupon bond can have a negative yield. No demand would exist for that zero-coupon bond. This non-realistic possibility is the main shortcoming of the Vasicek model. The CIR model for interest rate guarantees the positive value of the interest rate. In this sense, the CIR model of the interest rate is probably superior to the Vasicek one. However, the mathematics is usually simpler in the Vasicek framework.

One can also observe that the stock prices modelled as a GBM are always non negative. This feature reconciles with the real world feature of the common stock as being the limited liability security.

A typical approach to modelling the stochastic interest rate and portfolio containing cash, bonds and equities is used in Chapter 4.

2.1.3 Portfolio Process, Arbitrage and Completeness

Definition 2.1 : Portfolio Process

Let

- N = the number of different types of stocks;
- $h_i(t)$ = number of shares of type i held during the period $[t, t + \Delta t)$;
- $h(t)$ = the portfolio $[h_1(t), \dots, h_N(t)]$ held during the period $[t, t + \Delta t)$;
- $c(t)$ = the rate of consumption during the period $[t, t + \Delta t)$, and $c(t) \geq 0$;
- $S_i(t)$ = the price of one share of type i during the period $[t, t + \Delta t)$;
- $V(t)$ = the value of portfolio h at time t ;

Then,

- a portfolio process (or simply portfolio) is any process $\{h(t); t \geq 0\}$;
- the value process $V^h(t)$ corresponding to the portfolio h is given by

$$V^h(t) = \sum_{i=1}^N h_i(t) S_i(t);$$

- a portfolio consumption pair (h, c) is called self financing if the value process satisfies the condition

$$dV^h(t) = \sum_{i=1}^N h_i(t) dS_i(t) - c(t) dt;$$

A self financing portfolio is sometimes defined as the portfolio where purchasing new assets as well as consumption can be financed only by selling assets already held in the portfolio.

Definition 2.2 : Arbitrage

An arbitrage possibility exists on the financial market if there exists a self financed portfolio h such that

$$V^h(0) = 0, \text{ and } V^h(T) > 0 \quad P - a.s.$$

We say that the market is arbitrage free if there is no arbitrage possibility (Björk 1998). No arbitrage is a standard assumption in DCPS asset allocation models.

Definition 2.3 : Hedging and Completeness

We say that a T -claim χ can be replicated, or that it is reachable or hedgeable, if there exists a self financing portfolio h such that

$$V^h(T) = \chi, \quad P - a.s.$$

In this case, we say that h is the hedge against χ . We also say that h is a replicating or hedging portfolio. If every contingent claim is reachable, we say that the market is complete (Björk 1998).

The following result gives relation between the number of assets and no arbitrage and completeness in the model (Björk 1998).

Theorem 2.1: Meta theorem

Let M denote the number of underlying traded assets in the model excluding the risk free asset, and let R denote the number of random sources. Generally, we can have the following relations

1. the model is arbitrage free if and only if $M \leq R$;
2. the model is complete if and only if $M \geq R$;
3. the model is arbitrage free and complete if and only if $M = R$.

Let us define the risk neutral measure. It is also referred to as risk adjusted or martingale measure.

Definition 2.4 : Risk Neutral Measure

Let the rate of interest $r(t)$ be defined by (2.1). The risk-neutral measure Q is characterised by any of the following equivalent conditions

1. Under Q , every price process $\Pi(t)$ has the risk neutral valuation property

$$\Pi(t) = E_t^Q \left[e^{-\int_t^T r_s ds} \Pi(T) \right];$$

2. Under Q every price process $\Pi(t)$, be it underlying or derivative, has the short interest rate as its local rate of return, i.e. the Q -dynamics are of the form

$$d\tilde{\Pi}(t) = r_t \Pi(t) dt + \Pi(t) \sigma_{\Pi}(t) d\tilde{\varepsilon}(t),$$

3. Under Q every price process $\Pi(t)$, be it underlying or derivative, has the property that the normalised price process

$$\frac{\Pi(t)}{S_0(t)},$$

is a martingale, i.e., it has a vanishing drift coefficient, where $S_0(t)$ is defined by $dS_0(t) = r(t)S_0(t)dt$ with $S_0(0) = 1$.

The following two theorems establish relations between risk neutral probability measure and no arbitrage and completeness.

Theorem 2.2 : Fundamental theorem of asset pricing – part I

If a market has a risk neutral probability measure then it admits no arbitrage (Harrison and Pliska 1981). The opposite is true as well.

Theorem 2.3 : Fundamental theorem of asset pricing – part II

The risk neutral probability measure is unique if and only if every contingent claim can be hedged (Harrison and Pliska 1983).

These two theorems are important in a sense that the assumption of complete market guarantees us that the powerful tool of risk neutral probability measure is available. When modelling DCPS, one can easily end up with the incomplete market environment if for example the contributions have their own source of risk stemming from the member's uncertain salaries.

In this thesis we investigate discrete time models. However, a discrete time model can be deemed as an approximation of the continuous time one. The definitions and theorems given in this section are useful because we can appropriately position our models in the whole range of the papers investigating the similar problem such that we can compare our results with the relevant results of the other authors.

2.1.4 Some Features of Post-Retirement Period

When modelling DCPS in the post-retirement period, one has to consider the entire set of assumptions about a member's preferences and about the available post-retirement options.

Under annuitisation, members pool their assets and completely give up the control of their annuitized assets. Upon the death of the member, the residual assets are shared among surviving members. The amount transferred to surviving members of the pension plan is sometimes referred to as the survival credit or mortality drag (Blake 1996). Mortality drag is a positive feature for those who survive as it increases their rate of return and provides assets for those living longer than average. If part of the pension wealth is annuitised then no asset allocation and no additional trading of those assets is possible from the member's point of view.

Giving up one's assets in exchange for annuity means that no pension assets will be left behind to the heirs. Often, the member will wish to bequeath some of his assets. This is referred to as the bequest motive. In fact, there is evidence that very few people voluntarily annuitise all their assets (Mitchell and McCarthy (2002) and Finkelstein and Poterba (2000)). It is particularly true for those having large pension assets available and those who believe that they are in a worse health than an average member. The member's bequest motive could be an important determinant of the decumulation strategy.

The family can influence one's attitude towards risk as well as the bequest motive. For example, a married couple have the ability to pool their mortality risk and the decision to annuitise is less likely (Kotlikoff and Spivak (1981) and Brown and Poterba (2000)). The existence of income from other sources, such as social security, encourages the bequest motive as well (Brown (2001)).

Bodie, Merton and Samuelson (1992) investigate the possibility of a flexible retirement date. They investigate the effects of the labour–leisure choice on portfolio and consumption decisions over an individual’s lifecycle. A flexible time of retirement or allowing working part– or full–time in retirement can also change a member’s attitude towards risk and his optimal asset allocation and annuitisation strategy.

2.1.5 Risk Faced by the DCPS Member

We have already mentioned the individualisation of risks which happens when the pensioner is in DCPS and not in DBPS. Let us now consider some of the risks faced by the member of DCPS.

Two main risks in the pre–retirement period are the investment risk and the risk of inadequate contributions. The member bears the risk of the high volatility of return and lower than expected investment returns. The investment risk is particularly important a few years prior to the time of retirement, because not much time is left for asset prices to recover. Contribution rates can be inadequate but this fact is usually realised when it is too late to repair it. Unwanted contribution holidays caused by different reasons can diminish the retirement income significantly. Also, unwanted contribution holidays and poor return on investment can both happen at the same time. Namely, in years when the overall economy performs badly then both poor investment results as well as increased unemployment can be expected.

In the post–retirement period, risks largely depend on the chosen decumulation options.

If the retired member chooses to convert part of his pension wealth into an annuity, he bears the risk of unfavourable annuity prices, due to the lower than expected interest rate or due to the increase in the population’s longevity. Adverse selection among the annuitants also increases annuity prices. Inflation risk can be a very important source of uncertainty in retirement as the real value of the income in retirement can be significantly eroded by inflation. The member is also under the risk of not making use of his pension wealth if the annuitisation is soon followed by death and no bequest is arranged.

If the retired member chooses self investment of the significant part of his pension wealth, then investment risk remains important in the post-retirement period as well. Further, restricted investment opportunities, such as limited availability of index-linked bonds or legislative investment restrictions, can be a source of inadequate income in retirement. With self investment, the pensioner is generally under the risk of running out of pension assets, particularly if he underestimates his remaining lifetime.

During the whole period of DCPS, the member should be aware of the risk of high costs and profit margins in any financial arrangement, as well as possibly unfavourable taxation.

The properly chosen asset allocation and annuitisation strategies can decrease or eliminate some of these risks. However, for some risks, choosing the member's choice of the proper annuitisation strategy can be more important than the asset allocation strategy. Further, the criteria for properly chosen asset allocation and annuitisation strategies will not be related to the different risks to the same extent. Optimising to a certain criterion usually means handling one or more risks, but not all. So, we should always think of the optimal asset allocation and annuitisation strategies as dependent on the particular criterion or criteria. We can expect that different criteria will result in different optimal asset allocation and annuitisation strategies, which sometimes may even be contradictory.

2.1.6 Criteria for Optimality

Each optimal asset allocation strategy will be optimal according to the certain criterion. A number of criteria can be defined. The two commonly met criteria are:

- maximising member's expected utility, or in a mathematical form

$$\max_c E[U(f(t))],$$

- minimising member's expected disutility, or

$$\min_c E[D(f(t))],$$

where $f(t)$ is a function from which utility (disutility) is drawn and c is the control variable. The optimal value of the control variable is the one that provides maximised (minimised) expected utility (disutility).;

If we are optimising the pre-retirement period, the function $f(t)$ can be

- the final wealth or intertemporal wealth, or;
the pension target (pension ratio, the amount of annuity or some other target),
where the pension ratio PR is defined by

$$PR = \frac{\text{annual pension income}}{\text{final annual salary}}.$$

In the post-retirement period, the function $f(t)$ can be utility from consumption plus utility from the wealth bequeathed to heirs, for example

$$f = \int_0^T U_1(c_t, t) dt + U_2(W(T)),$$

where T is the time of death, U_1 is utility drawn from consumption in retirement and U_2 is utility drawn from the bequest.

A number of criteria are based on the maximising or minimising probabilities. Let again c be the control variable. The optimal value of the control variable is now the one that provides maximised (minimised) probability. For example, the problem can be defined

- maximising probability of reaching certain target, for example

$$\max_c P(W(T) \geq G_T),$$

where $W(T)$ is a pension wealth at the time of retirement and G_T is the minimum guarantee;

- minimizing probability of ruin in post-retirement period, for example

$$\min_c P\left(\inf_{0 \leq t \leq T} W(t) \leq 0 \mid W(0)\right),$$

where $W(0)$ is pension wealth at the time of retirement and T is random time of death;

- minimising the probability of shortfall below a certain lower bound given for example with the formula

$$\min_c P(W(t) \leq Tar(t))$$

where $W(t)$ is a pension wealth at time t and $Tar(t)$ is some target value that the member sees as a lower acceptable bound of his pension wealth.;

One can optimise asset allocation in the pre-retirement period in order to minimise the value at risk of a certain variable (pension ratio, pension wealth at retirement, etc.). In a mathematical form the problem can be defined for example as

$$\min_c (W_R), \text{ where}$$

$$P(W(T) \geq -W_R) = 1 - \varepsilon,$$

where $W(T)$ is pension wealth at retirement, W_R is value at risk calculated from the second equation. c is the control variable and the optimal value of the control variable is the one that provides minimal value at risk.

A number of other risk measures such as variance, semi-variance, can also be used, and optimal asset allocation and annuitisation can be determined according to these criteria.

Different criteria will probably result in different optimal asset allocation and annuitisation strategies. It opens a number of questions about the optimality and probably requires some subjective judgement before concluding which asset allocation and annuitisation strategies to apply.

In Chapter 1, we categorised the basic options for income from DCPS in retirement as: annuities, income drawdown and the combination of these two. Usually, each member will be eager to exercise some combination of income drawdown and annuities with particular amounts to be annuities and withdrawn from pension wealth. Each DCPS member will probably have his own preferences towards risk and bequest. Once the preferences of the pensioner are known, we can define the criteria for optimality. Only then we can try to find the optimal asset allocation and annuitisation strategy for that particular member.

2.1.7 Discrete Versus Continuous Time Models

Discrete time models for DCPS asset allocation and annuitisation can be deemed as the natural modelling approach for the pensioner's income and consumption. Indeed, the retirement income inflows and retirement consumption outflows from the pension

fund are usually made on a regular time basis, say monthly or annually, and one can assume that asset allocation and annuitisation strategies are examined and possibly improved on a regular time intervals. One can also expect that the required mathematical tools are simpler in a discrete time environment and obviously numerical solutions done using computers are expected to be more easily solved. However, in modelling optimal asset allocation and annuitisation, we have to model financial assets such as stocks and bonds. These assets do not stop changing its values during the regular and longer time intervals but quite opposite. There are many agents and the values of assets change at different and short time intervals and thus continuous time approach seems to be more appropriate approximation when modelling values of the assets.

The powerful tool of mathematical analysis in continuous time and the analytical results arising from continuous time models induce us to consider the continuous time approach as well. A great advantage of continuous time models is the possibility of an analytical solution to the problem. If we have an analytical solution to the problem then we can investigate this solution in many ways and get a clear idea about the changes of the solution when we change different parameters. If we use continuous time model then we can better model different volatility functions, we can better investigate possible heavy tail of the distribution. Strong dependence of the serial data can usually be modelled better using continuous time than discrete time approach. Also, continuous time models allow for modelling occasionally sudden but large jumps.

In the seminal papers, Merton (1969, 1971) set up the models for asset allocation strategy in the continuous time framework. These papers are reprinted in Merton (1990). He employs a general technique which has been widely used subsequently for developing intertemporal problems under uncertainty. Merton (1969) analyses the model with given initial wealth, and the case when income is generated by capital asset gains only. Asset prices are modelled using geometric Brownian motion process. Firstly, he examines a two assets problem then extends it to multi asset problem and finds the relevant optimality equations for consumption and for asset allocation. He particularly investigates constant relative risk aversion (CRRA) and constant absolute risk aversion (CARA) and finds explicit solutions for these cases. He also shows that, in the given framework and with many assets, if asset prices follow geometric Brownian motion, one can work with two assets only without a loss of generality. Merton (1971) further develops the model for more general utility functions, price

behaviour assumptions, and contributions into the fund from non-capital gains sources.

There is also a significant disadvantage to the continuous time models. Although there are many continuous time models in finance investigated and solved, we find that the solution of the continuous time model involving annuities and different constraints on investment and access to annuities becomes very hard or even mathematically intractable. For example, it is possible to have an analytical solution for a certain continuous time model with no constraints on short-selling, and introducing no borrowing constraint in the model can become mathematically extremely hard or even intractable problem. We can say that there is a mathematical barrier to the complexity of the model that can be solved in the continuous time framework.

On the other hand, discrete time models can be solved using computers. In recent years we have witnessed the fast development of computer hardware and software, and of parallel computing. Very powerful software, particularly for optimisation, has been developed as well. So, when we develop a discrete time model there are very powerful tools for obtaining a numerical solution. Even more, if we want to improve the model, for example to add certain constraints or to add annuities or one or more other variables, the improved version of the model still can be solvable. A shortcoming of the numerical solution on the computer is that we usually get one numerical solution for one choice of the values for each parameter. In order to get an idea about the solution for different values of the parameters, we need to get a number of solutions and to compare them numerically.

2.2 Models and Results in Pre-Retirement Period

The models for pension wealth development in the pre-retirement period are characterised by income from investment and from contribution, and possible outflow due to adverse investment results. Usually, no consumption of the pension wealth is allowed. The asset allocation strategy objective is to provide the appropriate wealth at the moment of retirement. The appropriate wealth at the moment of retirement means that the member will be in a position to obtain a satisfactory income in retirement from the accumulated pension wealth.

2.2.1 In Discrete Time

Knox (1993) develops a discrete time stochastic model for both inflation and a range of investment returns, and analyses different investment strategies and the distribution of retirement income that are to be obtained from DCPS. He stresses the importance of the investment performance of the fund. He recognises the risk return trade-off in investment, resulting in different levels of annuity income in retirement, where this income is expressed as a percentage of the final salary. Rather than suggesting the appropriate investment strategy, he suggests that each member should understand the risks of his particular DCPS.

The model for DCPS fund value developed by Ludvik (1994) incorporates the most important variables and develops a closed form formula for pension benefit as a fraction of the final salary. Pension benefit is modelled as an annuity after withdrawing a lump sum at retirement. Numerical investigations are done using Wilkie (1986) model. The Wilkie (1986) model is based on modelling financial variables using time series. Ludvik (1994) extends it to include time series for major investment classes and national average earnings. He investigates four investment strategies: 100% in equities, 100% in bonds, 100% in cash, and a deterministic lifestyle, which entails switching from 100% equities to 100% bonds over the last five years of accumulation phase. The criterion for comparing the strategies is downside volatility measuring floor level (the worst 5% percentile). He finds that bonds and cash are a superior strategy to the equity and deterministic lifestyle, although with a lower median.

Khorasanee (1995) examines the investment problem by analysing different investment strategies and comparing them. The models for annual investment returns are: iid log-normal random variables and a dividend yield model. He analyses the following investment strategies: investing the whole fund in equities, static investment strategy investing 75% and 50% in equities and the rest in index-linked bonds, and one-off switching to low risk asset close to retirement. He also investigates the use of derivative based investment products. The switching strategy is supported as appropriate for reducing investment risk associated with equities, particularly in a period close to the time of retirement. However, it is found that the equities are the most appropriate asset class for DCPS.

Booth and Yakoubov (2000) analyse deterministic lifestyle investment strategy close to retirement based on historical datasets. They use Wilkie's simulation model, where

parameters are determined by historical values from the available databases. A number of asset allocation strategies are analysed with respect to the post-retirement preferences towards the decumulation choice of the pension wealth. Funding is analysed for cash, for purchasing a fixed annuity and for purchasing an index-linked annuity at retirement. They find no evidence for supporting the superiority of a lifestyle investment strategy. However, they find strong evidence for supporting a well-diversified investment strategy until retirement rather than a one-off switch to low risk asset. They also conclude that the investment strategy close to retirement should be dependent on the required decumulation strategy.

Blake, Cairns and Dowd (2001) examine different models for investment returns on assets and different strategies for asset allocation in the accumulation phase of DCPS. They investigate the following asset allocation strategies: static asset allocation throughout the accumulation period, deterministic lifestyle strategy with gradual switch 10 years before retirement and stochastic dynamic lifestyle investment strategy. In the latter, randomness is involved via feedback control. The assumed decumulation in retirement is full annuitisation and the member's pension target is the pension ratio. The main criterion adopted in this paper is the Value-at-Risk (*VaR*). The formulae for different models are presented followed by a number of interesting numerical results based on historical values. They recognise that a DCPS can be risky compared with a DBPS. The high sensitivity of *VaR* estimates to the choice of asset allocation strategy is found. They find that the constant proportion asset allocation strategy with heavy investing in equities yields much better results than any dynamic strategy, including lifestyle investment strategy. Further, if the same retirement income is to be obtained, a bond based asset allocation strategy will require a considerably higher contribution rate compared with equity based investment strategy.

The dynamic programming approach in a discrete time framework is applied by Vigna (1999), Vigna and Haberman (2001), and Haberman and Vigna (2002). Vigna and Haberman (2001) investigate the model with the two assets, one low-risk and the other high-risk. The assets are modelled by assuming that annual asset returns are iid log-normally distributed, and that returns from different assets are uncorrelated. They develop a multi-period model for DCPS asset accumulation and determine the optimal investment strategy that minimises member's discounted future costs. The proposed total future cost at time t is of the form

$$G_t = \sum_{s=t}^N \gamma^{s-t} C(s),$$

where γ is intertemporal discount factor which can be seen as a psychological discount factor, and the “cost” at time t is defined as

$$C(t) = \theta_1 (f_t - F_t)^2 \text{ for } t = 1, 2, \dots, N-1, \text{ and}$$

$$C(N) = \theta_0 (f_N - F_N)^2,$$

where f_t is the fund value, and F_t is the target fund value. They allow θ_0 and θ_1 to be different, recognising the different weights given to the costs of the intertemporal fund deviations and the final fund deviation. They present the link in their approach with expected quadratic utility. The two assets model leads to conclusions that are supportive of the stochastic dynamic lifestyle strategy. They find that with low volatility it is optimal to invest in the high risk asset at the beginning of the accumulation period and then switch into the low risk asset once the fund value is close to the target. In the case of the higher volatility, it is optimal to diversify, with increasing diversification as volatility increases. The annuity risk, i.e. the risk of low conversion rate when purchasing annuity at retirement, is analysed as well. In the case of the pension ratio as a target, they find that the probability of failing the target tends to decrease as the time to retirement increase. Haberman and Vigna (2002) extend the previous model in three ways: by analysing n assets, introducing correlations between assets, and by improving the disutility function. The cost at time t is now defined as

$$C(t) = (F_t - f_t)^2 + \alpha (F_t - f_t) \text{ for } t = 0, 1, \dots, N-1, \text{ and}$$

$$C(N) = \theta \left[(F_N - f_N)^2 + \alpha (F_N - f_N) \right],$$

where $\alpha \geq 0$ and $\theta \geq 1$. Under these settings, the deviations of the fund above the target are not penalised to the same degree as the deviations below the target, and the risk profile of the individual is taken into consideration. They also analyse two new risk measures: the mean shortfall and the value-at-risk. These risk measures are introduced with respect to the net replacement ratio. The optimal asset allocation strategy for risk averse member is again the stochastic dynamic lifestyle strategy, and the time when the switch to low-risk asset begins depends on the risk aversion. They conclude that the optimal asset allocation into high-risk asset increases as α increases. The risk neutral member, i.e. the member whose $\alpha \rightarrow +\infty$, will not switch from riskier to lower risky assets according to this model. Haberman and Vigna (2002) find that different risk measures of the downside risk faced by the member of a DCPS give different and contradictory indications. The effect of changing the

correlation factor between assets does not appear to be of great significance. They suggest that the member's risk profile and the trade-off between different risk measures of the downside risks are important factors when defining the optimal asset allocation for DCPS.

2.2.2 In Continuous Time

Boulier, Trussant and Florens (1995) apply Merton's model in order to determine the optimal pre-retirement asset allocation. Although they concentrate on a defined benefit model, they stated the possible application for the defined contribution model. They set up the model with a constant risk free rate of return and one risky asset, and optimise the contributions and proportions invested into risky asset. An explicit solution is found for the case of the quadratic loss function. Further developments in the same directions are done by Siegmann and Lucas (1999). They calculate optimal policies for a loss function with constant relative risk aversion as well as one with constant absolute risk aversion.

Boulier, Huang and Taillard (2001) set up the model for DCPS where the guarantee in the form of the minimal fund value is given on the benefit. The rate of interest is modelled using the Vasicek framework, and the guarantee is a bond like liability. They assume two sources of randomness: one from the interest rate and the other from the stock itself. The assets available for investments are cash, bonds with the constant time to maturity and stocks. The rate of contribution is assumed to follow a simple exponential function. They maximise the expected utility of the excess of the fund over guarantee, where CRRA utility function is taken. In order to end up with a solvable maximisation problem, they make two important modifications to the model. Firstly, a loan corresponding to the contributions is taken and put into the fund at the outset. The loan is repaid by contributions. In other words, they construct the bond portfolio that replicates the future contributions. The discounted future contributions at time t are defined by

$$D(t) = \int_t^T c(s) B(t, s) ds ,$$

where T is time of retirement, $c(s)$ is the rate of contribution at time s , and $B(t, s)$ is the price at time t of the bond paying 1 at time s . The guarantee is also replicated by bonds, and its value at time t is

$$G(t) = \int_T^{T'} f(s) B(t, s) ds,$$

where T' is the time of death, $f(s)$ is the rate of pension income, and T and $B(t, s)$ as already defined. Using this technique, they define the equivalent problem where the contributions are converted into the initial wealth with no further contributions and the minimum guarantee of the pension wealth is expressed as a constraint on the fund being positive. The explicit solution of the optimal asset allocation in the pre-retirement period is found. Numerical application of the derived solution shows that although the amount of the stock investment increases smoothly, its proportion in the pension fund declines. They also find that the member will have a short position in cash until just a few years prior to retirement, and that the proportion invested in bonds would decline first slowly and then sharply a few years prior to retirement.

Deelstra, Grasselli and Koehl (2000) investigate optimal investment problem with initial wealth only and no further contribution, and where the stochastic interest rate follows the Cox–Ingersoll–Ross model. They explicitly expressed the asset allocation strategy which maximises the expected utility of the terminal wealth. They use the Cox, Huang (1989) methodology and find the explicit solution in the form of optimal proportions that should be invested in each asset in order to maximise CRRA utility drawn from the final wealth. The maximisation problem in this paper is closely related to the modified maximisation problem stated and solved by Boulier et al (2001). The difference is the model for stochastic interest rate, with the CIR framework probably being less easy to manage.

In related papers, Deelstra, Grasselli and Koehl (2002) and Deelstra, Grasselli and Koehl (2003) exploit their model and results from Deelstra et al (2000), now in the continuous time framework of the accumulation period for DCPS. Deelstra et al (2003) tackle the problem of optimal asset allocation in order to maximise the expected utility of the excess of the terminal wealth over the minimum guarantee. They assume the complete market, investing in cash, bonds and stock, CRRA utility function, and affine dynamics of the stochastic interest rate. An explicit solution of the optimal asset allocation is found under the assumption that a contribution process and the guarantee are not subject to its own sources of risk. The results include Vasicek as well as CIR stochastic interest rate models as special cases. Applying the model from Deelstra et al (2000), Deelstra et al (2002) move in the direction of obtaining the optimal guarantee that maximises the expected utility function of the benefit in DCPS.

The detailed derivation from simple to more general DCPS asset accumulation model and analysis of the resulting formulae is done by Cairns, Blake and Dowd (2004). The simple model includes one risk free asset and one risky asset, stochastic salary, and two cases of contribution, a single premium with no subsequent contribution and no initial wealth with contributions as a constant proportion of the salary. CRRA utility function is employed and they maximise the expected utility of the final wealth over the final salary. The terminal utility function is assumed to be of the form

$$u(W(T), Y(T)) = \begin{cases} \frac{1}{\gamma} \left(\frac{W(T)}{Y(T)} \right)^\gamma, & \text{where } \gamma < 1 \text{ and } \gamma \neq 0 \\ \log \left(\frac{W(T)}{Y(T)} \right), & \text{when } \gamma = 0. \end{cases} \quad (2.2)$$

where T is the time of retirement, $W(T)$ is the pension wealth at retirement and $Y(T)$ is the final salary. They find that, in the case of investing initial wealth with no further contributions, the optimal asset allocation is a constant and does not depend on the salary related risk. In this case, the member can do nothing but accept unhedgeable volatility of the salary. In the case of a regular contribution and hedgeable future salaries, the explicit optimal asset allocation is the lifestyle strategy. With unhedgeable salaries, the lifestyle strategy is still favourable. The analysis of the cost of sub-optimality shows that the member with a low degree of risk aversion can have a substantial cost of sub-optimality. The more general model includes a stochastic rate of interest, n risky assets, and introduction of the replacement ratio as the argument of utility function. The terminal utility is now either of the form (2.2) or in the case of drawing utility from the replacement ratio

$$u \left(\frac{W(T)}{a(T, r(T))}, Y(T) \right), \text{ where}$$

$a(t, r(t))$ is the market rate for life annuities. They show that the optimal asset allocation consists of three funds: one hedging salary risk, one hedging annuity risk and one satisfying member's risk appetite. They develop a partial differential equation for the case of hedgeable salaries, solve it when the rate of interest follows the Vasicek model and analyse it numerically. Again, support for the stochastic lifestyle strategy is found. However, optimal asset allocation requires cash borrowing that can be impractical in the real world. In the stochastic interest rate environment, the interest rate risk to the future annuity is present regardless of risk aversion.

The model for DCPS asset allocation with the stochastic salary and stochastic inflation is developed by Battocchio and Menoncin (2004). They assume the Vasicek interest rate model. The financial market consists of one riskless asset, one rolling bond, and one stock. The salary has its own source of risk. The risks from the interest rate, the stock and the salary are also the only sources of risk for inflation. The member's objective is to choose the asset allocation strategy in order to maximise the expected value of the terminal utility of the fund's real value. In the market structure with inflation, riskless asset becomes another risky asset. However, the market completeness is maintained with three risky assets. The optimal portfolio is represented as a sum of: preference-free hedging component, speculative component depending on portfolio Sharpe ratio and the inverse of the risk aversion index, and the hedging component depending on the state variable parameters. The closed form solution of the optimal asset allocation strategy is presented for the exponential (CARA) utility function. Using a numerical simulation, they show that the weights of stock and bond decrease with time, while the proportion of the riskless asset increases.

Many authors, including some of those referenced here, use the trick of adding the present value of future contributions and subtracting the present value of the future consumptions and/or guarantees at the time of retirement. It brings us to the models based on Merton (1969, 1971) with zero consumption. This type of problem is analysed more widely, and we can reference a number of papers with different designs and results about this type of model (e.g., Liu (2007), Korn and Krekel (2002)). However, the strategy of discounted future contribution depends on the possibility to replicate contributions with the available assets. Even in the complete markets, the mathematics can become very complex when applying this approach.

2.3 Models and Results in Post-Retirement Period

Post-retirement asset allocation strongly depends on the member's choice of how to spend his available pension wealth. We can say that the most important member's requirement is a safe, lifelong income stream providing him with a "reasonable" lifestyle in retirement. The proper asset allocation and annuitisation strategy will be the one that suits the above stated needs in the best way, given the initial pension wealth and overall conditions in a given market and population. Let us emphasise that in this thesis, we are primarily interested in the member's decisions on asset allocation and annuitisation strategies. By definition, annuitisation means paying a non refundable lump sum, i.e. giving up assets, in exchange for a guaranteed lifelong

income. Thus, annuitised assets are not subject to the member's asset allocation strategy any longer. However, the time of annuitisation and the possible partial or phased annuitisation will influence the optimal strategy for the allocation of the available assets. The question of optimal asset allocation strategy in the post-retirement period will be interesting for our investigation only if the member decides to keep control of part or all of his pension wealth after retirement.

2.3.1 In Discrete Time

The risk of outliving one's money, i.e. the risk of ruin in retirement as one of the most important post-retirement risks, is analysed by Milevsky, Ho and Robinson (1997). They develop a discrete time stochastic model for post-retirement wealth and use it to determine the optimal asset allocation, where optimisation is done such that the probability of ruin is minimised. They assume a random rate of return, a fixed initial pension wealth and desired level of consumption. The model is supported by empirical values and numerical results are presented. In the bond and equity portfolio, they found the optimal asset allocation to be 70–100% in equities. The member's risk of outliving his money is generally surprisingly high, particularly for low risk, low return investment policy. They find that retirees should consider their desired consumption, existing wealth, age and gender, before deciding how to allocate available assets. Their analysis shows that women face significantly greater risk of ruin in retirement than men.

Milevsky (1998) develops a model where the retiree defers annuitisation as long as he can to obtain a better rate of return from an investment than from a life annuity. He allows the rate of return on a life annuity to be the subject of mortality drag and cost and profit loading. The development of the discrete time model is followed by a stochastic continuous time model. Assuming annual consumption in the amount of the available annuity and a continuously compounded rate of return on the pension wealth δ , he finds that the member will run out of money at time t^* given by

$$t^* = \begin{cases} -\delta^{-1} \ln(1 - a_x \delta), & \text{for all } \delta < (a_x)^{-1} \\ \infty, & \text{for all } \delta \geq (a_x)^{-1} \end{cases},$$

showing that the member can safely beat the annuity if $\delta \geq (a_x)^{-1}$. The probability of successful deferral is analysed. Although no optimal asset allocation is developed, the model and criterion in the paper can be used for optimal asset allocation analysis. In a

closely related paper, Milevsky (2000) uses the same model for further analysis of the same question. He concludes that annuitisation of assets provides a unique and valuable longevity insurance and should be encouraged at the higher ages. It is pointed out that an adverse selection in life annuities acts as a deterrent to full annuitisation. Retirees with a (strong) bequest motive might be inclined to self annuitise during the early stages of retirement. Focusing on the strategy “consume term and invest the difference”, he finds that the pensioner can successfully defer annuitisation up to age 75–80.

Self-annuitisation and probability of consumption shortfall with respect to German insurance and stock market is analysed by Albrecht and Maurer (2002). They allow investment in three assets: stocks, bonds and real estate. They find that the self annuitisation strategy bears a substantial risk of outliving one’s wealth, providing that the amounts to be withdrawn every year are equal to the level annuity available on the insurance market. This appears to be particularly true for older members. Again, no optimal asset allocation strategy is found, but different asset allocation strategies are analysed and the importance and the superiority of the proper asset allocation is shown.

Mitchell, Poterba, Warshawsky and Brown (1999) investigate the market for annuities in the United States and the reasons why that market has historically been small. They find that the prices charged for a single premium immediate life annuity vary widely, that the effective transaction costs to participating in the individual annuity market have declined during this period, and that the specialised income tax liabilities that are associated with annuity income does not significantly affect the expected present discounted value of annuity payouts. They compute the expected utility that a consumer with random lifetime and an additively separable utility function would derive from following an optimal intertemporal consumption plan in the absence of an annuity market, and the same individual’s utility if he can purchase an actuarially fair nominal annuity.

Horneff, Maurer, Mitchell and Dus (2008) use a utility framework to compare the value of purchasing a standalone life annuity, versus a number of phased withdrawal strategies backed by a properly diversified investment portfolio, as well as combinations of these two tactics. They show that the appropriate mix depends on the retiree’s attitudes toward risk as well as the underlying economic and demographic assumptions. Then they compare standalone withdrawal rules versus immediate annuitisation of the entire portfolio. Consistent with previous studies, they show that

annuities are attractive as a standalone product when the retiree has sufficiently high risk aversion and lacks a bequest motive. Withdrawal plans dominate annuities for low/moderate risk preferences, because the retiree can gain by investing in the capital market and from “betting on death”. Finally, they examine combination/mixed strategies where retirees may both invest some of their assets and also purchase a payout annuity. In the case where the annuitisation decision occurs at the point of retirement, they find that annuities become appealing for those with moderate risk aversion, when retirees can hold both annuities and phased withdrawal plans as a mixed strategy. Withdrawal plans are now attractive for highly risk averse retirees.

Brown (2001) examines household decisions about whether or not to annuitise retirement sources. A lifecycle model of consumption, implemented with the use of dynamic programming techniques, is used to construct a utility based measure of annuity value for individuals and couples. He develops annuity equivalent wealth as a measure which essentially captures the maximum mark-up over the actuarially fair cost that an individual would be willing to pay. He finds that one percentage point increase in annuity equivalent measure corresponds to a one percentage point increase in the probability of planning to annuitise. Marital status appears to be a particularly important source of underlying variation in the annuity equivalent wealth measure and annuity decision and that the ability of the simple lifecycle model to predict annuity behaviour is the strongest among individuals. He recognises the existence of both lifecycleers and “myopes” in the population. He also casts doubt on the importance of the bequest motives in influencing annuity decisions.

Blake, Cairns and Dowd (2003) thoroughly analyse and compare three types of decumulation plans: purchased life annuity, equity-linked annuity with a level life annuity purchased at age 75, and equity-linked income-drawdown with a level life annuity purchased at age 75. The latter two plans are considered with equity exposure in the managed fund: 0%, 25%, 50%, 75% and 100%. The optimal retirement program among those proposed is the one which maximises the value function, where the value function is given by

$$V(s, f) = E \left[\sum_{t=s}^K e^{-\beta t} J_1(P(t)) + k_2 e^{-\beta(K+1)} J_2(D(K+1)) \mid F(s) = f, \text{ alive at } s \right],$$

where K is the curtate future lifetime, β measures the member’s subjective rate of time preference, k_2 specifying the desire for income and desire to make a bequest, $F(s)$ is pension fund value at time s , J_1 is utility drawn from the pension income and J_2 is utility drawn from a bequest. They find that for the central values chosen for the

bequest function, the best program does not usually involve a bequest. The best program is the one paying survival credits to the member. The optimal choice of a distribution program is not found to be sensitive to the member's weight attached to the bequest. The equity proportion chosen for the distribution programme has a considerably more important effect on the plan member's welfare than the distribution programme chosen, and a poor choice can lead to substantially reduced expected discounted utility. The optimal annuitisation age is found to be very sensitive to the plan member's degree of risk aversion, sensitive to the bequest motive and dependent on fund size.

Horneff, Maurer and Stamos (2008a) compute the optimal dynamic annuitisation and asset allocation policy for a retiree with Epstein–Zin (EIS) preferences as defined in Epstein and Zin (1989), uncertain investment horizon, a potential bequest motives, and pre-existing pension income. In their setting, the retiree can decide each year how much he consumes and how much he invests in stocks, bonds, and life annuities. The gradual annuitisation refers to the intertemporal asset allocation problem of equity, bonds, and life annuities in a setting, in which the annuities purchased to date provide constant payments for the retiree's remaining lifetime. The partial switch limits the freedom of choice given in the gradual annuitisation strategy. The partial switch restriction urges the retiree to purchase annuities only once, but it also gives him the freedom to decide when to switch and how much wealth to shift into annuities. The third possible annuitisation strategy is a complete switch, where no investments into stocks and bonds are allowed. They show that postponing the annuity purchase is no longer optimal in the gradual annuitisation case since investors are able to attain the optimal mix between liquid assets (stocks and bonds) and illiquid life annuities each year. In order to assess potential utility losses, they benchmark various restricted annuitisation strategies against the unrestricted gradual annuitisation strategy. Taking into account a reasonable parameterization of the asset model (e.g. risk free rate, magnitude of the equity premium, volatility of stocks, and cost structure of life annuities), their numerical assessment indicates for a moderately risk averse and endowed retiree that complete and partial switch restrictions cause the annuitisation age to be postponed for 10–15 years after retirement when annuities offer a higher mortality credit, or put differently, when annuities become cheaper. Although the partial switch strategy is less restrictive than the complete switch strategy, the partial switch occurs only slightly earlier than the complete switch tactic. The reason is that it is optimal to annuitise a high fraction of wealth later in life due to the increased mortality credit. Switching restrictions do not only cause annuitisation to be deferred but they can also reduce annuitisation. This is particularly severe in the cases with a

bequest: annuities are never purchased if complete switch restrictions are present. In contrast to the switching cases, the retiree already starts to purchase annuities at the beginning of the retirement phase, the age of 65, if he is allowed to follow the gradual annuitisation strategy. In the base case, he invests 30 percent of his wealth in annuities while keeping the remainder fully invested in stocks. Doing so, the investor is able to attain both the mortality credit of annuities and the equity premium of stocks. After the age of 65, he continues to repurchase annuities until full annuitisation is reached at about the age of 78. This is true for all RRA and EIS specifications considered, if no bequest motive is present. Welfare analysis evaluates the utility costs of restricted annuitisation and decumulation strategies compared to the case where gradual annuitisation is possible. This analysis is conducted for various degrees of RRA coefficients, EIS coefficients, the bequest motives, and initial endowments. For the CRRA case, we find utility losses of up to 30 percent of financial wealth if the retiree is forced to completely annuitise his wealth at the beginning of retirement. The utility loss is of a similar magnitude, if the retiree can only invest in stocks and bonds and has no access to annuities at all. They also conduct the welfare analysis for Epstein–Zin preferences in order to disentangle the implications of varying the RRA and EIS coefficients. If the EIS coefficient is relatively low, utility losses are below those of the CRRA case for all restricted strategies considered.

2.3.2 In Continuous Time

The idea of continuously distributed time of death, continuous evolution and adjustments of the portfolio of the pension wealth, and continuous income and consumption rates, lead us to the continuous time models for post-retirement asset allocation and annuitisation in DCPS.

The application of Merton's (1969) model for the maximisation of the utility of the retirement income, when the DCPS member can invest in equities and annuity is examined by Kapur and Orszag (1999). They set up the model as if an annuity is an investment although an annuity is not a tradable asset. They exclude bond investment after retirement because annuities are superior to bonds due to the mortality drag. They find that that in the case of a no bequest motive the optimal decision depends on risk aversion but that all individuals switch into annuities as they get older. When mortality drag is large enough, annuity investment is superior to the equities as well. In this framework and based on UK data, they argue that annuitising pension wealth is not optimal before age 80 for males and even later for females.

The member's right, but not the obligation to annuitise, as an option, is analysed by Milevsky and Young (2007). They analyse the optimal annuitisation and investment and consumption for utility maximising retiree facing a stochastic time of death under a variety of institutional pension and annuity arrangements. The model consists of a risk free rate of interest, one risky asset modelled via GBM, and annuity purchasing process. They firstly analyse the case where self investment, consumption and only full annuitisation at one distinct point of time is allowed. They argue that in this framework, the member's option to delay annuitisation has substantial value at younger ages. In their model, they distinguish the subjective and objective probability of survival. The insurance company calculates annuity rates using the objective probability of death. The value function in the model is defined by

$$U(w, t, T) = \sup_{\{c_s, \pi_s\}} E \left[\int_t^T e^{-r(s-t)} p_{x+t}^S u(c_s) ds + e^{-r(s-t)} p_{x+t}^S u\left(\frac{W_T}{\bar{a}_{x+T}^O}\right) \bar{a}_{x+T}^S \mid W_T = w \right],$$

where c_s is consumption prior annuitisation, π_s is the optimal asset allocation strategy prior annuitisation, T is the time of annuitisation, $u(x)$ is utility drawn from consumption, and superscripts S and O denote subjective and objective probabilities of death. They assume $u(x)$ to be a CRRA utility function. It is shown that the optimal annuitisation time is independent of one's wealth and can be regarded as some fixed time in the future. They find that in the case of equal subjective and objective force of mortality, the optimal age to annuitise is when the instantaneous force of mortality $\lambda_x^O = \lambda_x^S$ exceeds $(\mu - r)^2 / 2\sigma^2\gamma$. If the subjective force of mortality is different from the objective one, no matter higher or lower, then the optimal time of annuitisation increases. Using historical market parameters and realistic mortality estimates, they conclude that in this framework annuitisation is not optimal before at least age 70. They also investigate the case when annuitising in small amounts is allowed. In this case, they find that the member should obtain some basic level of annuities and then keep investing the rest at least until the wealth to income ratio exceeds a certain level. According to their results this would not occur before age 70.

A similar model and question as Milevsky and Young (2007) are analysed by Stabile (2006). For a general form of the utility function and for the level annuity, he characterises the optimal rules, and does not provide an explicit solution. However, for a CRRA utility function, the explicit value function as well as optimal consumption, optimal asset allocation and the optimal time of annuitisation are presented. Stabile (2006) provides a number of useful equations.

Gerrard, Haberman and Vigna (2004) investigate DCPS member's optimal investment strategy during the income drawdown period until the time of compulsory annuitisation. The pensioner invests the money in a typical Merton (1969) financial market. They assume a constant riskless rate of return, one risky asset whose price follows GBM, constant consumption per unit of time, and no bequest motive. It is assumed that the member wishes to minimise his expected disutility of quadratic loss function

$$L(t, X(t)) = (F(t) - X(t))^2, \text{ and} \\ K(T, X(T)) = \varepsilon (F(T) - X(T))^2$$

where $X(t)$ is the fund value at time t , and $F(t)$ is the target value of the fund that the member wishes to achieve at time t , and ε is constant. This choice of utility function implies the dependence of the asset allocation on the pension wealth. It is claimed to be a desirable feature in the context of the pension related problems. They aim to find the optimal asset allocation such that the value function

$$E_{0, x_0} \left[\int_0^T e^{-\rho s} L(s, X(s)) ds + e^{-\rho T} K(T, X(T)) \right],$$

is minimised, where ρ is subjective intertemporal discount factor, x_0 is pension wealth at retirement, and T is the time of annuitisation. The problem is explicitly solved for the certain targets in finite time and for a constant target in infinite time. Further analysis of the optimal asset allocation is done using a Monte Carlo simulation. They investigate the probability of ruin, the average time of ruin, given that ruin occurs, the probability of reaching the target (e.g. the desired level of annuity) at time T , the distribution of the annuity that can be bought at time T , compared to the target pursued, how the risk attitude of the individual can affect optimal choices and final results. The main conclusion is that for the member with not too high risk aversion, the income drawdown option should be preferred to immediate annuitisation, adopting an optimal asset allocation strategy with a sufficiently good (in terms of its risk-reward characteristics) risky asset. They find a relatively high probability of being worse off when adopting income drawdown for the member with the high risk aversion or for the member who aims at a target pension which is not too much higher than one he could receive with immediate annuitisation. In the extreme framework with no risky assets, they find the support for Yaari (1965) results that the immediate annuitisation seems to be optimal.

In a related paper, Gerrard, Haberman and Vigna (2003) use a somewhat similar framework to investigate income drawdown first in the case of no mortality and then include mortality and a bequest as well. However, now they penalise the variation of the consumption relative to the individual's ideal level of the income. They aim to minimise the consumer's expected discounted future loss

$$V_0(t, x) = \inf_{b(\cdot), y(\cdot) \in \mathbb{R}} E \left[\int_t^T e^{-\rho u} (b_1 - b(u))^2 du + \Psi(X(T)) \middle| X(t) = x \right],$$

where b_1 is the individual's ideal level of consumption, T being the time of the compulsory annuitisation, $\Psi(X(t))$ represent member's disutility after annuitisation. The optimisation takes place in variables $b(u)$ being the consumption in retirement and in the asset allocation strategy. In the case with no mortality, the effect of imposing the restrictions on the income drawdown and proportions of the fund invested in a risky asset is investigated and certain directions for tackling the problem are discussed. In the case that includes mortality and bequest, they introduce the idea of the double state variable, $(X(t), I(t))$, where $I(t)$ takes value 1 if the member is still alive and value 0 otherwise. They solve this type of stochastic optimal control problem, providing the proof of the verification theorem as well. This result is based on the work by Steffensen (2001). The authors believe that this approach of the double state variable stochastic optimal control problem is original in the actuarial literature. Stochastic simulations are carried out in order to investigate the behaviour of the results, such as the sensitivity of the optimal choices to the weight given to the bequest motive. Also, investigation is done on the probability of ruin. The results which they obtain show that inclusion of mortality in the model tends to decrease the volatility of the growth of the pension fund, of the optimal asset allocation and consumption. Including mortality in the model significantly decreases borrowing money from the bank as the optimal investment strategy. Further results lead to the conclusion that the ability to bequeath wealth affects only the riskiness of the optimal asset allocation, which seems to increase slightly with the increase of the bequest motive.

2.4 Lifecycle Models and Results

If we optimise asset allocation in order to maximise member's utility at the end of the accumulation period, drawn from the post-retirement consumption, we in fact take into account post-retirement asset allocation and possibly annuitisation. For example, maximising the replacement ratio is a form of a model that involves both pre-

retirement and post-retirement asset allocation. Here one makes the implicit assumption that the member plans to invest pension wealth into the annuity. The models presented so far in this thesis investigate only one period, either accumulation or decumulation period.

Let us firstly present two papers which actually have interdependency between two periods and their influence on the optimal asset allocation strategy are important characteristic of the model. Although these models are not directly related to the following chapters in this thesis, the way they develop the models is relevant and it is also interesting to see the definitions and the use of other possible criteria in retirement.

Arts and Vigna (2003) develop the model in a discrete time framework, with one low risk (bonds) and one high risk (equities) asset, the time of retirement is fixed, and the member chooses the decumulation option depending on the investment results. They assume that the member first starts investing contributions into the equity fund only, and when a “switch contribution” criterion depending on experienced investment results is satisfied he invests all future contribution into the bond fund. When the “switch funds” criterion is satisfied, also depending on the investment results, the member switches the equity fund into bonds. They investigate the best time to start investing contributions into bonds and the best time to switch the equity fund into the bonds. The switching criterion is to reach certain target fund values. “Switch contribution” can obviously happen in the accumulation period only, while “switch funds” can happen after “switch contribution”, either in the accumulation or in the decumulation period. At the time of retirement, there is the choice between income drawdown and fixed real annuity depending on if “switch fund” occurs before retirement. A number of numerical results are presented. They conclude that it seems to be important to consider both the period before and after retirement since an income drawdown option is available. They do not find equity investment to be risky in their framework and do not find strong support for the lifestyle investment strategy.

Another attempt in setting up the DCPS model that takes into account both periods is done by Lachance (2004). She investigates the optimal consumption and portfolio choice where the retirement date is adjustable as a function of market return. An adjustable retirement date means that the duration of the accumulation period can be appropriately adjusted, i.e. that labour supply is flexible. The idea of labour flexibility is developed by Bodie, Merton and Samuelson (1992), and empirically supported in EBRI’s Retirement Confidence Survey (2003). Lachance (2004) maximises expected

utility for pre-retirement and for post-retirement period, and retirement is efficient when retiring yields as least as much utility as continuing to work. She finds that the introduction of labour flexibility into the model results in a higher proportion of wealth invested in risky asset. She also finds that eagerness to retire motivates these workers to take more investment risk prior to retirement and that the incentive disappears upon retirement. She suggests that investment risk influences retirement security only as long as the worker is unable to adjust her labour supply in response to market shocks.

Chai, Horneff, Maurer and Mitchell (2009) derive optimal lifecycle portfolio asset allocations as well as annuity purchases trajectories for a consumer who can select his hours of work and also his retirement age. Using a realistically calibrated model with stochastic mortality and uncertain labour income, they extend the investment universe to include not only stocks and bonds, but also survival contingent payout annuities. Making labour supply endogenous raises older persons' equity share and substantially increases work effort of the young; it also affords significant lifetime welfare gains of 7% or more than 60% of first year earnings. Introducing annuities then generates even more realistic models which permit earlier retirement and higher participation by the elderly in financial markets.

Although we aim to work with a constant time of retirement, it is interesting to see that the papers investigating time of retirement in a different framework lead to qualitatively new results for the optimal asset allocation and annuitisation.

In this thesis, we investigate the post-retirement period only. The models involving the whole lifecycle can be used for getting the ideas and results about post-retirement asset allocation and annuitisation because in many models of this type the results are developed backwards year by year.

Charupat and Milevsky (2002) derive the optimal asset allocation that maximises utility from pension wealth at retirement and utility from consumption in retirement. They observe two separate models for the accumulation phase and for the decumulation phase, and then compare optimal asset allocation strategies. For the accumulation period, they follow the Merton (1971) model and find an optimal asset allocation for CRRA utility function. In the post-retirement period, they assume the similar economy of one riskless asset, one risky asset, and a CRRA utility function. The member can annuitise his pension wealth by a level annuity or by a variable annuity. The level annuity is backed by the riskless asset, and the variable annuity by the risky asset. They analyse two separate cases: one with the assumption of the

exponential distribution of death and the other under the assumption of the Gompertz mortality law. They find that the optimal proportion is the same for these two cases. Further, this optimal asset allocation is contrasted with its counterpart in the accumulation phase. They show that the optimal asset allocation remains the same upon transition to the payout phase. Although accumulation and decumulation periods are independently analysed, the results show that in this particular framework optimal asset allocation is constant and does not change throughout the whole duration of the lifecycle.

Cocco, Gomes and Maenhout (2005) develop a lifecycle model of consumption and portfolio choice with non tradable uncertain labour income and borrowing constraints. They assume CRRA utility function and one risk free and one risky asset and also allow for the presence of the bequest motive of the member. They calibrate the model realistically and analyse a number of realistic labour income possibilities. Given the quantitative focus of the article, they investigate what can reduce the average allocation to stocks and thus bring the empirical predictions of the model closer to what is observed in the data. They give a number of results regarding optimal asset allocation and optimal consumption depending on many different changes in the model set up. In terms of the lifecycle pattern of optimal asset allocation, the share invested in equities is roughly decreasing with age. With an increase in age, labour income becomes less important and the investor reacts optimally to this by shifting his financial portfolio towards the risk free asset. There is no annuity option in this model, but they realistically model pension fund, income and optimal consumption and asset allocation in post-retirement period.

Horneff and Maurer (2009) develop the model for the investor who can purchase and sell stocks, bonds, money market investments, and mortality contingent claims continuously. They assume a risky stock market and the Vasicek model for the interest rate, and stochastic wages. In order to maximise utility drawn from future consumptions and possibly bequest, the investor consumes optimally, and optimally purchases and sells stock, bonds, cash and life insurance. The investor can have a long or short position in the term life insurance, which is not realistic but can be deemed as a simplification of a reverse mortgage. They find that the demand for insurance products is growing with age. Then the investor will have more demand for annuities if his wealth is higher and the human capital is lower. They find a considerably small influence of the short rate on the demand of life insurance.

Horneff, Maurer, Mitchell and Stamos (2009) and Horneff, Maurer, and Stamos (2008) are two similar papers. Basically, the authors use the same models, with the difference that in Horneff et al (2009) they assume that the investor has access to variable annuities and in Horneff et al (2008) they assume access to the constant real payout lifetime annuities. Other assumptions are almost the same and we will concentrate on Horneff et al (2008) as it is more relevant to the work in this thesis. They observe the investor over lifecycle facing uninsurable income risk, ruin risk, equity investment risk and uncertain lifetime. They introduce an incomplete annuity market into the lifecycle model assuming that the investor has access to annuities anytime during his lifetime. The investor can convert his available assets into one risky, one riskless asset, and into annuities. Each year, he optimally chooses the allocation into equities, bonds, annuities and optimally chooses consumption. The investor has subjective survival probabilities, while annuities are calculated using objective survival probabilities. He aims to maximise his discounted utility drawn from future consumption and bequest, if a bequest motive is present. They use Epstein–Zin preferences as in Epstein and Zin (1989). For welfare analysis, Horneff et al (2008) compute the additional constant lifelong income (as a fraction of average labour income) an individual without access to annuity markets would need in order to attain the same expected utility as in the case with annuity markets, while Horneff et al (2009) calculate for a certain age the expected equivalent increase in financial wealth needed to compensate an individual lacking access to annuity products. The model for income and parameterisation is mostly the same as the ones used by Cocco et al (2005). For the sake of convenience, they use Gompertz law for mortality. Each year investment into risky, riskless assets and annuities is constrained to be nonnegative. Due to untradeable labour income, the irreversibility of annuity purchases and the short selling restrictions, the problem cannot be solved analytically, and they adopt the standard approach of dynamic stochastic programming to solve the investor's optimisation problem. They find that over time the annuity demand increases (age effect) for the following reasons. The mortality credit of annuities, the excess return above the bond return, increases with age. The sinking value of human capital results in a lower stock demand, as human capital is perceived as a closer substitute to a bond investment than to equity. Liquidity is also required to rebalance the portfolio. The demand for annuities also increases with the level of wealth on hand (wealth effect) because the investor does not require a high stock position in financial wealth in order to compensate for the investment in bond like human capital. In addition, the higher is the wealth in hand, the lower is the need for liquidity. Their welfare analysis reveals that loads, poor health, public pensions, and the bequest motives clearly reduce the willingness to participate in annuity markets. However,

none of them can really explain limited annuitisation in the market. Utility gains from purchasing annuities are still substantial. They suggest that behavioural factors might explain the remaining part of the “annuity puzzle”. This relates back to a bigger literature (Mitchell et al (1999) and Brown (2001)).

2.5 Our Position in Literature

The two main articles used as a starting point for the development of the models in this thesis are the models developed by Cocco, Gomes and Maenhout (2005) and Horneff, Maurer and Stamos (2008). These authors investigate the lifecycle model. We develop the models for the pensioner, retiring at age 65 with an uncertain and limited life time. If we observe the model investigated by Cocco et al (2005), for the individual age 65 and above, and if this individual has access to annuities, then we get to the starting point for the models in this thesis. Also, we observe the model investigated by Horneff, Maurer and Stamos (2008), for the individual age 65 and above, and if this individual has constant relative risk aversion utility function then we again get to the starting point for the models in this thesis.

In Chapter 3, we develop this model further by introducing nominal annuities and stochastic inflation. Inflation has been addressed by Brennan et al (2002) and Battocchio et al (2004). However, we develop a discrete time and state spaces stochastic inflation model based on the work of Wilkie (1986, 1995). We introduce the stochastic inflation model into the framework of a post-retirement model for the pensioner making optimal decisions regarding optimal consumption, optimal asset allocation, and optimal nominal and real annuitisation in order to maximise expected discounted utility derived from consumption and bequest in retirement. The idea of the individual’s maximising expected discounted utility derived from consumption and bequest have been investigated in the seminal papers by Merton (1969) and (1971).

In Chapter 4, we improve the basic model described above by introducing stochastic interest rate and annuities. Stochastic interest rate has been addressed by many authors and we have relied on the work of Boulier et al (2001) and Deelstra et al (2000). We actually make discrete time and space approximation of the bond market developed by Boulier et al (2001), and similar reasoning could be applied to the work of Deelstra et al (2000). We derive a discrete time and space stochastic interest rate model and

develop the bond market and derive the model such that the pensioner has access to three assets and annuities. The annuity rate is defined according to the bond prices.

In order to measure the pensioner's welfare, we apply constant equivalent measures used for example by Cocco et al (2005) and also the required equivalent wealth measure as used by Mitchell et al (1999) and Horneff, Maurer and Stamos (2008).

Chapter 3

The Inflation Risk Model

3.1 The Problem to be Solved

In this chapter, we model two assets world with the presence of inflation. We are interested in the post-retirement period of the member's life and we investigate financial gains/losses due to optimal/suboptimal behaviour in postretirement period.

3.1.1 Economic Environment

Let us first explain the economic environment that is represented by the model.

We assume two sources of randomness in our model: one from the risky rate on equity investment and the other one from inflation. We assume a constant real interest rate. We have two assets: one risky asset (equities) and one risk free asset (cash).

We assume that the pensioner has income coming from social security, and that this income is constant in real terms. Regarding consumption, we assume that he consumes part of his available assets at the beginning of each year, and we assume that this amount is not subject to inflation during the period shorter than one year. The pensioner draws utility from consumed amounts and possibly from amounts bequeathed to his heirs. He draws utility from the amounts in real terms only.

After receiving income and consuming part of his pension wealth and income, the pensioner can invest his available assets into the risky and risk free asset at his own discretion.

We assume the presence of real and nominal annuities on the market. Real annuity provides constant income in real terms, while nominal annuity provides constant income in nominal terms. Having access to each type of annuity at each age is the most general case and obviously, the pensioner can act optimally in this environment and obtain the biggest gains. However, we will investigate this market with different limitations in order to investigate the importance of these limitations. Depending on constraints, the pensioner can be in the market where only real or only nominal annuities are available. It can also be that only real annuities are available and only at certain ages. There are many combinations and we will choose what we think are appropriate to shed light on the significance of having access to each of them.

Under our assumption about the market, the pensioner who takes nominal annuities has no protection against the risk of inflation. His income from a nominal annuity will be subject to inflation risk and will diminish in time if inflation is positive. However, nominal annuities provide better income in both real and nominal terms in the early years after purchasing nominal annuities. As we will see later, the pensioner will still choose optimally some nominal annuities and expose himself to the risk of inflation.

We assume that the member can annuitise his pension wealth only, and that the very first income from an annuity is receivable after one year time. So, at the beginning of each year of life, he annuitises the available pension wealth, receives income from social security and annuities bought in earlier years, consumes part of the remaining amount and invests the rest.

We will present our model and results in real terms. However, one should be aware that this is just for presentational reasons, and we will actually convert from nominal to real values in order to present the results more clearly.

We work in discrete time. We assume that postretirement decumulation process starts at age $t = 65$, and finishes at age $t = 100$. We assume that the maximum member's age is 99, i.e. no member will be alive at age 100. The decumulation process lasts for 35 years. If the bequest motive exists, the pensioner aged 99 will consume part of his assets and the rest will be invested and bequeathed when he is going to die during that year. Otherwise, he will consume everything at age 99 and nothing will be left for investing. In the earlier periods, the pensioner consumes part of his available assets, uses one part for purchasing real annuities, one part for purchasing nominal annuities and invests the rest. We take the duration of one period to be one year. A slight modification would allow other lengths of each period. As we will see, the solution to

the problem follows the same pattern for different periods. That is why it is useful to investigate one representative period and then the solution for the whole problem can be derived from the solution of one representative period. We will always denote random variables with a $\sim$ sign above the name of variable. Graphical presentation of the most important variables appearing in our model is given as follows

State (information) variables						
W_t is pension wealth, Y_t is income,						
d_t^{NA} is percentage of the real income received from nominal annuity, I_{t-1} is inflation						
W_{65}	W_{66}	...	W_t	W_{t+1}	...	W_{100}
Y_{65}	Y_{66}	...	Y_t	Y_{t+1}	...	$Y_{100} = 0$
d_{65}^{NA}	d_{66}^{NA}	...	d_t^{NA}	d_{t+1}^{NA}	...	d_{100}^{NA}
I_{64}	I_{65}	...	I_{t-1}	I_t	...	I_{99}
Inflation						
$\tilde{I}_t$ is random inflation rate						
$\tilde{I}_{65}$	$\tilde{I}_{66}$	...	$\tilde{I}_t$	$\tilde{I}_{t+1}$	...	$\tilde{I}_{100}$
Returns						
r is constant interest rate, $\tilde{r}_t$ is random rate on risky asset						
r	r	...	r	r	...	—
$\tilde{r}_{65}$	$\tilde{r}_{66}$	...	$\tilde{r}_t$	$\tilde{r}_{t+1}$	...	—
Control (decision) variables						
C_t is consumption, α_t is proportion invested into equities,						
m_t^{NA} is proportion used for purchasing nominal annuities,						
m_t^{RA} is proportion used for purchasing real annuities						
C_{65}	C_{66}	...	C_t	C_{t+1}	...	—
α_{65}	α_{66}	...	α_t	α_{t+1}	...	—
m_{65}^{NA}	m_{66}^{NA}	...	m_t^{NA}	m_{t+1}^{NA}	...	—
m_{65}^{RA}	m_{66}^{RA}	...	m_t^{RA}	m_{t+1}^{RA}	...	—
Age during the decumulation process						
65	66	...	t	$t+1$	...	100
◆	◆	→	◆	◆	→	◆

We assume that the maximum pensioner's age is 100 years. However, we witness constantly increasing longevity in recent years and it is not unusual any more that the pensioner's age is more than 100 years. We recognise that this assumption in the thesis is at the variance with the empirical evidence. However, as we will see in the later text in the thesis, we investigate a number of numerical examples. Producing the

numerical results is time consuming and increasing the maximum pensioner's age to, for example, 115 years would require more time for calculation. Some other authors who investigate the problem of the pensioner's optimal annuitisation and asset allocation use the maximum pensioner's age of 100 years (Horneff, Maurer, Mitchell and Stamos (2009), Chai, Horneff, Maurer and Mitchell (2009), Horneff, Maurer and Stamos (2008)). When investigating consumption and portfolio choice over life cycle, but with no annuities, Cocco et al (2005) assume that the investor dies with probability 1 at age 100.

3.1.2 The Types of the Problem to be Investigated

We assume that the member can annuitise any part of the available pension wealth. We will assume that the member never annuitises any part of his income, only pension wealth available at the beginning of the year can be used for purchasing annuities.

The pensioner aims to maximise the expected discounted utility derived from consumption and possibly from bequeathing wealth by choosing the optimal consumption, asset allocation and annuitisation. Regarding annuitisation, we distinguish the strategies for the proportions of the pension wealth m_t^{RA} and m_t^{NA} to be annuitised. We group these assumptions into six types of problems to be investigated as follows:

- 3.1 Annuitising m_t^{NA} and m_t^{RA} parts of a pension wealth exogenously. In this type of the problem, the pensioner chooses a predetermined amount for purchasing real and nominal annuities and for given m_t^{NA} and m_t^{RA} the pensioner invests and consumes optimally. The control variables are $\{C_t, \alpha_t\}$, and m_t^{NA} and m_t^{RA} are determined exogenously and are usually suboptimal. The model can handle any assumption about predetermined values of m_t^{NA} and m_t^{RA} for $65 \leq t \leq 99$. We will investigate in more details the results with no annuitisation which is the special case of this type of problem.
- 3.2 m_t^{NA} is chosen optimally for some ages and exogenously for others, and m_t^{RA} is chosen exogenously for all ages $65 \leq t \leq 99$. For ages where m_t^{NA} is chosen endogenously, the member chooses a predetermined amount for purchasing real annuities and for given m_t^{RA} the member maximises the value function with respect to three control variables $\{C_t, \alpha_t, m_t^{NA}\}$. Otherwise, the control variables are $\{C_t, \alpha_t\}$. The model allows us to

calculate the results for any combination of exogenous/endogenous nominal annuitisation. All we need to know is for which age nominal annuitisation is endogenous, and for which it is exogenous, and for exogenous annuitisation ages we need to know m_t^{NA} . We will thoroughly investigate the results under the assumption that the pensioner purchases nominal annuities optimally at age 65 and no nominal annuities is available afterwards, and no real annuities.

- 3.3 m_t^{NA} is chosen exogenously for all ages, and m_t^{RA} is chosen optimally for some ages and exogenously for others. For ages where m_t^{RA} is chosen endogenously, the member maximises the value function with respect to three control variables $\{C_t, \alpha_t, m_t^{RA}\}$, and otherwise the control variables are $\{C_t, \alpha_t\}$. Similarly to the type of problem 2, we will thoroughly investigate the results under the assumption that the pensioner purchases real annuities optimally at age 65 and no real annuities is available afterwards, and no nominal annuities is bought at any age.
- 3.4 m_t^{NA} chosen endogenously and m_t^{RA} exogenously for all ages $65 \leq t \leq 99$. In this type of problem, the member chooses a predetermined amount for purchasing real annuities and for given m_t^{RA} the member maximises the value function with respect to three control variables $\{C_t, \alpha_t, m_t^{NA}\}$ at all ages.
- 3.5 m_t^{NA} chosen exogenously and m_t^{RA} endogenously for $65 \leq t \leq 99$. In this type of problem, the member chooses a predetermined amount for purchasing nominal annuities and for given m_t^{NA} the member maximises the value function with respect to the three control variables $\{C_t, \alpha_t, m_t^{RA}\}$ at all ages.
- 3.6 m_t^{NA} and m_t^{RA} are optimally chosen proportions for $65 \leq t \leq 99$. In this case, the member maximises the value function with respect to the four control variables, and control variables are $\{C_t, \alpha_t, m_t^{NA}, m_t^{RA}\}$ for all ages.

We have six groups of problems to be investigated, and these groups are differentiated by the assumption regarding exogenous/endogenous nominal/real annuitisation. When we have a particular assumption about the values of m_t^{NA} and m_t^{RA} for ages when m_t^{NA} and/or m_t^{RA} are exogenous we will refer to this assumption as a case. We can think of different cases as being different markets which are comparable and which differ in offering annuities only. Actually, market and case are equivalent expressions in this thesis. That is why we sometimes referred to cases as markets.

Although we will not investigate the results for many other combinations of optimal/suboptimal annuitisation strategies, we want to emphasize that the model in this chapter can be used for any exogenous/endogenous nominal/real annuitisation strategies. For example, if we assume that full compulsory annuitisation is imposed at a certain age then annuitisation occurs at the pensioner's discretion after retirement and before full compulsory annuitisation. Full compulsory annuitisation at a certain age can be deemed to be exogenous annuitisation with a proportion 100% at the age of compulsory annuitisation, and exogenous annuitisation with proportion 0% afterwards. We have witnessed this example in UK (Blake (1999)). Many countries do not impose compulsory annuitisation at any age.

We allow that the pensioner has a certain utility from the bequest. If 100% compulsory annuitisation happens, we exclude the bequest after that age since no pension wealth is left for bequeathing in the case of full annuitisation. In this context, it is sensible to assume that the bequest motive exists until the time of full annuitisation and not after that.

Regarding the amount to be annuitised at each age t , if exogenous annuitisation happens then it means that the member purchases real annuities for the amount of $m_t^{RA}W_t$, or nominal ones for the amount of $m_t^{NA}W_t$, and these annuitisation choices are suboptimal. Endogenous annuitisation happens if $m_t^{RA}W_t$ and/or $m_t^{NA}W_t$ are chosen optimally from the model.

We will write $\{cv_t\}$ to denote the $\{control\ variables_t\}$ at age t , such that we have the general notation for any type of problem. As we will see later, we work with control variables for values in money units and with control variables for scaled down values suitable for the calculations. In order to differentiate between the two we will denote with $\{CV_t\}$ the control variables for values in money units and with $\{cv_t\}$ the control variables for scaled down values.

3.2 The Model

3.2.1 Definitions and Notation

We use the following notation and definitions:

- W_t is the pension wealth at time t , just before income Y_t is received;
- Y_t is the variable denoting income at time t . We model income as

$$Y_t = \begin{cases} \bar{P}P_t & \text{for } 65 \leq t \leq 99 \\ 0 & \text{for } t = 100 \end{cases} \quad (3.1)$$

$\bar{P}$ is constant and is equal to the income at age 65, $P_{65} = 1$, and P_t will be defined later.

- C_t is consumption at the beginning of the period $[t, t+1]$ for $t = 65, 66, \dots, 99$, just after annuitisation and receiving income Y_t ;
- b_t is the factor which controls the pensioner's strength of the bequest motive. If no bequest motive exists then $b_t = 0$, for $t = 65, 66, \dots, 99$;
- d_t^{NA} is the percentage of the real income at time t received from nominal annuity bought before time t . We always assume that $d_{65}^{NA} = 0$. We refer to d_t^{NA} as nominal income coefficient;
- $\tilde{I}_t$ is the inflation rate during the period $[t, t+1]$ for $t = 65, 66, \dots, 99$. We model the inflation process as being approximated by the autoregressive scheme

$$\tilde{I}_{t+1} = \mu_I + (1 - \psi_I) I_t + \sigma_I \tilde{\varepsilon}_t$$

where μ_I , ψ_I and σ_I are constants, $\tilde{\varepsilon}_t \sim N(0, 1)$. I_{64} is known inflation rate during the year prior to retirement, The value of inflation rate I_t during the period $[t-1, t]$ is known at time t ;

- p_t – probability that the member aged t will survive until the age of $t+1$;
- r – risk free real interest rate, the constant and the same in all periods;
- $\tilde{r}_t$ – random variable denoting random real rate on risky asset during the period $[t, t+1]$, for $t = 65, 66, \dots, 99$. We assume that $[t, t+1]$ is one year period, and that

$$\text{Ln}(\tilde{r}_t) = \mu + \sigma \tilde{\varepsilon}_e(t) \quad (3.2)$$

where μ and σ are constants and $\tilde{\varepsilon}_e(t) \sim N(0, 1)$.

- α_t – the proportion of the wealth invested in the risky asset during the period $[t, t+1]$, for $t = 65, 66, \dots, 99$;
- m_t^{NA} – the proportion of the pension wealth used for purchasing nominal annuity at time t , for $t = 65, 66, \dots, 99$;
- m_t^{RA} – the proportion of the pension wealth used for purchasing real annuity at time t , for $t = 65, 66, \dots, 99$;

The control variables of the type of problem 3.6 are $\{C_t, \alpha_t, m_t^{NA}, m_t^{RA}\}$. Depending on assumptions they can also be $\{C_t, \alpha_t, m_t^{NA}\}$ for type of problem 3.4 and for 3.2 for those ages where nominal annuitisation is optimal. Control variables are $\{C_t, \alpha_t, m_t^{RA}\}$ for type of problem 3.5 and for 3.3 for those ages where real annuitisation is optimal, and $\{C_t, \alpha_t\}$ for type of problem 3.1 for all ages and 3.2 and 3.3 for those ages where we assume both real and nominal annuitisation to be exogenous. The state variables of the problem are $\{t, W_t, Y_t, d_t^{NA}, I_{t-1}\}_{t=65}^{99}$. We will skip explicitly writing the state variable t and write the state variables as $\{W_t, Y_t, d_t^{NA}, I_{t-1}\}_{t=65}^{99}$.

Let us also introduce the random variable

$$\tilde{r}_t^P = (1 - \alpha_t)r + \alpha_t \tilde{r}_t = r + \alpha_t(\tilde{r}_t - r)$$

denoting the random real rate on the portfolio during the period $[t, t+1]$, for $t = 65, 66, \dots, 99$.

In inflation risk model, we define all variables in real term and allow the inflation to influence annuity income from nominal annuities only. Thus, we assume that real interest rate is constant, return on equity investment is modelled in real terms, income from social security is constant in real terms, and consumption and pension wealth are always expressed in real terms. If the inflation risk is present then it would influence the annuity rate of nominal annuity and then future income from nominal annuities. From this point of view it may be interesting to explore the possible correlation between real return on equity investment and inflation. Inflation can influence both expected return on stock as well as its volatility. We acknowledge here that this may be interesting topic to explore and with different assumptions and results, but due to already complicate model and numerous results that we provide in this thesis, we leave this analysis for future research. In this thesis, we assume that real return on equity investment is not correlated with inflation.

The utility function is CRRA function, given by

$$u(x) = \frac{x^\gamma}{\gamma} \text{ for } \gamma < 1, \gamma \neq 0, \text{ and}$$

$$u(x) = \text{Log}(x) \text{ for } \gamma = 0.$$

3.2.2 Income process

Let us define income process more precisely. At age $t = 65$, income comes from the last salary only. After receiving his last salary, for ages $66 \leq t \leq 99$ the member's income consists of social security income and income from annuities bought at age 65 and afterwards. For reasons of simplicity, we will assume that income from the social security Y_t^{SS} for $66 \leq t \leq 99$ is constant in real terms. We also define $Y_{100} = 0$, as we actually assume that no pensioner aged 100 is alive.

We will distinguish three types of income in retirement, income from social security sources denoted by Y_t^{SS} , income from nominal annuities bought before time t denoted by Y_t^{NA} , and income from real annuities bought before time t denoted by Y_t^{RA} . Income from social security Y_t^{SS} and income from index-linked annuity Y_t^{RA} are real incomes to be received at time t . Income from nominal annuity Y_t^{NA} is income in real terms received at time t provided from nominal annuities bought before time t . It means that income Y_t^{NA} is adjusted for inflation up to time t . This can be written as

$$Y_t = Y_t^{SS} + Y_t^{NA} + Y_t^{RA}$$

for $66 \leq t \leq 99$. Y_{65} is defined in (3.1). Let us now define Y_t^{SS} , Y_t^{NA} and Y_t^{RA} , for $66 \leq t \leq 99$ more precisely.

We assume that the pension member receives his very first income from social security at age 66, and Y_t^{SS} will be defined as

$$Y_t^{SS} = replrate \cdot Y_{65}$$

for $66 \leq t \leq 99$ and *replrate* is the percentage of the last salary provided from the state in form of social security income after age 65. It is a constant real income until the end of pensioner's life. We also introduce the variable

$$\rho_t = \begin{cases} replrate & t = 65 \\ 1 & 66 \leq t \leq 99 \end{cases} \quad (3.3)$$

Now we assume the environment where purchasing real and nominal annuities from pension wealth is allowed at the member's discretion at age 65 and afterwards. Whenever the member purchases an annuity his wealth decreases by the amount used for purchasing that annuity, and his income in future periods increases from the newly provided annuity income. For reasons of simplicity, we assume that annuities provide

the very first instalment one year after purchasing annuities. Let us denote income from nominal and real annuities bought at age 65 with Y_{a65}^{NA} and Y_{a65}^{RA} respectively, at age 66 with Y_{a66}^{NA} and Y_{a66}^{RA} respectively, and so on until maximum age $t = 99$.

We define

$$a_{99}^{NA} = 1 \text{ and } a_{99}^{RA} = 1$$

and

$$a_t^{NA} = (1 + NALoadings) \sum_{i=1}^{99-t} \left(\left(\prod_{j=1}^i p_{t+j-1} \right) (1 + r + E[\tilde{I}_{t,i}])^{-i} \right) \quad (3.4)$$

and

$$a_t^{RA} = (1 + RALoadings) \sum_{i=1}^{99-t} \left(\left(\prod_{j=1}^i p_{t+j-1} \right) (1 + r)^{-i} \right) \quad (3.5)$$

for $t = 65, 66, \dots, 99$, where $NALoadings$ and $RALoadings$ are loadings on the actuarially fair nominal and real annuities depending on the market, and $E[\tilde{I}_{t,i}]$ is expected annual inflation rate at time t for the period of next i years. Now, we can write

$$Y_{at}^{NA} = \frac{m_t^{NA} W_t}{a_t^{NA}}, \text{ and}$$

$$Y_{at}^{RA} = \frac{m_t^{RA} W_t}{a_t^{RA}}.$$

Thus, if some annuities are bought at age 65, the real income at age 66 is

$$\begin{aligned} Y_{66} &= Y_{66}^{SS} + Y_{a65}^{NA} (1 + I_{65})^{-1} + Y_{a65}^{RA} \\ &= Y_{66}^{SS} + Y_{66}^{NA} + Y_{66}^{RA} \end{aligned}$$

Then some new annuities are bought at age 66, and the real income at age 67 is

$$\begin{aligned} Y_{67} &= Y_{67}^{SS} + \left(Y_{a65}^{NA} \prod_{k=65}^{66} (1 + I_k)^{-1} + Y_{a66}^{NA} (1 + I_{66})^{-1} \right) + (Y_{a65}^{RA} + Y_{a66}^{RA}) \\ &= Y_{67}^{SS} + Y_{67}^{NA} + Y_{67}^{RA} \end{aligned}$$

The same pattern repeats itself and at age $66 \leq t \leq 99$ the real income is

$$\begin{aligned} Y_t &= Y_t^{SS} + \left(Y_{a65}^{NA} \prod_{k=65}^{t-1} (1 + I_k)^{-1} + \dots + Y_{a(t-1)}^{NA} (1 + I_{t-1})^{-1} \right) + (Y_{a65}^{RA} + \dots + Y_{a(t-1)}^{RA}) \\ &= Y_t^{SS} + Y_t^{NA} + Y_t^{RA} \end{aligned}$$

where

$$Y_t^{NA} = Y_{a65}^{NA} \prod_{k=65}^{t-1} (1 + I_k)^{-1} + \dots + Y_{a(t-1)}^{NA} (1 + I_{t-1})^{-1}, \text{ and}$$

$$Y_t^{RA} = Y_{a65}^{RA} + \dots + Y_{a(t-1)}^{RA}.$$

For incomes in the two subsequent periods, we have the relation

$$Y_{t+1} = Y_{t+1}^{SS} + (Y_t^{NA} + Y_{at}^{NA})(1 + I_t)^{-1} + Y_t^{RA} + Y_{at}^{RA} \quad (3.6)$$

It can be seen from the last relation that we need to know what part of the income is subject to an inflation adjustment. We do this using the state variable d_t^{NA} denoting the percentage of the income at time t received from nominal annuity bought before time t , where all values are in real terms. Thus

$$d_t^{NA} = \frac{Y_t^{NA}}{Y_t}, \text{ and} \quad (3.7)$$

$$1 - d_t^{NA} = \frac{Y_t^{SS} + Y_t^{RA}}{Y_t} \quad (3.8)$$

for $65 \leq t \leq 99$. Now, we can write income in retirement at age t as $Y_t = (1 - d_t^{NA})Y_t + d_t^{NA}Y_t$. The first summand $(1 - d_t^{NA})Y_t$ is a constant real income consisting of real income from social security and from previously bought real annuities. The second one, $d_t^{NA}Y_t$, is a nominal income adjusted for inflation.

Using the nominal income coefficient d_t^{NA} the relation between the two subsequent periods becomes

$$Y_{t+1} = (1 - d_t^{NA})\rho_t Y_t + Y_{at}^{RA} + (d_t^{NA}Y_t + Y_{at}^{NA})(1 + I_t)^{-1} \quad (3.9)$$

for $65 \leq t \leq 99$, where ρ_t is defined in (3.3). The term ρ_t appears as a multiplicative factor, and it influences this and other equations where it appears for age 65 only. One can also see that ρ_t is not a factor in the term $d_t^{NA}Y_t$, and it is because $d_{65}^{NA} = 0$ and so ρ_t does not influence this term. The real income is represented by the term $(1 - d_t^{NA})\rho_t Y_t + Y_{at}^{RA}$, and the nominal income in real terms is given by $(d_t^{NA}Y_t + Y_{at}^{NA})(1 + I_t)^{-1}$. Using (3.7), we can write

$$d_{t+1}^{NA} = \frac{Y_{t+1}^{NA}}{Y_{t+1}} = \frac{(d_t^{NA}Y_t + Y_{at}^{NA})(1 + I_t)^{-1}}{Y_{t+1}}.$$

Now, using (3.9), we can write

$$d_{t+1}^{NA} = \frac{(d_t^{NA}Y_t + Y_{at}^{NA})(1+I_t)^{-1}}{(1-d_t^{NA})\rho_t Y_t + Y_{at}^{RA} + (d_t^{NA}Y_t + Y_{at}^{NA})(1+I_t)^{-1}} \quad (3.10)$$

for $65 \leq t \leq 99$. Earlier in this chapter, we introduce the constant $\bar{P}$. All equations in realistic amounts will be divided by this constant in order to work with smaller numbers when solving the problem numerically on the computer. Let us now express the equations of income process in terms of P variables. We said earlier that $\bar{P}$ is constant, equal to the income at age 65 and $P_{65} = 1$. Now, we define P_t^{SS} , P_t^{NA} , P_t^{RA} , P_{at}^{NA} and P_{at}^{RA} via equations $Y_t^{SS} = \bar{P}P_t^{SS}$, $Y_t^{NA} = \bar{P}P_t^{NA}$, $Y_t^{RA} = \bar{P}P_t^{RA}$, $Y_{at}^{NA} = \bar{P}P_{at}^{NA}$, $Y_{at}^{RA} = \bar{P}P_{at}^{RA}$, respectively.

The equivalent equations to equations (3.9) and (3.10) are given by

$$P_{t+1} = (1-d_t^{NA})\rho_t P_t + P_{at}^{RA} + (d_t^{NA}P_t + P_{at}^{NA})(1+I_t)^{-1}. \quad (3.11)$$

and

$$d_{t+1}^{NA} = \frac{(d_t^{NA}P_t + P_{at}^{NA})(1+I_t)^{-1}}{(1-d_t^{NA})\rho_t P_t + P_{at}^{RA} + (d_t^{NA}P_t + P_{at}^{NA})(1+I_t)^{-1}}. \quad (3.12)$$

and also

$$P_t = P_t^{SS} + P_t^{NA} + P_t^{RA}.$$

for $65 \leq t \leq 99$. The equation (3.1) is fully defined now. The variable ρ_t influences equations (3.11) and (3.12) for $t = 65$ only.

For representing the equations that follow later in this chapter in a more compact form, it will be useful to define

$$G_{t+1} = \frac{Y_{t+1}}{Y_t} \quad (3.13)$$

From (3.9) we get

$$\begin{aligned} G_{t+1} &= \frac{(1-d_t^{NA})\rho_t Y_t + Y_{at}^{RA} + (d_t^{NA}Y_t + Y_{at}^{NA})(1+I_t)^{-1}}{Y_t} \\ &= (1-d_t^{NA})\rho_t + \frac{Y_{at}^{RA}}{Y_t} + \left(d_t^{NA} + \frac{Y_{at}^{NA}}{Y_t}\right)(1+I_t)^{-1}, \end{aligned}$$

and using (3.4) and (3.5) we get

$$G_{t+1} = (1-d_t^{NA})\rho_t + \frac{m_t^{RA}W_t}{Y_t a_t^{RA}} + \left(d_t^{NA} + \frac{m_t^{NA}W_t}{Y_t a_t^{NA}}\right)(1+I_t)^{-1} \quad (3.14)$$

for $65 \leq t \leq 99$. Again, ρ_t influences the equation above for $t = 65$ only, otherwise $\rho_t = 1$ and ρ_t does not influence equation (3.14).

3.2.3 Mathematical Model for the Problem

We assume that the pensioner's maximum attainable age is $t = 99$. No pensioner will survive until age 100. Thus, $p_{99} = 0$ and there is no annuitisation at age 99. Let us start with the last age period $[99, 100]$. If the pensioner is alive at the beginning of this period, he draws utility from consuming part of his available financial wealth and possibly draws utility from bequeathing some assets. There is no income at the end of the period $[99, 100]$, i.e. $Y_{100} = 0$. Pensioner's value function (utility) at age 99 is

$$V_{99}(W_{99}, Y_{99}, d_{99}^{NA}, I_{98}) = \max_{\{C_{99}\}} E_{99} \left[u(C_{99}) + \delta(1 - p_{99}) b_{99} u(\tilde{W}_{100}) \right] \quad (3.15)$$

where

$$\tilde{W}_{100} = (W_{99} + Y_{99} - C_{99})(1 + r + \alpha_{99}(\tilde{r}_{99} - r)) \quad (3.16)$$

The pensioner maximises his value function at age 99 over all possible consumption C_{99} and investment decisions α_{99} . These two are the only control variables at this age as no annuitisation occurs. We assume that the control variables are subject to the no borrowing constraint. It means that the maximum amount the member can consume is his available pension wealth W_{99} and his income Y_{99} . The maximum amount he can invest in equities is $W_{99} + Y_{99} - C_{99}$. Mathematically,

$$0 \leq C_{99} \leq W_{99} + Y_{99}, \text{ and} \quad (3.17)$$

$$0 \leq \alpha_{99} \leq 100\%. \quad (3.18)$$

We will assume that the member's pension wealth $W_t \geq 0$ for $65 \leq t \leq 99$. Another sensible assumption would be $W_{99} + Y_{99} \geq 0$, or in the other words we assume that pension wealth can become negative up to the level of the income in that period. This assumption would be equivalent to the assumption that limited borrowing is allowed because the pensioner must consume certain money each period. Although, this is an interesting problem for investigation, we will keep the assumption $W_t \geq 0$ in our model.

The factor b_t controls the pensioner's strength of the bequest motive and where $b_t \geq 0$ for all t .

The formulae (3.15)–(3.18) can be used for developing formulae for value function in earlier period as well. In order to see changes in formulae when we move one period backwards, let us firstly see member's value function at age $t = 98$. We have

$$V_{98}(W_{98}, Y_{98}, d_{98}^{NA}, I_{97}) = \max_{\{CV_t\}_{t=98}^{99}} E_{98} \left[u(C_{98}) + \delta p_{98} u(\tilde{C}_{99}) + \right. \\ \left. \delta(1-p_{98}) b_t u(\tilde{W}_{99}) + \delta^2 p_{98} (1-p_{99}) b_t u(\tilde{W}_{100}) \right] \\ V_{98}(W_{98}, Y_{98}, d_{98}^{NA}, I_{97}) = \max_{\{CV_t\}_{t=98}^{99}} E_{98} \left[u(C_{98}) + \delta(1-p_{98}) b_t u(\tilde{W}_{99}) + \right. \\ \left. \delta p_{98} (u(\tilde{C}_{99}) + \delta(1-p_{99}) b_t u(\tilde{W}_{100})) \right]$$

Thus

$$V_{98}(W_{98}, Y_{98}, d_{98}^{NA}, I_{97}) = \max_{\{CV_t\}_{t=98}^{99}} E_{98} \left[u(C_{98}) + \delta(1-p_{98}) b_t u(\tilde{W}_{99}) + \right. \\ \left. \delta p_{98} V_{99}(\tilde{W}_{99}, \tilde{Y}_{99}, \tilde{d}_{99}^{NA}, \tilde{I}_{98}) \right]$$

where

$$\tilde{W}_{99} = \left((1 - m_{98}^{NA} - m_{98}^{RA}) W_{98} + Y_{98} - C_{98} \right) (1 + r + \alpha_{98} (\tilde{r}_{98} - r)) \quad (3.19)$$

Using (3.9) we have

$$\tilde{Y}_{99} = (1 - d_{98}^{NA}) Y_{98} + \frac{m_{98}^{RA} W_{98}}{a_{98}^{RA}} + \left(d_{98}^{NA} Y_{98} + \frac{m_{98}^{NA} W_{98}}{a_{98}^{NA}} \right) (1 + \tilde{I}_{98})^{-1} \quad (3.20)$$

Using (3.10)

$$\tilde{d}_{99}^{NA} = \frac{\left(d_{98}^{NA} Y_{98} + \frac{m_{98}^{NA} W_{98}}{a_{98}^{NA}} \right) (1 + \tilde{I}_{98})^{-1}}{\left((1 - d_{98}^{NA}) Y_{98} + \frac{m_{98}^{RA} W_{98}}{a_{98}^{RA}} + \left(d_{98}^{NA} Y_{98} + \frac{m_{98}^{NA} W_{98}}{a_{98}^{NA}} \right) (1 + \tilde{I}_{98})^{-1} \right)} \quad (3.21)$$

and the constraints are

$$0 \leq C_{98} \leq (1 - m_{98}^{RA} - m_{98}^{NA}) W_{98} + Y_{98} \quad (3.22)$$

$$0 \leq m_{98}^{NA} \leq 1, \quad 0 \leq m_{98}^{RA} \leq 1, \quad \text{and} \quad 0 \leq m_{98}^{NA} + m_{98}^{RA} \leq 1 \quad (3.23)$$

$$0 \leq \alpha_{98} \leq 100\% . \quad (3.24)$$

Here, we have used the Bellman principal of optimality and the law of iterated conditional expectations.

Now, one can derive value function for any age $65 \leq t \leq 99$. The value function for ages $65 \leq t_0 \leq 99$ is given by

$$V_{t_0}(X_{t_0}, Y_{t_0}, d_{t_0}^{NA}, I_{t_0-1}) = \max_{\{CV_t\}_{t=t_0}^{99}} E_{t_0} \left[u(C_{t_0}) + \delta(1-p_{t_0}) b_{t_0} u(\tilde{W}_{t_0+1}) + \right. \\ \left. \delta p_{t_0} \sum_{t=t_0+1}^{99} \left(\delta^{t-(t_0+1)} p_{t_0+1} u(\tilde{C}_t) + \delta^{t-t_0} p_{t_0+1} (1-p_t) b_t u(\tilde{W}_{t+1}) \right) \right] \quad (3.25)$$

Using Bellman's principal of optimality, which says that

$$\max_{\{CV_t\}_{t=t_0}^{99}} (Z) = \max_{\{CV_t\}_{t=t_0}} \left[\max_{\{CV_t\}_{t=t_0+1}^{99} \mid \text{outcome from } \{CV_t\}_{t=t_0}} (Z) \right]$$

we have

$$\begin{aligned} V_{t_0}(X_{t_0}, Y_{t_0}, d_{t_0}^{NA}, I_{t_0-1}) = & \max_{\{CV_t\}_{t=t_0}} \left[u(C_{t_0}) + E_{t_0} \left[\delta(1-p_{t_0}) b_{t_0} u(\tilde{W}_{t_0+1}) \right] + \right. \\ & \delta p_{t_0} E_{t_0} \left[\max_{\{CV_t\}_{t=t_0+1}^{99}} \left[\sum_{t=t_0+1}^{99} \left(\delta^{t-(t_0+1)} p_{t-(t_0+1)} p_{t_0+1} u(\tilde{C}_t) + \right. \right. \right. \\ & \left. \left. \left. \delta^{t-t_0} p_{t-(t_0+1)} p_{t_0+1} (1-p_t) b_t u(\tilde{W}_{t+1}) \right) \right] \right] \end{aligned}$$

and using the law of iterated conditional expectations

$$\begin{aligned} V_{t_0}(W_{t_0}, Y_{t_0}, d_{t_0}^{NA}, I_{t_0-1}) = & \max_{\{CV_t\}_{t=t_0}} \left[u(C_{t_0}) + E_{t_0} \left[\delta(1-p_{t_0}) b_{t_0} u(\tilde{W}_{t_0+1}) \right] + \right. \\ & \delta p_{t_0} E_{t_0} \left[\max_{\{CV_t\}_{t=t_0+1}^{99}} E_{t_0+1} \left[\sum_{t=t_0+1}^{99} \left(\delta^{t-(t_0+1)} p_{t-(t_0+1)} p_{t_0+1} u(\tilde{C}_t) + \right. \right. \right. \\ & \left. \left. \left. \delta^{t-t_0} p_{t-(t_0+1)} p_{t_0+1} (1-p_t) b_t u(\tilde{W}_{t+1}) \right) \right] \right] \end{aligned}$$

Thus,

$$\begin{aligned} V_t(W_t, Y_t, d_t^{NA}, I_{t-1}) = & \max_{\{CV_t\}} E_t \left[u(C_t) + \delta(1-p_t) b_t u(\tilde{W}_{t+1}) + \right. \\ & \left. \delta p_t V_{t+1}(\tilde{W}_{t+1}, \tilde{Y}_{t+1}, \tilde{d}_{t+1}^{NA}, \tilde{I}_t) \right] \end{aligned} \quad (3.26)$$

where

$$\tilde{W}_{t+1} = \left((1-m_t^{NA} - m_t^{RA}) W_t + Y_t - C_t \right) (1 + \tilde{r}_t^P) \quad (3.27)$$

$$\tilde{Y}_{t+1} = (1-d_t^{NA}) \rho_t Y_t + \frac{m_t^{RA} W_t}{a_t^{RA}} + \left(d_t^{NA} Y_t + \frac{m_t^{NA} W_t}{a_t^{NA}} \right) (1 + \tilde{I}_t)^{-1} \quad (3.28)$$

$$\tilde{r}_t^P = r + \alpha_t (\tilde{r}_t - r) \quad (3.29)$$

$$\rho_t = \begin{cases} replrate & t = 65 \\ 1 & 66 \leq t \leq 99 \end{cases} \quad (3.30)$$

$$\tilde{d}_{t+1}^{NA} = \frac{\left(d_t^{NA} Y_t + \frac{m_t^{NA} W_t}{a_t^{NA}} \right) (1 + \tilde{I}_t)^{-1}}{(1-d_t^{NA}) \rho_t Y_t + \frac{m_t^{RA} W_t}{a_t^{RA}} + \left(d_t^{NA} Y_t + \frac{m_t^{NA} W_t}{a_t^{NA}} \right) (1 + \tilde{I}_t)^{-1}} \quad (3.31)$$

with the constraints

$$0 \leq C_t \leq (1-m_t^{NA} - m_t^{RA}) W_t + Y_t \quad (3.32)$$

$$0 \leq m_t^{NA} \leq 1, 0 \leq m_t^{RA} \leq 1, \text{ and } 0 \leq m_t^{NA} + m_t^{RA} \leq 1 \quad (3.33)$$

$$0 \leq \alpha_t \leq 100\% \quad (3.34)$$

for $65 \leq t \leq 99$.

3.3 Solution to the Problem

Let us present the solution to the problem defined in the previous section. We will show in detail the solution assuming endogenous m_t^{NA} and m_t^{RA} . Other solutions for the types of problem explained in Section 3.1.2 follow the same suit, and we will give the explanations about the changes needed to obtain these solutions.

3.3.1 Solution for Endogenous m_t^{NA} and m_t^{RA}

The analytical solution to the problem (3.26)–(3.34) cannot be found in the current literature. Assuming stochastic inflation, we have two sources of randomness. If we assume deterministic inflation then we stay with only one source of randomness but the problem is still unsolved analytically in the current literature.

The usual approach to this type of problem nowadays is a numerical solution using computers. There are two main approaches. The first one is deriving first order condition equations and then solving them numerically. The second approach is finding the maximum using numerical mathematics, i.e. solving the equations given above directly. We apply the latter.

Observing equations (3.26)–(3.34) and the constraints accompanying them, one can see that we need to solve a problem of nonlinear optimization with constraints. In this particular problem, we have four control variables. The constraints are analytical function. We solve this problem in Mathematica 5.2 using Gauss Quadrature for the approximating random variables and cubic splines for interpolating the value function. The Gauss Quadrature method is used in many papers and we refer here to the paper Tauchen and Hussey (1991). The N –points Gauss Quadrature rules are a discrete approximation to a density function determined by the method of using moments up through $2N-1$, where N is the number of pairs of points and weights used for the approximation. Gauss Quadrature rules are close to the minimum norm rule and possess several optimum properties (Davis and Rabinowitz (1975)). They are the best that can be done with N points using moments as a criterion, because if two

probability distributions have the same moments up through $2N$, and if one of the distributions is a discrete distribution concentrated on N points, then the two distributions must coincide (Norton and Arnold (1985)). We will use the Gauss Quadrature rules throughout this thesis whenever we need the discrete approximation of a density function.

In this setup of the problem, we assume that m_t^{NA} and m_t^{RA} , where $65 \leq t \leq 99$, are not predetermined and depend on the evolution of the process. Let us explain the way we will solve the problem.

Before solving the problem itself, we need to explain the solution we are aiming to obtain. We assume that inflation can take a finite number of values for each age t . We denote the possible states of inflation as $(I_{t;k})_{k=1}^{n_I}$, where N_I is the number of possible values of inflation for each age $65 \leq t \leq 99$, and where $I_{t;k}$ for $65 \leq t \leq 99$ and $1 \leq k \leq N_I$ are predefined allowed values of inflation. As a solution we get

$$\begin{aligned} & \left(C_t^*(W_t, Y_t, d_t^{NA}, I_{t-1;k}); \alpha_t^*(W_t, Y_t, d_t^{NA}, I_{t-1;k}); m_t^{NA*}(W_t, Y_t, d_t^{NA}, I_{t-1;k}); \right. \\ & \left. m_t^{RA*}(W_t, Y_t, d_t^{NA}, I_{t-1;k}); V_t(W_t, Y_t, d_t^{NA}, I_{t-1;k}) \right) \end{aligned} \quad (3.35)$$

for $65 \leq t \leq 99$, $W_t \geq 0$ and $Y_t \geq 0$, $0 \leq d_t^{NA} \leq 1$, and $I_{t-1;k}$ in the domain of inflation rate. In other words, we get the solution for continuous values of the variables W_t and Y_t and d_t^{NA} in their domains, and for discrete predefined values of the variable $I_{t-1;k}$ in the predefined domain of inflation values.

Let us assume that we have solution for time $t+1$ and we need to go one step back in order to find the solution for time t . It means that we have obtained

$$\begin{aligned} & \left\{ \left(C_i^*(W_i, Y_i, d_i^{NA}, I_{i-1;m}); \alpha_i^*(W_i, Y_i, d_i^{NA}, I_{i-1;m}); m_i^{NA*}(W_i, Y_i, d_i^{NA}, I_{i-1;m}); \right. \right. \\ & \left. \left. m_i^{RA*}(W_i, Y_i, d_i^{NA}, I_{i-1;m}); V_i(W_i, Y_i, d_i^{NA}, I_{i-1;m}) \right) \right\}_{i=t+1}^{99} \end{aligned} \quad (3.36)$$

for $65 \leq t \leq 99$, $W_t \geq 0$ and $Y_t \geq 0$, $0 \leq d_t^{NA} \leq 1$, and $I_{t-1;m}$ in the domain of the inflation rate. Having this solution we want to find the solution

$$\begin{aligned} & \left(C_t^*(W_t, Y_t, d_t^{NA}, I_{t-1;k}); \alpha_t^*(W_t, Y_t, d_t^{NA}, I_{t-1;k}); m_t^{NA*}(W_t, Y_t, d_t^{NA}, I_{t-1;k}); \right. \\ & \left. m_t^{RA*}(W_t, Y_t, d_t^{NA}, I_{t-1;k}); V_t(W_t, Y_t, d_t^{NA}, I_{t-1;k}) \right) \end{aligned} \quad (3.37)$$

for $W_t \geq 0$, $Y_t \geq 0$, $0 \leq d_t^{NA} \leq 1$, and $I_{t-1;k}$ in the domain of inflation rate values, where $C_t^*(W_t, Y_t, d_t^{NA}, I_{t-1;k})$ is optimal consumption, $\alpha_t^*(W_t, Y_t, d_t^{NA}, I_{t-1;k})$ is optimal asset allocation, $m_t^{NA*}(W_t, Y_t, d_t^{NA}, I_{t-1;k})$ optimal nominal annuitisation, $m_t^{RA*}(W_t, Y_t, d_t^{NA}, I_{t-1;k})$ optimal real annuitisation, and $V_t(W_t, Y_t, d_t^{NA}, I_{t-1;k})$ is the value function for those optimal control variables. It means, we want to determine (3.37) which maximizes the value function below

$$V_t(W_t, Y_t, d_t^{NA}, I_{t-1;k}) = \max_{\{C_t, \alpha_t, m_t^{NA}, m_t^{RA}\}} \left[u(C_t) + E_t \left[\delta(1-p_t) b_t u(\tilde{W}_{t+1}) + \delta p_t V_{t+1}(\tilde{W}_{t+1}, \tilde{Y}_{t+1}, \tilde{d}_{t+1}^{NA}, \tilde{I}_t) \right] \right]$$

For simplicity and for reasons of grasping the solution more easily for other control variables assumptions, we will write $\{CV_t\}$ (abbreviation for control variables) instead of $\{C_t, \alpha_t, m_t^{NA}, m_t^{RA}\}$. Let us now, for reason of explicit derivation of formulae, rewrite the last equation in a more explicit form as

$$V_t(W_t, Y_t, d_t^{NA}, I_{t-1;k}) = \max_{\{CV_t\}} \left[u(C_t) + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\delta(1-p_t) b_t u(W_{t+1}(r_t, I_t)) + \delta p_t V_{t+1}(W_{t+1}(r_t, I_t), Y_{t+1}(I_t), d_{t+1}^{NA}(I_t), I_t) \right) dF(I_t) dF(r_t) \right] \quad (3.38)$$

It is possible to decrease the number of state variables from four to three. For this reason, we will now make the transformations that will allow us to work with only three state variables. The state variable that is going to be excluded is income Y_t . Using the results from Appendix 1, we know that

$$V_t(W_t, Y_t, d_t^{NA}, I_{t-1;k}) = \left(\frac{Y_t}{\bar{y}} \right)^\gamma V_t \left(W_t \frac{\bar{y}}{Y_t}, \bar{y}, d_t^{NA}, I_{t-1;k} \right)$$

for any $\bar{y} > 0$. Introducing this relation into equation (3.38) we get

$$\begin{aligned} \left(\frac{Y_t}{\bar{y}} \right)^\gamma V_t \left(W_t \frac{\bar{y}}{Y_t}, \bar{y}, d_t^{NA}, I_{t-1;k} \right) &= \max_{\{CV_t\}} \left[u(C_t) + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\delta(1-p_t) b_t \left(\frac{Y_{t+1}(I_t)}{\bar{y}} \right)^\gamma u \left(W_{t+1}(r_t) \frac{\bar{y}}{Y_{t+1}(I_t)} \right) + \right. \right. \\ &\quad \left. \left. \delta p_t \left(\frac{Y_{t+1}(I_t)}{\bar{y}} \right)^\gamma V_{t+1} \left(W_{t+1}(r_t) \frac{\bar{y}}{Y_{t+1}(I_t)}, \bar{y}, d_{t+1}^{NA}(I_t), I_t \right) \right) dF(I_t) dF(r_t) \right] \end{aligned}$$

Using (3.27) and skipping writing dependent variables one get

$$\begin{aligned} \left(\frac{Y_t}{\bar{y}}\right)^\gamma V_t \left(W_t \frac{\bar{y}}{Y_t}, \bar{y}, d_t^{NA}, I_{t-1;k} \right) &= \max_{\{c_t\}} \left[\left(\frac{Y_t}{\bar{y}}\right)^\gamma u \left(\frac{C_t}{Y_t} \bar{y} \right) + \right. \\ &\quad \left. \delta \int_0^\infty \int_{-\infty}^\infty \left(\frac{Y_{t+1}}{\bar{y}}\right)^\gamma \left((1-p_t) b_t u \left(\left((1-m_t^{RA} - m_t^{NA}) \frac{W_t}{Y_t} \bar{y} + \bar{y} - \frac{C_t}{Y_t} \bar{y} \right) (1+r_t^P) \frac{Y_t}{Y_{t+1}} \right) + p_t \cdot \right. \right. \\ &\quad \left. \left. V_{t+1} \left(\left((1-m_t^{RA} - m_t^{NA}) W_t + Y_t - C_t \right) (1+r_t^P) \frac{\bar{y}}{Y_{t+1}}, \bar{y}, d_{t+1}^{NA}, I_t \right) \right) \right] dF(I_t) dF(r_t) \end{aligned}$$

where $r_t^P = r + \alpha_t (r_t - r)$. Let us define

$$w_t = \frac{W_t}{Y_t} \bar{y} \text{ and } c_t = \frac{C_t}{Y_t} \bar{y}, \text{ and} \quad (3.39)$$

Multiplying both sides by $\left(\frac{\bar{y}}{Y_t}\right)^\gamma$ and introducing (3.39) we have

$$\begin{aligned} V_t(w_t, \bar{y}, d_t^{NA}, I_{t-1;k}) &= \max_{\{c_t\}} \left[u(c_t) + \right. \\ &\quad \left. \int_0^\infty \int_{-\infty}^\infty \left(\frac{Y_{t+1}}{Y_t}\right)^\gamma \left(\delta (1-p_t) b_t u \left(\left((1-m_t^{RA} - m_t^{NA}) w_t + \bar{y} - c_t \right) (1+r_t^P) \frac{Y_t}{Y_{t+1}} \right) + \right. \right. \\ &\quad \left. \left. \delta p_t V_{t+1} \left(\left((1-m_t^{RA} - m_t^{NA}) w_t + \bar{y} - c_t \right) (1+r_t^P) \frac{Y_t}{Y_{t+1}}, \bar{y}, d_{t+1}^{NA}, I_t \right) \right) \right] dF(I_t) dF(r_t) \end{aligned}$$

where $\{c_t\}$ is now $\{c_t, \alpha_t, m_t^{NA}, m_t^{RA}\}$. We will actually derive our solution for $Y_t = \bar{y}$ and $w_t \geq 0$, and control variables $\{c_t, \alpha_t, m_t^{NA}, m_t^{RA}\}$ and then use the transformation from Appendix 1 to get solution (3.37) for any $W_t \geq 0$, $Y_t \geq 0$, $0 \leq d_t^{NA} \leq 1$, and $I_{t-1;k}$ in the domain of inflation values. Using (3.28) we have

$$Y_{t+1} = (1-d_t^{NA}) \rho_t Y_t + \frac{m_t^{RA} W_t}{a_t^{RA}} + \left(d_t^{NA} Y_t + \frac{m_t^{NA} W_t}{a_t^{NA}} \right) (1+I_t)^{-1}$$

We have defined in (3.13)

$$G_{t+1} = \frac{P_{t+1}}{P_t} = \frac{\bar{P} P_{t+1}}{\bar{P} P_t} = \frac{Y_{t+1}}{Y_t}.$$

Thus, we have that

$$G_{t+1} = (1-d_t^{NA}) \rho_t + \frac{m_t^{RA} w_t}{\bar{y} a_t^{RA}} + \left(d_t^{NA} + \frac{m_t^{NA} w_t}{\bar{y} a_t^{NA}} \right) (1+I_t)^{-1}$$

and

$$d_{t+1}^{NA} = \frac{\left(d_t^{NA} + \frac{m_t^{NA} w_t}{\bar{y} a_t^{NA}}\right) (1 + I_t)^{-1}}{\left(1 - d_t^{NA}\right) \rho_t + \frac{m_t^{RA} w_t}{\bar{y} a_t^{RA}} + \left(d_t^{NA} + \frac{m_t^{NA} w_t}{\bar{y} a_t^{NA}}\right) (1 + I_t)^{-1}}$$

Introducing these relations into the previous equation one get

$$\begin{aligned} V_t(w_t, \bar{y}, d_t^{NA}, I_{t-1;k}) = \max_{\{c_t\}} & \left[u(c_t) + \right. \\ & \delta \int_0^\infty \int_{-\infty}^\infty G_{t+1}^\gamma \left((1 - p_t) b_t u \left(\left((1 - m_t^{RA} - m_t^{NA}) w_t + \bar{y} - c_t \right) \frac{(1 + r_t^P)}{G_{t+1}} \right) + \right. \\ & \left. \left. p_t V_{t+1} \left(\left((1 - m_t^{RA} - m_t^{NA}) w_t + \bar{y} - c_t \right) \frac{(1 + r_t^P)}{G_{t+1}}, \bar{y}, d_{t+1}^{NA}, I_t \right) \right) dF(I_t) dF(r_t) \right] \end{aligned} \quad (3.40)$$

As we are looking for the numerical solution using the Gauss Quadrature method, the continuous random variable $\tilde{r}_t$ is approximated with the discrete random variable

$$dis \tilde{r}_t \sim \begin{pmatrix} r_{t;1} & r_{t;2} & \cdots & r_{t;n_r-1} & r_{t;n_r} \\ p_{r;1} & p_{r;2} & \cdots & p_{r;n_r-1} & p_{r;n_r} \end{pmatrix} \quad (3.41)$$

Let us assume that wealth gets only the values on the wealth grid $(w_{t,i})_{i=1}^{n_w}$ and nominal income coefficient d_t^{NA} takes values from the set $(d_{t,j}^{NA})_{j=1}^{n_d}$, where $d_{t,1}^{NA} = 0$ and $d_{t,n_d}^{NA} = 1$. Let us assume that we model inflation as a discrete state autoregressive process. We denote the states for inflation as $(I_{t-1,k})_{k=1}^{n_I}$ and the transitional matrix as $(p_{I;k,m})_{(k,m)=(1,1)}^{(n_I,n_I)}$

Thus, we actually find and save into the file the solution

$$\begin{aligned} & \left\{ \left(c_t^* (w_{t,i}, \bar{y}, d_{t,j}^{NA}, I_{t-1;k}) \right); \alpha_t^* (w_{t,i}, \bar{y}, d_{t,j}^{NA}, I_{t-1;k}) \right\}; m_t^{NA*} (w_{t,i}, \bar{y}, d_{t,j}^{NA}, I_{t-1;k}); \\ & m_t^{RA*} (w_{t,i}, \bar{y}, d_{t,j}^{NA}, I_{t-1;k}); V_t (w_{t,i}, \bar{y}, d_{t,j}^{NA}, I_{t-1;k}) \Big\}_{(i,j,k)=(1,1,1)}^{(n_w, n_d, n_I)} \end{aligned} \quad (3.42)$$

of the following equation

$$\begin{aligned}
V_t(w_{t;i}, \bar{y}, d_{t;j}^{NA}, I_{t-1;k}) &= \max_{\{c_{t;i,j,k}, \alpha_{t;i,j,k}, m_{t;i,j,k}^{NA}, m_{t;i,j,k}^{RA}\}} \left[u(c_{t;i,j,k}) + \right. \\
&\delta \sum_{l=1}^{n_t} \sum_{m=1}^{n_l} G_{t+1;i,j,k,m}^\gamma ((1-p_t) b_t u \left(\left((1-m_{t;i,j,k}^{RA} - m_{t;i,j,k}^{NA}) w_{t;i} + \bar{y} - c_{t;i,j,k} \right) \frac{(1+r_{t;i,j,k,l}^P)}{G_{t+1;i,j,k,m}} \right) + \quad (3.43) \\
&\left. p_t V_{t+1} \left(\left((1-m_{t;i,j,k}^{RA} - m_{t;i,j,k}^{NA}) w_{t;i} + \bar{y} - c_{t;i,j,k} \right) \frac{(1+r_{t;i,j,k,l}^P)}{G_{t+1;i,j,k,m}}, \bar{y}, d_{t+1;i,j,k,m}^{NA}, I_{t;k,m} \right) \right] p_{l;k,m} p_{r;l}
\end{aligned}$$

where

$$1 + r_{t;i,j,k,l}^P = 1 + r + \alpha_{t;i,j,k} (r_{t;l} - r),$$

and

$$G_{t+1;i,j,k,m} = (1 - d_{t;j}^{NA}) \rho_t + \frac{m_{t;i,j,k}^{RA} w_{t;i}}{\bar{y} a_t^{RA}} + \left(d_{t;j}^{NA} + \frac{m_{t;i,j,k}^{NA} w_{t;i}}{\bar{y} a_{t;k}^{NA}} \right) (1 + I_{t;k,m})^{-1},$$

and

$$d_{t+1;i,j,k,m}^{NA} = \frac{\left(d_{t;j}^{NA} + \frac{m_{t;i,j,k}^{NA} w_{t;i}}{\bar{y} a_{t;k}^{NA}} \right) (1 + I_{t;k,m})^{-1}}{(1 - d_{t;j}^{NA}) \rho_t + \frac{m_{t;i,j,k}^{RA} w_{t;i}}{\bar{y} a_t^{RA}} + \left(d_{t;j}^{NA} + \frac{m_{t;i,j,k}^{NA} w_{t;i}}{\bar{y} a_{t;k}^{NA}} \right) (1 + I_{t;k,m})^{-1}}.$$

Having the set of solutions (3.42) in hands, for each $k = 1, \dots, n_l$ we use cubic splines to interpolate the optimal consumption through the points $\{c_t^*(w_{t;i}, \bar{y}, d_{t;j}^{NA}, I_{t-1;k})\}_{(i,j)=(1,1)}^{(n_w, n_d)}$, the optimal asset allocation through the points $\{\alpha_t^*(w_{t;i}, \bar{y}, d_{t;j}^{NA}, I_{t-1;k})\}_{(i,j)=(1,1)}^{(n_w, n_d)}$, the optimal nominal annuitisation through the points $\{m_t^{NA*}(w_{t;i}, \bar{y}, d_{t;j}^{NA}, I_{t-1;k})\}_{(i,j)=(1,1)}^{(n_w, n_d)}$, and the optimal real annuitisation through the points $\{m_t^{RA*}(w_{t;i}, \bar{y}, d_{t;j}^{NA}, I_{t-1;k})\}_{(i,j)=(1,1)}^{(n_w, n_d)}$, and the value function through the points $\{V_t(w_{t;i}, \bar{y}, d_{t;j}^{NA}, I_{t-1;k})\}_{(i,j)=(1,1)}^{(n_w, n_d)}$. Then we have

$$\begin{aligned}
&\left\{ \left(c_t^*(w_t, \bar{y}, d_t^{NA}, I_{t-1;k}); \alpha_t^*(w_t, \bar{y}, d_t^{NA}, I_{t-1;k}); m_t^{NA*}(w_t, \bar{y}, d_t^{NA}, I_{t-1;k}); \right. \right. \\
&\left. \left. m_t^{RA*}(w_t, \bar{y}, d_t^{NA}, I_{t-1;k}); V_t(w_t, \bar{y}, d_t^{NA}, I_{t-1;k}) \right) \right\}_{k=1}^{N_t} \quad (3.44)
\end{aligned}$$

for $w_t \geq 0$, $0 \leq d_t^{NA} \leq 1$. Now, using (3.39) and the results from Appendix 1, we get

$$C_t^*(W_t, Y_t, d_t^{NA}, I_{t-1;k}) = \frac{Y_t}{\bar{y}} c_t^*(w_t, \bar{y}, d_t^{NA}, I_{t-1;k}), \text{ for } k = 1, \dots, n_l \quad (3.45)$$

$$\alpha_t^*(W_t, Y_t, d_t^{NA}, I_{t-1;k}) = \alpha_t^*(w_t, \bar{y}, d_t^{NA}, I_{t-1;k}), \text{ for } k = 1, \dots, n_l \quad (3.46)$$

$$m_t^{NA*}(W_t, Y_t, d_t^{NA}, I_{t-1;k}) = m_t^{NA*}(w_t, \bar{y}, d_t^{NA}, I_{t-1;k}), \text{ for } k = 1, \dots, n_I \quad (3.47)$$

$$m_t^{RA*}(W_t, Y_t, d_t^{NA}, I_{t-1;k}) = m_t^{RA*}(w_t, \bar{y}, d_t^{NA}, I_{t-1;k}), \text{ for } k = 1, \dots, n_I \quad (3.48)$$

$$V_t(W_t, Y_t, d_t^{NA}, I_{t-1;k}) = \left(\frac{Y_t}{\bar{y}} \right)^\gamma V_t(w_t, \bar{y}, d_t^{NA}, I_{t-1;k}), \text{ for } k = 1, \dots, n_I \quad (3.49)$$

for $W_t \geq 0$ and $Y_t \geq 0$, $0 \leq d_t^{NA} \leq 1$, and $I_{t-1;k}$ in the domain of values for the inflation rate. Thus, we have the solution to the problem (3.26)–(3.34) in the form

$$\begin{aligned} & \left(C_t^*(W_t, Y_t, d_t^{NA}, I_{t-1;k}); \alpha_t^*(W_t, Y_t, d_t^{NA}, I_{t-1;k}); m_t^{NA*}(W_t, Y_t, d_t^{NA}, I_{t-1;k}); \right. \\ & \left. m_t^{RA*}(W_t, Y_t, d_t^{NA}, I_{t-1;k}); V_i(W_t, Y_t, d_t^{NA}, I_{t-1;k}) \right)_{k=1}^{N_I} \end{aligned} \quad (3.50)$$

for $65 \leq t \leq 99$, $W_t \geq 0$ and $Y_t \geq 0$, $0 \leq d_t^{NA} \leq 1$, and $I_{t-1;k}$ takes discrete values in the domain of values for the inflation rate.

3.3.2 Solution for Exogenous m_t^{RA} and Endogenous m_t^{NA}

In order to develop the way to solve the remaining three cases explained in 3.1.2 we can use the results from 3.3.1. The algorithm for solving the problem (3.26)–(3.34) for the remaining types of problem in 3.1.2 is similar to the one used for solving the problem of endogenous m_t^{RA} and m_t^{NA} , and actually this is a special case of it. Let us now explain how we can solve the case of endogenous m_t^{NA} and exogenous m_t^{RA} .

Now, in equations (3.38) we write $\{C_t, \alpha_t, m_t^{NA}\}$ instead of $\{CV_t\}$. m_t^{RA} is now exogenous and it means it is not derived from the model itself but predefined earlier. Then, we follow the same steps as in the previous section. The only difference is that we know the value of m_t^{RA} and we are using its value in equation (3.38) and all the following equations, instead of finding its value from the model. As a result, we end up with the solution to the problem (3.26)–(3.34) as follows

$$\begin{aligned} & \left(C_t^*(W_t, Y_t, d_t^{NA}, I_{t-1,k}); \alpha_t^*(W_t, Y_t, d_t^{NA}, I_{t-1,k}); \right. \\ & \left. m_t^{NA*}(W_t, Y_t, d_t^{NA}, I_{t-1,k}); V_i(W_t, Y_t, d_t^{NA}, I_{t-1,k}) \right)_{k=1}^{N_I} \end{aligned} \quad (3.51)$$

for $65 \leq t \leq 99$, $W_t \geq 0$, $Y_t \geq 0$, $0 \leq d_t^{NA} \leq 1$, $I_{t-1;k}$ in the domain of values of the inflation rate, where $C_t^*(W_t, Y_t, d_t^{NA}, I_{t-1;k})$, $\alpha_t^*(W_t, Y_t, d_t^{NA}, I_{t-1;k})$ and $m_t^{NA*}(W_t, Y_t, d_t^{NA}, I_{t-1;k})$ are optimal consumption, optimal asset allocation and optimal percentage for purchasing nominal annuities. $V_t(W_t, Y_t, d_t^{NA}, I_{t-1;k})$ is the value function for those optimal control variables.

3.3.3 Solution for Endogenous m_t^{RA} and Exogenous m_t^{NA}

This case is mathematically the same as the previous one. The only difference is the change in the control variables. Instead of $\{C_t, \alpha_t, m_t^{NA}\}$ now we find the maximum value function by controlling the variables $\{C_t, \alpha_t, m_t^{RA}\}$. Now, the solution to the problem (3.26)–(3.34) is

$$\begin{aligned} & \left(C_t^*(W_t, Y_t, d_t^{NA}, I_{t-1;k}); \alpha_t^*(W_t, Y_t, d_t^{NA}, I_{t-1;k}); \right. \\ & \left. m_t^{RA*}(W_t, Y_t, d_t^{NA}, I_{t-1;k}); V_t(W_t, Y_t, d_t^{NA}, I_{t-1;k}) \right)_{k=1}^{N_t} \end{aligned} \quad (3.52)$$

for $65 \leq t \leq 99$, $W_t \geq 0$, $Y_t \geq 0$, $0 \leq d_t^{NA} \leq 1$, and $I_{t-1;k}$ in the domain of values for the inflation rate, where $C_t^*(W_t, Y_t, d_t^{NA}, I_{t-1;k})$, $\alpha_t^*(W_t, Y_t, d_t^{NA}, I_{t-1;k})$ and $m_t^{RA*}(W_t, Y_t, d_t^{NA}, I_{t-1;k})$ are optimal consumption, optimal asset allocation and optimal percentage for purchasing real annuities. $V_t(W_t, Y_t, d_t^{NA}, I_{t-1;k})$ is the value function for those optimal control variables.

3.3.4 Solution for Exogenous m_t^{RA} and m_t^{NA}

Again, the solution is very similar. The control variables are $\{C_t, \alpha_t\}$ now. The solution to the problem (3.26)–(3.34) is

$$\left(C_t^*(W_t, Y_t, d_t^{NA}, I_{t-1,k}); \alpha_t^*(W_t, Y_t, d_t^{NA}, I_{t-1,k}); V_t(W_t, Y_t, d_t^{NA}, I_{t-1,k}) \right)_{k=1}^{N_t} \quad (3.53)$$

for $65 \leq t \leq 99$, $W_t \geq 0$, $Y_t \geq 0$, $0 \leq d_t^{NA} \leq 1$, and $I_{t-1;k}$ in the domain of inflation values, where $C_t^*(W_t, Y_t, d_t^{NA}, I_{t-1;k})$, $\alpha_t^*(W_t, Y_t, d_t^{NA}, I_{t-1;k})$ are optimal consumption and optimal asset allocation. $V_t(W_t, Y_t, d_t^{NA}, I_{t-1;k})$ is the value function for those optimal control variables.

3.3.5 Solution for Other Endogenous/Exogenous m_t^{RA} and m_t^{NA}

We can also assume that m_t^{NA} and/or m_t^{RA} are endogenous or exogenous variables, i.e. that m_t^{NA} and/or m_t^{RA} are derived optimally and/or sub optimally from the model for each age $65 \leq t \leq 99$. For example, we can assume that at certain age m_t^{NA} and/or m_t^{RA} are endogenous and at some other age, these two variables are exogenous. Solving our problem (3.26)–(3.34) for any other assumption follows the same suit. The only difference is that we put m_t^{NA} and/or m_t^{RA} in the control variable or give them

predefined values depending on the endogenous/exogenous assumptions about the variables m_t^{NA} and/or m_t^{RA} .

3.3.6 Parallel Computation in Mathematica

As we said earlier, the problem (3.26)–(3.34) is converted into a form suitable for solving it on the computer. Actually, we solve the problem on eight computers and use the technique of parallel computations. It means that we transform the problem into a form suitable for distributing similar tasks to many processors, in our case 14 processors, where each processor solves one set of tasks and returns the results. Each processor works on the computer where Mathematica 5.2 is installed. In these circumstances, one computer is the master and others are slaves. The master computer runs Mathematica front-end and one kernel, while the others use Mathematica kernels only. The slave computers have two processors and one kernel runs on each of them, thus running two kernels at the same time on each slave computer.

All programming and storage of data is done on the master computer. We develop the programs using standard Mathematica 5.2 as the main programming language, supported with Parallel Toolkit 2.0 and Global Optimisation 5.2 package.

Parallel Toolkit 2.0 is an addition to Mathematica 5.2, which provides the tool for distributing tasks to many processors, which are run on the computers where Mathematica 5.2 and Parallel Toolkit 2.0 are installed. We also need to connect the computers in the appropriate way such that each processor recognises the master computer. Once we have this setup, we write the programs on the master computer which sets up the model and all the variables, procedures and functions. We use the standard Mathematica 5.2 as the front-end on the master computer. Parallel Toolkit 2.0 provides us with the tool to send tasks to other processors such that they receive the task, solve it and return the results to the master computer. The results are then collected to the master computer and regrouped such that they give us one full solution. The solution is stored on the master computer and available for further analysis. Parallel Toolkit 2.0 runs Mathematica kernel on the remote computers only, and we actually do not see these calculations, we only send tasks and get results. Using this technique, we have parallel computations which means that by running the problem on 14 processors we obtain the solution 14 times faster compared to running the same problem on just one computer.

The Global Optimisation 5.2 package provides a suite of tools for solving nonlinear optimization problems, as well as a variety of other applications such as finding the roots or zeros of a non-analytic function. As the authors say, the package is easier, simpler, and more robust than most optimisation tools, and we find it works quite efficiently. The range of functions for solving nonlinear optimisation problems uses different techniques. In order to find the best solution in the shortest time, we combined two main functions, GlobalSearch and MultiStartMin. Generally, both functions find the minimum of a nonlinear function of n variables with equality and inequality constraints. Both functions use a multiple-start generalised hill-climbing algorithm designed to work with or without constraints. The function itself starts the calculation from the random point in the domain of the solution and then finds the optimal solution and multiple-start algorithm means that the function starts from a predefined number of randomly chosen starting points. Then the function chooses the best solution obtained amongst solutions for each starting point. MultiStartMin handles highly nonlinear problems better and in order to solve this subset of problems, it handles constraints differently and is thus slower than GlobalSearch, particularly as problems get larger. Both functions are robust to noisy functions and local minima.

Let us now give some more details of the solution of our problem. On the computer we solve equation (3.43) for the values on the wealth grid $(w_{t,i})_{i=1}^{n_w}$, where $n_w = 51$. Nominal income coefficient d_t^{NA} takes values from the set $(d_{t,j}^{NA})_{j=1}^{n_d}$, where $n_d = 8$. The inflation grid $(I_{t,k})_{k=1}^{n_I}$ takes $n_I = 15$ values, and transitional matrix for inflation is $(I_{t;k,m})_{(k,m)=(1,1)}^{(n_I,n_I)}$. Grids, states and transitional matrix for inflation are presented in 3.4.1.

We calculate and store in the file the set of solutions (3.42) of equation (3.43). This is the point where we use parallel computing. We need to calculate the solution of equation (3.43) $n_w \times n_d \times n_I$ times.

One calculation is measured in seconds depending on the complexity of the calculation. We firstly apply the GlobalSearch algorithm for two random starting points. If we get two same solutions, we assume that this solution is correct and practice has shown that it is the correct assumption. If we get two different solutions, or only one solution, or no solution then we apply the MultiStartMin algorithm for three random starting points. Now, if we get one solution using the GlobalSearch algorithm then we compare the expected value functions obtained using different algorithms and take as the solution the one providing the highest expected value function. If no solution is obtained using the GlobalSearch algorithm then we take as

the solution the one providing the highest value function using the MultiStartMin algorithm. This approach has proved to lead to good solutions in each instance.

Rarely it is possible that two random starting points for the GlobalSearch algorithm and three random starting points for the MultiStartMin algorithm are not enough because we get unsatisfactory solutions in some instances. Then we increase the number of starting points. It means that we solve the problem a bit more slowly but we increase the quality of the solution. In some cases, useful criteria for finding the good solution are the precisions of the required solution in terms of number of decimal places up to which we compare resulting expected value function. Usually, we use 10^{-8} for ages 90–99, 10^{-7} for ages 80–89, 10^{-6} for ages 70–79, and 10^{-5} for ages 65–69. We determine these precision limits from experience such that we get smooth curves of the expected value functions, asset allocation functions, and annuitisation functions. If we do not get satisfactory solutions in terms of smooth curves, then we can try to improve the solutions by decreasing the powers mentioned earlier and we get solutions that are more precise. Thus, we have two main assumptions that can be changed in order to improve the quality of the solutions and these are the number of random starting points and the required precisions of the value functions.

Depending on the complexity of the instance, the time needed for obtaining one solution varies from 1 to 20 seconds. Rarely it can take more time but never more than 100 seconds. One calculation takes 10 seconds on average. If we work with deterministic inflation and if we work with 51 points of wealth grid and 8 points of d_t^{NA} grid then the calculation for one year of age takes $10 \cdot 51 \cdot 8 = 4000$ seconds or approximately one hour. However, we use parallel computation and the set of calculation tasks is distributed to 14 processors. As a result, it takes us just a few minutes to obtain the solution for one year of age. The time needed for the cubic splines, and storing results on the hard drive is not significant. If we work with stochastic inflation and if we take the inflation grid to have 15 points then our calculation is 15 times longer.

With the appropriate programs in Mathematica, we can make the full set of solutions for one setup of assumptions in one run of the program. In practice, we start the program, leave it for a couple of hours, and get solutions stored on the computer. Solutions are stored in excel files. As we will see later, stochastic inflation is not always necessary and in that case we store on the computer one excel file for each age and for five functions, optimal consumption, optimal asset allocation, optimal nominal annuitisation, optimal real annuitisation and derived optimal value function.

Altogether, it is $35 \cdot 5 = 175$ files. If we work with stochastic inflation then we have 15 times more, or 2625 files as the solution for one assumption of parameters. We save and name these files in a predetermined way and then they are easily manageable.

Transformation of the solutions from the form (3.44) into the form (3.50) is done afterwards when we have the set of the solution (3.44) stored on the master computer. This transformation is not time consuming once when we have the set of solutions (3.44) stored in files.

Solutions can be easily used for the analysis afterwards. Producing tables and graphs in excel is not time consuming once the appropriate excel files together with macros are made. We do not need parallel computing for this part of obtaining the results.

3.3.7 Check of Accuracy of Numerical Calculations in Mathematica

Once we have the solution saved in the excel files, we check that these solutions are accurate. In order to check the accuracy of the solution we make 2,000 simulations with the same assumptions as used for producing results. As a result, we get 2,000 random realisations of the paths of optimal consumption, optimal asset allocation, optimal nominal annuitisation, optimal real annuitisation, pension wealth income and paths of all other variables of interest. Equation (3.25) shows explicitly the expected value function as a discounted sum of utilities derived from future consumption and bequest. This equation can also be applied to the sample of random realisations. By analogy with the set of equations (3.26)–(3.34), we can write the set of equations for random realisations

$$V_{t;n}(W_t, Y_t, d_t^{NA}, I_{t-1;n}) = \sum_{i=t}^{99} \left(\delta^{i-t} \left(\prod_{k=t}^{i-1} p_k \right) \left(u(C_{i;n}) + \delta(1-p_i) b_i u(W_{i+1;n}) \right) \right) \quad (3.54)$$

where

$$W_{i+1;n} = \left((1 - m_{i;n}^{RA} - m_{i;n}^{NA}) W_{i;n} + Y_{i;n} - C_{i;n} \right) (1 + r_{i;n}^P) \quad (3.55)$$

$$Y_{i+1;n} = (1 - d_{i;n}^{NA}) \rho_t Y_{i;n} + \frac{m_{i;n}^{RA} W_{i;n}}{d_{i;n}^{RA}} + \left(d_{i;n}^{NA} Y_{i;n} + \frac{m_{i;n}^{NA} W_{i;n}}{d_{i;n}^{NA}} \right) (1 + I_{i;n})^{-1} \quad (3.56)$$

$$r_{i;n}^P = r + \alpha_{i;n} (r_{i;n} - r) \quad (3.57)$$

$$\rho_t = \begin{cases} replrate & t = 65 \\ 1 & 66 \leq t \leq 99 \end{cases} \quad (3.58)$$

$$d_{i+1,n}^{NA} = \frac{\left(d_{i,n}^{NA} Y_{i,n} + \frac{m_{i,n}^{NA} W_{i,n}}{a_{i,n}^{NA}} \right) (1 + I_{i,n})^{-1}}{\left(1 - d_{i,n}^{NA} \right) \rho_i Y_{i,n} + \frac{m_{i,n}^{RA} W_{i,n}}{a_{i,n}^{RA}} + \left(d_{i,n}^{NA} Y_{i,n} + \frac{m_{i,n}^{NA} W_{i,n}}{a_{i,n}^{NA}} \right) (1 + I_{i,n})^{-1}} \quad (3.59)$$

for $65 \leq t \leq 99$, for $(W_{t,n}, Y_{t,n}, d_{t,n}^{NA}, I_{t-1;k;n}) = (W_t, Y_t, d_t^{NA}, I_{t-1;k})$, for $t \leq i \leq 100$ and $n = 1, \dots, 2,000$, and where $C_{i,n}$, $\alpha_{i,n}$, $m_{i,n}^{NA}$ and $m_{i,n}^{RA}$ are optimal consumption, asset allocation and nominal and real annuitisation calculated from functions (3.45)–(3.49). $C_{i,n}$, $\alpha_{i,n}$, $m_{i,n}^{NA}$ and $m_{i,n}^{RA}$ depend on $(W_{i,n}, Y_{i,n}, d_{i,n}^{NA}, I_{i-1;k;n})$. $r_{i,n}$ and $I_{i,n}$ are random realisation from the stochastic simulation based on the assumptions in Table 3.3 and 3.4. Index n represents each random realisation. Thus, we get 2,000 values of discounted utilities derived from random realisations of consumption and bequest.

If our calculations using equations (3.26)–(3.34) are correct then the following equations should be valid

$$V_t(W_t, Y_t, d_t^{NA}, I_{t-1;k}) \approx \text{Mean}_{n=1, \dots, 2000} \left[V_{t,n}(W_t, Y_t, d_t^{NA}, I_{t-1;k}) \right] \quad (3.60)$$

We make calculations and check if the differences are very small. Usually, it appears to be less than 2% for 2000 random realisations. This variability depends on the assumptions, and particularly and significantly depends on the assumption of availability of annuities. If we have more annuitisation then the difference in equation (3.60) is lower than 2%, sometimes it is less than 0.1%. The difference in equation (3.60) will also decrease with an increase of the random sample, but we can say from analyses not presented here that 2,000 random realisations is enough to get insignificant differences and to see all of the basic rules as expected. So in all examples of random realisation we use the same number of $n = 2,000$ random realisation

In Section 3.4.7, we make left-tail analysis of the values of the function $V_{t,n}(W_t, Y_t, d_t^{NA}, I_{t-1;k})$. For this purpose we use the same realisations of the stochastic simulations as we use for the check of accuracy of the numerical calculations. We believe that this number of random realisations provides us with reasonably good results for the left-tail analysis as well. The deeper analysis of the pensioner's left-tail risk would require more than 2,000 random realisations. However, we investigated a couple of examples with 10,000 random realisations and no new results significantly different for the purposes of the investigation in this thesis were obtained.

So, we calculate the value function using a set of equations (3.26)–(3.34) and calculate the mean discounted utility derived from future consumption and bequest using equations (3.54)–(3.59). Then, for fixed $(W_t, Y_t, d_t^{NA}, I_{t-1;k})$, we compare the two and check if these two values are close to each other. Our criterion for the evaluation of accuracy of the results is to have $\text{Mean}_{n=1, \dots, 2000} [V_{t;n}(W_t, Y_t, d_t^{NA}, I_{t-1;k})]$ sometimes higher and sometimes lower than $V_t(W_t, Y_t, d_t^{NA}, I_{t-1;k})$, for different choices of $(W_t, Y_t, d_t^{NA}, I_{t-1;k})$, and that this difference is never higher than 2%. All our results have passed this test.

3.4 The Results

We now present the results obtained using the model developed in this chapter. We choose the results that are in our opinion the most representative and shed light on important assumptions, variables, and other parts of the model and the ways it influences results. However, the model can be used for solving a wide variety of problems and the results presented here are just some chosen examples. One can produce results for virtually any combinations of suboptimal and optimal asset allocation, and nominal and real annuitisation strategies. It can be done by defining m_t^{NA} and m_t^{RA} in an appropriate way. We also give results that can be compared with the results in Chapter 4. In Chapter 5, we will compare the results from this chapter and from Chapter 4.

When we have one particular assumption about the pensioner's exogenous/endogenous annuitisation for each age we refer to this particular example as a Case. If we assume exogenous annuitisation, then we also need the assumed proportion to be annuitised. As we said earlier, we sometimes use the term market instead of the term case. Market and case have the same meaning of the economic environment for the pensioner. We will concentrate on six cases and each of these can be connected to a type of problem described in Section 3.1.2.

Case 3.1 is connected to the type of problem 3.1 and we will assume no annuities at any age, and mathematically it means the assumption that $m_t^{NA} = 0$ and $m_t^{RA} = 0$, for $65 \leq t \leq 99$. Case 3.2 is connected to type of problem 3.2 and we will assume here optimal nominal annuitisation at age 65 only, no nominal annuitisation at other ages, and no real annuitisation at any age. Mathematically, Case 3.2 assumption is m_{65}^{NA} is endogenous, $m_t^{NA} = 0$ for $66 \leq t \leq 99$, and $m_t^{RA} = 0$, for $65 \leq t \leq 99$. Case 3.3 is calculated under the assumption of no nominal annuities at any age, optimal real

annuities at age 65 only and no real annuities afterwards. Mathematically, Case 3.3 assumption is $m_t^{NA} = 0$, for $65 \leq t \leq 99$, m_{65}^{RA} is endogenous, $m_t^{RA} = 0$ for $66 \leq t \leq 99$. Case 3.4 assumption is optimal nominal annuitisation at any age and no real annuitisation, or mathematically m_t^{NA} is endogenous and $m_t^{RA} = 0$, for $65 \leq t \leq 99$. Case 3.5 is connected to type of problem 3.5 and we assume no nominal annuities at any age and optimal real annuitisation at all ages. Mathematically, $m_t^{NA} = 0$ and m_t^{RA} is endogenous, for $65 \leq t \leq 99$. The best option for the pensioner is optimal nominal and real annuitisation at any age and this assumption will be investigated as Case 3.6, which is related to the type of problem 3.6.

In order to show clearly the assumption about the control variables for each case to be investigated, we present the assumptions regarding annuitisation in the following table. Consumption and asset allocation are always optimal.

Case	Nominal Annuitisation at age 65	Nominal Annuitisation at ages 66–99	Real Annuitisation at age 65	Real Annuitisation at ages 66–99
Case 3.1	Exogenous, $m_{65}^{NA} = 0$	Exogenous, $m_t^{NA} = 0$ for $66 \leq t \leq 99$	Exogenous, $m_{65}^{RA} = 0$	Exogenous, $m_t^{RA} = 0$ for $66 \leq t \leq 99$
Case 3.2	Endogenous, m_{65}^{NA} is optimal	Exogenous, $m_t^{NA} = 0$ for $66 \leq t \leq 99$	Exogenous, $m_{65}^{RA} = 0$	Exogenous, $m_t^{RA} = 0$ for $66 \leq t \leq 99$
Case 3.3	Exogenous, $m_{65}^{NA} = 0$	Exogenous, $m_t^{NA} = 0$ for $66 \leq t \leq 99$	Endogenous, m_{65}^{RA} is optimal	Exogenous, $m_t^{RA} = 0$ for $66 \leq t \leq 99$
Case 3.4	Endogenous, m_{65}^{NA} is optimal	Endogenous, m_t^{NA} is optimal for $66 \leq t \leq 99$	Exogenous, $m_{65}^{RA} = 0$	Exogenous, $m_t^{RA} = 0$ for $66 \leq t \leq 99$
Case 3.5	Exogenous, $m_{65}^{NA} = 0$	Exogenous $m_t^{NA} = 0$ for $66 \leq t \leq 99$	Endogenous, m_{65}^{RA} is optimal	Endogenous, m_t^{RA} is optimal for $66 \leq t \leq 99$
Case 3.6	Endogenous, m_{65}^{NA} is optimal	Endogenous, m_t^{NA} is optimal for $66 \leq t \leq 99$	Endogenous, m_{65}^{RA} is optimal	Endogenous, m_t^{RA} is optimal for $66 \leq t \leq 99$

Table 3.1 The assumptions about nominal and real annuitisation for each case.

3.4.1 Parameter Values

We chose to investigate thoroughly six Cases just defined. For each Case, we find the optimal solution for RRA coefficient γ taking values -1 , -4 and -9 , and the bequest motive coefficient b_t , for $65 \leq t \leq 99$ taking values 0 and 1. All together, we have six Cases and six combinations of coefficients, and overall 36 solutions. The values of other parameters in all of the basic numerical solutions are as follows:

– Income at age 65	$Y_{65} = 33,320.90$
– Replacement ratio	$\rho_{65} = 0.68212$, $\rho_t = 1$ for $65 \leq t \leq 99$
– Wealth at age 65	$W_{65} = 200,000$
– Risk free interest rate	$r = 0.02$
– Inflation	to be defined in 3.4.1.2
– Real rate on risky investment	$Ln(\tilde{r}_t) \sim N(\mu, \sigma^2)$ $\mu = 0.0474187$, $\sigma = 0.14731$ $E[\tilde{r}_t] = 0.06$, $StD[\tilde{r}_t] = 0.157$
– Survival (Mortality) table	Interim life table produced by The Government Actuary's Department for United Kingdom Males, based on data for years 2002–2004
– Discount factor	$\delta = 0.96$

The parameter values are chosen in accordance with the parameter values chosen by many authors who investigated similar problems, for example Cocco et al (2005).

Two particularly coefficients, income at age 65 and replacement ratio are very precise numbers. We develop these two numbers in accordance with the work of Cocco et al (2005). They fitted a third-order polynomial to the age dummies and propose the three labour income processes depending on the individual's education, for the individuals no high-school, for the individuals with high-school and with college education. They obtain the average income from age 20 to age 75. Income at age 65 is obtained using the following function

$$18,127 \cdot (0.002 \cdot age^3 - 0.0323 \cdot age^2 + 0.1682 \cdot age - 2.1700)$$

The polynomial in the previous equation is the same as proposed by Cocco et al (2005) for the individual with high-school education. The amount of the coefficient 18,127 is taken such that the pensioner's first yearly salary at age 21 was 20,500. We take higher value of the first salary than the one taken by Cocco et al (2005), in order

to get slightly higher income at age 65 such that the individual investigated here is slightly richer person than the individual with high-school education in Cocco et al (2005) and in the same time slightly poorer person than the individual with college education in Cocco et al (2005). So, we investigate the pensioner with the income at age 65 in between the average individual with high-school and college education. The value of the replacement ratio is the same as those proposed Cocco et al (2005) for the individual with high-school education.

The choice of the amount of the pension wealth at age 65 is also based on the work by Cocco et al (2005). They investigate similar values of the pension wealth at age 65.

We acknowledge here that these assumptions about the pensioner's income at age 65 and his pension wealth are above the average values and it means that we investigate the richer pensioner than average. The results presented up to Section 3.4.3 are not dependent on the pensioner's last salary and pension wealth at age 65. In the sections afterwards, the values of the pensioner income at age 65, replacement ratio and pension wealth at age 65 are important for the numerical results obtained. However, as we will see later, our choice of these parameters results in a quite wide range of the optimal decisions depending on the pensioner's attitude towards risk and bequest. The wide range of the results is also one reason for choosing these values of the pensioner wealth and income at age 65. Sensitivity analysis based on the model and the results developed here would give us the answers for the pensioners with different pension wealth and income at age 65. In this thesis we will focus our investigation in Sections 3.4.3 and onwards on these, richer pensioners.

3.4.1.1 Grids

When making numerical calculations one needs to approximate all continuous variables with discrete ones. We solve equation (3.43) on the computer and from this equation one can see that we have four continuous variables which need to be approximated by the discrete ones. These are: wealth, nominal income coefficient, rate of return on equities and inflation.

Wealth is approximated with values on the wealth grid $(w_{t,i})_{i=1}^{n_w}$, where $n_w = 51$ and $0 \leq w_{t,i} \leq 40$ and $w_{t,i}$ are taken such that the grid is denser for smaller values. The wealth grid points are given in the following table.

wealth grid $(w_{t,i})_{i=1}^{n_w}$, $n_w = 51$, $65 \leq t \leq 99$	0.000	0.019	0.040	0.061	0.084	0.109	0.135	0.162	0.192	0.223
	0.257	0.293	0.332	0.373	0.418	0.467	0.519	0.576	0.637	0.704
	0.777	0.857	0.944	1.040	1.146	1.263	1.393	1.537	1.697	1.878
	2.080	2.309	2.568	2.864	3.202	3.591	4.041	4.566	5.179	5.903
	6.763	7.791	9.030	10.538	12.390	14.686	17.563	21.210	25.890	31.975
	40.000									

Table 3.2 Wealth grid

The range for the wealth grid is 0 to 40, and one can see that it is not an equally spaced grid. The differences between points are smaller for smaller values of wealth and become larger as we approach 40. This is done because of the curvature of the value function. The value function is negative and an increasing function of wealth and, if all other variables are constant, for smaller values the value function increases very steeply. As wealth increases, the value function increases more slowly. After a certain value, it becomes almost a flat horizontal line. Due to this characteristic shape of the value function it is important to have more wealth grid points for smaller wealth in order to capture the behaviour of the value function for these wealth values. For the larger values of pension wealth, we can capture the value function behaviour with a less dense wealth grid. Actually, we need larger values of wealth to ensure a stable solution while reasonable wealth values to be analysed will be in the range 0 to 4.

The nominal income coefficient d_t^{NA} takes values from the set $(d_{t,j}^{NA})_{j=1}^{n_d}$, where $n_d = 8$ and $d_{t,j}^{NA}$ are equally spaced on the interval $[0,0.91]$, i.e. taking values $\{0,0.13,0.26,...,0.78,0.91\}$. It appears to be a good enough grid because d_t^{NA} in reality will not take values above 0.5. Again, we use larger values for reasons of solution stability only.

The rate on equities is approximated with $n_r = 15$ points. The first and the third rows in the following table present possible states of real rate on equities, i.e. the values of $r_{t,l}$ in the first row of (3.41) and the second and fourth rows give the probabilities for attaining those states, i.e. the values of $p_{t,l}$ in the second row of (3.41), where $1 \leq l \leq n_r$.

Rate on equities – state	0.661	0.685	0.727	0.786	0.859	0.943	1.035	1.131
Rate on equities – probability	0.044	0.203	0.871	3.394	10.028	19.715	24.680	20.287
Rate on equities – state	1.227	1.319	1.403	1.476	1.535	1.577	1.601	
Rate on equities – probability	11.850	5.441	2.176	0.828	0.319	0.124	0.040	

Table 3.3 Distribution of discrete approximation of rate on risky investment.

With this approximation of equity rate, we have that $E[\tilde{r}_t] = 0.05914$ and $StD[\tilde{r}_t] = 0.154443$.

3.4.1.2 Inflation

We assume that consumption, wealth, and real annuities are all in real terms. It means that inflation does not influence their values through the time. Income partly comes from nominal annuities and thus is affected by inflation. When we investigate nominal annuities, and all other processes are in real terms then the constant nominal income provided by nominal annuity is changing in real terms. Thus, for a nominal annuity, we need to adjust nominal income with inflation in order to get real income.

We have defined the inflation process in 3.2.1. Let us now define the inflation process numerically. It is defined as a discrete time–state space stochastic process, $\tilde{I}_{t+1} = \mu + (1 - \psi)I_t + \sigma_I \tilde{\varepsilon}_t$, where $\mu_I = 0.024$, $\psi_I = 0.6$, $\sigma_I = 0.02$, $\tilde{\varepsilon}_t \sim N(0,1)$, and here $E[I_{t,\infty}] = 0.04$ and $StD[I_{t,\infty}] \approx 0.016$. The approximation follows Tauchen (1986). Assuming fifteen possible states, $n_I = 15$ of yearly inflation, we get the following states and transition matrix

	0.05	0.25	0.61	1.10	1.72	2.42	3.20	4.00	4.80	5.58	6.28	6.90	7.39	7.75	7.95
0.05	1.84	4.86	9.18	14.71	19.59	20.40	15.73	8.72	3.51	1.08	0.28	0.07	0.02	0.00	0.00
0.25	1.66	4.43	8.51	13.96	19.14	20.62	16.49	9.50	3.98	1.27	0.34	0.08	0.02	0.01	0.00
0.61	1.37	3.73	7.37	12.61	18.20	20.81	17.75	10.95	4.90	1.67	0.47	0.12	0.03	0.01	0.00
1.10	1.03	2.87	5.92	10.73	16.63	20.65	19.28	13.06	6.42	2.40	0.74	0.21	0.06	0.02	0.00
1.72	0.70	2.01	4.35	8.48	14.36	19.75	20.62	15.69	8.67	3.62	1.23	0.37	0.11	0.04	0.01
2.42	0.42	1.26	2.90	6.14	11.53	17.83	21.17	18.42	11.64	5.53	2.11	0.71	0.23	0.08	0.02
3.20	0.23	0.71	1.75	4.05	8.49	14.94	20.41	20.55	15.04	8.21	3.57	1.35	0.48	0.17	0.05
4.00	0.11	0.36	0.96	2.43	5.72	11.52	18.21	21.37	18.21	11.52	5.72	2.43	0.96	0.36	0.11
4.80	0.05	0.17	0.48	1.35	3.57	8.21	15.04	20.55	20.41	14.94	8.49	4.05	1.75	0.71	0.23
5.58	0.02	0.08	0.23	0.71	2.11	5.53	11.64	18.42	21.17	17.83	11.53	6.14	2.90	1.26	0.42
6.28	0.01	0.04	0.11	0.37	1.23	3.62	8.67	15.69	20.62	19.75	14.36	8.48	4.35	2.01	0.70
6.90	0.00	0.02	0.06	0.21	0.74	2.40	6.42	13.06	19.28	20.65	16.63	10.73	5.92	2.87	1.03
7.39	0.00	0.01	0.03	0.12	0.47	1.67	4.90	10.95	17.75	20.81	18.20	12.61	7.37	3.73	1.37
7.75	0.00	0.01	0.02	0.08	0.34	1.27	3.98	9.50	16.49	20.62	19.14	13.96	8.51	4.43	1.66
7.95	0.00	0.00	0.02	0.07	0.28	1.08	3.51	8.72	15.73	20.40	19.59	14.71	9.18	4.86	1.84

Table 3.4 The transition matrix for inflation process. The values in the first row are possible random states of the inflation rate in the current year, the first column are possible known states of inflation rate in the previous year. The values crossing row and column in the table are probabilities of transition from known rate in the first column to random rate in the first row. The values are in percentages.

All values in the table are in percentages. The values in the Table 3.3 are rounded to two decimal places for presentation purposes only, and we actually work with as many decimal places as we need. Values 0.00% that appear in the table have positive values but are less than 0.01%.

If we are at the beginning of the year, then the first row are possible states of inflation rate in the previous year, and the first column are possible states of random yearly inflation rate in the coming year. The data in the table are probabilities for moving from the state in the first row to the state in the first column.

One can see that the sum of values in Table 3.3 in each row is equal to 100. If we observe one particular row then the values in the table in this row show probabilities for all possible states after one year and thus the sum of probabilities for all possible states must be one.

For example, if the pensioner's age is exactly 70 and he knows that the inflation in the previous year was 2.42%, then during the year when his age will move from 70 to 71 the inflation will be 4.80% with a probability of 11.64%.

In equation (3.4) we need an expected annual inflation rate at time t for the period of next i years, denoted as $E[\tilde{I}_{t,i}]$. The following table show some values of $E[\tilde{I}_{t,i}]$ for different inflation rates during the year prior to attaining age t and for different periods of the following i years.

	1	2	3	4	5	6	7	8	9	10	15	20	25	30	35
0.05	2.17	2.64	2.95	3.16	3.31	3.42	3.50	3.57	3.62	3.66	3.78	3.84	3.88	3.90	3.92
0.25	2.24	2.70	2.99	3.20	3.34	3.45	3.53	3.59	3.64	3.67	3.79	3.85	3.88	3.91	3.92
0.61	2.39	2.80	3.08	3.26	3.40	3.49	3.57	3.62	3.67	3.70	3.81	3.86	3.90	3.92	3.93
1.10	2.59	2.96	3.20	3.36	3.48	3.56	3.62	3.67	3.71	3.74	3.84	3.88	3.91	3.93	3.94
1.72	2.87	3.16	3.36	3.49	3.58	3.65	3.70	3.74	3.77	3.80	3.87	3.91	3.93	3.95	3.96
2.42	3.21	3.41	3.55	3.64	3.71	3.76	3.80	3.82	3.85	3.86	3.92	3.94	3.96	3.97	3.98
3.20	3.59	3.70	3.77	3.82	3.86	3.88	3.90	3.92	3.93	3.94	3.97	3.98	3.99	4.00	4.00
4.00	4.00	4.01	4.01	4.01	4.01	4.02	4.02	4.02	4.02	4.02	4.02	4.02	4.02	4.02	4.02
4.80	4.41	4.31	4.25	4.20	4.17	4.15	4.13	4.12	4.11	4.10	4.08	4.06	4.06	4.05	4.05
5.58	4.79	4.60	4.47	4.38	4.32	4.27	4.24	4.21	4.19	4.18	4.13	4.10	4.09	4.08	4.07
6.28	5.13	4.85	4.66	4.53	4.44	4.38	4.33	4.29	4.26	4.24	4.17	4.13	4.11	4.10	4.09
6.90	5.41	5.05	4.82	4.66	4.55	4.47	4.41	4.36	4.32	4.29	4.20	4.16	4.13	4.11	4.10
7.39	5.61	5.20	4.94	4.75	4.63	4.53	4.46	4.41	4.37	4.33	4.23	4.18	4.15	4.13	4.11
7.75	5.76	5.31	5.02	4.82	4.68	4.58	4.50	4.44	4.40	4.36	4.25	4.19	4.16	4.14	4.12
7.95	5.83	5.37	5.06	4.85	4.71	4.60	4.52	4.46	4.41	4.38	4.26	4.2	4.17	4.14	4.13

Table 3.5 The expected annual inflation rate for the period of next i years. The values in the first row are duration of next i years, and in the first column are possible known states of inflation rate in the previous year. The values that cross row and column in the table are expected annual inflation for the duration of i years assuming a known rate in the first column in the previous year. The values, apart from the first row, are in percentages.

The values in the first column are possible inflation rates during the year prior to the point of time when we calculate $E[\tilde{I}_{t,i}]$. Values in the first row represent the number of years i . All values in the table apart from the first row are in percentages.

For example, if the pensioner's age is 65 exactly, then $t = 65$. We assume that, for example, the inflation during the previous year was 1.72%. Then, expected inflation during pensioner's age $[65, 66]$ is 2.87%. Also, the expected annual inflation during pensioner's age $[65, 75]$ is 3.80%.

3.4.1.3 Survival Rates

We assume that the pensioner's subjective survival rates are the same as the survival rates used for the calculation of annuity rate. We assume that the pensioner's survival (mortality) rates follow Interim life table produced by The Government Actuary's Department for United Kingdom males, based on the data for years 2002–2004. We present in Table 3.6, for a given pensioner's age, the values of probability that the pensioner will survive at least one year more.

Age	Probability	Age	Probability	Age	Probability
65	98.345%	78	93.734%	91	79.842%
66	98.183%	79	93.001%	92	77.916%
67	97.987%	80	92.331%	93	76.015%
68	97.779%	81	91.530%	94	74.818%
69	97.513%	82	90.808%	95	71.908%
70	97.309%	83	90.135%	96	70.490%
71	96.960%	84	89.222%	97	68.573%
72	96.631%	85	88.068%	98	66.677%
73	96.274%	86	86.314%	99	0.000%
74	95.813%	87	85.084%		
75	95.349%	88	83.718%		
76	94.855%	89	82.307%		
77	94.305%	90	81.416%		

Table 3.6 Survival rates – Interim life table produced by The Government Actuary's Department for United Kingdom Males, based on the data for years 2002–2004.

3.4.2 Optimal Consumption, Asset Allocation and Annuitisation

Before investigating the simulations, we present the main results as functions of age and wealth, and of wealth and nominal income coefficient.

In our investigation, inflation is assumed to be random. However, this assumption significantly influences only some of the results. In a number of results, random inflation has a negligible influence. For reasons of a clearer presentation, we will often present results obtained using the assumption of constant inflation and when random inflation brings new information and conclusions we will also present these results. If we present results with constant inflation then we assume that $I_{t,k} = 4\%$ for $64 \leq t \leq 99$ and $1 \leq k \leq n_t$.

In Section 3.4.2, we present deterministic numerical values of the control variables obtained by solving equations (3.26)–(3.34). These results are not dependent on the realisation of random variables but on their distributions only. In subsequent sections, we will concentrate on the analysis of the value function, and on the results obtained from simulations of the random paths of stochastic equity return and inflation.

Regarding the results in this section, we will usually have three–dimensional surfaces where one dimension is pension wealth, the second one is age or nominal income coefficient and the third dimension is the value of optimal consumption or optimal proportion of asset allocation or optimal annuitisation. We will also have two–dimensional graphs where again one dimension is pension wealth and the other is the proportion to be annuitised. So, we should read these figures as results of calculations for given wealth and possibly one other variable. If the x and y axes are wealth and age then the surface shows the possible paths of the value presented on the surface for one particular realisation of random variables. For example, in Figure 3.2 the surface gives us the value of optimal asset allocation for a particular level of pension wealth and age.

3.4.2.1 Case 3.1 – Dependence on Wealth and Age

Let us first present optimal consumption in Case 3.1. One can see that optimal consumption is always an increasing function of age and wealth. The more pension wealth the pensioner possesses the more he consumes, which is an expected result. The increase in optimal consumption as the pensioner’s age increases while the wealth is fixed can be explained by the decreasing incentive to save as age increases. The older pensioner has a lower expected remaining lifetime and their incentive to save decreases, and so consumption increases. The pensioner with no bequest motive loses the incentive to save and his consumption increases faster with age. The pensioner with a bequest motive still loses the incentive to save, but due to the utility from the bequest this decrease is lower compared to the pensioner with no bequest motive. One can see that the shapes of the surfaces in Figure 3.1 are similar. For the more risk averse pensioner, the surface of optimal consumption moves downwards and becomes flatter at early ages. A similar characteristic of the surfaces is seen when the bequest motive is introduced. The nominal income coefficient is constant and $d_t^{NA} = 0$, for all ages $65 \leq t \leq 99$ because there is no nominal annuitisation in Case 3.1 and Figure 3.1 refers to this assumption.

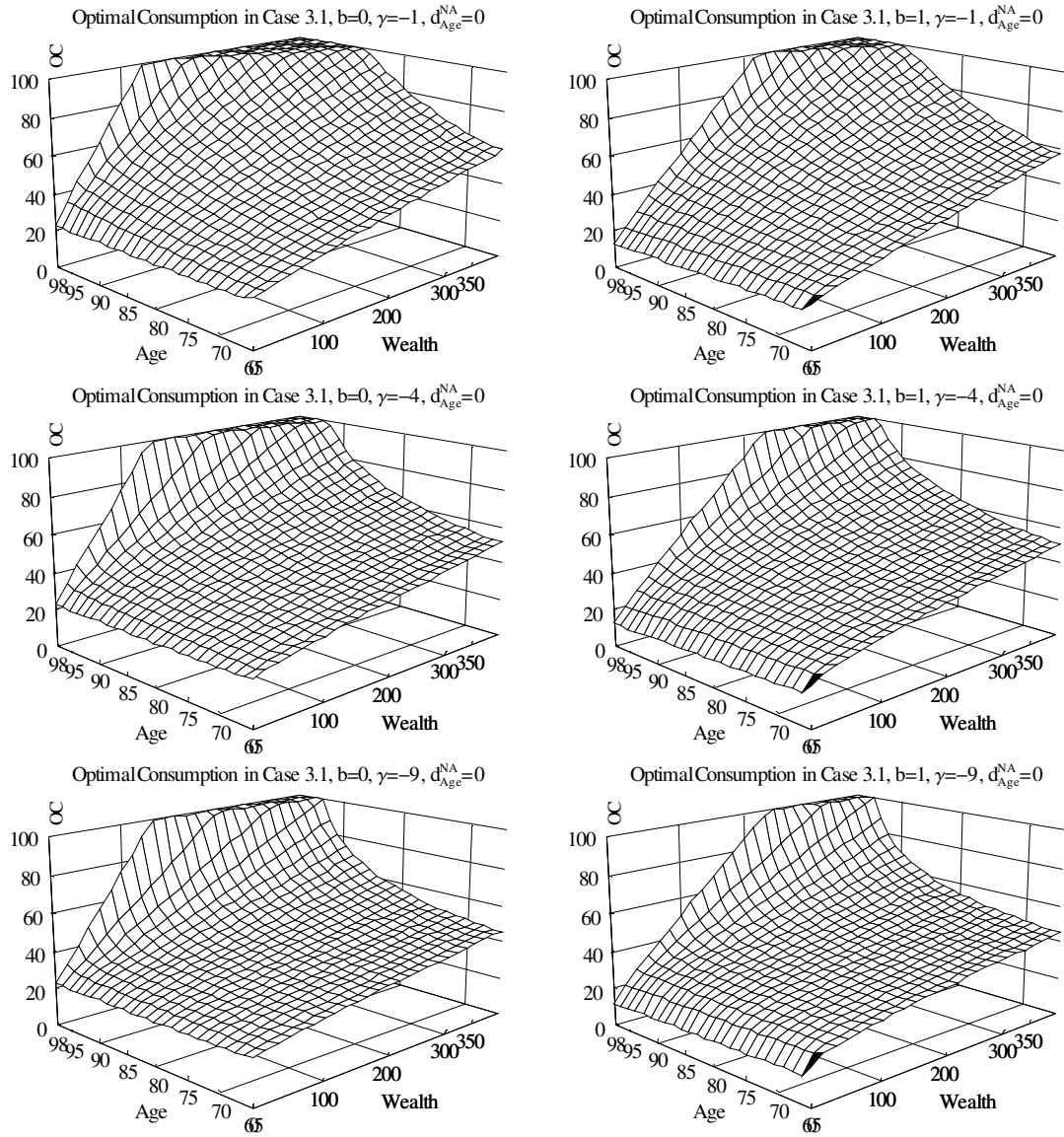


Figure 3.1 Optimal consumption in Case 3.1, for the values of RRA coefficient γ taking values -1 , -4 , and -9 , and for bequest motive coefficient b_t taking values 0 and 1 . Wealth and Optimal Consumption values are in thousands.

Optimal asset allocation as a function of wealth and age in Case 3.1 is presented in Figure 3.2.

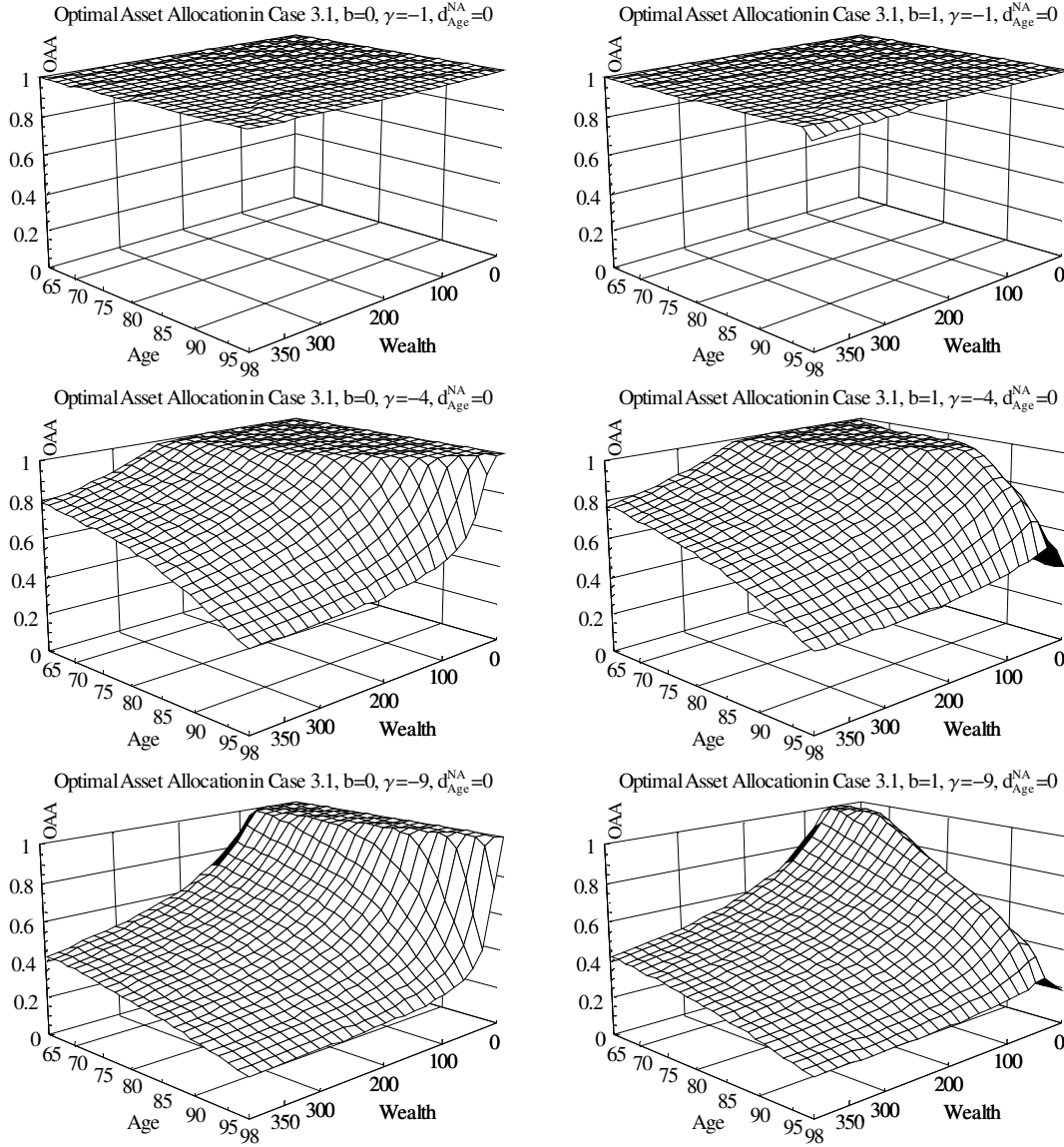


Figure 3.2 Optimal asset allocation in Case 3.1, for the values of RRA coefficient γ taking values -1 , -4 , and -9 , and for bequest motive coefficient b_t taking values 0 and 1. Wealth values are in thousands, Optimal asset allocation values are in percentages.

We can see that the less risk averse pensioner with RRA coefficient $\gamma = -1$ invests all of his available assets into the risky asset for all ages and for all reasonable values of pension wealth. For the more risk averse pensioner, for $\gamma = -4$ and $\gamma = -9$ and no bequest motive, the pensioner would invest a lower percentage of pension wealth into the risky asset at older ages. Also, if more wealth is available then a lower percentage of pension wealth is invested into the risky asset. Optimal allocation into risky asset decreases faster for the more risk averse pensioner with RRA coefficient $\gamma = -9$. For the pensioner with a bequest motive, we can see a very similar optimal asset allocation for wealth above 50,000 approximately. Below these values, the pensioner

with a bequest motive would invest less into the risky asset due to the risk of not gaining utility derived from the bequest.

In Figure 3.2, for the pensioner with no bequest motive, we observe that optimal asset allocation decreases for higher values of the pension wealth and also for the later ages. This is due to the implicit possession of the risk free assets in a form of income from social security. We recall here that the pensioner in Case 3.1 has no access to annuities but has constant, risk free income from social security. Firstly, for a given age, optimal asset allocation decreases as pension wealth increases. If we understand the risk free income from social security as a form of risk free asset already in the possession of the pensioner then the pensioner will adjust the optimal asset allocation of his pension wealth according to the amount of the pension wealth. For the lower values of the pension wealth, he will optimally invest more into equities in order to balance the implied risk free assets in already in possession. As his pension wealth increases, the relative amount of the implied risk free asset decreases, and the pensioner optimally invests a lower part of his pension wealth into equities. Secondly and similarly, for a given amount of the pension wealth, optimal asset allocation decreases as the age of the pensioner increases. The reason lies in the fact that the amount of the implied risk free asset in a form of income from social security is larger for a younger pensioner. So, an older pensioner will have a smaller amount of the risk free asset already in a possession and consequently he will invest a higher portion of his pension wealth into risk free asset and a lower portion into risky asset. Similar findings are done by Cocco et al (2005) and Merton (1971).

For the pensioner with a bequest motive, we find decreasing proportion of the pension wealth optimally invested into equities as the pensioner gets older. For the higher values, say higher than 50,000, we also find decreasing proportion of the pension wealth optimally invested into equities as the pension wealth increases. However, for the pensioner with bequest motive and for lower values of the pension wealth, we find decreasing optimal equity allocation as the pension wealth decreases from say 50,000 to 0. The reason is that income from social security is implied risk free asset in a possession of the pensioner but he cannot transfer these assets to his heirs. So, for a lower value of the pension wealth the pensioner needs protection in a form of risk free investment for the pension wealth to be bequeathed to his heirs. So, as the pension wealth decreases and the age of the pensioner is constant, optimal asset allocation for the pensioner with a bequest motive will decrease in order to protect the amount to be bequeathed.

3.4.2.2 Cases 3.2 and 3.3 – Dependence on Wealth and Age

In Cases 3.2 and 3.3, the pensioner has access to nominal and real annuities respectively, at age 65 only and no other annuitisation is possible.

We can use formulae (3.39) and (3.45) and the surfaces in Figure 3.1 to read consumption in Case 3.2 and 3.3. Actually, we use the same technique for reading consumption for all cases with annuitisation. Figure 3.1 shows consumption in the case with no annuities. It means that we assume that income is constant and equal to social security income. If we assume that $Y_{65} = 33,320.90$ and that $\rho_{65} = 0.68212$, then income from social security is $Y_t^{SS} = 22,728.85$ for $66 \leq t \leq 99$. Thus, Figure 3.1 shows optimal consumption under the assumption that real income after age 65 is $Y_t^{3.1} = Y_t^{SS} = 22,728.85$. Now, if we have annuitisation then the income in real terms is changed. If we denote income in Case 3.3 with $Y_t^{3.3}$ then using (3.39) we have

$$W_t^{3.1} = W_t^{3.3} \frac{Y_t^{3.1}}{Y_t^{3.3}}.$$

Now using (3.45) we have

$$C_t^{3.3*}(W_t^{3.3}, Y_t^{3.3}, d_t^{NA}, I_{t-1;k}) = \frac{Y_t^{3.3}}{Y_t^{3.1}} C_t^{3.1*}\left(W_t^{3.3} \frac{Y_t^{3.1}}{Y_t^{3.3}}, Y_t^{3.1}, d_t^{NA}, I_{t-1;k}\right).$$

So, if we want to determine the optimal consumption $C_t^{3.3*}(W_t^{3.3}, Y_t^{3.3}, d_t^{NA}, I_{t-1;k})$ in Case 3.3 using Figure 3.1, we firstly need to calculate value $W_t^{3.1}$. Then we read optimal consumption $C_t^{3.1*}(W_t^{3.1}, Y_t^{3.1}, d_t^{NA}, I_{t-1;k})$ from the surface in Figure 3.1, and then multiply $C_t^{3.1*}(W_t^{3.1}, Y_t^{3.1}, d_t^{NA}, I_{t-1;k})$ with

$$\frac{Y_t^{3.3}}{Y_t^{3.1}}.$$

Using this technique, we can determine the optimal consumption from Figure 3.1 for any value of income. Obviously, there will be many different optimal consumption paths depending on annuitisation and income from nominal or real annuities. One should be aware that optimal real income is constant in Case 3.3, while in Case 3.2 income decreases in real terms with age. It is also worth mentioning that in Case 3.1 pension wealth for any particular pensioner decreases slowly with age, while in Case 3.2 and 3.3 optimally it usually decreases sharply at age 65 and then takes low values at later ages. This feature can be seen in Figure 3.12.

Regarding optimal asset allocation, we find that annuities are better options for the pensioner with no bequest motive than the riskless asset for all reasonable combinations of pension wealth and income. We find that for the more risk averse pensioner, it is optimal to keep part of his pension wealth in the riskless asset if the pension wealth is about 50,000 or more. However, it is optimal for the more risk averse pensioner to annuitise such a large part of his pension wealth at age 65 so that his remaining pension wealth after age 65 is below the level of the pension wealth where keeping riskless asset is optimal.

In Figure 3.3, we choose to show the graphs for three combinations of RRA and bequest motive coefficients. The graphs for other combinations of RRA and bequest motive coefficients have similar patterns.

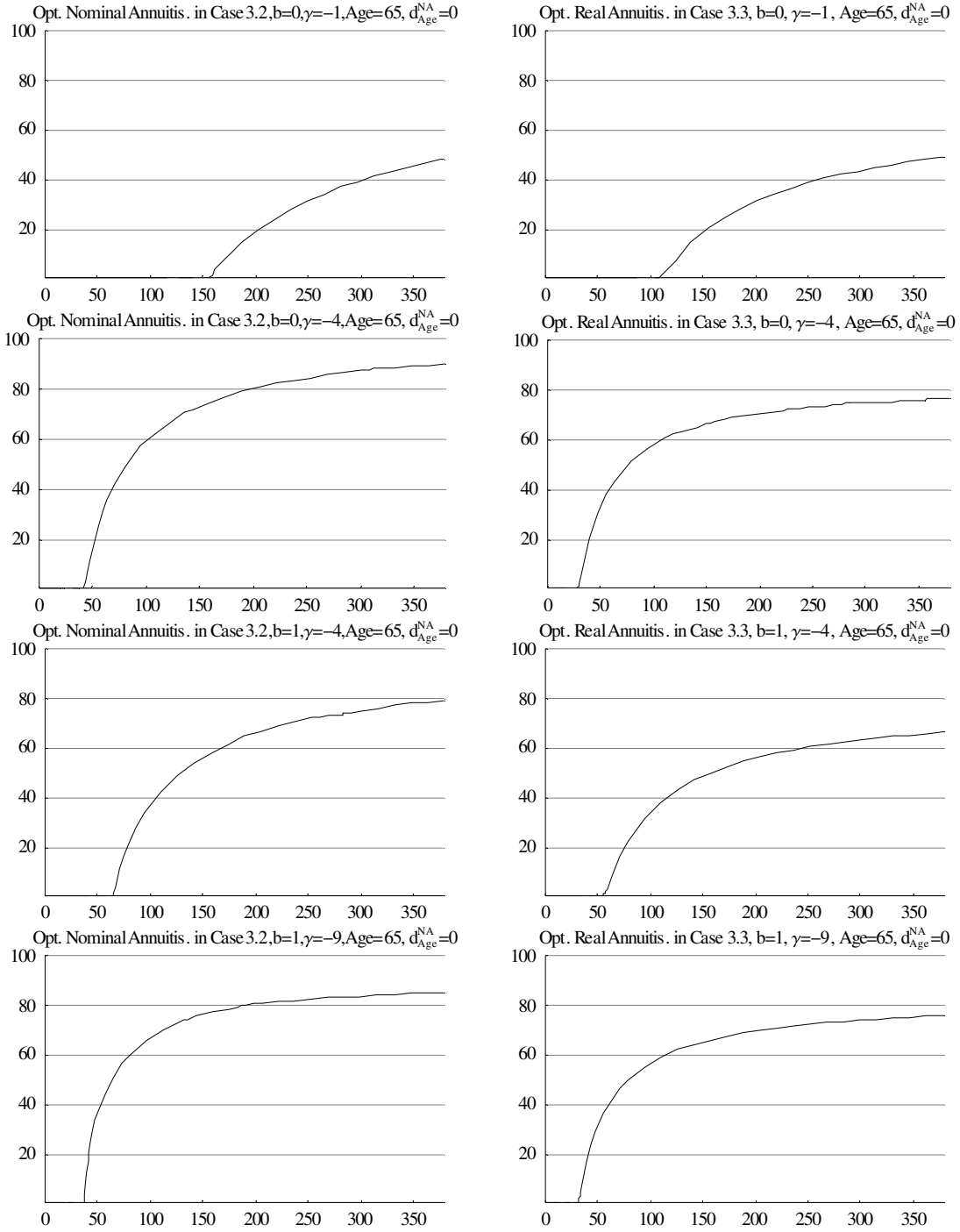


Figure 3.3 x-axis shows pension wealth in thousands, and y-axis show percentages of pension wealth annuitised at age 65 in Cases 3.2 and 3.3 and for three combinations of RRA and bequest motive coefficients, i.e. $\gamma = -1$ and $b = 0$, $\gamma = -4$ and $b_l = 1$, and $\gamma = -9$ and $b_l = 1$.

If we observe the example for $\gamma = -1$ and $b = 0$ in the left upper corner of Figure 3.3, we can see that taking any nominal annuity becomes optimal when pension wealth is larger than 170,000. At the same time, for the pensioner who decides to take real

annuities at age 65 it becomes optimal to take any real annuity when his pension wealth is about 120,000.

If we observe the more risk averse pensioner who has RRA and bequest motive coefficients $\gamma = -9$ and $b = 1$, then if he is taking nominal annuities it is optimal for him to start taking them when his pension wealth exceeds 45,000, while for the pensioner who is going to take real annuity this point is somewhere around 35,000. If the pensioner in the market modelled via Case 3.2 possesses the amount of 100,000 of pension wealth, then it is optimal to annuitise about 65% of this amount. If the pensioner with the same amount of pension wealth is in the Case 3.3 market, then it is optimal for him to annuitise about 55% of his pension wealth at age 65.

If we observe the pensioner in Case 3.2 having RRA and bequest motive coefficients $\gamma = -4$ and $b = 0$, then it is optimal for him to start purchasing annuities with the pension wealth of about 45,000, while for the pensioner in the same case and with the same attitude towards risk but with the bequest motive it is optimal to start purchasing annuities when his pension wealth is about 60,000. Also, we observe that the percentage of the pension wealth to be optimally annuitised for the pensioner with no bequest motive increasing faster than for the pensioner with the bequest motive. Similarly, in Case 3.3 and for $\gamma = -4$, the demand for annuities starts for the lower values of the pension wealth and then increases faster for the pensioner with no bequest motive than for the pensioner with the bequest motive.

The following general conclusions can be drawn from Figure 3.3 and from other related examples not presented here. Firstly, when we compare the pensioners with the same RRA and bequest motive coefficients but one in Case 3.2 and the other in Case 3.3, we observe that the curve representing optimal annuitisation in Case 3.2 as a function of pension wealth increases similarly or more quickly as a function of pension wealth than in Case 3.3. Secondly, taking any annuity becomes optimal for the pensioner for larger amounts of pension wealth in Case 3.2 than in Case 3.3. So, we find that the pensioner having access to nominal annuities at age 65 only has more demand for annuities than the pensioner having access to real annuities at age 65 only. The reason for this finding lies in the decreasing, in real terms, income from the nominal annuities in Case 3.2. So the pensioner optimally purchases more annuities in Case 3.2 compared to Case 3.3 in order to compensate for the lower income from annuities in real terms in the later years.

Observing the changes in the pensioner's risk aversion, we find that the more risk averse pensioner optimally takes any annuity with less pension wealth. In other words, the curve of optimal annuitisation moves leftwards on the x-axis as the pensioner's risk aversion increases. Fourthly, for the more risk averse pensioner optimal annuitisation at age 65 increases faster as a function of pension wealth at age 65. The third and fourth conclusions actually show that the more risk averse pensioner annuitises more in both Cases 3.2 and 3.3 compared to the less risk averse pensioner. Annuities are a form of protection against equity risk and the more risk averse pensioner has more demand from annuities.

When comparing the two pensioner in the same case, either Case 3.2 or Case 3.3, and with the same risk aversion, but one without and the other with the bequest motive, we observe that the pensioner with no bequest motive optimally purchases any annuity for the smaller values of the pension wealth. Also, if the demand for annuities exists, the pensioner with no bequest motive optimally purchases more annuities than the pensioner with the bequest motive. Thus, we find that the pensioner with the bequest motive has a lower demand for annuities in both Cases 3.2 and 3.3 compared to the pensioner with the same attitude to risk but with no bequest motive. This is expected result, because the pensioner with the bequest motive has the desire to bequeath assets to his heirs and due to this desire he purchases fewer annuities and keeps more pension wealth available for bequeathing.

3.4.2.3 Cases 3.4, 3.5 and 3.6 – Dependence on Wealth and Age, no bequest

Regarding the optimal consumption in Cases 3.4, 3.5 and 3.6 and for the pensioner with no bequest motive, the surfaces takes slightly higher values than the values of the optimal consumption presented in Figure 3.1. However in Cases 3.4, 3.5 and 3.6 we find similar shapes of the surfaces to each other and just moved slightly up compared to surfaces in Case 3.1 and we will not present them graphically here. We give here a couple of observations. For one chosen set of the parameters, the surfaces of optimal consumption in Cases 3.4, 3.5 and 3.6 have very similar, almost identical shapes and values to each other. If we compare the surfaces of optimal consumption in Cases 3.4 and change RRA coefficient only, then we observe slightly lower values of optimal consumption for the more risk averse pensioner compared to the less risk averse pensioner. However, the differences between values of optimal consumption when we change the RRA coefficient in Case 3.4 are lower than those presented in Figure 3.1. The same conclusions when we allow the changes to the RRA coefficient only, are drawn in Cases 3.5 and 3.6. Again, when we have one surface of optimal

consumption, we can use the same technique as explained in 3.4.2.2 to read optimal consumption for any income. Changing income is important here because income is changed whenever the pensioner purchases either nominal or real annuities. So these conclusions are valid only for the pensioners with the same income.

Similar values of the optimal consumption in Cases 3.4, 3.5 and 3.6, for the same values of the pension wealth and income, age of the pensioners, pensioners' RRA and bequest coefficient, and for the same value of income at that age, are the consequences of the fact that the newly bought annuities, either nominal or real or combined, are a form of the risk free investment providing similar added value to the pensioner. If we observe the pensioners with the same RRA and bequest coefficients, the same age and the amount of the pension wealth and income, but in Cases 3.4, 3.5 and 3.6, then at the beginning of the year just before asset allocation all these pensioners also have the same amount of the risk free asset, implied from their risk free income. So, all these pensioners have the same states at the beginning of the year and the same amount of the risk free asset already in a possession. Regarding investment and annuitisation, annuitisation is a form of risk free investment. The pensioner in Case 3.1 can invest in risk free asset and in equities, while the pensioners in Cases 3.4, 3.5 and 3.6 can invest in risk free asset and into annuities as well. Purchasing annuities is a form of risk free investment with a better return than cash. It seems that nominal and real annuities bear similar added value to the pensioner, and thus, the pensioners in different cases will optimally choose similar amounts to be invested into risky investment and into cash and annuities as risk free investment. The pensioners in different cases will optimally choose similar amounts to be consumed as well, as the pensioner consumes the part of his assets not invested into equities, cash and annuities. That is why the optimal consumption graphs are similar in different Cases 3.4, 3.5 and 3.6. However, we state here that small differences exist, and we find that the highest consumption values are in Case 3.6, just slightly smaller values are in Case 3.5 and then a bit lower values in Case 3.4.

We find that optimal consumption is higher in Cases 3.4, 3.5 and 3.6 than in Case 3.1, and it is due to the better return from annuity than cash investment. We should bear in mind that annuities provide better return than risk free asset due to the survival credits.

If the pensioner has no bequest motive ($b_t = 0$) and purchases optimally either nominal or real annuities whenever during retirement, he invests all his remaining pension wealth into the risky asset for almost all ages and pension wealth values. We

observe that only at ages above 95 and the highest values of pension wealth considered, the pensioner would only invest optimally small percentage into riskless asset. Thus, in Cases 3.4, 3.5 and 3.6, the surface representing optimal asset allocation against the pensioner's wealth and age is a flat surface at the level of 100%, apart from the ages above 95 and the highest values of pension wealth considered. We can conclude that annuities in the case with no bequest motive are the preferred investment for the pensioner compared to the riskless asset.

Regarding optimal nominal and real annuitisation, we can say that, under the same assumptions about the parameters, the surfaces of the optimal nominal annuitisation in Case 3.4 and the surfaces of the optimal real annuitisation in Case 3.5 have similar shapes and values. Optimal nominal annuitisation in Case 3.4 is very similar to optimal real annuitisation in Case 3.5, but the shape of the surfaces are very similar. In Case 3.6, the surface representing the sum of optimal nominal and real annuitisation has a similar shape and values as optimal nominal annuitisation in Case 3.4, again of course if we compare the cases with the same assumptions about the parameters.

Very similar values of optimal nominal annuitisation in Case 3.4, optimal real annuitisation and 3.5, and the sum of optimal nominal and real annuitisation in Case 3.6 shows that the pensioner optimally annuitises similar proportion of his available pension wealth regardless of the types of the annuity available. Nominal annuity in Case 3.4 provides higher income in the early years after purchasing them and decreasing income in real term afterwards, and the real annuity provides constant income in real term. From the observation that the pensioner will optimally convert similar part of pension wealth into nominal annuity in Case 3.4 as into real annuity in Case 3.5 and also as into combination of the two in Case 3.6 shows that any of these annuities provides similar added value. Actually, access to annuities is important added value as risk free investment is not optimal to the pensioner. From Figure 3.8 later in this section, we see that in Case 3.6 the demand for nominal annuities exists when both nominal and real annuities are available. And we see that the demand for nominal annuities exists for older ages.

We will present here in detail the surfaces in Case 3.4 only.

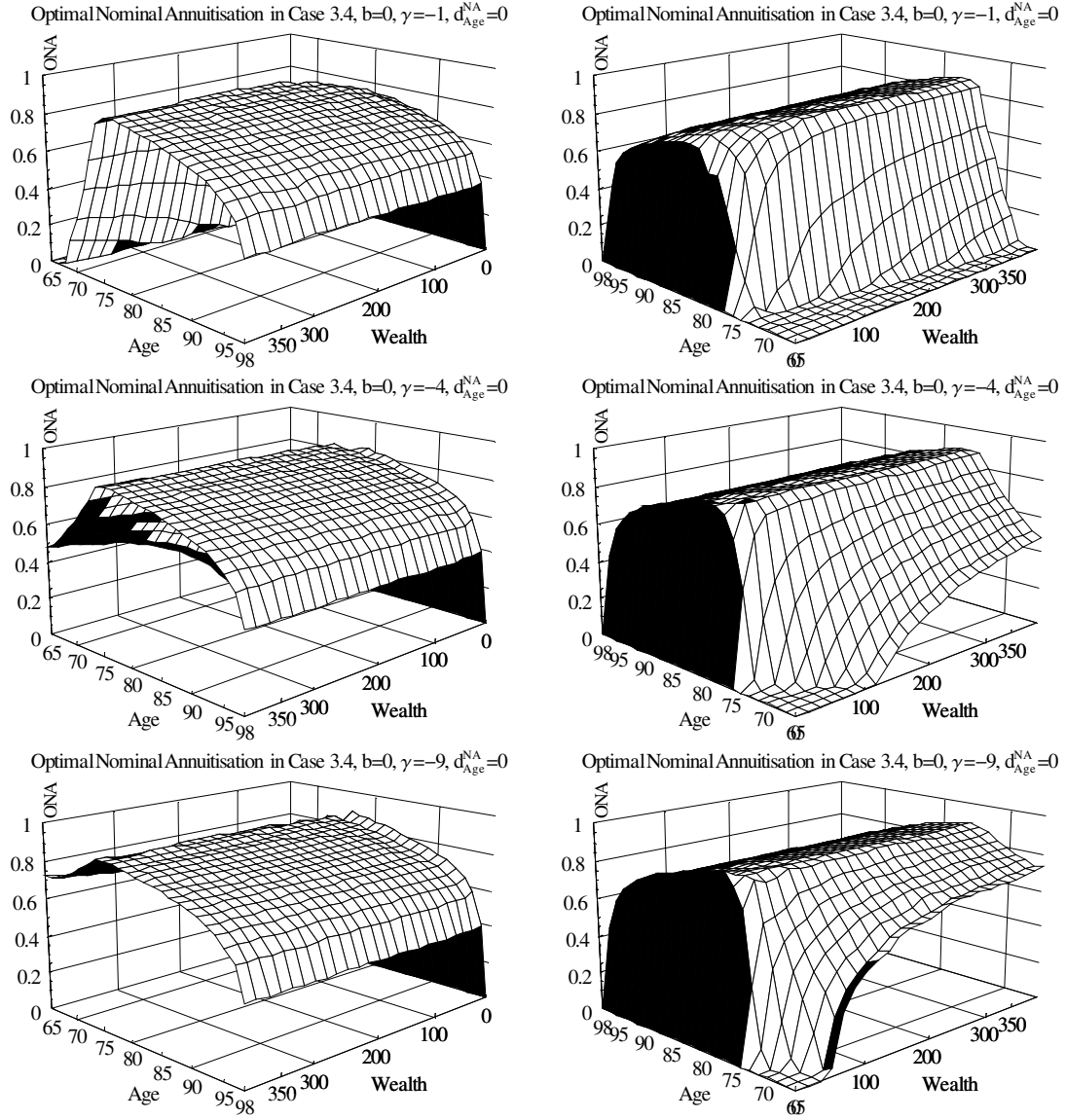


Figure 3.4 Percentages of the pension wealth optimally annuitised in Cases 3.4 for three values of RRA coefficients, $\gamma = -1$, $\gamma = -4$, $\gamma = -9$, and with no bequest motive, i.e. $b_t = 0$. The two surfaces in the same row present the surfaces of the same function viewed from different points. Pension wealth values are in thousands.

From the plots on the left side of Figure 3.4, we can see that the shapes of the surfaces for ages 75 and above are very similar irrespective of the risk aversion of the pensioner. Significant differences can be seen in the plots on the right side in Figure 3.4, which clearly depict the optimal annuitisation for earlier ages. There is an obvious connection with the risk aversion of the pensioner. For the less risk averse pensioner, with RRA coefficient $\gamma = -1$, optimally there is no annuitisation in the early years of retirement, and then there is a steep increase in the degree of optimal

annuitisation. This steep increase starts slightly earlier for the pensioner with more pension wealth.

The more risk averse pensioner will optimally annuitise part of his pension wealth even at age 65, if his pension wealth reaches appropriate level. For this pensioner, the increasing age and pension wealth is followed by the increasing optimal annuitisation until ages 75–80. After these ages, optimal annuitisation does not significantly depend on wealth any longer.

We can see in Figure 3.4 that in Case 3.4 with no bequest motive, the level of risk aversion influences the degree of optimal annuitisation up to ages 75–80, but not after these ages.

3.4.2.4 Cases 3.4, 3.5 and 3.6 – Dependence on Wealth and Age, with a bequest

Regarding optimal consumption in Cases 3.4, 3.5 and 3.6 and with a bequest motive, we have very similar patterns as in Cases 3.4, 3.5 and 3.6 and no bequest motive.

However, the pensioner's optimal asset allocation in Cases 3.4, 3.5 and 3.6 and with a bequest motive is quite different compared to optimal asset allocation in Cases 3.4, 3.5 and 3.6 with no bequest motive. Optimal asset allocation in Cases 3.4, 3.5 and 3.6 and with a bequest motive depends on the pensioner's level of risk aversion significantly. For $\gamma = -1$, which is not shown here, the pensioner invests all his pension wealth into the risky asset for all considered ages when the amounts of pension wealth is below 350,000. In Figure 3.5, we present the surfaces of optimal asset allocation for the pensioner with a bequest motive and with the level of RRA coefficients $\gamma = -4$ $\gamma = -9$.

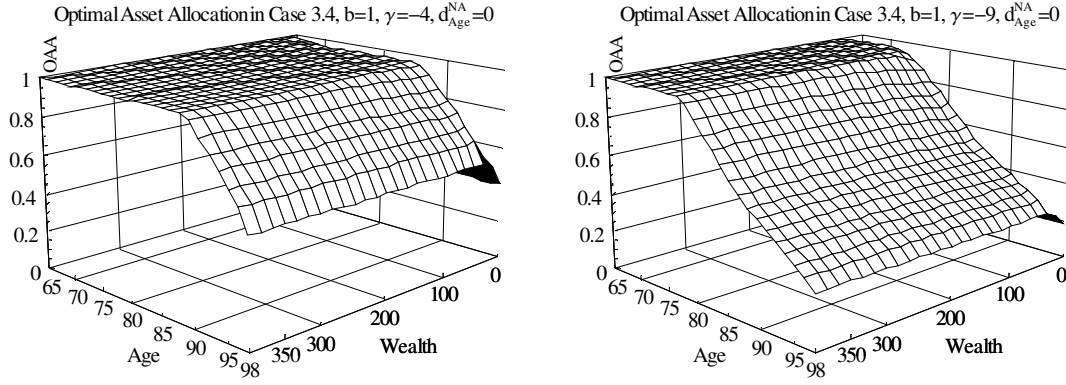


Figure 3.5 Optimal asset allocation in Case 3.4, for the value of RRA coefficients $\gamma = -4$ $\gamma = -9$, and for bequest motive coefficient $b_t = 1$. Pension wealth values are in thousands.

Under the assumption of $\gamma = -4$ $\gamma = -9$ shown in Figure 3.5, the pensioner will invest less in the risky asset as his risk aversion increases, and as his age and pension wealth increase. For these pensioners, annuities are not a better choice than the riskless asset, as it is the case for the pensioner with no bequest motive and for the less risk averse pensioner.

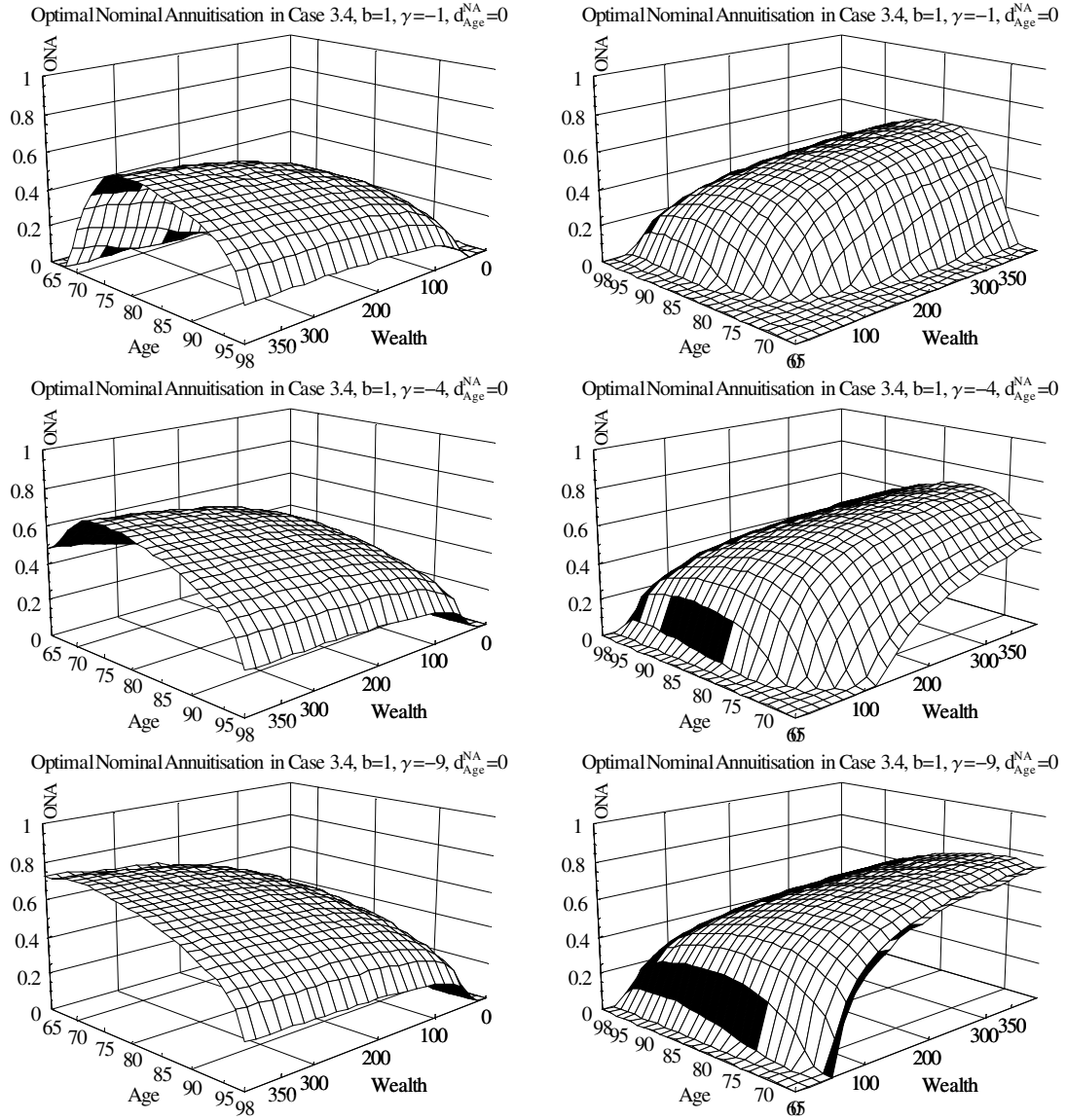


Figure 3.6 Percentages of pension wealth optimally annuitised in Case 3.4 for three values of RRA coefficients $\gamma = -1$, $\gamma = -4$, $\gamma = -9$, and $b_t = 1$. Two plots in the same row present the surfaces of the same function viewed from the different points of view. Pension wealth values are in thousands

In Figure 3.6, we present optimal nominal annuitisation for the pensioners with a bequest motive and with different levels of risk aversion in Case 3.4.

As in Section 3.4.2.3, for the no bequest assumption, one can see in Figure 3.6 that the surfaces have a similar shape for ages after 75. Before this age, optimal annuitisation decreases for less risk averse pensioners. Now, for low values of pension wealth no annuitisation is optimal. Thus, when a significant part of pension wealth is converted into annuities once, then it is optimal not to annuitise anymore.

As we noted earlier, optimal annuitisation and asset allocation surfaces as functions of age and wealth are very similar for both Cases 3.4 and 3.5. We present in Figure 3.7 one example of the optimal real annuitisation function in Case 3.5.

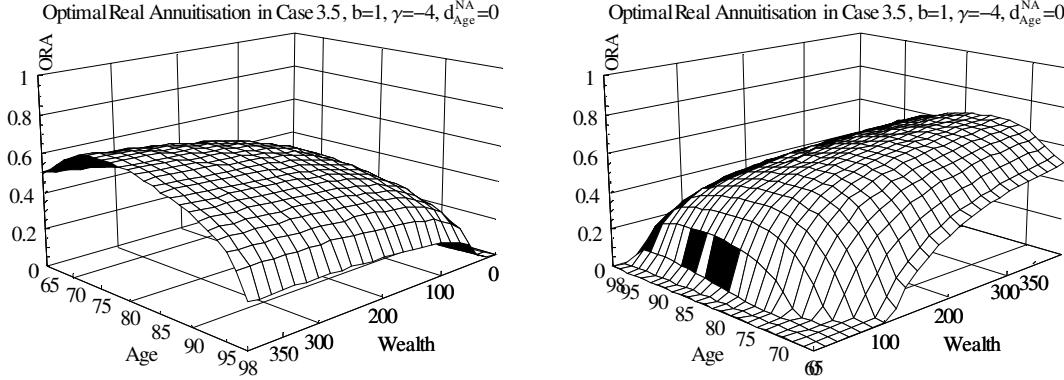


Figure 3.7 Optimal real annuitisation as function of age and wealth in Case 3.5, for $b_t = 1$ and $\gamma = -4$. Pension wealth values are in thousands.

We can compare surfaces in Figure 3.7 with the two surfaces in the middle row in Figure 3.6. Clearly, the surfaces on both left and right sides are similar. The only small difference that we can observe is that the surfaces in Case 3.4 have slightly lower values.

3.4.2.5 Cases 3.6 – Distribution of nominal and real annuities

As we said earlier, there is a little difference between optimal consumption, optimal asset allocation in Cases 3.4, 3.5 and 3.6. Regarding annuitisation, we also find small differences between optimal nominal, real and the sum of optimal nominal and real annuitisation in Cases 3.4, 3.5 and 3.6, respectively.

In Section 3.4.5, we measure the value added from the access to annuities in terms of discounted expected utility. We will see from the analysis in Section 3.4.5 that having access to both nominal and real annuities at all ages brings little extra benefits to the pensioner compared to Cases 3.4 and 3.5 where the pensioner has access to nominal and real annuities respectively only. Although both nominal and real annuities exist extensively in practice, we find that both of them provide similar additional discounted expected utility to the pensioner. However, it is interesting to observe the optimal real and optimal nominal annuitisation surfaces in Case 3.6. In order to compare the surfaces with those in Case 3.4 and 3.5, we choose $b = 0$ and $b = 1$, and $\gamma = -4$ and $\gamma = -9$, and present the optimal real and optimal nominal annuitisation surfaces in Case 3.6 in Figure 3.8.

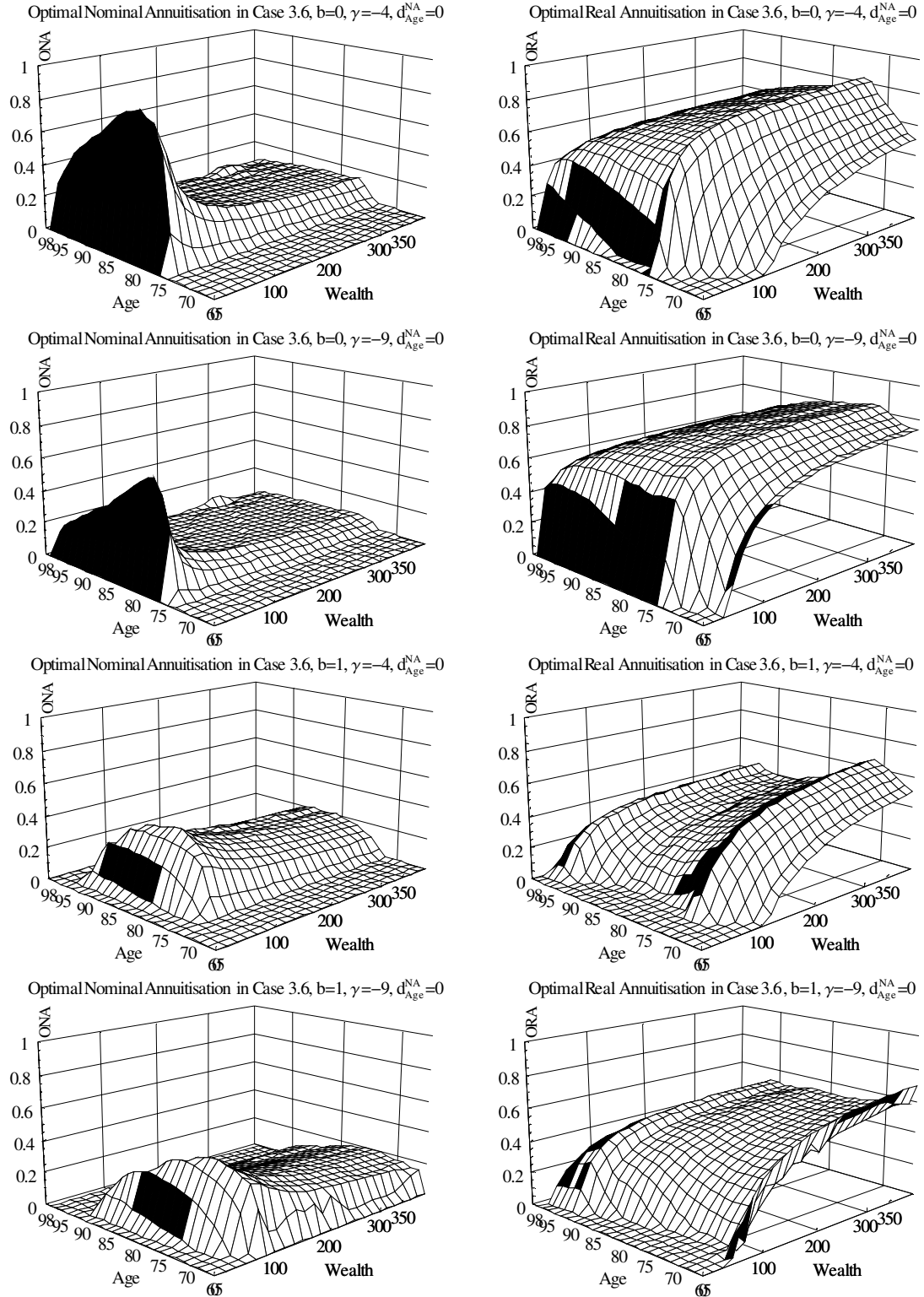


Figure 3.8 Optimal nominal and real annuitisation as function of age and wealth in Case 3.6, for the values of RRA coefficients $\gamma = -4$ $\gamma = -9$, and for the values of the bequest coefficient $b_t = 0$ and $b_t = 1$. Pension wealth values are in thousands.

In Figure 3.8, we observe that both nominal and real annuitisation is optimal for a certain values of age and wealth. It means that the pensioner having access to both nominal and real annuities will optimally purchase both types of annuities for those values of age and pension wealth. If we observe carefully the shapes of surfaces on the left and right hand sides of Figure 3.8, we can see the following. If we add values on the z -axis of the optimal nominal annuitisation to the values on the z -axis of the optimal real annuitisation for the same age and wealth, then we get a function with a very similar shape as the plots on the right hand side Figures 3.4, 3.6 and 3.7 for the appropriate value of relative risk aversion and a bequest motive coefficients. It means that in Case 3.6 we have the similar values of the overall percentage of optimal annuitisation as in Cases 3.4 and 3.5, and it is just a question of how much the nominal annuitisation is preferred over real annuitisation and vice versa.

Regarding optimal nominal and real annuitisation in Figure 3.8 as a function of the pension wealth, we observe that, for higher values of the pension wealth, the pensioner optimally converts a higher proportion of his pension wealth into real annuities than into nominal annuities. The pensioner will optimally convert more pension wealth into nominal annuities only for lower values of his pension wealth. If the pensioner has no bequest motive, then as his pension wealth increases the percentage of the optimal nominal annuitisation decreases. If the pensioner has a bequest motive, his optimal nominal annuitisation increases slightly as pension wealth increases and then decreases after attaining a maximum percentage for the value of the pension wealth about 80,000–120,000.

Regarding optimal nominal and real annuitisation in Figure 3.8 as a function of age, we observe that if the pensioner has no a bequest motive, the demand for nominal annuities exists for ages 75 and above only. However, if the pensioner has a bequest motive we observe the demand for nominal annuities starts at earlier ages. If $\gamma = -9$ and $b_t = 1$ then purchasing at least some nominal annuities is optimal at age 66. We also observe that after the earliest pensioner's age when the demand for nominal annuities exists, during a couple of the following years the optimal percentage of the nominal annuitisation increases, then it attains a certain maximum, and then the percentage of the optimal nominal annuitisation decreases again.

It is not easy to find a clear general reasoning for the distribution of the optimal nominal and real annuitisation in Case 3.6. One should also bear in mind that the results presented in Figure 3.8 are conditional on the assumed income from social security. The reasons for the observed distribution should be sought in the

characteristics of the income provided by nominal and real annuities. For the same amount of the pension wealth converted into annuities, the nominal annuity provides higher income in early years after purchasing annuities and then income decreases in real terms. Real annuities provides a lower income compared to nominal annuities in early years after purchasing but then income is constant and after a couple of years it is higher than income from nominal annuities bought for the same amount of the pension wealth. Thus, we have two effects, a higher income from nominal annuities in early years and lower income a couple of years after purchasing them compared to real annuities. In early years of retirement, nominal annuities are less preferable compared to real ones as the effect of decreasing income in real terms overweight the effect of the higher income in early years due to possible long period of receiving income from annuities. Thus, it is optimal to purchase nominal annuities later during the retirement. Regarding more demand for nominal annuities for lower values of the pension wealth, if any demand for nominal annuities exists, the effect of the higher income in early years after purchasing nominal annuities overweight the effect of the decreasing income in real terms. For a lower amount of pension wealth, a lower amount is converted into annuities and a lower additional income is provided. In this case, the percentage of the additional income from nominal annuities is higher than the percentage of the increase of the income provided from real annuities, and also percentage of the overall decrease of income in real terms is lower.

So far in Section 3.4.2, we have investigated the behaviour of the optimal consumption, optimal asset allocation, optimal nominal annuitisation and optimal real annuitisation (given by functions (3.45)–(3.48)) for different cases and we have presented results for these functions when age and pension wealth changes and other variables stay the same. However, these functions depend on $(W_t, Y_t, d_t^{NA}, I_{t,k})$. In order to obtain an idea of how these functions depend on income, nominal income coefficient and inflation, we will present more results in the rest of Section 3.4.2.

3.4.2.6 Dependence on Income

Regarding optimal consumption, we have explained in Section 3.4.2.2 the way in which the optimal consumption changes with the changes of income variable.

Regarding optimal asset allocation and optimal annuitisation, the changes of the value of the income variable Y_t does not change the general shape of the surfaces. It only changes the scale on the wealth axis. This conclusion can be drawn from the relations given in (3.46)–(3.48). If the income variable Y_t increases and we keep other variables

the same, then the surfaces of optimal asset allocation and optimal annuitisation will be pulled on the wealth axis towards larger values, keeping their overall shape. In contrast, if the income variable Y_t decreases while keeping other variables the same, then the surfaces will be squeezed again keeping their overall shape.

As we noted in 3.4.2.2, income changes if annuitisation is present. In all of our figures above, the assumed income at age 65 is $Y_{65} = 33,320.90$, $\rho_{65} = 0.68212$ and afterwards income is $Y_t = 22,728.85$ for $66 \leq t \leq 99$. If we have any annuitisation then $Y_t > 22,728.85$ for $66 \leq t \leq 99$. According to the previous paragraph, if the level of income is increased then the surface of optimal asset allocation and optimal annuitisation will be pulled towards larger values on the wealth axis, while their shapes are kept the same.

3.4.2.7 Dependence on the Nominal Income Coefficient

The dependence on the nominal income coefficient d_t^{NA} changes with age and wealth. Sometimes these changes are not significant, but sometimes they are, depending on the assumptions regarding the other parameter values.

The nominal income coefficient d_t^{NA} in Cases 3.1, 3.3 and 3.5 keeps its value equal to zero for all ages $65 \leq t \leq 99$, because only real income is present. However, in Cases 3.2, 3.4 and 3.6 the nominal income coefficient is not equal to zero once the pensioner purchases nominal annuities.

In detailed figures not shown here, we observe the optimal consumption as a function of nominal income coefficient d_t^{NA} only, and note that it decreases slightly as d_t^{NA} increases. One should be aware that if the nominal income coefficient is positive then income in real terms is larger than income from social security because the pensioner has bought some nominal annuities. So, if one part of the pensioner's income comes from the nominal annuities, optimal consumption depends on the positive nominal income coefficient and on the level of the real income above the level of the income from social security only.

Regarding optimal asset allocation, the positive value of the nominal income coefficient can influence optimal asset allocation in Cases 3.4 and 3.6. Not shown here, but we again find that the optimal asset allocation is increasing slightly if the nominal income coefficient increases, and if we keep the values of all other variables the same.

Figure 3.9 presents the dependence of the optimal nominal annuitisation on wealth and the nominal income coefficient. Four surfaces presented in Figure 3.9 show optimal nominal annuitisation for pensioners aged 66, 71, 76 and 81 years.

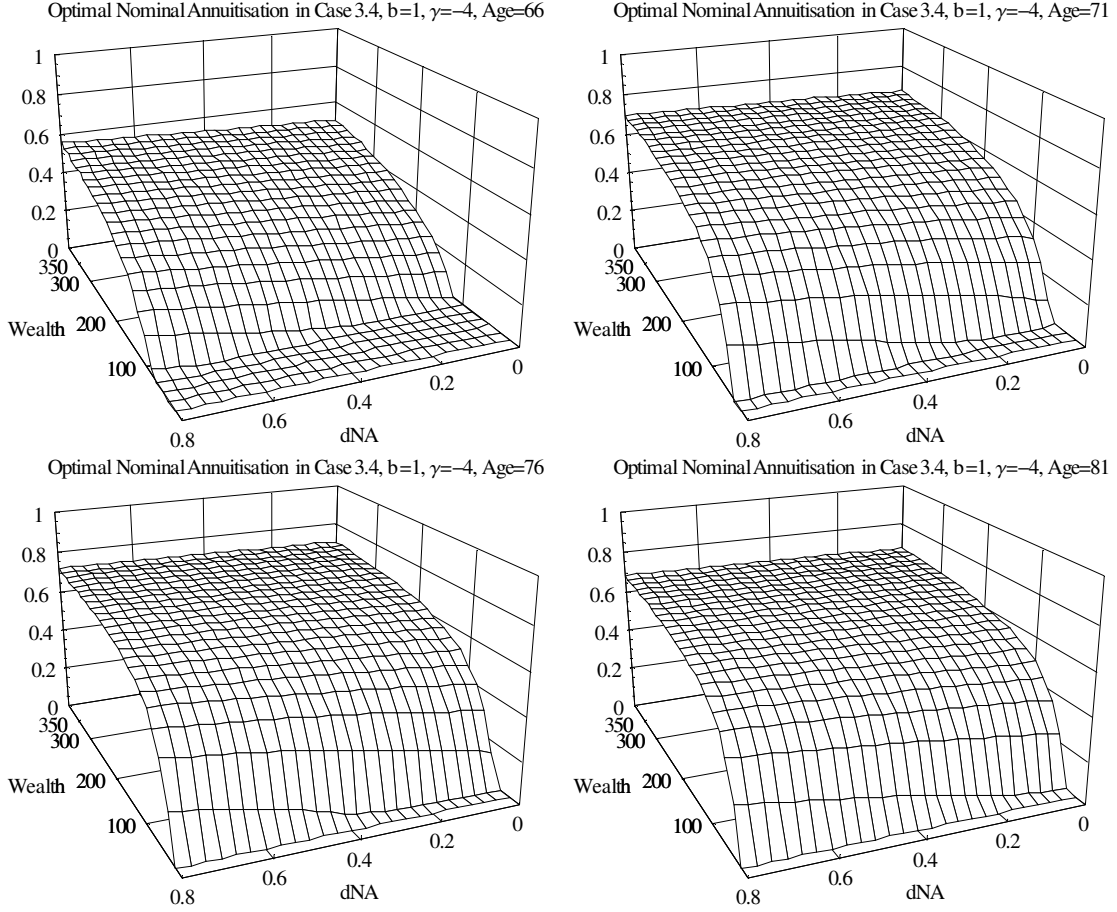


Figure 3.9 Optimal nominal annuitisation in Case 4, for $b_t = 1$ and $\gamma = -4$, for different pensioner's ages, and as a function of the nominal income coefficient d_t^{NA} and wealth.

Figure 3.9 shows that optimal nominal annuitisation usually does not depend strongly on d_t^{NA} . We can see in Figure 3.9 that optimal nominal annuitisation increases more steeply for the lower pension wealth values. The surfaces in Figure 3.9 assume the same income of $Y_t = 22,728.85$. However, if the pensioner has a positive nominal income coefficient it means that he has already bought some nominal annuities and that his income is not the same as in the years before purchasing nominal annuities. If one wants to calculate the exact value of the optimal nominal annuitisation for a given income, he needs to calculate it using equations (3.39) and (3.47) and the values from the surfaces in Figure 3.9. The aim of presenting Figure 3.9 is to give an idea of optimal nominal annuitisation as a function of the pension wealth, age and the nominal income coefficient.

3.4.2.8 Change of Control Variables due to Stochastic Inflation

In this section, we observe optimal consumption, optimal asset allocation and optimal annuitisation as a function of the stochastic inflation. Before presenting any result, we need to understand how income and nominal annuity rate change with the change of the value of the inflation, when inflation is random.

If inflation during the year before the annuitisation is lower, then the nominal annuity rate defined in (3.4) is larger and income from nominal annuity is smaller. However, the lower inflation during the year before nominal annuitisation will be on average followed by the lower inflation in the following years as well. The lower is inflation in the years after the year of nominal annuitisation the smaller is the decrease of income in real terms. It means that if we start with lower inflation then income from any nominal annuity will be lower but the decrease of income in real terms due to inflation will be slower. The opposite is true as well. If the inflation during the year before nominal annuitisation is higher, then income in nominal terms from any nominal annuity will be higher as well but this income in real terms will decrease more quickly because of higher inflation in the following years. So, the lower/higher inflation in the year prior to the nominal annuitisation the higher/lower is the nominal annuity rate and the lower/higher is income from nominal annuity. The lower/higher inflation in the year prior to the nominal annuitisation is on average followed by the lower/higher inflation in the years after nominal annuitisation. The effects of lower/higher nominal annuity income and slower/faster decrease of nominal annuity income in real terms due to (on average) lower/higher inflation in the years following nominal annuitisation seem to cancel each other out. In almost all of our investigated examples, when we allow the inflation to be random, we do not find significant changes in any of the control variables of the problem (3.26)–(3.34).

Regarding optimal consumption, asset allocation and annuitisation, we compare the data for a given age of the pensioner, and for a given reasonable value of income, and for different values of inflation in the year prior to the observed age. The differences depend on the nominal income coefficient, and of the pensioner's preferences towards risk and bequest. We present here the observations from the calculated optimal values, and we do not present the optimal values itself.

For any chosen preferences towards risk and bequest and for the nominal income coefficient $d_t^{NA} = 0$, we find almost no differences in the level of the optimal consumption. In Case 3.2, if the nominal income coefficient takes values

$0 \leq d_t^{NA} \leq 0.2$, we find the differences to be less than 0.1%. In Case 3.4, if the nominal income coefficient $0 \leq d_t^{NA} \leq 0.2$, we find the differences to be less than 1% for almost all combination of RRA coefficient and bequest motive. We observe that the differences of the values of optimal consumption increase as the nominal income coefficient increases. We also observe that for a given value of the nominal income coefficient optimal consumption decreases as inflation increases. The last observation is a consequence of the irreversibility of the converting pension wealth into nominal annuity. If a nominal annuity is purchased in the earlier years of the retirement then nominal income coefficient is positive. It means the part of the pensioner's income, coming from nominal annuity, decreases in real terms every year. If the value of inflation is higher, then the pensioner expects his income to decrease faster in following years. If we observe a future income from nominal annuity as an asset in a possession of the pensioner, then the pensioner actually expects that his overall wealth will decrease faster in real terms in following years. Thus the pensioner will optimally consume less in order to balance out the lower expected income in real terms in the following years, or in the other words the lower expected discounted value of future income from nominal annuity.

Regarding optimal asset allocation, we again find very small differences. If the pensioner has no bequest motive, then it is optimal for him to annuitise one part of his pension wealth and to keep the rest of his available assets in the risky asset. Similar to the conclusions for the constant inflation, we find that for the pensioner with no bequest motive annuities are the preferred investment than riskless asset. If the pensioner is a more risk averse person and he has the bequest motive then it is optimal for him to keep part of his pension wealth in both riskless and risky asset. We find that in Case 3.2 with a bequest motive and if the nominal income coefficient $0 \leq d_t^{NA} \leq 0.2$, the differences in optimal asset allocation are less than 1% when the level of the pension wealth is larger than 30,000. Below this level of pension wealth the differences can be up to 3%. In Case 3.4 with a bequest motive, optimal asset allocation is 100% in equities up to a certain age. For example, that age for $\gamma = -9$ is about 77. For this pensioner it is optimal to keep part of his pension wealth in the riskless asset. After that age, for the different levels of inflation the differences in the proportions kept in the equities are less than 1%.

When we observe the calculated data about the optimal nominal annuitisation and in the presence of the stochastic inflation the conclusions are again that the differences in the level of optimal nominal annuitisation due to the differences of the level of inflation are less than 1%.

As we can see, introduction of the random inflation in the model has very limited influence on any of the control variables. The differences are so small that they cannot be seen from the figures and it is for this reason that we do not present these results in graphic form.

3.4.3 A Typical Example of Simulation

In order to give to an idea how the model developed in this chapter can be used for the analysis of the different paths of the values important for the pensioner, we present here one typical solution to the problem in graphic form. Firstly, we have to know the case where the pensioner acts optimally, to know the pensioner's preferences towards risk and bequest and all other parameters relevant to the pensioner. Then we make 2,000 simulations of the random samples of the equity returns and inflation rates for ages 65–99. Then we calculate 2,000 simulations of the random samples for ages 65–99 of optimal consumption, optimal asset allocation, optimal nominal and real annuitisation, pension wealth and income. All these calculated data are kept in the excel files and can be used for different analysis. We present here the pensioner in Case 3.4 and with a bequest motive. In Figure 3.10 and 3.11, we depict the mean values of these optimal values calculated using simulations, together with 0.05 and 0.95 quantiles for optimal consumption, asset allocation and nominal annuitisation. We assume that pension wealth at age 65 is $W_{65} = 200,000$, income at age 65 is $Y_{65} = 33,321$, inflation is stochastic and the value of inflation prior to the year of retirement is equal to 4%. The following four graphs depict mean values and 5% and 95% quantiles values of the pensioner's optimal behaviour.

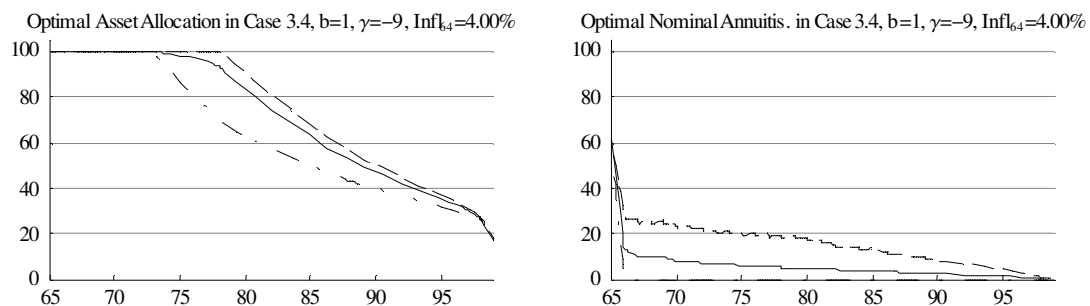


Figure 3.10 Mean (full line), 5% (dash and dot line) and 95% (dash line) of optimal asset allocation on the left hand side graph and mean (full line), 5% (dash and dot line) and 95% (dash line) of optimal nominal annuitisation on the right hand side graph, for the pensioner in Case 3.4, with bequest motive and with RRA coefficient $\gamma = -9$.

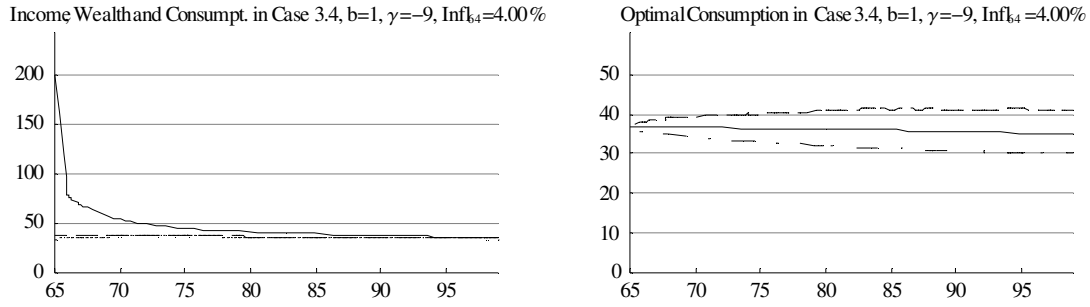


Figure 3.11 Mean income (dash line with shorter dashes), wealth (dash line with longer dashes) and consumption (full line) on the left hand side graph, and mean (full line), 5% (dash and dot line) and 95% (dash line) quantiles of consumption on the right hand side. The pensioner is in Case 3.4, with bequest motive $b_t = 1$ and with RRA coefficient $\gamma = -9$.

In our model, the pensioner draws utility from consumption and from bequeathing the wealth to his heirs. The paths presented in Figures 3.10 and 3.11 are not the actual (realised) paths, but the mean value and the quantiles values of the paths. Depending on each particular random realisation of inflation and equity rates we get different paths.

If not emphasised otherwise, we assume constant inflation in the following sections. In Section 3.4.7, we will again investigate the consequences of random inflation.

3.4.4 Criteria for Comparing Results

The main aim of this study is to investigate importance of access to annuities, either nominal, real, or both. We need to compare different results and get an insight into the gains from access to annuities, either nominal or real, or both. A range of conclusions is possible. At one extreme, we may conclude that losses due to the lack of the availability of a certain class of annuities remain significant although the pensioner behaves optimally regarding consumption, asset allocation and available annuitisation. If we draw this conclusion then access to the class or classes of unavailable annuities is very important for the pensioner. At the other extreme, we may conclude that losses due to the lack of a certain class of annuities can be significantly decreased by the pensioner's optimal consumption, asset allocation and optimal annuitisation using available annuities if any. If we draw this conclusion then access to the other class or classes of annuities is not very important for the pensioner. Of course, many conclusions will lie somewhere between these two.

Gains from access to annuities can be measured in different ways. In this thesis, we optimise the pensioner's behaviour with respect to the maximum derived utility from consumption. However, at the same time, we are interested in other risks to which the pensioner is exposed. Thus, we will investigate the left tail of the distribution of possible random realisations for the pensioner. As we will see, a combination of these measures will give us an idea of the importance of access to annuities for the pensioner. Conclusions are not straightforward and one needs both to observe value function and carry out a left tail analysis to come to an understanding of the benefits and risks. However, the criterion for the optimisation of the pensioner's behaviour is the maximisation of derived utility only and not the minimising of the left tail risk of the possible less than expected result of random realisation. Thus, we here analyse the left tail of the distribution, or worse than expected random realisations for the pensioner in order to shed some light on this obviously important risk for the pensioner.

In our model, the pensioner wishes to maximise the expected utility derived from future consumption and, if there is a bequest motive, expected utility derived from bequeathing assets to heirs. When we have two different utilities that result from the two different examples, it is not directly clear how significant that difference is. One solution is to transform that difference into money terms and then to compare them. We classify our criteria for measuring the differences between two comparable examples in two groups. The first one is the group where we compare expected discounted utilities. In this group, we have two criteria: constant equivalent consumption – *CEC* measure and required equivalent wealth – *REW* measure. The second group of criteria is the group where we measure risks of possible worse than expected realisations of utility drawn from consumption and bequest in retirement. The second group of criteria consists of Value at Risk – VaR_α and Conditional Value at Risk – $CVaR_\alpha$, for $0 < \alpha < 1$. The second group of criteria is based on the distribution of random utility derived from future consumption and bequest.

The constant equivalent consumption – *CEC* measure is based on finding the constant amount of consumption such that the expected utility derived from the optimal consumption as a result of the calculations is the same as the expected utility derived from the stream of those constant consumption. This criterion can be applied in the case with no bequest motive. This measure is widely accepted and examples of analysis using *CEC* measure can be found in Cocco, Gomes, and Maenhout (2005).

We now explain in more detail how we use *CEC* criterion. At age 65, we know pension wealth, income at that age, inflation, and nominal income coefficient $d_{65}^{NA} = 0$. Based on that information we calculate the expected discounted utility derived from the stream of future random consumption. Future random consumption depends on the realisation of the stochastic processes for risky rate on equities and for random inflation. We denote that the stream of future random consumption (as earlier in this chapter) by $\tilde{C}_t$ for $65 \leq t \leq 99$. Avoiding writing the dependent variables and using the formula (3.25) and the derivation before this formula we can write the following

$$V_{65} = E_{65} \left[\sum_{t=65}^{99} \left(\delta^{t-65} \left(\prod_{j=65}^{t-1} p_j \right) \frac{(\tilde{C}_t)^\gamma}{\gamma} \right) \right] \quad (3.61)$$

We need to emphasise here that $\tilde{C}_{65}$ is not random but it is a control variable. However, we keep the notation as if it is random variable in order to have a more compact form of the formula (3.61). We can see that there is no bequest motive in equation (3.61) while in equation (3.25) there is a possibility that the pensioner has a bequest motive. It is not possible to include in a proper way the pensioner's utility from the bequest motive in a *CEC* measure.

Let us now assume that the constant stream of consumption C_{CEC} for $65 \leq t \leq 99$ produces the same expected utility at age 65 then we have

$$V_{65} = \sum_{t=65}^{99} \left(\delta^{t-65} \left(\prod_{j=65}^{t-1} p_j \right) \frac{(C_{CEC})^\gamma}{\gamma} \right) \quad (3.62)$$

where V_{65} is defined in (3.61). Taking the term C_{CEC} on the left side and the other terms on the right side of the equation, we obtain

$$C_{CEC} = \left(\frac{\mathcal{N}_{65}}{\sum_{t=65}^{99} \left(\delta^{t-65} \left(\prod_{j=65}^{t-1} p_j \right) \right)} \right)^{\frac{1}{\gamma}}. \quad (3.63)$$

Now, introducing equation (3.61) into the last equation we get

$$C_{CEC} = \left(E \left[\sum_{t=65}^{99} \frac{\delta^{t-65} \left(\prod_{j=65}^{t-1} p_j \right) \frac{1}{\gamma}}{\sum_{t=65}^{99} \left(\delta^{t-65} \left(\prod_{j=65}^{t-1} p_j \right) \frac{1}{\gamma} \right)} (\tilde{C}_t)^\gamma \right] \right)^{\frac{1}{\gamma}}$$

We leave the denominator γ in this last formula so that we see that constant equivalent consumption is in some sense normalised expected discounted utility derived from future consumption. In numerical calculations we cancel out the common term γ and get the following formula for the constant equivalent consumption

$$C_{CEC} = \left(E \left[\sum_{t=65}^{99} \frac{\delta^{t-65} \left(\prod_{j=65}^{t-1} p_j \right) (\tilde{C}_t)^\gamma}{\sum_{t=65}^{99} \left(\delta^{t-65} \left(\prod_{j=65}^{t-1} p_j \right) \right)} \right] \right)^{\frac{1}{\gamma}} \quad (3.64)$$

In our numerical calculations, we actually calculate the values V_{65} for different values of the state variables. So, equation (3.63) can be used as the definition of C_{CEC} . However, we want to emphasise that a CEC measure recognises utility from consumption only and we prefer to define it by equation (3.64).

We apply the CEC measure for comparing two cases while the values of the parameters and the pensioner's preferences towards risk and bequest are the same. Having calculated C_{CEC} for the two different comparable cases, we then compare C_{CEC} values. The pensioner will prefer the case where C_{CEC} is higher. Also, we determine in money terms how much one case is more favourable than the other.

A required equivalent wealth – REW measure is the second measure of the value function that we apply. The same idea of this type of measure is employed in a welfare analysis in Horneff, Maurer, Mitchell, and Stamos (2009). However, the concept of the equivalent wealth is used by other authors as well, for example Mitchell et al (1999). They define it as an equivalent increase in financial wealth needed to compensate an individual lacking access to annuity products. We define it as the required equivalent wealth needed to provide the pensioner with the same expected derived utility in different cases, where cases differ in availability of a certain class/classes of annuities.

Having solved the problem (3.26)–(3.34) we get the value function

$$V_{65}(W_{65}, Y_{65}, d_{65}^{NA}, I_{64;k}) \quad (3.65)$$

for $k = 1, \dots, n_I$, $W_{65} \geq 0$ and $Y_{65} \geq 0$, $d_{65}^{NA} = 0$, and $I_{64;k}$ in the domain of the values for the inflation rates.

Function $V_{65}(W_{65}, Y_{65}, d_{65}^{NA}, I_{64;k})$ is an increasing function with respect to variable W_{65} . For given values $V_{65}(W_{65}, Y_{65}, 0, I_{64;k})$, $Y_{65} \geq 0$ and $I_{64;k}$ we can calculate the inverse function with respect to variable W_{65} . We calculate required equivalent wealth in the following way. If we have one case and given values of ${}_1W_{65} \geq 0$ and $Y_{65} \geq 0$, $d_{65}^{NA} = 0$, and given $I_{64;k}$ in the domain of the values for the inflation rates, we calculate ${}_1V_{65}({}_1W_{65}, Y_{65}, 0, I_{64;k})$. Then, if we have another value function ${}_2V_{65}({}_2W_{65}, Y_{65}, 0, I_{64;k})$ in another comparable case where ${}_2W_{65}$ is unknown variable, then we can calculate ${}_2W_{65}$ such that

$${}_2V_{65}({}_2W_{65}, Y_{65}, 0, I_{64;k}) = {}_1V_{65}({}_1W_{65}, Y_{65}, 0, I_{64;k}) \quad (3.66)$$

Thus, we get the amount of wealth in the second case such that the expected discounted utility is the same in both cases. Then we compare ${}_1W_{65}$ and ${}_2W_{65}$, and we can conclude which one of two cases is more favourable. If ${}_1W_{65} > {}_2W_{65}$ then the pensioner in the second case can derive the same utility as the pensioner in the first case but with the lower value of wealth. Then we can say that the second case is more favourable for the pensioner. Again, we determine in money terms how much one case is more favourable than the other. If the opposite is true, i.e. if ${}_1W_{65} < {}_2W_{65}$, then the first market is more favourable for the pensioner. If ${}_1W_{65} = {}_2W_{65}$, then the pensioner is indifferent between the two cases in terms of expected discounted utility derived from future consumption and bequest.

The value at Risk and Conditional Value at Risk from random discounted utility derived from future consumption and bequest, Var_{α} and $CVaR_{\alpha}$, are measures of the pensioner's left tail risk. Now, we consider the discounted utility from future consumption and bequest as a random variable and investigate its characteristics. In order not to create confusion with the value function, we introduce a new random variable $\tilde{D}(W_{65}, Y_{65}, 0, I_{64;k})$ such that

$$\tilde{D}(W_{65}, Y_{65}, 0, I_{64;k}) = \sum_{t=65}^{99} \left(\delta^{t-65} \left(\prod_{j=65}^{t-1} p_j \right) \left(\frac{(\tilde{C}_t)^\gamma}{\gamma} + \delta(1-p_t) b_t \frac{(\tilde{W}_{t+1})^\gamma}{\gamma} \right) \right) \quad (3.67)$$

for $k = 1, \dots, n_t$, $W_{65} \geq 0$ and $Y_{65} \geq 0$, $d_{65}^{NA} = 0$, $I_{65;k}$ in the domain of values of the inflation rate, and where $\tilde{C}_t$ and $\tilde{W}_{t+1}$, for $65 \leq t \leq 99$, are random variables, resulting from the process (3.26)–(3.34). Again, $\tilde{C}_{65}$ is a control variable but we keep the notation as if it is a random variable to get a compact form of equation (3.67). We cannot find *PDF* or *CDF* of the random variable $\tilde{D}(W_{65}, Y_{65}, 0, I_{64;k})$ analytically, but we can make a number of random realisations of this random variable and use this to calculate Var_α and $CVaR_\alpha$. One property that we expect to be satisfied in all examples when we calculate the right hand side of (3.67) numerically comes from the very first definition of the value function and $\tilde{D}(W_{65}, Y_{65}, 0, I_{64;k})$ and it is

$$V_{65}(W_{65}, Y_{65}, 0, I_{64;k}) \approx E[\tilde{D}(W_{65}, Y_{65}, 0, I_{64;k})].$$

This is the result that we use in Section 3.3.7.

Once we have a random variable $\tilde{D}(W_{65}, Y_{65}, 0, I_{64;k})$ in a numeric form, we can use it to calculate the approximate values of Var_α and $CVaR_\alpha$ which will give us an idea of the pensioner's left tail risk of discounted utility from future consumption and bequest. We define this measure more precisely in Section 3.4.7 and investigate its use.

3.4.5 Application of *CEC* and *REW* Measures

In this section, we investigate the relations amongst the results from *CEC* and *REW* measures for different cases and for different parameter setup. The parameters which we change here are the RRA coefficient γ and a bequest motive coefficient b_t . In order to focus our investigation on the differences in the results in different cases, we assume in this section that inflation is constant and equal to 4%. Further investigation of the effects of the stochastic inflation will be presented in Section 3.4.6.

Before presenting the results using *CEC* and *REW* measures, we show two examples of mean consumption and mean wealth paths. In Figures 3.12 and 3.13, we show mean pension wealth and mean consumption development during the retirement for the pensioner who optimally consumes, allocates assets and annuitises in Cases 3.1–3.6. The mean is calculated from the sample of 2,000 simulated pension wealth and consumption paths. We assume $\gamma = -9$ and $b_t = 0$, pension wealth at age 65 is 200,000, inflation is constant and equal to 4%, and other parameters are as stated in Section 3.4.1.

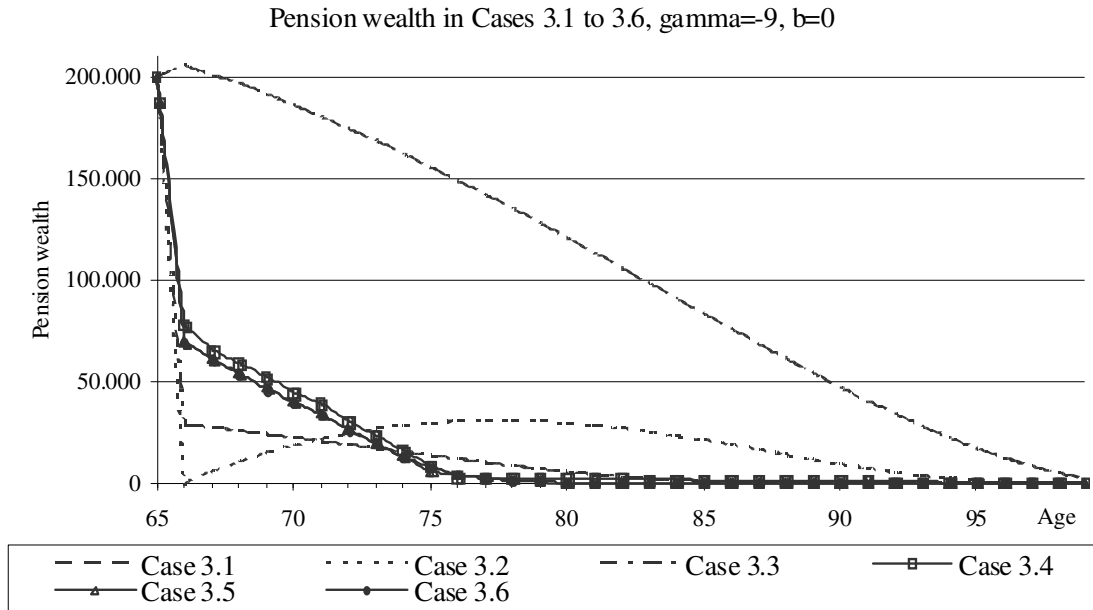


Figure 3.12 Mean pension wealth development in the retirement period in Cases 3.1–3.6, for $\gamma = -9$ and $b_t = 0$

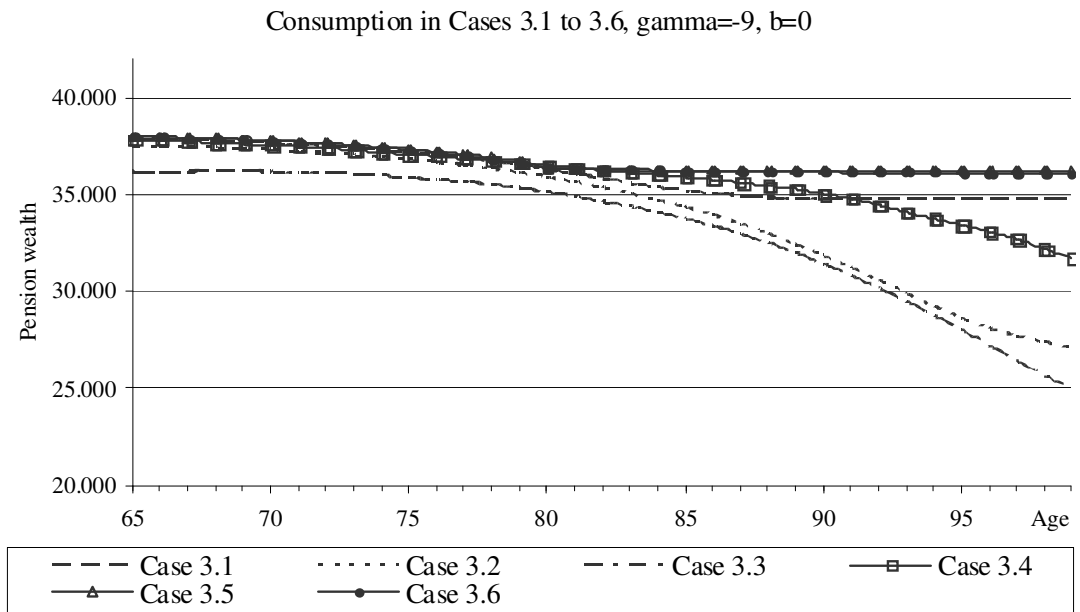


Figure 3.13 Mean optimal consumption development in the retirement in Cases 3.1–3.6, for $\gamma = -9$ and $b_t = 0$

Figures 3.12 and 3.13 give an idea of very different mean wealth and mean consumption paths in Cases 3.1–3.6. Then the *CEC* and *REW* are measures which summarise into a single number the complexity of these future developments. We can see that both mean pension wealth and mean consumption paths are very different from case to case.

In Case 3.2, where optimal nominal annuitisation is allowed at age 65 only and no real annuities are allowed, it is optimal to annuitise almost all pension wealth (more than 97% of the available pension wealth). However, some of the pension wealth is saved again afterwards, which means that the pensioner does not spend all his income from social security and nominal annuities during retirement, but saves one part in the early pension years and then consumes it afterwards. The pensioner in this example uses his only opportunity to purchase annuity, or in the other words he uses the only opportunity to benefit from the survival credits. Apart from the survival credits the pensioner protects himself from the decreasing income in real terms during retirement.. As he has access to annuities at age 65 only, he converts almost all his pension wealth into nominal annuities at that age. This is interesting result because the pensioner optimally purchases more annuities than he needs for the consumption, and actually uses annuity as a risk free investment for saving and not only as an instrument for providing income in retirement. Also, income from nominal annuity is beneficial as risk free investment in early years of retirement because income from nominal annuities is increased by inflation rate in the early years compared to the risk free investment. We emphasise here that increase in pension wealth is not in a contradiction with the assumption, stated in Section 3.1.2, that the pensioner never annuitises any part of his income. The pension wealth can increase if it is optimal for the pensioner not to consume all his income in earlier years, but annuities are bought from the available pension wealth at the beginning of the year only. An increase in the pension wealth can also happen if the return on investment is significantly better than expected. However, in all our examples where purchasing annuities is allowed wherever during retirement, the mean consumption will always be higher than the mean income. Increase in the mean pension wealth happens only in the cases where no annuities are allowed after retirement such as in the Case 3.2 in Figure 3.13.

The mean wealth and optimal consumption in Cases 3.4, 3.5 and 3.6 are almost the same. It means that whichever option the pensioner chooses amongst these three, he would have very similar mean pension wealth and mean optimal consumption paths. We also see that the Case 3.1 is the worst one in terms of consumption. It provides a lower consumption than any other case at all ages.

Figures 3.14 and 3.15 show the same results as Figures 3.12 and 3.13 but for $\gamma = -1$ and $b_t = 1$.

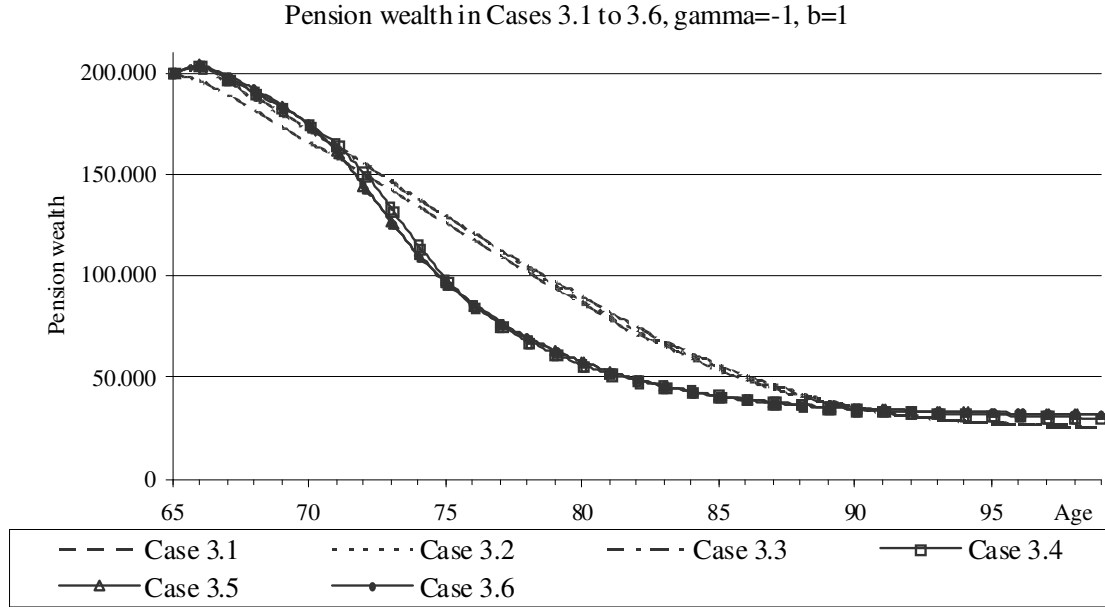


Figure 3.14. Mean pension wealth development in the retirement in Cases 3.1–3.6, for $\gamma=-1$ and $b_t=1$

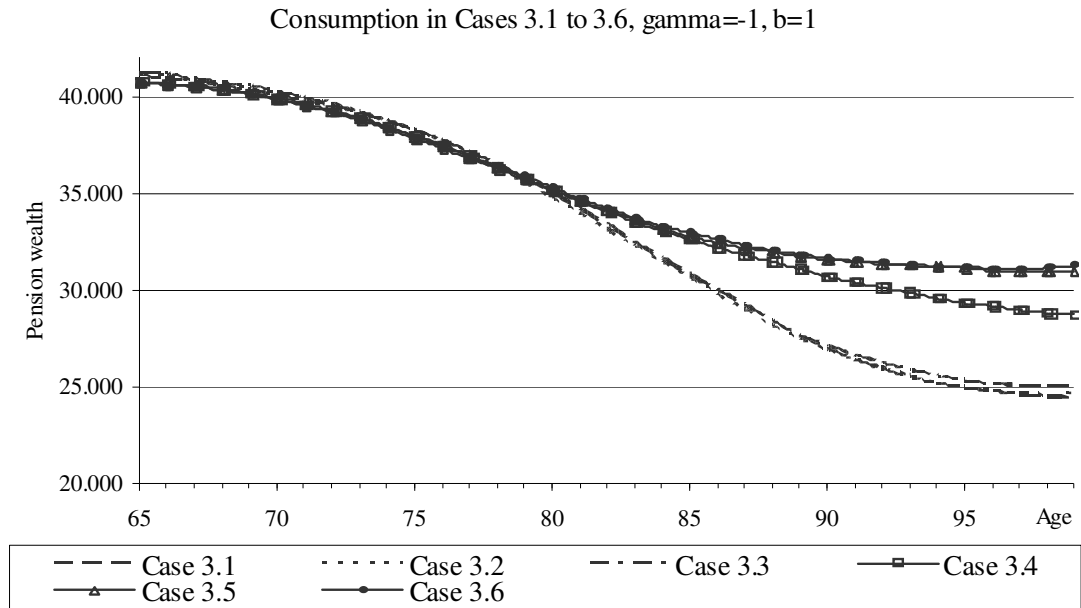


Figure 3.15 Mean optimal consumption development in the retirement in Cases 3.1–3.6, for $\gamma=-1$ and $b_t=1$

Comparing Figures 3.12 and 3.13 and Figures 3.14 and 3.15, we observe less differences of the mean pension wealth and mean optimal consumption paths in the latter two. Regarding mean pension wealth in Figure 3.14, we observe two groups of the patterns of mean pension wealth developments. In the first group are the mean pension wealth paths for Cases 3.4, 3.5 and 3.6, and in the second one are the mean

pension wealth paths for Cases 3.1, 3.2 and 3.3. We observe the similar patterns in Figure 3.15, apart from mean optimal consumption path in the later years of retirement when income decreases due to the influence of the inflation.

Table 3.7 shows the *CEC* measure for different bequest and RRA coefficients. All *CEC* values are calculated at age 65.

In Table 3.8, we compare the results for *CEC* more directly, by considering the change relative to the no annuity case, Case 3.1. Thus, Table 3.8 shows the percentage changes calculated using the formula

$$\frac{\frac{row(i)}{Case\ 3.j}C_{CEC} - \frac{row(i)}{Case\ 3.1}C_{CEC}}{\frac{row(i)}{Case\ 3.1}C_{CEC}}$$

for $1 \leq i \leq 6$ and $2 \leq j \leq 6$.

	Bequest and RRA coefficients	No annuity	Optimal NA at 65, no NA afterwards, no RA	Optimal RA at 65, no RA afterwards, no NA	Optimal NA at 65 and afterwards, no RA	Optimal RA at 65 and afterwards, no NA	Optimal RA and NA at 65 and afterwards
		Case 3.1	Case 3.2	Case 3.3	Case 3.4	Case 3.5	Case 3.6
1	$b_t = 0$ $\gamma = -1$	37,597	37,627	37,749	38,098	38,120	38,121
2	$b_t = 0$ $\gamma = -4$	35,706	36,777	37,192	37,261	37,383	37,383
3	$b_t = 0$ $\gamma = -9$	33,981	36,360	37,003	36,909	37,098	37,100
4	$b_t = 1$ $\gamma = -1$	35,976	35,976	35,980	36,128	36,139	36,144
5	$b_t = 1$ $\gamma = -4$	34,956	35,818	36,016	36,078	36,141	36,142
6	$b_t = 1$ $\gamma = -9$	33,355	35,396	35,693	35,727	35,780	35,782

Table 3.7 *CEC* measure in amounts – Values in the cell show *CEC* measure for different cases and different pensioner's preferences towards risk and bequest. Assumed interest rate during the year prior to retirement is 2.00%. Initial pension wealth is 200,000 money units.

	Bequest and RRA coefficients	Optimal NA at 65, no NA afterwards, no RA	Optimal RA at 65, no RA afterwards, no NA	Optimal NA at 65 and afterwards, no RA	Optimal RA at 65 and afterwards, no NA	Optimal RA and NA at 65 and afterwards
		Case 3.2	Case 3.3	Case 3.4	Case 3.5	Case 3.6
1	$b_t = 0$ $\gamma = -1$	0.08%	0.41%	1.33%	1.39%	1.39%
2	$b_t = 0$ $\gamma = -4$	3.00%	4.16%	4.36%	4.70%	4.70%
3	$b_t = 0$ $\gamma = -9$	7.00%	8.89%	8.62%	9.17%	9.18%
4	$b_t = 1$ $\gamma = -1$	0.00%	0.01%	0.42%	0.45%	0.47%
5	$b_t = 1$ $\gamma = -4$	2.47%	3.03%	3.21%	3.39%	3.40%
6	$b_t = 1$ $\gamma = -9$	6.12%	7.01%	7.11%	7.27%	7.28%

Table 3.8 *CEC* measure in percentages – The values in cells show percentage difference between the Case in the header of the column and Case 3.1, for the values of *CEC* measure in amounts given in Table 3.7.

Table 3.9 shows the *REW* measures for one set of parameters such that the pension wealth values in a particular row give the pensioner the same expected discounted utility derived from future consumption. Benchmark wealth is in Case 3.1 and it is 200,000. Again, all the calculations assume that the pensioner's age is 65.

Similarly to the case for *CEC* measure, we can compare the *REW* measure with Case 3.1. Table 3.10 shows the percentage changes calculated according to the following formula

$$\frac{\frac{row(i)W_{65}^{Case\ 3.1} - row(i)W_{65}^{Case\ 3.j}}{row(i)W_{65}^{Case\ 3.1}}}{}$$

For $1 \leq i \leq 6$ and $2 \leq j \leq 6$. This is the same formulae as for *CEC* measure but with a negative sign in order to get positive percentages.

	Bequest and RRA coefficients	No annuity	Optimal NA at 65, no NA afterwards, no RA	Optimal RA at 65, no RA afterwards, no NA	Optimal NA at 65 and afterwards, no RA	Optimal RA at 65 and afterwards, no NA	Optimal RA and NA at 65 and afterwards
		Case 3.1	Case 3.2	Case 3.3	Case 3.4	Case 3.5	Case 3.6
1	$b_t = 0$ $\gamma = -1$	200,000	199,551	197,773	192,877	192,586	192,576
2	$b_t = 0$ $\gamma = -4$	200,000	183,438	178,007	176,767	175,311	175,303
3	$b_t = 0$ $\gamma = -9$	200,000	162,277	155,052	155,826	153,756	153,737
4	$b_t = 1$ $\gamma = -1$	200,000	200,000	199,934	197,729	197,562	197,498
5	$b_t = 1$ $\gamma = -4$	200,000	186,460	183,798	182,744	181,942	181,928
6	$b_t = 1$ $\gamma = -9$	200,000	167,101	163,941	163,196	162,704	162,672

Table 3.9 *REW* measure in amounts – Values in the cell show wealth needed in Case shown in the column to obtain the same utility as 200,000 in Case 3.1.

	Bequest and RRA coefficients	Optimal NA at 65, no NA afterwards, no RA	Optimal RA at 65, no RA afterwards, no NA	Optimal NA at 65 and afterwards, no RA	Optimal RA at 65 and afterwards, no NA	Optimal RA and NA at 65 and afterwards
		Case 3.2	Case 3.3	Case 3.4	Case 3.5	Case 3.6
1	$b_t = 0$ $\gamma = -1$	0.22%	1.11%	3.56%	3.71%	3.71%
2	$b_t = 0$ $\gamma = -4$	8.28%	11.00%	11.62%	12.34%	12.35%
3	$b_t = 0$ $\gamma = -9$	18.86%	22.47%	22.09%	23.12%	23.13%
4	$b_t = 1$ $\gamma = -1$	0.00%	0.03%	1.14%	1.22%	1.25%
5	$b_t = 1$ $\gamma = -4$	6.77%	8.10%	8.63%	9.03%	9.04%
6	$b_t = 1$ $\gamma = -9$	16.45%	18.03%	18.40%	18.65%	18.66%

Table 3.10 *REW* measure in percentages – Values in the cells show percentage difference between Case 3.1 and the Case in the header of the column, for the values in the Cases given in Table 3.9.

In Tables 3.7–3.10 we clearly see the importance of using annuities. Each case with annuities is more favourable in terms of both the *CEC* and *REW* measure.

If we compare Tables 3.8 and 3.10, we find that the ratio of the values in the cells in Table 3.10 and the value in the relevant cells in Table 3.8 is between 2.5 and 2.8. So, we find that for a given pensioner's preferences, both *CEC* and *REW* measures give similar results in terms of ratios between any two values in a single row in Table 3.8, and the ratio between the values in the relevant cells in Table 3.10. In percentage terms, the values obtained using *REW* measure are higher than the relevant values obtained using *CEC* measure. The results of *CEC* and *REW* measures in percentages give us relative gains and losses for the pensioner due to the introduction of a certain class of annuities. However as we noted earlier, *CEC* measure includes the utility from consumption and does not include the utility from a bequest. *CEC* measure does not give us the appropriate result in the cases with a bequest motive. Thus, one should observe Tables 3.9 and 3.10 when investigating rows 4–6.

Depending on the pensioner's preferences, the importance of access to annuities varies significantly. The biggest difference is between rows 3 and 4. We can say that these two parameter combinations are the two most extreme investigated as all of the other results are somewhere between these results. It is interesting to see from Figures 3.13 and 3.15 that all of the mean optimal consumption paths for $\gamma = -9$ and $b_t = 0$ apart from Case 1 are almost the same, and for $\gamma = -1$ and $b_t = 1$ almost all are the same up to about age 80. However, we see in Tables 3.7–3.10 that the results lead to much larger differences in terms of both *CEC* and *REW* measures for $\gamma = -9$ and $b_t = 0$ than for $\gamma = -1$ and $b_t = 1$. For $\gamma = -9$ and $b_t = 0$, it is optimal to annuitise more than 97% of pension wealth in Case 3.2, about 84% in Case 3.3, and in Cases 3.4, 3.5 and 3.6 mean optimal annuitisation is about 65% at age 65 and then about 7%–15% afterwards. For $\gamma = -1$ and $b_t = 1$, it is optimal not to annuitise at all in Case 3.2, to annuitise about 4% in Case 3.3, and in Cases 3.4, 3.5 and 3.6 it is optimal to start annuitisation at age 70 and to annuitise mostly 5% afterwards. Due to the lower differences amongst cases regarding optimal annuitisation for $\gamma = -1$ and $b_t = 1$ than for $\gamma = -9$ and $b_t = 0$, we observe the lower differences in the mean optimal consumption paths and consequently the lower differences in *CEC* and *REW* measures.

Case 3.4 is always more advantageous than Case 3.2 because there are weaker constraints on the annuitisation in Case 3.4 than in Case 3.2. Similarly Case 3.5 is always more advantageous than Case 3.3. In Case 3.6, the pensioner has no

constraints on the annuitisation strategy and obviously, this is the most favourable case for the pensioner. The results in Tables 3.8 and 3.10 confirm this as we observe that for the lower constraints on the annuitisation the higher are the values of *CEC* and *REW* measures.

From Tables 3.7–3.10, we can say that generally Cases 3.4–3.6 result in similar values of *CEC* and *REW* measures. From detailed results not presented here, we also observe that the pensioners with different preferences will behave similarly in these cases regarding mean optimal asset allocation. Also, if we observe the sum of the mean optimal nominal and real annuitisation in Case 3.6, the mean optimal nominal annuitisation in Case 3.4, and the mean optimal real annuitisation in Case 3.5, then we find a similar optimal annuitisation strategy.

In terms of both *CEC* and *REW* measures, Case 3.6 and Case 3.5 are the best cases for the pensioner. For $\gamma = -1$ and $b_t = 1$, the pensioner purchases real annuities only and is indifferent between Cases 3.5 and 3.6. For the other combinations of the parameters γ and b_t , the losses of Case 3.5 compared to Case 3.6 are almost negligible. In almost all of our examples, the pensioner in Case 3.6 optimally purchases significantly more real annuities than the nominal ones and real annuities are optimally bought earlier in the retirement than the nominal ones. These are the reason that the differences between Cases 3.5 and 3.6 are so small.

Case 3.4 is always inferior compared to Cases 3.5. We find the largest differences for $\gamma = -9$ and $b_t = 0$ where the losses are about 0.55% according to the *CEC* measure and 1.03% according to the *REW* measure. With the introduction of the bequest motive the losses decrease. Mean optimal consumption in Cases 3.4 and 3.5 seems to be very similar in the early years of retirement. However, we observe that mean optimal consumption in Case 3.4 decreases below the values of mean optimal consumption in Case 3.5 in the later years of the retirement due to the effects of inflation on the income from nominal annuities. The more income is received in nominal terms the stronger is the effect of the erosion of the income in real terms, and consequently optimal consumption decreases in the later years of retirement due to this effect. Thus, if the pensioner in Case 3.4 optimally purchases more nominal annuities and if he purchases them earlier in retirement then his income will decrease more in real terms, and the difference between Case 3.4 and Case 3.5 will be larger in terms of *CEC* and *REW* measures.

As we have already noted Case 3.3 is inferior compared to Case 3.5 due to the stronger constraints in Case 3.3. Optimal asset allocation and annuitisation strategies differ significantly. The common conclusions for any pensioner's preferences investigated are the following. The pensioner purchases more annuities in Case 3.3 than in Case 3.5 because he uses the only opportunity to purchase annuities at age 65. As more annuitisation is done at age 65, the pensioner has a lower pension wealth available for investment in retirement in Case 3.3 than in Case 3.5. Consequently, because annuities are a kind of riskless investment, the pensioner invests the higher proportion of the available pension wealth in equities in Case 3.3 than in Case 3.5. However, the optimal asset allocation and annuitisation strategy in Case 3.3 provides the pensioner with a lower gains than in Case 3.5 losses in terms of *CEC* and *REW* measures in all of the examples we investigated. The gains in Case 3.5 compared to Case 3.3 are higher if the pensioner has no bequest motive and if the pensioner is less risk averse. For example, the pensioner in Case 3.3 and with preferences $\gamma = -1$ and $b_t = 0$ optimally annuitises about 31% of his pension wealth at age 65, and in Case 3.5 he optimally starts annuitisation at age 69 and optimally annuitises up to 40% of his available pension wealth in a single year during the rest of the retirement. The difference between Case 3.3 and 3.5 for this pensioner in terms of *CEC* measure is 2.6%. The pensioner in Case 3.3 and with preferences $\gamma = -1$ and $b_t = 1$ optimally annuitises about 4% of his pension wealth at age 65, and in Case 3.5 he optimally starts annuitisation at age 70 and optimally annuitises up to 6% of available pension wealth in a single year during the rest of the retirement. The difference between Case 3.3 and 3.5 for this pensioner in terms of *CEC* measure is 1.19%. If the pensioner in Case 3.3 has the preferences $\gamma = -9$ and $b_t = 0$ he optimally annuitises about 84% of his pension wealth at age 65, and in Case 3.5 he optimally annuitises at age 65 about 65% and much lower percentages of his available pension wealth afterwards. The difference between Case 3.3 and 3.5 are now 0.65%. We conclude that if annuitisation is less attractive for the pensioner, depending on his risk and bequest preferences, then the differences in terms of *CEC* and *REW* measures are larger.

In Cases 3.3 and 3.5 the pensioner is always better off than in Cases 3.2 and 3.4 respectively, and we can say that real annuities provide gains in terms of *CEC* and *REW* measures compared to nominal annuities. However, there is no simple conclusion if we compare Case 3.3 and Case 3.4. In the third row in Tables 3.8 and 3.10, where the pensioner has preferences $\gamma = -9$ and $b_t = 0$, Case 3.3 is more preferable than Case 3.4. It means that the availability of optimal real annuities at age 65 only is more favourable for the pensioner than optimal nominal annuities at any age for this choice of γ and b_t . However, for all other pensioner's preferences, Case

3.4 is more favourable than Case 3.3. The pensioner with no bequest motive, for $\gamma = -9$ gains 0.38% in terms of *REW* measure in Case 3.3 compared to Case 3.4, for $\gamma = -4$ he loses 0.62%, and for $\gamma = -1$ he loses 2.6% in Case 3.3 compared to Case 3.4. If the pensioner has the bequest motive, he is always better off in Case 3.4 than in Case 3.3 and the range of the gains in terms of *REW* measure is from 0.37% for $\gamma = -9$ to 1.11% for $\gamma = -1$. Thus, the more risk averse the pensioner is the lower are the gains in Case 3.4 compared to Case 3.3 in terms of *REW* measure, and if no bequest motive is present, the more risk averse pensioner experiences even lower gains in Case 3.4 compared to Case 3.3.

When comparing Cases 3.2 and 3.3, we see in Tables 3.8 and 3.10 that Case 3.3 is always more favourable than Case 3.2 in terms of *CEC* and *REW* measures. The less risk averse pensioner optimally converts a larger part of the pension wealth into annuities in Case 3.3 than in Case 3.2, while the more risk averse pensioner optimally converts smaller part of the pension wealth into annuities in Case 3.3 than in Case 3.2. If the pensioner has no bequest motive and if we observe the values of RRA coefficient in the range from -1 to -9 , then the optimal nominal annuitisation in Case 3.2 ranges from 9% to 97%, while in Case 3.3 the range of optimal real annuitisation is from 31% to 84%. If the pensioner has the bequest motive, then optimal nominal annuitisation in Case 3.2 ranges from 0% to 80 %, while in Case 3.3 the range of optimal real annuitisation is from 4% to 69%. At the same time, we observe the larger differences in terms of *CEC* and *REW* measures for the less risk averse pensioner. This again shows that the right choice of the optimal asset allocation and optimal annuitisation, together with optimal consumption is crucially important for attaining the best results in terms of *CEC* and *REW* measures. We conclude here that the gains in Case 3.3 compared to Case 3.2 decreases with a decrease of the pensioner's risk aversion and decreases with the introduction of the bequest motive.

Case 3.1 is inferior to any other case, apart for the value of parameters $\gamma = -1$ and $b_t = 1$. The availability of any kind of annuity investigated in this thesis is always beneficial to the pensioner. For $\gamma = -9$ and $b_t = 0$, the pensioner gains 23.13% in terms of *REW* measures in Case 3.6 compared to Case 3.1. For $\gamma = -1$ and $b_t = 0$, he gains 3.71%, for $\gamma = -1$ and $b_t = 1$, he gains 1.25%. The more risk averse the pensioner is, the larger are the gains in terms of *CEC* and *REW* measures in Case 3.6 compared to Case 3.1. The pensioner with the bequest motive obtains the lower gains compared to the pensioner with the same RRA coefficient and with no bequest motive.

3.4.6 Expected Discounted Utility and Stochastic Inflation

In this section, we investigate the effects of stochastic inflation on the value function and on *CEC* and *REW* measures. Random inflation will affect Cases 3.2, 3.4 and 3.6 because inflation influences the demand for nominal annuities and income from nominal annuities only. We noted in the previous section that in Case 3.6 the majority of the annuities bought are real annuities. Small differences in the results in Case 3.5 and 3.6 show that nominal annuities do not have a significant influence in Case 3.6. As we generally do not find significant change of the optimal asset allocation and optimal annuitisation strategies with the introduction of the random instead of constant inflation, we will focus our investigation in this section on Cases 3.2 and 3.4 only and compare them with Cases 3.1, 3.3 and 3.5.

As we note in Section 3.4.2.8, the lower/higher value of inflation in the year prior to the nominal annuitisation the higher/lower is the nominal annuity rate. It is then followed by a lower/higher income from nominal annuity and slower/faster decrease of this income in real terms due to (on average) lower/higher inflation in the following years.

We find that in all our investigated examples, the results in terms of the value function are changing just slightly when we allow inflation to be random.

In order to give an idea of how the expected discounted utility changes with the change of the value of inflation in the year prior to retirement, we present in Table 3.11 the values of the expected discounted utility for different cases and for chosen possible states of inflation in the year prior to retirement. We also present the values of the expected discounted utility with the assumption of the constant inflation. In Table 3.11, we show the results for $\gamma = -9$ and $b_i = 0$, initial pension wealth of 200,000 money units and other parameters as defined in Section 3.4.1.

In Table 3.11, we choose five out of 15 possible states of the inflation rate in the year preceding retirement. The first and the last values of the inflation are the most extreme allowed values that can be attained with the lowest probability. The second and the fourth values of inflation are the fifth and eleventh possible values when inflation states are ordered from the lowest to the highest value and these values of inflation can be attained with reasonably high probability. The third one is the mean value of the inflation.

	Inflation in the year prior to retirement	No annuity	Optimal NA at 65, no NA afterwards, no RA	Optimal RA at 65, no RA afterwards, no NA	Optimal NA at 65 and afterwards, no RA	Optimal RA at 65 and afterwards, no NA
		Case 3.1	Case 3.2	Case 3.3	Case 3.4	Case 3.5
Random Inflation						
1	0.05%	-55,522.87	-30,442.34	-25,789.68	-26,549.71	-25,200.31
2	1.72%	-55,522.87	-30,444.07	-25,789.68	-26,550.19	-25,200.31
3	4.00%	-55,522.87	-30,447.12	-25,789.68	-26,551.05	-25,200.31
4	6.28%	-55,522.87	-30,451.23	-25,789.68	-26,552.27	-25,200.31
5	7.95%	-55,522.87	-30,455.97	-25,789.68	-26,553.82	-25,200.31
Constant Inflation						
6	4.00%	-55,522.87	-30,197.95	-25,789.68	-26,385.99	-25,200.31

Table 3.11 Expected discounted utility – Values in the cells show expected discounted utility at age 65 in different cases, for $\gamma = -9$ and $b_t = 0$, initial pension wealth $W_{65} = 200.000$, last salary income $Y_{65} = 33.321$ and for different values of the inflation rate in the year preceding retirement.

As we expect, random inflation affects Cases 3.2 and 3.4 only. The effect of the random inflation has a decreasing effect on the expected discounted utility in Cases 3.2 and 3.4 for all presented values of inflation. In Case 3.2, we find the decrease of about 250 units of utility, and in Case 3.4 the decrease of about 166 utility units. Comparing expected discounted utility for random inflation in Case 3.2, we find the differences between expected discounted utility of only 13 utility units. In Case 3.4, these differences are less than 5 utility units. Expected discounted utility with a stochastic inflation is always lower than with a constant inflation. Thus, we find that stochastic inflation results in the loss of utility units compared with the results for constant inflation. Changing the value of the inflation rate in the year prior to the retirement affects expected discounted utility less than the introduction of the stochastic inflation instead of the constant one. However, we find that increasing inflation rate results in increasing expected discounted utility in both Cases 3.2 and 3.4. We can see from Table 3.11 that the pensioner in Cases 3.2 and 3.4 will attain a lower expected discounted utility in the presence of the stochastic inflation than in a case where inflation is constant, and also the degree of losses is not significantly dependent on the value of the inflation rate in the year prior to retirement.

In Table 3.12, we present expected discounted utility values from Table 3.11 in terms of the *REW* measure.

	Inflation in the year prior to retirement	No annuity	Optimal NA at 65, no NA afterwards, no RA	Optimal RA at 65, no RA afterwards, no NA	Optimal NA at 65 and afterwards, no RA	Optimal RA at 65 and afterwards, no NA
		Case 3.1	Case 3.2	Case 3.3	Case 3.4	Case 3.5
Random Inflation						
1	0.05%	200,000	162,562	155,052	155,953	153,756
2	1.72%	200,000	162,583	155,052	155,957	153,756
3	4.00%	200,000	162,599	155,052	155,964	153,756
4	6.28%	200,000	162,591	155,052	155,971	153,756
5	7.95%	200,000	162,575	155,052	155,976	153,756
Constant Inflation						
6	4.00%	200,000	162,277	155,052	155,826	153,756

Table 3.12 *REW* measure in amounts – Values in the cells show *REW* at age 65 in different cases, for $\gamma = -9$ and $b_t = 0$, initial pension wealth $W_{65} = 200.000$, last salary income $Y_{65} = 33.321$ and for different values of the inflation rate in the year preceding retirement.

In Table 3.12, we see that the *REW* measure shows lower values with stochastic than with the constant inflation in both Cases 3.2 and 3.4. When the value of the inflation rate before retirement changes from the lowest to the highest possible value, the decrease in terms of *REW* measure ranges from 306 to 322 money units in Case 3.2 when we compare stochastic versus constant inflation results. We observe that, due to stochastic inflation, the *REW* measure in Case 3.2 firstly increases when the inflation rate in the year prior to retirement increases from the lowest values to the mean value of inflation and then decreases slightly as the value of inflation increase further. In Case 3.4, the values of the differences in terms of *REW* measure range from 127 to 131 money units and we observe monotone increase in terms of *REW* measure as the value of the inflation rate in the year prior to retirement increases from the lowest to the highest value.

In Table 3.13, we present *REW* measure in percentage, using the same techniques as in the previous section.

	Inflation in the year prior to retirement	Optimal NA at 65, no NA afterwards, no RA	Optimal RA at 65, no RA afterwards, no NA	Optimal NA at 65 and afterwards, no RA	Optimal RA at 65 and afterwards, no NA
		Case 3.2	Case 3.3	Case 3.4	Case 3.5
Random Inflation					
1	0.05%	18.72%	22.47%	22.02%	23.12%
2	1.72%	18.71%	22.47%	22.02%	23.12%
3	4.00%	18.70%	22.47%	22.02%	23.12%
4	6.28%	18.70%	22.47%	22.01%	23.12%
5	7.95%	18.71%	22.47%	22.01%	23.12%
Constant Inflation					
6	4.00%	18.86%	22.47%	22.09%	23.12%

Table 3.13 *REW* measure in percentages – Values in the cells show *REW* in percentages at age 65 in different cases, for $\gamma = -9$ and $b_t = 0$, initial pension wealth $W_{65} = 200.000$, last salary income $Y_{65} = 33.321$ and for different values of the inflation rate in the year preceding retirement.

The results in Table 3.13 show that in Case 3.2, the pensioner loses 0.14–0.16 % in terms of *REW* measure and 0.07–0.08% in Case 3.4 due to the introduction of the stochastic inflation.

In Table 3.14 and 3.15 we present the same results as in Tables 3.12 and 3.13 but now for the pensioner with the bequest motive. The values of the other parameters are the same.

	Inflation in the year prior to retirement	No annuity	Optimal NA at 65, no NA afterwards, no RA	Optimal RA at 65, no RA afterwards, no NA	Optimal NA at 65 and afterwards, no RA	Optimal RA at 65 and afterwards, no NA
		Case 3.1	Case 3.2	Case 3.3	Case 3.4	Case 3.5
Random Inflation						
1	0.05%	200,000	167,133	163,941	163,225	162,704
2	1.72%	200,000	167,147	163,941	163,230	162,704
3	4.00%	200,000	167,166	163,941	163,237	162,704
4	6.28%	200,000	167,180	163,941	163,244	162,704
5	7.95%	200,000	167,186	163,941	163,249	162,704
Constant Inflation						
6	4.00%	200,000	167,101	163,941	163,196	162,704

Table 3.14 *REW* measure in amounts – Values in the cells show *REW* at age 65 in different cases, for $\gamma = -9$ and $b_t = 1$, initial pension wealth $W_{65} = 200.000$, last salary income $Y_{65} = 33.321$ and for different values of the inflation rate in the year preceding retirement.

	Inflation in the year prior to retirement	Optimal NA at 65, no NA afterwards, no RA	Optimal RA at 65, no RA afterwards, no NA	Optimal NA at 65 and afterwards, no RA	Optimal RA at 65 and afterwards, no NA
		Case 3.2	Case 3.3	Case 3.4	Case 3.5
Random Inflation					
1	0.05%	16.43%	18.03%	18.39%	18.65%
2	1.72%	16.43%	18.03%	18.39%	18.65%
3	4.00%	16.42%	18.03%	18.38%	18.65%
4	6.28%	16.41%	18.03%	18.38%	18.65%
5	7.95%	16.41%	18.03%	18.38%	18.65%
Constant Inflation					
6	4.00%	16.45%	18.03%	18.40%	18.65%

Table 3.15 *REW* measure in percentages – Values in the cells show *REW* in percentages at age 65 in different cases, for $\gamma = -9$ and $b_t = 1$, initial pension wealth $W_{65} = 200.000$, last salary income $Y_{65} = 33.321$ and for different values of the inflation rate in the year preceding retirement.

Similar to the no bequest case, in Cases 3.2 and 3.4 with the bequest motive the introduction of the stochastic inflation results in slightly higher values of *REW* measure in amounts compared to the examples with the constant inflation. In both Cases 3.2 and 3.4, we now observe increase of the values of the *REW* measure as the inflation rate in the year preceding retirement increases from the lowest to the highest possible values. The range of the differences due to the introduction of stochastic

inflation in terms of *REW* measure in amounts is from 33 to 85 in Case 3.2 and from 29 to 53 in Case 3.4. Regarding the differences in terms of *REW* measure in percentages, gains in Cases 3.2 and 3.4 are lower for less than 4% if inflation is stochastic compared to the constant inflation.

The pensioner with RRA coefficient $\gamma = -9$ will optimally annuitise the largest amount of his pension wealth and will annuitise earlier in retirement compared to the less risk averse pensioners. Stochastic inflation affects nominal annuities only, and due to the highest demands for annuities, the pensioner with RRA coefficient $\gamma = -9$ will be most affected with the stochastic inflation. However, from the results presented in this section, we see that the effects of the stochastic inflation compared to the examples with the constant inflation are not significant for this pensioner. The pensioners with the lower risk aversion will be even less affected with the introduction of the stochastic inflation.

We conclude that stochastic inflation in the model compared to the constant inflation brings small differences in terms of expected discounted utility. The results with constant inflation are very similar to the ones with the stochastic inflation in terms of *REW* measure. We observe slightly lower gains in Cases 3.2 and 3.4 when stochastic inflation is present than in the example with constant inflation.

3.4.7 Left Tail Analysis of the Random Utility

The results presented in Sections 3.4.5 and 3.4.6 are based on the value function which is calculated using numerical mathematics. The value function depends on the four state variables and it is a deterministic function. Whenever *CEC* and *REW* measures have been applied in these sections, these measures were based on the deterministic function. We have used stochastic simulations in Sections 3.4.5 and 3.4.6 in order to observe the mean and quantiles of the pension wealth path, optimal consumption path, optimal asset allocation path, optimal annuitisation path, and paths of the other variables that we find interesting for explaining the results of *CEC* and *REW* measures. Pension wealth path, optimal consumption path, optimal asset allocation path, optimal annuitisation path are all stochastic processes. In this section, we investigate the consequences of the worse than expected market realisation. When we say worse than expected, we mean on the fundamental random variables, equity and inflation random variables. For reasons of simplicity, we will assume that inflation is constant in this section and that it takes a mean value of 4%. Thus, in this

section we investigate the consequences of the worse than expected realisation of the random equity return.

The value function is defined as expected value of the discounted utility derived from future consumption and bequest. We defined equation (3.67) as the discounted utility derived from the future random consumption and bequest. In contrast to the value function, the function $\tilde{D}(W_{65}, Y_{65}, 0, I_{64;k})$ defined in (3.67) is a random variable.

Stochastic simulations provide us with realisations of the random paths of the pension wealth, optimal consumption, optimal asset allocation, and optimal annuitisation. Each realisation of the stochastic simulation gives one particular realisation of these random variables. From the pensioner's point of view, each realisation of the stochastic simulation gives him one possible development of the state and control variables during retirement.

Also, each realisation of the stochastic simulation gives one particular realisation of the discounted derived utility. The pensioner will be concerned about the possibility that he ends up with the lower than expected utility from future consumption. In order to investigate this risk, we analyse in this section the left tail of the random utility derived from future consumption and bequest.

We have optimisation with respect to expected discounted utility only. The importance of the left tail risk is recognised in the model through the concave shape of the utility function. Namely, let C_t be a certain amount of the consumption and let ΔC_t be some other value of consumption such that $0 < \Delta C_t < C_t$. Then, the amount of utility units lost is larger if the pensioner consumes $C_t - \Delta C_t$ than the amount of extra utility units gained if the pensioner consumes $C_t + \Delta C_t$. Thus, the importance of the left tail risk is already included in the model.

The left tail risk of the lower than expected derived utility is just a consequence of the pensioner's attitude to risk represented in his utility function. We employ Var_α and $CVaR_\alpha$ measure in order to see the degree of the important pensioner's risk of the lower than expected derived utility in retirement due to the worse than expected market conditions during his retirement.

However, left tail risk is obviously very important for the pensioner and it would be of interest to investigate modified utility function such that less than expected consumption is punished even more in terms of lost utility units. In that way, we

would be in a position to control $CVaR_\alpha$ better and define a utility function balanced against $CVaR_\alpha$. The analysis in this section aims to raise the question of the possible maximization of the pensioner's expected derived utility as criterion but with constraints on $CVaR_\alpha$.

3.4.7.1 The Definition of VaR_α and $CVaR_\alpha$ measures

The random variable $\tilde{D}(W_{65}, Y_{65}, 0, I_{64;k})$ depends on the control variable $\tilde{C}_{65}$, and on the stochastic processes $\tilde{C}_t$ for $66 \leq t \leq 99$ and $\tilde{W}_{t+1}$ for $65 \leq t \leq 99$. These stochastic processes depend on random variable $\tilde{r}_t$ and stochastic process $\tilde{I}_t$ for $65 \leq t \leq 99$, random equity and inflation rates respectively.

We can measure different characteristics of the random discounted utility $\tilde{D}(W_{65}, Y_{65}, 0, I_{64;k})$. As we have noted, we will focus on the left tail analysis of the random variable $\tilde{D}(W_{65}, Y_{65}, 0, I_{64;k})$. In order to have understandable results, we need to convert discounted utility from utility units into the money units. As in Section 3.4.5 and 3.4.6 we rely on the idea of required equivalent wealth for presenting VaR_α and $CVaR_\alpha$ in money terms.

As we noted earlier, $V_{65}(W_{65}, Y_{65}, 0, I_{64;k}) = E[\tilde{D}(W_{65}, Y_{65}, 0, I_{64;k})]$. Now, we define random variable $\tilde{W}_{65}$, such that

$$V_{65}(\tilde{W}_{65}, Y_{65}, 0, I_{64;k}) = \tilde{D}(W_{65}, Y_{65}, 0, I_{64;k}) \quad (3.68)$$

The uniqueness of random variable $\tilde{W}_{65}$ comes from the fact that value function V_{65} is a strictly increasing function in variable W_{65} . The existence of random variable $\tilde{W}_{65}$ should be mathematically proved, but we believe that for this thesis it is enough to say that for each random realisation of random variable $\tilde{D}(W_{65}, Y_{65}, 0, I_{64;k})$, we could find the realisation of random variable $\tilde{W}_{65}$.

The random variable $\tilde{W}_{65}$ gives us the wealth that the pensioner needs such that the mean value of all possible random discounted utilities with initial wealth $\tilde{W}_{65}$ is equal to the random discounted utility $\tilde{D}(W_{65}, Y_{65}, 0, I_{64;k})$. The cumulative distribution function (abbreviation *CDF*) of the random variable $\tilde{W}_{65}$ can be defined in the following way. If *CDF* of the random variable $\tilde{D}(W_{65}, Y_{65}, 0, I_{64;k})$ is given by

$$P_{\tilde{D}}(\tilde{D}(W_{65}, Y_{65}, 0, I_{64;k}) \leq x),$$

for $x \in (-\infty, \infty)$ then CDF of the random variable $\tilde{W}_{65}$ is defined by

$$P_{\tilde{W}_{65}}(\tilde{W}_{65} \leq y) = P_{\tilde{D}}\left(V_{65}(\tilde{W}_{65}, Y_{65}, 0, I_{64;k}) \leq x \text{ such that } x = V_{65}(y, Y_{65}, 0, I_{64;k})\right)$$

for y in the domain of the solutions of equation $x = V_{65}(y, Y_{65}, 0, I_{64;k})$. Equation $x = V_{65}(y, Y_{65}, 0, I_{64;k})$ will have a solution in a certain interval. For values of y smaller than the lowest value in the interval of the solutions of equation $x = V_{65}(y, Y_{65}, 0, I_{64;k})$ we define $P_{\tilde{W}_{65}}(\tilde{W}_{65} \leq y) = 0$, and for higher than the highest value in that interval we define $P_{\tilde{W}_{65}}(\tilde{W}_{65} \leq y) = 1$. Thus, CDF of random variable $\tilde{W}_{65}$ is fully defined.

Having defined random variable $\tilde{W}_{65}$, we can investigate the left tail of possible future random realisations of discounted utility in money terms.

We now can define VaR_{α} and $CVaR_{\alpha}$ measures, as left tail measures of the random variable $\tilde{W}_{65}$. Firstly, we define VaR_{α} , as follows

$$VaR_{\alpha} = \inf \left\{ W \in \mathbb{R} : P[\tilde{W}_{65} \geq W] \leq 1 - \alpha \right\} \quad (3.69)$$

or

$$VaR_{\alpha} = \inf \left\{ W \in \mathbb{R} : P\left[\tilde{D}(W_{65}, Y_{65}, 0, I_{64;k}) \geq V_{65}(W, Y_{65}, 0, I_{64;k})\right] \leq 1 - \alpha \right\} \quad (3.70)$$

for $0 < \alpha < 1$. The value of VaR_{α} gives us the following information. For the pensioner with pension wealth W_{65} , there is an $\alpha\%$ probability that unfavourable market realisations in the future will result in a lower or same discounted utility that would have been obtained as expected discounted utility with the pension wealth VaR_{α} . In other words, VaR_{α} is the $\alpha\%$ worst pension wealth due to less favourable than expected market conditions in the future, where pension wealth is measured using equation (3.68).

In our investigation, we make 2,000 stochastic simulations of the developments from age 65 to age 99 of all random variables in the model. For the purpose of deeper investigation of the pensioner's left-tail risk, more than 2,000 random realisations may be appropriate. However, the results presented here are not very dependent on the number of the random realisations and we believe that it is appropriate to use here the same realisation of the stochastic simulations that we use for the check of accuracy of the numerical calculations.

In order to calculate VaR_α from the realisations of the stochastic simulation we use formula (3.67). For each realisation of the stochastic simulation, we obtain optimal consumption and pension wealth for each age. Substituting these values in equation (3.67), we obtain 2,000 realisations of discounted derived utility. So, we obtain the sample of 2,000 random realisations of discounted derived utility $\tilde{D}(W_{65}, Y_{65}, 0, I_{64;k})$. In our analysis we investigate VaR_α for $\alpha \in \{0.01, 0.05, 0.10, 0.25\}$. We obtain the value of VaR_α in the following way. Firstly, we calculate 2,000 random realisations of the variable $\tilde{W}_{65}$ using formula (3.68). Then, we order these 2,000 random realisations of the variable $\tilde{W}_{65}$ in an increasing array. Then $VaR_{0.01}$ is the twentieth member of the ordered array, $VaR_{0.05}$ is the hundredth member, $VaR_{0.10}$ is the two hundredth member, and $VaR_{0.25}$ is the five hundredth member of the ordered array.

Conditional Value at Risk is a measure of risk that has advantages over Value at Risk. $CVaR_\alpha$ is able to quantify dangers beyond VaR_α , and moreover it is a coherent measure of risk (Rockafellar and Uryasev (2002)). We define $CVaR_\alpha$ in the simplest way. $CVaR_\alpha$ is defined as mean shortfall, or in a mathematical definition as

$$CVaR_\alpha = Mean[\bar{W}_{65} | \bar{W}_{65} < VaR_\alpha], \quad (3.71)$$

where $\bar{W}_{65}$ are random realisations of random variable $\tilde{W}_{65}$ that satisfy the condition $\bar{W}_{65} < VaR_\alpha$.

In the same way as for VaR_α , we calculate $CVaR_\alpha$ for $\alpha \in \{0.01, 0.05, 0.10, 0.25\}$. We use 2,000 random realisations of the random variable $\tilde{W}_{65}$ from the ordered array already obtained for the calculation of VaR_α . $CVaR_{0.01}$ is calculated as the mean of the first nineteen members of the ordered array, $CVaR_{0.05}$ is the mean of the first ninety nine members, $CVaR_{0.10}$ is the mean of the first one hundred ninety nine members, and $CVaR_{0.25}$ is the mean of the first four hundred ninety nine members of the ordered array.

3.4.7.2 VaR_α and $CVaR_\alpha$ measures – Results

We will present the results for $\alpha = 0.10$. The results for the other values of α , not presented here, have different values but the pattern is the same and the same conclusions can be drawn. As we noted, we aim to shed light on the pensioner's left tail risk and we leave a deeper analysis for future work.

Firstly, we present some examples of the graphs representing values of the random variable $\tilde{W}_{65}$ on the x-axis and frequencies of this random variable on the y-axis where frequencies are taken from the random sample of 2,000 random realisations. The left vertical straight line represents $CVaR_{0.10}$, and the right one $VaR_{0.10}$.

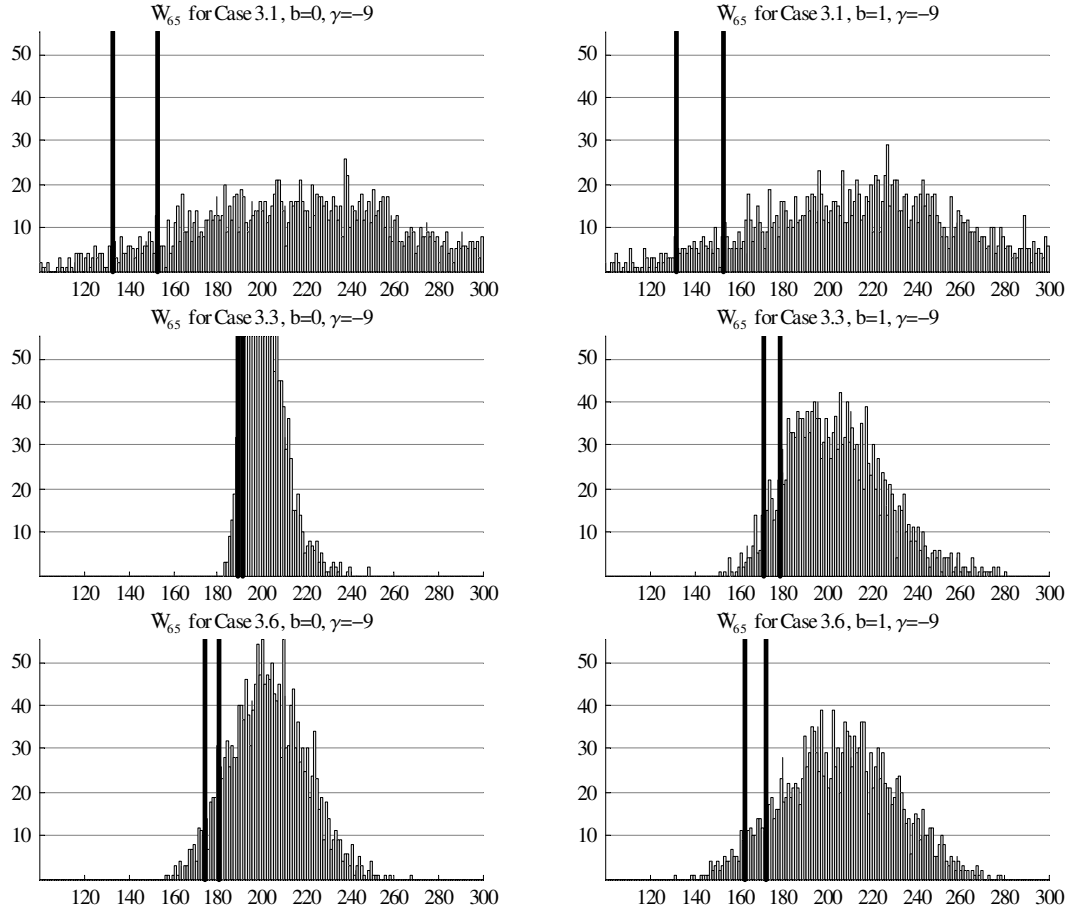


Figure 3.16 Histogram of the random sample of 2,000 random realisation of $\tilde{W}_{65}$ for $W_{65} = 200,000$, in Cases 3.1, 3.3 and 3.6, for $b_t = 0$ and $b_t = 1$ and for $\gamma = -9$. The left one vertical straight line represents $CVaR_{0.10}$, and the right one $VaR_{0.10}$.

Although the pensioner in the Case 3.1 behaves optimally in terms on maximising utility, he can end up with very different discounted derived utility depending on the random realisation of equity rates. His pension wealth at the time of retirement is $W_{65} = 200,000$ and he expects to end up his retirement with the value of 200,000 money units of the random variable $\tilde{W}_{65}$. However, the distribution has a very wide range and he can end up in significantly higher or lower discounted derived utility than he expects.

In Case 3.3, we have a quite different situation. In the case with no bequest, the pensioner optimally annuitises about 85% of his available pension wealth and he is less exposed to the risk of the randomness of equity rate. The distribution on the left histogram in the middle row in Figure 3.16 has the lowest range of all the graphs, and $CVaR_{0.10}$ and $VaR_{0.10}$ are within the histogram. If the pensioner has a bequest motive then at age 65 he optimally annuitises about 70% of the pension wealth. Due to the lower annuitisation, the pensioner with the bequest motive is more exposed to the risk of the randomness of equity rate and the histogram in the middle row on the right hand side in Figure 3.16 shows a wider range of the values of the random variable $\tilde{W}_{65}$. It means that the pensioner with the bequest motive has a less stable single outcome of the derived utility compared with the pensioner with no bequest motive.

In Case 3.6, the sum of optimal real and nominal annuitisation for the pensioner without bequest motive at age 65 is about 65% and less than 10% afterwards, and for the pensioner with the bequest motive it is about 60% at age 65 and less than 5% afterwards. Optimally nominal annuitisation is very low for both pensioners and does not influence the results significantly. Again, it seems that the pensioner with no bequest motive has the lower left tail risk of the lower than expected realisation of the discounted derived utility in retirement than the pensioner with bequest motive.

Comparing graphs on the left hand side of Graph 3.16, we find that the pensioner with no bequest motive in Case 3.1 has by far the widest range of possible outcomes, by far the lowest range on outcomes in Case 3.3 and the Case 3.6 is in between. The right hand side present the pensioner with a bequest motive and we observe that in the Case 3.1 the possible outcomes in terms of the random variable $\tilde{W}_{65}$ are very unstable, while in Cases 3.3 possible outcomes are slightly more concentrated around the mean value than in the Case 3.6.

In Figure 3.17, we present more examples of the histograms of the approximate distributions of the random variable $\tilde{W}_{65}$, but for different values of the RRA coefficient. Again, we include on each graph two vertical lines representing $CVaR_{0.10}$ (left one vertical line), and $VaR_{0.10}$ (right one vertical line). Frequencies of the random variable $\tilde{W}_{65}$ are taken from the random sample of 2,000 realisations of stochastic simulations.

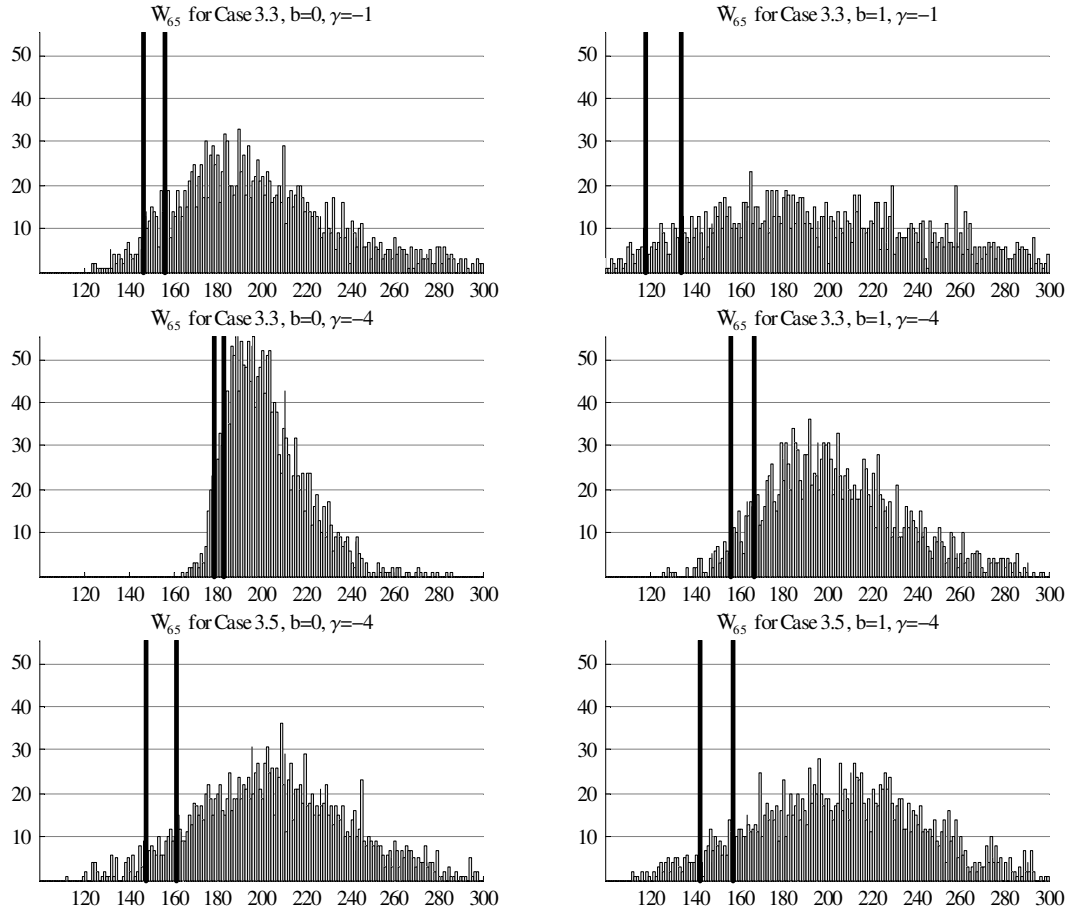


Figure 3.17 Histogram of the random sample of 2,000 random realisation of $\tilde{W}_{65}$ for $W_{65} = 200,000$, in Case 3.3, for $b_t = 0$ and $b_t = 1$ and for $\gamma = -1$, and for Cases 3.3 and 3.5, for $b_t = 0$ and $b_t = 1$ and for $\gamma = -4$. The left one vertical straight line represents $CVaR_{0.10}$, and the right one $VaR_{0.10}$.

We observe in Figure 3.17 that the pensioner has very different distributions of discounted utility presented in terms of the distribution of random variable $\tilde{W}_{65}$. We observe in the left hand side histogram in the middle for Case 3.3, the pensioner with RRA coefficient $\gamma = -4$ and with no bequest motive has a short left tail and both $VaR_{0.10}$ and $CVaR_{0.10}$ values are closest to the initial pension wealth of 200,000. In this Case, and for these values of parameters, the pensioner optimally purchases the largest amount of annuities, compared to any other histogram presented in Figure 3.17. If the pensioner has the bequest motive, then he optimally purchases fewer annuities compared to the pensioner with no bequest motive, and he is more exposed to the equity rate risk. Thus, the distribution of random variable $\tilde{W}_{65}$ has a wider range of values for the pensioner with a bequest motive. In Case 3.5, the pensioner optimally purchases annuities during retirement and at age 65 he optimally purchases less annuities compared to the pensioner in Case 3.3. So, the pensioner in Case 3.5 has the distribution of random variable $\tilde{W}_{65}$ with a longer left tail.

In Table 3.16, we present the values of $CVaR_\alpha$ for the pensioners with different preferences towards risk and bequest and in different cases. We assume that each pensioner has pension wealth of $W_{65} = 200,000$ at age 65. The results are obtained from the sample of 2,000 realisations of the random variable $\tilde{W}_{65}$. We emphasise that the results in Table 3.16 contain random errors due to the limited size of the sample. However, we can observe interesting relations between the values of $CVaR_{0.10}$.

	Bequest and RRA coefficients	No annuity	Optimal NA at 65, no NA afterwards, no RA	Optimal RA at 65, no RA afterwards, no NA	Optimal NA at 65 and afterwards, no RA	Optimal RA at 65 and afterwards, no NA	Optimal RA and NA at 65 and afterwards
		Case 3.1	Case 3.2	Case 3.3	Case 3.4	Case 3.5	Case 3.6
1	$b_t = 0$ $\gamma = -1$	118,636	136,089	146,233	118,173	119,988	119,447
2	$b_t = 0$ $\gamma = -4$	114,256	180,511	177,946	144,957	148,005	146,200
3	$b_t = 0$ $\gamma = -9$	134,755	190,107	188,984	171,052	174,802	174,311
4	$b_t = 1$ $\gamma = -1$	113,266	112,786	117,918	114,873	112,180	111,091
5	$b_t = 1$ $\gamma = -4$	109,981	160,736	156,500	139,810	141,968	142,702
6	$b_t = 1$ $\gamma = -9$	130,983	170,454	170,993	165,202	162,503	161,929

Table 3.16 $CVaR_{0.10}$ – Values in the cells show the values of $CVaR_{0.10}$ for different pensioner's preferences towards risk and bequest and in different cases. Pensioner is at age 65. Pension wealth is 200,000. The values of $CVaR_{0.10}$ are calculated from the sample of 2,000 random realisations.

Firstly, we emphasise that all $CVaR_{0.10}$ values in Table 3.16 are calculated for the value of the pension wealth of 200,000. The pensioners in different cases have different expected discounted utility and in Table 3.9 we present the values of initial pension wealth that provide the same expected discounted utility for the pensioners in different cases. Thus, the conclusions drawn from the values in Table 3.16 cannot be simply compared with the conclusions drawn from Tables 3.7 – 3.10.

If both REW and $CVaR_{0.10}$ measures show better results in one case than in another, then we can conclude that the pensioner in the first case is better off in terms of both measures. We can conclude that the pensioner in the first case benefits in terms of expected discounted utility and we can measure this benefit in terms of REW

measure. At the same time, the pensioner decreases left tail risk of the lower than expected realisation of his discounted utility in retirement.

If we compare the pensioner with no bequest motive and for the value of RRA coefficient $\gamma = -4$, in Cases 3.3 and 3.1, then from Table 3.10 we observe that the pensioner is 11.00% better off in terms of *REW* measure. Applying the optimal strategy such that *REW* measure is maximised, the pensioner at the same time decreases the risk of the lower than expected discounted utility during retirement. In terms of $CVaR_{0.10}$ measure, the pensioner in Case 3.3 increases the mean value of the worst 10% realisation of discounted utility from 134,755 to 188,984.

However, if the pensioner in one case is better off in terms of *REW* measure and worse off in terms of the $CVaR_{0.10}$ measure than in another case, all we can conclude is that the pensioner achieved the better result in terms of the criterion that he wanted to maximise. According to the $CVaR_{0.10}$ measure, the pensioner is worse off in the first case, but it is the consequence of the pensioner's optimal strategy.

For example, the pensioner with the bequest motive and for the value of RRA coefficient $\gamma = -9$, gains 0.26% in terms of *REW* measure in Cases 3.6 compared to Case 3.3. At the same time, the mean value of 10% worst discounted utilities in terms of the random variable $\tilde{W}_{65}$ decreases from 170,993 in Case 3.3 to 161,929 in Case 3.6.

Chapter 4

The Interest Rate Risk Model

4.1 The Problem to be Solved

In this chapter, we investigate the three assets model in the post-retirement period where annuities are available. We assume that the pensioner can invest his pension wealth into risk free deposit, bonds and stocks, and apart from that he can purchase annuities as irreversible investments. We work in a discrete time framework, where one time unit is one year and we assume that the member rebalances his wealth at the moment when his age increases for one year.

4.1.1 Economic Environment

We model the market consisting of four possible options for converting wealth available for investment. Firstly, there are three assets: risk free assets – one year bond, low risk asset – T_t year rolling bond, and high risk asset – equities. Then, as we investigate the post-retirement period we allow annuities to be the fourth possible option into which available wealth can be converted. However, one should always have in mind that annuities are irreversible investments, and due to irreversibility they differ crucially from first mentioned set of three assets.

We emphasise that there is no inflation in this chapter and thus all amounts are in real terms.

We assume that the retirement age is 65 and that the pensioner receives his last salary at that age. At age 66 he receives the first income from social security which continues at the beginning of each year of pensioner's life until his death. We assume

that income from social security is constant. We also assume that the pensioner has certain pension wealth at the time of retirement.

The pensioner draws utility from consuming part of his available assets at the beginning of the year. Available assets consist of pension wealth and received income. If a bequest motive exists then besides drawing utility from consuming the pensioner draws utility from bequeathing assets to heirs. We assume that the remaining assets are bequeathed to heirs at the end of the year in which the pensioner dies.

At the beginning of the year, the pensioner receives income and interest, then he consumes part of his available assets and invests the rest into three assets and annuities. The investment into three assets is done at the pensioner's discretion apart from no borrowing constraint. We assume that no borrowing is allowed to the pensioner for both assets and annuities. We will investigate different constraint on purchasing annuities. Sometimes it will be at the pensioner's discretion at all age, sometimes at the pensioner's discretion at certain ages and limited at some other ages, and sometimes it will be limited for all ages.

In our model, we assume two sources of randomness: random interest rate and random rate on equity investment. On the other side, we have bonds and equities. Risk free investment is not influenced by randomness because we assume that interest rate changes annually. We can say that we have two sources of randomness and two assets depending on that randomness.

We summarise the model to be investigated in this chapter and present the most important variables graphically.

We work in the discrete time. We assume that the postretirement decumulation process starts at age $t = 65$, and finishes at age $t = 100$. The decumulation process lasts for 35 years. If a bequest motive exists, then the pensioner aged 99 will consume part of his assets and the rest will be invested and bequeathed when he dies during that year. Otherwise, he will consume everything at age 99 and nothing will be left for investing. In the earlier periods, the pensioner consumes part of his available assets, uses one part for purchasing annuities and invests the rest into three available assets. As we will see, the solution to the problem follows the same pattern for different periods. Hence, it is useful to investigate one representative period and then the solution to the whole problem can be derived from the solution of one representative

period. The graphical presentation of the most important variables in this problem is given as follows

State (information) variables						
W_t is pension wealth, Y_t is income, r_{t-1} is known interest rate during previous year						
W_{65}	W_{66}	...	W_t	W_{t+1}	...	W_{100}
Y_{65}	Y_{66}	...	Y_t	Y_{t+1}	...	$Y_{100} = 0$
r_{64}	r_{65}	...	r_{t-1}	r_t	...	r_{99}
Random variables						
$\tilde{r}_t$ is random interest rate, $\tilde{r}_t^e$ is random rate on stock investment						
$\tilde{r}_{65}$	$\tilde{r}_{66}$	...	$\tilde{r}_t$	$\tilde{r}_{t+1}$	...	—
$\tilde{r}_{65}^e$	$\tilde{r}_{66}^e$	...	$\tilde{r}_t^e$	$\tilde{r}_{t+1}^e$	...	—
Control (decision) variables						
C_t is consumption, α_t^e is proportion invested into equities, α_t^b is proportion invested into bonds, m_t is proportion used for purchasing annuities						
C_{65}	C_{66}	...	C_t	C_{t+1}	...	—
α_{65}^e	α_{66}^e	...	α_t^e	α_{t+1}^e	...	—
α_{65}^b	α_{66}^b	...	α_t^b	α_{t+1}^b	...	—
m_{65}	m_{66}	...	m_t	m_{t+1}	...	—
Age during the decumulation process						
65	66	...	t	$t+1$	...	100

4.1.2 The Types of the Problem to be Investigated

We assume that the member annuitises part of the available pension wealth. We will assume that the member never annuitises any part of his income, only part of his pension wealth.

The pensioner aims to maximise the expected discounted utility derived from consumption and a possible bequest by choosing the optimal consumption, asset allocation and annuitisation. Regarding annuitisation, we distinguish the assumptions for the proportions of the pension wealth m_t to be annuitized. We group these assumptions into three groups of the types of the problem to be investigated as follows:

4.1 Annuitising m_t part of pension wealth exogenously for all ages $65 \leq t \leq 99$.

Under this assumption, the pensioner firstly chooses in a predetermined way how much to annuitise and for a given m_t he consumes and invests optimally the remaining part of pension wealth. The control variables are $\{C_t, \alpha_t^e, \alpha_t^b\}$, m_t is determined exogenously and is suboptimal. The model can handle any assumption about predetermined values of m_t for $65 \leq t \leq 99$. We will investigate in more detail the results with no annuitisation which is the special case of exogenous annuitisation. For the no annuities assumption we will have $m_t = 0$ for $65 \leq t \leq 99$.

4.2 Annuitising m_t part of pension wealth exogenously for some ages and endogenously for the others. In this case, the control variables are $\{C_t, \alpha_t^e, \alpha_t^b\}$ for ages where annuities are chosen exogenously and $\{C_t, \alpha_t^e, \alpha_t^b, m_t\}$ for ages where annuities are chosen endogenously. The model allows us to calculate the results for any combination of exogenous/endogenous annuitisation. All we need to know is for which age annuitisation is endogenous, and for which it is exogenous, and for exogenous annuitisation ages we need to know the value of m_t . We will thoroughly investigate the results under the assumption that the pensioner optimally annuitises at age 65 and no annuities are available afterwards.

4.3 m_t is the optimally chosen proportion for all ages $65 \leq t \leq 99$. In this case, the member maximises the value function with respect to the four control variables, and control variables are $\{C_t, \alpha_t^e, \alpha_t^b, m_t\}$.

With this definition of types of problems to be analysed, we have three groups of problems. When we have a particular assumption about the values of m_t for ages when m_t is exogenous we will refer to this assumption as a Case. As in Chapter 3, we can think of different cases as being different markets. When other parameters are fixed, cases are comparable and differ in the annuity offered in the market only. Hence, we sometimes referred to cases as markets.

We will analyse in more details Case 4.1 where we will assume no annuitisation and it is an interesting problem of type 4.1. An interesting problem of type 4.2 to be analysed in more detail will be Case 4.2 where we will assume optimal annuitisation at age 65 only and no annuitisation afterwards.

Regarding the amount to be annuitised at each age t , if exogenous annuitisation happens then it means that the member purchases annuities for the amount of $m_t W_t$

and this annuitisation choice is usually suboptimal. Endogenous annuitisation happens if $m_t W_t$ is chosen optimally from the model.

As in Chapter 3, we will write $\{cv_t\}$ to denote the $\{\text{control variables}_t\}$ at age t , such that we have the general notation for control variables for each exogenous/endogenous annuitisation assumption. As we will see later, we work with control variables for values in money units and with control variables for scaled down values suitable for calculations. In order to differentiate the two we will denote with $\{CV_t\}$ the control variables for values in money units and with $\{cv_t\}$ the control variables for scaled down values.

Before presenting the full model we need to define the model for the bond market which is the part of the interest rate risk model.

4.2 Bond Market Model

We model the real interest rate as an autoregressive discrete time and discrete state space process. The process is an approximation of Vasicek continuous time–space autoregressive process presented in Vasicek (1977). As the Vasicek model provides bond prices for an implied bond market, we can compare bond prices on the bond market obtained in our model with the Vasicek one.

We choose Vasicek model for interest rate for developing bond prices as the simple one and the one which is used in the analysis of optimal asset allocation problems by some other authors (Boulier et al (2001)). It is a type of one factor short rate model where interest rate movements are driven by one source of market risk. We use it for modelling real interest rate. The shortcoming of Vasicek model is the positive probability of the negative value of interest rate. However, we will use one and ten years rolling bonds in the interest rate risk model. Due to mean reverting characteristic of the interest rate, even for the negative value of real interest rate, there will be a certain demand for index–linked bonds. It is possible to derive the bond market model using the interest rate which does not allow the negative values of the interest rate, for example Cox–Ingersoll–Ross model (Cox et al (1985)). Although CIR model may be more appropriate, and the one and ten years rolling bonds market model can be developed using CIR model, it would be also computationally more demanding.

In our model we assume that the discrete time interval is one year. We will show below the technique to transform the continuous time Vasicek process into a discrete time one.

We assume that real interest rate can take finite number of values in a reasonable range. As the Vasicek process transformed into discrete time is still a continuous state space process we use the technique from Tauchen and Hussey (1991) and as a result we get a process with discrete time–state space.

Once we obtain a discrete time–state process for real interest rate we can model bond prices as the expected present value of future incomes from the bond. As we assume a zero coupon bond, it means that the bond price is expected present value of one money unit that will be due in T years time, where T years is the bond duration.

Following the Vasicek approach, we can also introduce a market price of risk. As a final result we get the approximation of the bond market.

4.2.1 The Main Parts of the Vasicek Model

The Vasicek model is used for modelling interest rate where time and state spaces are continuous. It is a continuous time AR (1) process given by

$$d\tilde{r}_t = (a - b\tilde{r}_t)dt - \sigma_{\tilde{r}}d\tilde{W}_{\tilde{r}}(t) \quad (4.1)$$

where $\hat{r}_0$ is the initial value of the interest rate, a , b and $\sigma_{\tilde{r}}$ are non–negative constants and $\tilde{W}_{\tilde{r}}(t)$ is Brownian motion. We use notation $\hat{r}_t$ for interest rate from Vasicek model in order to avoid the confusion with interest rate afterwards in this thesis. As throughout the whole thesis, $\sim$ above variable denotes it is a random variable.

We know that $\tilde{r}_t$ is a normally distributed random variable and that the conditional expectation and variance of the process given current level $\hat{r}_0$ are

$$E\left[\tilde{r}_T\right] = \frac{a}{b} + \left(\hat{r}_0 - \frac{a}{b}\right)e^{-bT} \quad (4.2)$$

$$Var\left[\tilde{r}_T\right] = \frac{\sigma_{\tilde{r}}^2}{2b}(1 - e^{-2bT}) \quad (4.3)$$

for $T \geq 0$.

The stochastic differential equation of the bond investments is given by

$$\frac{d\tilde{B}(T-t, \hat{r}_t)}{B(T-t, \hat{r}_t)} = (\hat{r}_t + \sigma_B(T-t, \hat{r}_t) \lambda_r) dt + \sigma_B(T-t, \hat{r}_t) d\tilde{W}_{\hat{r}}(t) \quad (4.4)$$

where t is the time such that $0 \leq t \leq T$, T is bond duration, $B(T, T) = 1$, and

$$\lambda_r = \frac{\mu_B(T-t, \hat{r}_t) - \hat{r}_t}{\sigma_B(T-t, \hat{r}_t)}.$$

λ_r is referred to as bond's market price of risk and is constant.

The function $\sigma_B(T-t, \hat{r}_t)$ is given by

$$\sigma_B(T-t, \hat{r}_t) = \frac{1 - e^{-b(T-t)}}{b} \sigma_{\hat{r}} \quad (4.5)$$

for $T-t \geq 0$.

If we work with zero-coupon bonds and assume that we are interested in current value of the bonds maturing at time T and with current interest rate is $\hat{r}_0$, then $t = 0$ and the price of the zero-coupon bond is given by

$$B(T, \hat{r}_0) = \text{Exp} \left[-\frac{\sigma_{\hat{r}} \lambda_r}{b} \left(\frac{1 - e^{-bT}}{b} - T \right) \right] \cdot \text{Exp} \left[\left(\frac{1 - e^{-bT}}{b} - T \right) \left(\frac{a}{b} - \frac{1}{2} \left(\frac{\sigma_{\hat{r}}}{a} \right)^2 \right) - \frac{1 - e^{-bT}}{b} \hat{r}_0 - \frac{\sigma_{\hat{r}}^2}{4a^3} (1 - e^{-bT})^2 \right] \quad (4.6)$$

4.2.2 Discrete Time Space Approximation of the Vasicek Model

In order to approximate Vasicek model in discrete time and continuous space we observe the process

$$\Delta \tilde{R}_t = (a_d - b_d R_t) \Delta t - \sigma_{dR} \tilde{\mathcal{E}}_R(t) \quad (4.7)$$

where $\tilde{\mathcal{E}}_R(t) \sim N(0, 1)$ are independent random variables with normal distribution, for $t \in \mathbb{N}$. In order to have similar results from the continuous time and discrete time process we fit the parameters a_d , b_d and σ_{dR} into the Vasicek model (4.1).

Let us derive formula for R_T using equation (4.7). We have

$$\begin{aligned}\tilde{R}_1 - R_0 &= a_d - b_d R_0 - \sigma_{dR} \tilde{\epsilon}_R(1) \text{ and} \\ \tilde{R}_1 &= a_d + (1 - b_d) R_0 - \sigma_{dR} \tilde{\epsilon}_R(1).\end{aligned}$$

Then

$$\begin{aligned}\tilde{R}_2 &= a_d + (1 - b_d) \tilde{R}_1 - \sigma_{dR} \tilde{\epsilon}_R(2) \\ &= a_d + (1 - b_d) (a_d + (1 - b_d) R_0 - \sigma_{dR} \tilde{\epsilon}_R(1)) - \sigma_{dR} \tilde{\epsilon}_R(2) \\ &= a_d \sum_{k=0}^{2-1} (1 - b_d)^k + (1 - b_d)^2 R_0 - \sigma_{dR} \sum_{k=1}^2 (1 - b_d)^{2-k} \tilde{\epsilon}_R(k)\end{aligned}$$

Continuing the similar reasoning gives us the relation

$$\tilde{R}_T = a_d \sum_{k=0}^{T-1} (1 - b_d)^k + (1 - b_d)^T R_0 - \sigma_{dR} \sum_{k=1}^T (1 - b_d)^{T-k} \tilde{\epsilon}_R(k), \text{ for } \forall T \in \mathbb{N} \quad (4.8)$$

Knowing that the sum of normally distributed random variables is again normally distributed random variable we have that

$$\begin{aligned}\tilde{\epsilon}_r(T) &= \sigma_{dr} \sum_{k=1}^T (1 - b_d)^{T-k} \tilde{\epsilon}_r(k) \sim N\left(0, \sigma_{dr}^2 \sum_{k=1}^T (1 - b_d)^{2(T-k)}\right), \text{ or} \\ \tilde{Z}_R(T) &\sim N\left(0, \sigma_{dR}^2 \sum_{k=0}^{T-1} (1 - b_d)^{2k}\right)\end{aligned}$$

Now, we can easily derive

$$E[\tilde{R}_T] = \frac{a_d}{b_d} + \left(R_0 - \frac{a_d}{b_d}\right) (1 - b_d)^T \quad (4.9)$$

and

$$Var[\tilde{R}_T] = \sigma_{dR}^2 \frac{1 - (1 - b_d)^{2T}}{b_d (2 - b_d)} \quad (4.10)$$

Let us determine the coefficients a_d , b_d and σ_{dR} such that equations (4.2) and (4.9), and (4.3) and (4.10) respectively, gives the same values. From the first two equations, by equating the expectations, we have that

$$b_d = 1 - e^{-b} \quad (4.11)$$

and

$$a_d = a \frac{1 - e^{-b}}{b} \quad (4.12)$$

Now, from the second pair of equations, by equating variances, we get

$$\sigma_{dR} = \sigma_{\hat{r}} \sqrt{\frac{1 - e^{-2b}}{2b}} \quad (4.13)$$

The discrete time version of the Vasicek process given in (4.7) is now fully defined and the appropriate parameter values are given in (4.11)–(4.13). We have the discrete time and continuous state AR (1) process such that $\tilde{R}_t$ is normally distributed and the conditional expectation and variation of this random variable is the same as the conditional expectation and variance for the Vasicek process given in (4.1). Thus, we here defined the discrete time and continuous state space approximation of the Vasicek process (4.1).

Tauchen and Hussey (1991) gives the technique for approximating continuous state discrete time space AR(1) process with a discrete state and time spaces process. We apply this technique to the process (4.7).

In order to deploy the technique from Tauchen and Hussey (1991), we need to choose the density function $\omega(y)$, and the number N denoting the number of Quadrature points. Let the density function $\omega(y)$ be the density function of the random variable with the distribution

$$N\left(\frac{a_d}{b_d}, \sigma_{dR}\right). \quad (4.14)$$

This choice is based on the proposal in Tauchen and Hussey (1991), where the authors say that this choice works well in most examples.

Let us denote with $\tilde{r}_t$ random variable which has discrete time and space states and which approximate random variable $\tilde{R}_t$. It is autoregressive process

$$\tilde{r}_{t+1} - r_t = (a_d - b_d r_t) \Delta t - \sigma_{dr} \tilde{Z}_r(t) \quad (4.15)$$

where a_d , b_d and σ_{dr} are constants and $\tilde{Z}_r(t)$ is random variable to be defined later, where a_d , b_d are defined in (4.11)–(4.12), and $\sigma_{dr} = \sigma_{dR}$ where σ_{dR} is defined in (4.13)

Let the number of Quadrature points be N . The bigger the number of points the better is approximation. However, the choice of $N = 15$ provides quite good behavior and we show the analysis of this behavior in Appendix 3.

Based on this choice we choose abscissa points, i.e. the possible states of the interest rate are constants $r_{t;1}, r_{t;2}, \dots, r_{t;N}$, such that the probabilities derived using this technique satisfies the condition $P[\tilde{r}_{t+1} = r_{t+1;i} | \tilde{r}_t = r_{t;1}] = P[\tilde{r}_{t+1} = r_{t+1;i} | \tilde{r}_t = r_{t;N}] < 0.02$ for $1 \leq i \leq N$ and that the points are derived from Gauss Quadrature with these ending points. We also derive the weights $w_1, \dots, w_N$, for these choice of abscissa and the density function $\omega(y)$.

Let us also define the function $f(y|r_0)$ as the density function for the random variable with the distribution

$$N\left(\frac{a_d}{b_d} + \left(r_0 - \frac{a_d}{b_d}\right)(1-b_d), \sigma_{dr}\right) \quad (4.16)$$

Having determined the abscissa points, the weighting function and the function $f(y|r_0)$, we can apply the Tauchen and Hussey (1991) technique as follows. Let

$$s(r_j) = \sum_{i=1}^N \frac{f(r_i | r_j)}{\omega(r_j)} w_j \quad (4.17)$$

and let

$$\pi_{jk}^N = \frac{f(r_k | r_j)}{s(r_j) \omega(r_k)} w_k \quad (4.18)$$

Then according to Tauchen and Hussey (1991), we have

$$\{\pi_{jk}^N\}_{(j,k)=(1,1)}^{(N,N)} = \{p_{jk}\}_{(j,k)=(1,1)}^{(N,N)} = P[\tilde{r}_{t+1} = r_{t+1;k} | \tilde{r}_t = r_{t;j}] \quad (4.19)$$

The random variable $\tilde{Z}_r(t)$ is now defined via its transitional matrix $\{p_{jk}\}_{(j,k)=(1,1)}^{(N,N)}$.

4.2.3 Numerical Derivation of the Bond prices

In Section 4.1.2, we defined the discrete time-state AR (1) process which approximates the Vasicek model. In order to use this process as an approximation of the interest rate, we need to derive the zero-coupon bond prices from this process and get the model for the bond market.

We first derive the price of the zero-coupon bond with no market price of risk. As usual, it is defined as expected present value of one unit payout after time T . Thus, we have

$$\bar{B}(T, r_0) = E \left[e^{-\tilde{r}_1} e^{-\tilde{r}_2} \dots e^{-\tilde{r}_T} \right] \quad (4.20)$$

where $\tilde{r}_1$ is a random variable denoting random interest during the first year, $\tilde{r}_2$ is a random variable denoting random interest during the second year knowing $\tilde{r}_1$, and so on. In order to allow for the existence of the market price of risk, we use the idea from the equation (4.6) and introduce the market price of risk by multiplying the bond price with no market price of risk with the similar factor as in the continuous time Vasicek model. Let the constant λ_r represents the market price of risk in the Vasicek bond market model. Then we get the equation for the price of a zero-coupon bond as follows

$$B(T, r_0) = e^{-\frac{\sigma_r \lambda_r}{b} \left(\frac{1}{b} (1 - e^{-bT}) - T \right)} E \left[e^{-\tilde{r}_1} e^{-\tilde{r}_2} \dots e^{-\tilde{r}_T} \right] \quad (4.21)$$

Let us explain how we can calculate numerically the bond price in discrete time-state spaces. Following the main formula for the expected value we have that

$$\begin{aligned} B(1, r_0 = r_{0;j}) &= e^{-\frac{\sigma_r \lambda_r}{b} \left(\frac{1 - e^{-b}}{b} - 1 \right)} E \left[e^{-\tilde{r}_1} \right] \\ &= e^{-\frac{\sigma_r \lambda_r}{b} \left(\frac{1 - e^{-b}}{b} - 1 \right)} \sum_{k=1}^N e^{-r_{1;k}} p_{jk} \end{aligned}$$

For the bond of the duration two years we have

$$B(2, r_0 = r_{0;j}) = e^{-\frac{\sigma_r \lambda_r}{b} \left(\frac{1 - e^{-2b}}{b} - 2 \right)} E \left[e^{-\tilde{r}_1} e^{-\tilde{r}_2} \right] = e^{-\frac{\sigma_r \lambda_r}{b} \left(\frac{1 - e^{-2b}}{b} - 2 \right)} \sum_{k_1=1}^N \left(\sum_{k_2=1}^N e^{-r_{1;k_1}} e^{-r_{2;k_2}} p_{k_1 k_2} \right) p_{jk_1}$$

or

$$B(2, r_0 = r_{0;j}) = e^{-\frac{\sigma_r \lambda_r}{b} \left(\frac{1 - e^{-2b}}{b} - 2 \right)} \sum_{k_1=1}^N e^{-r_{1;k_1}} \left(\sum_{k_2=1}^N e^{-r_{2;k_2}} p_{k_1 k_2} \right) p_{jk_1}$$

The same pattern is applied for longer durations. However, we can see that the part of the second sum is the same as the sum for the bond with one year duration. Apart from the coefficient for the market price of risk the difference is in the indices only. Using this observation, one can firstly calculate 1 year duration bond prices for all possible states for r_0 and then use these results to obtain the results for the bond with duration of two years. This feature is important when the calculation is applied on the computer. If we define

$$\bar{B}(1, r_1 = r_{1;k_1}) = \sum_{k_2=1}^N e^{-r_{2;k_2}} p_{k_1 k_2} \quad (4.22)$$

Then one can write

$$B(2, r_0 = r_{0;j}) = e^{-\frac{\sigma_r \lambda_r}{b} \left(\frac{1-e^{-2b}}{b} - 2 \right)} \sum_{k_1=1}^N e^{-\bar{r}_{1;k_1}} \bar{B}(1, r_1 = r_{1;k_1}) p_{jk_1}$$

Similarly, if we define

$$\bar{B}(2, r_1 = r_{1;k_1}) = \sum_{k_2=1}^N e^{-r_{2;k_2}} \bar{B}(1, r_2 = r_{2;k_2}) p_{k_1 k_2}$$

then

$$B(3, r_0 = r_{0;j}) = e^{-\frac{\sigma_r \lambda_r}{b} \left(\frac{1-e^{-3b}}{b} - 3 \right)} \sum_{k_1=1}^N e^{-\bar{r}_{k_1}} \bar{B}(2, r_1 = r_{1;k_1}) p_{jk_1}.$$

Following this pattern, we get an inductive formula for bond prices which significantly reduces computing time.

However, we calculate bond prices $B(T, r_0 = r_{0;j})$, for $0 \leq T \leq 35$ and $1 \leq j \leq N$ only once and then use the results. So, it is important to calculate it in reasonable time only once.

4.3 The Model

Let us define the model that will be investigated in this thesis.

4.3.1 Definitions and Notation

We use the following definitions and notation:

- W_t is the pension wealth at time t , just before income Y_t is received;
- Y_t is the variable denoting income at time t . We model income as

$$Y_t = \begin{cases} \bar{P}P_t & \text{for } 65 \leq t \leq 99 \\ 0 & \text{for } t = 100 \end{cases} \quad (4.23)$$

$\bar{P}$ is constant and is equal to the income at age 65, $P_{65} = 1$, and P_t will be defined later.

- C_t is consumption at the beginning of the period $[t, t+1]$ for $t = 65, 66, \dots, 99$, just after annuitisation and receiving income Y_t ;
- b_t is the factor which controls the pensioner's strength of the bequest motive. If no bequest motive exists then $b_t = 0$, for $t = 65, 66, \dots, 99$;
- $\tilde{r}_t$ is the random real interest rate during the period $[t, t+1]$ for $t = 65, 66, \dots, 99$. We model the real interest rate as autoregressive process

$$\tilde{r}_{t+1} - r_t = (a_d - b_d r_t) \Delta t - \sigma_{dr} \tilde{\varepsilon}_r(t) \quad (4.24)$$

where a_d , b_d and σ_{dr} are constants and random variable $\tilde{\varepsilon}_r(t)$ is defined via its transitional matrix $\{p_{jk}\}_{(j,k)=(1,1)}^{(N,N)}$, as explained in Section 4.2. r_{64} is known interest rate during the year prior to retirement. The value of real interest rate r_t during the period $[t-1, t]$ is known at time t ;

- p_t – probability that the member aged t will survive until the age of $t+1$;
- $\bar{r}_t$ – variable denoting deterministic rate of return on one year risk free investment during the period $[t, t+1]$, for $t = 65, 66, \dots, 99$;
- $\tilde{r}_t^e$ – random variable denoting random real rate on equities during the period $[t, t+1]$, for $t = 65, 66, \dots, 99$. We assume that $[t, t+1]$ is one year period, and that

$$Ln(\tilde{r}_t^e) = \mu_e + \sigma_e \tilde{\varepsilon}_e(t) \quad (4.25)$$

where μ_e and σ_e are constants and $\tilde{\varepsilon}_e(t) \sim N(0, 1)$;

- $\tilde{r}_t^b$ – random variable denoting random real rate on bond investment during the period $[t, t+1]$, for $t = 65, 66, \dots, 99$;
- α_t^e – the proportion of the wealth invested in the equities during the period $[t, t+1]$, for $t = 65, 66, \dots, 99$;
- α_t^b – the proportion of the wealth invested in the bonds during the period $[t, t+1]$, for $t = 65, 66, \dots, 99$;
- m_t – the proportion of the pension wealth used for purchasing annuity at time t , for $t = 65, 66, \dots, 99$;
- $B(T, r_{t-1})$ – the price of the zero-coupon bond at time t maturing after T years and with r_{t-1} being experienced interest rate during the period $[t-1, t]$, for $t = 65, 66, \dots, 99$. $B(T, r_{t-1})$ is defined in (4.21);

The control variables of the most general type of the problem are $\{c_t, \alpha_t^e, \alpha_t^b, m_t\}_{t=65}^{99}$, and the state variables of the problem are $\{t, W_t, Y_t, r_{t-1}\}_{t=65}^{99}$. We will skip explicitly writing the state variable t and write state variables as $\{W_t, Y_t, r_{t-1}\}_{t=65}^{99}$. As we will see, we will decrease the number of control variables from three to two.

Regarding risk free investment, we will assume that the member invest in risk free deposit with duration of one year. The rate on one-year risk free investment is calculated as follows

$$\bar{r}_t = \frac{1}{B(1, r_{t-1})} - 1.$$

Regarding low risk investment, we will assume that the pensioner aged t invests in bonds with the duration of Υ_t years, for $65 \leq t \leq 99$. It means that at age t , the pensioner invests in Υ_t -years bonds at the beginning of the year and at the end of year he sells the bonds with $\Upsilon_t - 1$ years to maturity, rebalances his portfolio and then again purchases bonds with the duration of Υ_t years, and so on. According to this strategy, at the beginning of the period $[t, t+1]$, the member invests the amount of $\alpha_t^b ((1-m_t)W_t + Y_t - C_t)$ into bonds and purchases them for the price of $B(\Upsilon_t, r_{t-1})$, where r_{t-1} is real interest rate during the previous year. At the end of year, he possesses in his bond portfolio the amount of

$$\frac{\alpha_t^b ((1-m_t)W_t + Y_t - C_t)}{B(\Upsilon_t, r_{t-1})} B(\Upsilon_t - 1, r_t) = \alpha_t^b ((1-m_t)W_t + Y_t - C_t) \frac{B(\Upsilon_t - 1, r_t)}{B(\Upsilon_t, r_{t-1})}.$$

Thus, we can write that, observed at time t , the rate of return on bond investment during the year $[t-1, t]$ is

$$1 + \tilde{r}_t^b = \frac{B(\Upsilon_t - 1, \tilde{r}_t)}{B(\Upsilon_t, r_{t-1})}. \quad (4.26)$$

In the main results we will assume that $\Upsilon_t = 10$, for $65 \leq t \leq 99$. It means that we will assume that the pensioner invests in 10-year rolling bonds. However, we make it more general in the model such that it is possible to use the model with the assumption of different duration of rolling bonds and that duration can depend on age.

Let us now introduce the random variable $\tilde{r}_t^P$, representing rate of return on portfolio investment during the year $[t, t+1]$

$$\begin{aligned} \tilde{r}_t^P &= (1 - \alpha_t^e - \alpha_t^b) \bar{r}_t + \alpha_t^e \tilde{r}_t^e + \alpha_t^b \tilde{r}_t^b \\ &= \bar{r}_t + \alpha_t^e (\tilde{r}_t^e - \bar{r}_t) + \alpha_t^b (\tilde{r}_t^b - \bar{r}_t) \\ &= \bar{r}_t + \alpha_t^e (\tilde{r}_t^e - \bar{r}_t) + \alpha_t^b \left(\frac{B(\Upsilon_t - 1, \tilde{r}_t)}{B(\Upsilon_t, r_{t-1})} - 1 - \bar{r}_t \right) \end{aligned}$$

for $t = 65, 66, \dots, 99$.

In interest rate risk model, we assume that all variables are in real terms. Real interest rate is modelled based on Vasicek model, and from this model we develop the market of bonds providing return in real terms. We assume in this thesis that the real interest rate, and also derived bond market, is not correlated with the stock market. This assumption is a simplification of the real world in order to have more compact set of results. Introduction of the correlation between the market of bonds providing real return and the market of stock providing real return would bring the new results. However, as we will see later in this Chapter and in Chapter 5, many different results are obtained and although the analysis of the possible correlation between real interest rate and equities would give us interesting results, it would also give us even more results and affect our focus on the obtained results. Also, the model would be more complicated, and calculation time will increase. However, we acknowledge that investigating correlation between real interest rate and stock return is important. We also acknowledge that introduction of correlation between real interest rate and stock return in the interest rate risk model is possible and computationally feasible. We leave this analysis for further research and hope that the results in this thesis will be a good basis for the further research in this direction.

We assume that the member wishes to maximise expected utility from his future consumption and possibly a bequest. The utility function is CRRA function, given by

$$u(x) = \frac{x^\gamma}{\gamma} \text{ for } \gamma < 1, \gamma \neq 0 \text{ and,}$$

$$u(x) = \text{Log}(x) \text{ for } \gamma = 0.$$

4.3.2 Income Process

In this section we present all details of the income process. At age $t = 65$, income comes from the last salary only. Afterwards, for ages $66 \leq t \leq 99$, the member's income consists of social security income and income from annuities bought at age 65 and afterwards. For the simplicity reasons, we will assume that income from the social security Y_t^{ss} for $66 \leq t \leq 99$ is constant in real terms. We also define $Y_{100} = 0$.

We will distinguish two types of income in retirement, income from social security sources denoted with Y_t^{ss} and income from annuities bought before time t denoted with Y_t^A . This can be written as

$$Y_t = Y_t^{SS} + Y_t^A$$

for $66 \leq t \leq 99$. Y_{65} is defined in (4.23). Let us now define Y_t^{SS} and Y_t^A for $66 \leq t \leq 99$ more precisely.

We assume that the first income from social security is received at age 66, and Y_t^{SS} will be defined as

$$Y_t^{SS} = \text{replrate} \cdot Y_{65},$$

for $66 \leq t \leq 99$ and *replrate* is the percentage of the last salary provided from the state in form of social security income after age 65. It is a constant income until the end of pensioner's life. The following variable will be of use in the later discussion

$$\rho_t = \begin{cases} \text{replrate} & t = 65 \\ 1 & 66 \leq t \leq 99 \end{cases} \quad (4.27)$$

Now we assume the environment where purchasing annuities from pension wealth is allowed at the member's discretion at age 65 and afterwards. Whenever a member purchases annuities his pension wealth decreases by the amount used for purchasing those annuities, and his income in future periods increases by the newly provided annuity income. For simplicity reasons, we assume that annuities provide the first instalment one year after purchasing annuities. Let us denote income in the form of annuities bought at age 65 with Y_{a65}^A , at age 66 with Y_{a66}^A , and so on until maximum age $t = 99$.

We assume that the annuitised pension fund is invested into bonds and thus we have

$$a_t = (1 + \text{Loadings}) \sum_{i=1}^{99-t} \left(\left(\prod_{j=1}^i p_{t+j-1} \right) B(i, r_{t-1}) \right) \quad (4.28)$$

for $t = 65, 66, \dots, 99$, where *Loadings* is loadings on the actuarially fair annuities depending on the market. Now, we can write

$$Y_{at}^A = \frac{m_t W_t}{a_t}. \quad (4.29)$$

Thus, if some annuities are bought at age 65, then income at age 66 is

$$\begin{aligned} Y_{66} &= Y_{66}^{SS} + Y_{a65}^A \\ &= Y_{66}^{SS} + Y_{66}^A \end{aligned}$$

Then some new annuities are bought at ages 66 and 67 then income at age 67 is

$$\begin{aligned} Y_{67} &= Y_{67}^{SS} + (Y_{a65}^A + Y_{a66}^A) \\ &= Y_{67}^{SS} + Y_{67}^A \end{aligned}$$

The same pattern repeats itself and, for $66 \leq t \leq 99$, income at age t is

$$\begin{aligned} Y_t &= Y_t^{SS} + (Y_{a65}^A + \dots + Y_{a(t-1)}^A), \\ &= Y_t^{SS} + Y_t^A \end{aligned}$$

where

$$Y_t^A = Y_{a65}^A + \dots + Y_{a(t-1)}^A.$$

We also have the relation

$$Y_{t+1} = \rho_t Y_t + Y_{at}^A \quad (4.30)$$

for $65 \leq t \leq 99$, where ρ_t is defined in (4.27). ρ_t will always appear as multiplicative factor. Due to its definition in equation (4.27) ρ_t influences this and other equations where it appears for $t = 65$ only.

In order to work with smaller numbers when solving the problem on a computer, we introduce the constant $\bar{P}$. Let us now express the equations of the income process in terms of P variables. $\bar{P}$ is constant, equal to the income at age 65 and $P_{65} = 1$. Now, we define P_t^{SS} , P_t^A and P_{at}^A via equations $Y_t^{SS} = \bar{P} P_t^{SS}$, $Y_t^A = \bar{P} P_t^A$ and $Y_{at}^A = \bar{P} P_{at}^A$, respectively. We have

$$P_t = P_t^{SS} + P_t^A \quad (4.31)$$

where $t = 65, 66, \dots, 99$. The equivalent equation to equation (4.30) is given by

$$P_{t+1} = \rho_t P_t + P_{at}^A \quad (4.32)$$

for $65 \leq t \leq 99$. Equation (4.23) is now fully defined. Let us also define

$$G_{t+1} = \frac{Y_{t+1}}{Y_t} = \frac{P_{t+1}}{P_t} \quad (4.33)$$

From (4.30) we have

$$G_{t+1} = \rho_t + \frac{Y_{at}^A}{Y_t},$$

and using (4.29) we get

$$G_{t+1} = \rho_t + \frac{m_t W_t}{Y_t a_t} \quad (4.34)$$

for $65 \leq t \leq 99$.

4.3.3 Mathematical Model for the Problem

We will assume that the member's pension wealth is always non-negative, i.e. $W_t \geq 0$ for $65 \leq t \leq 99$. Other assumptions about pension wealth are possible and effectively it would mean that limited or unlimited borrowing is allowed.

We assume that the member's maximum attainable age is $t = 99$. $p_{99} = 0$ and there is no annuitisation at age 99. Let us start with the last age period $[99, 100]$. If the member is alive at the beginning of this period, he draws utility from consuming part of his available financial wealth and possibly draws utility from bequeathing some assets. Income at the end of the period $[99, 100]$ is $Y_{100} = 0$. The member's value function at age 99 is

$$V_{99}(W_{99}, Y_{99}, r_{98}) = \max_{\{CV_{99}\}} E_{99} \left[u(C_{99}) + \delta(1 - p_{99}) b_{99} u(\tilde{W}_{100}) \right] \quad (4.35)$$

where

$$\tilde{W}_{100} = (W_{99} + Y_{99} - C_{99}) \left(1 + \bar{r}_{99} + \alpha_{99}^e (\tilde{r}_{99}^e - \bar{r}_{99}) + \alpha_{99}^b \left(\frac{B(\Upsilon_{99} - 1, \tilde{r}_{99})}{B(\Upsilon_{99}, r_{98})} - 1 - \bar{r}_{99} \right) \right) \quad (4.36)$$

The member maximises his value function at age 99 over all possible consumptions C_{99} and investment decisions α_{99}^e and α_{99}^b . These three are the only control variables at this age as no annuitisation occurs. We assume that after retirement, control variables are subject to the no-borrowing constraint. It means that the maximum amount the member can consume is $W_{99} + Y_{99}$, and it is also the maximum amount he can invest. Mathematically,

$$0 \leq C_{99} \leq W_{99} + Y_{99}, \text{ and} \quad (4.37)$$

$$0 \leq \alpha_{99}^e \leq 1, \quad 0 \leq \alpha_{99}^b \leq 1 \text{ and } 0 \leq \alpha_{99}^e + \alpha_{99}^b \leq 1 \quad (4.38)$$

In order to get an idea how we move backward year by year we first show the member's value function at age $t = 98$. We have

$$V_{98}(W_{98}, Y_{98}, r_{97}) = \max_{\{CV_t\}_{t=98}^{99}} E_{98} \left[u(C_{98}) + \delta p_{98} u(\tilde{C}_{99}) + \right. \\ \left. \delta(1 - p_{98}) b_{98} u(\tilde{W}_{99}) + \delta^2 p_{98} (1 - p_{99}) b_{99} u(\tilde{W}_{100}) \right]$$

or

$$V_{98}(W_{98}, Y_{98}, r_{97}) = \max_{\{CV_t\}_{t=98}^{99}} E_{98} \left[u(C_{98}) + \delta(1-p_{98})b_{98}u(\tilde{W}_{99}) + \right. \\ \left. \delta p_{98} \left(u(\tilde{C}_{99}) + \delta(1-p_{99})b_{99}u(\tilde{W}_{100}) \right) \right]$$

Thus

$$V_{98}(W_{98}, Y_{98}, r_{97}) = \max_{\{CV_{98}\}} E_{98} \left[u(C_{98}) + \delta(1-p_{98})b_{98}u(\tilde{W}_{99}) + \delta p_{98}V_{99}(\tilde{W}_{99}, Y_{99}, \tilde{r}_{98}) \right]$$

where

$$\tilde{W}_{99} = ((1-m_{98})W_{98} + Y_{98} - C_{98})(1 + \tilde{r}_{98}^P) \quad (4.39)$$

using (4.30) we have

$$\tilde{r}_{98}^P = \bar{r}_{98} + \alpha_{98}^e (\tilde{r}_{98}^e - \bar{r}_{98}) + \alpha_{98}^b \left(\frac{B(\Upsilon_{98} - 1, \tilde{r}_{98})}{B(\Upsilon_{98}, r_{97})} - 1 - \bar{r}_{98} \right) \quad (4.40)$$

$$Y_{99} = Y_{98} + \frac{m_{98}W_{98}}{a_{98}} \quad (4.41)$$

$$G_{99} = 1 + \frac{m_{98}W_{98}}{a_{98}Y_{98}} \quad (4.42)$$

and the constraints are

$$0 \leq C_{98} \leq (1-m_{98})W_{98} + Y_{98} \quad (4.43)$$

$$0 \leq \alpha_{98}^e \leq 1, \quad 0 \leq \alpha_{98}^b \leq 1 \quad \text{and} \quad 0 \leq \alpha_{98}^e + \alpha_{98}^b \leq 1 \quad (4.44)$$

$$0 \leq m_{98} \leq 1. \quad (4.45)$$

Here, we used the Bellman principal of optimality and the law of iterated conditional expectations.

Now, one can derive value function for any age $65 \leq t \leq 99$. The value function for ages $65 \leq t_0 \leq 99$ is given as

$$V_{t_0}(W_{t_0}, Y_{t_0}, r_{t_0-1}) = \max_{\{CV_t\}_{t=t_0}^{99}} E_{t_0} \left[u(C_{t_0}) + \delta(1-p_{t_0})b_{t_0}u(\tilde{W}_{t_0+1}) + \right. \\ \left. \delta p_{t_0} \sum_{t=t_0+1}^{99} \left(\delta^{t-(t_0+1)} p_{t-(t_0+1)} p_{t_0+1} u(\tilde{C}_t) + \delta^{t-t_0} p_{t-(t_0+1)} p_{t_0+1} (1-p_t)b_t u(\tilde{W}_{t+1}) \right) \right] \quad (4.46)$$

Using Bellman's principal of optimality which says that

$$\max_{\{CV_t\}_{t=t_0}^{99}} (Z) = \max_{\{CV_{t_0}\}} \left[\max_{\{CV_t\}_{t=t_0+1}^{99} \mid \text{outcome from } \{CV_{t_0}\}} (Z) \right]$$

we have

$$V_{t_0}(W_{t_0}, Y_{t_0}, r_{t_0-1}) = \max_{\{CV_{t_0}\}} \left[u(C_{t_0}) + E_{t_0} \left[\delta(1-p_{t_0}) b_{t_0} u(\tilde{W}_{t_0+1}) \right] + \right. \\ \left. \delta p_{t_0} E_{t_0} \left[\max_{\{CV_t\}_{t=t_0+1}^{99}} \left[\sum_{t=t_0+1}^{99} \left(\delta^{t-(t_0+1)} p_{t_0+1} u(\tilde{C}_t) + \right. \right. \right. \right. \\ \left. \left. \left. \delta^{t-t_0} p_{t_0+1} (1-p_t) b_t u(\tilde{W}_{t+1}) \right) \right] \right] \right]$$

and using the law of iterated conditional expectations

$$V_{t_0}(W_{t_0}, Y_{t_0}, r_{t_0-1}) = \max_{\{CV_{t_0}\}} \left[u(C_{t_0}) + E_{t_0} \left[\delta(1-p_{t_0}) b_{t_0} u(\tilde{W}_{t_0+1}) \right] + \right. \\ \left. \delta p_{t_0} E_{t_0} \left[\max_{\{CV_t\}_{t=t_0+1}^{99}} E_{t_0+1} \left[\sum_{t=t_0+1}^{99} \left(\delta^{t-(t_0+1)} p_{t_0+1} u(\tilde{C}_t) + \right. \right. \right. \right. \quad (4.47) \\ \left. \left. \left. \delta^{t-t_0} p_{t_0+1} (1-p_t) b_t u(\tilde{W}_{t+1}) \right) \right] \right] \right]$$

Thus,

$$V_t(W_t, Y_t, r_{t-1}) = \max_{\{CV_t\}} E_t \left[u(C_t) + \delta(1-p_t) b_t u(\tilde{W}_{t+1}) + \delta p_t V_{t+1}(\tilde{W}_{t+1}, Y_{t+1}, \tilde{r}_t) \right] \quad (4.48)$$

where

$$\tilde{W}_{t+1} = ((1-m_t)W_t + Y_t - C_t)(1 + \tilde{r}_t^P) \quad (4.49)$$

$$Y_{t+1} = \rho_t Y_t + \frac{m_t W_t}{a_t} \quad (4.50)$$

$$\tilde{r}_t^P = \bar{r}_t + \alpha_t^e (\tilde{r}_t^e - \bar{r}_t) + \alpha_t^b \left(\frac{B(Y_t - 1, \tilde{r}_t)}{B(Y_t, r_{t-1})} - 1 - \bar{r}_t \right) \quad (4.51)$$

$$\rho_t = \begin{cases} replrate & t = 65 \\ 1 & 66 \leq t \leq 99 \end{cases} \quad (4.52)$$

with the constraints

$$0 \leq C_t \leq (1-m_t)W_t + Y_t \quad (4.53)$$

$$0 \leq \alpha_t^e \leq 1, \quad 0 \leq \alpha_t^b \leq 1, \quad \text{and} \quad 0 \leq \alpha_t^e + \alpha_t^b \leq 1 \quad (4.54)$$

$$0 \leq m_t \leq 1. \quad (4.55)$$

for $65 \leq t \leq 99$.

4.4 Solution to the Problem

Let us present the solution to the problem defined in the previous section. We will show the solution for different assumptions about m_t as explained in Section 4.1.2.

4.4.1 Solution for Endogenous m_t

The analytical solution to the problem (4.48)–(4.55) cannot be found in the current literature. Further, the random real interest rate and random rate of return on equity investment can be correlated.

The usual approach to this type of problems nowadays is a numerical solution using computers. We approach this problem by finding the maximum in equation (4.48) using numerical mathematics.

By observing equations (4.48)–(4.55) and the constraints accompanying them, one can see that we need to solve the problem of nonlinear optimization with constraints. In this particular problem we have four control variables. The constraints are analytical functions. We solve this problem in Mathematica 5.2 using the Gauss Quadrature for approximating the interest rate and rate on equity investment and cubic splines for interpolating the value function. Let us explain the way we solve the problem.

Let us assume that we have a solution for ages $t+1$ and onwards and we need to go one step backward aiming to find the solution for time t . It means that we have obtained

$$\left\{ \left(C_i^*(W_i, Y_i, r_{i-1;m}); \alpha_i^{e*}(W_i, Y_i, r_{i-1;m}); \alpha_i^{b*}(W_i, Y_i, r_{i-1;m}); m_i^*(W_i, Y_i, r_{i-1;m}); V_i(W_i, Y_i, r_{i-1;m}) \right) \right\}_{i=t+1}^{99} \quad (4.56)$$

for $W_i \geq 0$, $Y_i \geq 0$, and $r_{i-1;m}$ in the domain of interest rate values, where $C_i^*(W_i, Y_i, r_{i-1;m})$, $\alpha_i^{e*}(W_i, Y_i, r_{i-1;m})$, $\alpha_i^{b*}(W_i, Y_i, r_{i-1;m})$ and $m_i^*(W_i, Y_i, r_{i-1;m})$ are optimal consumptions, optimal equity and bond allocations and optimal annuitisation, and $V_i(W_i, Y_i, r_{i-1;m})$ is the value function for those optimal control variables. Having this solution in hand, we want to derive the solution at time t . It means that we want to determine $C_t^*(W_t, Y_t, r_{t-1;j})$, $\alpha_t^{e*}(W_t, Y_t, r_{t-1;j})$, $\alpha_t^{b*}(W_t, Y_t, r_{t-1;j})$ and $m_t^*(W_t, Y_t, r_{t-1;j})$ which maximises the value function below

$$V_t(W_t, Y_t, r_{t-1;j}) = \max_{\{C_t, \alpha_t^e, \alpha_t^b, m_t\}} \left[u(C_t) + E_t \left[\delta(1-p_t)b_t u(\tilde{W}_{t+1}) + \delta p_t V_{t+1}(\tilde{W}_{t+1}, Y_{t+1}, \tilde{r}_t) \right] \right]$$

which can be written in more explicit form as

$$V_t(W_t, Y_t, r_{t-1;j}) = \max_{\{C_t, \alpha_t^e, \alpha_t^b, m_t\}} \left[u(C_t) + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\delta(1-p_t) b_t u(W_{t+1}(r_t, r_{t+1}^e)) + \right. \right. \\ \left. \left. \delta p_t V_{t+1}(W_{t+1}(r_t, r_{t+1}^e), Y_{t+1}, r_t) \right) dF(r_t) dF(r_{t+1}^e) \right] \quad (4.57)$$

It is possible to decrease the number of state variables from three to two. We will now make the transformations that will allow us to work with only two state variables. The state variable that is going to be excluded is income Y_t . Using the results from Appendix 2, we know that

$$V_t(W_t, Y_t, r_{t-1;j}) = \left(\frac{Y_t}{\bar{y}} \right)^\gamma V_t \left(W_t \frac{\bar{y}}{Y_t}, \bar{y}, r_{t-1;j} \right) \quad (4.58)$$

for any constant $\bar{y} > 0$. Introducing this relation into equation (4.57) one get

$$\left(\frac{Y_t}{\bar{y}} \right)^\gamma V_t \left(W_t \frac{\bar{y}}{Y_t}, \bar{y}, r_{t-1;j} \right) = \max_{\{C_t, \alpha_t^e, \alpha_t^b, m_t\}} \left[u(C_t) + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\delta(1-p_t) b_t u(W_{t+1}(r_t, r_{t+1}^e)) + \right. \right. \\ \left. \left. \delta p_t \left(\frac{Y_{t+1}}{\bar{y}} \right)^\gamma V_{t+1} \left(W_{t+1}(r_t, r_{t+1}^e) \frac{\bar{y}}{Y_{t+1}}, \bar{y}, r_t \right) \right) dF(r_t) dF(r_{t+1}^e) \right]$$

Using (4.49) and skipping writing dependent variables one get

$$\left(\frac{Y_t}{\bar{y}} \right)^\gamma V_t \left(W_t \frac{\bar{y}}{Y_t}, \bar{y}, r_{t-1;j} \right) = \max_{\{C_t, \alpha_t^e, \alpha_t^b, m_t\}} \left[u(C_t) + \right. \\ \left. \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\delta(1-p_t) b_t u(((1-m_t)W_t + Y_t - C_t)(1+r_t^p)) + \delta p_t \left(\frac{Y_{t+1}}{\bar{y}} \right)^\gamma \cdot \right. \right. \\ \left. \left. V_{t+1} \left(((1-m_t)W_t + Y_t - C_t)(1+r_t^p) \frac{\bar{y}}{Y_{t+1}}, \bar{y}, r_t \right) \right) dF(r_t) dF(r_{t+1}^e) \right]$$

where

$$1+r_t^p = 1 + \bar{r}_{t;j} + \alpha_t^e (r_t^e - \bar{r}_{t;j}) + \alpha_t^b \left(\frac{B(\Upsilon_t - 1, r_t)}{B(\Upsilon_t, r_{t-1;j})} - 1 - \bar{r}_{t;j} \right)$$

and rearranging terms in this equation and using (4.58) we have

$$\begin{aligned} \left(\frac{Y_t}{\bar{y}}\right)^\gamma V_t\left(\frac{W_t}{Y_t}\bar{y}, \bar{y}, r_{t-1;j}\right) &= \max_{\{C_t, \alpha_t^e, \alpha_t^b, m_t\}} \left[\left(\frac{Y_t}{\bar{y}}\right)^\gamma u\left(\frac{C_t}{Y_t}\bar{y}\right) + \right. \\ &\quad \delta \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{Y_{t+1}}{\bar{y}}\right)^\gamma \left((1-p_t)b_t u\left(\left((1-m_t)\frac{W_t}{Y_t}\bar{y} + \bar{y} - \frac{C_t}{Y_t}\bar{y}\right)(1+r_t^p)\frac{Y_t}{Y_{t+1}}\right) + p_t \cdot \right. \\ &\quad \left. \left. V_{t+1}\left(\left((1-m_t)\frac{W_t}{Y_t}\bar{y} + \bar{y} - \frac{C_t}{Y_t}\bar{y}\right)(1+r_t^p)\frac{Y_t}{Y_{t+1}}, \bar{y}, r_t\right) \right) dF(r_t) dF(r_t^e) \right] \end{aligned}$$

Let us define

$$w_t = \frac{W_t}{Y_t} \bar{y} \text{ and } c_t = \frac{C_t}{Y_t} \bar{y} \quad (4.59)$$

Multiplying both sides by $\left(\frac{\bar{y}}{Y_t}\right)^\gamma$ and introducing (4.59) into the previous equation we have

$$\begin{aligned} V_t(w_t, \bar{y}, r_{t-1;j}) &= \max_{\{C_t, \alpha_t^e, \alpha_t^b\}} \left[u(c_t) + \right. \\ &\quad \delta \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{Y_{t+1}}{Y_t}\right)^\gamma \left((1-p_t)b_t u\left(\left((1-m_t)w_t + \bar{y} - c_t\right)(1+r_t^p)\frac{Y_t}{Y_{t+1}}\right) + p_t \cdot \right. \\ &\quad \left. \left. V_{t+1}\left(\left((1-m_t)w_t + \bar{y} - c_t\right)(1+r_t^p)\frac{Y_t}{Y_{t+1}}, \bar{y}, r_t\right) \right) dF(r_t) dF(r_t^e) \right] \end{aligned}$$

We will actually derive our solution for $Y_t = \bar{y}$ and $w_t \geq 0$, and control variables c_t , α_t^e , α_t^b and m_t and then use the transformation from Appendix 2 to get solution C_t , α_t^e , α_t^b and m_t for any $Y_t \geq 0$ and $W_t \geq 0$. Using (4.50), we have

$$Y_{t+1} = \rho_t Y_t + \frac{m_t W_t}{a_t}$$

We have defined in (4.34)

$$G_{t+1} = \rho_t + \frac{m_t W_t}{Y_t a_t} = \rho_t + \frac{m_t w_t}{\bar{y} a_t}$$

Introducing these relations into the previous equation one get

$$\begin{aligned}
V_t(w_t, \bar{y}, r_{t-1;j}) = & \max_{\{c_t, \alpha_t^e, \alpha_t^b\}} \left[u(c_t) + \right. \\
& \delta \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_{t+1}^{\gamma} \left((1-p_t) b_t u \left(\left((1-m_t) w_t + \bar{y} - c_t \right) \frac{(1+r_t^P)}{G_{t+1}} \right) + p_t \cdot \right. \\
& \left. \left. V_{t+1} \left(\left((1-m_t) w_t + \bar{y} - c_t \right) \frac{(1+r_t^P)}{G_{t+1}}, \bar{y}, r_t \right) \right) \right] dF(r_t) dF(r_t^e)
\end{aligned} \quad (4.60)$$

When finding numerical solution on the computer we need to approximate each continuous variable with a discrete one. We use the Gauss Quadrature method in order to approximate the continuous random variable $\tilde{r}_t^e$ with the appropriate discrete random variable as follows

$$dis \tilde{r}_t^e \sim \begin{pmatrix} r_{t;1}^e & r_{t;2}^e & \cdots & r_{t;n_{re}-1}^e & r_{t;n_{re}}^e \\ p_{re;1} & p_{re;2} & \cdots & p_{re;n_{re}-1} & p_{re;n_{re}} \end{pmatrix} \quad (4.61)$$

Let us assume that wealth takes only the values on the wealth grid $(w_{t;i})_{i=1}^{n_w}$. We model the interest rate as a discrete state autoregressive process. We denote the states for the real interest rate as $(r_{t;k})_{k=1}^{n_r}$ and the transitional matrix as $(p_{r;j,m})_{(j,m)=(1,1)}^{(n_r,n_r)}$, such that $p_{r;j,m}$ is the probability that during one year period the interest rate will move from state $r_{t;j}$ to state $r_{t+1;m}$.

Thus, we actually find and save into the file the solution

$$\begin{aligned}
& \left\{ (C_t^*(w_{t;i}, \bar{y}, r_{t-1;j}); \alpha_t^{e*}(w_{t;i}, \bar{y}, r_{t-1;j}); \alpha_t^{b*}(w_{t;i}, \bar{y}, r_{t-1;j}); \right. \\
& \left. m_t^*(w_{t;i}, \bar{y}, r_{t-1;j}); V_t(w_{t;i}, \bar{y}, r_{t-1;j})) \right\}_{(i,j)=(1,1)}^{(n_w,n_r)}
\end{aligned} \quad (4.62)$$

of the following equation

$$\begin{aligned}
V_t(w_{t;i}, \bar{y}, r_{t-1;j}) = & \max_{\{c_{t;i,j}, \alpha_{t;i,j}^e, \alpha_{t;i,j}^b, m_{t;i,j}\}} \left[u(c_{t;i,j}) + \right. \\
& \delta \sum_{l=1}^{n_{re}} \sum_{m=1}^{n_r} G_{t+1,i,j}^{\gamma} \left((1-p_t) b_t u \left(\left((1-m_{t;i,j}) w_{t;i} + \bar{y} - c_{t;i,j} \right) \frac{(1+r_{t;j,m,l}^P)}{G_{t+1,i,j}} \right) + \right. \\
& \left. \left. p_t V_{t+1} \left(\left((1-m_{t;i,j}) w_{t;i} + \bar{y} - c_{t;i,j} \right) \frac{(1+r_{t;j,m,l}^P)}{G_{t+1,i,j}}, \bar{y}, r_{t;m} \right) \right) \cdot p_{r;j,m} p_{re;l} \right]
\end{aligned} \quad (4.63)$$

where

$$1 + r_{t;j,m,l}^P = 1 + \bar{r}_{t;j} + \alpha_{t;i,j}^e (r_{t;l}^e - \bar{r}_{t;j}) + \alpha_{t;i,j}^b \left(\frac{B(\Upsilon_t - 1, r_{t;m})}{B(\Upsilon_t, r_{t-1;j})} - 1 - \bar{r}_{t;j} \right)$$

and

$$G_{t+1;i,j} = 1 + \frac{m_{t,i,j} w_{t,i}}{\bar{y} a_{t;j}}$$

and

$$\bar{r}_{t;j} = \frac{1}{B(1, r_{t-1;j})} - 1.$$

Having the set of solutions (4.62) in hands, for each $j = 1, \dots, n_r$ we use cubic splines to interpolate the consumption through the points $\{c_t^*(w_{t;i}, \bar{y}, r_{t-1;j})\}_{i=1}^{n_w}$, optimal asset allocation through the points $\{\alpha_t^{e*}(w_{t;i}, \bar{y}, r_{t-1;j})\}_{i=1}^{n_w}$ and $\{\alpha_t^{b*}(w_{t;i}, \bar{y}, r_{t-1;j})\}_{i=1}^{n_w}$, optimal annuitisation $\{m_t^*(w_{t;i}, \bar{y}, r_{t-1;j})\}_{i=1}^{n_w}$, and the value function $\{V_t(w_{t;i}, \bar{y}, r_{t-1;j})\}_{i=1}^{n_w}$ calculated in these optimal points. Thus, we have

$$\left\{ \left(c_t^*(w_t, \bar{y}, r_{t-1;j}); \alpha_t^{e*}(w_t, \bar{y}, r_{t-1;j}); \alpha_t^{b*}(w_t, \bar{y}, r_{t-1;j}); m_t^*(w_t, \bar{y}, r_{t-1;j}); V_t(w_t, \bar{y}, r_{t-1;j}) \right) \right\}_{j=1}^{n_r} \quad (4.64)$$

for $w_t \geq 0$, and $r_{t-1,j}$ taking discrete values for $j = 1, \dots, n_r$. Finally, using (4.59) and the results from Appendix 2

$$C_t^*(W_t, Y_t, r_{t-1;j}) = \frac{Y_t}{\bar{y}} c_t^*(w_t, \bar{y}, r_{t-1;j}), \text{ for } j = 1, \dots, n_r \quad (4.65)$$

$$\alpha_t^{e*}(W_t, Y_t, r_{t-1;j}) = \alpha_t^{e*}(w_t, \bar{y}, r_{t-1;j}), \text{ for } j = 1, \dots, n_r \quad (4.66)$$

$$\alpha_t^{b*}(W_t, Y_t, r_{t-1;j}) = \alpha_t^{b*}(w_t, \bar{y}, r_{t-1;j}), \text{ for } j = 1, \dots, n_r \quad (4.67)$$

$$m_t^*(W_t, Y_t, r_{t-1;j}) = m_t^*(w_t, \bar{y}, r_{t-1;j}), \text{ for } j = 1, \dots, n_r \quad (4.68)$$

$$V_t(W_t, Y_t, r_{t-1;j}) = \left(\frac{\bar{y}}{Y_t} \right)^\gamma V_t(w_t, \bar{y}, r_{t-1;j}), \text{ for } j = 1, \dots, n_r \quad (4.69)$$

for $W_t \geq 0$ and $Y_t \geq 0$, and $r_{t-1,j}$ in the domain of the real interest rate.

4.4.2 Solution for Exogenous m_t

In order to develop the solution for exogenous m_t we can use the results from Section 4.4.1. The algorithm for solving the problem (4.48)–(4.55) for exogenous m_t is very

similar to the one used for solving the case of endogenous m_t . Mathematically, it is actually sub-case of the problem with endogenous m_t . Let us now explain how we can solve the case of exogenous m_t .

In equations (4.48) we now write $\{C_t, \alpha_t^e, \alpha_t^b\}$ instead of $\{CV_t\}$. Let us assume that we have the solution

$$\left\{ \left(C_i^*(W_i, Y_i, r_{i-1;m}); \alpha_i^{e*}(W_i, Y_i, r_{i-1;m}); \alpha_i^{b*}(W_i, Y_i, r_{i-1;m}); V_i(W_i, Y_i, r_{i-1;m}) \right) \right\}_{i=t+1}^{99} \quad (4.70)$$

for $W_t \geq 0$, $Y_t \geq 0$ and $r_{i-1;m}$ in the domain of real interest rate, where $C_i^*(W_i, Y_i, r_{i-1;m})$, $\alpha_i^{e*}(W_i, Y_i, r_{i-1;m})$ and $\alpha_i^{b*}(W_i, Y_i, r_{i-1;m})$ are optimal consumptions and optimal equity and bond allocation. $V_i(W_i, Y_i, r_{i-1;m})$ is the value function for those optimal control variables. Having this solution in hand, we want to derive the solution at time t . It means, we want to determine $C_t^*(W_t, Y_t, r_{t-1;j})$, $\alpha_t^{e*}(W_t, Y_t, r_{t-1;j})$ and $\alpha_t^{b*}(W_t, Y_t, r_{t-1;j})$ which maximizes the value function (4.48). Thus, we have the control variables $\{C_t, \alpha_t^e, \alpha_t^b\}$ instead of having the control variables $\{C_t, \alpha_t^e, \alpha_t^b, m_t\}$ in Section 4.4.1.

Then we follow the same steps already explained in Section 4.4.1 up to the equation (4.61). Now, using computers we find the solution

$$\left\{ \left(c_t^*(w_{t;i}, \bar{y}, r_{t-1;j}); \alpha_t^{e*}(w_{t;i}, \bar{y}, r_{t-1;j}); \alpha_t^{b*}(w_{t;i}, \bar{y}, r_{t-1;j}); V_t(w_{t;i}, \bar{y}, r_{t-1;j}) \right) \right\}_{(i,j)=(1,1)}^{(n_w, n_r)} \quad (4.71)$$

of equation (4.63).

Having the set of solutions (4.71) in hands, for each $j = 1, \dots, n_r$ we use cubic splines to interpolate the optimal consumption through the points $\{c_t^*(w_{t;i}, \bar{y}, r_{t-1;j})\}_{i=1}^{n_w}$, optimal asset allocation through the points $\{\alpha_t^{e*}(w_{t;i}, \bar{y}, r_{t-1;j})\}_{i=1}^{n_w}$ and $\{\alpha_t^{b*}(w_{t;i}, \bar{y}, r_{t-1;j})\}_{i=1}^{n_w}$ and the value function through the points $\{V_t(w_{t;i}, \bar{y}, r_{t-1;j})\}_{i=1}^{n_w}$. Then we have

$$\left\{ \left(c_t^*(w_t, \bar{y}, r_{t-1;j}); \alpha_t^{e*}(w_t, \bar{y}, r_{t-1;j}); \alpha_t^{b*}(w_t, \bar{y}, r_{t-1;j}); V_t(w_t, \bar{y}, r_{t-1;j}) \right) \right\}_{j=1}^{n_r} \quad (4.72)$$

For $w_t \geq 0$, and $r_{t-1;j}$ taking discrete values for $j = 1, \dots, n_r$. Then, using (4.59) and the results from Appendix 2

$$C_t^*(W_t, Y_t, r_{t-1;j}) = \frac{Y_t}{\bar{y}} c_t^*(w_t, \bar{y}, r_{t-1;j}), \text{ for } j = 1, \dots, n_r \quad (4.73)$$

$$\alpha_t^{e*}(W_t, Y_t, r_{t-1;j}) = \alpha_t^{e*}(w_t, \bar{y}, r_{t-1;j}), \text{ for } j = 1, \dots, n_r \quad (4.74)$$

$$\alpha_t^{b*}(W_t, Y_t, r_{t-1;j}) = \alpha_t^{b*}(w_t, \bar{y}, r_{t-1;j}), \text{ for } j = 1, \dots, n_r \quad (4.75)$$

$$V_t(W_t, Y_t, r_{t-1;j}) = \left(\frac{\bar{y}}{Y_t} \right)^\gamma V_t(w_t, \bar{y}, r_{t-1;j}), \text{ for } j = 1, \dots, n_r \quad (4.76)$$

for $W_t \geq 0$ and $Y_t \geq 0$, and $r_{t-1;j}$ in the domain of the real interest rate.

4.4.3 Solution for Endogenous/Exogenous m_t

We apply endogenous/exogenous solutions to each possible type of problem 1–3 explained in Section 4.1.2. Whenever we solve the problem for one particular pensioner, we firstly have to know what his decision is regarding endogenous/exogenous annuitisation in each year during his retirement. If in a certain year the pensioner decides to annuitise exogenously then we also need to know which part of the pension wealth at that age he wants to annuitise. Of course, his decision needs to be in line with the availability of annuities due to possible market limitations.

Once we have this information, we can find the solution of optimal consumption, asset allocation and annuitisation starting from the last possible age period and calculating year by year backwards. When calculating one particular year we comply with the information about endogenous/exogenous annuitisation and if exogenous then we also comply with the amount chosen to be annuitised. Thus, applying the endogenous/exogenous solutions just derived, we are able to deal with any combination of yearly endogenous/exogenous annuitisation in retirement.

4.4.4 Check of Accuracy of Numerical Calculations in Mathematica

Once we have the numerical solution saved in the excel files, we need to check the accuracy of the numerical solution. In order to prove the accuracy of the results, we make stochastic simulation and obtain 2,000 random realisations with the same assumptions as used for getting the deterministic results. Now, we get 2,000 random realisations of the paths for all variables in the model. Equation (4.46) explicitly shows the formula for expected discounted utility derived from future consumption and bequest. This most important formula of this thesis can also be used for testing the accuracy of the results. Similarly to the set of equations (4.48)–(4.55), we can write the set of equations for random realisations

$$V_{t;n}(W_t, Y_t, r_{t-1;j}) = \sum_{i=t}^{99} \left(\delta^{i-t} \left(\prod_{k=t}^{i-1} p_k \right) \left(u(C_{i;n}) + \delta(1-p_i) b_i u(W_{i+1;n}) \right) \right) \quad (4.77)$$

where

$$W_{i+1;n} = ((1-m_{i;n})W_{i;n} + Y_{i;n} - C_{i;n})(1+r_{i;n}^P) \quad (4.78)$$

$$Y_{i+1;n} = \rho_i Y_{i;n} + \frac{m_{i;n} W_{i;n}}{a_{i;n}} \quad (4.79)$$

$$r_{i;n}^P = \bar{r}_{i;n} + \alpha_{i;n}^e (r_{i;n}^e - \bar{r}_{i;n}) + \alpha_{i;n}^b \left(\frac{B(\Upsilon_i - 1, r_{i;n})}{B(\Upsilon_i, r_{i-1;n})} - 1 - \bar{r}_{i;n} \right) \quad (4.80)$$

$$\rho_i = \begin{cases} replrate & i = 65 \\ 1 & 66 \leq i \leq 99 \end{cases} \quad (4.81)$$

for $65 \leq t \leq 99$, $t \leq i \leq 100$ and $n = 1, \dots, 2,000$, $(W_{t,n}, Y_{t;n}, r_{t-1;j,n}) = (W_t, Y_t, r_{t-1;j})$, and where $C_{i,n}$, $\alpha_{i,n}^e$, $\alpha_{i,n}^b$ and $m_{i,n}$ are optimal consumption, asset allocation and annuitisation calculated from functions (4.65)–(4.69). $C_{i,n}$, $\alpha_{i,n}^e$, $\alpha_{i,n}^b$ and $m_{i,n}$ are functions of $(W_{i,n}, Y_{i;n}, r_{i-1;j,n})$, and $r_{i;n}$ and $r_{i;n}^e$ are random realisations of the stochastic simulations based on the assumptions presented in Table 3.3, 4.2 and 4.3. As a result, we get 2,000 discounted utilities derived from future consumption and bequest, if the bequest motive exists.

If the calculations using equations (4.48)–(4.55) are correct then the following equations must be valid

$$V_t(X_t, Y_t, r_{t-1;j}) \approx \underset{n=1, \dots, 2000}{Mean} [V_{t,n}(W_t, Y_t, r_{t-1;j})] \quad (4.82)$$

We make calculations and check if the equation (4.82) is approximately satisfied. The difference appears to be less than 2% for 2,000 random realisations. This variability depends on the assumptions, and in particular significantly depends on the assumption regarding the availability of annuities. If more pension wealth is converted into annuities, then the difference in equation (4.82) is less than 2%, and sometimes it is less than 0.1%. The difference in equation (4.82) will also decrease with the increase of the random sample, but we can say from an analysis not presented here that 2,000 random realisations are sufficient to get very small differences and to see all the basic rules as expected. In all examples of random realisation we use the sample of the same size of $n = 2,000$ random realisations.

In Section 4.5.6, we make left-tail analysis of the values of the function $V_{t,n}(W_t, Y_t, r_{t-1;j})$, and in Chapter 5 we make more investigations using random

realisations. For this purpose we use the same realisations of the stochastic simulations as we use for the check of accuracy of the numerical calculations. We believe that this number of random realisations provides us with reasonably good results for the left-tail analysis in this chapter and for the analysis of the results in Chapter 5. The deeper analysis of the pensioner's left-tail risk would require more than 2,000 random realisations.

Thus, we calculate the value function using a set of equations (4.48)–(4.55) and calculate the mean discounted utility derived from future consumptions and a bequest using equations (4.77)–(4.81). Then, for a given $(W_t, Y_t, r_{t-1;j})$, we compare the two and check if these two values are close to each other. The criterion for checking the accuracy of the results is to have $\text{Mean}_{n=1, \dots, 2000} [V_{t;n}(W_t, Y_t, r_{t-1;j})]$ sometimes higher and sometimes lower than $V_t(W_t, Y_t, r_{t-1;j})$ and that this difference is never higher than 2%. All our results passed this test.

4.5 The Results

In Section 4.5, we present the numerical results of the problem defined in this chapter. We will show the results for different values of the parameters and will choose the most representative results in our opinion.

We investigate three types of the problem differentiated by constraints on the control variables as it is presented in Section 4.1.2. We refer to the problem with the particular constraint on m_t for $65 \leq t \leq 99$ as Case. In each Case, we firstly assume that m_t is either exogenous or endogenous and then, for those ages where m_t is exogenous we also assume the values of m_t such that $0 \leq m_t \leq 1$.

Case 4.1 in Section 4.5 is related to the type of problem numbered 4.1 in Section 4.1.2. In Case 4.1, we investigate the pensioner with no access to annuities, or in other words, we investigate the problem where m_t is exogenous and its value is $m_t = 0$ for $65 \leq t \leq 99$. Case 4.2 in Section 4.5 is related to the type of problem numbered 4.2 in Section 4.1.2. In Case 4.2, the pensioner has access to annuities at age 65 only and he purchases annuities optimally at age 65. In mathematical terms, we assume that m_{65} is endogenous and exogenous otherwise and that $m_t = 0$ for $66 \leq t \leq 99$. Case 4.3 is the most general one. In Case 4.3, the pensioner can optimally invest in equities, bonds and cash and optimally purchase annuities whenever in retirement. Case 4.3 is related

to the type of problem numbered 4.3 in Section 4.1.2. Any suboptimal behaviour in terms of investment and annuitisation decisions can be investigated as well.

In Table 4.1, we show the main assumptions about the control variable m_t for $65 \leq t \leq 99$.

Case	Annuitisation at age 65	Annuitisation at ages 66–99
Case 4.1	Exogenous, $m_{65} = 0$	Exogenous, $m_t = 0$ for $66 \leq t \leq 99$
Case 4.2	Endogenous, m_{65} is optimal	Exogenous, $m_t = 0$ for $66 \leq t \leq 99$
Case 4.3	Endogenous, m_{65} is optimal	Endogenous, m_t is optimal for $66 \leq t \leq 99$

Table 4.1 The assumptions about annuitisation in each case.

4.5.1 Parameter Values

For each Case 4.1, 4.2 and 4.3, we find the optimal solution for RRA coefficient γ taking values -1 , -4 and -9 , and the bequest motive coefficient b_t being constant and taking values 0 or 1 for $65 \leq t \leq 99$. All together, we have three cases and six combinations of coefficients, overall 18 solutions. The values of other parameters used in the basic numerical solutions are as follows:

- Income at age 65 $Y_{65} = 33,320.90$
- Wealth at age 65 $W_{65} = 200,000$
- Interest rate $\tilde{r}_{t+1} - r_t = (a_d - b_d r_t) \Delta t - \sigma_{dr} \tilde{\epsilon}_r(t)$
 $a_d = 0.00902377$, $b_d = 0.451188$,
 $\sigma_{dr} = 0.0152622$, $\tilde{\epsilon}_t \sim N(0,1)$,
 $E[\tilde{r}_t] = 0.02$, $StD[r_t] \approx 0.0172195$
- Market price of risk $\lambda = 0.1528$
- Rate on risky investment $Ln(\tilde{r}_t) \sim N(\mu, \sigma^2)$
 $\mu = 0.0474187$, $\sigma = 0.14731$
 $E[\tilde{r}_t^e] = 0.06$, $StD[\tilde{r}_t^e] = 0.157$
- Survival (Mortality) table Interim life table produced by The Government Actuary's Department for United Kingdom Males, based on data for years 2002–2004
- Discount factor $\delta = 0.96$

The values of the parameters are used by other authors who have investigated similar problems. The same value of the volatility of the interest rate is used by Boulier et al (2001). The values of the other parameters are used for example by Cocco et al (2005). We will not repeat it, but the similar comments about the chosen values of the income at age 65, replacement ratio and pension wealth as at the beginning of Section 3.4.1 are also applicable here.

4.5.1.1 Grids

In order to solve the problem (4.48)–(4.55) numerically we need to approximate all the continuous variables with discrete ones. We solve equation (4.63) on the computer. From this equation, we see that three continuous variables need to be approximated by the discrete ones. These are: pension wealth, interest rate and rate of return on equity investment. Bond prices are calculated from the interest rate model.

In Chapter 4, we use the same pension wealth and the same approximation of the rate of return on equities as in Chapter 3.

Interest rate is approximated with $n_r = 15$ points. The values in Table 4.2 are possible states of interest rate, i.e. the values of $r_{t;j}$, where $1 \leq i \leq n_r$ and $65 \leq t \leq 99$. All values in Table 4.2 are in percentages.

Interest rate – state	–2.44	–2.21	–1.81	–1.25	–0.56	0.22	1.09	2.00
Interest rate – state	2.91	3.78	4.56	5.25	5.81	6.21	6.44	

Table 4.2 The possible states of the interest rate (in percentages)

The transitional matrix for interest rate, i.e. the values of $(p_{r;j,m})_{(j,m)=(1,1)}^{(n_r,n_r)}$, such that $p_{r;j,m}$ is a probability that during one year period the interest rate will move from state $r_{t;j}$ to state $r_{t+1;m}$, where $1 \leq j \leq n_r$, $1 \leq m \leq n_r$ and $65 \leq t \leq 99$. We present the transitional matrix for interest rate in Table 4.3.

The values in Table 4.3 are rounded to two decimal places for presentation purposes only. In our calculation we work with all decimal digits. Values 0.00% that appear in Table 4.3 have positive values but are less than 0.01%.

	-2.44	-2.21	-1.81	-1.25	-0.56	0.22	1.09	2.00	2.91	3.78	4.56	5.25	5.81	6.21	6.44
-2.44	1.67	4.61	9.21	15.59	21.38	21.84	15.45	7.28	2.32	0.53	0.10	0.02	0.00	0.00	0.00
-2.21	1.48	4.12	8.41	14.67	20.88	22.26	16.49	8.17	2.73	0.66	0.13	0.02	0.00	0.00	0.00
-1.81	1.17	3.34	7.08	13.02	19.77	22.73	18.28	9.86	3.59	0.94	0.19	0.04	0.01	0.00	0.00
-1.25	0.83	2.43	5.42	10.72	17.84	22.76	20.52	12.46	5.11	1.49	0.34	0.07	0.02	0.00	0.00
-0.56	0.51	1.56	3.72	8.05	14.98	21.75	22.58	15.89	7.55	2.54	0.67	0.16	0.04	0.01	0.00
0.22	0.27	0.87	2.25	5.40	11.44	19.29	23.57	19.65	11.06	4.38	1.34	0.35	0.09	0.03	0.01
1.09	0.13	0.43	1.19	3.20	7.82	15.51	22.63	22.71	15.38	7.28	2.61	0.80	0.23	0.07	0.02
2.00	0.05	0.18	0.55	1.68	4.76	11.18	19.64	23.90	19.64	11.18	4.76	1.68	0.55	0.18	0.05
2.91	0.02	0.07	0.23	0.80	2.61	7.28	15.38	22.71	22.63	15.51	7.82	3.20	1.19	0.43	0.13
3.78	0.01	0.03	0.09	0.35	1.34	4.38	11.06	19.65	23.57	19.29	11.44	5.40	2.25	0.87	0.27
4.56	0.00	0.01	0.04	0.16	0.67	2.54	7.55	15.89	22.58	21.75	14.98	8.05	3.72	1.56	0.51
5.25	0.00	0.00	0.02	0.07	0.34	1.49	5.11	12.46	20.52	22.76	17.84	10.72	5.42	2.43	0.83
5.81	0.00	0.00	0.01	0.04	0.19	0.94	3.59	9.86	18.28	22.73	19.77	13.02	7.08	3.34	1.17
6.21	0.00	0.00	0.00	0.02	0.13	0.66	2.73	8.17	16.49	22.26	20.88	14.67	8.41	4.12	1.48
6.44	0.00	0.00	0.00	0.02	0.10	0.53	2.32	7.28	15.45	21.84	21.38	15.59	9.21	4.61	1.67

Table 4.3 Transitional matrix for interest rate (values in percentages)

The values in the first column in Table 4.3 are the known values of interest rate during the year before the observed year. The values in the first row in Table 4.3 are the possible values of interest rate during the observed year. Now, the value in Table 4.3 crossing one row and one column (apart from the values in the first column and row) is the probability that interest rate during the observed year will move from the state given in the first column to the state given in the first row. The sum of values in each row is 100 (excluding the value in the first column and apart from values in the first row). These are expected results as the probabilities in each single row are the probabilities for each possible new state of the interest rate.

4.5.1.2 Bonds

The model for the bond market is developed in Section 4.2. For the purpose of the numerical solution, we assume that the duration of the rolling bonds is 10 years. In this section, we present numerical values of derived one year, nine year and ten year bonds, the values of risk free rate, the values of possible states and the transitional matrix of low risk rate. The parameters for the interest rate are given at the beginning of Section 4.5.1.

We firstly show the numerical values of the prices of one year, nine year and ten year bonds. In Table 4.4, we give the bond prices providing 100 money units at maturity for different values of the known interest rate in the previous year.

	Known interest rate in the previous year	Price of one year bond	Price of nine year bond	Price of ten year bond
1	−2.44%	100.00	84.39	82.35
2	−2.21%	99.96	84.24	82.19
3	−1.81%	99.79	83.95	81.91
4	−1.25%	99.54	83.53	81.50
5	−0.56%	99.22	82.97	80.96
6	0.22%	98.82	82.30	80.30
7	1.09%	98.38	81.54	79.56
8	2.00%	97.91	80.73	78.77
9	2.91%	97.44	79.94	77.99
10	3.78%	97.00	79.20	77.27
11	4.56%	96.61	78.55	76.64
12	5.25%	96.30	78.02	76.12
13	5.81%	96.05	77.62	75.73
14	6.21%	95.89	77.35	75.47
15	6.44%	95.80	77.21	75.33

Table 4.4 Zero coupon bond prices for durations of one, nine and ten years.

In Table 4.5, we present rates of return on a one year bond, i.e. risk free rates.

Interest rate in the previous year	−2.44	−2.21	−1.81	−1.25	−0.56	0.22	1.09	2.00
Return on one year bond investment	0.00	0.04	0.21	0.46	0.79	1.19	1.65	2.14
Interest rate in the previous year	2.91	3.78	4.56	5.25	5.81	6.21	6.44	
Return on one year bond investment	2.63	3.09	3.51	3.85	4.11	4.29	4.38	

Table 4.5 Risk free rates (values in percentages)

In Table 4.5, in the first and the third row we present the values of the known interest rate in the previous year. In the second and fourth row we present the rates obtained from investment in the risk free asset, under the assumption that the value of the interest rate in the previous year is given in the cell above.

Table 4.6 shows rates of investment return in the low risk asset, i.e. in the ten year rolling bonds. Here, we apply formula (4.26) to the values from the last two columns in Table 4.4.

	-2.44	-2.21	-1.81	-1.25	-0.56	0.22	1.09	2.00	2.91	3.78	4.56	5.25	5.81	6.21	6.44
-2.44	2.49	2.30	1.95	1.43	0.76	-0.06	-0.98	-1.96	-2.92	-3.83	-4.61	-5.25	-5.74	-6.06	-6.24
-2.21	2.68	2.49	2.14	1.62	0.95	0.13	-0.80	-1.77	-2.74	-3.65	-4.43	-5.08	-5.56	-5.89	-6.07
-1.81	3.03	2.84	2.49	1.97	1.29	0.47	-0.46	-1.44	-2.41	-3.31	-4.11	-4.75	-5.24	-5.56	-5.74
-1.25	3.55	3.36	3.01	2.49	1.81	0.98	0.05	-0.94	-1.92	-2.83	-3.62	-4.27	-4.76	-5.09	-5.27
-0.56	4.25	4.05	3.70	3.17	2.49	1.66	0.72	-0.27	-1.26	-2.17	-2.97	-3.63	-4.12	-4.45	-4.63
0.22	5.10	4.90	4.55	4.02	3.33	2.49	1.54	0.54	-0.45	-1.37	-2.18	-2.84	-3.33	-3.67	-3.85
1.09	6.08	5.88	5.52	4.99	4.29	3.45	2.49	1.48	0.48	-0.45	-1.27	-1.93	-2.43	-2.77	-2.95
2.00	7.14	6.94	6.57	6.04	5.33	4.48	3.51	2.49	1.48	0.54	-0.28	-0.95	-1.46	-1.80	-1.99
2.91	8.21	8.00	7.64	7.09	6.38	5.52	4.54	3.51	2.49	1.54	0.71	0.04	-0.47	-0.82	-1.01
3.78	9.22	9.02	8.64	8.10	7.38	6.51	5.52	4.49	3.45	2.49	1.66	0.97	0.46	0.11	-0.08
4.56	10.12	9.92	9.54	8.99	8.27	7.39	6.40	5.35	4.31	3.34	2.50	1.81	1.29	0.94	0.75
5.25	10.87	10.66	10.28	9.73	9.00	8.11	7.12	6.06	5.01	4.04	3.19	2.50	1.97	1.62	1.43
5.81	11.44	11.23	10.85	10.29	9.56	8.67	7.67	6.61	5.55	4.57	3.72	3.02	2.50	2.14	1.95
6.21	11.83	11.62	11.24	10.68	9.94	9.05	8.04	6.98	5.92	4.94	4.08	3.38	2.85	2.50	2.30
6.44	12.04	11.83	11.45	10.89	10.15	9.26	8.25	7.18	6.12	5.14	4.28	3.58	3.05	2.69	2.50

Table 4.6 Rates on low risk investment, i.e. investment in ten year rolling bonds (values in percentages).

The values in Table 4.6 show rates on low risk investments during a one year period. We read Table 4.6 in the following way. At the beginning of the year we know the interest rate in the previous year and we read this value in the first column in Table 4.6. If interest rate during the following year appears to be the value written in the first row in Table 4.6, then rate on a ten year rolling bond investment during the observed year is read in the cell crossing those row and column. We see that values in Table 4.6 are decreasing in each row. It is to be expected. Namely, when we know the interest rate during the previous year then we read only that row and the ten year bond price is fixed in one row. Each column represent the possible realisation of interest rate in the year to come and nine year bond prices decrease with an increase of the interest rate observed during the coming year. Also, we observe that the values on the diagonal of Table 4.6 are almost the same and roughly 2.50%. This is an expected result as the mean value of real interest rate is 2% and we have an increase of about 25% due to the market price of risk.

We use the same survival table as in Chapter 3.

4.5.2 Optimal Consumption, Asset Allocation and Annuitisation

In this section, we present optimal consumption, optimal equity and bond allocation and optimal annuitisation as functions dependent on age, pension wealth and interest rate. The figures below represent the functions that are calculated using numerical mathematics and stored on the computer. Of course, we present here only the results which we believe are the most interesting and informative.

Almost all the figures in this section have age and wealth on x and y axes and the value in money units or proportions on z-axis. Proportions vary from zero to one. The figures show three dimensional surfaces. Figures in this section show the values of the control variables, i.e. optimal behaviour, of the pensioner for any combination of the value of x and y axes. All these figures are deterministic and do not depend on one particular realisation of random interest and equity rates.

If the x and y axis are age and wealth, then the pensioner's optimal decisions during the retirement, regarding the value on the z axis, is a single line on the surface. In other words, at a certain age and with a certain pension wealth the pensioner can read on the surface the value of his optimal decision.

If x and y axis are interest rate in the previous year and wealth, then the pensioner's optimal decision regarding the value on z axis is a single point on the surface depending on his wealth and on the value of the interest rate in the previous year.

We emphasise here that all figures with age as one variable present optimal values on the z-axis assuming that the interest rate in the previous year is 2.00%. In this chapter we derive and analyse the interest rate risk model where the value of the interest rate is changing and optimal values depend on the known interest rate in the previous year. The interest rate in the previous year is a state variable in the model. If we want to present all optimal results, we should actually present 15 surfaces for each single surface presented here and these 15 surfaces will show the optimal values for each possible interest rate in the previous year. However, showing all these surfaces would be impossible due to the limited space in the thesis. All we want to show here is an idea about the shape of the surfaces and about the values of the control variables. Sometimes the optimal values for other values of the interest rate in the previous year are significantly different but we will not show all these results here. More results depending on the interest rate will be presented in the later text in this chapter and in Chapter 5. Actually, the pensioner's optimal values of the control variables are below

or above the values shown on the surfaces if interest rate in the previous year is not equal to 2.00%.

4.5.2.1 Case 4.1 – Dependence on Wealth and Age

We firstly present the dependence of optimal consumption, equity and bond allocation on wealth and age in Case 4.1. There is no annuitisation in Case 4.1.

Optimal consumption is presented in Figure 4.1. We observe that the shapes of the surfaces in Figure 4.1 are similar. There is always an increase in optimal consumption with an increase of wealth and age. These two features of optimal consumption are to be expected. The more wealth the pensioner possesses the more he consumes. Also, if the pensioner has the same wealth at two different ages he consumes more at an older age because he has fewer years to live and then less incentive to save.

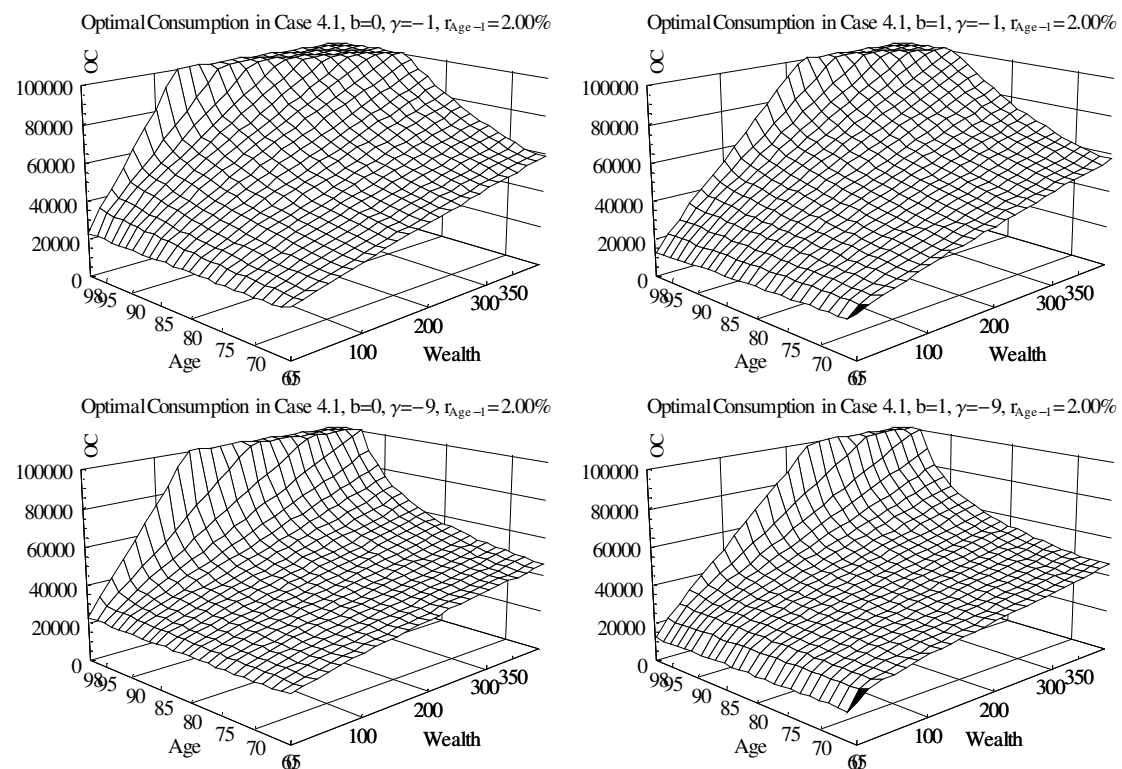


Figure 4.1 Optimal consumption in Case 4.1, for RRA coefficient γ taking values -1 , and -9 , and for bequest motive coefficient b_i taking values 0 and 1 for $65 \leq t \leq 99$. Interest rate in the previous year is 2.00%. Values of wealth are in thousands.

It seems that optimal consumption is not significantly influenced by the bequest motive. If we compare the two upper (or two lower) surfaces in Figure 4.1, we

observe that the values on the left hand side surface are slightly higher than the values on the right hand side surface. Thus, the pensioner with the bequest motive tends to consume slightly less than the pensioner with the same age and wealth and the same preferences apart from not having a bequest motive.

The less risk averse pensioner consumes significantly more if he possesses more wealth. For example, if we compare the upper left hand side surfaces where $\gamma = -1$ and $b_t = 0$ and the lower left hand side surface where $\gamma = -9$ and $b_t = 0$, we observe that for each fixed age, the values on the upper surface increase faster than the values on the lower surface.

Optimal equity allocation, presented in Figure 4.2, is significantly influenced by the pensioner's risk aversion. If $\gamma = -1$, 100% investment in equities is almost always optimal. For a more risk averse pensioner, i.e. under assumption $\gamma = -9$, optimal equity investment decreases with age and with wealth, when wealth is about 50,000 or more. We observe a steep decrease for the values of the pension wealth from about 50,000 up to about 200,000. For the pensioner with the bequest motive and the value of RRA coefficient $\gamma = -9$, then his optimal equity investment increases for the values of pension wealth from 0 to about 50,000. When certain values are attained optimal equity investment becomes a decreasing function of wealth. For the pensioner with no bequest motive optimal equity investment either decreases or has a constant value equal to 100%.

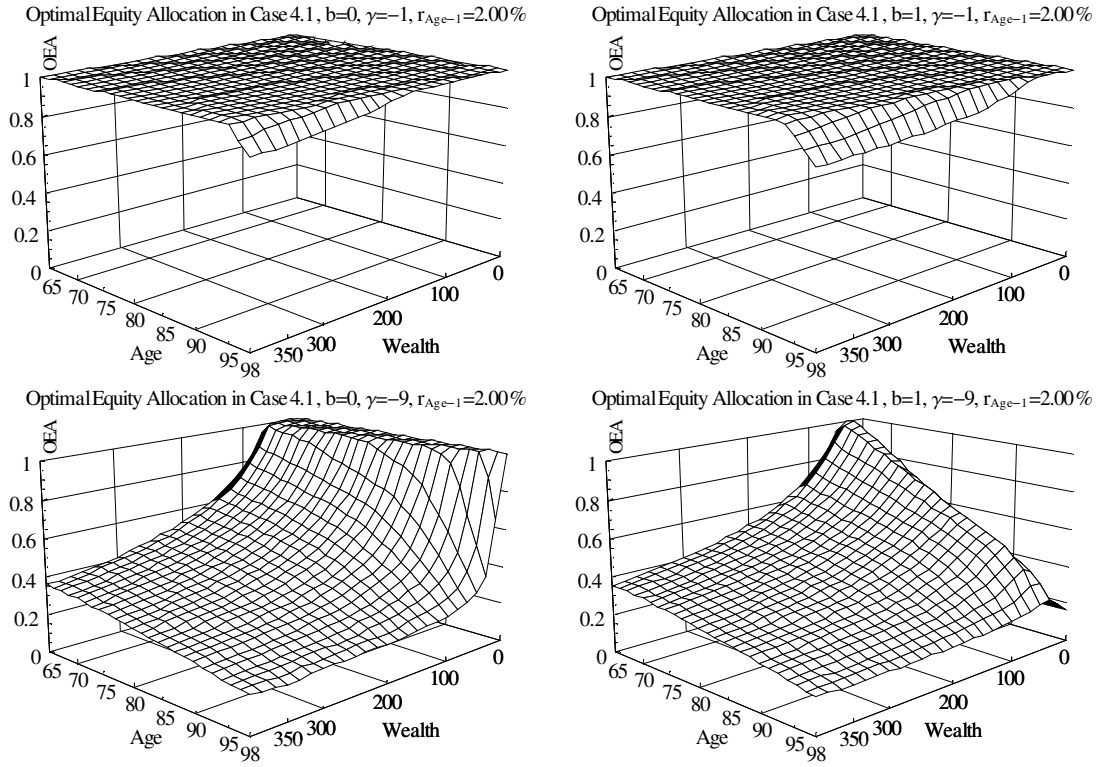


Figure 4.2 Optimal equity allocation in Case 4.1, for RRA coefficient γ taking values -1 , and -9 , and for bequest motive coefficient b_t taking values 0 and 1 for $65 \leq t \leq 99$. Interest rate in the previous year is 2.00% . Values of wealth are in thousands. Values of optimal equity allocation are proportions from 0 to 1

Under the assumption stated at the beginning of Section 4.5, optimal bond allocation is always equal to one minus optimal equity investment. It means that it is never optimal to invest in the risk free asset. Later we will give examples when it is optimal to invest one part of wealth into the risk free asset. Surfaces of optimal bond allocation are given in Figure 4.3.

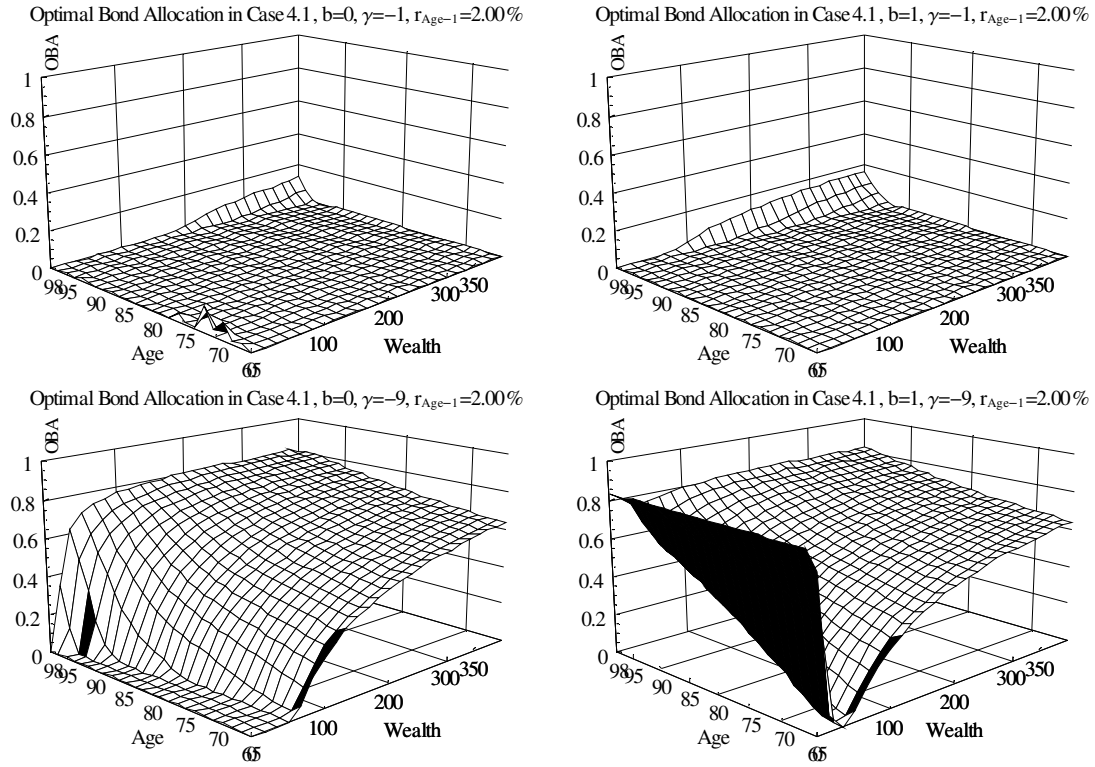


Figure 4.3 Optimal bond allocation in Case 4.1, for RRA coefficient γ taking values -1 , and -9 , and for bequest motive coefficient b_t taking values 0 and 1 for $65 \leq t \leq 99$. Interest rate in the previous year is 2.00% . Values of wealth are in thousands. Values of optimal bond allocation are proportions from 0 to 1 .

4.5.2.2 Case 4.2 – Dependence on Wealth and Age

In Figure 4.4, we present optimal consumption in Case 4.2. We can see that the respective surfaces in Figure 4.1 and Figure 4.4 are almost the same. In Case 4.2, optimal annuitisation is allowed at age 65 only. We can see from equation (4.63) that solutions are calculated backwards year by year. Thus all consumption after age 65 is the same. At age 65 we have a different option for the pensioner, i.e. he optimally annuitises his pension wealth at that age. We observe later in Figure 4.5 that the pensioner optimally uses this opportunity to annuitise, but Figure 4.4 shows that using this option does not influence optimal consumption at age 65 significantly. However, annuitisation at age 65 will influence the pensioner's consumption afterwards because his income is increased and pension wealth is decreased due to annuitisation at age 65. Figure 4.4 shows optimal consumption as a function of wealth and age under assumption of constant income after age 65.

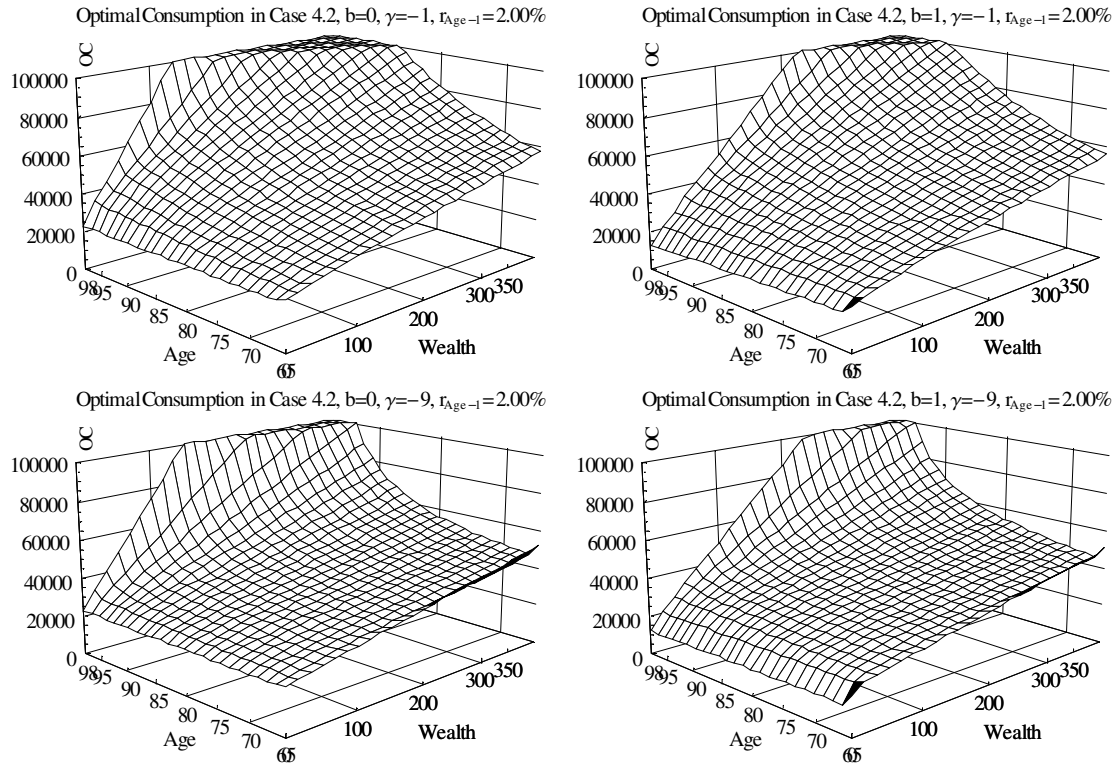


Figure 4.4 Optimal consumption in Case 4.2, for RRA coefficient γ taking values -1 , and -9 , and for bequest motive coefficient b_t taking values 0 and 1 for $65 \leq t \leq 99$. Interest rate in the previous year is 2.00% . Values of wealth are in thousands.

Figure 4.4 shows optimal consumption under the assumption that income at age 65 is $Y_{65} = 33,320.90$, replacement ratio is $\rho_{65} = 0.68212$, and then income is $Y_t = 22,728.85$ for $66 \leq t \leq 99$. We know that in Case 4.2, the level of income increases after age 65. Now, we need to apply the same technique as explained in Section 3.4.2.2. Values of optimal consumption shown in Figure 4.4 should be taken as the basis for calculating optimal consumption in Case 4.2. Knowing pension wealth and income after purchasing annuities, knowing values in Figure 4.4 and using equation (4.65) for $66 \leq t \leq 99$ one can calculate optimal consumption for any income in Case 4.2 after optimally purchasing annuities. Following the same suit as in Section 3.4.2.2, Figure 4.4 shows optimal consumption under the assumption that income after age 65 is $Y_t^{4.1} = Y_t^{SS} = 22,728.85$. Now, if we have annuitisation at age 65 then income after age 65 is higher. If we denote income in Case 4.2 with $Y_t^{4.2}$ then using (4.59) we have

$$W_t^{4.1} = W_t^{4.2} \frac{Y_t^{4.1}}{Y_t^{4.2}}.$$

Now using (4.65) we have

$$C_t^{4.2*}(W_t^{4.2}, Y_t^{4.2}, r_{t-1;j}) = \frac{Y_t^{4.2}}{Y_t^{4.1}} C_t^{4.1*}\left(W_t^{4.2} \frac{Y_t^{4.1}}{Y_t^{4.2}}, Y_t^{4.1}, r_{t-1;j}\right).$$

In order to calculate optimal consumption $C_t^{4.2*}(W_t^{4.2}, Y_t^{4.2}, r_{t-1;j})$ in Case 4.2, we firstly read value $W_t^{4.1}$ on wealth axis in Figure 4.4. Then, we read optimal consumption for this value of wealth and age on the surface, and then multiply this value with

$$\frac{Y_t^{4.2}}{Y_t^{4.1}}.$$

Using this technique, we can determine from Figure 4.4. optimal consumption for a given age, pension wealth and any level of income.

For the low risk averse pensioner in Case 4.2, it is optimal to invest all his available assets in equities for all ages from 65 onwards and for all reasonable levels of pension wealth. In this thesis, we investigate the pensioner who has between 100,000 and 350,000 money units of pension wealth at age 65. For this pensioner, regardless of annuitisation showed in Figure 4.5, it is optimal to invest all his pension wealth in equities. For the more risk averse pensioner, it is optimal to invest all his remaining pension wealth, after annuitisation, into equities. The more risk averse pensioner with no bequest motive will optimally continue investing only into equities during the whole retirement period. However the more risk averse pensioner with a bequest motive will decrease optimal investment into equities during retirement period and the demand for bond investment increases as this pensioner getting older.

Regarding optimal annuitisation at age 65 for the pensioner in Case 4.2, we can see from Figure 4.5 that optimal annuitisation at age 65 depends significantly on both pension wealth at age 65 and on the interest rate observed during the year prior to retirement. Comparing the two upper surfaces ($\gamma = -1$) and the two lower surfaces ($\gamma = -9$) in Figure 4.5, we observe that optimal annuitisation is lower for the less risk averse pensioner. From the lower left hand side surface, we see that the more risk averse pensioner with no bequest motive annuitises around 90% of his pension wealth for almost all wealth and interest rate values at age 65. If the pensioner with RRA coefficient $\gamma = -9$ has the bequest motive, we can see on the lower right hand side surface that his optimal annuitisation is influenced by pension wealth, particularly for the level of pension wealth up to about 100,000 money units. For the less risk averse pensioner, with RRA coefficient $\gamma = -1$, the choice of optimal annuitisation at age 65 changes more with changes of wealth and interest rate than for the more risk averse

pensioner. If $\gamma = -1$ and $b_t = 1$ present optimal annuitisation is less than 55%, and decreases with decrease in the values of pension wealth and of interest rate as well.

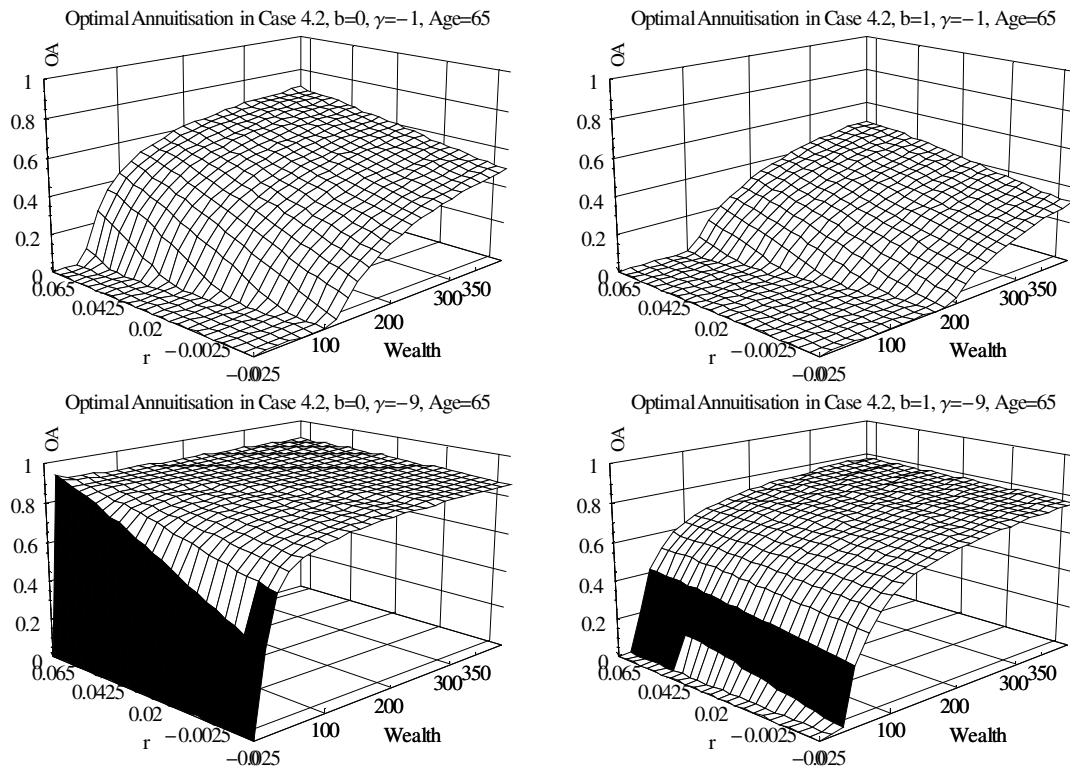


Figure 4.5 Optimal annuitisation in Case 4.2, for RRA coefficient γ taking values -1 , and -9 , for bequest motive coefficient b_t taking values 0 and 1 for $65 \leq t \leq 99$, and for the different values of interest rate in the previous year Values of wealth are in thousands. Values of optimal annuitisation are proportions from 0 to 1 .

4.5.2.3 Case 4.3 – Dependence on Wealth and Age

Figure 4.6 is very similar to Figures 4.1 and 4.4 which means that optimal consumption for the same level of income is not significantly influenced by the pensioner's constraints on annuitisation. We emphasise that optimal consumption in Figure 4.6 is calculated for the same level of income for all ages. However, due to the purchase of annuities, income increases after age 65. Again, we use the same technique as in Section 4.5.2.2 to calculate optimal consumption for any value of income. In the analysis not presented here, we observed that increased income only moves the surfaces in Figure 4.6 up and does not change the shape of the surfaces.

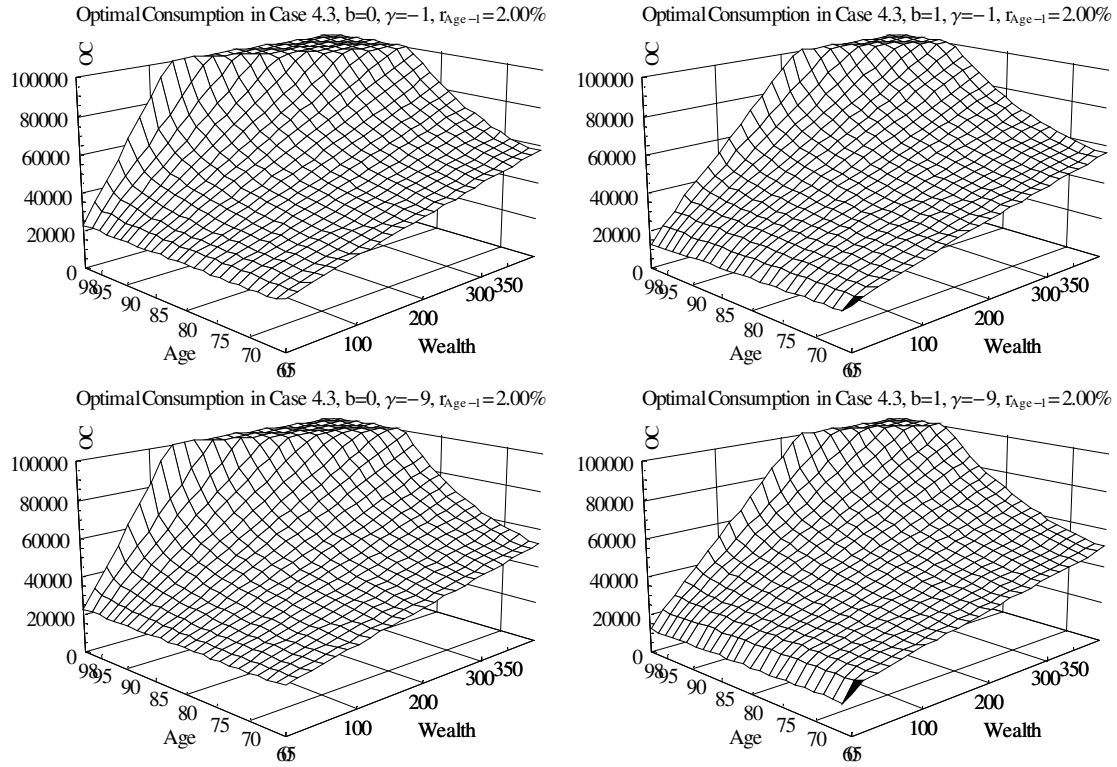


Figure 4.6 Optimal consumption in Case 4.3, for RRA coefficient γ taking values -1 , and -9 , and for bequest motive coefficient b_t taking values 0 and 1 for $65 \leq t \leq 99$. Interest rate in the previous year is 2.00% . Values of wealth are in thousands.

Optimal asset allocation in Case 4.3 with no bequest assumption is full investment in equities for all reasonable values of pension wealth and age. Less than full equity investment is optimal for higher wealth and age but for this combination the pension wealth is already mostly annuitised and thus the pension wealth available for investment is already significantly decreased.

If the pensioner has the bequest motive, optimal equity and bond investment is influenced by the pensioner's risk aversion. In Figure 4.7, on the left hand side surface, we see that for the less risk averse pensioner with $\gamma = -1$, optimal equity investment is 100% in equities. However, if the pensioner has RRA coefficient $\gamma = -9$, then the left hand side surface in Figure 4.7 shows that, apart for very low values of pension wealth, optimal equity allocation is decreasing with age and is not significantly influenced by the amount of pension wealth available for investment.

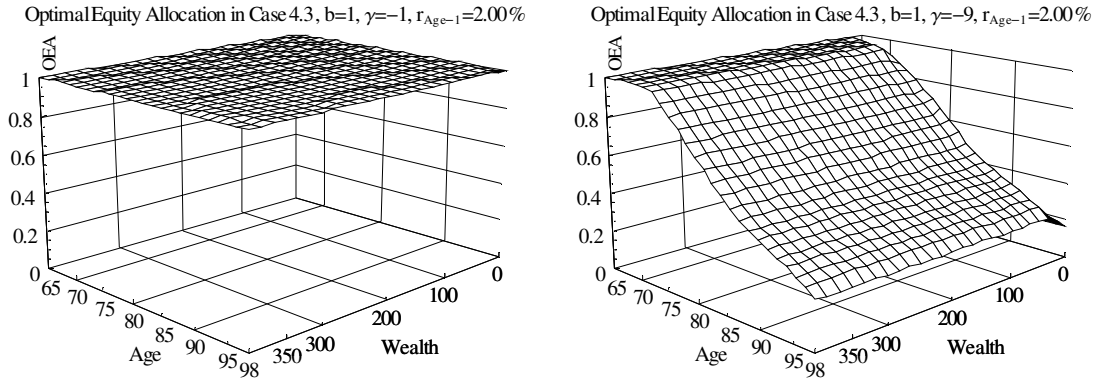


Figure 4.7 Optimal equity allocation in Case 4.3, for RRA coefficient γ taking values -1 , and -9 , and for bequest motive coefficient b_t taking value 1 for $65 \leq t \leq 99$. Interest rate in the previous year is 2.00%. Values of wealth are in thousands. Values of optimal equity allocation are proportions from 0 to 1

Figure 4.8 shows that all pension wealth not invested into equities is optimally invested in bonds.

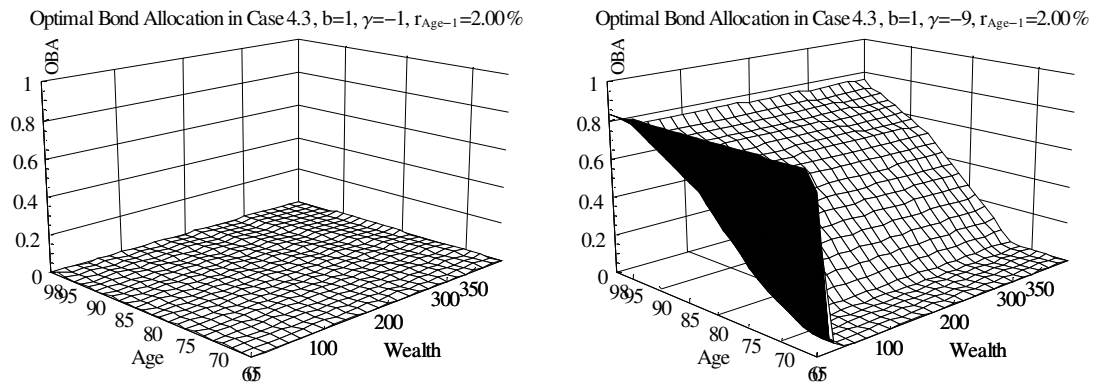


Figure 4.8 Optimal bond allocation in Case 4.3, for the values of RRA coefficient γ taking values -1 , and -9 , and for bequest motive coefficient b_t taking values 1 for $65 \leq t \leq 99$. Values of wealth are in thousands. Optimal bond allocation values are proportions from 0 to 1.

Regarding optimal annuitisation in Case 4.3 presented in Figure 4.9, we have significantly different shapes of the surfaces. If we observe fixed age and different values of pension wealth then optimal annuitisation changes significantly for all ages for the pensioner with the bequest motive only. For the pensioner with no bequest motive, optimal annuitisation for a given age changes significantly with changes of the values of wealth just for some ages. On the two upper surfaces, we observe that the less risk averse pensioner will defer annuitisation for a couple of years after retirement. How many years this pensioner will optimally defer annuitisation depends

on pension wealth, risk aversion and a bequest motive. On two lower surfaces in Figure 4.9, we observe that the more risk averse pensioner will optimally annuitise part of his pension wealth at the time of retirement for all reasonable values of pension wealth.

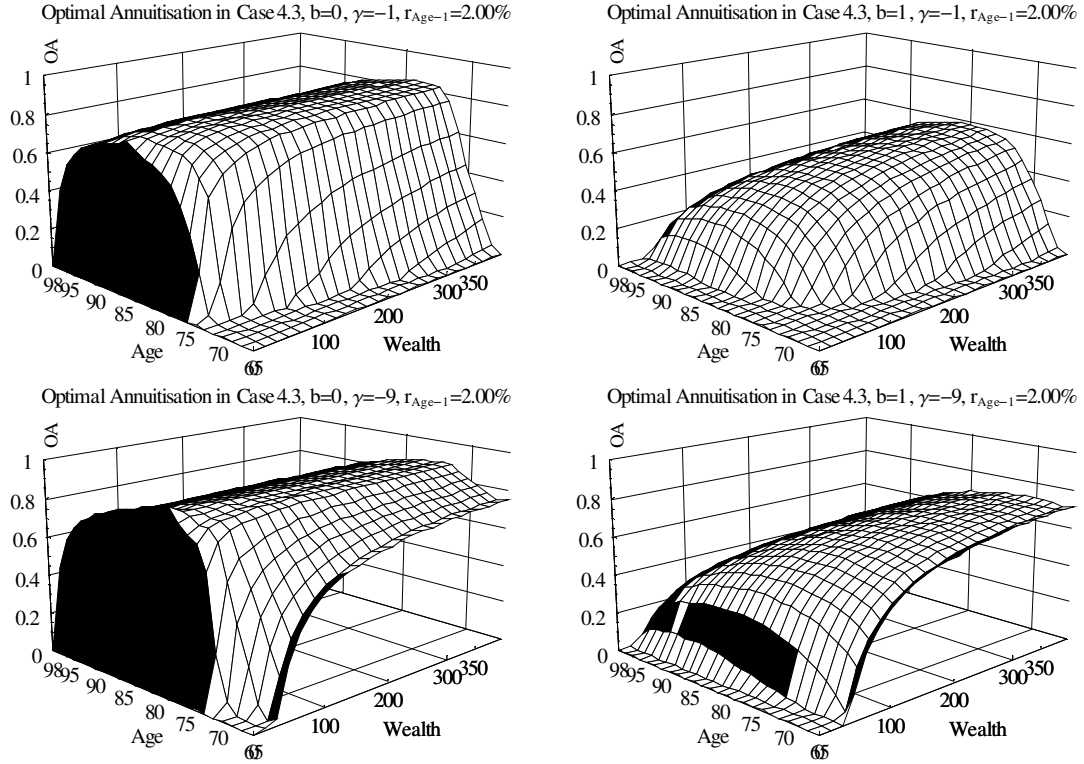


Figure 4.9 Optimal annuitisation in Case 4.3, for the values of RRA coefficient γ taking values -1 , and -9 , and for bequest motive coefficient b_t taking values 0 and 1 for $65 \leq t \leq 99$. Values of wealth are in thousands. Optimal annuitisation values are proportions from 0 to 1 .

Again, we should be aware that Figures 4.7, 4.8 and 4.9 show results when the income at age 65 is $Y_{65} = 33,320.90$, replacement ratio is $\rho_{65} = 0.68212$, and income is $Y_t = 22,728.85$ for $66 \leq t \leq 99$. As result of purchasing annuities, income increases and precise reading of the values on the mentioned figures above should be done using similar technique as explained in Section 4.5.2.2 and using equations(4.59), and (4.65)–(4.69).

4.5.2.4 Dependence on Income

Similarly to our conclusions earlier in Section 4.5, an increase/decrease of the value of income variable Y_t pulls/squeezes the surfaces of optimal consumption towards larger/lower values on the pension wealth axis and moves the whole surface up/down.

The overall shape of the surfaces stays the same. This conclusion can be drawn from the relations given in (4.59) and (4.65)–(4.68). In Figure 4.10, we present optimal consumption for the pensioner with risk preferences $\gamma = -1$ and $b_t = 0$, and $\gamma = -9$ and $b_t = 1$, and for two different levels of income, $Y_{t,1} = 22,728.85$ and $Y_{t,2} = 1.5 \cdot Y_{t,1} = 34,039.27$. This is increase of the value of income for 50%.

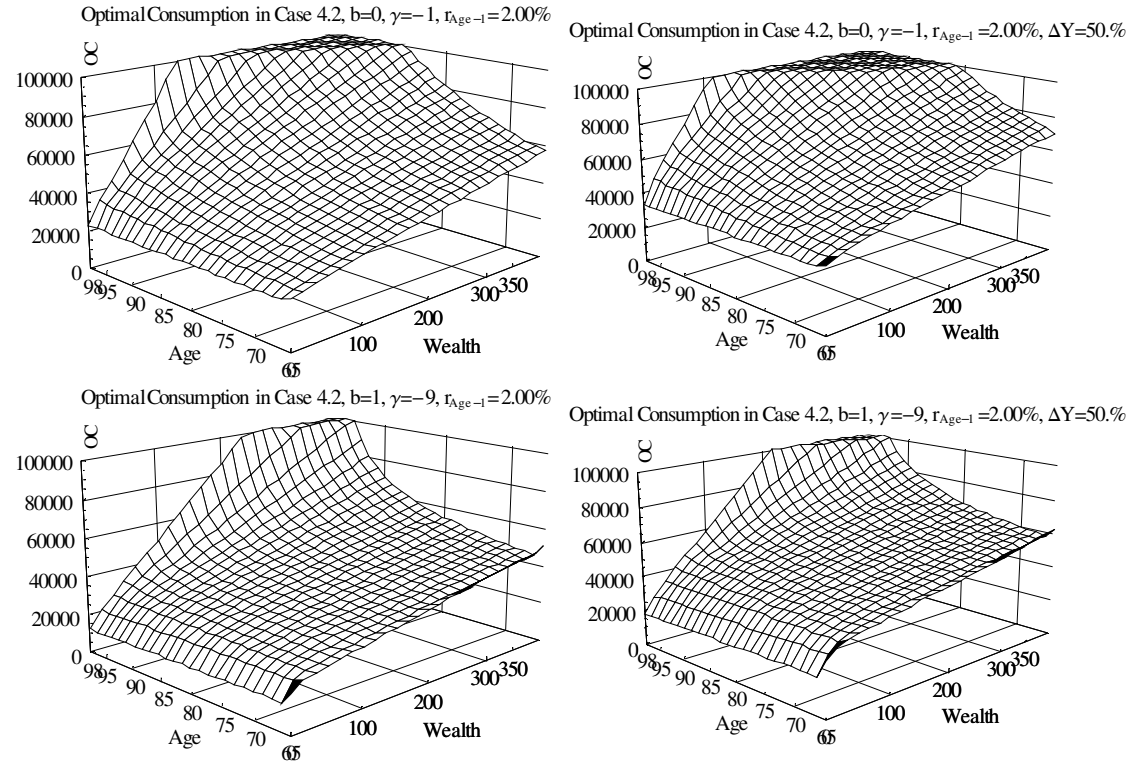


Figure 4.10 Optimal consumption in Case 4.2, for the pensioners with $\gamma = -1$ and $b_t = 0$, and $\gamma = -9$ and $b_t = 1$. Interest rate in the previous year is 2.00%. Income on the left hand side surfaces is $Y_{t,1} = 22,728.85$, and income on the right side surfaces is $Y_{t,2} = 1.5 \cdot Y_{t,1} = 34,039.27$. Values of wealth are in thousands. Optimal annuitisation values are proportions from 0 to 1.

In Figure 4.10, we can observe the effect of the change of income. Income on the left hand side surfaces is $Y_{t,1} = 22,728.85$ and these two surfaces are the same ones as in Figure 4.4 for the appropriate values of RRA and bequest motive coefficients. The surfaces on the right hand side in Figure 4.10 have income increased by 50% and all other parameters are the same as for the surfaces on the left hand side.

If the income variable Y_t increases and other variables remain the same, then the surfaces of optimal annuitisation and equity and bond allocation will be pulled on the wealth axis towards larger values, while keeping its shape. Similarly, if income variable Y_t decreases keeping other variables the same, then the surfaces will be

squeezed, again keeping their shape. In Figure 4.11, the surfaces in each row differ in the value of income.

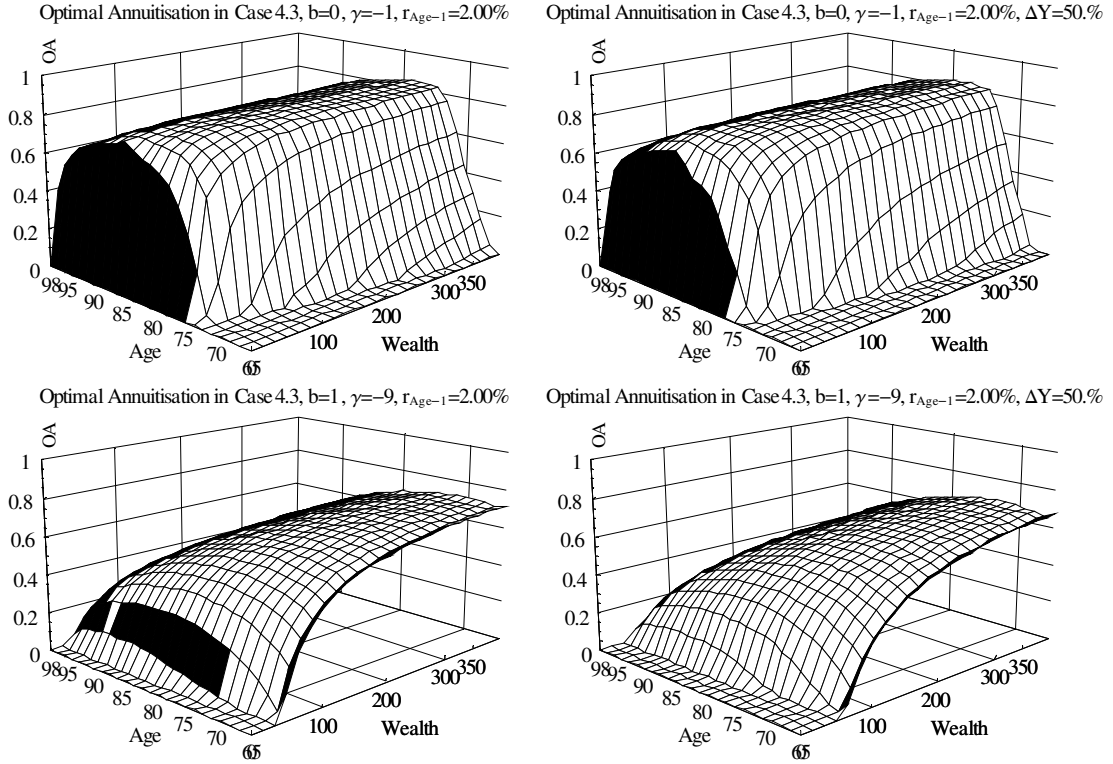


Figure 4.11 Optimal annuitisation in Case 4.3, for the pensioners with $\gamma = -1$ and $b_t = 0$, and $\gamma = -9$ and $b_t = 1$. Interest rate in the previous year is 2.00%. Income on the left hand side surfaces is $Y_{t,1} = 22,728.85$, and income on the right side surfaces is $Y_{t,2} = 1.5 \cdot Y_{t,1} = 34,039.27$. Values of wealth are in thousands

We observe in Figure 4.11, in either the upper or lower pair of surfaces, that the value of optimal annuitisation for a given age and wealth on the left hand side surface is the same as the value on the right hand side surface for the same age but for a 50% larger value of pension wealth.

4.5.2.5 Dependence on the value of Interest Rate

Optimal consumption does not show a significant dependence on the known interest rate in the previous year. The pensioner in any one of the investigated cases will experience relative differences of up to 3% in the values of optimal consumption. Optimal consumption is closely related to optimal asset allocation and annuitisation and the changes in optimal consumption can be explained in the context of optimal asset allocation and annuitisation only. We need to observe income at the beginning

of the year as implied risk free asset in a possession of the pensioner. As interest rate increases the value of this implied risk free asset decreases. On the other side, the pensioner expects better return on his pension wealth only if optimal asset allocation for a given value of pension wealth includes investment in risk free investment, bonds or annuities. So, if the pensioner possesses pension wealth such that it is optimal to invest 100% or almost 100% into equities than his optimal consumption decreases because his implied risk free assets decrease in value and his perspective of investment returns stays the same. However, if optimal asset allocation includes a higher proportion of risk free or bond investment or annuities, then his expected return on investment increases the value of interest rate increases. If this is the case then, for enough high values of pension wealth, his optimal consumption increases as the value of interest rate increases because higher expected return due to the higher expected return on risk free investment, bonds or annuities provides him with a higher overall wealth (including implied assets from future income). As a result of these combined effects the patterns of changes in optimal consumption as the value of interest rate increases are different from case to case. For example, in Case 4.1 for the more risk averse pensioner with no bequest motive, for a lower value of pension wealth, it is optimal to invest 100% in equities and optimal consumption decreases as the value of interest rate increases. However, for the higher values of the pension wealth for this pensioner, it is optimal to increase consumption as interest rate increases. If we observe the more risk averse pensioner with a bequest motive in Case 4.1, then it is optimal to increase consumption for very low values of the pension wealth, then for a certain range of the higher values of the pension wealth it is optimal to decrease consumption, and then after that range for further higher values of the pension wealth it is optimal to increase consumption as the value of interest rate increases. Thus, for a given case and the pensioner's preferences towards risk and bequest, and for a given value of the pension wealth, we need to observe optimal asset allocation and annuitisation and to draw conclusion if the optimal consumption will increase or decrease as the value of the interest rate increases.

In Figure 4.12, we present the changes of optimal equity allocation in Case 4.1 due to the changes of the value of the interest rate in the year preceding the pensioner's ages of 65 and 80.

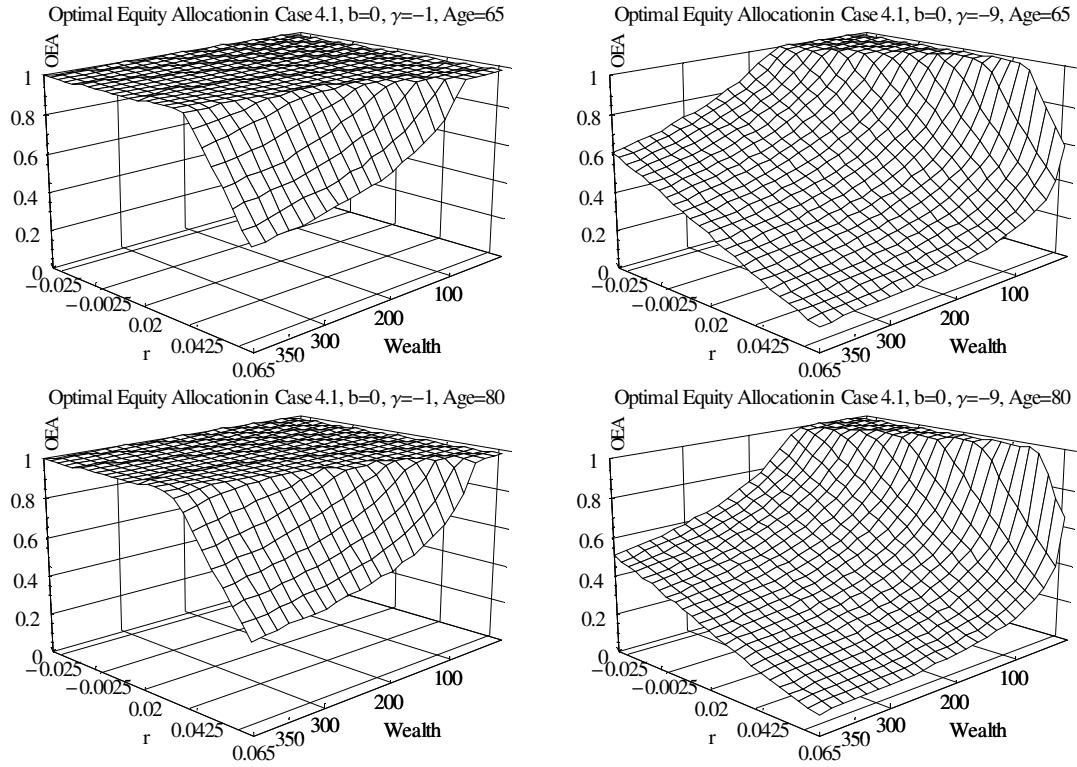


Figure 4.12 Dependence of optimal equity allocation on interest rate for ages 65 and 80 in Case 4.1, RRA coefficient γ taking values -1 , and -9 , $b_t = 0$. Wealth values are in thousands. Optimal equity allocation values are proportions from 0 to 1.

In the upper right hand side surface in Figure 4.12, optimal equity allocation changes from 20% to 90% for pension wealth of about 200,000 units. On the upper left hand side surface the optimal equity allocation changes from 70% to 100% for pension wealth of about 200,000 units and it is 100% for small pension wealth values. We observe that the upper and the lower surfaces on the left hand side and the upper and the lower surfaces on the right hand side have a similar shapes and that the surfaces have a slightly lower position for the higher ages. The differences in values are up to about 10%.

In Figure 4.13, we present the changes of optimal equity allocation in Cases 4.3 with the changes of the value of the interest rate in the year preceding the pensioner's ages of 65 and 80.

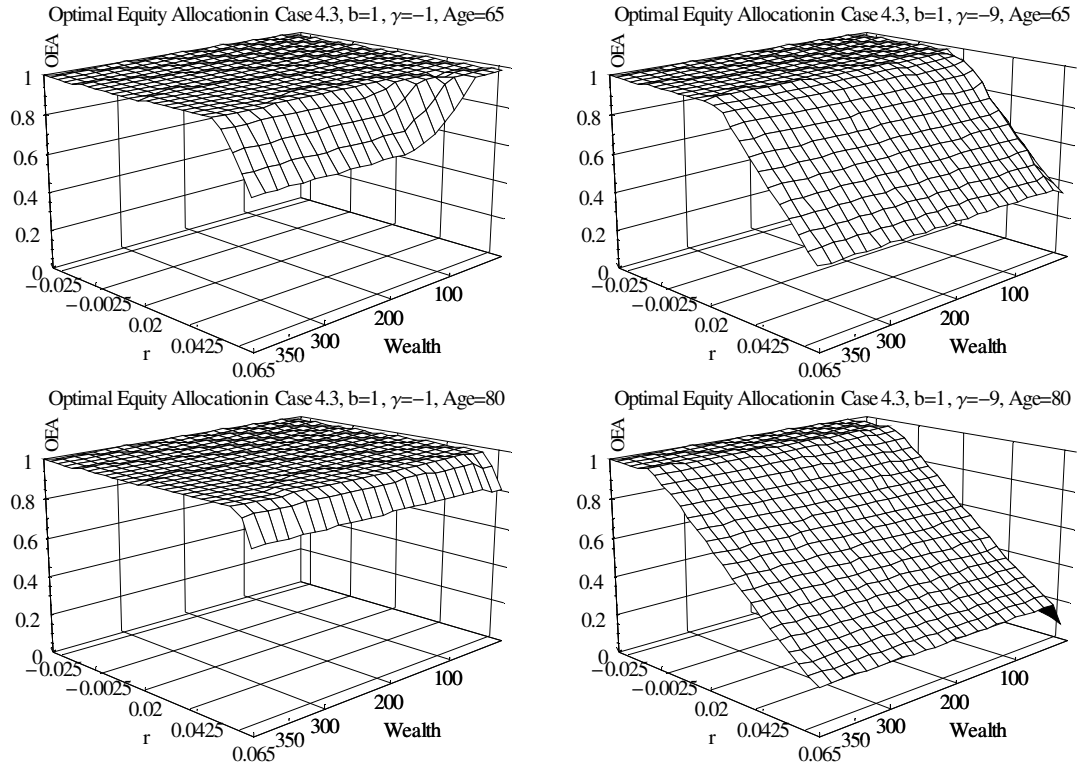


Figure 4.13 Dependence of optimal equity allocation on interest rate for ages 65 and 80 in Case 4.3, RRA coefficient γ taking values -1 , and -9 , with a bequest. Wealth values are in thousands. Optimal equity allocation values are proportions from 0 to 1.

In Figure 4.13 we see that in Case 4.3 the optimal equity allocation surfaces, as functions of interest rate and wealth, change significantly with age. These changes depend on the combinations of the risk and bequest parameters.

For $\gamma = -1$, for the lower values of pension wealth the pensioner at age 65 optimally invests all available pension wealth into equities for all values of interest rate. For the values of pension wealth larger than 50,000, he optimally invests less in equities for the higher values of interest rate and 100% for the lower values of interest rate. However, the same pensioner invests similarly at age 80 for all pension wealth values. Thus, we have two surfaces with different patterns on the left hand side in Figure 4.13.

For the less risk averse pensioner, with $\gamma = -9$, we find that both surfaces have similar shapes with the following characteristics. Optimal equity allocation does not depend on pension wealth apart from very small pension wealth values. The percentage of the pension wealth invested into equities significantly depends on the known interest rate in the year preceding the year of investment and on the age of the pensioner. The pensioner with $\gamma = -9$ and $b_t = 1$ will optimally invest into equities

40–100% at age 65, while the same pensioner will optimally invest into equities 20–100% of his pension wealth at age 80. We observe that the pensioner aged 65 will invest roughly 20% more into equities than the pensioner aged 80.

In Figures 4.12 and 4.13, we find that interest rate significantly influences optimal equity allocation for different combinations of parameters. It is the expected result because bond prices and the annuity rate directly depend on the known interest rate during the previous year. For the higher values of interest rate, the bond prices are lower, and consequently the bonds and annuities become more attractive. We can observe on all surfaces that optimal equity allocation decreases as the value of the interest rate increases. However, the degree of the changes of optimal equity allocation with the changes of the values of interest rate depends significantly on the value of the parameters.

In Figure 4.14, we show optimal annuitisation for the two pensioners, one with RRA coefficient $\gamma = -1$ and the other with RRA coefficient $\gamma = -9$, and both pensioners have the bequest motive. Optimal annuitisation is presented for ages 65 and 75.

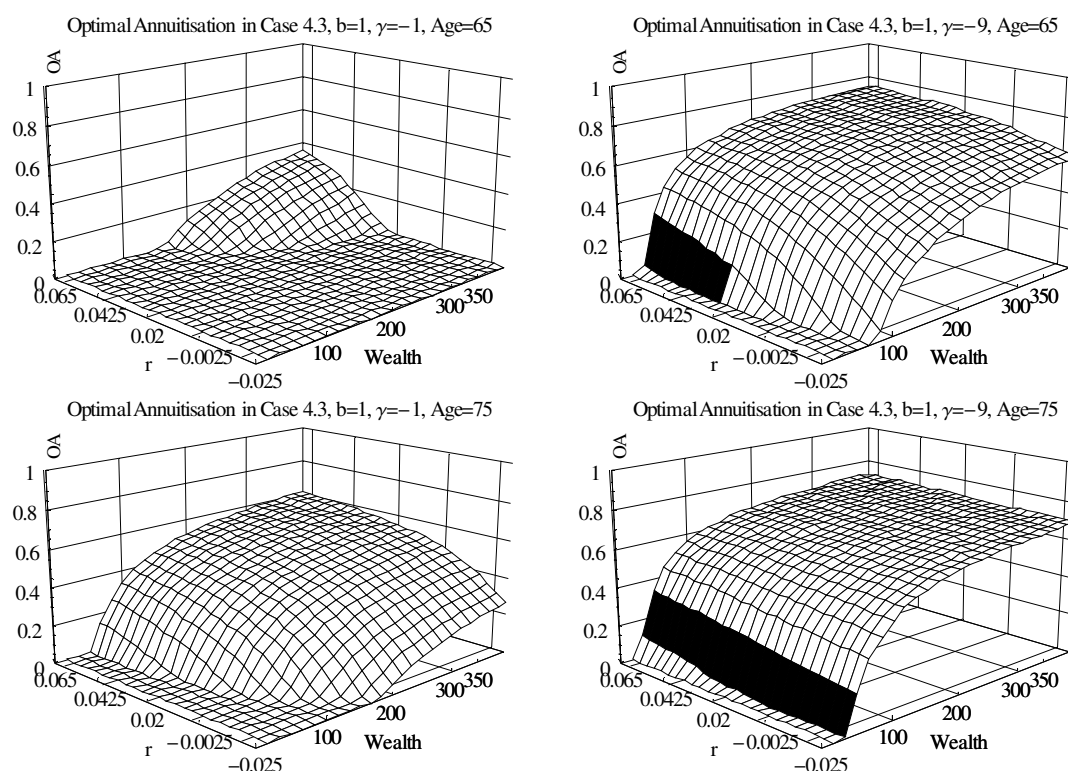


Figure 4.14 Dependence of optimal annuitisation on interest rate for ages 65 and 75 in Case 4.3, RRA coefficient γ taking values -1 , and -9 , with bequest. Wealth values are in thousands. Values of optimal annuitisation are proportions from 0 to 1.

For the investigated values of pension wealth, the pensioner with $\gamma = -1$ and $b_t = 1$ will optimally annuitise at age 65 only if the interest rate in the previous year is favourable. Otherwise, it is optimal for this pensioner to defer annuitisation. The same pensioner at age 75 will optimally annuitise part of his pension wealth if his pension wealth is reasonably large. Again, annuities are more attractive for this pensioner if the value of the interest rate is attractive.

The pensioner with $\gamma = -9$ and $b_t = 1$ optimally annuitises part of his pension wealth for all reasonable values of pension wealth. However, we observe the decrease of the values of optimal annuitisation as interest rate decreases. It means that the pensioner defers annuitisation partly if interest rate is not favourable and purchases annuities in the later years of retirement when he expects the values of interest rate to be favourable. At age 75, optimal annuitisation does not significantly depend on interest rate. The pensioner with $\gamma = -1$ and $b_t = 1$ will optimally annuitise at age 75 depending on his available pension wealth only

In Figure 4.14, we observe that at age 65, the pensioner will optimally annuitise a certain part of his pension wealth. If interest rate is favourable, he annuitises more and if not then he defers annuitisation partly or in full. This is the general pattern of optimal annuitisation in Case 4.3 for all investigated combinations of the values of the parameters. However, at later ages annuities are more advantageous for the pensioner, due to the higher value of mortality drag. Thus, at later ages the pensioner's demand for annuities increases and is less sensitive to the value of the interest rate during the year before annuitisation. The largest advantage in deferring annuitisation is in the early years in retirement.

More results related to changing interest rate in the year prior to retirement will be presented in Sections 4.5.5 and 4.5.7 and also in Chapter 5. In Chapter 5 we compare the results in Chapter 3 for Cases 3.1, 3.3 and 3.5 and the results in Chapter 4 for Cases 4.1, 4.2 and 4.3.

4.5.3 The Typical Example of Simulation

In this chapter, we determine the control variables (optimal consumption, asset allocation and annuitisation) such that the pensioner's expected derived utility is maximised. All control variables are functions and we obtain these functions using numerical mathematics. Once calculated, control variables are stored on the computer

and ready for further investigation. Control variables allow us to make stochastic simulations and to investigate different realisations of the stochastic simulations.

For each case and for each combination of parameters γ and b_t given in Section 4.5.1, we make 2,000 random realisations of interest and equity rate. Then we calculate 2,000 random realisations of control variables and derived utility, as well as all other variables of interest. By investigating these random realisations, we get a clearer idea about the pensioner's optimal behaviour. From the sample of random realisations, we calculate mean value, quantiles and other statistics of any interesting variable.

In this section we present one typical solution obtained from stochastic simulations of different random paths of interest rate, equity rates, and paths for all other variables in the model. We assume that pension wealth at age 65 is $W_{65} = 200.000$, income at age 65 is $Y_{65} = 33.321$, interest rate prior to retirement is 2.00%. The size of the sample of random realisations is 2,000. The following four graphs show the mean values and 0.05 and 0.95 quantiles of the pensioner's optimal behaviour.

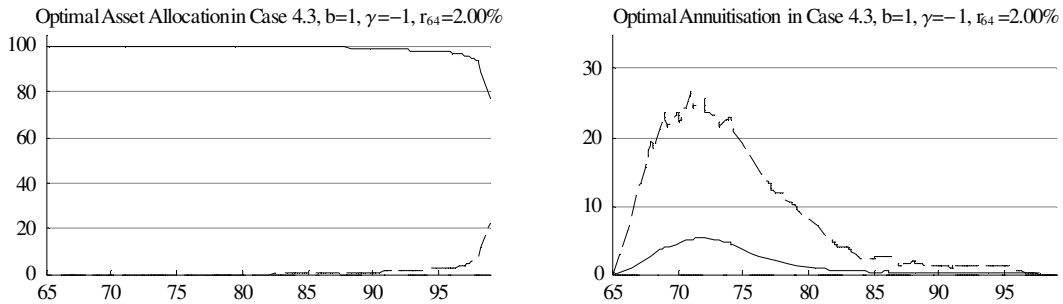


Figure 4.15 Mean optimal asset allocation and mean optimal annuitisation for the pensioner in Case 4.3, with $b_t = 1$ and $\gamma = -1$. Mean optimal equity allocation (solid line), mean optimal bond allocation (dashed line), optimally no cash in the left hand side graph. Mean (solid line) and 95% quantile (dashed line) of optimal annuitisation in the right hand side graph.

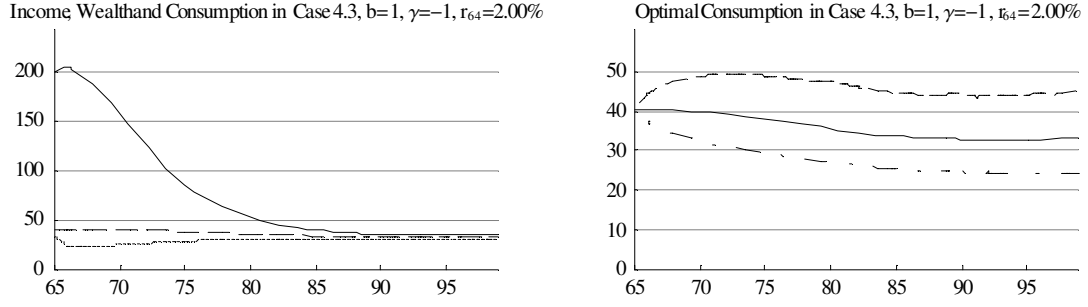


Figure 4.16 Mean income, mean wealth and mean optimal consumption for the pensioner in Case 4.3, with bequest motive $b_t=1$ and for RRA coefficient $\gamma=-1$. Mean income (dash line with shorter dashes), mean wealth (full line) and mean optimal consumption (dash line with longer dashes) in the left hand side graph, and mean (full line), 5% (dash line) and 95% (dash and dot line) quantiles of optimal consumption in the right hand side graph.

4.5.4 Criteria for Comparing Results

In this chapter we apply the same criteria as in Chapter 3. We have two groups of criteria (measures). In the first group, we have Constant Equivalent Consumption – CEC , and Required Equivalent Wealth – REW measure. We apply these criteria to expected discounted utility derived from consumption and bequest. The second group of measures consists of Value at Risk – VaR_α and Conditional Value at Risk – $CVaR_\alpha$, for $0 < \alpha < 1$. We apply the criteria from the second group to the random discounted utility derived from consumption and bequest. If we have the results in terms of utility units then the degree of pensioner's gains or loss is not clear. That is why we convert and present all measures in money terms.

Regarding CEC measure, we follow the derivation in Section 3.4.4 and obtain the same formula

$$C_{CEC} = \left(\frac{\mathcal{N}_{65}}{\sum_{t=65}^{99} \left(\delta^{t-65} \left(\prod_{j=65}^{t-1} p_j \right) \right) \right)^{\frac{1}{\gamma}}. \quad (4.83)$$

and

$$C_{CEC} = \left(E \left[\frac{\sum_{t=65}^{99} \delta^{t-65} \left(\prod_{j=65}^{t-1} p_j \right) (\tilde{C}_t)^\gamma}{\sum_{t=65}^{99} \left(\delta^{t-65} \left(\prod_{j=65}^{t-1} p_j \right) \right)} \right] \right)^{\frac{1}{\gamma}} \quad (4.84)$$

where $\tilde{C}_t$ is random consumption at age t , for $65 \leq t \leq 99$, and $\tilde{C}_{65}$ is optimal consumption at age 65. Thus, $\tilde{C}_{65}$ is control variable.

We can calculate the *CEC* measure for any case and for any reasonable values of the parameters and as a result we get a single non-negative number. The pensioner is better off if C_{CEC} is higher. If we determine the values of the *CEC* measure for two comparable examples, then the pensioner is better off in the example where the value of the *CEC* measure is higher. Now, we can also observe the difference between the two values of the *CEC* measure and get an idea of how much better off the pensioner is in one example to the next. Two comparable examples can be the results in different cases, while all other assumptions are the same or the two results with different values of interest rate in the year before retirement and all other assumptions the same.

Required equivalent wealth – *REW* measure is the second measure. Having solved the problem (4.48)–(4.55) we get the value function

$$V_{65}(W_{65}, Y_{65}, r_{64;j}) \quad (4.85)$$

$W_{65} \geq 0$ and $Y_{65} \geq 0$, and $r_{64;j}$ for $j=1, \dots, n_r$ in the domain of the value for interest rate.

Function $V_{65}(W_{65}, Y_{65}, r_{64;j})$ is increasing function with respect to variable W_{65} . For given values $V_{65}(W_{65}, Y_{65}, r_{64;j})$, $Y_{65} \geq 0$ and $r_{64;j}$ we can calculate the inverse function with respect to variable W_{65} . *REW* measure can be calculated from two comparable examples only. Let us suppose that two comparable examples are two different cases, while all other assumptions in examples are the same. Let us suppose that we have expected discounted utility ${}_1V_{65}({}_1W_{65}, Y_{65}, r_{64;j})$ in the first case, where ${}_1W_{65} \geq 0$ and $Y_{65} \geq 0$, and $r_{64;j}$ are known values. Then, we can calculate ${}_2W_{65}$ such that

$${}_2V_{65}({}_2W_{65}, Y_{65}, r_{64;j}) = {}_1V_{65}({}_1W_{65}, Y_{65}, r_{64;j}) \quad (4.86)$$

where ${}_2V_{65}({}_2W_{65}, Y_{65}, r_{64;j})$ is expected discounted utility function in the second case. Thus, we get the amount of wealth in the second case such that expected discounted utility is the same in both cases. Now, we compare ${}_1W_{65}$ and ${}_2W_{65}$, and we can conclude which one of two comparable cases is favourable for the pensioner. If ${}_1W_{65} > {}_2W_{65}$ then the pensioner in the second case can derive the same utility as the pensioner in the first case but with the lower value of initial pension wealth. So, the second case is more favourable for pensioner. We get in money term how much one case is more favourable than the other. If the opposite is true, i.e. if ${}_1W_{65} < {}_2W_{65}$, then the first case is more favourable for the pensioner. If ${}_1W_{65} = {}_2W_{65}$, then the pensioner is, in terms of expected discounted utility derived from future consumption and bequest, indifferent between the two cases.

We prefer to use *REW* measure compared to *CEC* measure because *REW* measure takes into account the expected discounted utility from both consumption and bequest while *CEC* measure takes into account expected discounted utility from consumption only. However, we will show the results in terms of both *CEC* and *REW* measures.

Value at Risk and Conditional value at risk are measures of the pensioner's left tail risk. We calculate VaR_α and $CVaR_\alpha$ from a random utility derived from discounted future random consumption and bequest. We follow the same steps as in Chapter 3, and introduce a new random variable $\tilde{D}(W_{65}, Y_{65}, r_{64;j})$ such that

$$\tilde{D}(W_{65}, Y_{65}, r_{64;j}) = \sum_{t=65}^{99} \left(\delta^{t-65} \left(\prod_{i=65}^{t-1} p_i \right) \left(\frac{(\tilde{C}_t)^\gamma}{\gamma} + \delta(1-p_t)b_t \frac{(\tilde{W}_{t+1})^\gamma}{\gamma} \right) \right) \quad (4.87)$$

where $W_{65} \geq 0$ and $Y_{65} \geq 0$, $r_{64;j}$ for $j=1, \dots, n_r$, in the domain the values of interest rate, and where $\tilde{C}_{65}$ is control variable, $\tilde{C}_t$ are random variables for $66 \leq t \leq 99$. The *PDF* or *CDF* of the random variable $\tilde{D}(W_{65}, Y_{65}, r_{64;j})$ cannot be found analytically. However, we can make a number of random realisations of this random variable and then calculate approximate values of VaR_α and $CVaR_\alpha$ from the random realisation. Obviously, one property that we expect to be satisfied in all examples when we calculate right hand side of (4.87) numerically comes from the very first definition of the value function and $\tilde{D}(W_{65}, Y_{65}, r_{64;j})$ and it is

$$V_{65}(W_{65}, Y_{65}, r_{64;j}) \approx E[\tilde{D}(W_{65}, Y_{65}, r_{64;j})].$$

This is the rule that we use in Section 4.4.4 for checking the accuracy of the results.

Once we have a random variable $\tilde{D}(W_{65}, Y_{65}, r_{64;j})$ in the form of a random sample, we can use it to calculate approximate values of VaR_α and $CVaR_\alpha$ as measures of the risk that the pensioner derives lower than expected discounted utility. In Section 4.5.6, we define this measure precisely and investigate the results.

4.5.5 CEC and REW Measures Applied

In this section, we investigate expected discounted utility using *CEC* and *REW* measures for different cases and for different values of the parameter. The parameters that we change here are the RRA coefficient γ and the bequest motive coefficient b_t . In order to focus on the analysis of the different cases, we firstly investigate the results for the value of the interest rate in the year prior to retirement $r_{64} = 2.00\%$. In the second part of this section, we present some results for different values of the interest rate. As we note at the end of this section, a deeper investigation related to the different values of interest rate is done in Chapter 5.

Before investigating *CEC* and *REW* measures, we present the mean consumption and mean wealth paths for $\gamma = -9$ and $b_t = 0$ and for $\gamma = -1$ and $b_t = 1$.

In Figure 4.17, we show the mean values of pension wealth paths in different Cases, for the pensioner with $\gamma = -9$ and $b_t = 0$, and for initial pension wealth 200,000 and interest rate in the year before retirement $r_{64} = 2.00\%$.

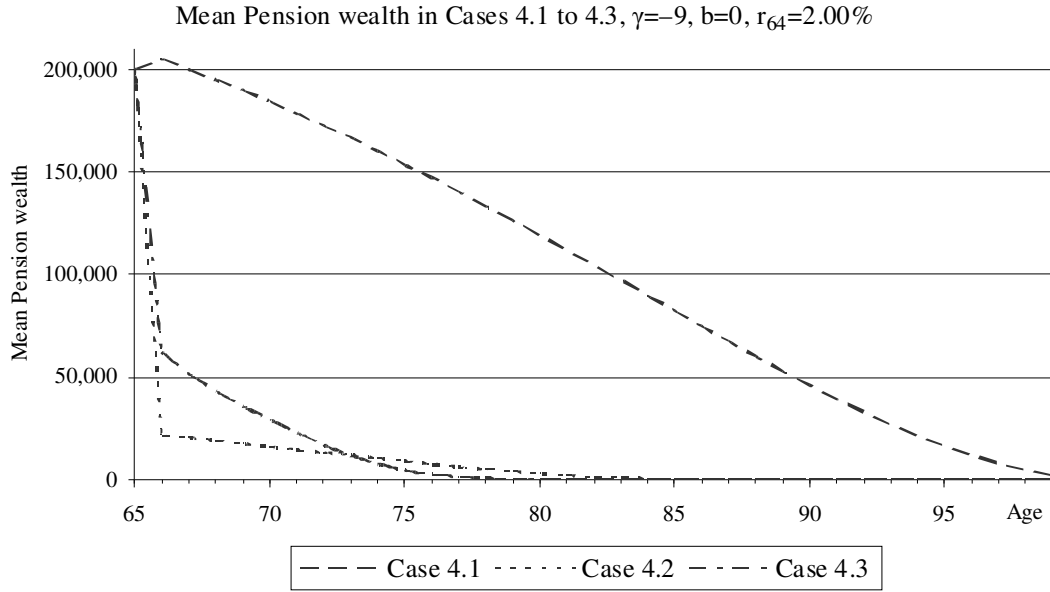


Figure 4.17 Mean pension wealth development in the retirement in Cases 4.1, 4.3 and 4.3, for $\gamma = -9$ and $b_t = 0$. The value of the interest rate during the year prior to retirement is 2.00%. Initial pension wealth is 200,000.

In Figure 4.18, we present the mean values of optimal consumption in different Cases, for the pensioner with $\gamma = -9$ and $b_t = 0$, and for initial pension wealth 200,000 and interest rate in the year before retirement $r_{64} = 2.00\%$.

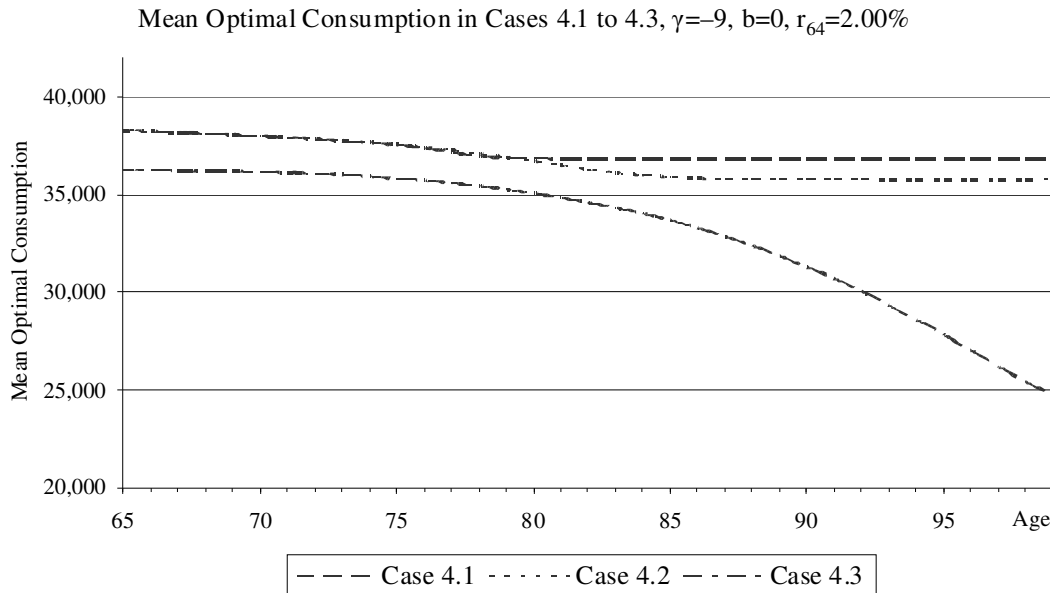


Figure 4.18 Mean optimal consumption development in the retirement in Cases 4.1. 4.2 and 4.3, for $\gamma = -9$ and $b_t = 0$. The value of the interest rate during the year prior to retirement is 2.00%. Initial pension wealth is 200,000.

We observe very different paths of the values of mean pension wealth and consumption. *CEC* and *REW* measures summarise into a single number the complexity of these future developments.

We observe that both mean pension wealth and mean consumption paths are very different from case to case. In Case 4.2, where optimal annuitisation is allowed at age 65 only, it is optimal to annuitise almost 90% of pension wealth. None of the pension income is saved afterwards, which means that after age 65 the pensioner consumes all his income from social security and annuities. The mean values of pension wealth and consumption in Case 4.3 have similar paths as in Case 4.2, but less annuitisation is done at age 65 in Case 4.3 and some annuitisation is done afterwards as well. In Figure 4.18, we observe that Case 4.1 is the worst one in terms of mean values of optimal consumptions.

In Figures 4.19 and 4.20, we present the same results as in figures 4.17 and 4.19 but now for the pensioner with $\gamma = -1$ and $b_t = 1$.

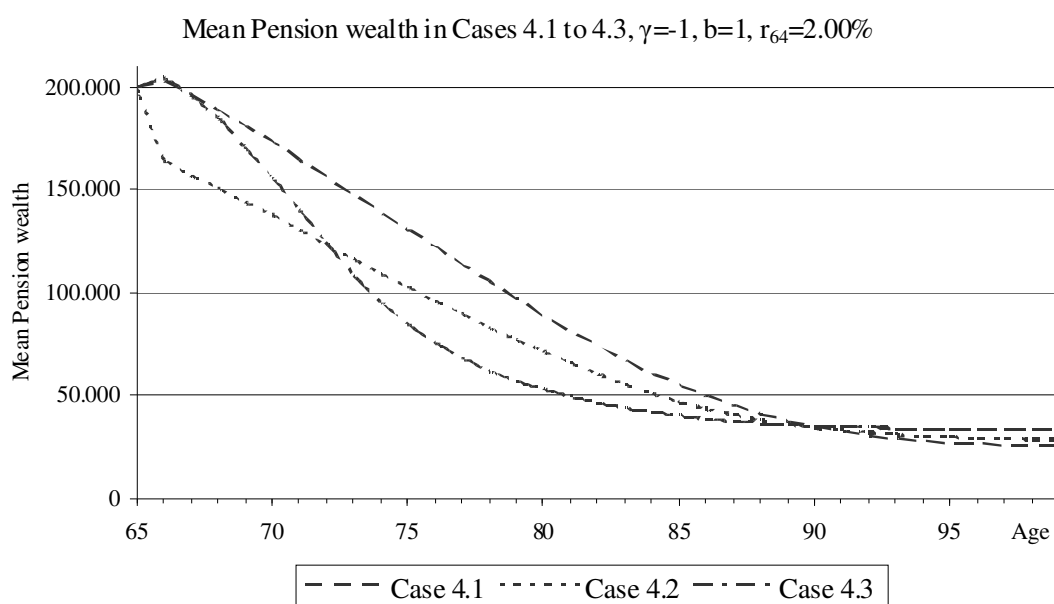


Figure 4.19 Mean pension wealth development in the retirement in Cases 4.1, 4.2 and 4.3, for $\gamma = -1$ and $b_t = 1$. The value of the interest rate during the year prior to retirement is 2.00%. Initial pension wealth is 200,000.

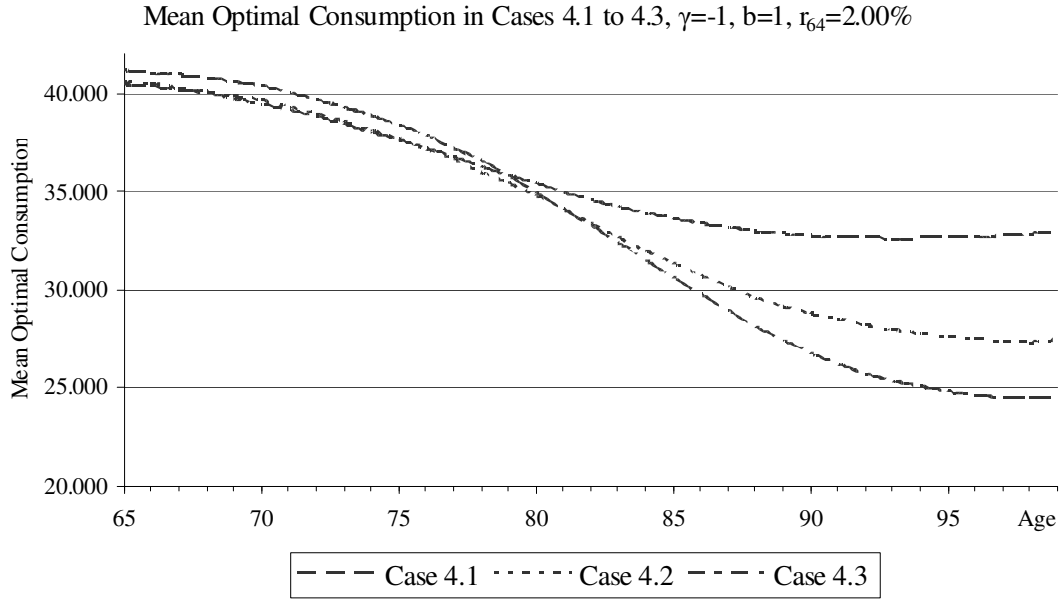


Figure 4.20 Mean optimal consumption development in the retirement in Cases 4.1, 4.2 and 4.3, for $\gamma = -1$ and $b_t = 1$. The value of the interest rate during the year prior to retirement is 2.00%. Initial pension wealth is 200,000.

The differences between mean pension wealth paths in Figure 4.19 are less than in Figure 4.17. There is a fewer annuitisation in both Cases 4.2 and 4.3 in Figure 4.19. For example, optimal annuitisation in Case 4.2 in Figure 4.19 is 18%. Regarding mean optimal consumption paths, we observe a wider range of values in Figure 4.20 than in Figure 4.18. If we observe case by case, then in each case mean optimal consumption is higher in Figure 4.20 than in Figure 4.18 for the lower ages and lower for the higher ages..

Table 4.7 shows the *CEC* measure for different bequest and RRA coefficients. The value of the interest rate in the year prior to retirement is equal to 2.00% for each calculated value. Initial pension wealth is 200,000 money units and all values of *CEC* measure are calculated at age 65.

In Table 4.8 we present the relative changes of *CEC* measure in Table 4.7 in Cases 4.2 and 4.3 to Case 4.1, for the same values of γ and b_t . Percentages changes presented in Table 4.8 are calculated using the formula

$$\frac{\frac{row(i)}{Case\ 4.j} C_{CEC} - \frac{row(i)}{Case\ 4.1} C_{CEC}}{\frac{row(i)}{Case\ 4.1} C_{CEC}}$$

for $1 \leq i \leq 6$ and $2 \leq j \leq 3$.

	Bequest and RRA parameters	No annuity	Optimal annuities at 65, no annuities	Optimal annuities at 65 and afterwards
		Case 4.1	Case 4.2	Case 4.3
1	$b_t = 0 \quad \gamma = -1$	37,597	37,958	38,322
2	$b_t = 0 \quad \gamma = -4$	35,761	37,583	37,752
3	$b_t = 0 \quad \gamma = -9$	34,205	37,457	37,541
4	$b_t = 1 \quad \gamma = -1$	35,977	36,041	36,237
5	$b_t = 1 \quad \gamma = -4$	35,046	36,328	36,438
6	$b_t = 1 \quad \gamma = -9$	33,641	36,114	36,185

Table 4.7 *CEC* measure in amounts – Values in the cell show *CEC* measure for different cases and different pensioner's preferences towards risk and bequest. Assumed interest rate during the year prior to retirement is 2.00%. Initial pension wealth is 200,000 money units. Pensioner's age is 65.

	Bequest and RRA parameters	Optimal annuities at 65, no annuities	Optimal annuities at 65 and afterwards
		Case 2	Case 3
1	$b_t = 0 \quad \gamma = -1$	0.96%	1.93%
2	$b_t = 0 \quad \gamma = -4$	5.10%	5.57%
3	$b_t = 0 \quad \gamma = -9$	9.51%	9.75%
4	$b_t = 1 \quad \gamma = -1$	0.18%	0.72%
5	$b_t = 1 \quad \gamma = -4$	3.66%	3.97%
6	$b_t = 1 \quad \gamma = -9$	7.35%	7.56%

Table 4.8 *CEC* measure in percentages – The values in cells show percentage difference between the case in the header of the column and Case 4.1, for the values of *CEC* measure in amounts given in Table 4.8.

Table 4.9 shows *REW* measures for one set of parameters such that all pension wealth in one row give to the pensioner the same expected discounted utility derived from future consumption and bequest. The benchmark wealth is in Case 4.1 and it is 200,000. The value of the interest rate in the year prior to retirement is 2.00%. Again, all calculations are done for the pensioner aged 65.

Similarly to *CEC* measure in percentages, we develop *REW* measure in percentages. Using values from Table 4.9 and the following formula

$$\frac{\frac{row(i)}{Case\ 4.1}W_{65} - \frac{row(i)}{Case\ 4.j}W_{65}}{\frac{row(i)}{Case\ 4.1}W_{65}}$$

for $1 \leq i \leq 6$ and $2 \leq j \leq 3$ we calculate the values presented in table 4.10. This is similar formula as for *CEC* measure in percentages, but with a negative sign in order to get positive percentages.

	Bequest and RRA parameters	No annuity	Optimal annuities at 65 only	Optimal annuities at 65 and afterwards
		Case 4.1	Case 4.2	Case 4.3
1	$b_t = 0 \ \gamma = -1$	200,000	194,880	189,941
2	$b_t = 0 \ \gamma = -4$	200,000	173,900	171,607
3	$b_t = 0 \ \gamma = -9$	200,000	153,270	152,165
4	$b_t = 1 \ \gamma = -1$	200,000	199,050	196,196
5	$b_t = 1 \ \gamma = -4$	200,000	181,005	179,441
6	$b_t = 1 \ \gamma = -9$	200,000	163,139	162,164

Table 4.9 *REW* in amounts – Values in the cell show wealth needed in Case shown in the column to obtain the same utility as 200,000 in Case 4.1. The value of the interest rate during the year prior to retirement is 2.00%. Initial pension wealth is 200,000.

	Bequest and RRA parameters	Optimal annuities at 65 only	Optimal annuities at 65 and afterwards
		Case 4.2	Case 4.3
1	$b_t = 0 \ \gamma = -1$	2.56%	5.03%
2	$b_t = 0 \ \gamma = -4$	13.05%	14.20%
3	$b_t = 0 \ \gamma = -9$	23.37%	23.92%
4	$b_t = 1 \ \gamma = -1$	0.48%	1.90%
5	$b_t = 1 \ \gamma = -4$	9.50%	10.28%
6	$b_t = 1 \ \gamma = -9$	18.43%	18.92%

Table 4.10 *REW* measure in percentages – Values in the cells show percentage difference between Case 4.1 and the cases shown in the first column. The values in Cases are taken from Table 4.10. The value of the interest rate during the year prior to retirement is 2.00%. Initial pension wealth is 200,000 money units. Pensioner's age is 65.

Again, as in the analysis of Tables 3.8 and 3.10 in Chapter 3, the differences are larger in Table 4.10 than in Table 4.8. The *CEC* measure does not include utility derived from a bequest, only from consumption. Thus, if we observe the pensioner with a bequest motive then we should use the results from Table 4.10 only.

Table 4.8 and 4.10 gives the results assuming the values of interest rate in the year prior to retirement to be 2.00%. The numbers are different but the conclusions in Case 3.1, 3.3 and 3.5 in Chapter 3 are similar to the conclusions in Cases 4.1, 4.2 and 4.3 here. As all the results in these two tables are based on the expected discounted utility and all assumptions apart from the difference in interest rate are the same, it is not surprising that the conclusions from Table 4.8 and 4.10 will be similar to the conclusions from Table 3.8 and 3.10, respectively.

We clearly see the importance of having access to annuities. In Tables 4.8 and 4.10 we observe that Cases 4.2 and 4.3 results are always more preferable than Case 4.1. Access to annuities always brings extra expected discounted utility for the pensioner. The exact amount of extra expected discounted utility either measured using *CEC* or *REW* measures significantly depends on the pensioner's preferences towards risk and the bequest motive.

Comparing any pair of the rows in Table 4.10, we find the largest difference between the results in the rows 3 and 4. In both Case 4.2 and 4.3, the pensioner with $\gamma = -9$ and $b_t = 0$, have the highest gains and the pensioner with $\gamma = -1$ and $b_t = 1$ has the lowest gains. In Figures 4.17 and 4.18, we present mean pension wealth and mean optimal consumption for the pensioner with $\gamma = -9$ and $b_t = 0$, and in Figures 4.19 and 4.20 for the pensioner with $\gamma = -1$ and $b_t = 1$. We observe in Figure 4.18 that up to age 80, mean optimal consumption in Cases 4.2 and 4.3 is almost the same, and after age 80 we observe a lower mean optimal consumption in Case 4.2. Mean optimal annuitisation for this pensioner in Case 4.2 is about 87%, and in Case 4.3 it is about 67% at age 65 and mostly 17% afterwards. Thus, this pensioner annuitises a significant part of his pension wealth at age 65 in both Cases 4.2 and 4.3. On the other hand, we observe in Figure 4.19 that the pensioner with $\gamma = -1$ and $b_t = 1$ optimally annuitises a smaller amounts of his pension wealth. His mean optimal annuitisation in Case 4.2 is about 18%, and in Case 4.3 he optimally defers annuitisation at the very beginning of the retirement and then annuitises less than 5% of his available pension wealth. We emphasise that we state here the values of mean optimal annuitisation in Cases 4.2 and 4.3. Each optimal annuitisation depends on the random interest rate and

on pension wealth and income. However, we can conclude that the pensioner with the higher demand for annuities benefits more in terms of the *CEC* and *REW* measure.

We observe in Table 4.10 that the overall gains, in terms of the *CEC* or *REW* measures, are lower for the less risk averse pensioner. If we observe the pairs of the results in the rows such that the RRA coefficients are the same, then we can see that the existence of the bequest motive results in a lower increase of the pensioner's gains in terms of the *CEC* or *REW* measures. We can conclude that the more risk averse pensioner will benefit more from access to annuities than the less risk averse pensioner. Also, the pensioner with no bequest motive will benefit more from access to annuities than the pensioner with the bequest motive.

Case 4.3 is always more favourable than Case 4.2. It is an expected result because the constraints on annuitisation are stricter in Case 4.2 compared to Case 4.3. We observe that the differences between Cases 4.2 and 4.3 are larger for the less risk averse pensioner.

In Table 4.10, we find that if $b_t = 0$ and $\gamma = -1$, then the pensioner's access to annuities whenever in retirement brings him 5.03% gains. If $b_t = 1$ and $\gamma = -1$ then the pensioner gains 1.90% in terms of the *REW* measure. At the same time, the pensioner has 2.47% and 1.42% better results in Case 4.3 compared to Case 4.2, respectively.

For the more risk averse pensioner, for $b_t = 0$ and $\gamma = -9$, the gains in Case 4.3 compared to Case 4.1 is 23.92% and the gains in Case 4.3 compared to Case 4.2 are 0.55%. Observing other combinations of the pensioner's risk and bequest preferences, we find the following The more risk averse pensioner benefits more in Case 4.3 compared to Case 4.1 in terms of the *REW* measure and at the same time the benefit for the more risk averse pensioner in Case 4.3 compared to Case 4.2 is lower than the benefits for the less risk averse pensioner. The same pattern of higher gains occurs for the pensioner with the bequest motive compared to the pensioner with no bequest motive.

Purchasing fewer annuities is followed by a lower income and a higher pension wealth. The pattern for income repeats itself for mean consumption. Purchasing fewer annuities is followed by a lower mean consumption in later years. Generally, increasing the bequest motive and lowering the pensioner's risk aversion is followed by a lower levels of annuitisation. Thus, increasing the bequest motive, and

decreasing the pensioner's risk aversion will be followed by a higher mean optimal consumption in the early years of retirement and a lower mean optimal consumption in the later years of retirement.

In Table 4.11 we present the pensioner's gains in terms of *REW* measure when the value of the interest rate in the year prior to retirement changes. We show these results for different combinations of the pensioner's preferences towards risk and bequest.

Interest rate	$b_t = 0 \quad \gamma = -1$		$b_t = 0 \quad \gamma = -9$		$b_t = 1 \quad \gamma = -1$		$b_t = 1 \quad \gamma = -9$	
	Case 4.2	Case 4.3	Case 4.2	Case 4.3	Case 4.2	Case 4.3	Case 4.2	Case 4.3
-2.44	1.04%	4.96%	21.64%	23.08%	0.02%	1.87%	17.21%	18.14%
-2.21	1.09%	4.96%	21.73%	23.12%	0.02%	1.87%	17.28%	18.17%
-1.81	1.19%	4.97%	21.88%	23.18%	0.04%	1.87%	17.39%	18.24%
-1.25	1.34%	4.97%	22.10%	23.28%	0.07%	1.87%	17.55%	18.34%
-0.56	1.55%	4.98%	22.38%	23.41%	0.12%	1.88%	17.75%	18.46%
0.22	1.83%	5.00%	22.69%	23.57%	0.20%	1.88%	17.98%	18.61%
1.09	2.17%	5.01%	23.03%	23.74%	0.32%	1.89%	18.21%	18.77%
2.00	2.56%	5.03%	23.37%	23.92%	0.48%	1.90%	18.43%	18.92%
2.91	2.98%	5.05%	23.67%	24.09%	0.66%	1.91%	18.62%	19.05%
3.78	3.40%	5.07%	23.93%	24.25%	0.85%	1.92%	18.80%	19.18%
4.56	3.79%	5.10%	24.13%	24.38%	1.04%	1.94%	18.94%	19.28%
5.25	4.12%	5.14%	24.29%	24.48%	1.21%	1.94%	19.05%	19.37%
5.81	4.35%	5.19%	24.40%	24.55%	1.30%	1.95%	19.13%	19.43%
6.21	4.49%	5.22%	24.47%	24.60%	1.36%	1.96%	19.18%	19.47%
6.44	4.57%	5.24%	24.50%	24.63%	1.39%	1.97%	19.21%	19.50%

Table 4.11 *REW* measure in percentages – Values in the cells show percentage difference between Case 4.1 and the Case shown in the column header. The pensioner's preferences towards risk and bequest are given in the very first row. The values of interest rate in the year prior to retirement are given in the very first column. Pension wealth is 200,000, pensioner's age is 65.

We see in Table 4.11 that interest rate in the year prior to the year of retirement influences the pensioner's expected discounted utility drawn from consumption and bequest during retirement. If the value of interest rate is lower than the annuity factor a_t is higher and income from annuity is lower. Thus, the pensioner will be keener to purchase more annuities if the value of interest rate is higher. In each column in Table 4.11 we have better results for the pensioner if the value of the interest rate is higher. This happens because annuitisation is almost always advantageous for the pensioner and if annuitisation occurs at a time of good value of the interest rate, then

annuitisation is even more advantageous. We say “almost always advantageous” because the pensioner with preferences $b_t = 1$ and $\gamma = -1$ and in Case 4.2 will convert a very small part of his pension wealth into annuities if the value of the interest rate is low. Case 4.3 is always beneficial for the pensioner, because at a certain age mortality will be high enough such that even with unfavourable values of the interest rate at age 65 there will be a demand for annuities at later ages.

As we expect in Table 4.11, the gains in terms of *REW* measure in Case 4.3 are always larger than in Case 4.2. The more risk averse pensioner will annuitise a significant part of his pension wealth at an earlier age, and if the value of the interest rate is unfavourable he will partly defer annuitisation. The pensioner with RRA coefficient $\gamma = -1$ in Case 4.3 will completely defer annuitisation if the value of the interest rate is not favourable, but he will eventually attain almost the same gains for any value of the interest rate at age 65.

In Figure 4.5, we have presented optimal annuitisation in Case 4.2 as a function of the value of pension wealth and interest rate during the year prior to retirement. In Figure 4.5 for $b_t = 0$, $\gamma = -1$, we observe significant differences in the values of optimal annuitisation at age 65. If pension wealth has the values of about 200,000 money units, the optimal annuitisation ranges from 30% to more than 60%. So, the pensioner in Case 4.2 with low risk aversion will choose very different optimal annuitisation depending on the value of the interest rate during the year prior to retirement. Using optimal annuitisation policy, he will gain extra utility from 1.04% to 4.57% (a range of 3.53%) in terms of the *REW* measure depending on the known value of the interest rate. In Case 4.3, we again have a quite different optimal annuitisation policy. Using optimal annuitisation at any age, he will be able to avoid the risk of unfavourable interest rate at age 65 and gains from 4.96% to 5.24% (a range of 0.28%) in terms of the *REW* measure depending on the value of the interest rate at age 65. So we can say that the less risk averse pensioner in Case 4.3 has the possibility to control the risk of unfavourable interest rates at age 65 quite well.

If we now observe the pensioner in Case 4.2 with lower values of coefficient of RRA γ and no bequest, then he will annuitise more at age 65. For the pensioner with no bequest and RRA coefficient $\gamma = -4$, not presented in Table 4.11, the range is 4.55%, and for no bequest assumption and RRA coefficient $\gamma = -9$ the range is 2.86%. The pensioner in Case 4.3 will have slightly wider range of gains as RRA coefficient γ decreases. In Case 4.3 for the pensioner with no bequest and RRA coefficient $\gamma = -4$, not presented in Table 4.11, the range is 1.68%, and the pensioner with no bequest

and RRA coefficient $\gamma = -9$ will have the range of gains of 1.55%. So, for the more risk averse pensioner we have more annuitisation at age 65 in Case 4.2 and the range of gains will firstly slightly increase with the increase of risk aversion and then decrease. The same pattern regarding range of gains repeats itself in Case 4.3.

If we observe the pensioner with the bequest motive then, as we already know, he will optimally annuitise a smaller part of his pension wealth compared to the pensioner with the same level of risk aversion and no bequest motive. Also, the gains from access to annuities will be smaller for the pensioner with the bequest motive. As the direct consequence of this fact, the ranges of the gains depending on the interest rate during the year prior to retirement are smaller compared to no bequest cases. We have that the ranges in Case 4.2 are 1.37%, 3.22% and 2.00% for the values of RRA coefficient $\gamma = -1$, $\gamma = -4$ and $\gamma = -9$, respectively. In Case 4.3, the ranges are 0.10%, 1.57% and 1.36% for $\gamma = -1$, $\gamma = -4$ and $\gamma = -9$, respectively. Again, we observe here that the ranges of gains are not monotonic function of RRA coefficient γ and we observe the same patterns as in the results for the pensioner with no bequest motive.

We will show more results related to changing interest rate in the year prior to retirement in Chapter 5 where we compare chosen results in Chapter 3 for Cases 3.1, 3.3 and 3.5 and results in Chapter 4 for Cases 4.1, 4.2 and 4.3, and where we also investigate in more detail results for a chosen set of pensioner's preferences.

In Chapter 5, we will focus our investigation on optimal consumption, asset allocation annuitisation and expected derived utility in a single case. In Chapter 4, we almost always try to compare the results between different cases. However, if we observe the pensioner at age 65 then it is interesting to investigate how much he is going to gain or lose due to the value of the interest rate in the year prior to retirement. We have presented some results of this kind in Table 4.11, but in Chapter 5 we will investigate this problem more thoroughly.

4.5.6 Left tail Analysis of Discounted Utility

So far in this section, apart from Figures 4.15–4.20, we have presented the results that are based on expected discounted utility. All these results are obtained without stochastic simulations, but are based on the exact results from the model. Figures 4.15–4.20 are presented in order to give an idea about the development of pension

wealth, optimal consumption, annuitisation, and asset allocation paths during retirement. In 4.5.6, we investigate the realisations from stochastic simulations of the random variables and investigate discounted utility as a random variable. Thus, all results in Section 4.5.6 are based on stochastic simulations, and stochastic simulations are based on the values of the variables given in Section 4.5.1 and on the solutions derived in Section 4.4.

Similarly as in Section 3.4.7, we aim to present only the basic results related to discounted utility when observed as a random variable and to give a possible way of measuring pensioner's left tail risk of lower than expected realisation of discounted utility derived from consumption and bequest. In this thesis, we have no optimisation with respect to VaR_α or $CVaR_\alpha$ as a criterion. We have optimisation with respect to expected discounted utility only. The importance of the left tail risk is recognised in the concave shape of the utility function. We find optimal control variables such that the maximum of expected utility derived from consumption and bequest is attained. Thus, the aim of the analysis in this section is to open the question of the importance of the possible maximization of derived utility as criterion but with the constraints on VaR_α or $CVaR_\alpha$.

We have defined discounted utility derived from future consumption and bequest as a random variable in equation (4.87). The value function is the expected value of discounted derived utility. However, we are interested in the left tail of discounted utility as it shows the risk of the worse than expected possible outcomes of the pensioner's random discounted utility. In order to have the results in money terms we will convert discounted utility in money terms first and then present the results in money terms.

4.5.6.1 The Definition of VaR_α and $CVaR_\alpha$ measure

Random variable $\tilde{D}(W_{65}, Y_{65}, r_{64;j})$, defined in (4.87) depends on the control variable $\tilde{C}_{65}$, random variables $\tilde{C}_t$ for $66 \leq t \leq 99$ and $\tilde{W}_{t+1}$ for $65 \leq t \leq 99$. These two random variables further depend on random variables $\tilde{r}_t$ and $\tilde{r}_t^e$ for $65 \leq t \leq 99$, random interest and equity rates respectively. Also, random variables $\tilde{C}_t$ and $\tilde{W}_t$ depend on decisions $C_i^*(W_i, Y_i, r_{i-1;j})$, $\alpha_i^{e*}(W_i, Y_i, r_{i-1;j})$, $\alpha_i^{b*}(W_i, Y_i, r_{i-1;j})$ and $m_i^*(W_i, Y_i, r_{i-1;j})$ for $65 \leq i < t$, $j = 1, \dots, n_r$, for $W_i \geq 0$ and $Y_i \geq 0$, and $r_{i-1;j}$ in the domain of the values of the interest rate.

Thus, we have random variable $\tilde{D}(W_{65}, Y_{65}, r_{64;j})$ depending on decisions, random variables and known interest rate during the year prior to retirement.

We noted earlier that $V_{65}(W_{65}, Y_{65}, r_{64;j}) = E[\tilde{D}(W_{65}, Y_{65}, r_{64;j})]$. Now, we define random variable $\tilde{W}_{65}$, such that

$$V_{65}(\tilde{W}_{65}, Y_{65}, r_{64;j}) = \tilde{D}(W_{65}, Y_{65}, r_{64;j}) \quad (4.88)$$

The random variable $\tilde{W}_{65}$ is unique because value function V_{65} is strictly increasing function in variable W_{65} . Regarding the existence of random variable $\tilde{W}_{65}$ we rely in this thesis on the fact that for each random realisation of random variable $\tilde{D}(W_{65}, Y_{65}, r_{64;j})$, we have found the realisation of random variable $\tilde{W}_{65}$.

The value of random variable $\tilde{W}_{65}$ is the value of pension wealth that the pensioner needs such that the mean value of all possible random discounted utilities with initial wealth $\tilde{W}_{65}$ is equal to the random discounted utility $\tilde{D}(W_{65}, Y_{65}, r_{64;j})$. The cumulative distribution function (abbreviation *CDF*) of the random variable $\tilde{W}_{65}$ can be defined in the following way. If the *CDF* of the random variable $\tilde{D}(W_{65}, Y_{65}, r_{64;j})$ is given by

$$P_{\tilde{D}}(\tilde{D}(W_{65}, Y_{65}, r_{64;j}) \leq x),$$

for $x \in (-\infty, \infty)$ then *CDF* of the random variable $\tilde{W}_{65}$ is defined by

$$P_{\tilde{W}_{65}}(\tilde{W}_{65} \leq y) = P_{\tilde{D}}(V_{65}(\tilde{W}_{65}, Y_{65}, r_{64;j}) \leq x \text{ such that } x = V_{65}(y, Y_{65}, r_{64;j}))$$

for y in the domain of the solutions of equation $x = V_{65}(y, Y_{65}, r_{64;j})$. Equation $x = V_{65}(y, Y_{65}, r_{64;j})$ will have a solution for a certain interval. For the values of y smaller than the lowest value in the interval we define $P_{\tilde{W}_{65}}(\tilde{W}_{65} \leq y) = 0$, and for higher than the highest value of the interval we can define $P_{\tilde{W}_{65}}(\tilde{W}_{65} \leq y) = 1$. Thus, the *CDF* of random variable $\tilde{W}_{65}$ is fully defined.

Having defined the random variable $\tilde{W}_{65}$, we can investigate the left tail of possible future random realisations of discounted utility in money term. It is important here to have this transformation from utility into pension wealth, as utility itself does not give a clear idea of the meanings of the results to the pensioner.

Now, we can define α Value at Risk (abb. VaR_{α}) and α Conditional Value at Risk (abb. $CVaR_{\alpha}$) measures, as left tail measures for random variable $\tilde{W}_{65}$. We define VaR_{α} , as follows

$$VaR_\alpha = \inf \left\{ W \in \mathbb{R} : P \left[\tilde{W}_{65} \geq W \right] \leq 1 - \alpha \right\} \quad (4.89)$$

or

$$VaR_\alpha = \inf \left\{ W \in \mathbb{R} : P \left[\tilde{D}(W_{65}, Y_{65}, r_{64;j}) \geq V_{65}(W, Y_{65}, r_{64;j}) \right] \leq 1 - \alpha \right\} \quad (4.90)$$

for $0 < \alpha < 1$. The value of VaR_α gives us the following information. For the pensioner with pension wealth W_{65} , there is a $\alpha\%$ probability that possible unfavourable market realisations in the future will result with lower or the same discounted utility that would have been obtained as expected discounted utility with the pension wealth VaR_α . In other words, VaR_α is the $\alpha\%$ worst pension wealth due to less favourable than expected market conditions in the future.

Similar to the technique for obtaining VaR_α in Chapter 3, we again make 2,000 stochastic simulations for ages 65 to age 99 for all random variables in the model in Chapter 4. . For the purpose of deeper investigation of the pensioner's left-tail risk, more than 2,000 random realisations may be appropriate. However, the results presented here are not very dependent on the number of the random realisations and we believe that it is appropriate to use here the same realisation of the stochastic simulations that we use for the check of accuracy of the numerical calculations.

For each realisation of the stochastic simulation, we obtain optimal consumption and pension wealth for each age. Substituting these values in equation(4.87), we obtain 2,000 realisations of discounted derived utility $\tilde{D}(W_{65}, Y_{65}, r_{64;j})$. We determine the values of VaR_α for $\alpha \in \{0.01, 0.05, 0.10, 0.25\}$. We obtain the value of VaR_α in the following way. Firstly, we calculate 2,000 random realisations of the random variable $\tilde{W}_{65}$ using formula (4.88). Then, we order these 2,000 random realisations in an increasing array. Then $VaR_{0.01}$ is the twentieth member of the ordered array, $VaR_{0.05}$ is the hundredth member, $VaR_{0.10}$ is the two hundredth member, and $VaR_{0.25}$ is the five hundredth member of the ordered array.

$CVaR_\alpha$ is defined as the mean shortfall, or in a mathematical definition as

$$CVaR_\alpha = Mean \left[\bar{W}_{65} \mid \bar{W}_{65} < VaR_\alpha \right], \quad (4.91)$$

where $\bar{W}_{65}$ are random realisations of random variable $\tilde{W}_{65}$ that satisfy the condition $\bar{W}_{65} < VaR_\alpha$.

In order to calculate $CVaR_\alpha$ for $\alpha \in \{0.01, 0.05, 0.10, 0.25\}$, we use 2,000 random realisations of the random variable $\tilde{W}_{65}$ in the ordered array already obtained for the

calculation of Var_{α} . $CVaR_{0.01}$ is calculated as the mean of the first nineteen members of the ordered array, $CVaR_{0.05}$ is the mean of the first ninety nine members, $CVaR_{0.10}$ is the mean of the first one hundred ninety nine members, and $CVaR_{0.25}$ is the mean of the first four hundred ninety nine members of the ordered array.

4.5.6.2 Var_{α} and $CVaR_{\alpha}$ measures – the Results

We present here the results for $\alpha = 0.10$. The results for the other values of α , not presented here, have different values but the pattern is the same and the same conclusions can be drawn. As we noted, we aim to shed light on the pensioner's left tail risk and we leave a more thorough analysis for future work.

In Figure 4.21, we present graphs showing the histogram of the random variable $\tilde{W}_{65}$, and also the values of $Var_{0.10}$ and $CVaR_{0.10}$. The histogram is made from the sample of 2,000 random realisations of random variable $\tilde{W}_{65}$. The left bold vertical line in each graph in the following figures shows the value of $CVaR_{0.10}$, while the right one shows the value of $Var_{0.10}$.

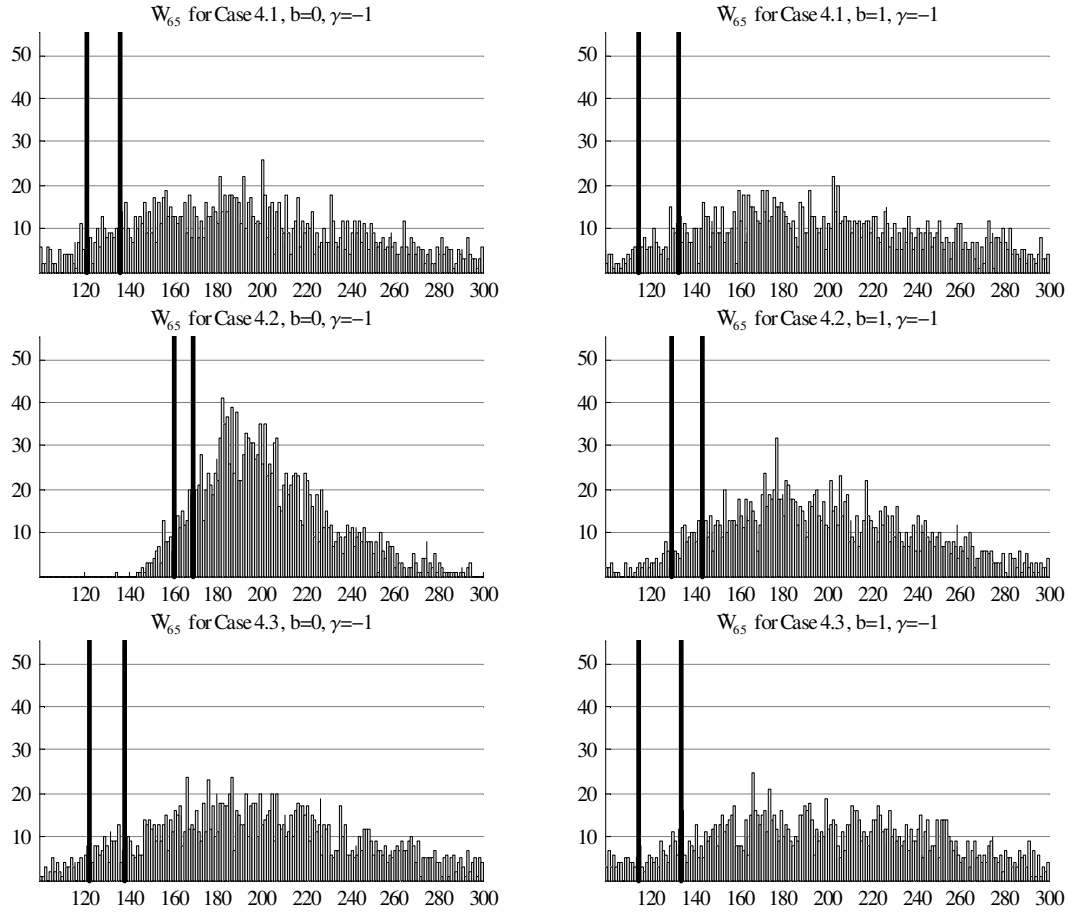


Figure 4.21 Histogram of the random sample of 2,000 random realisation of $\tilde{W}_{65}$ for $W_{65} = 200,000$, in Cases 4.1, 4.2 and 4.2, for $b_t = 0$ and $b_t = 1$, for $65 \leq t \leq 99$, and for $\gamma = -1$. The left one vertical straight line represents the value of $CVaR_{0.10}$, and the right one the value of $VaR_{0.10}$. The value of interest rate in the year prior to retirement is equal to 2.00%.

In Figure 4.21, we observe that the pensioner with RRA coefficient $\gamma = -1$ has a very wide range of possible outcomes of the random variable $\tilde{W}_{65}$. Cases 4.1 and 4.3 are quite similar, meaning that although the pensioner purchases annuities during retirement in Case 4.3 and increases his derived utility he decreases the left tail risk of the derived discounted utility just slightly. The pensioner with $\gamma = -1$, in Case 4.2, has a lower left tail spread of the random realisations of random variable $\tilde{W}_{65}$, and both $VaR_{0.10}$ and $CVaR_{0.10}$ lines are positioned on values higher than in Cases 4.1 and 4.3. Thus, the pensioner with $\gamma = -1$ has a lower left tail risk in Case 4.2 than in Cases 4.3. The reason for the lower left tail risk lies in the fact that the pensioner in Case 4.2 optimally purchases more annuities at an earlier age, and the pensioner is exposed to less risk of possible unfavourable developments of random equity and interest rates compared to Cases 4.1 and 4.3 where either no annuities are bought or annuities are bought later in the retirement period.

In Case 4.2 with no bequest, both $VaR_{0.10}$ and $CVaR_{0.10}$ are positioned more to the right than in the Case 4.2 with a bequest. Again, the pensioner with no bequest motive in Case 4.2 optimally purchases more annuities at age 65 than the pensioner with bequest motive in Case 4.2 and thus $VaR_{0.10}$ and $CVaR_{0.10}$ take higher values.

In Figure 4.22, we present the same group of graphs as in Figure 4.21 but now for the pensioner with the value of RRA coefficient $\gamma = -9$.

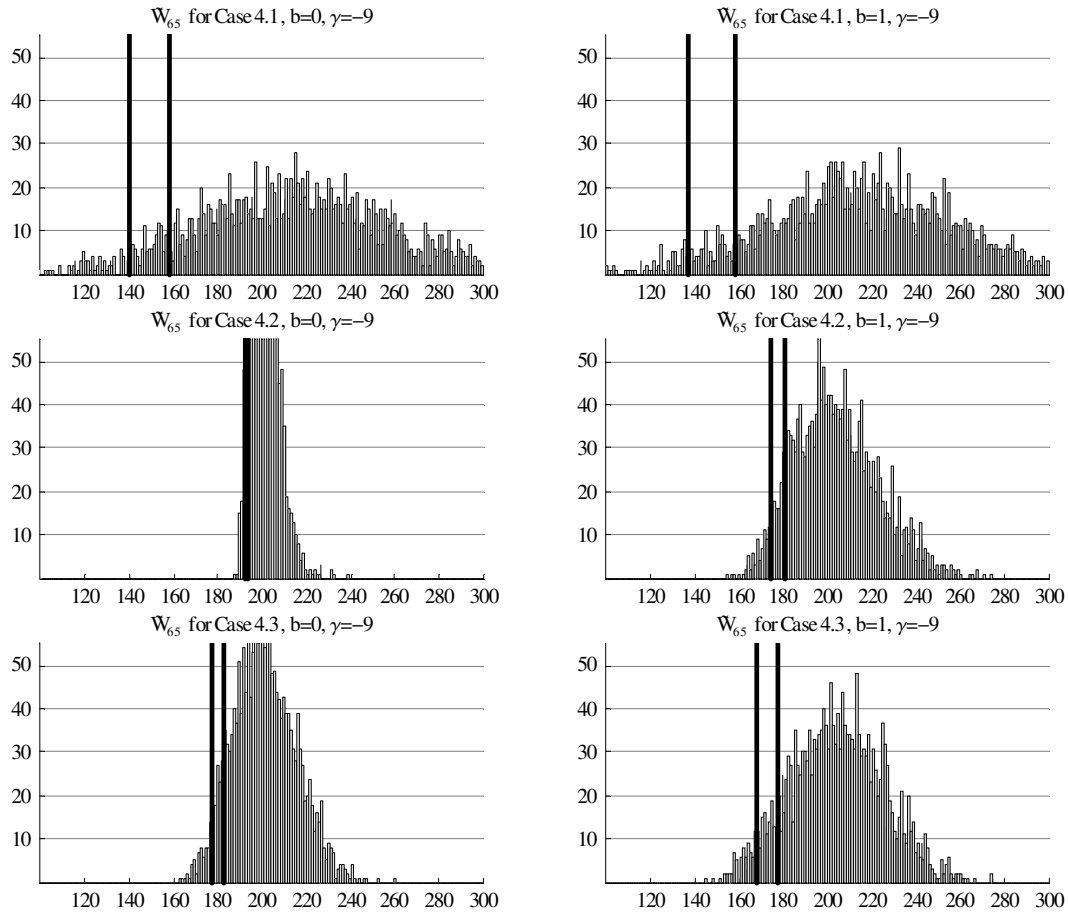


Figure 4.22 Histogram of the random sample of 2,000 random realisation of $\tilde{W}_{65}$ for $W_{65} = 200,000$, in Cases 4.1, 4.2 and 4.2, for $b_t = 0$ and $b_t = 1$, for $65 \leq t \leq 99$, and for $\gamma = -9$. The left one vertical straight line represents the value of $CVaR_{0.10}$, and the right one the value of $VaR_{0.10}$. The value of interest rate in the year prior to retirement is equal to 2.00%.

In Figure 4.22, the histograms have very different shapes as well as the values of $VaR_{0.10}$ and $CVaR_{0.10}$. In Case 4.1, histograms' spread are again quite wide and the values of $VaR_{0.10}$ and $CVaR_{0.10}$ are in similar positions. For the pensioner in Case 4.2 and with no bequest, the range of values on the histogram is very narrow and the $VaR_{0.10}$ and $CVaR_{0.10}$ lines cannot be differentiated from the histogram itself. The

reason for such a shape of the histogram lies in the fact that it is optimal for this pensioner to convert more than 87% of his pension wealth to annuities at age 65, and so very few assets are left to be under the influence of the randomness of equity and interest rates. In Case 4.2 with a bequest, the histogram's spread is wider because this pensioner optimally converts a lower amount of pension wealth into annuities than the pensioner with no bequest motive.

We can make the general conclusion that the pensioner in Case 4.2 optimally annuitises a significant part of his pension wealth at age 65 because he uses that single opportunity to annuitise. As a consequence, a lower amount of pension wealth is left under the pensioner's control and this lower amount of pension wealth is subject to interest and equity rate risk. Thus, it is not surprising that $VaR_{0.10}$ and $CVaR_{0.10}$ have the highest values observing three histograms in either left or right column in Figures 4.21 and 4.22.

At age 65, the pensioner with $\gamma = -9$ in Case 4.3 optimally purchases fewer annuities than the pensioner with $\gamma = -9$ in Case 4.2. As a consequence of this optimal strategy, the pensioner with $\gamma = -9$ in Case 4.3 is less exposed to equity and interest rate risks than the pensioner with $\gamma = -9$ in Case 4.1, but is more exposed to these risks than the pensioner with $\gamma = -9$ in Case 4.2. Thus, we obtain the widest range of values of $\tilde{W}_{65}$ in the histogram and the lowest values of $VaR_{0.10}$ and $CVaR_{0.10}$ in Case 4.1, the lowest range of values of $\tilde{W}_{65}$ and the highest values of $VaR_{0.10}$ and $CVaR_{0.10}$ in Case 4.2. In Case 4.3 we obtain results that are somewhere in between the first two.

In Table 4.12, we present the values of $CVaR_{0.10}$ for the pensioner at age 65, in different cases, with and without bequest motive, for the value of RRA coefficient $\gamma = -1$. Pension wealth at age 65 is 200,000. The values of $CVaR_{0.10}$ are presented for the five chosen values of interest rate in the year preceding retirement. The first and the fifth values of interest rate are the two extreme values investigated, the second and the fourth values are moderately different from the mean value of the interest rate, and the third value of the interest rate is the mean value of the interest rate.

Interest rate prior retirement	$\gamma = -1, b_t = 0$			$\gamma = -1, b_t = 1$		
	Case 4.1	Case 4.2	Case 4.3	Case 4.1	Case 4.2	Case 4.3
-2.44	120,506	146,046	120,568	114,186	117,215	112,650
-0.56	119,613	150,319	119,514	113,566	123,178	115,539
2.00	119,748	160,315	123,511	115,416	129,515	115,556
4.56	121,240	168,315	127,621	113,574	136,188	114,628
6.44	124,383	172,677	146,629	118,601	142,738	129,962

Table 4.12 $CVaR_{0.10}$ – Values in the cells show the values of $CVaR_{0.10}$ in different cases for the pensioner's preferences towards risk and bequest stated in the first row, and for the different values of interest rate during the year preceding retirement. Pensioner is at age 65. Pension wealth is 200,000. The values of $CVaR_{0.10}$ are calculated from the sample of 2,000 random realisations.

In Table 4.12, we observe that the pensioner with $\gamma = -1$ in Case 4.1 has slightly increasing but similar values of $CVaR_{0.10}$ as the value of the poor interest rate increases. The values of $CVaR_{0.10}$ are the lowest compared to other cases. Thus, the pensioner with access to annuities gains in terms of expected discounted utility and at the same time gains in terms of the lower left tail risk. We observe that in Case 4.1, both for $\gamma = -1, b_t = 0$ and for $\gamma = -1, b_t = 1$, the values of $CVaR_{0.10}$ are similar but not increasing or decreasing as the value of interest rate increases. The reason for this pattern is that optimal asset allocation is 100% in equities and no annuitisation is present. Thus, the different values of the interest rate do not influence discounted utility.

In Case 4.2, it is optimal for the pensioner with $\gamma = -1$ to annuitise the highest proportion of pension wealth at age 65 and this leads to the lowest left tail risk. In this case, we observe the fastest increase of the values of $CVaR_{0.10}$ as the value of the interest rate increases. The reason is that the pensioner optimally increases the proportion of annuitised pension wealth as the value of the interest rate increases, and then he is less exposed to the risk of random equity and interest rate.

The pensioner in Case 4.3 has significantly lower values of $CVaR_{0.10}$ than in Case 4.2, particularly if the pensioner has no bequest motive. However, the pensioner in Case 4.3 has a just slightly lower left tail risk compared to the pensioner in Case 4.1 for all but the very high values of interest rate at age 65. The reason for the sharp increase of $CVaR_{0.10}$ for very high values of interest rate at age 65 in Case 4.3 is that it is optimal for this pensioner to annuitise a significantly higher part of his pension wealth at age

65 for very high values of interest rate at age 65. The pensioner with $\gamma = -1$ in Case 4.3 gains in terms of *REW* measure compared to Case 4.1, but also slightly in terms of left tail risk. However, the conclusion is not that clear if we compare Cases 4.2 and 4.3. The pensioner $\gamma = -1$ gains in terms of *REW* measure, but at the same time his left tail risk is higher in Case 4.3 compared to Case 4.2. We observe that in Case 4.3 for $\gamma = -1$, $b_t = 0$ and for the values of the interest rate -2.44% and -0.56% , we get similar values of $CVaR_{0.10}$. The reason is that optimal asset allocation is 100% and that for the chosen value of the pension wealth at age 65 it is optimal to differ annuitisation. So for the values of the interest rate -2.44% and -0.56% , the influence of the value of the interest rate to $CVaR_{0.10}$ in Case 4.3 for $\gamma = -1$, $b_t = 0$ decreases and we observe unexpected pattern of the higher value of $CVaR_{0.10}$ for the values of the interest rate -0.56% than for -2.44% . In Case 4.3 for $\gamma = -1$, $b_t = 1$, optimal asset allocation is 100% in equities, and deferred annuitisation for the values of the interest rate of -2.44% , -0.56% , 2.00% and 4.56% and as a result the influence of the value of the interest rate to $CVaR_{0.10}$ decreases to a level that we observe similar values of $CVaR_{0.10}$ for stated values of the interest rate during the year before retirement.

In Table 4.13, we give the same group of results but now for the pensioner with RRA coefficient $\gamma = -9$.

Interest rate prior retirement	$\gamma = -9, b_t = 0$			$\gamma = -9, b_t = 1$		
	Case 4.1	Case 4.2	Case 4.3	Case 4.1	Case 4.2	Case 4.3
-2.44	132,772	188,679	165,854	132,150	170,640	162,425
-0.56	134,481	189,739	168,807	132,513	172,442	163,889
2.00	136,003	191,831	179,088	137,269	174,049	168,306
4.56	144,479	193,526	183,340	142,992	176,436	172,586
6.44	143,166	194,242	187,499	144,305	178,950	174,577

Table 4.13 $CVaR_{0.10}$ – Values in the cells show the values of $CVaR_{0.10}$ in different cases for the pensioner's preferences towards risk and bequest stated in the first row, and for the different values of interest rate during the year preceding retirement. Pensioner is at age 65. Pension wealth is 200,000. The values of $CVaR_{0.10}$ are calculated from the sample of 2,000 random realisations.

In Table 4.13, we observe the same patterns in Cases 4.1 and 4.2, but the values of $CVaR_{0.10}$ are now significantly higher than in Case 4.1 and these values increase now as the values of interest rate at age 65 increases. The reason is that the pensioner with $\gamma = -9$ in Case 4.3 optimally annuitises a higher proportion of pension wealth at age

65 than the less risk averse pensioner. We conclude that this pensioner significantly gains from annuitisation both in terms of *REW* measure and in terms of lowering the left tail risk. Again comparing Cases 4.2 and 4.3, the more risk averse pensioner gains in terms of *REW* measure but his left tail risk is higher in Case 4.3.

4.5.7 Sensitivity Analysis

In this section, we present the pensioner's gains from access to annuities, in terms of the *REW* measure in percentages, but now for different values of the chosen variables. We present the results for the new values of the following variables: last salary income (and income from social security afterwards), pension wealth at age 65, mean value of random equity rate, mean value of random interest rate, and market price of risk. We change the value of a single variable only and the results presented below show the way and the level of the changes in the results due to the change of the value of that single variable.

The aim of presenting these results is to explore the sensitivity of the main results in this thesis, the pensioner's gains from access to annuities, and to the values of the above mentioned variables.

For each new value of the variable, we present the pensioner's gains from access to annuities, in terms of *REW* measure in percentages, in different cases, for different pensioner's preferences towards risk and bequest, and for different values of interest rate during the year before retirement. We also present the percentage differences of the pensioner's gains for the new value of the chosen variable compared to the pensioner's gains presented in Table 4.11. We obtain these percentage differences simply by subtracting the pensioner's gains for the new value of the chosen variable from the pensioner's gain in Table 4.11.

4.5.7.1 Increasing Last Salary Income for 50%

In Section 4.5.1, we have defined the value of the last salary income at age 65 to be $Y_{65} = 33,321$, replacement ratio is $\rho_{65} = 0.68212$, and income from social security is $Y_t^{SS} = 22,729$ for $66 \leq t \leq 99$. Increasing income by 50%, we get the value of last salary income at age 65 of $Y_{65} = 49,981$ and income from social security is $Y_t^{SS} = 34,093$ for $66 \leq t \leq 99$. The value of the replacement ratio is kept

$\rho_{65} = 0.68212$. In Table 4.14, we present the pensioner's gains for the higher value of the last salary income.

Interest rate	$b_t = 0 \quad \gamma = -1$		$b_t = 0 \quad \gamma = -9$		$b_t = 1 \quad \gamma = -1$		$b_t = 1 \quad \gamma = -9$	
	Case 4.2	Case 4.3	Case 4.2	Case 4.3	Case 4.2	Case 4.3	Case 4.2	Case 4.3
The pensioner's gains in terms of <i>REW</i> measure in percentages								
-2.44	0.11%	3.46%	17.47%	19.43%	0.00%	0.91%	12.42%	13.68%
-0.56	0.34%	3.48%	18.40%	19.81%	0.00%	0.91%	13.08%	14.05%
2.00	0.99%	3.50%	19.67%	20.43%	0.00%	0.92%	13.92%	14.58%
4.56	1.94%	3.53%	20.66%	20.99%	0.07%	0.93%	14.53%	15.00%
6.44	2.66%	3.55%	21.14%	21.31%	0.21%	0.91%	14.87%	15.25%
Differences of the pensioner's gains above compared to the gains in Table 4.11								
-2.44	-0.93%	-1.50%	-4.17%	-3.65%	-0.02%	-0.96%	-4.79%	-4.46%
-0.56	-1.21%	-1.51%	-3.98%	-3.60%	-0.12%	-0.97%	-4.67%	-4.41%
2.00	-1.57%	-1.53%	-3.69%	-3.49%	-0.48%	-0.98%	-4.51%	-4.34%
4.56	-1.84%	-1.57%	-3.47%	-3.38%	-0.98%	-1.01%	-4.41%	-4.28%
6.44	-1.91%	-1.69%	-3.36%	-3.32%	-1.18%	-1.06%	-4.35%	-4.25%

Table 4.14 *REW* measure in percentages for the 50% higher value of last salary income – Values in the cells show percentage difference between Case 4.1 and the Case shown in the column header. The pensioner's preferences towards risk and bequest are given in the very first row. The values of interest rate in the year prior to retirement are given in the very first column. Pension wealth is 200,000, pensioner's age is 65.

Income from social security is a form of annuity already in a possession of the pensioner. From that point of view, we observed the expected result of the lower pensioner's gains from annuities. In further results not presented here, we observe a lower level of optimal annuitisation in Case 4.2, and lower and later during retirement optimal annuitisation in Case 4.3.

In Case 4.2, the less risk averse pensioner has a small gains from access to annuities according to the results in Table 4.11, and if his income increases by 50%, the gains are even lower. If this pensioner has the bequest motive then there is no demand for annuities for the lower values of interest rate during the year before retirement. In Case 4.3, the gains for the less risk averse pensioner again do not depend on the value of the interest rate during the year before retirement, but the gains are smaller in Table 4.14 compared to the gains in Table 4.11.

For the more risk averse pensioner the differences in gains in Table 4.14 do not seem to depend significantly on the value of the interest rate during the year before

retirement. The range of the differences is slightly larger in Case 4.2 than in Case 4.3. We observe that the differences in the pensioner's gains are larger for about 0.5% to 1% for the pensioner with the bequest motive than for the pensioner with no bequest motive.

4.5.7.2 Increasing Pension Wealth at age 65 for 50%

In Section 4.5.1, we have defined the value of pension wealth at age 65 to be $W_{65} = 200,000$. Increasing pension wealth at age 65 by 50%, we get the value pension wealth at age 65 of $W_{65} = 300,000$. In Table 4.15, we present the pensioner's gains for the higher value of pension wealth at age 65.

Interest rate	$b_t = 0 \quad \gamma = -1$		$b_t = 0 \quad \gamma = -9$		$b_t = 1 \quad \gamma = -1$		$b_t = 1 \quad \gamma = -9$	
	Case 4.2	Case 4.3	Case 4.2	Case 4.3	Case 4.2	Case 4.3	Case 4.2	Case 4.3
The pensioner's gains in terms of <i>REW</i> measure in percentages								
-2.44	2.64%	6.64%	25.23%	26.31%	0.78%	3.23%	21.42%	22.12%
-0.56	3.32%	6.68%	25.79%	26.57%	1.15%	3.26%	21.84%	22.38%
2.00	4.55%	6.76%	26.55%	26.96%	1.89%	3.32%	22.38%	22.75%
4.56	5.94%	6.96%	27.13%	27.31%	2.77%	3.47%	22.77%	23.03%
6.44	6.70%	7.23%	27.41%	27.50%	3.20%	3.65%	22.98%	23.20%
Differences of the pensioner's gains above compared to the gains in Table 4.11								
-2.44	1.60%	1.68%	3.59%	3.24%	0.76%	1.37%	4.21%	3.98%
-0.56	1.77%	1.69%	3.41%	3.16%	1.03%	1.38%	4.09%	3.92%
2.00	1.99%	1.73%	3.18%	3.05%	1.41%	1.42%	3.94%	3.83%
4.56	2.15%	1.86%	3.00%	2.94%	1.72%	1.54%	3.83%	3.75%
6.44	2.13%	1.99%	2.90%	2.87%	1.81%	1.68%	3.77%	3.70%

Table 4.15 *REW* measure in percentages for the 50% higher value of pension wealth at age 65 – Values in the cells show percentage difference between Case 4.1 and the Case shown in the column header. The pensioner's preferences towards risk and bequest are given in the first row. The values of interest rate in the year prior to retirement are given in the first column. Pension wealth is 300,000, pensioner's age is 65.

The pensioner with a higher amount of pension wealth at age 65, other values of the variables being the same, benefits more from access to annuities. Again, income from social security is a form of annuity income, and as the pensioner has a higher pension wealth at age 65, he has a relatively lower value of income from social security, and thus gains more from access to annuities.

For the less risk averse pensioner the increase in gains in Table 4.15 compared to the gains in Table 4.11 are lower compared to the more risk averse pensioner. The differences in gains for the less risk averse pensioner are less dependent on the value of the interest rate during the year before retirement in Case 4.3 than in Case 4.2. We observe that the differences in gains for the less risk averse pensioner are lower in Case 4.2 than in Case 4.3 for the lower values of interest rate before retirement, and higher for the higher values of the interest rate before retirement. We also observe that the differences in gains are slightly higher for the less risk averse pensioner with no bequest motive compared with the less risk averse pensioner with a bequest motive.

The more risk averse pensioner has quite stable differences in gains compared in Tables 4.15 and 4.11 for all values of the interest rate during the year before retirement. However, we observe slightly higher differences in gains for the more risk averse pensioner with a bequest motive compared to the more risk averse pensioner with no bequest motive.

We observe different patterns in the differences in gains for the less and more risk averse pensioners for different values of the interest rate during the year before retirement. The less risk averse pensioner's differences in gains increases, while the differences in gains for the more risk averse pensioner decreases as the value of the interest rate increases.

4.5.7.3 Decreasing of the Mean Value of Equity Rate to 4%

In Section 4.5.1, we have defined the mean value of equity rate to be $E[\tilde{r}_t^e] = 0.06$. In Section 4.5.7.3, we present in Table 4.16 the pensioner's gains in terms of *REW* measure in percentages, if the mean value of equity rate is $E[\tilde{r}_t^e] = 0.04$.

Interest rate	$b_t = 0 \quad \gamma = -1$		$b_t = 0 \quad \gamma = -9$		$b_t = 1 \quad \gamma = -1$		$b_t = 1 \quad \gamma = -9$	
	Case 4.2	Case 4.3	Case 4.2	Case 4.3	Case 4.2	Case 4.3	Case 4.2	Case 4.3
The pensioner's gains in terms of <i>REW</i> measure in percentages								
-2.44	7.85%	9.35%	27.92%	28.25%	3.08%	4.19%	21.07%	21.34%
-0.56	8.82%	9.61%	28.37%	28.52%	3.69%	4.34%	21.38%	21.57%
2.00	10.36%	10.45%	28.91%	28.93%	4.67%	4.86%	21.80%	21.88%
4.56	11.32%	11.32%	29.24%	29.24%	5.21%	5.27%	22.12%	22.16%
6.44	11.67%	11.67%	29.37%	29.37%	5.44%	5.48%	22.29%	22.32%
Differences of the pensioner's gains above compared to the gains in Table 4.11								
-2.44	6.81%	4.39%	6.28%	5.17%	3.06%	2.33%	3.86%	3.21%
-0.56	7.27%	4.62%	6.00%	5.11%	3.57%	2.46%	3.63%	3.11%
2.00	7.80%	5.42%	5.55%	5.01%	4.19%	2.95%	3.37%	2.96%
4.56	7.53%	6.22%	5.10%	4.86%	4.16%	3.33%	3.19%	2.88%
6.44	7.10%	6.43%	4.87%	4.74%	4.05%	3.51%	3.08%	2.83%

Table 4.16 *REW* measure in percentages for the mean value of equity rate of 4% – Values in the cells show percentage difference between Case 4.1 and the Case shown in the column header. The pensioner's preferences towards risk and bequest are given in the first row. The values of interest rate in the year prior to retirement are given in the first column. Pension wealth is 200,000, pensioner's age is 65.

If the mean value of the equity rate is lower than the demand for the less risky investments, bond and cash as well as annuities, increases. Due to increasing demand for annuities, the pensioner facing lower mean value of equity rate gains more from access to annuities.

We observe in Table 4.16 that the differences in gains are the highest for the pensioner with no bequest motive. This pensioner, regardless of his attitude towards risk, has the highest increase in demand for annuities as the mean value of equity rate decreases. This is particularly true for the pensioner in Case 4.2, who has access to annuities only once at age 65.

For the less risk averse pensioner, we again observe increasing differences in gains as the value of the interest rate before retirement increases. However, we observe a sharper increase of the differences in gains in Case 4.3, and a more modest increase of the differences in gains in Case 4.2 in Table 4.16 compared to the differences in gains in Table 4.15. For the more risk averse pensioner, we observe a decrease of the differences in gains in Table 4.16.

4.5.7.4 Increasing the Mean Value of the Interest Rate to 4%

In Section 4.5.1, we have defined the parameters for the interest rate as follows $a_d = 0.00902377$, $b_d = 0.451188$, $\sigma_{dr} = 0.0152622$, $E[\tilde{r}_t] = 0.02$, $StD[r_t] \approx 0.0172195$. We increase the mean value of the interest rate to 4% by changing the values of the parameters as follows: $a_d = 0.0180475$, $b_d = 0.451188$, $\sigma_{dr} = 0.0152622$, $E[\tilde{r}_t] = 0.04$, $StD[r_t] \approx 0.0164822$. Thus, by changing the parameters values we increase the mean value of the interest rate, but also slightly decrease the standard deviation of values of the interest rate. In Table 4.17, we present the pensioner's gains in terms of *REW* measure in percentages for these new values of the parameters for the interest rate.

Interest rate	$b_t = 0 \quad \gamma = -1$		$b_t = 0 \quad \gamma = -9$		$b_t = 1 \quad \gamma = -1$		$b_t = 1 \quad \gamma = -9$	
	Case 4.2	Case 4.3	Case 4.2	Case 4.3	Case 4.2	Case 4.3	Case 4.2	Case 4.3
The pensioner's gains in terms of <i>REW</i> measure in percentages								
-2.44	11.30%	12.72%	26.34%	26.64%	5.34%	6.48%	20.00%	20.25%
-0.56	12.21%	12.97%	26.63%	26.80%	5.96%	6.64%	20.19%	20.37%
2.00	13.63%	13.75%	27.01%	27.03%	6.89%	7.16%	20.44%	20.54%
4.56	14.53%	14.54%	27.22%	27.22%	7.39%	7.52%	20.63%	20.69%
6.44	14.83%	14.83%	27.29%	27.29%	7.59%	7.69%	20.73%	20.78%
Differences of the pensioner's gains above compared to the gains in Table 4.11								
-2.44	10.26%	7.76%	4.70%	3.56%	5.33%	4.61%	2.79%	2.11%
-0.56	10.66%	7.98%	4.26%	3.38%	5.84%	4.76%	2.44%	1.91%
2.00	11.07%	8.72%	3.64%	3.11%	6.42%	5.26%	2.01%	1.62%
4.56	10.75%	9.44%	3.09%	2.85%	6.35%	5.58%	1.70%	1.41%
6.44	10.26%	9.59%	2.79%	2.66%	6.19%	5.72%	1.51%	1.28%

Table 4.17 *REW* measure in percentages for the mean value of interest rate of 4% – Values in the cells show percentage difference between Case 4.1 and the Case shown in the column header. The pensioner's preferences towards risk and bequest are given in the first row. The values of interest rate in the year prior to retirement are given in the first column. Pension wealth is 200,000, pensioner's age is 65.

Changing the value of parameters of the interest rate such that the mean value of the interest rate increases, results in a lower annuity rate as well as a higher return on bond and cash investment. As annuity rate decreases, income from annuity increases. As a result, the pensioner's gains from access to annuities increases.

For the less risk averse pensioner, we observe a significant increase of benefits from access to annuities in Table 4.17 compared to the gains in Table 4.11. The differences

in gains are particularly high for the less risk averse pensioner with no bequest motive. His demand for annuities significantly increases in both Cases 4.2 and 4.3. The less risk averse pensioner with a bequest motive also has a significant increase in the gains in Table 4.17 compared to Table 4.11, but his demand for annuities does not increase as much as for the less risk averse with no bequest motive. The presence of the bequest motive limits the demand for annuities.

The more risk averse pensioner does not experience such significant changes in the gains in Table 4.17 compared to the gains in Table 4.11. The extra gains are lower compared to the less risk averse pensioner. In Table 4.17, we observe the lowest differences in gains for the more risk averse pensioner with the bequest motive.

The less risk averse pensioner's differences in gains, when the mean value of the interest rate increases, are higher than the differences in gains for the more risk averse pensioner in Case 4.3 but lower in Case 4.2. Also, the pensioner with no bequest motive increases the gains from access to annuities more when the mean value of the interest rate increases, than the pensioner with a bequest motive.

4.5.7.5 Decreasing the Market Price of Risk to 0.01528

In Section 4.5.1, we have defined the value of market price of risk to be $\lambda = 0.1528$. In Table 4.18, we present the pensioner's gains in terms of *REW* measure in percentages if the value of the market price of risk is significantly lower and equal to 0.01528.

Interest rate	$b_t = 0 \quad \gamma = -1$		$b_t = 0 \quad \gamma = -9$		$b_t = 1 \quad \gamma = -1$		$b_t = 1 \quad \gamma = -9$	
	Case 4.2	Case 4.3	Case 4.2	Case 4.3	Case 4.2	Case 4.3	Case 4.2	Case 4.3
The pensioner's gains in terms of <i>REW</i> measure in percentages								
-2.44	0.24%	4.02%	20.23%	21.99%	0.00%	1.40%	16.29%	17.44%
-0.56	0.52%	4.03%	21.03%	22.35%	0.00%	1.40%	16.91%	17.80%
2.00	1.18%	4.05%	22.14%	22.90%	0.04%	1.41%	17.71%	18.32%
4.56	2.11%	4.07%	23.02%	23.41%	0.30%	1.42%	18.30%	18.74%
6.44	2.79%	4.08%	23.46%	23.70%	0.56%	1.42%	18.61%	18.98%
Differences of the pensioner's gains above compared to the gains in Table 4.11								
-2.44	-0.80%	-0.94%	-1.42%	-1.09%	-0.02%	-0.47%	-0.92%	-0.69%
-0.56	-1.03%	-0.95%	-1.34%	-1.06%	-0.12%	-0.48%	-0.84%	-0.66%
2.00	-1.38%	-0.98%	-1.23%	-1.01%	-0.44%	-0.49%	-0.72%	-0.60%
4.56	-1.68%	-1.03%	-1.12%	-0.96%	-0.74%	-0.52%	-0.64%	-0.54%
6.44	-1.78%	-1.16%	-1.05%	-0.93%	-0.83%	-0.55%	-0.60%	-0.51%

Table 4.18 *REW* measure in percentages for the value of market price of risk of 0.01528 – Values in the cells show percentage difference between Case 4.1 and the Case shown in the column header. The pensioner's preferences towards risk and bequest are given in the first row. The values of interest rate in the year prior to retirement are given in the first column. Pension wealth is 200,000, pensioner's age is 65.

If the value of the market price of risk is lower, then the return on bond and cash investment decreases and also annuity rates are less attractive for the pensioner. In further results not presented here, we show that the demand for bonds decreases and the demand for cash investment increases if the market price of risk decreases.

The pensioner's demand for annuities slightly decreases and the gains from access to annuities also decrease slightly. We observe a lower decrease of the pensioner's gains from access to annuities for the pensioner with a bequest motive compared to the pensioner with no bequest motive.

Chapter 5

Comparing the Results Between the Models

5.1 The Connection between the Models

The problems in Cases 3.1, 3.3 and 3.5 in Chapter 3 and the problems in Cases 4.1, 4.2 and 4.3 in Chapter 4 respectively, differ in the assumption regarding interest rate. In the inflation risk model in Chapter 3, we model the interest rate as a deterministic variable taking a constant value. In the interest rate risk model in Chapter 4, the interest rate is modelled as stochastic process taking random values and based on stochastic interest rate we introduce bonds as an investment available for both the pensioner and the annuity provider. In mathematical terms, if we let the variability and the market price of risk of the interest rate tend to zero in the interest rate model in Chapter 4, then the problem in Cases 3.1, 3.3 and 3.5 in the inflation risk model becomes the same as the problem in Cases 4.1, 4.2 and 4.3 in the interest rate risk model.

In Chapters 3 and 4 we compare the results between different cases, where cases are differentiated by the assumptions regarding the pensioner's access to annuities. In this chapter, we focus on the results from the interest rate risk model depending on the introduction of stochastic interest rate. We focus on the results within a single case depending on the value of the interest rate in the year prior to investment and annuitisation. We also compare these results with the results from the appropriate example in the inflation risk model. Thus, in Chapter 5, we make a more thorough analysis of the outcomes related to the introduction of the stochastic interest rate and to the introduction of the third asset.

Cases 3.2, 3.4 and 3.6 in Chapter 3 cannot be compared with the interest rate risk model. In these cases, the pensioner has access to nominal annuities and the inflation influences the results, while in the interest rate risk model in Chapter 4 the pensioner has no access to nominal annuities and is not subject to inflation risk. In the interest

rate risk model, all values are in real terms and thus we assume no dependence on inflation. In Chapter 5 we will investigate the results from Chapter 3 where inflation is irrelevant and compare those results with the results from Chapter 4. Thus, nominal annuities and inflation are irrelevant for this chapter.

The model in Chapter 4 represents a more realistic view of the real world regarding a pensioner's experience with real annuities because we allow for two sources of randomness: equity and interest rate. We want to investigate the results in the interest rate risk model in more detail so that we recognise the new results related to the stochastic interest rate. In Chapter 4, the result depending on the value of the interest rate in the year prior asset allocation and annuitisation is a single result as interest rate is a constant. From this point of view we will try to isolate the effects of the randomness of equities which can also be investigated in the inflation risk model, and the effects of randomness of the value of the interest rate that can be investigated in the interest rate risk model only. In this chapter we investigate the results where introduction of the stochastic interest rate is important and also the extent to which it is important.

The point where we concentrate is the variability of interest rate introduced in Chapter 4. We investigate the consequences of this extra variability regarding optimal consumption, optimal asset allocation, optimal annuitisation and value function. In the inflation risk model, the annuity rates of real annuities depend on the pensioner's age and on a single value of the interest rate for all ages. In the interest rate risk model, we use the interest rate in the year prior to the observed year as a state variable and the future development of the value of the interest rate depends on this state. Thus, the annuity rate in the interest rate risk model depends on the value of the interest rate in the year prior to annuitisation. In the interest rate risk model, besides the randomness of interest rate, we have introduced the market price of risk that slightly increases the return on one year investment in a rolling bond compared to the mean value of the interest rate during one year, all conditional on the value of the interest rate during the previous year. The points stated in this paragraph are very important for explaining the results in this chapter.

Although the interest rate risk model in Cases 4.1, 4.2 and 4.3 is an improvement over the inflation risk model in Cases 3.1, 3.3 and 3.5, one should carefully choose what to compare amongst the results from the inflation risk model in Cases 3.1, 3.3 and 3.5 and interest rate risk models. We must be aware that the inflation risk model in Cases 3.1, 3.3 and 3.5 and the interest rate risk model are developed under different

assumptions and the numerical results in Chapter 5 should be compared very carefully. The interest rate risk model is a richer model and gives new results. We compare results side by side in order to see the differences due to the introduction of the stochastic interest rate and the market price of risk. We can observe the inflation risk model in Cases 3.1, 3.3 and 3.5 to be an extreme case of the interest rate risk model in Cases 4.1, 4.2 and 4.3, respectively, when the variability of the interest rate and the market price of risk tend to be zero.

If we do not state otherwise, we assume the same parameter values as stated in Chapter 3 and 4 in Sections 3.2.1 and 3.4.1 and Sections 4.3.1 and 4.5.1.

5.2 Control Variables and Value Function

The pensioner's optimal consumption, asset allocation and annuitisation strategy in the interest rate risk model, as well as the gains from access to annuities depend on the value of the interest rate during the previous year. However, the results again significantly depend on the pensioner's preferences towards risk and bequest, and also on the market price of risk. In Chapter 5, we investigate the pensioners with the value of RRA coefficient $\gamma = -1$ and $\gamma = -9$, and the pensioners with and with no bequest motive. Regarding market price of risk, throughout Chapter 5 we assume the same value as it is defined in Section 4.5.1.

We assume 15 possible states of the interest rate in the interest rate risk model. The results depend on the value of the interest rate during the previous year, and we choose five representative states of the interest rate: the two extreme (the lowest and the highest value) symmetric states, two symmetric states that are attainable with a reasonable probability, and mean value state. If we order the states of the interest rate defined in Section 4.5.1, from the lowest to the highest then amongst all possible states of the interest rate in the previous year we choose the first, fifth, eighth, eleventh and fifteenth states with the values $\{-2.44\%, -0.56\%, 2.00\%, 4.56\%, 6.44\%\}$. In Section 5.2, we present the results under the assumption that retirement starts at the pensioner's age 65, income at age 65 $Y_{65} = 33,320.90$, and income from social security is $Y_t^{ss} = 22,728.85$ for $66 \leq t \leq 99$. The results obtained from stochastic simulations and the results concerning expected discounted utility are all obtained with the assumption that pension wealth at age 65 is 200,000.

In Section 5.2, we present and analyse the results obtained from stochastic simulations, and also include some examples of the numerical values of the control variable functions depending on the pensioner's age and wealth. Regarding stochastic simulations, we produce a sample of 2,000 realisations of the paths of optimal consumption, optimal asset allocation and optimal annuitisation. From the sample of 2,000 random realisation paths we calculate and present on the graph the mean, 5% quantile and 95% quantile values for each age. We use the same random realisations that we use for the check of accuracy of the numerical calculations and for the left-tail analysis. We will not present here in the thesis, but we checked a couple of examples of the results obtained from the more than 2,000 random realisations and the results were not dependent on the number of the random realisations.

For each figure, we will emphasise if the results represent the values of an optimal control variable function or the numerical realisations of stochastic simulations. However, the results based on stochastic simulations show us the differences of the mean, 5% quantile and 95% quantile values of optimal control variables for each age for the pensioner with given pension wealth and income at age 65 and the value of the interest rate during the year preceding retirement. Now, observing the differences of the mean, 5% quantile and 95% quantile values of optimal control variables for each age, we can determine for how many years and in which way in each year, the value of the interest rate during the year preceding retirement influences the pensioner's optimal decisions. We cannot make these conclusions from the numerical values of the control variable deterministic functions that depend on the pensioner's age and wealth.

5.2.1 Optimal Consumption

The values of optimal consumption change just slightly as the value of the interest rate in the previous year changes. In Section 4.5.2, we have presented the surfaces of optimal consumption for the value of the interest rate in the previous year equal to 2.00%. When we change the value of the interest rate, the surfaces of optimal consumption are very similar. In order to see the small differences in the values of the pensioner's optimal consumption clearly, in Section 5.2.1 we present the graphs of the development of the mean, 5% quantile and 95% quantile values obtained from stochastic simulations.

In Figure 5.1, we show the mean, 5% and 95% quantiles paths of optimal consumption in Case 3.1 and Case 4.1 for the pensioner with $b_t = 1$, $\gamma = -9$ and for different values of the interest rates during the year before retirement.

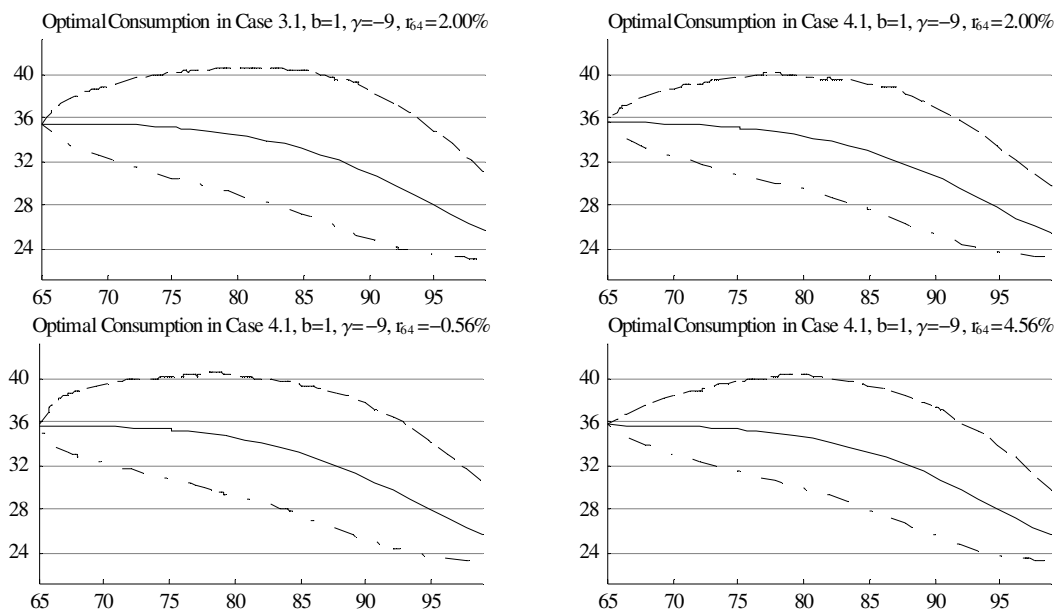


Figure 5.1 Optimal consumption – mean, 5% and 95% quantile in Case 3.1 (upper left hand side graph), and in Case 4.1 (other graphs) for the pensioner with bequest motive and the value of RRA coefficient $\gamma = -9$, for the different values of interest rates in the year before retirement in Case 4.1. Mean consumption (full line), 5% quantile of consumption (dash and dot line, lower line) and 95% quantile of optimal consumption (dash line with longer dashes, upper line). The numerical values in graphs are calculated from 2,000 random realisations.

The differences in the graphs in Figure 5.1 are very small. We observe an increase of the values of mean optimal consumptions in Case 4.1 as the value of the interest rate in year before retirement increases. The differences between the values of 95% and 5% quantiles for a given age are the largest and almost the same in Case 3.1 (upper left hand side graph) and in Case 4.1 for $r_{64} = -0.56\%$ (lower left hand side graph). In graphs in Case 4.1 for $r_{64} = 2.00\%$ (upper right hand side graph) and in Case 4.1 for $r_{64} = 4.56\%$ (lower right hand side graph), we can see that the differences between the values of 95% and 5% quantiles for a given age are decreasing as the value of the interest rate in the year prior to retirement increases.

For the less risk averse pensioner with $\gamma = -1$ compared with the more risk averse pensioner, we observe even smaller differences of the values of mean, 5% and 95% quantiles paths of optimal consumption in Case 3.1 and in Case 4.1 for different

values of interest rates. We have similar changes in the value of mean, 5% and 95% quantiles paths of optimal consumption for the pensioner with and with no bequest motive.

In Figure 5.2, we show mean, 5% and 95% quantiles paths of optimal consumption in Case 3.3 and Case 4.2 for the pensioner with $b_t = 1$, $\gamma = -9$ and for different values of interest rates during the year before retirement.

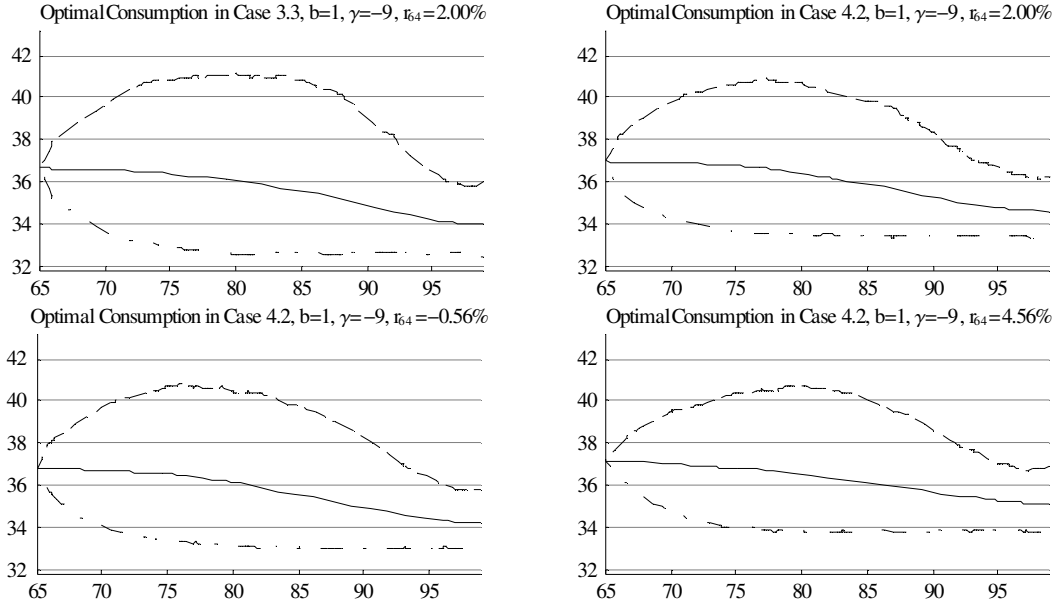


Figure 5.2 Optimal consumption – mean, 5% and 95% quantile in Case 3.3 (upper left hand side graph), and Case 4.2 (other graphs) for the pensioner with bequest motive and RRA coefficient $\gamma = -9$, for the different values of interest rates in the year before retirement in Case 4.2. Mean (full line), 5% quantile (dash and dot line, lower line) and 95% quantile of optimal consumption (dash line with longer dashes, upper line). The numerical values in graphs are calculated from 2,000 random realisations.

We observe the same pattern in Figure 5.2 as in Figure 5.1. However, the differences in graphs in Figure 5.2 are clearer. We emphasise that we use a smaller range of values on y-axis on Figure 5.2 than in Figure 5.1. From Figure 5.2 and from the other numerical solution not presented here, we find that the changes of the value of the mean, 5% and 95% of optimal consumption paths have a regular behaviour apart from the random error which is due to the limited size of the random sample.

Regarding the values of mean optimal consumption paths in Figure 5.2, we observe the lowest values in Case 3.3 (upper left hand side graph) and in Case 4.2 for $r_{64} = -0.56\%$ (lower left hand side graph). In Case 4.2 for $r_{64} = 4.56\%$ (lower right

hand side graph) the values of mean optimal consumption paths are the highest in Figure 5.2. We find that in Case 4.2, the values of mean optimal consumption paths increase as the value of the interest rate in the year before retirement increases.

Then we observe that the range between 95% and 5% quantiles is the largest in the graph showing consumption paths in Case 3.3 where the 5% quantile line is the lowest and the 95% quantile line is the highest one amongst all the graphs in Figure 5.2. Although we have one less source of risk in Case 3.3 than in Case 4.2, this observation can be justified. The first reason for less variability in Case 4.2 is that slightly more annuities are bought in Case 4.2 than in Case 3.3 and this will be presented in Section 5.2.3. The next reason lies in the possibility (that will also be clearer from the rest of Section 5.2.3) for the pensioner to behave optimally according to the state of the interest rate in the year prior investment and annuitisation decisions during the whole retirement period. The pensioner uses this opportunity optimally and achieves better results in terms of optimal consumption.

Amongst the graphs for Case 4.2, we see that 5% and 95% quantiles lines move upwards with the changes of the value of the interest rate during the year preceding retirement. We find that when the value of the interest rate during the year preceding retirement increases, the values of 5% quantiles increase more compared to the increase of the value of mean and 95% quantile of optimal consumption.

In Figure 5.3, we present mean, 5% and 95% quantiles paths of optimal consumption for the same pensioner as in Figures 5.1 and 5.2, but now in Case 3.5 and Case 4.3 where the pensioner has access to annuities during the whole retirement period.

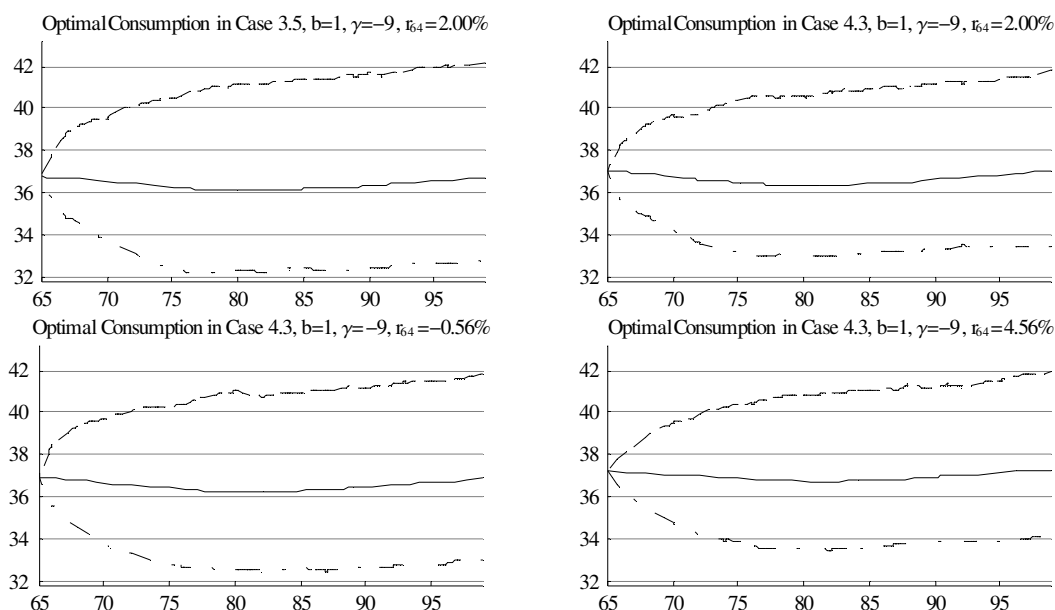


Figure 5.3 Optimal consumption – mean, 5% and 95% quantile in Case 3.5 (upper left hand side graph), and Case 4.3 (other graphs) for the pensioner with bequest motive and RRA coefficient $\gamma = -9$, for the different values of interest rates in the year before retirement in Case 4.3. Mean (full line), 5% quantile (dash and dot line, lower line) and 95% quantile of optimal consumption (dash line with longer dashes, upper line). The numerical values in graphs are calculated from 2,000 random realisations.

In Figure 5.3, one can see a similar movement of the data on the graphs as we have observed in Figure 5.2. In Case 3.5, the range between the 5% and 95% quantile lines is larger than in all examples in Case 4.3. It means that the pensioner in Case 4.3 uses the opportunity of optimal annuitisation according to bond prices and gets lower variability of consumption. When comparing the mean and 5% and 95% quantile lines in Case 4.3 for the different values of the interest rate during the year before retirement, we observe a small movements of all these lines upwards as the value of interest rate increases and it is the result of the better pensioner's state if the value of the interest rate is higher. If the value of the interest rate is higher during the year before retirement, the pensioner will purchase more annuities and at the better annuity rate earlier in retirement.

An interesting, and maybe an unexpected, observation in each graph in Figure 5.3 is the decreasing mean consumption lines for the earlier years of retirement and the increasing mean during the later years of retirement. Observing the values of the mean optimal consumption paths, we find that the values change moderately during retirement, and that there is no steep decrease or increase of the values of mean optimal consumptions paths. In the results not presented here, the same shape of the

mean optimal consumption paths can be observed for the values of RRA coefficient $\gamma = -4$ and $\gamma = -1$. We find a U-shape of the mean optimal consumption path obtained from stochastic simulations in Cases 3.5, 3.6 and 4.3 and only if the pensioner has a bequest motive and for all investigated values of RRA coefficient. The reason for this shape of the mean optimal consumption path is the possibility for the pensioner to keep the overall value his assets, pension wealth and implied asset from income, during the whole period of retirement and even to increase it slightly at older ages due to survival credits. In Cases 4.1 and 4.2 and any investigated pensioner's preferences towards risk and bequest, we observe that the values of mean optimal consumption paths from stochastic simulations decrease as the pensioner gets older.

In Case 4.1, the pensioner has no access to annuities and he optimally invests his pension wealth and consumes his income and part of his pension wealth throughout the whole period of retirement. However, his income is constant and its implied value decreases as the pensioner gets older and as his pension wealth decreases as well he optimally consumes lower amounts as he gets older. Also, at older ages the pensioner possesses a lower pension wealth and variability of his consumption decreases because of the lower variability of his pension wealth.

In Case 4.2, the pensioner has access to annuities at age 65 only and he uses that opportunity for purchasing annuities optimally. The pensioner in this case has constant income after age 65 and his asset implied form income decreases as he gets older. His pension wealth decreases during the retirement and at some age his pension wealth will approach certain minimal value and will not decrease anymore because the pensioner has a bequest motive. So, his pension wealth will approach a certain minimal value and his optimal consumption will keep decreasing. However, comparing with the pensioner in Case 4.1, the pensioner in Case 4.2 will have smaller variability of his consumption as his pension wealth is on average will take lower values, due to purchased annuities at age 65. When pension wealth approaches a minimum value the variability of consumption will be more or less constant until the maximum possible age of the pensioner.

In Case 4.3, we have qualitatively different opportunity for the pensioner. He can purchase annuities whenever in retirement and take advantages of increasing survival credits at older ages. The pensioner in Case 4.3 optimally purchasing annuities at earlier years of retirement and his pension wealth decreases and his income increases. However, survival credits in the early years of retirement are not high enough to

prevent his overall assets, pension wealth and assets implied in income, from decreasing. So, we observe small decrease in the mean optimal consumption during early years of retirement. Again, we observe that the pensioner will keep a certain minimum value of the pension wealth due to a bequest motive. Interesting optimal behaviour of the pensioner in Case 4.3 happens at older ages. If the return on the pension wealth is below or at average the pensioner will not purchase more annuities at the end of that year and his overall all assets will decrease and optimal consumption will decrease. However, if the return on investment is above average, the pensioner will optimally purchase new annuities and take advantage of increasing survival credits at older ages. Thus, the pensioner in Case 4.3 will purchase new annuities until very late ages and increase his income and also his asset implied in income. At older ages survival credits are so high that they provide the opportunity for the pensioner in Case 4.3 to increase his overall assets at older ages and as a result we observe increase in the mean optimal consumption. Also, we observe increase of the 95% quantile line because of the possibility of significantly increase in the pension wealth and asset implied from income if we have a couple of years of better than expected returns on investments. However, we emphasise that the mean consumption line increases as the pensioner in Case 4.3 getting older. For one particular random realisation, it is possible that optimal consumption decreases if returns on investments are not above average and purchasing more annuities is not optimal at older ages because of the minimal pension wealth the this pensioner keeps due to his bequest motive.

Regarding the changes of the values of mean, 5% and 95% optimal consumption path in Case 4.3 in Figure 5.3, we observe the same pattern as in Figure 5.2. As the value of the interest rate during the year before retirement increases, we find that all three lines move upwards, and that 5% optimal consumption line moves upwards to a greater extent than the mean and 95% optimal consumption lines.

5.2.2 Optimal Asset Allocation

In Case 3.1, the pensioner with the value of RRA coefficient $\gamma = -1$ optimally invests at each age 100% of pension wealth in equities for any investigated amount of pension wealth.

The pensioner with $\gamma = -1$ in Case 4.1 optimally invests 100% of pension wealth in equities at each age and for all but very high values of the interest rate during the year prior to the investment decision. For example, if the value of the interest rate is 6.44%

at age 65, this pensioner optimally invests about 60% into equities. However, this value of the interest rate has a low probability and very soon the interest rate moves closer to the mean and then 100% into equities is optimal again.

In Figure 5.4, we show the value of the mean, 5% and 95% quantiles of optimal equity allocation obtained from 2,000 random realisations of stochastic simulation for the pensioner with no bequest motive and RRA coefficient $\gamma = -9$, in Cases 3.1 and 4.1, and for the different values of interest rate during the year prior to retirement.

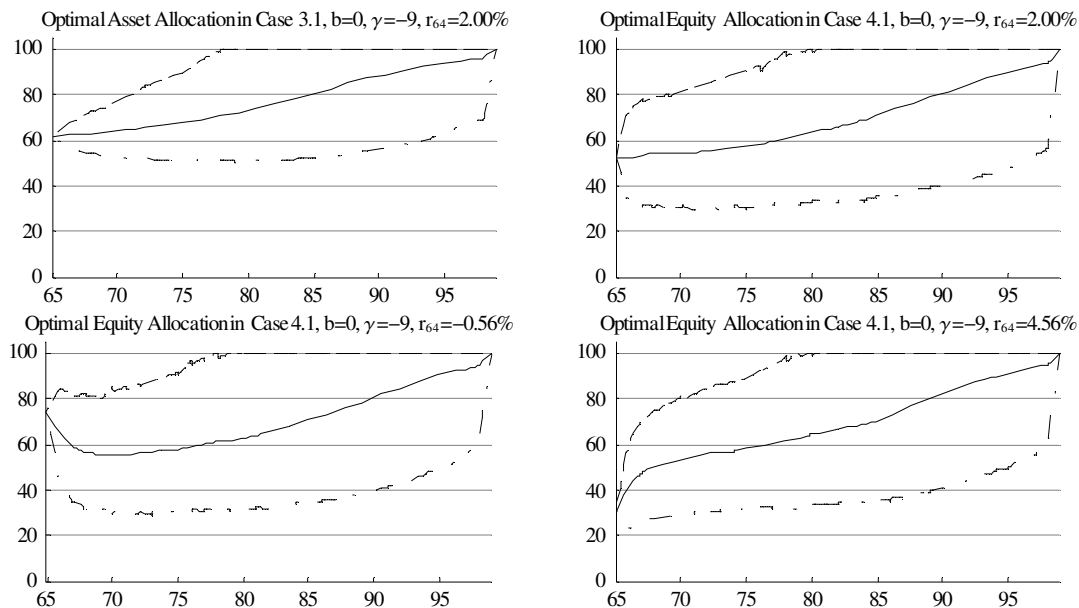


Figure 5.4 Optimal equity allocation – mean, 5% and 95% quantile in Case 3.1 (upper left hand side graph), and Case 4.1 (other graphs) for the pensioner with no bequest motive and RRA coefficient $\gamma = -9$, for the different values of interest rates in the year before retirement in Case 4.1. Mean (full line), 5% quantile (dash and dot line, lower line) and 95% quantile of optimal equity allocation (dash line with longer dashes, upper line). The numerical values in graphs are calculated from 2,000 random realisations.

The pensioner with no bequest will generally have a mean value of optimal equity allocation that increases with age. However, one should bear in mind that in the later years of retirement the pension wealth available for investment is very low. Thus, the investment strategy for the pensioner with no bequest motive is actually interesting for earlier retirement ages only, say up to age 85 or 90.

The first observation is that the value of the mean optimal equity allocation is larger in Case 3.1 than in Case 4.1. The reason is that bond investment in Case 4.1 offers, on average, a better return than a constant interest rate in Case 3.1. The lower left hand

side graph represents the pensioner in Case 4.1 when the value of the interest rate during the year before retirement is -0.56% . The demand for bonds for this pensioner is lower due to the lower value of the interest rate during the year before retirement and the pensioner optimally invests more into equities. Up to age 72, this pensioner invests on average more into equities than the pensioner when the value of the interest rate before retirement is 2.00% and after that age both pensioners have very similar graphs in Figure 5.4. It seems that the effects of the lower than average interest rate before retirement last for about 5–7 years regarding optimal equity investment. A similar situation, but with a higher demand for bonds at the early ages, is for the pensioner experiencing a value of the interest rate before retirement of 4.56% . This pensioner's optimal equity allocation (lower right hand side graph) is on average lower for about the first 5–7 years of retirement compared to the optimal equity allocation of the pensioner experiencing the value of the interest rate before retirement of 2.00% (upper right hand side graph). After the first 5–7 years of retirement these two pensioners have very similar graphs in Figure 5.4.

It is interesting that in the upper right hand side graph and two lower graphs in Figure 5.4 that the 5% and 95% quantile lines are also very similar after age of 72. It means that the variability of the optimal equity investment in Case 4.1 is influenced by the interest rate at age 65 only for the first few years during retirement.

Another interesting observation in Figure 5.4 can be found when we compare the 5% and 95% quantile lines in the graph showing Case 3.1 and the graphs showing Case 4.1. The distance between the 5% and 95% quantile lines is larger in Case 4.1 than in Case 3.1 for each age. It means that optimal equity allocation is more variable in Case 4.1 than in Case 3.1. The reason for more variability in Case 4.1 is the randomness of the interest rate. The pensioner in Case 4.1 optimally invests in equities not only based on current values of pension wealth, income and age, but also based on the value of the interest rate experienced in the year prior to the optimal decisions. Due to there being one more source of risk in Case 4.1 compared to Case 3.1, more variability in the pensioner's optimal equity allocation is observed in Case 4.1. If the interest rate in the year prior to the time of investment decision is lower then the higher proportion of pension wealth is invested in bonds and less into equities, and vice versa. The pensioner whose optimal equity allocation is presented in Figure 5.4 has no demand for cash.

We can connect the observation regarding variability of optimal equity allocation in the previous paragraph with the observation in Section 5.2.1 that optimal consumption

is slightly less variable in the interest rate risk model compared with the appropriate values in the inflation risk model. We can conclude that the pensioner in the interest rate risk model in Case 4.1 has a more proactive optimisation strategy and in using it he makes decisions that are more variable than in the inflation risk model in Case 3.1. However, using these optimal decisions the pensioner in the interest rate risk model achieves a lower variability of the optimal consumption.

In Figure 5.5, we present the same results as in Figure 5.4 but now for the pensioner with a bequest motive.

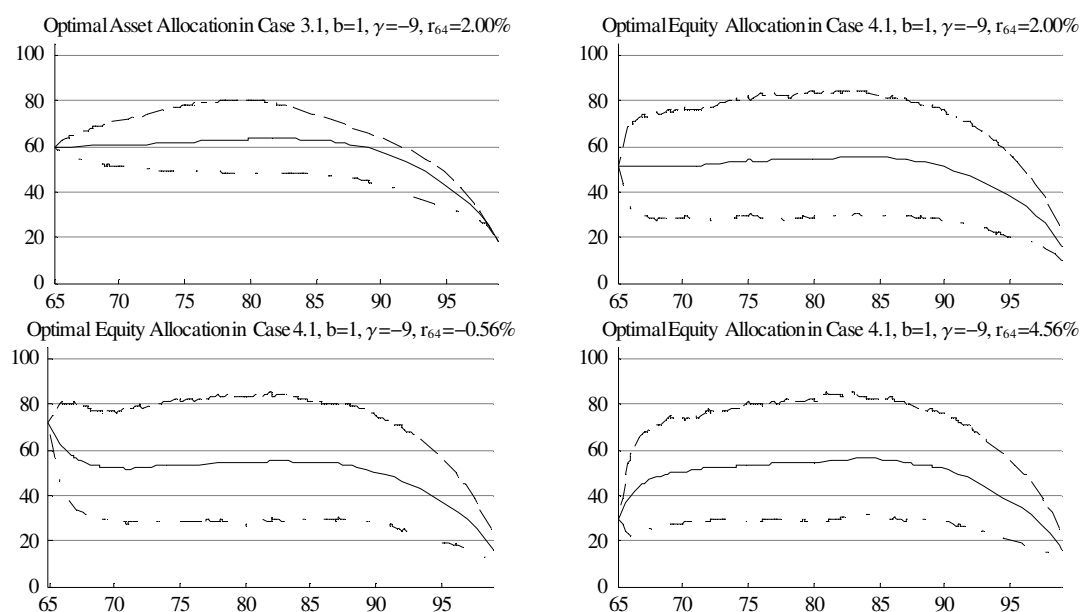


Figure 5.5 Optimal equity allocation – mean, 5% and 95% quantile in Case 3.1 (upper left hand side graph), and Case 4.1 (other graphs) for the pensioner with bequest motive and RRA coefficient $\gamma = -9$, for the different values of interest rates in the year before retirement in Case 4.1. Mean (full line), 5% quantile (dash and dot line, lower line) and 95% quantile of optimal equity allocation (dash line with longer dashes, upper line). The numerical values in graphs are calculated from 2,000 random realisations.

In Cases 3.1 and 4.1, the pensioner with a bequest motive and RRA coefficient $\gamma = -9$ has a moderately increasing mean optimal equity allocation from around age 70 to around age 85 and then a decreasing mean optimal equity investment until the maximum possible pensioner's age. The reason for this shape of mean optimal equity investment line in each graph in Figure 5.5 is the amount of pension wealth in later years of the retirement period. Thus, the pensioner with a bequest motive optimally keeps part of his pension wealth during the whole retirement period. From the data not

presented here, we find that the amount of pension wealth is usually larger than the income. In the later years of retirement, the amount of pension wealth is already decreased and the pensioner draws utility from keeping a certain amount of pension wealth available for the bequest. So, the pensioner with a bequest motive keeps the remaining pension wealth in a less risky portfolio in the later years of retirement.

Observations regarding the values of the mean, 5% and 95% quantile lines of optimal equity allocation in the graphs in Figure 5.5 are similar to the observations of the graphs in Figure 5.4 and we will not repeat them. We only emphasise again that the duration of the period while significant differences in the values of mean, 5% and 95% quantile of optimal equity allocation due to the different values of interest rate in the year preceding retirement in Case 4.1 in Figure 5.5 is about 5–7 years. The duration of this period is similar as in Figure 5.4 for the pensioner with no bequest motive.

In Figure 5.6, we present the numerical values of mean, 5% and 95% of optimal equity allocation in Cases 3.3 and 4.2 for the pensioner with a bequest motive and with the value of RRA coefficient $\gamma = -9$.

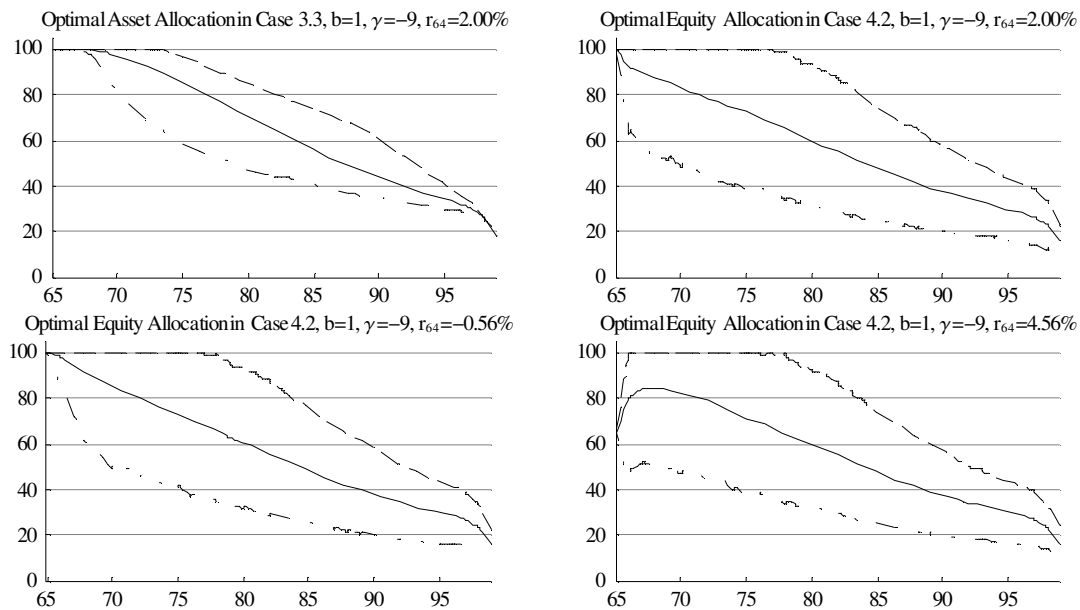


Figure 5.6 Optimal equity allocation – mean, 5% and 95% quantile in Case 3.3 (upper left hand side graph), and Case 4.2 (other graphs) for the pensioner with bequest motive and RRA coefficient $\gamma = -9$, for the different values of interest rates in the year before retirement in Case 4.2. Mean (full line), 5% quantile (dash and dot line, lower line) and 95% quantile of optimal equity allocation (dash line with longer dashes, upper

line). The numerical values in graphs are calculated from 2,000 random realisations.

In Case 3.3, the pensioner optimally purchases annuities instead of investing in cash and during the first few years of retirement he optimally invests all the available pension wealth in equities. Thus, in Case 3.3 the pensioner uses annuities as a substitution for the risk free asset. Figure 5.6 relates to the pensioner with a bequest motive, and again he is keen to keep a certain part of his pension wealth for the bequest and his risky investments decrease at later ages.

Regarding the pensioner in Case 4.2 in Figure 5.6, he has a similar pattern of optimal equity investment observed as a function of the interest rate during the year prior to retirement as the pensioner in Case 4.1 in Figure 5.5. During the first few years of retirement, the value of the interest rate in the year before retirement influences the pensioner's decisions regarding optimal investment in equities. After say 5 years, this influence disappears and the pensioner's decisions in terms of the values of the mean, 5% and 95% quantiles of optimal equity allocation are more or less the same for any value of the interest rate during the year before retirement.

Comparing the two upper graphs in Figure 5.6, we observe same pattern as earlier in this section. In Case 4.2 when the value of the interest rate during the year before retirement is 2.00%, the mean optimal equity allocation line is a few percentage points below the mean optimal equity allocation line in Case 3.3. Also, the distance between the mean and 5% quantile lines and the distance between the mean and 95% quantile lines of optimal equity investment in Case 3.3 are smaller than the relevant distances in Case 4.2.

In results not presented here, we find that the pensioner with no bequest motive and with the same attitude towards risk will optimally invest all available pension wealth in equities throughout the whole retirement period. This means that annuities are a better investment than risk free asset for this pensioner, and he uses the opportunity of access to annuities at age 65 so that nothing is invested in the risk free asset after retirement.

In Figure 5.7, we present the values of the mean, 5% and 95% quantiles of optimal equity allocation for the pensioner with a bequest motive and with the value of RRA coefficient $\gamma = -9$, in Cases 3.5 and 4.3.

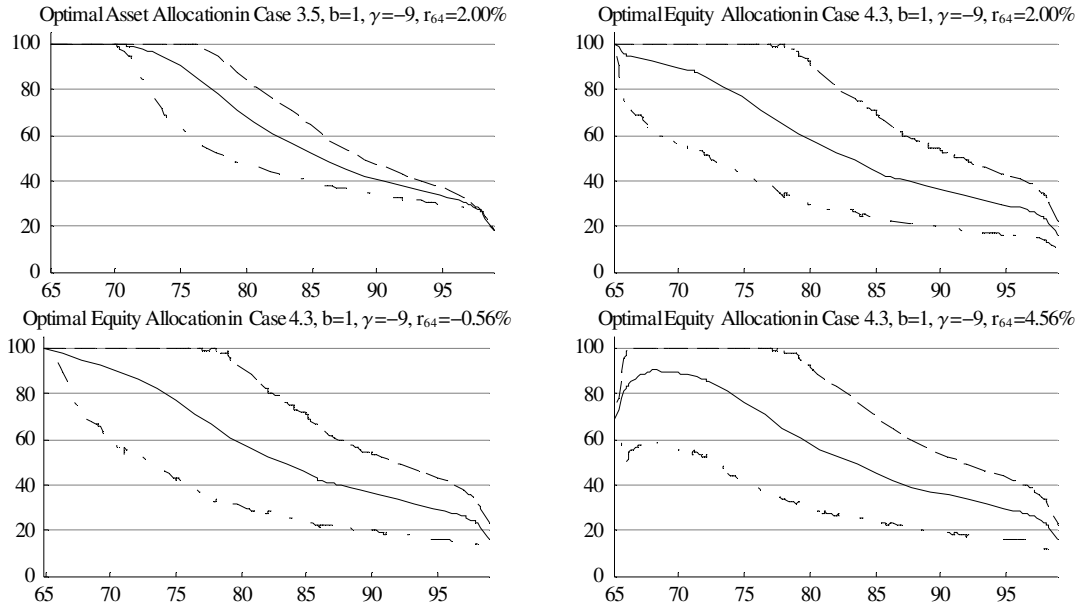


Figure 5.7 Optimal equity allocation – mean, 5% and 95% quantile in Case 3.5 (upper left hand side graph), and Case 4.3 (other graphs) for the pensioner with bequest motive and RRA coefficient $\gamma = -9$, for the different values of interest rates in the year before retirement in Case 4.3. Mean (full line), 5% quantile (dash and dot line, lower line) and 95% quantile of optimal equity allocation (dash line with longer dashes, upper line). The numerical values in graphs are calculated from 2,000 random realisations.

In Cases 3.5 and 4.3, the pensioner with preferences $b_t = 1$ and $\gamma = -9$ optimally invests in equities using a large proportion of his pension wealth at an early retirement age and afterwards the proportion of pension wealth invested in equities decreases. However, a significant part of pension wealth is annuitised at age 65, which we will present in Table 5.1 in Section 5.2.3. Due to this fact, the optimal equity allocation strategy remains similar as in Cases 3.3 and 4.2 with the difference that the optimal asset has higher equities allocation in early ages and then the decrease of mean optimal equity allocation is steeper in Cases 3.5 and 4.3. Other characteristics observed for Cases 3.3 and 4.2 remain the same.

We present in Figure 5.8 the surfaces of optimal equity allocations for the pensioner with a bequest motive and with the value of RRA coefficient $\gamma = -9$, in Cases 3.5 and 4.3. The surfaces in Figure 5.8 are deterministic functions that are the solutions of the optimisation problem in Chapter 4. The upper left hand side surface in Figure 5.8 is very similar to the right hand side surface in Figure 3.5, and the upper right hand side surface is already presented in Figure 4.7.

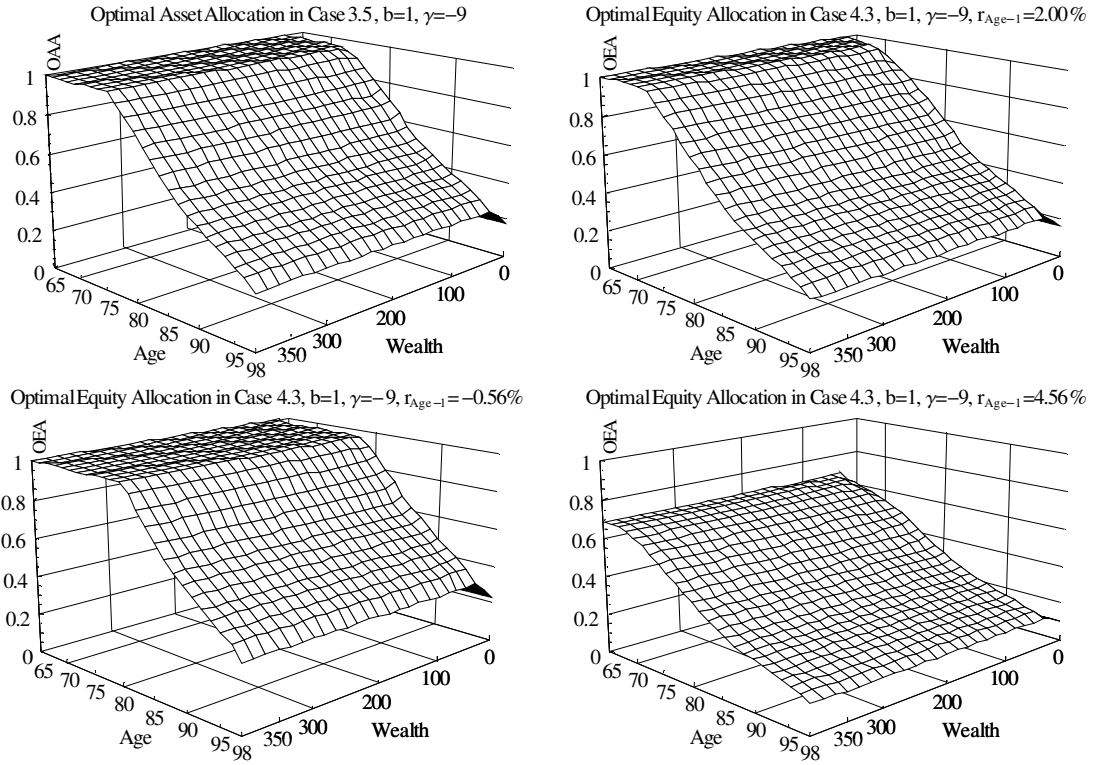


Figure 5.8 Optimal equity allocation – in Case 3.5 (upper left hand side graph), and Case 4.3 (other graphs) for the pensioner with bequest motive and RRA coefficient $\gamma = -9$, for the different values of interest rates in the year before retirement in Case 4.3. The numerical values in the surfaces are calculated from the deterministic functions of optimal control variables.

In Figure 5.8, we observe similar surfaces in Case 3.5 and in Case 4.3 when the value of the interest rate in the year prior to equity investment is 2.00%. The lower left hand side surface, when the value of the interest rate in the year prior equity investment is -0.56% , is moved upwards and is steeper compared to the upper right side surface. The lower right hand side surface, when the value of the interest rate in the year prior equity investment is 4.56% , is shifted downwards and is less steep compared to the upper right side surface. Thus, an increase in the value of the interest rate in the year prior equity investment results in the downward movement of the surface of the optimal equity allocation and also in a less steep surface.

The downward movement of the value of the mean, 5% and 95% as the value of the interest rate before retirement decreases is also observed in Figure 5.7 in the first few years of retirement. The previous paragraph explains the reasons for this observation in Figure 5.7. After a few years in retirement, the value of the interest rate does not depend on the value of the interest rate in the year prior to retirement and the graphs in Figure 5.7 are almost the same after say 5 years. In Figure 5.8, we observe the

differences in optimal equity allocation due to the differences of the value of the interest rate in the year prior to the equity investment are similar for any age. Also, it seems that the optimal equity allocation does not change its value for the pensioners with different amounts of pension wealth. However, from Figure 5.8 we cannot get an idea of how many years during the retirement period the pensioner's optimal equity allocation is going to be influenced by the value of the interest rate during the year before retirement. This can only be seen in the figures presenting the realisations of stochastic simulations. Thus, for presenting the results regarding optimal control variables, the majority of the results presented in this chapter are obtained from stochastic simulations.

5.2.3 Optimal Annuitisation

In this thesis, we investigate two main annuitisation policies, optimal annuitisation at age 65 only with no annuities afterwards, and optimal annuitisation at any age during retirement. In Cases 3.3 and 4.2, the pensioner optimally annuitises at age 65 only and as a result we obtain a single number only. In Cases 3.5 and 4.3, the pensioner optimally annuitises at any age, and as a result we obtain optimal annuitisation for each age. In Cases 3.5 and 4.3, optimal annuitisation at each age depends on the development of the random variables in the earlier years. In Section 5.2.3, we investigate pensioner's optimal annuitisation strategy but now with a particular emphasis on the dependence of optimal annuitisation on the value of the interest rate.

In Table 5.1, we present optimal annuitisation percentages at age 65 in Case 4.2 for the pensioner aged 65 and for the different values of the interest rate during the year before retirement. Pension wealth is 200,000, income from the last salary is $Y_{65} = 33,320.90$, and income from social security is $Y_t^{ss} = 22,728.85$ for $66 \leq t \leq 99$. The values of the optimal annuitisation percentages presented in Table 5.1 are the deterministic solution of the problems in Chapters 3 and 4.

Interest rate	$b_t = 0 \quad \gamma = -1$		$b_t = 0 \quad \gamma = -9$		$b_t = 1 \quad \gamma = -1$		$b_t = 1 \quad \gamma = -9$	
	Case 3.3	Case 4.2	Case 3.3	Case 4.2	Case 3.3	Case 4.2	Case 3.3	Case 4.2
-2.44		30.16%		84.15%		3.23%		69.36%
-0.56		36.54%		85.43%		9.64%		70.18%
2.00	31.04%	46.33%	84.33%	87.36%	4.36%	18.75%	69.73%	71.39%
4.56		55.50%		89.15%		27.29%		72.41%
6.44		60.93%		90.19%		31.80%		72.94%

Table 5.1 Optimal annuitisation – in Case 3.3 and in Case 4.2, for the pensioner with different preferences towards risk and bequest, and for different values of interest rate in the year preceding retirement. Initial pension wealth is 200,000. The numerical values in the cells are calculated from the deterministic functions of optimal control variables.

For the pensioner with $b_t = 0 \quad \gamma = -1$, and for the pensioner with $b_t = 1 \quad \gamma = -9$, the values of optimal annuitisation in Case 3.3, presented in Table 5.1 are also presented in Figure 3.3 in Chapter 3. For the pensioners in Case 4.2 with the different combinations of the values of b_t and γ , optimal annuitisation percentages are also presented in Figure 4.5 in Chapter 4.

In Table 5.1, we observe that the pensioner who purchases annuities optimally at age 65 has more demand for annuities in the interest rate risk model in Case 4.2 than in the inflation risk model in Case 3.3. We know from the definition of the annuity factor in the inflation risk and interest rate risk models that the annuity factor in the interest rate risk model is slightly lower for the mean value of the interest rate in the previous year, and provides a slightly better annuity income. In the inflation risk model, we calculate the annuity rate using a constant value of the interest rate. In the interest rate risk model, the annuity rate is calculated using bond prices with the appropriate duration. As the market price of risk increases the bond prices, we get a slightly more attractive annuity factor in the three than in the inflation risk model, if the value of the interest rate in the year preceding retirement is 2.00%. In all examples shown in Table 5.1, the pensioner in the inflation risk model optimally annuitises similar amount as the pensioner in the interest rate risk model when the value of the interest rate is – 2.44%.

In Table 5.1 in Case 4.2, we observe a wide range of the percentages representing optimal annuitisation of the pensioner with RRA coefficient $\gamma = -1$. Thus, for the less risk averse pensioner in Case 4.2, the decisions regarding optimal annuitisation significantly depend on the value of the interest rate during the year before retirement.

The more risk averse pensioner in the interest rate risk model has a narrower range of the percentages of optimal annuitisation in Case 4.2. Further, for any value of the interest rate in the year before retirement, this pensioner annuitises a significant part of his pension wealth.

From the values in Table 5.1 and from other results not presented here, we find that annuities, as a protection from future uncertain development of random equity and interest rates, are important for the pensioner in Case 4.2. The more risk averse is the pensioner, the higher is the percentage of optimal annuitisation at age 65 in Case 4.2, for a given value of the interest rate. Also, the more risk averse is the pensioner, the narrower is the range of the percentages of optimal annuitisation at age 65 in Case 4.2, for different values of the interest rate.

Now we will investigate optimal annuitisation in Case 3.5 and Case 4.3. The pensioner in Cases 3.5 and 4.3 firstly makes an optimal annuitisation decision at age 65. That decision is a single number depending on state variables at age 65. In Case 3.5, the pensioner's decision regarding optimal annuitisation in later years of retirement is conditional on the development of the equity rate experienced in the previous years of retirement and consequently on all other variables depending on the equity rate. These developments are summarised in the values of the state variables at the moment of optimal annuitisation. In Case 4.3, the pensioner's decision regarding optimal annuitisation in later years of retirement is conditional on the development of both equity and interest rate experienced during retirement. Similarly to Case 3.5, all developments, from age 65 to the moment of annuitisation, of the variables depending on the values of random equity and interest rates are summarised into the values of the state variables at the moment of optimal annuitisation.

The pensioner in Case 4.3 optimally annuitises depending on the value of the interest rate in the year prior to the year of the annuitisation decision. If the value of the interest rate in the prior year is lower then the pensioner's demand for annuities will be lower. He will hope that the value of the interest rate in the coming years will be better and he will purchase more annuities in the following years. Conversely, if the value of the interest rate in the preceding year is higher, then the pensioner purchases more annuities at that point of time and less afterwards.

In Figure 5.9, we present the values of mean, 5% and 95% quantiles of optimal annuitisation in Cases 3.5 and 4.3 for the pensioner with no bequest motive and with the value of RRA coefficient $\gamma = -1$.

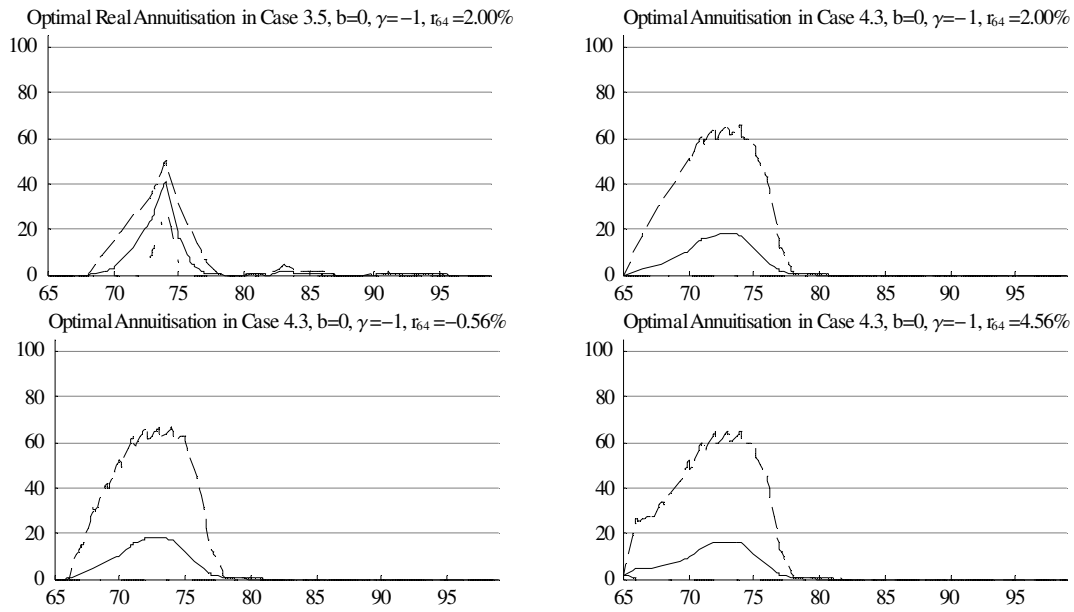


Figure 5.9 Optimal annuity allocation – mean, 5% and 95% quantile in Case 3.5 (upper left hand side graph), and Case 4.3 (other graphs) for the pensioner with no bequest motive and RRA coefficient $\gamma=-1$, for the different values of interest rates in the year before retirement in Case 4.3. Mean (full line), 5% quantile (dash and dot line, lower line) and 95% quantile of optimal equity allocation (dash line with longer dashes, upper line). The numerical values in graphs are calculated from 2,000 random realisations.

Firstly, we observe that optimal annuitisation is significantly different in the inflation risk model in Case 3.5 compared to the interest rate risk model in Case 4.3. In the inflation risk model, in the upper left hand side graph in Figure 5.9, we have no annuitisation at the very early ages of the retirement period and then we have steep increase. Then, at age 73, there is the peak, and then a steep decrease. The distances between the 5% quantile and mean lines and also between the mean and 95% quantile lines are much smaller in the inflation risk model than in any graph representing optimal annuitisation in the interest rate risk model. The other graphs representing optimal annuitisation in the interest rate risk model have similarities. The mean optimal annuitisation does not show a steep increase or decrease and no sharp peak exists. The highest value of mean optimal annuitisation in Case 4.3 is lower than in Case 3.5 and at the same time the 95% quantile line attains larger values in Case 4.3 than in Case 3.5. In Case 4.3, the 5% quantile line is equal to the x-axis which means that there is at least 5% probability that no annuitisation will occur at any age. So, we find that the shapes and the values of mean of optimal annuitisation are similar in all

graphs in Case 4.3 but very different compared to Case 3.5. The same conclusion is valid for the shapes and the values of 5% and 95% quantiles of optimal annuitisation.

If we compare the three graphs in Figure 5.9 representing optimal annuitisation in the interest rate risk model, we observe similar values of the mean, 5% and 95% quantile lines for ages after 70. However, up to the pensioner's age of 70, if the value of the interest rate during the year before retirement is higher than expected, shown in the lower right hand side graph in Figure 5.9, then the pensioner optimally purchases more annuities at age 65 and there is a steeper increase of the 95% quantile line during the first two or three years of retirement. If the value of the interest rate in the year prior to retirement decreases then the pensioner optimally decreases or even defers annuitisation during the early ages of retirement. It means that if the value of the interest rate during the year prior to retirement decreases, the pensioner with no bequest motive and RRA coefficient $\gamma = -1$ in the interest rate risk model purchases fewer annuities in the first five years of retirement in order to use the advantages of possible achieving a better annuity rate in the following years. If the value of the interest rate during the year prior to retirement is very low, then it is optimal for this pensioner to defer annuitisation.

In Figure 5.10, we present the same type of results as in Figure 5.9, but now for the pensioner with a bequest motive and with the value of RRA coefficient $\gamma = -9$. In order to present the most important part of the graphs in more details, we rescale the y-axis such that the values of optimal annuitisation percentages of pension wealth from 0% to 35% are shown.

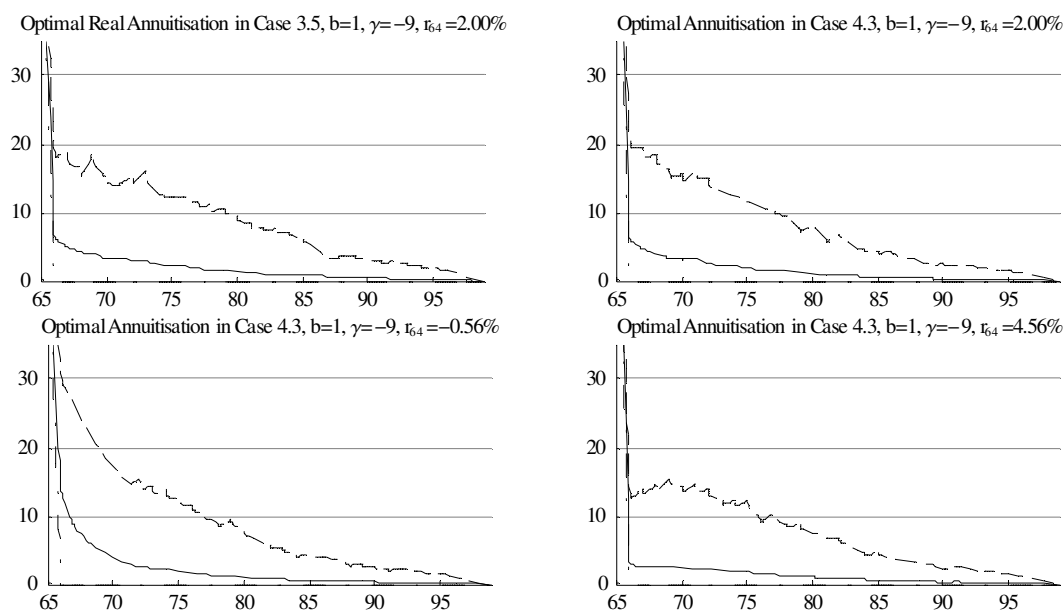


Figure 5.10 Optimal annuity allocation – mean, 5% and 95% quantile in Case 3.5 (upper left hand side graph), and Case 4.3 (other graphs) for the pensioner with bequest motive and RRA coefficient $\gamma = -9$, for the different values of interest rates in the year before retirement in Case 4.3. Mean (full line), 5% quantile (dash and dot line, lower line) and 95% quantile of optimal equity allocation (dash line with longer dashes, upper line). The numerical values in graphs are calculated from 2,000 random realisations

We observe fewer differences between the inflation risk model in Case 3.5 (the upper left hand side graph in Figure 5.10) and the interest rate risk model in Case 4.3 for the average value of the interest rate before retirement (the upper right hand side graph in Figure 5.10) compared to the differences in the equivalent graphs in Figure 5.9. Thus, the graph presenting the mean, 5% and 95% quantile lines of optimal annuitisation in the inflation risk model and the graph presenting the same lines in the interest rate risk model for the average value of the interest rate before retirement show fewer differences for the more risk averse pensioner with a bequest motive than for the less risk averse pensioner with no bequest motive. In results not shown on these graphs, we find that the pensioner with a bequest motive and $\gamma = -9$ optimally annuitises about 63% at age 65 according to the interest rate risk model for the value of the interest rate before retirement of 2.00%, and about 60% according to the inflation risk model.

Regarding the graphs representing the results in the interest rate risk model in Figure 5.10, we clearly observe the differences of the mean and 95% quantile lines of optimal annuitisation up to age 75. It is optimal to annuitise about 54%, 63% and 67%

of the initial pension wealth of 200,000 for the value of initial interest rate during the year before retirement of -0.56% , 2.00% and 4.56% , respectively. Then, the higher optimal annuitisation at age 65 is followed by lower annuitisation in the coming years and vice versa. For instance, if the value of the interest rate before retirement is -0.56% , then the 95% quantile of optimal annuitisation at age 68 is around 23%, and the mean optimal annuitisation is around 6%. If the value of the interest rate before retirement is 2.00% , then the 95% quantile at age 68 is around 19% and the mean is around 4%, while for the value of the interest rate of 4.56% before retirement, the 95% quantile is about 14.5% and the mean is about 2.5%. Thus, we find the following pattern for the pensioner's optimal annuitisation in terms of mean, 5% and 95% quantile values. For the higher values of the interest rate during the year prior to retirement, the pensioner with a bequest motive and with $\gamma = -9$ optimally purchases more annuities at age 65 and fewer annuities afterwards.

In Figures 5.9 and 5.10, we observe a pattern that is actually the same for the pensioner who has any preferences towards risk and bequest. In the interest rate risk model, we find that the pensioner optimally annuitises a lower amount of pension wealth, or completely defers annuitisation, at age 65 if the value of the interest rate during the year before retirement is unfavourable. After age 65, the pensioner waits for a year with a favourable value of the interest rate and annuitises more when the annuity rate is favourable. However, after several years of retirement, the annuity rate becomes favourable due to the mortality drag as well and the pensioner optimally does not defer annuitisation for a too long period. In examples that we have investigated, we observe that the pensioner partly or completely defers annuitisation during a maximum of the first eight years of the retirement period.

In Figure 5.11, we present the surfaces of optimal annuitisation for the pensioner with a bequest motive and with the value of RRA coefficient $\gamma = -9$, in Cases 3.5 and 4.3. The upper left hand side surface in Figure 5.11 is very similar to the lower right hand side surface in Figure 3.6, and the upper right hand side surface in Figure 5.11 is already presented in the lower right hand side in Figure 4.9.

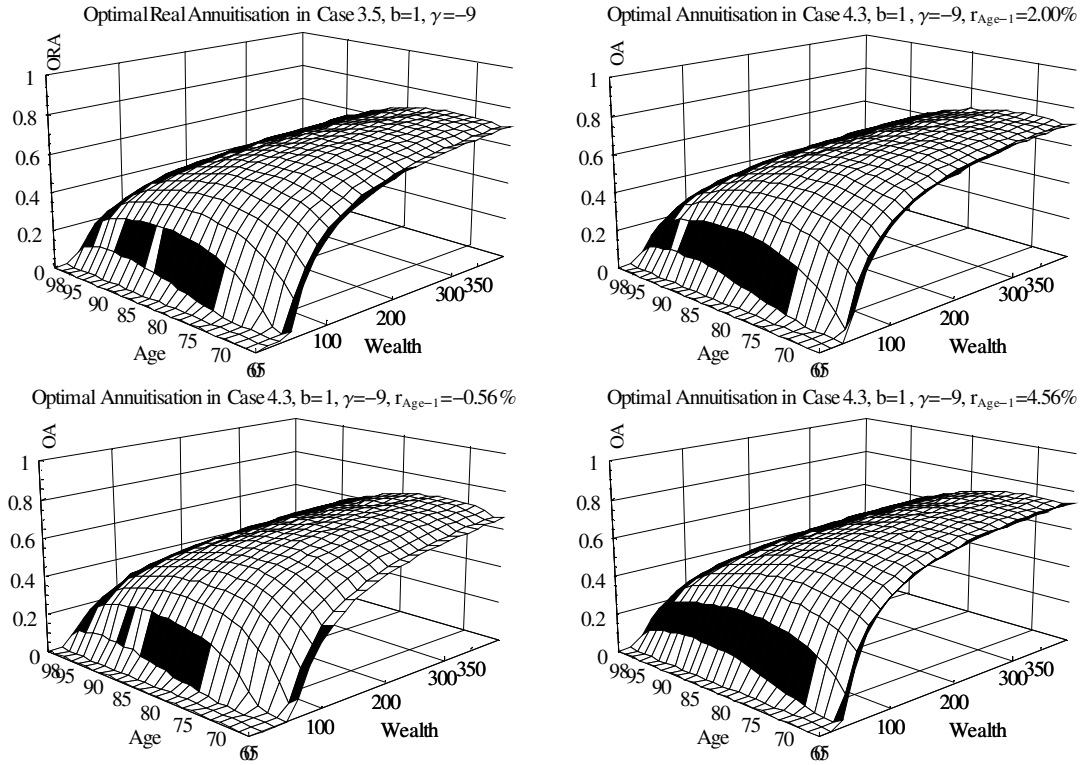


Figure 5.11 Optimal annuitisation – in Case 3.5 (upper left hand side graph), and Case 4.3 (other graphs) for the pensioner with bequest motive and RRA coefficient $\gamma = -9$, for the different values of interest rates in the year before retirement in Case 4.3. The numerical values are in the surfaces graphs are calculated from the deterministic functions of optimal control variables.

In Figure 5.11, we observe that the optimal annuitisation as a deterministic function of age, wealth and the value of the interest rate during the year before annuitisation has very similar values for ages above say 75, for fixed pension wealth and for different values of the interest rate before annuitisation. From this observation, we can conclude that the only reason for the position of the 95% quantile line above the mean line after age 75 in Figure 5.10 is that the pensioner has different amounts of pension wealth due to the investment and annuitisation realisations up to age 75. However, with this conclusion from Figure 5.11, we cannot determine how the length of the period during which the value of the interest rate in the year before retirement influences pensioner's optimal annuitisation decisions.

In Section 5.2.2, we concluded that the pensioner experiences more variability of optimal equity allocation in the interest rate risk model compared to the appropriate cases in the inflation risk model. We can confirm this conclusion regarding optimal annuitisation as the second control variable. Actually, we can conclude that both

optimal asset allocation and annuitisation control variables will be more variable in random samples in the interest rate risk model than in the comparable cases in inflation risk model, but this more active pensioner's optimal policies will result in a slightly less variable optimal consumption in the observed random sample.

5.2.4 Expected Discounted Utility and Adjusted REW

In Section 5.2.4, we present the expected discounted utility drawn from consumption and bequest in different cases for the pensioners with different preferences towards risk and bequest. All the numerical results presented in this section are deterministic calculated using the value function.

We present the values of expected discounted utility, and also expected discounted utility in terms of required equivalent wealth. However, the *REW* measure in this section is a modification of the *REW* measure in Chapters 3 and 4, which we used to compare required equivalent wealth for comparing expected discounted utility in different cases. Now, *REW* measure shows the required equivalent wealth that will provide the pensioner with the same expected discounted utility for a given value of the interest rate during the year prior to retirement as the expected discounted utility he would have obtained if the value of the interest rate in the year prior to retirement were 2.00%. In order to differentiate the *REW* measure in Chapter 5 from the *REW* measure in Chapters 3 and 4, we refer to the *REW* measure in Chapter 5 as “the adjusted *REW* measure”.

In Table 5.2, we present the values of the adjusted *REW* measure and the values of the expected discounted utility in Cases 3.1 and 4.1, for pensioners with different preferences towards risk and bequest and for the different values of the interest rate in the year prior to the time of retirement.

Interest rate	$b_t = 0 \quad \gamma = -1$		$b_t = 0 \quad \gamma = -9$		$b_t = 1 \quad \gamma = -1$		$b_t = 1 \quad \gamma = -9$	
	Case 3.1	Case 4.1	Case 3.1	Case 4.1	Case 3.1	Case 4.1	Case 3.1	Case 4.1
Required equivalent wealth within case								
-2.44		200,003		202,897		200,005		203,115
-0.56		200,002		202,011		200,004		202,137
2.00	200,000	200,000	200,000	200,000	200,000	200,000	200,000	200,000
4.56		199,993		197,285		199,988		197,184
6.44		199,784		195,236		199,719		195,084
Expected discounted utility								
-2.44		-34.67		-54,201		-36.23		-63,176
-0.56		-34.67		-53,634		-36.23		-62,428
2.00	-34.67	-34.67	-55,523	-52,328	-36.23	-36.23	-65,641	-60,776
4.56		-34.67		-50,558		-36.23		-58,597
6.44		-34.65		-49,227		-36.21		-56,983

Table 5.2 The adjusted *REW* measure and expected discounted utility – in Cases 3.1 and 4.1, for the pensioners with different preferences towards risk and bequest, and for the different values of interest rate in the year prior retirement.

We observe in Table 5.2, that the value of expected discounted utility in Case 3.1 is always lower or equal to the lowest value of the values of the expected discounted utility attained in Case 4.1.

The pensioner with the value of RRA coefficient $\gamma = -1$ has almost the same values of expected discounted utility in Case 3.1 in the inflation risk model and for all investigated values of interest rates in the year preceding the time of retirement in the interest rate risk model. There are no annuities in Cases 3.1 and 4.1 presented in Table 5.2 and only the optimal asset allocation differentiates the obtained values of expected discounted utility. If the demand for bonds and cash exist then we get different results. For $\gamma = -1$, only for very high values of interest rates in the year preceding the time of retirement does some demand for bonds exist and then we get slightly better results than in Case 3.1 in the inflation risk model. Otherwise, no demand for bonds and cash exist and the values of expected discounted utilities are the same. Small differences actually exist but these differences are beyond the second decimal place in the numerical values and cannot be seen in the presented results. These small differences are confirmed with the small differences in the adjusted *REW* measure.

If we observe examples for the pensioner with preferences towards risk represented by $\gamma = -9$, we observe the pensioner who has demand for bonds at almost all ages. As a result of this demand, the value of expected discounted utilities in the interest rate

risk model is always higher than in the inflation risk model. The numerical values of the results should not be compared directly because the inflation risk and interest rate risk models are not the same. In the interest rate risk model we have a random interest rate, the market price of risk and investments in one year bonds (risk free asset) and 10 year bonds (low risk asset). The pensioner makes optimal cash, bonds and equities investment decisions knowing the value of the interest rates in the year preceding the year when making investments. Due to this fact and due to the presence of market price of risk, the pensioner in the interest rate risk model gets a higher return from risk free and bond investment than 2.00%, which is the return on the risk free investment in the inflation risk model. It would be possible to fit risk free rate and a given value of the market price in the interest rate model such that one year bond prices for the average risk free interest rate in the previous year is 2.00%. If this is the case we would get the results that are more comparable. Although we investigate only one value of the market price of risk, apart from the results in Section 4.5.7.5, we want to keep market prices as a variable that can take different values. In this sense, fitting the return on one year bond investment in the interest rate risk model would be valid for just one choice of the market price of risk. We are actually not interested in comparing a limited number of the numerical results in the inflation risk and interest rate risk models, but to develop the complete model with the three available assets and investigate qualitatively new results from the more rich model.

The combination of optimal investment based on the dependence on the known value of the interest rate and the existence of the market price of risk obviously gives the pensioner an opportunity for attaining a higher expected discounted utility.

The pensioner with $\gamma = -9$ always has demand for bonds. If the value of the interest rate in the year preceding retirement is higher, bond prices are lower at age 65, the returns from bonds are higher and the pensioner obtains a higher value of expected discounted utility. If the value of the interest rate in the year prior to retirement is higher, then, on average, the value of the interest rate in the early years of retirement is higher and the pensioner gains in terms of expected discounted utility and the adjusted *REW* measure.

Regarding the range of the values of the adjusted *REW* measure in Table 5.2, we observe that for the pensioner with $\gamma = -1$, the range is very narrow and we can say that this pensioner is almost indifferent to the value of the interest rate in the year prior to retirement. However, the pensioner with $\gamma = -9$ and no bequest motive has the range of the values of the adjusted *REW* measure of 7.661, and for the pensioner

with $\gamma = -9$ and with a bequest motive this range is 8.031 money units. Thus, we conclude that in Case 4.1, the more risk averse the pensioner, the more important is the state of interest rate in the year preceding the time of retirement.

In Table 5.3, we present the values of the adjusted *REW* measure and the values of expected discounted utility in Cases 3.3 and 4.2.

Interest rate	$b_t = 0 \quad \gamma = -1$		$b_t = 0 \quad \gamma = -9$		$b_t = 1 \quad \gamma = -1$		$b_t = 1 \quad \gamma = -9$	
	Case 3.3	Case 4.2	Case 3.3	Case 4.2	Case 3.3	Case 4.2	Case 3.3	Case 4.2
Required equivalent wealth within case								
-2.44		203,196		207,167		200,943		205,948
-0.56		202,116		204,415		200,727		203,666
2.00	200,000	200,000	200,000	200,000	200,000	200,000	200,000	200,000
4.56		197,440		195,575		198,837		196,209
6.44		195,630		192,804		197,873		193,683
Expected discounted utility								
-2.44		-34.54		-25,960		-36.23		-35,355
-0.56		-34.47		-24,847		-36.21		-34,091
2.00	-34.53	-34.34	-25,790	-23,110	-36.22	-36.16	-35,669	-32,099
4.56		-34.17		-21,432		-36.08		-30,093
6.44		-34.05		-20,415		-36.02		-28,791

Table 5.3 The adjusted *REW* measure and expected discounted utility – in Cases 3.3 and 4.2, for the pensioners with different preferences towards risk and bequest, and for the different values of interest rate in the year prior retirement.

In Table 5.3, we give the results under the assumption that the pensioner optimally purchases annuities at age 65 only, and has no access to annuities afterwards. Note that the ways that we calculate the annuity factor in the inflation risk model and interest rate risk model are not the same. In the inflation risk model in Cases 3.3 and 3.5, the annuity factor is calculated under the assumption of a constant interest rate and in the interest rate risk model it is calculated using bond prices with the appropriate durations. Annuity loadings are the same. In the inflation risk model in Cases 3.3 and 3.5 annuity factor is always the same for a given age and in the interest rate risk model it depends on the age and on the value of the interest rate in the year preceding the time of annuitisation. Thus, apart from the two ways that cash and bond prices influence expected discounted utility explained after Table 5.2, this is another reason why different bond prices lead to different results in the inflation risk and interest rate risk models.

Table 5.3 shows that the expected discounted utility in Case 3.3 is close to, but not always lower, than the lowest value of expected discounted utility obtained in Case 4.2.

For the pensioner with no bequest motive and with RRA coefficient $\gamma = -1$, the range of the values of the adjusted *REW* measure is 7,566 money units. This pensioner has a demand for annuities and he attains the best results in terms of the adjusted *REW* measure using access to annuities. However, the advantage of annuities depends on the value of the interest rate before retirement. The pensioner with the same RRA coefficient but with a bequest motive has a smaller demand for annuities and the range of the values of the adjusted *REW* measure is 3,070 money units. The pensioner with a bequest motive experiences the same risk of interest rate as the pensioner with no bequest motive, but this risk has lower consequences on the expected discounted utility.

The same pattern repeats itself for the pensioner with RRA coefficient $\gamma = -9$. The difference between the values of the adjusted *REW* measure for the smallest and the highest values of interest rate in the year preceding retirement is 14.363 if no bequest motive exists, and 12.265 money units if the bequest motive exists. We observe that the more risk averse pensioner is significantly exposed to the risk of interest rate in the year preceding the time of retirement. However, annuities for the pensioner with $\gamma = -9$ in Case 4.2 and with pension wealth of 200,000 are good options for any value of the interest rate in the year preceding retirement. This pensioner optimally annuitises from 84% to more than 90% of their pension wealth if no bequest motive exists and from 69% to 73% if a bequest motive exists.

In Table 5.4, we present the values of the adjusted *REW* measure and the values of expected discounted utility in Cases 3.5 and 4.3.

Interest rate	$b_t = 0 \quad \gamma = -1$		$b_t = 0 \quad \gamma = -9$		$b_t = 1 \quad \gamma = -1$		$b_t = 1 \quad \gamma = -9$	
	Case 3.5	Case 4.3	Case 3.5	Case 4.3	Case 3.5	Case 4.3	Case 3.5	Case 4.3
Required equivalent wealth within case								
-2.44		200,163		205,256		200,083		205,032
-0.56		200,109		203,376		200,056		203,206
2.00	200,000	200,000	200,000	200,000	200,000	200,000	200,000	200,000
4.56		199,830		196,229		199,915		196,497
6.44		199,262		193,697		199,562		194,089
Expected discounted utility								
-2.44		-34.02		-24,667		-35.97		-34,230
-0.56		-34.02		-23,938		-35.97		-33,248
2.00	-34.19	-34.01	-25,200	-22,647	-36.07	-35.97	-34,904	-31,542
4.56		-34.00		-21,239		-35.96		-29,717
6.44		-33.96		-20,318		-35.94		-28,491

Table 5.4 The adjusted *REW* measure and expected discounted utility – in Cases 3.5 and 4.3, for the pensioners with different preferences towards risk and bequest, and for the different values of interest rate in the year prior retirement

In Case 4.3, the pensioner has the opportunity to decrease the risk of unfavourable interest rate value before retirement by deferring annuitisation partly or completely to the later years of retirement when the annuity factor is better due to mortality drag and when the pensioner hopes that the value of the interest rate would be more favourable. We have concluded in Chapter 4 that Case 4.3 is the most favourable for the pensioner in terms of expected discounted utility.

The pensioner with the RRA coefficient $\gamma = -1$ will be in a position to almost completely control the risk of unfavourable interest rates in Case 4.3. The range of the adjusted *REW* measure values for the pensioner with no bequest motive is 901 and for the pensioner with a bequest motive it is 521 money units.

For the less risk averse pensioner with RRA coefficient $\gamma = -9$, annuities are a more preferable option that even with unfavourable interest rate values, he optimally annuitises a significant part of his pension wealth at age 65. However, unfavourable interest rate value results in an unfavourable annuity factor and in lower gains from annuitisation. The less risk averse pensioner in Case 4.3 only partly defers annuitisation and has limited success in decreasing the risk of unfavourable interest rate before retirement. The pensioner with $\gamma = -9$ and with no bequest motive has a range of values of the adjusted *REW* measure of 11.559, and if the pensioner has a bequest motive the range is 10.943 money units. We observe that although the

pensioner in Case 4.3 still has a wide range of the adjusted *REW* measure values, it is a lower range than in Case 4.2.

We also make one more observation regarding the comparison of the values of expected discounted utility in Tables 5.2, 5.3 and 5.4. If we compare the values in each table for a given pensioner's preferences towards risk and bequest and for a given value of the interest rate before retirement, we find increasing values in each triple of compared values. The results in Tables 5.2, 5.3 and 5.4 are the results in Cases 3.1 and 4.1, Cases 3.3 and 4.2, and Cases 3.5 and 4.3, respectively. Thus, we expect to observe increasing values in each triple of the compared values because, in Cases 3.1 and 4.1 the pensioner has no access to annuities, in Cases 3.3 and 4.2 the pensioner has access to annuities with a constraint (access to annuities at age 65 only), and in Cases 3.5 and 4.3 the pensioner has access to annuities with no constraint (access to annuities at any age during retirement).

Chapter 6

Conclusions

6.1 Aims and Objectives of the Thesis

In this thesis, we aim to extend the models investigated so far in the literature and to explore numerically their properties. The starting point for the development of our model is the following one. A retirement period starts at age 65 and lasts until the random moment of a pensioner's death. The pensioner possesses a pension wealth at age 65 and also has income from social security during retirement. The pensioner can keep his pension wealth in cash (risk free asset), equities (risky asset) and annuities (irreversible risk free asset). The pensioner draws utility from consuming money during retirement and, if the bequest motive exists, from bequeathing money to his heirs. The pensioner optimally invests and annuitises available pension wealth aiming to maximise an expected discounted utility drawn from consumption in retirement and a bequest.

Based on this framework we develop two models. The first one is the model with stochastic inflation. In this model we investigate pensioner's gains from having access to real and nominal annuities. The second one is the model with a random interest rate and no inflation. In the second model, we investigate the benefits of the pensioner's access to annuities. As no inflation is present in the second model, the pensioner has access to real annuities only.

The objective of the thesis is to understand better how inflation influences the pensioner's benefits from access to annuities, and also to understand better the benefits to the pensioner from access to annuities in the presence of stochastic interest rate. We develop the models that are extensions of the current models, but at the same time the model we develop in the thesis can be the basis for further extensions and research. We also develop the measures such that the pensioner's benefits can be

measured, and investigate numerical values of the pensioner's gains from access to annuities in terms of these measures.

Thus, we want to understand in which way and by how much, the introduction of stochastic inflation in the two assets model influences the pensioner's gains from his access to annuities, and also, the pattern and the level of the pensioner's gains from his access to annuities in the three assets model in the presence of the stochastic interest rate. We develop and investigate two distinct models, the first one addressing the inflation risk and the second one addressing the interest rate risk, and do not attempt in this thesis to address the both risks in the single model.

6.2 The Model

In this thesis, we develop the models based on the model investigated by Cocco, Gomes and Maenhout (2005) with the introduction of annuities that is similar to Horneff, Maurer and Stamos (2008). However, we improve this model further in two directions. In Chapter 3 we introduce nominal annuities and random inflation and in Chapter 4 we introduce stochastic interest rate in the model. We investigate the model for the pensioner who retires at age 65. Both models are precisely developed mathematically and allow for further developments as noted in Section 6.4. However, the model inherits all the limitations of the models investigated in Cocco, Gomes and Maenhout (2005) and Horneff, Maurer and Stamos (2008).

The main limitation of our research is the lack of the behavioural motives of the pensioner and the household. There is some evidence that households understand their own limitations and avoid financial strategies for which they feel unqualified to judge (Campbell (2006)). Pensioners who do not feel qualified enough to participate in the investment and annuity markets will try to avoid possible mistakes. We do not try to deeper understand the pensioner's motivation for annuitisation. We assume that the whole psychological or sociological motivation of the pensioner in a particular society is included in the utility function. We assume that a single aim of the pensioner is to draw a maximised expected utility from his future consumptions. Even with this single pensioner's goal, we assume that the utility function for each pensioner for both consumption and a bequest can be modelled using a single parameter. It is obviously not the true and each particular pensioner will have his own utility function depending on many factors. Another important limitation comes from the fact that the conclusion are drawn from the investigation of the numerical solution where a number of values

of the parameters and probability distributions are assumed and although we tried to investigate different possible values of the most important variables, it is still a limited range of possible solutions. We are not able to derive an analytical solution to the main problems stated in the thesis. However, the model and the solution developed in the thesis allow for sensitivity analysis of many variables and it is just a question of time needed for a calculation of numerical results for different values of the parameters. We should also be aware that the probability distribution of inflation in the inflation risk model and the probability distribution of the interest rate in the interest rate risk model do not include possible longer tails of these distributions. Longer tails can be particularly interesting for possible higher values of the inflation and interest rate. Also, related to the assumed probability distribution of the inflation and interest rate, we assume that the only state variables in the inflation risk and interest rate risk models are the value of the inflation and interest rate during the previous year, respectively. So we assume that the values of inflation and interest rate before two or more years do not influence the probability distribution of the inflation and interest rate in the coming year. This is a questionable assumption and even more if we want to extend our models in terms of the stronger serial dependence of the inflation and interest rate the time needed for the calculation of the numerical solution would probably increase exponentially. The reason is that we would need to increase a number of state variables which results in significant increase of the computational time. Thus, we can say that our model do not allow for the stronger serial dependence of the inflation and interest rate. One more limitation of the model investigated in the thesis is related to problem that we tried to address in the left tail analysis. The left tail risk is implicitly included in the utility function by its concave shape. However, we question if it is good enough for modelling pensioner's optimal decisions because the pensioner has no opportunity to actively recover from possible worse than expected experience during the retirement. He is not in a position to earn more and all he can do is to spend his assets in appropriate way. So, it is possible that the pensioner's decisions would be much more influenced by significant decrease or increase of the pension wealth or income than expected. In the models in this thesis we have no tool for modelling the possible additional decisions influenced by significant decrease or increase of the pension wealth or income than expected.

In Chapter 3, we develop the two assets model with annuities, one asset being cash and the other equities, and with nominal and real annuities. In Chapter 4, we develop the interest rate risk model with annuities, one asset being a one year bond, the second one is rolling bond with a constant duration and the third asset being equities, and with annuities. Inflation in the inflation risk model in Chapter 3 influences income

from nominal annuities only, while all other variables in money units are in real terms. In the interest rate risk model all variables in money units are in real terms.

6.2.1 The Inflation Risk Model

Using an improvement of the well-known two assets model, the inflation risk model in Chapter 3 allows us to investigate the pensioner's optimal decisions and expected discounted utility in the presence of income from nominal and real annuities in retirement. The only variable having a value in nominal terms is nominal income and this variable is converted into real terms. The value function depends on the state representing nominal income as a proportion of the overall income and adjusts it each year for inflation. There are four state variables in this model at each age: the value of pension wealth, income, the nominal income coefficient and the value of inflation during the previous year. In the most general form of the model, the pensioner optimises consumption, asset allocation and nominal and real annuitisation in order to draw his maximised expected discounted utility from consumption and bequest.

6.2.2 The Interest Rate Risk Model

In Chapter 4, we introduce a stochastic interest rate to the well known two assets model and develop the interest rate risk model, where the third asset is a bond, introduced as a consequence of the introduction of stochastic interest rates. Other authors (Boulier et al (2001) and Deelstra et al (2000)) have developed the continuous time models with three assets. If we assume no annuitisation in our model, then our model is a discrete time approximation of these two continuous time models. Furthermore, we introduce annuities in the interest rate risk model. Due to the discrete time framework, in our interest rate risk model we can investigate different constraints and investigate annuitisation that is not possible in the continuous time models. In the most general form of the model, the pensioner optimises consumption, asset allocation and annuitisation in order to draw maximised expected discounted utility from consumption and bequest.

We model stochastic interest rate as a discrete time and space approximation of the Vasicek model for interest rate. An one year bond is the riskless asset, a rolling bond with constant duration is the low risk asset and equities are the third asset available for investment. In order to get an actuarially fair annuity factor, we calculate it using bonds with the appropriate duration.

6.3 Main Results

We find that the risk of inflation and risk of random interest rate have different consequences for the pensioner. Generally, if we introduce the inflation risk into the two assets model then optimal variables do not significantly change and the expected discounted utility drawn from consumption and bequest decreases slightly. Regarding the risk of random interest rate, it influences optimal values of control variables significantly and the value of expected discounted utility increases. Expected discounted utility decreases due to the randomness of interest rate and it increases due to the presence of the market price of risk and due to the availability of the third asset.

6.3.1 The Inflation Risk Model

6.3.1.1 Constant Inflation

For reasonable values of pension wealth and income from social security (values stated in Section 3.4.1), we find that, in the inflation risk model in Case 3.1, it is optimal for the pensioner with no bequest motive to keep a large part of his pension wealth in equities. For the less averse pensioner with no bequest it is optimal to keep all pension wealth in equities. For the pensioner with a bequest motive in Case 3.1 in the inflation risk model, it is still optimal to keep significant part of his pension wealth in equities, but the amounts are lower than for the pensioner with no bequest motive. Increasing pensioner's risk aversion and an introduction of the bequest motive result in a lower optimal equity asset allocation.

For reasonable values of pension wealth and income from social security, we find that in the inflation risk model in all cases where the pensioner has access to annuities, it is optimal for the pensioner with no bequest motive to keep all his, non annuitised, pension wealth in equities. For the less averse pensioner with no bequest it is optimal to keep all his pension wealth in equities due to his attitude to risk. This pensioner does not convert a large percentage of his pension wealth to annuities in the early years of retirement. The more risk averse pensioner will convert large part of his pension wealth to annuities at age 65 and the rest of his pension wealth will be invested in equities. Thus, we find that for the pensioner with no bequest motive, annuities are the preferred investment compared to the risk free asset. This is not surprising because annuities provide a higher return due to the effect of mortality drag.

The pensioner with a bequest motive has a higher demand for annuities than for risk free asset for the most combinations of the pensioner's preferences towards risk. It is optimal for the pensioner with a bequest motive to keep part of his pension wealth until death. We find that only the pensioner with very low risk aversion and bequest motive keeps almost all his pension wealth in equities until the very late years of retirement, if the pensioner is alive at that age. For the more risk averse pensioner with a bequest motive it is optimal to increase the risk free investment as a proportion of his pension wealth in the later years of retirement. Thus, it is optimal for the pensioner with a bequest motive to keep part of his pension wealth until death and to decrease the riskiness of the pension wealth in the later years of retirement. The level of the decrease of the riskiness of the pension wealth in the later years of retirement depends on the pensioner's risk aversion.

Regarding optimal annuitisation, we find that the more risk averse pensioner in Cases 3.2 and 3.3 (access to nominal or real annuities respectively at age 65 only, respectively) purchases significantly more annuities compared to the less risk averse pensioner. This is particularly true for the pensioner in Case 3.2. The pensioner in Case 3.2 with no bequest motive and with a very low risk aversion will have almost no demand for annuities. If the pensioner in Case 3.2 with no bequest motive has a very high level of risk aversion then almost full annuitisation is optimal for him. The pensioner in Cases 3.2 and 3.3, as in other cases, prefers annuities towards risk free assets as annuities provides the higher return compared to risk free asset due to the survival credits. However, he can purchase annuities at age 65 only. Optimally, he converts the part of the pension wealth into annuities at age 65 so that he has no demand for risk free asset afterwards during the retirement. In Case 3.2, only nominal annuities are available and the pensioner receives a decreasing income from nominal annuities in real terms during the retirement. The low risk averse pensioner in Case 3.2 accepts the risk or equity return and the demand for the nominal annuity is very low. However, for the more risk averse pensioner in Case 3.2 seeks for the protection from the equity risk and the demand for nominal annuities increases steeply with the increase of the pensioner's risk aversion. The pensioner with a very high risk aversion in Case 3.2 will optimally purchase so high amount of annuities that he will save part of annuity income in early years of retirement and consume it afterwards. In the same time, the pensioner has a higher income from nominal annuities in real terms in later years of retirement as well. The pensioner with no bequest motive in Case 3.3 will optimally purchase annuities so that he always consumes all annuity income. As the real annuities provides the protection from equity risk and keeps income constant

there is a higher demand for real than for nominal annuities for the lower risk averse pensioner. However, as the pensioner's risk aversion increases in Case 3.3, his demand for the real annuities will increase slower than in Case 3.2, because in Case 3.3 the pensioner's implied risk free investment in a form of annuity will decrease slower during retirement.

If the pensioner in Case 3.2 has a bequest motive then we observe a lower demand for annuities compared to the pensioner with no bequest motive for all investigated levels of risk aversion. The less risk averse pensioner in Case 3.3 with a bequest motive has a higher demand for annuities than in Case 3.2, and a lower demand in Case 3.3 than in Case 3.2 if the pensioner is more risk averse. If the pensioner has a bequest motive, we find the similar conclusions about the optimal annuitisation in Cases 3.2 and 3.3. His demand for annuities is lower because the pensioner aims to keep a part of his pension wealth until the end of his life due to the bequest motive.

If the pensioner can purchase only nominal, or only real, or both nominal and real annuities any time during retirement, which is investigated in Cases 3.4, 3.5 and 3.6 respectively, we find that optimal nominal, real and the sum of nominal and real annuitisation respectively are similar for the different combinations of the pensioner's preferences towards risk and bequest.

For the pensioner in the earlier years of retirement, optimal annuitisation in Cases 3.4, 3.5 and 3.6 significantly depends on his preferences towards risk and bequest. For the lower risk averse pensioner it is optimal to defer annuitisation for 2–3 years and then to annuitise significant parts of his pension wealth soon after deferment of annuitisation. For the more risk averse pensioner it is optimal to annuitise a significant proportion of his pension wealth at the very beginning of retirement and then to keep annuitising smaller amounts of the remaining pension wealth. The lower risk averse pensioner is prepared to take equity risk during the early years of retirement because equity return is better compared to annuity, but after a couple of years survival credits become significant and the demand for annuities increases.

If the pensioner has no bequest motive, we find that optimal annuitisation depends significantly up to age 75. The pensioner with a lower risk averse purchases a lower amounts of annuities at very early ages and his demand for annuities increases steeper as he approaches age 75 compared to the pensioner with a higher risk aversion. At ages 75 and above, the demand for annuities does not depends significantly on the pensioner's risk aversion if the pensioner has no a bequest motive.

The pensioner with a bequest motive will always have a lower demand for annuities than the pensioner with no bequest motive, both having the same preferences towards risk. If the pensioner has a bequest motive, again he takes more risky investment strategy at very early ages, which results with a decrease or deferment of the annuitisation for the pensioner with a lower risk aversion.

We find that optimal nominal, real and the sum of nominal and real annuitisation in Cases 3.4, 3.5 and 3.6 respectively, do not significantly depend on the pensioner's preferences towards risk for ages above 75 if the pensioner has no a bequest motive. We can conclude that any type of annuities is more or less equally good investment for the pensioner with no bequest motive at later ages regardless of his preferences towards risk. The reason for very low dependence on the pensioner's risk aversion is that survival credit is so high that even a lower risk averse pensioner purchases significant amount of annuities and optimal annuitisation for the pensioners with different risk aversion differs just slightly.

For the pensioner with a bequest motive, we find that optimal nominal, real and the sum of nominal and real in Cases 3.4, 3.5 and 3.6 respectively depend significantly on the pensioner's preferences towards risk during the ages 75 and above. The pensioner's bequest motive decreases his demand for annuities and in the same time the more risk averse pensioner purchases more annuities compared to the pensioner with a lower risk aversion.

In Case 3.6, the pensioner has access to both nominal and real annuities during the whole retirement period. For different values of the pensioner's RRA and bequest motive coefficients, and for the different values of the pensioner's age and wealth, we find all possible combination of the demand for nominal and real annuities. For some values no demand for any annuity exists, for some values the pensioner has a demand only for real or only for nominal annuities, and for some other values he has a demand for both real and nominal annuities. It is very complicate to draw the general reasoning for the observed distribution of the nominal and real annuities in Case 3.6. It is particularly tricky question because from the analysis of the gains in terms of *REW*, we observe that access to both type of annuities are very small in Case 3.6 compared to Case 3.5. It seems that the pensioner will optimally purchase some nominal annuities but in terms of *REW* measure it would provide him with very low additional gains. It means that the pensioner in Case 3.6 will make optimally choice of optimal nominal and/or real annuitisation but he will obtain just slight gains of

expected discounted utility compared to the pensioner who has opportunity to optimally annuitise real annuities only.

If the pensioner has no bequest motive then we find that it is optimal for the pensioner to start purchasing nominal annuities at age 73–75, then after these ages the percentage of the optimal nominal annuitisation increases, and then it decreases again. We also find that if the demand for nominal annuities at a given pensioner's age exists then the percentage of the optimal nominal annuitisation is higher for lower values of the pension wealth and this percentage decreases as income wealth increases. In the same time, the percentage of the optimal real annuitisation increases as the pension wealth increases. We believe that the reasons for the demand for nominal annuities compared to real annuities should be sought in a higher income provided from nominal annuities during a couple of years after purchasing them, its constant decrease in real terms and eventually a lower income in real terms compared to the constant income in real terms provided by real annuities purchased at the same age and for the same amount of the pension wealth. We believe that the demand for the nominal annuities does not exist at earlier ages because of the long period of decreasing income from nominal annuities in real terms during the retirement. If the pensioner purchases nominal annuities very early during retirement period then the income from nominal annuities in real terms will decrease later during the retirement so that the gains of the higher income from nominal annuities in a years after purchasing them is overweight. Regarding a higher percentage of the optimal nominal annuitisation compared to the optimal real annuitisation for low values of the pension wealth, we believe that the reason for this observation lies in a fact that for a lower amount of pension wealth a lower amount of additional income can be provided using annuitisation. Thus, although income from nominal annuities decreases in real terms, the decrease of the overall income is relatively low and the gains from the higher income in early years after purchasing annuities overweight the losses in later years after purchasing nominal annuities.

We observe similar properties of the optimal nominal versus optimal real annuitisation in Case 3.6 for the pensioner with a bequest motive. The important difference, compared to the pensioner with no bequest motive, is that the pensioner with a bequest motive will not purchase any annuities for very low values of the pension wealth due to his bequest motive. However, if the demand for nominal annuities exists then this pensioner will optimally purchase only nominal annuities or more nominal annuities compared to real annuities for lower values of the pension wealth. As the amount of pension wealth increases for a given age, the percentage of

the optimal nominal annuitisation decreases and the percentage of the optimal real annuitisation increases. We also observe that no demand exists for nominal annuities at very early retirement years. However, for the pensioner with a bequest motive and with the value of the RRA $\gamma = -9$ it is only during the first 2 or 3 years of the retirement. So we find that the demand for nominal annuities for the pensioner with a bequest motive starts earlier compare to the pensioner with no bequest motive. Although we have different values of the optimal nominal and optimal real annuitisation for the pensioner with and with no bequest motive, we find that the same pattern of deferred optimal nominal annuitisation at the beginning of the retirement period, and of decreasing demand for nominal and increasing demand for real annuities as the pension wealth increases for a given age. Thus, we believe that the same general reasoning is applicable for the pensioner with a bequest motive as stated for the pensioner with no bequest motive.

Cases in the inflation risk model differ in access to the class/classes of annuities and in constraints regarding at which ages the pensioner can access annuities. We find that, in terms of *REW*, annuitisation is beneficial to the pensioner in each case and in all but one combination of the parameters representing the pensioner's preferences towards risk and bequest motive. We find that only the pensioner with a very low level of risk aversion and with a bequest motive in Case 3.2 does not have a demand for annuities. The level of a pensioner's benefit from annuitisation significantly depends on his preferences towards risk and bequest. The more risk averse pensioner has more demand for annuities and he benefits more in terms of expected discounted utility. Also, the pensioner with no bequest motive compared with the pensioner with a bequest motive and for the same level of risk aversion, will have more demand for annuities and will benefit more in terms of expected discounted utility. The more risk averse pensioner in Case 3.2 will gain significant benefits from nominal annuitisation at age 65. However, the pensioner in Case 3.2 always gains lower benefits compared to the pensioner with access to any other class/classes of annuities. The pensioner with access to nominal annuities whenever in retirement (Case 3.4) benefit more than the pensioner with access to real annuities at age 65 only (Case 3.3) for all investigated combinations of the risk aversion and bequest motive parameters, apart for the pensioner with no bequest and with high risk aversion. We find that the differences in gains between the pensioners in Case 3.4 and Case 3.3 are larger for the lower level of risk aversion and for no bequest motive. Cases 3.5 and 3.6 provide the highest and very similar levels of benefit to pensioners for all combinations of risk aversion and both with and without a bequest motive. We find that for the investigated values of the parameters, real annuitisation in Case 3.6 is more favourable to the

pensioner than nominal annuitisation. The differences in the level of benefits between Cases 3.5 and 3.4 increase with the increase of the pensioner's risk aversion. These differences are higher for the pensioners with no bequest motive than for the pensioner with a bequest motive. When comparing the levels of benefits between Cases 3.5 and 3.3, we find that the differences are smaller for the more risk averse pensioner, and again differences are smaller for the pensioner with a bequest motive than for the pensioner with no bequest motive.

6.3.1.2 Stochastic Inflation

As we want to isolate the effect of stochastic inflation, we have firstly investigated different cases and different combinations of the parameters for constant inflation and then we have investigated the same examples but with stochastic inflation. Stochastic inflation influences Cases 3.2, 3.4 and 3.6 when the pensioner has income from nominal annuities, as it is the only variable in nominal terms.

We observe that in both deterministic and stochastic inflation in Case 3.6, the pensioner optimally purchases much more real annuities than the nominal ones. As stochastic inflation has the effects on the nominal annuities only, the effects of stochastic inflation in Case 3.6 are almost negligible.

However, we find that stochastic inflation has minimal effects in Cases 3.2 and 3.4 as well. Regarding optimal control variables, the differences are very small. We find that for different pension wealth and for different amounts of nominal income as a proportion of overall income all control variables vary just slightly. We find that for the pensioner with a high level of risk aversion, for whom the benefits from nominal annuitisation are the largest, the differences in the benefits in constant inflation framework and in stochastic inflation framework are almost negligible. We find that the risk of uncertain inflation results in a small decrease in the pensioner's benefits compared to the case of constant inflation.

We believe that the higher/lower annuity income from nominal annuities at times when the value of inflation is higher/lower, and the faster/slower decrease of the nominal income in real terms in the years that follow the year when inflation is higher/lower cancel each other out. Namely, if the value of random inflation is higher/lower in the year prior to nominal annuitisation, then the nominal annuity rate is lower/higher. Income from nominal annuity bought at that moment is higher/lower. However, as the value of inflation in the year prior to nominal annuitisation is

higher/lower, then the value of inflation in the coming years is going to be higher/lower on average. Due to, on average, higher/lower inflation in the coming years, the nominal annuity income decreases its value in real terms faster/slower. As we measure expected discounted utility, these effects seem to cancel each other out and we find influences of inflation very small in terms of expected discounted utility. This is actually a consequence of a feature of a mean reverting AR(1) model for inflation.

6.3.2 The Interest Rate Risk Model

In the interest rate risk model, the pensioner can invest in cash, bonds and equities and the interest rate is stochastic. We find that the value of the interest rate significantly influences the pensioner's optimal control variables as well as the gains from access to annuities. We firstly focus our analysis on comparing results between cases and in this way we investigate the pensioner's gains from access to annuities. Then, we focus on comparing results within a case and in such way we investigate the influence of the value of the interest rate during the year before retirement on the pensioner's optimal decisions and gains from annuities in retirement.

6.3.2.1 Comparing Results between Cases

As in the inflation risk model, we focus our analysis on reasonable values of the variables, as stated in Section 4.5.1. For these values of the variables, we find that optimal allocation in the ten year rolling bond is always preferable for the pensioner compared to the allocation in cash. For the pensioner with no bequest motive in Cases 4.2 and 4.3, annuities are preferable compared to both bonds and cash as all pension wealth is optimally either annuitised or kept in equities. The pensioner with no bequest motive in Case 4.3 optimally annuitises all pension wealth before the last possible age, if still alive at that age. Due to the utility drawn from bequeathing assets to his heirs, the pensioner with a bequest motive keeps a part of his pension wealth until the end of his life. For the pensioner with a bequest motive it is optimal to reduce the investment risk during later years of retirement by increasing the proportion of the pension wealth invested in bonds.

Regarding optimal annuitisation in Case 4.2 (access to annuities at age 65 only), we find that the more risk averse pensioner optimally annuitises a significantly larger part of his pension wealth than the less risk averse pensioner. We also find that the optimal

proportion of annuitised pension wealth in Case 4.2 depends more on the value of the interest rate during the year before retirement for the less risk averse pensioner than for the more risk averse pensioner.

In Case 4.3 (access to annuities at any age), we find significant differences in optimal annuitisation strategies for ages 65 to about 75. We find that the pensioner with no bequest motive and with any level of risk aversion optimally converts all his pension wealth to annuities by age 80. In the early years of retirement, we find significant differences in optimal annuitisation for pensioners with no bequest motive and different attitudes to risk aversion. The more risk averse pensioner will optimally purchase a significant amounts of annuities at age 65 and will only partly defer annuitisation if the value of the interest rate during the year before retirement is not favourable. The less risk averse pensioner will completely defer annuitisation if the value of the interest rate during the year before retirement is not favourable, but if the value of the interest rate is favourable he will purchase a significant amounts of annuities at age 65. However, in the early years of retirement but after age 65, the more risk averse pensioner with no bequest motive will purchase a fewer annuities than the less risk averse pensioner with no bequest motive because the first pensioner has already bought more annuities at age 65. Similar trends will be observed for the pensioner with a bequest motive. However, we find that the pensioner with a bequest motive keeps part of his pension wealth until death and annuitises a smaller portion of his pension wealth.

In terms of expected discounted utility, the pensioner with no bequest motive benefits more from annuitisation than the pensioner with a bequest motive.

The more risk averse pensioner optimally purchases more annuities, and in Case 4.3 earlier during retirement, and this pensioner benefits significantly more from access to annuities than the less risk averse pensioner. We find this to be true for all investigated values of the interest rate during the year before retirement.

Regarding the differences in gains from access to annuities in Cases 4.2 and 4.3, we find that these differences are larger for the less risk averse pensioner compared to the more risk averse pensioner with a same bequest motive. These differences are also larger for the pensioner with no bequest motive compared to the pensioner with a bequest motive, both pensioners having the same attitude towards risk.

The difference between the benefits from annuitisation in Case 4.2 and 4.3 depend on the value of the interest rate during the year before retirement. For lower values of the interest rate during the year before retirement, the difference between benefits from annuitisation in Case 4.2 and 4.3 are larger for the less risk averse pensioner. As the values of the interest rate during the year before retirement increase we find that these differences become smaller for all investigated levels of risk aversion and for the high values of interest rate during the year before retirement, the differences are almost negligible.

6.3.2.2 Sensitivity Analysis

We have performed a sensitivity analysis of the pensioner's gains from access to annuities by changing the values of different variables. If the pensioner has a higher income from his last salary and also from social security, keeping the replacement ratio the same, his gains from access to annuities decreases. Income from social security is a form of annuity income and if the pensioner possesses a higher income from social security he optimally annuitises lower amounts of pension wealth and annuitises them later in retirement and benefits less from access to annuities compared to the pensioner with a lower income from social security. If the pensioner has a higher pension wealth at age 65, then he has a relatively lower income from social security, and he gains more from access to annuities compared to the pensioner with the lower pension wealth at age 65. If the mean value of equity rates decreases, the annuities are more favourable for the pensioner and the gains from access to annuities increases. If the mean value of the interest rate increases, then the demand for annuities increases and the pensioner's gains from access to annuities increases. If the market price of risk decreases, the demand for annuities and for rolling bonds decreases, and the demand for equities and cash increases. As a result, we observe a small decrease of the pensioner's gains from access to annuities.

6.3.2.3 Comparing Results within Case

For the different values of interest rate during the year before retirement, we find small differences in the pensioner's optimal consumption. We find that the pensioner adjusts his optimal investment strategy and, if he has access to annuities, his optimal annuitisation strategy according to the value of the interest rate before retirement. We find significant differences in the values of optimal investment and annuitisation for the different value of the interest rate before retirement. Applying this an optimal

strategy, the pensioner keeps the values of optimal consumption less variable, and in that way attains the highest expected discounted utility.

In Cases 4.2 and 4.3, for the pensioner with no bequest motive, optimal equity investment is preferred over bonds and cash for all but very high values of the interest rate during the year before investment. The pensioner with no bequest motive in Case 4.3 only, when the value of the interest rate during the year before investment is extremely high, has some demand for bonds during the early years of retirement.

For the pensioner with a bequest motive in Cases 4.2 and 4.3, optimal equity allocation decreases as the value of the interest rate during the year before investment increases. The range of optimal equity allocation as a function of the value of the interest rate during the year before investment is quite large for this pensioner.

The range of the percentages of optimal annuities purchased as a function of the value of the interest rate during the year before annuitisation and the pensioner's age decreases as the pensioner gets older. It seems that optimal annuitisation does not depend significantly on the value of the interest rate during the year before annuitisation for the less risk averse pensioner, aged around 80 and above. The pensioner's age limit when optimal annuitisation does not any more depend significantly on the value of the interest rate during the year before annuitisation decreases to less than 75 for the more risk averse pensioner with a bequest motive. We find that the range of the values of optimal annuitisation, as a function of the value of the interest rate during the year before annuitisation, is larger for the pensioner with no bequest motive than for the pensioner with a bequest motive. We also find that this range is larger for the less risk averse pensioner.

In Case 4.2, we find significant differences in optimal annuitisation at age 65 depending on the value of the interest rate during the year before retirement. We find that the range of the values of optimal annuitisation at age 65 in Case 4.2 as a function of the value of the interest rate during the year before retirement is larger for the less risk averse pensioner compared to the more risk averse pensioner with the same attitude towards the bequest. Also, the range is larger for the pensioner with no bequest motive compared to the pensioner with a bequest motive, both having the same risk aversion.

The pensioner in Case 4.3 defers annuitisation partly or completely, depending on the value of the interest rate during the year before retirement. For the pensioner aged 65,

we find that the values of the mean and 5% and 95% quantiles of optimal annuitisation changes during the first five to eight years of retirement as the value of the interest rate during the year before retirement changes.

Optimal annuitisation, optimal equities and bond allocation all change at the same time as a function of the value of the interest rate during the previous year. If annuitisation is allowed and the demand for annuities and bonds exists, we find that as the value of the interest rate during the previous year increases, the demand for both annuities and bond investment increases while the value of optimal equities allocation decreases.

In Cases 4.1, 4.2 and 4.3, the pensioner faces equity and interest rate risks, while in Cases 3.1, 3.3 and 3.5, he faces equity rate risk only. When comparing Cases 3.1, 3.3 and 3.5 and Cases 4.1, 4.2 and 4.3, respectively, we find that the variability of optimal asset allocation and annuitisation is higher in Cases 4.1, 4.2 and 4.3. At the same time, the variability of optimal consumption is very similar or even lower in Cases 4.1, 4.2 and 4.3. Thus, we conclude that the pensioner in a more risky environment in Cases 4.1, 4.2 and 4.3 has enough space to act optimally regarding optimal asset allocation and annuitisation such that the mean and 5% and 95% quantiles of optimal consumption are kept similar or even closer together compared to the less risky environment in Cases 3.1, 3.3 and 3.5.

Regarding expected discounted utility (measured via the adjusted *REW* measure) drawn from the pensioner's consumption and bequest in retirement as a function of the value of the interest rate during the year before retirement, we find the following results.

The less risk averse pensioner is almost indifferent toward the value of the interest rate before retirement in Cases 4.1 and 4.3. One reason for this result is that the less risk averse pensioner optimally invests all his pension wealth in equities and investment results are not affected by changes in the value of the interest rate. The second reason applies to Case 4.3: this pensioner optimally purchases annuities during the first ten to fifteen years of retirement and optimally he purchases small amount of annuities at age 65. So this pensioner purchases annuities when the value of the interest rate or when mortality drag is favourable and thus avoids the risk of interest rate. The less risk averse pensioner with a bequest motive in Case 4.2 is slightly more exposed to interest rate risk, but still we find small differences in terms of adjusted *REW* measure.

However, the less risk averse pensioner with no bequest motive in Case 4.2 obtains a wider range of results in terms of adjusted *REW* measure for the different values of the interest rate during the year before retirement. This pensioner optimally purchases annuities at age 65 whatever the value of the interest rate. So this pensioner benefits if the value of the interest rate before retirement is favourable because then the annuity rate is more favourable as well.

The more risk averse is the pensioner, the more influence has the value of the interest rate before retirement on the expected discounted utility. The more risk averse pensioner has a higher demand for both bonds and annuities during the whole retirement period and he benefits from the more favourable value of the interest rate before retirement. In all cases we find significant differences in terms of expected discounted utility for different values of the interest rate before retirement. The differences in terms of adjusted *REW* measure are the smallest in Case 4.1, as the value of the interest rate before retirement influences bond investment only. In Case 4.3, the more risk averse pensioner has a significant demand for annuities at age 65 for any value of the interest rate before retirement, but he can partly defer annuitisation during the early years of retirement and decrease the level of interest rate risk before retirement. The differences in terms of adjusted *REW* measure are higher in Case 4.3 compared to Case 4.1, but smaller than in Case 4.2. In Case 4.2, the more risk averse pensioner optimally purchases a significant amount of annuities at age 65 for any value of the interest rate before retirement and all he can do is to purchase fewer annuities if the value of the interest rate before retirement is unfavourable. That is why this pensioner in Case 4.2 has the widest range of the values of expected discounted utilities for different values of the interest rate before retirement.

Comparing pensioners with the same attitude towards risk, one with a bequest motive and the other with no bequest motive, we find the following results in terms of adjusted *REW* measure as a function of the value of the interest rate before retirement. In Case 4.1, the pensioner with a bequest motive is slightly more exposed to the risk of an unfavourable interest rate before retirement. The reason lies in the fact that in Case 4.1 it is optimal for the pensioner with a bequest motive to invest slightly less in equities. However, in Cases 4.2 and 4.3, the differences of expected discounted utility for the pensioner with a bequest motive are lower compared to the differences for the pensioner with no bequest motive. The pensioner with no bequest motive has a higher demand for annuities and he is more exposed to the risk of changes to the interest rate before retirement.

6.4 Future Research

As we have already noted, the two models developed in this thesis build on existing models. However, by improving the existing models by introducing stochastic inflation in Chapter 3 and stochastic interest rate in Chapter 4, we have not come to the limits of the development of the model. The main obstacle for further development of the model can be the speed of the numerical calculation on the computer. However, we witness increasing speed and capacity of processors and also the development of parallel and high performance computing centres in many countries. The possibility to numerically solve the equations in the more complex model, which is an extension of our model, makes the results in this thesis an excellent basis for future work.

We will mention some of the possible directions for further research based on the results presented here.

Different constraints on consumption, asset allocation and annuitisation can be imposed and investigated. For example, one can assume that the pensioner can borrow money and also set a constraint on the borrowing. The model would be numerically solvable.

We model the inflation rate and interest rate using AR(1) model. Another inflation and interest rate model can be used for developing the values of inflation and bond prices. It can be particularly interesting in the interest rate risk model. We base our interest rate model on the Vasicek model. However, the Vasicek model has long left tail as well as right tail and it is not skewed distribution. It would seem more sensible to base the interest rate model on the Cox–Ingersoll–Ross model because in CIR model left tail is short and limited to zero value and right tail is long. It seems to be better representation of the real interest rate. It is possible to develop the discrete time and space interest rate model based on the CIR model and to derive the bond market for that model. Once the bond market is derived, we just introduce new bond prices in our model.

An important extension will be allowing for longevity risk. We assume that the pensioner's maximum lifetime is 100. It can be increased in the model. We can introduce the pensioner's subjective and objective probabilities of survival in the model. Increasing the maximum lifetime and differentiating subjective and objective probabilities of survival can be particularly interesting in today's world of constantly

improving survival probabilities for pensioners and of longevity risk. We observe a single pensioner from his retirement age until his death. However, the tables we use are the 2002–2004 tables of the one year survival probability of the person alive at a given age. So it would be interesting to project the mortality table for the future years and in the same time to increase the maximum lifetime. Then we would get the better pensioner's survival probabilities in the future years until his death. We observe systematic downward trends in mortality rates especially at older ages and taking this trend into account would result in higher survival probabilities at older ages and new results.

It would be interesting to investigate the effects of the introduction of the correlation between random variables in each model. In the inflation risk model it would be interesting to investigate the correlation between inflation and equity return and the correlation between interest rate and equity return in the interest rate model .

Also, it is possible to introduce inflation in the interest rate risk model and get a single general model that we can use for the simultaneous investigation of stochastic inflation and interest rate. Then, the inflation risk model would be a special case of the general model if interest rate variability approaches zero, and the interest rate model would be a special case of the general model if the mean of the inflation rate is zero and variability of the inflation rate approaches zero. If one develops the general model, then it would be also possible to introduce and investigate the effects of the correlation between the inflation and interest rate risk. However, we recognise that in the model including both inflation and interest rate risks would have one more state variable and that would result in significant increase of the computational time. In the same time handling the numerical results would be more demanding.

Income from social security is assumed to be a constant in real terms in this thesis and this assumption can be relaxed. Introduction of possible random shocks in consumption due to the costs of the pensioner's health care or loan repayments, for example, would be one more improvement of the model making it more realistic.

In the model developed in this thesis, we assume that the pensioner has constant relative risk aversion utility function and no pensioner's target consumption of pension wealth is assumed. There are many possible variations regarding the pensioner's utility function that would be realistic and that could give us interesting answers. Thus, we could assume that the pensioner has an Epstein–Zin or some other utility function and investigate the numerical results for that pensioner. In Sections

3.4.6 and 4.5.6, we have defined and calculated numerically the pensioner's left tail risk. This measure seems important for the pensioner as (in retirement) the pensioner is spending the wealth that he saved from earning during the period of life before retirement. The pensioner gets older and is less willing and able to work and he has no possibility to recover from the poor financial experience in retirement. Thus, the model in this thesis could be improved such that it better recognises the pensioner's left tail risk.

We witness a number of discussions about the extending individual's retirement date. In our model we assume that the individual is retired at age 65 and after that age he receives income from social security only. The model allows for investigating the individual who choose to work after age 65 as well. Instead of the constant income from social security one can extend our model such that the individual works a few years after age 65 and receives either salary only or a combination of maybe part or the whole income from social security and salary as well. If we assume that the pensioner receives only part or no income from social security during additional working years then we can introduce a higher income from social security when the individual retires after age 65. In the same time it would be sensible to assume that the individual has access to annuities from age 65 onwards. The retirement age in this model could be an exogenously chosen age by the individual or it can be endogenously chosen as a function of the pension wealth and income from social security and annuities. In the interest rate model, the retirement age could also be connected to the value of the interest rate such that the individual chooses the retirement date optimally as a function of the pension wealth, income from social security and also as a function of the annuity rates available. Then we would probably have earlier retirement age if the value of the pension wealth is more than expected, but also it could be earlier if the pension wealth is below expected but the annuity rate is favourable and retirement becomes optimal. Introducing exogenously chosen retirement age is easier and one can use *REW* measure to compare the pensioner's gains from working additional years. However, if one develops the model with flexible retirement age then probably some kind of the utility from not working needs to be introduced because we can assume that the individual has additional utility from leisure. Bodie et al (1992), Lachance (2004) and Chai et al (2009) develop the models with flexible retirement age and besides utility from consumption they introduce utility from leisure as well.

The results in this thesis can be used for the investigation of optimal asset allocations in the preretirement period as well. We have determined the value functions at age 65

for the pensioner having access to a given class/classes of annuities in retirement. Using these value functions, one can extend our results for the individual in the preretirement period such that this individual makes optimal decisions knowing his time of retirement and the availability of annuities in retirement.

Developing the life cycle model with flexible retirement age, optimal asset allocation in preretirement period and optimal postretirement asset allocation and annuitisation could be the further development of the model in this thesis. Bodie et al (1992), Lachance (2004) and Chai et al (2009) investigate flexible retirement age. One can use their ideas and results and the model developed in this thesis, particularly the interest rate model, and develop the lifecycle model with three assets, with access to annuities from a given age and with flexible retirement age.

Appendices

A.1 Appendix 1 – Income as State Variable in the Inflation risk Model

In this Appendix, we prove the relations amongst the solutions of the problem (3.26)–(3.34) for different values of the income variable. We find that we can solve the problem (3.26)–(3.34) for one single value of the income variable and for different values of other variables and then transform this solution to obtain the solution for any value of the income. It is very useful for numerical solution because using this techniques, we decrease the number of state variables such that the income variable becomes a constant. Then, we can obtain a numerical solution for only one value of income and then investigate different values of income using the technique from this Appendix. It is also useful because we can derive the solution for a smaller range of values for different variables which is faster and more controllable and then convert this solution back into the nominal amounts in pounds.

We will prove the relations (3.45)–(3.49). This is the solution of the most general case of the problem (3.26)–(3.34). If real annuities (RA) or/and nominal annuities (NA) are exogenous then it is easy to see that transformation of the income variable is just special cases of the general solution presented here. Here, we exclude writing index k that appears in (3.45)–(3.49) in the inflation variable and just assume that inflation variable takes values in the domain of the inflation variable.

We apply mathematical induction in order to prove the relations (3.45)–(3.49). Let us start from equations (3.15)–(3.18), and let us prove that the relations (3.45)–(3.49) are valid for $t = i$ where $i = 99$.

For some fixed income Y_{99} and wealth W_{99} we have the solution

$$\text{optimal consumption:} \quad C_{99}^* (W_{99}, Y_{99}, d_{99}^{NA}, I_{98}) \quad (\text{A.1.1})$$

$$\text{optimal asset allocation:} \quad \alpha_{99}^* (W_{99}, Y_{99}, d_{99}^{NA}, I_{98}) \quad (\text{A.1.2})$$

$$\text{optimal NA:} \quad m_i^{NA*} (W_{99}, Y_{99}, d_{99}^{NA}, I_{98}) = 0 \quad (\text{A.1.3})$$

$$\text{optimal RA:} \quad m_i^{RA*} (W_{99}, Y_{99}, d_{99}^{NA}, I_{98}) = 0 \quad (\text{A.1.4})$$

value function:
$$V_{99}(W_{99}, Y_{99}, d_{99}^{NA}, I_{98}) \quad (\text{A.1.5})$$

for $W_{99} \geq 0$ and $Y_{99} \geq 0$, $0 \leq d_{99}^{NA} \leq 1$, and I_{98} in the domain of inflation values. This solution exists because we are looking for the maximum of the continuous function on the compact set. The solution is unique as well. Equations (A.1.3) and (A.1.4) say that no annuitisation occurs at age 99.

Let us now assume that we have a new income variable $\bar{Y}_{99} = kY_{99}$ and a new pension wealth variable $\bar{W}_{99} = kW_{99}$, for some positive constant $k \in \mathbb{R}^+$. Let us introduce a new random variable $\tilde{\bar{W}}_{100}$ defined in (A.1.7), and let $\bar{C}_{99}$ and $\bar{\alpha}_{99}$ be the new control variables. Thus, we have the problem (3.15)–(3.18) again but now with wealth and income variables, $\bar{W}_{99}$ and $\bar{Y}_{99}$ respectively. The problem can be written as

$$V_{99}(\bar{W}_{99}, \bar{Y}_{99}, d_{99}^{NA}, I_{98}) = \max_{\{\bar{\alpha}_{99}, \bar{C}_{99}\}} E_{99} \left[u(\bar{C}_{99}) + \delta(1 - p_{99}) b_{99} u(\tilde{\bar{W}}_{100}) \right] \quad (\text{A.1.6})$$

$$\tilde{\bar{W}}_{100} = (\bar{W}_{99} + \bar{Y}_{99} - \bar{C}_{99})(1 + r + \bar{\alpha}_{99}(\tilde{r}_{99} - r)) \quad (\text{A.1.7})$$

$$0 \leq \bar{C}_{99} \leq \bar{W}_{99} + \bar{Y}_{99}, \text{ and} \quad (\text{A.1.8})$$

$$0 \leq \bar{\alpha}_{99} \leq 100\% . \quad (\text{A.1.9})$$

As no annuitisation occurs at age 99, one can derive from formula (3.10) the following

$$\tilde{\bar{d}}_{100}^{NA} = \frac{\bar{d}_{99}^{NA} (1 + \tilde{I}_{99})^{-1}}{1 - \bar{d}_{99}^{NA} + \bar{d}_{99}^{NA} (1 + \tilde{I}_{99})^{-1}}$$

From (3.7) we can see that $\bar{d}_{99}^{NA} = d_{99}^{NA}$, and now we can write

$$\tilde{\bar{d}}_{100}^{NA} = \frac{d_{99}^{NA} (1 + \tilde{I}_{99})^{-1}}{1 - d_{99}^{NA} + d_{99}^{NA} (1 + \tilde{I}_{99})^{-1}}$$

and thus

$$\tilde{\bar{d}}_{100}^{NA} = \tilde{d}_{100}^{NA} .$$

However, one can observe that value function at age 99 does not depend on $\tilde{d}_{100}^{NA}$. Now, we can write equation (A.1.6) as follows

$$\begin{aligned}
V_{99}(\bar{W}_{99}, \bar{Y}_{99}, d_{99}^{NA}, I_{98}) &= \max_{\{\bar{C}_{99}, \bar{\alpha}_{99}\}} E_{99} \left[u(\bar{C}_{99}) + \delta(1-p_{99})b_{99}u(\bar{W}_{99}(1+r+\bar{\alpha}_{99}(\tilde{r}_{99}-r))) \right] \\
&= k^\gamma \max_{\{\bar{C}_{99}, \bar{\alpha}_{99}\}} E_{99} \left[u\left(\frac{\bar{C}_{99}}{k}\right) + \delta(1-p_{99})b_{99}u(W_{99}(1+r+\bar{\alpha}_{99}(\tilde{r}_{99}-r))) \right]
\end{aligned}$$

As we assume that

$$E_{99} \left[u\left(\frac{\bar{C}_{99}}{k}\right) + \delta(1-p_{99})b_{99}u(W_{99}(1+r+\bar{\alpha}_{99}(\tilde{r}_{99}-r))) \right]$$

attains its maximum for the set of solutions (A.1.1)–(A.1.5), and then

$$\max_{\{\bar{C}_{99}, \bar{\alpha}_{99}\}} E_{99} \left[u\left(\frac{\bar{C}_{99}}{k}\right) + \delta(1-p_{99})b_{99}u(W_{99}(1+r+\bar{\alpha}_{99}(\tilde{r}_{99}-r))) \right]$$

is maximized for

$$\begin{aligned}
\frac{\bar{C}_{99}(\bar{W}_{99}, \bar{Y}_{99}, d_{99}^{NA}, I_{98})}{k} &= C_{99}^*(W_{99}, Y_{99}, d_{99}^{NA}, I_{98}), \text{ and} \\
\bar{\alpha}_{99}(\bar{W}_{99}, \bar{Y}_{99}, d_{99}^{NA}, I_{98}) &= \alpha_{99}^*(W_{99}, Y_{99}, d_{99}^{NA}, I_{98}).
\end{aligned}$$

As we said earlier, the solution is unique and thus, we can conclude that the optimal solution on the problem (A.1.6)–(A.1.9) must be

$$\begin{aligned}
\bar{C}_{99}^*(\bar{W}_{99}, \bar{Y}_{99}, d_{99}^{NA}, I_{98}) &= kC_{99}^*(W_{99}, Y_{99}, d_{99}^{NA}, I_{98}), \\
\bar{\alpha}_{99}^*(\bar{W}_{99}, \bar{Y}_{99}, d_{99}^{NA}, I_{98}) &= \alpha_{99}^*(W_{99}, Y_{99}, d_{99}^{NA}, I_{98}).
\end{aligned}$$

It means that the following statement is valid:

if $k \in \mathbb{R}^+$ and

$$\text{wealth:} \quad \bar{W}_{99} = kW_{99} \quad (\text{A.1.10})$$

$$\text{income:} \quad \bar{Y}_{99} = kY_{99} \quad (\text{A.1.11})$$

then the solution to the problem (3.26)–(3.34) satisfies the following rules

$$\text{annuitisation coefficient:} \quad \tilde{d}_{100}^{NA} = \tilde{d}_{100}^{NA} \quad (\text{A.1.12})$$

$$\text{optimal consumption:} \quad \bar{C}_{99}^*(\bar{W}_{99}, \bar{Y}_{99}, d_{99}^{NA}, I_{98}) = kC_{99}^*(W_{99}, Y_{99}, d_{99}^{NA}, I_{98}) \quad (\text{A.1.13})$$

$$\text{optimal asset allocation:} \quad \bar{\alpha}_{99}^*(\bar{W}_{99}, \bar{Y}_{99}, d_{99}^{NA}, I_{98}) = \alpha_{99}^*(W_{99}, Y_{99}, d_{99}^{NA}, I_{98}) \quad (\text{A.1.14})$$

$$\text{optimal NA:} \quad \bar{m}_i^{NA*}(\bar{W}_{99}, \bar{Y}_{99}, d_{99}^{NA}, I_{98}) = m_i^{NA*}(W_{99}, Y_{99}, d_{99}^{NA}, I_{98}) \quad (\text{A.1.15})$$

$$\text{optimal RA:} \quad \bar{m}_i^{RA*}(\bar{W}_{99}, \bar{Y}_{99}, d_{99}^{NA}, I_{98}) = m_i^{RA*}(W_{99}, Y_{99}, d_{99}^{NA}, I_{98}) \quad (\text{A.1.16})$$

$$\text{value function:} \quad V_{99}(\bar{W}_{99}, \bar{Y}_{99}, d_{99}^{NA}, I_{98}) = k^\gamma V_{99}(W_{99}, Y_{99}, d_{99}^{NA}, I_{98}) \quad (\text{A.1.17})$$

for $W_{99} \geq 0$ and $Y_{99} \geq 0$, $0 \leq d_{99}^{NA} \leq 1$, and I_{98} in the domain of inflation values. Thus, we proved that the relations (3.45)–(3.49) are valid for $t = i$ where $i = 99$. Let us now assume that the relations equivalent to the relations (A.1.10)–(A.1.17) are valid for $t = i + 1$, for some $65 \leq i + 1 \leq 98$. Thus, we assume that if $k \in \mathbb{R}^+$ and

$$\text{wealth:} \quad \bar{W}_{i+1} = kW_{i+1} \quad (\text{A.1.18})$$

$$\text{income:} \quad \bar{Y}_{i+1} = kY_{i+1} \quad (\text{A.1.19})$$

then the solution to the problem (3.26)–(3.34) satisfies the following rules

$$\text{annuitisation coefficient:} \quad \tilde{d}_{i+2}^{NA} = \tilde{d}_{i+2}^{NA} \quad (\text{A.1.20})$$

$$\text{optimal consumption:} \quad \bar{C}_{i+1}^* (\bar{W}_{i+1}, \bar{Y}_{i+1}, d_{i+1}^{NA}, I_i) = kC_{i+1}^* (W_{i+1}, Y_{i+1}, d_{i+1}^{NA}, I_i) \quad (\text{A.1.21})$$

$$\text{optimal asset allocation:} \quad \bar{\alpha}_{i+1}^* (\bar{W}_{i+1}, \bar{Y}_{i+1}, d_{i+1}^{NA}, I_i) = \alpha_{i+1}^* (W_{i+1}, Y_{i+1}, d_{i+1}^{NA}, I_i) \quad (\text{A.1.22})$$

$$\text{optimal NA:} \quad \bar{m}_{i+1}^{NA*} (\bar{W}_{i+1}, \bar{Y}_{i+1}, d_{i+1}^{NA}, I_i) = m_{i+1}^{NA*} (W_{i+1}, Y_{i+1}, d_{i+1}^{NA}, I_i) \quad (\text{A.1.23})$$

$$\text{optimal RA:} \quad \bar{m}_{i+1}^{RA*} (\bar{W}_{i+1}, \bar{Y}_{i+1}, d_{i+1}^{NA}, I_i) = m_{i+1}^{RA*} (W_{i+1}, Y_{i+1}, d_{i+1}^{NA}, I_i) \quad (\text{A.1.24})$$

$$\text{value function:} \quad V_{i+1} (\bar{W}_{i+1}, \bar{Y}_{i+1}, d_{i+1}^{NA}, I_i) = k^\gamma V_{i+1} (W_{i+1}, Y_{i+1}, d_{i+1}^{NA}, I_i) \quad (\text{A.1.25})$$

for $W_{i+1} \geq 0$ and $Y_{i+1} \geq 0$, $0 \leq d_{i+1}^{NA} \leq 1$, and I_i in the domain of the inflation values.

Let us now assume that $t = i$ for some $65 \leq i \leq 99$. We have the following equations

$$V_i (W_i, Y_i, d_i^{NA}, I_{i-1}) = \max_{\{C_i, \alpha_i, m_i^{NA}, m_i^{RA}\}} E_i \left[u(C_i) + \delta(1 - p_i) b_i u(\tilde{W}_{i+1}) + \right. \\ \left. \delta p_i V_{i+1} (\tilde{W}_{i+1}, \tilde{Y}_{i+1}, \tilde{d}_{i+1}^{NA}, \tilde{I}_i) \right] \quad (\text{A.1.26})$$

and

$$V_i (\bar{W}_i, \bar{Y}_i, d_i^{NA}, I_{i-1}) = \max_{\{\bar{C}_i, \bar{\alpha}_i, \bar{m}_i^{NA}, \bar{m}_i^{RA}\}} E_i \left[u(\bar{C}_i) + \delta(1 - p_i) b_i u(\tilde{\bar{W}}_{i+1}) + \right. \\ \left. \delta p_i V_{i+1} (\tilde{\bar{W}}_{i+1}, \tilde{\bar{Y}}_{i+1}, \tilde{\bar{d}}_{i+1}^{NA}, \tilde{\bar{I}}_i) \right] \quad (\text{A.1.27})$$

and also assuming the relations $\bar{W}_i = kW_i$ and $\bar{Y}_i = kY_i$ for some $k \in \mathbb{R}^+$.

Let us firstly derive the formulae for $\tilde{d}_{i+1}^{NA}$ and $\tilde{d}_{i+1}^{NA}$. From (3.10) we have

$$\tilde{d}_{i+1}^{NA} = \frac{\left(\bar{d}_i^{NA} \bar{Y}_i + \frac{\bar{m}_i^{NA*} \bar{W}_i}{a_i^{NA}} \right) (1 + \tilde{I}_i)^{-1}}{\left((1 - \bar{d}_i^{NA}) \rho_i \bar{Y}_i + \frac{\bar{m}_i^{RA*} \bar{W}_i}{a_i^{RA}} + \left(\bar{d}_i^{NA} \bar{Y}_i + \frac{\bar{m}_i^{NA*} \bar{W}_i}{a_i^{NA}} \right) (1 + \tilde{I}_i)^{-1} \right)}$$

From equation (3.7) and our assumption $\bar{W}_i = kW_i$ and $\bar{Y}_i = kY_i$ for some $k \in \mathbb{R}^+$ we easily see that $\bar{d}_i^{NA} = d_i^{NA}$. Knowing this fact and dividing the equation above by k we get

$$\tilde{d}_{i+1}^{NA} = \frac{\left(d_i^{NA}Y_i + \frac{\bar{m}_i^{NA*}W_i}{a_i^{NA}}\right)(1+\tilde{I}_i)^{-1}}{(1-d_i^{NA})\rho_iY_i + \frac{\bar{m}_i^{RA*}W_i}{a_i^{RA}} + \left(d_i^{NA}Y_i + \frac{\bar{m}_i^{NA}W_i}{a_i^{NA}}\right)(1+\tilde{I}_i)^{-1}} \quad (\text{A.1.28})$$

Similarly, we can derive the following formula for $\tilde{d}_{i+1}^{NA}$

$$\tilde{d}_{i+1}^{NA} = \frac{\left(d_i^{NA}Y_i + \frac{m_i^{NA*}W_i}{a_i^{NA}}\right)(1+\tilde{I}_i)^{-1}}{(1-d_i^{NA})\rho_iY_i + \frac{m_i^{RA*}W_i}{a_i^{RA}} + \left(d_i^{NA}Y_i + \frac{m_i^{NA}W_i}{a_i^{NA}}\right)(1+\tilde{I}_i)^{-1}} \quad (\text{A.1.29})$$

Thus, $\tilde{d}_{i+1}^{NA}$ depends on $(d_i^{NA}, W_i, Y_i, \bar{m}_i^{NA*}, \bar{m}_i^{RA*}, \tilde{I}_i, a_i^{NA}, a_i^{RA})$ and $\tilde{d}_{i+1}^{NA}$ depends on $(d_i^{NA}, W_i, Y_i, m_i^{NA*}, m_i^{RA*}, \tilde{I}_i, a_i^{NA}, a_i^{RA})$. The only difference in the variables on which $\tilde{d}_{i+1}^{NA}$ and $\tilde{d}_{i+1}^{NA}$ depend on are the control variables $\bar{m}_i^{NA*}$ and $\bar{m}_i^{RA*}$, and m_i^{NA*} and m_i^{RA*} respectively.

Equation (A.1.26) can be written as

$$\begin{aligned} V_i(W_i, Y_i, d_i^{NA}, I_{i-1}) = & \max_{\{C_i, \alpha_i, m_i^{NA}, m_i^{RA}\}} E_i \left[u(C_i) + \right. \\ & \delta(1-p_i)b_i u \left(\left((1-m_i^{RA}-m_i^{NA})W_i + Y_i - C_i \right) (1+r+\alpha_i(\tilde{r}_i-r)) \right) + \\ & \left. \delta p_i V_{i+1} \left(\left((1-m_i^{RA}-m_i^{NA})W_i + Y_i - C_i \right) (1+r+\alpha_i(\tilde{r}_i-r)), \tilde{Y}_{i+1}, \tilde{d}_{i+1}^{NA}, \tilde{I}_i \right) \right] \end{aligned} \quad (\text{A.1.30})$$

and equation (A.1.27) as

$$\begin{aligned} V_i(\bar{W}_i, \bar{Y}_i, d_i^{NA}, I_{i-1}) = & \max_{\{\bar{C}_i, \bar{\alpha}_i, \bar{m}_i^{NA}, \bar{m}_i^{RA}\}} E_i \left[u(\bar{C}_i) + \right. \\ & \delta(1-p_i)b_i u \left(\left((1-\bar{m}_i^{RA}-\bar{m}_i^{NA})\bar{W}_i + \bar{Y}_i - \bar{C}_i \right) (1+r+\alpha_i(\tilde{r}_i-r)) \right) + \\ & \left. \delta p_i V_{i+1} \left(\left((1-\bar{m}_i^{RA}-\bar{m}_i^{NA})\bar{W}_i + \bar{Y}_i - \bar{C}_i \right) (1+r+\alpha_i(\tilde{r}_i-r)), \tilde{\bar{Y}}_{i+1}, \tilde{\bar{d}}_{i+1}^{NA}, \tilde{\bar{I}}_i \right) \right] \end{aligned}$$

and using (A.1.25) and assumption $\bar{W}_i = kW_i$ and $\bar{Y}_i = kY_i$, we get

$$\begin{aligned}
V_i(\bar{W}_i, \bar{Y}_i, d_i^{NA}, I_{i-1}) = & k^y \max_{\{\bar{C}_i, \bar{\alpha}_i, \bar{m}_i^{NA}, \bar{m}_i^{RA}\}} E_i \left[u \left(\frac{\bar{C}_i}{k} \right) + \right. \\
& \delta(1-p_i) b_i u \left(\left((1-m_i^{RA} - m_i^{NA}) W_i + Y_i - \frac{\bar{C}_i}{k} \right) (1+r+\alpha_i(\tilde{r}_i-r)) \right) + \\
& \left. \delta p_i V_{i+1} \left(\left((1-m_i^{RA} - m_i^{NA}) W_i + Y_i - \frac{\bar{C}_i}{k} \right) (1+r+\alpha_i(\tilde{r}_i-r)), \tilde{Y}_{i+1}, \tilde{d}_{i+1}^{NA}, \tilde{I}_i \right) \right]
\end{aligned} \quad (A.1.31)$$

Let the set of solution $\alpha_i^*(W_i, Y_i, d_i^{NA}, I_{i-1})$, $C_i^*(W_i, Y_i, d_i^{NA}, I_{i-1})$, $m_i^{NA*}(W_i, Y_i, d_i^{NA}, I_{i-1})$ and $m_i^{RA*}(W_i, Y_i, d_i^{NA}, I_{i-1})$ maximises equation (A.1.30) then, knowing formulae (A.1.28) and (A.1.29), the set of equations

$$\begin{aligned}
\frac{\bar{C}_i(\bar{W}_i, \bar{Y}_i, d_i^{NA}, I_{i-1})}{k} &= C_i^*(W_i, Y_i, d_i^{NA}, I_{i-1}) \\
\bar{\alpha}_i(\bar{W}_i, \bar{Y}_i, d_i^{NA}, I_{i-1}) &= \alpha_i^*(W_i, Y_i, d_i^{NA}, I_{i-1}) \\
\bar{m}_i^{NA}(\bar{W}_i, \bar{Y}_i, d_i^{NA}, I_{i-1}) &= m_i^{NA*}(W_i, Y_i, d_i^{NA}, I_{i-1}) \\
\bar{m}_i^{RA}(\bar{W}_i, \bar{Y}_i, d_i^{NA}, I_{i-1}) &= m_i^{RA*}(W_i, Y_i, d_i^{NA}, I_{i-1})
\end{aligned}$$

maximises equation (A.1.31).

Based on mathematical induction we can conclude that if k is positive constant $k \in \mathbb{R}^+$ and if

$$\text{wealth:} \quad \bar{W}_t = kW_t \quad (A.1.32)$$

$$\text{income:} \quad \bar{Y}_t = kY_t \quad (A.1.33)$$

then the solution to the problem (3.26)–(3.34) satisfies the following rules

$$\text{optimal consumption:} \quad C_t^*(\bar{W}_t, \bar{Y}_t, d_t^{NA}, I_{t-1}) = kC_t^*(W_t, Y_t, d_t^{NA}, I_{t-1}) \quad (A.1.34)$$

$$\text{optimal asset allocation:} \quad \alpha_t^*(\bar{W}_t, \bar{Y}_t, d_t^{NA}, I_{t-1}) = \alpha_t^*(W_t, Y_t, d_t^{NA}, I_{t-1}) \quad (A.1.35)$$

$$\text{optimal NA:} \quad m_t^{NA*}(\bar{W}_t, \bar{Y}_t, d_t^{NA}, I_{t-1}) = m_t^{NA*}(W_t, Y_t, d_t^{NA}, I_{t-1}) \quad (A.1.36)$$

$$\text{optimal RA:} \quad m_t^{RA*}(\bar{W}_t, \bar{Y}_t, d_t^{NA}, I_{t-1}) = m_t^{RA*}(W_t, Y_t, d_t^{NA}, I_{t-1}) \quad (A.1.37)$$

$$\text{value function:} \quad V_t(\bar{W}_t, \bar{Y}_t, d_t^{NA}, I_{t-1}) = k^y V_t(W_t, Y_t, d_t^{NA}, I_{t-1}) \quad (A.1.38)$$

for $W_t \geq 0$ and $Y_t \geq 0$, $0 \leq d_t^{NA} \leq 1$, and I_{t-1} in the domain of inflation values, and for t such that $65 \leq t \leq 99$.

Now we see from (A.1.34)–(A.1.38) that these relations are valid for any combination of the values of income and wealth if they satisfy relations (A.1.32) and (A.1.33). Optimal values are actually optimal functions depending on certain variables. So, the

set of equations (A.1.32)–(A.1.38) can be deemed as a set of characteristics which optimal functions satisfy. We can present the result just obtained in equivalent form as follows.

If $\bar{Y}_t$ and Y_t are two different positive values for income then the solution to the problem (3.26)–(3.34) satisfies the following rules

optimal consumption:
$$C_t^*(W_t, \bar{Y}_t, d_t^{NA}, I_{t-1}) = \frac{\bar{Y}_t}{Y_t} C_t^*\left(W_t \frac{Y_t}{\bar{Y}_t}, Y_t, d_t^{NA}, I_{t-1}\right) \quad (\text{A.1.39})$$

optimal asset allocation:
$$\alpha_t^*(W_t, \bar{Y}_t, d_t^{NA}, I_{t-1}) = \alpha_t^*\left(W_t \frac{Y_t}{\bar{Y}_t}, Y_t, d_t^{NA}, I_{t-1}\right) \quad (\text{A.1.40})$$

optimal NA:
$$m_t^{NA*}(W_t, \bar{Y}_t, d_t^{NA}, I_{t-1}) = m_t^{NA*}\left(W_t \frac{Y_t}{\bar{Y}_t}, Y_t, d_t^{NA}, I_{t-1}\right) \quad (\text{A.1.41})$$

optimal RA:
$$m_t^{RA*}(W_t, \bar{Y}_t, d_t^{NA}, I_{t-1}) = m_t^{RA*}\left(W_t \frac{Y_t}{\bar{Y}_t}, Y_t, d_t^{NA}, I_{t-1}\right) \quad (\text{A.1.42})$$

value function:
$$V_t(W_t, \bar{Y}_t, d_t^{NA}, I_{t-1}) = \left(\frac{\bar{Y}_t}{Y_t}\right)^\gamma V_t\left(W_t \frac{Y_t}{\bar{Y}_t}, Y_t, d_t^{NA}, I_{t-1}\right) \quad (\text{A.1.43})$$

and also, if y is positive constant and $w_t = W_t \frac{y}{Y_t}$ then the solution to the problem (3.26)–(3.34) satisfies the following rules

optimal consumption:
$$C_t^*(W_t, Y_t, d_t^{NA}, I_{t-1}) = \frac{Y_t}{y} C_t^*(w_t, y, d_t^{NA}, I_{t-1}) \quad (\text{A.1.44})$$

optimal asset allocation:
$$\alpha_t^*(W_t, Y_t, d_t^{NA}, I_{t-1}) = \alpha_t^*(w_t, y, d_t^{NA}, I_{t-1}) \quad (\text{A.1.45})$$

optimal NA:
$$m_t^{NA*}(W_t, Y_t, d_t^{NA}, I_{t-1}) = m_t^{NA*}(w_t, y, d_t^{NA}, I_{t-1}) \quad (\text{A.1.46})$$

optimal RA:
$$m_t^{RA*}(W_t, Y_t, d_t^{NA}, I_{t-1}) = m_t^{RA*}(w_t, y, d_t^{NA}, I_{t-1}) \quad (\text{A.1.47})$$

value function:
$$V_t(W_t, Y_t, d_t^{NA}, I_{t-1}) = \left(\frac{Y_t}{y}\right)^\gamma V_t(w_t, y, d_t^{NA}, I_{t-1}) \quad (\text{A.1.48})$$

for $W_t \geq 0$ and $Y_t \geq 0$, $0 \leq d_t^{NA} \leq 1$, and I_{t-1} in the domain of inflation values, and for t such that $65 \leq t \leq 99$.

A.2 Appendix 2 – Income as State Variable in the Interest Rate Risk Model

We will now prove the relation between solutions (4.65)–(4.69) of the problem (4.48)–(4.55) for different values of income variable. We will prove the general relations amongst solutions if we change the income variable values, and then using

this result we will show that it is possible to transform the solution for constant income into any value of income. Using this result it is possible to solve the problem for one income value, and it means that it is not necessary to have income variable as state variable. Thus, we decrease the number of states variable for one. It is very useful for a numerical solution because using these results we can work with a more suitable value of income and then make transformation for any required income values. Again, we prove the most general case with four control variables and other cases are then special cases of the general solution.

We exclude writing index j that appears in (4.65)–(4.69) as subscript in interest rate variable and just assume that interest rate variable takes values in the domain of the interest rate variable.

We apply mathematical induction in order to prove the relations (4.65)–(4.69).

Let us firstly prove that the relations (4.65)–(4.69) are valid for $t = i$ where $i = 99$. For some fixed income Y_{99} and wealth W_{99} we have the solution

$$\text{optimal consumption:} \quad C_{99}^*(W_{99}, Y_{99}, r_{98}) \quad (\text{A.2.1})$$

$$\text{optimal equity allocation:} \quad \alpha_{99}^{e*}(W_{99}, Y_{99}, r_{98}) \quad (\text{A.2.2})$$

$$\text{optimal bond allocation:} \quad \alpha_{99}^{b*}(W_{99}, Y_{99}, r_{98}) \quad (\text{A.2.3})$$

$$\text{optimal annuitisation:} \quad m_{99}^*(W_{99}, Y_{99}, r_{98}) = 0 \quad (\text{A.2.4})$$

$$\text{value function:} \quad V_{99}(W_{99}, Y_{99}, r_{98}) \quad (\text{A.2.5})$$

for $W_{99} \geq 0$ and $Y_{99} \geq 0$, and r_{98} in the domain of the real interest rate. This solution exists because we are looking for the maximum of the continuous function on the compact set. The solution is unique as well. As we said there is no annuitisation at this age.

Let us now assume that we have some other income $\bar{Y}_{99} = kY_{99}$ and wealth $\bar{W}_{99} = kW_{99}$, for some positive constant $k \in \mathbb{R}^+$. Let us introduce a new variable $\tilde{\bar{W}}_{100}$, and a new control variables $\bar{C}_{99}$, $\bar{\alpha}_{99}^e$ and $\bar{\alpha}_{99}^b$. There is no annuitisation during the last period, so again $\bar{m}_{99} = 0$. Now the problem equivalent to the problem (4.48)–(4.55) but with wealth $\bar{W}_{99}$ and income $\bar{Y}_{99}$ can be written as

$$V_{99}(\bar{W}_{99}, \bar{Y}_{99}, r_{98}) = \max_{\{\bar{C}_{99}, \bar{\alpha}_{99}^e, \bar{\alpha}_{99}^b\}} E_{99} \left[u(\bar{C}_{99}) + \delta(1 - p_{99}) b_{99} u(\tilde{\bar{W}}_{100}) \right]$$

where

$$\tilde{\bar{W}}_{100} = \left((1 - m_{99}) \bar{W}_{99} + \bar{Y}_{99} - \bar{C}_{99} \right) (1 + \tilde{r}_{99}^P) \quad (\text{A.2.6})$$

$$\tilde{r}_{99}^P = \bar{r}_{99} + \bar{\alpha}_{99}^e (\tilde{r}_{99}^e - \bar{r}_{99}) + \bar{\alpha}_{99}^b \left(\frac{B(\Upsilon_t - 1, \tilde{r}_{99})}{B(\Upsilon_t, r_{98})} - 1 - \bar{r}_{99} \right) \quad (\text{A.2.7})$$

$$Y_{100} = 0 \quad (\text{A.2.8})$$

and the constraints are

$$0 \leq \bar{C}_{99} \leq (1 - m_{99}) \bar{W}_{99} + \bar{Y}_{99} \quad (\text{A.2.9})$$

$$0 \leq \bar{\alpha}_{99}^e \leq 1, \quad 0 \leq \bar{\alpha}_{99}^b \leq 1, \quad \text{and} \quad 0 \leq \bar{\alpha}_{99}^e + \bar{\alpha}_{99}^b \leq 1 \quad (\text{A.2.10})$$

$$m_{99} = 0. \quad (\text{A.2.11})$$

Now, we can write

$$\begin{aligned} V_{99}(\bar{W}_{99}, \bar{Y}_{99}, r_{98}) &= \max_{\{\bar{C}_{99}, \bar{\alpha}_{99}^e, \bar{\alpha}_{99}^b\}} E_{99} \left[u(\bar{C}_{99}) + \delta(1 - p_{99}) b_{99} u \left(\left((1 - m_{99}) \bar{W}_{99} + \bar{Y}_{99} - \bar{C}_{99} \right) (1 + \tilde{r}_{99}^P) \right) \right] \\ &= k^\gamma \max_{\{\bar{C}_{99}, \bar{\alpha}_{99}^e, \bar{\alpha}_{99}^b\}} E_{99} \left[u \left(\frac{\bar{C}_{99}}{k} \right) + \delta(1 - p_{99}) b_{99} u \left(\left((1 - m_{99}) \bar{W}_{99} + \bar{Y}_{99} - \frac{\bar{C}_{99}}{k} \right) (1 + \tilde{r}_{99}^P) \right) \right] \end{aligned}$$

Knowing that k^γ is positive constant and that the control variables $\{\bar{C}_{99}^*, \bar{\alpha}_{99}^{e*}, \bar{\alpha}_{99}^{b*}\}$ which provide the optimal solution are unique, from the equation above we can conclude that $\{\bar{C}_{99}^*, \bar{\alpha}_{99}^{e*}, \bar{\alpha}_{99}^{b*}\} = \{kC_{99}^*, \alpha_{99}^{e*}, \alpha_{99}^{b*}\}$. It means that

$$\begin{aligned} \frac{\bar{C}_{99}^* (\bar{W}_{99}, \bar{Y}_{99}, r_{98})}{k} &= C_{99}^* (W_{99}, Y_{99}, r_{98}), \\ \bar{\alpha}_{99}^{e*} (\bar{W}_{99}, \bar{Y}_{99}, r_{98}) &= \alpha_{99}^{e*} (W_{99}, Y_{99}, r_{98}), \\ \bar{\alpha}_{99}^{b*} (\bar{W}_{99}, \bar{Y}_{99}, r_{98}) &= \alpha_{99}^{b*} (W_{99}, Y_{99}, r_{98}), \quad \text{and also} \\ \bar{V}_{99} (\bar{W}_{99}, \bar{Y}_{99}, r_{98}) &= k^\gamma V_{99} (\bar{W}_{99}, \bar{Y}_{99}, r_{98}). \end{aligned}$$

It means that if $k \in \mathbb{R}^+$ and

$$\text{wealth:} \quad \bar{W}_{99} = kW_{99} \quad (\text{A.2.12})$$

$$\text{income:} \quad \bar{Y}_{99} = kY_{99} \quad (\text{A.2.13})$$

then the solution to the problem (4.48)–(4.55) satisfies the following rules

$$\text{optimal consumption:} \quad \bar{C}_{99}^* (\bar{W}_{99}, \bar{Y}_{99}, r_{98}) = kC_{99}^* (W_{99}, Y_{99}, r_{98}) \quad (\text{A.2.14})$$

$$\text{optimal equity allocation:} \quad \bar{\alpha}_{99}^{e*} (\bar{W}_{99}, \bar{Y}_{99}, r_{98}) = \alpha_{99}^{e*} (W_{99}, Y_{99}, r_{98}) \quad (\text{A.2.15})$$

$$\text{optimal bond allocation:} \quad \bar{\alpha}_{99}^{b*} (\bar{W}_{99}, \bar{Y}_{99}, r_{98}) = \alpha_{99}^{b*} (W_{99}, Y_{99}, r_{98}) \quad (\text{A.2.16})$$

$$\text{optimal annuitisation:} \quad \bar{m}_{99}^* (\bar{W}_{99}, \bar{Y}_{99}, r_{98}) = m_{99}^* (W_{99}, Y_{99}, r_{98}) \quad (\text{A.2.17})$$

$$\text{value function:} \quad \bar{V}_{99} (\bar{W}_{99}, \bar{Y}_{99}, r_{98}) = k^\gamma V_{99} (W_{99}, Y_{99}, r_{98}) \quad (\text{A.2.18})$$

for $W_{99} \geq 0$ and $Y_{99} \geq 0$, and r_{98} in the domain of the interest rate.

Let us now assume that the relations equivalent to the relations (A.2.12)–(A.2.18) are valid for $t = i + 1$, for some $65 \leq i + 1 \leq 98$. Thus, we assume that if $k \in \mathbb{R}^+$ and

$$\text{wealth:} \quad \bar{W}_{i+1} = kW_{i+1} \quad (\text{A.2.19})$$

$$\text{income:} \quad \bar{Y}_{i+1} = kY_{i+1} \quad (\text{A.2.20})$$

then the solution to the problem (4.48)–(4.55) satisfies the following rules

$$\text{optimal consumption:} \quad \bar{C}_{i+1}^* (\bar{W}_{i+1}, \bar{Y}_{i+1}, r_i) = kC_{i+1}^* (W_{i+1}, Y_{i+1}, r_i) \quad (\text{A.2.21})$$

$$\text{optimal equity allocation:} \quad \bar{\alpha}_{i+1}^{e*} (\bar{W}_{i+1}, \bar{Y}_{i+1}, r_i) = \alpha_{i+1}^{e*} (W_{i+1}, Y_{i+1}, r_i) \quad (\text{A.2.22})$$

$$\text{optimal bond allocation:} \quad \bar{\alpha}_{i+1}^{b*} (\bar{W}_{i+1}, \bar{Y}_{i+1}, r_i) = \alpha_{i+1}^{b*} (W_{i+1}, Y_{i+1}, r_i) \quad (\text{A.2.23})$$

$$\text{optimal annuitisation:} \quad \bar{m}_{i+1}^* (\bar{W}_{i+1}, \bar{Y}_{i+1}, r_i) = m_{i+1}^* (W_{i+1}, Y_{i+1}, r_i) \quad (\text{A.2.24})$$

$$\text{value function:} \quad V_{i+1} (\bar{W}_{i+1}, \bar{Y}_{i+1}, r_i) = k^\gamma V_{i+1} (W_{i+1}, Y_{i+1}, r_i) \quad (\text{A.2.25})$$

for $W_{i+1} \geq 0$ and $Y_{i+1} \geq 0$, and r_i in the domain of the interest rate.

Let us now assume that $t = i$ for some $65 \leq i \leq 99$. We will prove that if for some $k \in \mathbb{R}^+$ we define $\bar{W}_i = kW_i$ and $\bar{Y}_i = kY_i$ then the relations (4.65)–(4.69) are valid. We have the following equations

$$V_i (W_i, Y_i, r_{i-1}) = \max_{\{C_i, \alpha_i^e, \alpha_i^b, m_i\}} E_i \left[u(C_i) + \delta(1-p_i)b_i u(\tilde{W}_{i+1}) + \delta p_i V_{i+1} (\tilde{W}_{i+1}, Y_{i+1}, \tilde{r}_i) \right] \quad (\text{A.2.26})$$

and

$$V_i (\bar{W}_i, \bar{Y}_i, r_{i-1}) = \max_{\{\bar{C}_i, \bar{\alpha}_i^e, \bar{\alpha}_i^b, \bar{m}_i\}} E_i \left[u(\bar{C}_i) + \delta(1-p_i)b_i u(\tilde{\bar{W}}_{i+1}) + \delta p_i V_{i+1} (\tilde{\bar{W}}_{i+1}, \bar{Y}_{i+1}, \tilde{r}_i) \right] \quad (\text{A.2.27})$$

where $\bar{W}_i = kW_i$ and $\bar{Y}_i = kY_i$ for some $k \in \mathbb{R}^+$.

Using (4.49), equation (A.2.26) can be written as

$$\begin{aligned} V_i (W_i, Y_i, r_{i-1}) = \max_{\{C_i, \alpha_i^e, \alpha_i^b, m_i\}} E_i \left[u(C_i) + \right. \\ \left. \delta(1-p_i)b_i u\left(\left((1-m_i)W_i + Y_i - C_i\right)(1+\tilde{r}_i^P)\right) + \right. \\ \left. \delta p_i V_{i+1} \left(\left((1-m_i)W_i + Y_i - C_i\right)(1+\tilde{r}_i^P), Y_{i+1}, \tilde{r}_i\right) \right] \end{aligned} \quad (\text{A.2.28})$$

and using (A.2.25) and assumption $\bar{W}_i = kW_i$ and $\bar{Y}_i = kY_i$, (A.2.27) can be written as

$$\begin{aligned}
V_i(\bar{W}_i, \bar{Y}_i, r_{i-1}) = & k^\gamma \max_{\{\bar{C}_i, \bar{\alpha}_i^e, \bar{\alpha}_i^b, \bar{m}_i\}} E_i \left[u \left(\frac{\bar{C}_i}{k} \right) + \right. \\
& \delta(1-p_i) b_i u \left(\left((1-\bar{m}_i) W_i + Y_i - \frac{\bar{C}_i}{k} \right) (1+\tilde{r}_i^P) \right) + \quad (\text{A.2.29}) \\
& \left. \delta p_i V_{i+1} \left(\left((1-\bar{m}_i) W_i + Y_i - \frac{\bar{C}_i}{k} \right) (1+\tilde{r}_i^P), Y_{i+1}, \tilde{r}_i \right) \right]
\end{aligned}$$

Knowing that k^γ is positive constant and that the control variables $\{\bar{C}_i^*, \bar{\alpha}_i^{e*}, \bar{\alpha}_i^{b*}, \bar{m}_i^*\}$ which provide the optimal solution are unique, from equations (A.2.28) and (A.2.29) we can conclude that $\{\bar{C}_i^*, \bar{\alpha}_i^{e*}, \bar{\alpha}_i^{b*}, \bar{m}_i^*\} = \{kC_i^*, \alpha_i^{e*}, \alpha_i^{b*}, m_i^*\}$ are optimal control variables for equation (A.2.27). It means the solution to the problem (4.48)–(4.55) for $\bar{W}_i = kW_i$ and $\bar{Y}_i = kY_i$ for some $k \in \mathbb{R}^+$ is given by

$$\begin{aligned}
\frac{\bar{C}_i^*(\bar{W}_i, \bar{Y}_i, r_{i-1})}{k} &= C_i^*(W_i, Y_i, r_{i-1}) \\
\bar{\alpha}_i^{e*}(\bar{W}_i, \bar{Y}_i, r_{i-1}) &= \alpha_i^{e*}(W_i, Y_i, r_{i-1}) \\
\bar{\alpha}_i^{b*}(\bar{W}_i, \bar{Y}_i, r_{i-1}) &= \alpha_i^{b*}(W_i, Y_i, r_{i-1}) \\
\bar{m}_i^*(\bar{W}_i, \bar{Y}_i, r_{i-1}) &= m_i^*(W_i, Y_i, r_{i-1})
\end{aligned}$$

Based on the mathematical induction we have just proved that if $k \in \mathbb{R}^+$ and if

$$\text{wealth:} \quad \bar{W}_t = kW_t \quad (\text{A.2.30})$$

$$\text{income:} \quad \bar{Y}_t = kY_t \quad (\text{A.2.31})$$

then the solution to the problem (4.48)–(4.55) satisfies the following rules

$$\text{optimal consumption:} \quad C_t^*(\bar{W}_t, \bar{Y}_t, r_{t-1}) = kC_t^*(W_t, Y_t, r_{t-1}) \quad (\text{A.2.32})$$

$$\text{optimal equity allocation:} \quad \alpha_t^{e*}(\bar{W}_t, \bar{Y}_t, r_{t-1}) = \alpha_t^{e*}(W_t, Y_t, r_{t-1}) \quad (\text{A.2.33})$$

$$\text{optimal bond allocation:} \quad \alpha_t^{b*}(\bar{W}_t, \bar{Y}_t, r_{t-1}) = \alpha_t^{b*}(W_t, Y_t, r_{t-1}) \quad (\text{A.2.34})$$

$$\text{optimal annuitisation:} \quad m_t^*(\bar{W}_t, \bar{Y}_t, r_{t-1}) = m_t^*(W_t, Y_t, r_{t-1}) \quad (\text{A.2.35})$$

$$\text{value function:} \quad V_t(\bar{W}_t, \bar{Y}_t, r_{t-1}) = k^\gamma V_t(W_t, Y_t, r_{t-1}) \quad (\text{A.2.36})$$

for $W_t \geq 0$ and $Y_t \geq 0$, and r_{t-1} in the domain of the interest rate and for any t such that $65 \leq t \leq 99$.

Set of equation (A.2.30)–(A.2.36) can also be written in the following useful form. If W_t is wealth, and $\bar{Y}_t$ and Y_t are two different positive values of income then the solution to the problem (4.48)–(4.55) satisfies the following rules

$$\text{optimal consumption:} \quad C_t^*(W_t, \bar{Y}_t, r_{t-1}) = \frac{\bar{Y}_t}{Y_t} C_t^*\left(W_t \frac{Y_t}{\bar{Y}_t}, Y_t, r_{t-1}\right) \quad (\text{A.2.37})$$

$$\text{optimal equity allocation:} \quad \alpha_t^{e*}(W_t, \bar{Y}_t, r_{t-1}) = \alpha_t^{e*}\left(W_t \frac{Y_t}{\bar{Y}_t}, Y_t, r_{t-1}\right) \quad (\text{A.2.38})$$

$$\text{optimal bond allocation:} \quad \alpha_t^{b*}(W_t, \bar{Y}_t, r_{t-1}) = \alpha_t^{b*}\left(W_t \frac{Y_t}{\bar{Y}_t}, Y_t, r_{t-1}\right) \quad (\text{A.2.39})$$

$$\text{optimal annuitisation:} \quad m_t^*(W_t, \bar{Y}_t, r_{t-1}) = m_t^*\left(W_t \frac{Y_t}{\bar{Y}_t}, Y_t, r_{t-1}\right) \quad (\text{A.2.40})$$

$$\text{value function:} \quad V_t(W_t, \bar{Y}_t, r_{t-1}) = \left(\frac{\bar{Y}_t}{Y_t}\right)^\gamma V_t\left(W_t \frac{Y_t}{\bar{Y}_t}, Y_t, r_{t-1}\right) \quad (\text{A.2.41})$$

for $W_t \geq 0$ and $Y_t \geq 0$, and r_{t-1} in the domain of the interest rate and for $65 \leq t \leq 99$.

Also, if y is positive constant and $w_t = W_t \frac{y}{Y_t}$ then the solution to the problem (4.48)–(4.55) satisfies the following rules

$$\text{optimal consumption:} \quad C_t^*(W_t, Y_t, r_{t-1}) = \frac{Y_t}{y} C_t^*(w_t, y, r_{t-1}) \quad (\text{A.2.42})$$

$$\text{optimal equity allocation:} \quad \alpha_t^{e*}(W_t, Y_t, r_{t-1}) = \alpha_t^{e*}(w_t, y, r_{t-1}) \quad (\text{A.2.43})$$

$$\text{optimal bond allocation:} \quad \alpha_t^{b*}(W_t, Y_t, r_{t-1}) = \alpha_t^{b*}(w_t, y, r_{t-1}) \quad (\text{A.2.44})$$

$$\text{optimal annuitisation:} \quad m_t^*(W_t, Y_t, r_{t-1}) = m_t^*(w_t, y, r_{t-1}) \quad (\text{A.2.45})$$

$$\text{value function:} \quad V_t(W_t, Y_t, r_{t-1}) = \left(\frac{Y_t}{y}\right)^\gamma V_t(w_t, y, r_{t-1}) \quad (\text{A.2.46})$$

for $W_t \geq 0$ and $Y_t \geq 0$, and r_{t-1} in the domain of the interest rate and for $65 \leq t \leq 99$.

A.3 Appendix 3 – Bond Prices Analysis

In Appendix 3, we firstly derive the formula for the exact value of bond prices in discrete time and continuous state space. Then, we compare bond prices derived from the Vasicek model (continuous time and state spaces), from the first approximation of the Vasicek model (discrete time and continuous state spaces) and from the second approximation of the Vasicek model (discrete time and state spaces). In the interest rate risk model in Chapter 4 we use the second approximation of the Vasicek model as we work in discrete time and state spaces. This Appendix is intended to give the idea of the changes in bond prices due to the approximation. We will not try to evaluate the quality of approximation by any criteria, just to give comparable bond prices values.

A.3.1 Exact Bond Prices for Discrete Time and Continuous Space AR(1) Process

Equation (4.20) for the discrete time and continuous state spaces AR(1) process defined in (4.7) can be solved exactly. Having solved equation (4.20), we multiply it by the factor

$$e^{-\frac{\sigma_r \lambda_r}{b} \left(\frac{1-e^{-Tb}}{b} - T \right)} \quad (\text{A.3.1})$$

for $T \in \mathbb{N}$ and get the exact bond prices in the first approximation of the Vasicek model, where we have discrete time and continuous state spaces. Although we will not use this solution in the numerical solution of the main problem of the optimal asset allocation, it will be of use as an indication that the approximation of bond market in Chapter 4 is acceptable.

For $T = 1$ equation (4.20) in discrete time and continuous state spaces can be written as

$$\begin{aligned} \bar{B}(1, r_0) &= E \left[e^{-r_1} \right] \\ &= \int_{-\infty}^{\infty} e^{-r_1} f(r_1 | r_0) dr_1 \end{aligned} \quad (\text{A.3.2})$$

Knowing that r_1 is normally distributed with mean and variance defined in (4.9) and (4.10) respectively, we have that

$$B(1, r_0) = \frac{1}{2\pi\sqrt{\text{Var}[r_1 | r_0]}} \int_{-\infty}^{\infty} e^{-r_1} e^{-\frac{(r_1 - E[r_1 | r_0])^2}{2\text{Var}[r_1 | r_0]}} dr_1 \quad (\text{A.3.3})$$

Then, knowing that we can write

$$\begin{aligned} \bar{B}(2, r_0) &= E \left[e^{-r_1} e^{-r_2} | r_0 \right] \\ &= E \left[e^{-r_2} E \left[e^{-r_1} | r_0 \right] | r_0 \right] \\ &= E \left[e^{-r_2} \bar{B}(1, r_0) | r_0 \right] \end{aligned}$$

As we know that r_2 is normal random variable, we can derive the solution of the last equation. Having the solution $\bar{B}(2, r_0)$ and multiplying it with factor defined in (A.3.1) for $T = 2$ we get the bond price with the duration of two years for any for $r_0 \in \mathbb{R}$.

Continuing this process, we can calculate any $\bar{B}(T, r_0)$, for $T \in \mathbb{N}$. Multiplying $\bar{B}(T, r_0)$ with factor defined in (A.3.1) we get bond prices for any duration and any $r_0 \in \mathbb{R}$.

A.3.2 Examples of Comparable Bond Prices

We can calculate bond prices derived from the Vasicek model, bond prices derived from the first approximation of the Vasicek model where the time space is discrete and the state space is continuous and bond prices from the second approximation of the Vasicek model where the time and state spaces are both discrete. There is a requirement to have certain relations between bond prices if we want to have sound model. One way to check the soundness of the bond market model is to compare bond prices derived using the three models for interest rate. We expect these bond prices to have similar values. The second important thing we need to have in order to deem the bond prices model sound is to have the same pattern when bond prices are compared in each model. It means that we expect decreasing bond prices as the value of the interest rate during the previous year increases.

Tables A.3.1 below shows the prices of zero-coupon bonds with the duration of five and ten years and different values of the interest rate during the previous year, for discrete time and state spaces, for discrete time and continuous state space, and for the Vasicek model. The model in Chapter 4 allows any duration of zero-coupon rolling bond, but our main results will be for the duration of 10 years. We assume that the number of states $N=15$, and that the end points for the abscissa are -2.44% and 6.44% . Other parameter values are chosen to be the same or similar to the values used in the numerical results in Chapter 4.

	Interest rate	Duration 5 year			Duration 10 year		
		Discrete time/state spaces, numerical solution	Discrete time continuous state spaces	Continuous time and state spaces, Vasicek	Discrete time/state spaces, numerical solution	Discrete time continuous state spaces	Continuous time and state spaces, Vasicek
1	-2.44%	92,96	93,78	95,55	82,35	83,20	84,84
2	-2.21%	92,79	93,53	95,21	82,19	82,97	84,52
3	-1.81%	92,49	93,10	94,60	81,91	82,57	83,96
4	-1.25%	92,04	92,50	93,77	81,50	82,01	83,19
5	-0.56%	91,44	91,77	92,75	80,96	81,33	82,24
6	0.22%	90,73	90,93	91,59	80,30	80,54	81,16
7	1.09%	89,92	90,02	90,34	79,56	79,70	79,99
8	2.00%	89,06	89,08	89,05	78,77	78,83	78,80
9	2.91%	88,21	88,15	87,79	77,99	77,97	77,62
10	3.78%	87,42	87,27	86,59	77,27	77,15	76,51
11	4.56%	86,73	86,48	85,50	76,64	76,41	75,50
12	5.25%	86,17	85,79	84,57	76,12	75,77	74,64
13	5.81%	85,74	85,24	83,83	75,73	75,26	73,95
14	6.21%	85,46	84,84	83,30	75,47	74,90	73,46
15	6.44%	85,30	84,62	83,00	75,33	74,69	73,18

Table A.3.1 Bond prices for the parameters for the Vasicek model are as follows $a = 0.012$, $b = 0.6$ $\sigma = 0.02$ and $\lambda_r = 0,1528$. Then $a_d = 0.00902377$, $b_d = 0.451188$ and $\sigma_d = 0.0152622$.

We see that long term expected values $a/b = 0.02$ as well as $a_d/b_d = 0.02$, as we expected. When we compare bond prices with the same duration in each row we see similar values. For both chosen durations, we can see the biggest range of bond prices is for the Vasicek model and the lowest is for discrete time and state spaces. However, observing the columns for the first and for the second approximation of the Vasicek model we can say that bond prices behave quite reasonably in terms of changes as function of the value of the interest rate during the previous year.

In Table A.3.2 we present the values of the rates of return on 10 year rolling bonds during one year assuming the value of the interest rate during the previous year being -1.25% and 2.00% .

	Interest Rate	$\frac{B(9, r_1)}{B(10, -1.25\%)} - 1$ in %			$\frac{B(9, r_1)}{B(10, 2.00\%)} - 1$ in %		
		Discrete time/state spaces, numerical solution	Discrete time continuous state spaces	Continuous time and state spaces, Vasicek	Discrete time/state spaces, numerical solution	Discrete time continuous state spaces	Continuous time and state spaces, Vasicek
1	-2.44%	3,55	3,96	4,51	7,14	8,15	10,33
2	-2.21%	3,36	3,67	4,11	6,94	7,85	9,91
3	-1.81%	3,01	3,17	3,42	6,57	7,33	9,18
4	-1.25%	2,49	2,47	2,47	6,04	6,61	8,18
5	-0.56%	1,81	1,62	1,30	5,33	5,72	6,95
6	0.22%	0,98	0,65	-0,02	4,48	4,71	5,54
7	1.09%	0,05	-0,41	-1,45	3,51	3,61	4,03
8	2.00%	-0,94	-1,49	-2,92	2,49	2,48	2,48
9	2.91%	-1,92	-2,57	-4,37	1,48	1,36	0,95
10	3.78%	-2,83	-3,59	-5,74	0,54	0,30	-0,49
11	4.56%	-3,62	-4,51	-6,97	-0,28	-0,66	-1,79
12	5.25%	-4,27	-5,31	-8,03	-0,95	-1,49	-2,91
13	5.81%	-4,76	-5,94	-8,88	-1,46	-2,15	-3,81
14	6.21%	-5,09	-6,40	-9,48	-1,80	-2,62	-4,44
15	6.44%	-5,27	-6,66	-9,83	-1,99	-2,89	-4,80

Table A.3.2 Rates on 10 year rolling bonds during one year assuming the value of the interest rate during the previous year is -1.25% and 2.00% , and the value of interest the rate in the following year given in the first column. The other parameters are the same as in the example in Table A.3.1.

We suppose here that at the beginning of the year we know the value of the interest rate in the previous year and that the 10 year zero coupon bond is priced according to that value. This known value of the interest rate is written in the header, and we present examples for the two value $r_0 = -1.25\%$ and $r_0 = 2.00\%$. Then we suppose that during the following year the value of the interest rate r_1 appears to be as in the first column. At the end of the year we have the price of the 9 year bond and calculate the rate of return on 10 year rolling bonds by

$$\frac{B(9, r_1)}{B(10, r_0)} - 1.$$

We can see in Table A.3.2 that the rates of return on 10 year rolling bond investment are the highest for the Vasicek model, the lower for the first approximation and the

lowest for the second approximation. It means that in our examples, the variability of bond investment rates is lower compared to the Vasicek model. However, at the same time we can see a regular behaviour of returns for both approximations. If σ takes lower values than 0.02, then we get the rates on ten years rolling bond investment using approximations that are more similar to the rates calculated from the Vasicek model.

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